BK
TP.HCM

Cryptography and Network Security

Chapter 3

Public Key Cryptography

Lectured by
Nguyễn Đức Thái

Outline

- Number theory overview
- Public key cryptography
- RSA algorithm

Prime Numbers

- A prime number is ***an integer*** that can only be divided without remainder by positive and negative values of **itself** and **1**.
- Prime numbers play a critical role both in number theory and in cryptography.

Relatively Prime Numbers & GCD

- two numbers ***a***, ***b*** are relatively prime if they have no common divisors apart from 1
- Example: 8 & 15 are relatively prime since factors of 8 are 1,2,4,8 and of 15 are 1,3,5,15 and 1 is the only common factor
- Conversely can determine the Greatest Common Divisor by comparing their prime factorizations and using least powers
- Example: $300=2^2 x 3^1 x 5^2$

  $18=2^1 x 3^2$

  hence $GCD(18,300)=2^1 x 3^1 x 5^0=6$

Fermat's Theorem

- Fermat's theorem states the following: If ***p*** is prime and is ***a*** positive integer not divisible by ***p***, then

$$a^{p-1} = 1 \pmod{p}$$

- also known as Fermat's Little Theorem
- also have: $a^p = a \pmod{p}$
- useful in public key and primality testing

Public Key Encryption

- ***Asymmetric encryption*** is a form of cryptosystem in which ***encryption*** and ***decryption*** are performed using the different keys
  - a public key
  - a private key.

- It is also known as ***public-key encryption***

Public Key Encryption

- Asymmetric encryption transforms ***plaintext*** into ***ciphertext*** using a *one of two keys* and *an encryption algorithm*.
- Using the paired key and a decryption algorithm, the plaintext is recovered from the ciphertext
- Asymmetric encryption can be used for ***confidentiality***, ***authentication***, or ***both***.
- The most widely used public-key cryptosystem is **RSA**.
- The difficulty of attacking RSA is based on the difficulty of finding the prime factors of a composite
- number.

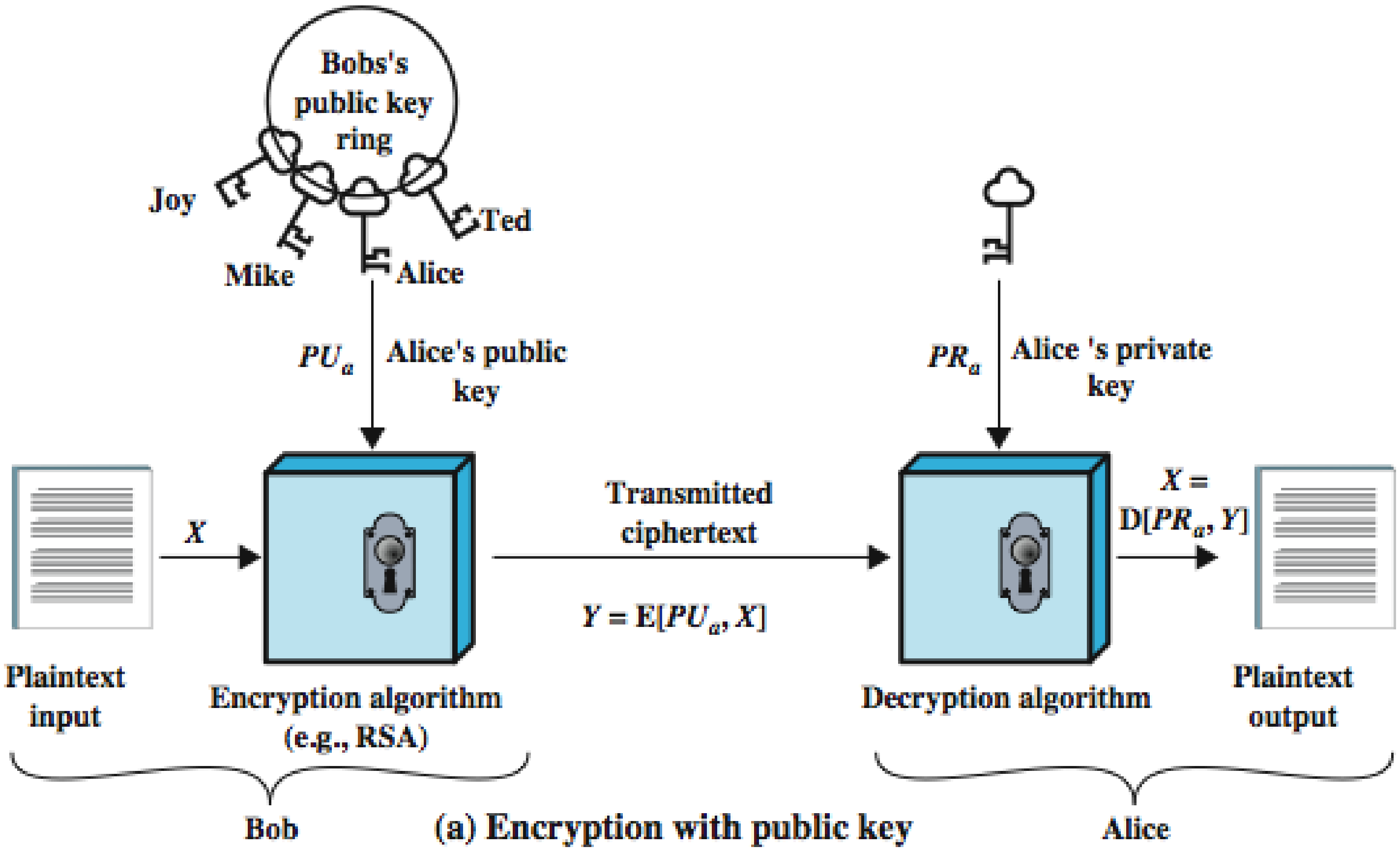
Why Public Key Cryptography?

