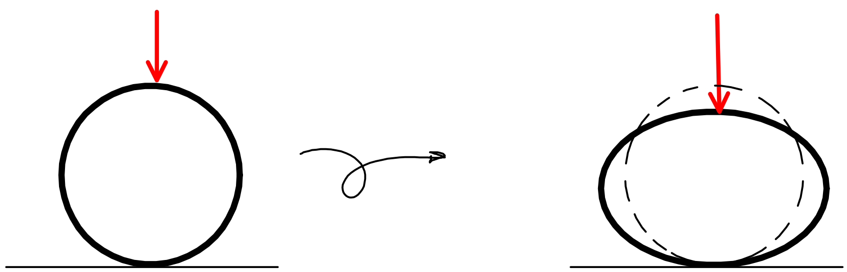
Chapter 3: Rigid Bodies; Equivalent Systems of forces

3.1 -3.3 Introduction, Internal & External Forces

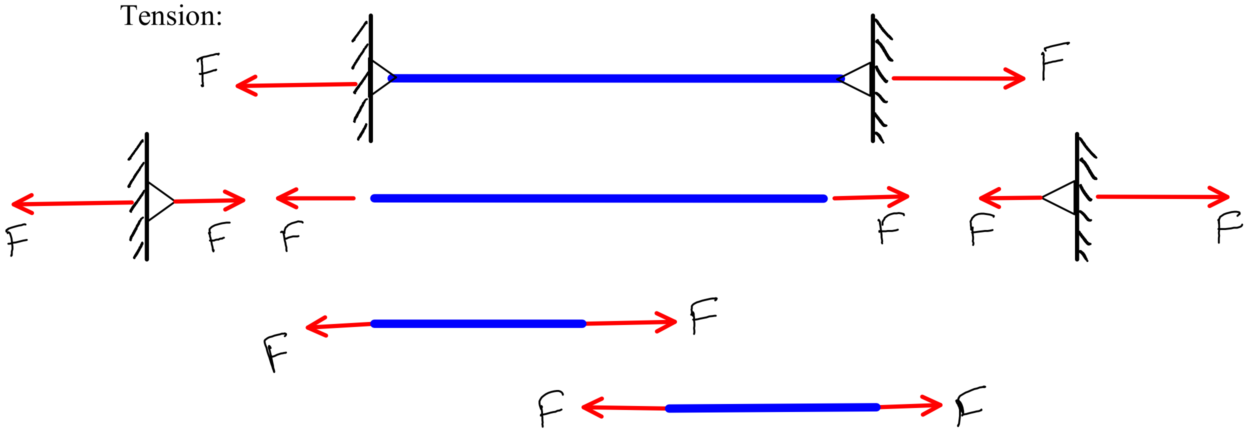
- Rigid Bodies: Bodies in which the relative position of any two points does not change.
  Note:
  - In real life no body is perfectly rigid. We can approximate the behavior of most structures with rigid bodies because the deformations are usually small and negligible.
  - Even in cases when the deformations are not negligible, we can still apply the principles of equilibrium and statics to the deformed or the current configuration of the body to find out what forces the body is carrying.



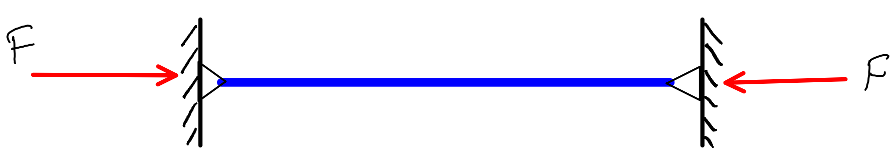
- External and Internal forces:
  - External forces on a rigid body are due to causes that are external to the body. They can cause the body to move or remain at rest as a whole.
    For example: force of gravity, applied external force on an object.

  - Internal forces develop in any body (not just rigid bodies) that keep all the particles of a body together.
    For example: Internal TENSION or COMPRESSION in a bar, bending moment in a beam.

Tension:

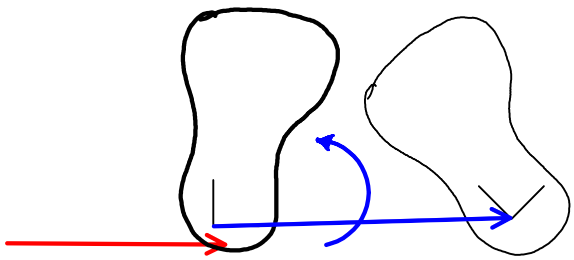


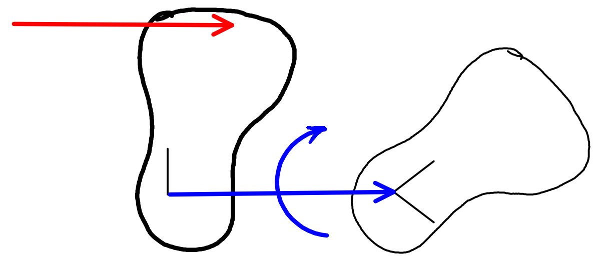
Similarly compression.



3.4 -3.8 Moment of a force; Vector CROSS Product

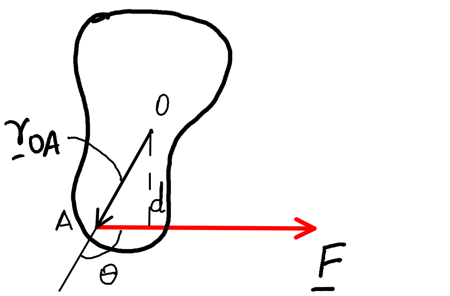
- Any force that is applied to a rigid body causes the body to translate or ROTATE or both.
- The tendency to ROTATE is caused by MOMENTS generated by the force.





- In general, a force **F** generates a moment about any point O which is offset by some distance from the line of action of **F**.

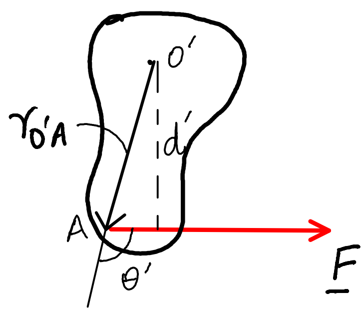
  Note:
  - Moment of **F** about O.
  - Moment is a VECTOR.



$$M_O = Fd$$

$$M_O = r_{OA} F \sin\theta$$

$$\boxed{\underline{M}_O = \underline{r}_{OA} \times \underline{F}}$$



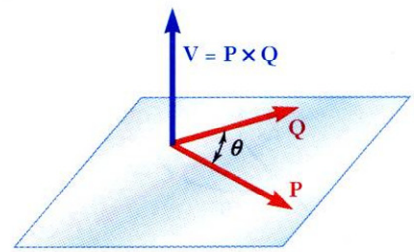
$$M_{O'} = Fd'$$

$$M_{O'} = r_{O'A} F \sin\theta'$$

$$\boxed{\underline{M}_{O'} = \underline{r}_{O'A} \times \underline{F}}$$

Vector Cross Product:

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
  1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
  2. Magnitude of $\boldsymbol{V}$ is $V = PQ \sin\theta$
  3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.