- Developed to address ***two key issues***:
  - **key distribution** – how to have secure communications in general without having to trust a KDC with your key
  - **digital signatures** – how to verify a message comes intact from the claimed sender
- Public invention due to Whitfield Diffie & Martin Hellman at Stanford University in 1976
  - known earlier in classified community

Public Key Cryptography

- **public-key/two-key/asymmetric cryptography involves the use of two keys:**
  - a ***public-key***, which may be known by anybody, and can be used to **encrypt messages**, and **verify signatures**
  - a related ***private-key***, known only to the recipient, used to **decrypt messages**, and **sign** (create) **signatures**
- **Infeasible to determine private key from public**
- **is asymmetric because**
  - those who encrypt messages or verify signatures **cannot** decrypt messages or create signatures

Public Key Cryptography

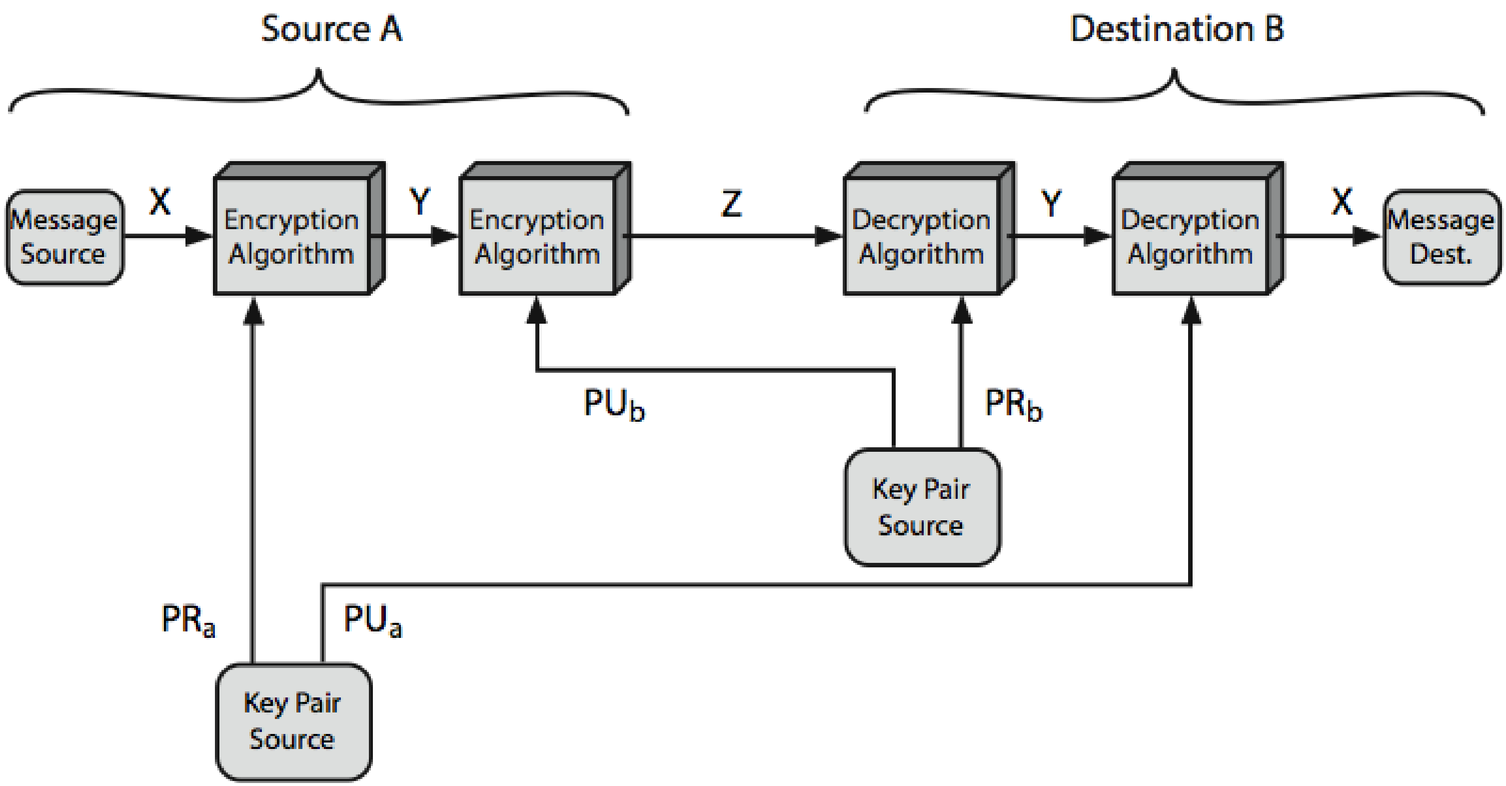


(a) Encryption with public key

Symmetric vs. Public Key

Conventional Encryption	Public-Key Encryption
Needed to Work: 1. The same algorithm with the same key is used for encryption and decryption. 2. The sender and receiver must share the algorithm and the key. *Needed for Security:* 1. The key must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus samples of ciphertext must be insufficient to determine the key.	*Needed to Work:* 1. One algorithm is used for encryption and decryption with a pair of keys, one for encryption and one for decryption. 2. The sender and receiver must each have one of the matched pair of keys (not the same one). *Needed for Security:* 1. One of the two keys must be kept secret. 2. It must be impossible or at least impractical to decipher a message if no other information is available. 3. Knowledge of the algorithm plus one of the keys plus samples of ciphertext must be insufficient to determine the other key.

Public Key Cryptosystems



Public Key Applications

- can classify uses into ***3 categories***:
  - **encryption/decryption** (provide secrecy)
  - **digital signatures** (provide authentication)
  - **key exchange** (of session keys)
- some algorithms are suitable for all uses, others are specific to one

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Elliptic Curve	Yes	Yes	Yes
Diffie-Hellman	No	No	Yes
DSS	No	Yes	No

Public Key Requirements

- **Public-Key algorithms rely on two keys where:**
  - it is computationally ***infeasible*** to find decryption key knowing only algorithm & encryption key
  - it is computationally ***easy to en/decrypt messages*** when the relevant (en/decrypt) key is known
  - either of the two related keys can be used for encryption, with the other used for decryption (for some algorithms)

Public Key Requirements

- **need a trap-door one-way function**
- **one-way function has**
  - $Y = f(X)$ easy
  - $X = f^{-1}(Y)$ infeasible
- **a trap-door one-way function has**
  - $Y = f_k(X)$ easy, if k and X are known
  - $X = f_k^{-1}(Y)$ easy, if k and Y are known
  - $X = f_k^{-1}(Y)$ infeasible, if Y known but k not known
- **a practical public-key scheme depends on a suitable trap-door one-way function**

Security of Public Key Schemes

- Like symmetric encryption, a public-key encryption scheme is ***vulnerable to a brute-force attack***
- The difference is, keys used are too large (>512bits)
- Requires the use of ***very large numbers***
- ***Slow*** compared to private key schemes

RSA

- by **R**ivest, **S**hamir & **A**dleman of MIT in 1977
- best known & widely used public-key scheme
- based on exponentiation in a finite (Galois) field over integers modulo a prime
  - **Note**: exponentiation takes $O((\log n)^3)$ operations (easy!)
- uses ***large integers*** (eg. 1024 bits)
- ***security*** due to cost of factoring large numbers
  - **Note:** factorization takes $O(e^{\log n \log \log n})$ operations (hard!)

RSA En/decryption

- **to encrypt a message M the sender:**
  - obtains **public key** of recipient PU={e,n}
  - computes: $C = M^e \bmod n$, where $0 \leq M < n$
- **to decrypt the ciphertext C the owner:**
  - uses their private key PR={d,n}
  - computes: $M = C^d \bmod n$
- **note that the message M must be smaller than the modulus n (block if needed)**

RSA Key Setup

Each user generates a public/private key pair by:

1. selecting two **large primes** at random: p, q
2. computing their system modulus n = p.q
   - note ø(n)=(p-1)(q-1)
3. selecting *at random* the encryption key e
   - where 1<e<ø(n), GCD(e,ø(n))=1
4. solve following equation to find decryption key d
   - e.d = 1 mod ø(n) and 0≤d≤n
5. publish their public encryption key: PU={e,n}
6. keep secret private decryption key: PR={d,n}

For more details, see references:

[1] pages 278-280

[2] Chapter 8: Security in Computer Networks

Why RSA works

- because of Euler's Theorem:
  - $a^{ø(n)} \bmod n = 1$ where $\gcd(a,n) = 1$
- in RSA have:
  - $n = p.q$
  - $ø(n) = (p-1)(q-1)$
  - carefully chose e & d to be inverses mod $ø(n)$
  - hence $e.d = 1 + k.ø(n)$ for some k
- hence :

$$C^d = M^{e.d} = M^{1+k.ø(n)} = M^1.(M^{ø(n)})^k$$

$$= M^1.(1)^k = M^1 = M \bmod n$$

RSA Example - Key Setup

1. Select primes: $p = 17$ & $q = 11$
2. Calculate $n = pq = 17 \times 11 = 187$
3. Calculate $ø(n)=(p-1)(q-1)=16x10=160$
4. Select e: gcd(e,160) = 1; choose $e = 7$
5. Determine d: $de = 1 \bmod 160$ and $d < 160$

   Value is d = 23 since 23x7 = 161 = 10x160 + 1

1. Publish public key PU = {7,187}
2. Keep secret private key PR = {23,187}

Efficient Operation using Public Key

- To speed up the operation of the RSA algorithm using the public key, a specific **choice of e** is usually made.
  - The most common choice is 65537 ($2^{16} + 1$);
  - Two other popular choices are 3 and 17.
- Each of these choices has only two 1 bits, so the number of multiplications required to perform exponentiation is minimized.
- However, with a ***very small public key***, such as e = 3, RSA becomes ***vulnerable*** to a simple attack.
- Suppose we have three different RSA users who all use the value e = 3 but have unique values of n, namely (n1, n2, n3)
- If user A sends the same encrypted message M to all three users, then the three ciphertexts are $C_1 = M_3$ mod n1,
- $C_2 = M_3$ mod n2, and $C_3 = M_3$ mod n3. It is likely that n_1, n_2, and n_3 are pairwise relatively prime

Efficient Operation using Public Key

- Suppose we have three different RSA users who all use the value e = 3 but have unique values of n, namely (n1, n2, n3)
- If user A sends the same encrypted message M to all three users, then the three ciphertexts are
  - $C1 = M3 \bmod n_1$,
  - $C2 = M3 \bmod n_2$, and
  - $C3 = M3 \bmod n_3$.
- It is likely that n_1, n_2, and n_3 are pairwise relatively prime
- Therefore, one can use the Chinese remainder theorem (CRT) to compute $M3 \bmod (n_1 n_2 n_3)$

RSA Security

- Four possible approaches to attacking the RSA algorithm are
  1. **Brute force**: This involves trying all possible private keys.
  2. **Mathematical attacks**: There are several approaches, all equivalent in effort tofactoring the product of two primes.
  3. **Timing attacks**: These depend on the running time of the decryption algorithm.
  4. **Chosen ciphertext attacks**: This type of attack exploits properties of the RSA algorithm.

Summary

- Definition of prime number
- Relatively prime numbers
- Public key cryptography
  - Public key
  - Private key
- RSA algorithm
  - Key setup
  - Security

References

1. Cryptography and Network Security, Principles and Practice, William Stallings, Prentice Hall, Sixth Edition, 2013
2. Computer Networking: A Top-Down Approach 6th Edition, Jim Kurose, Keith Ross, Pearson, 2013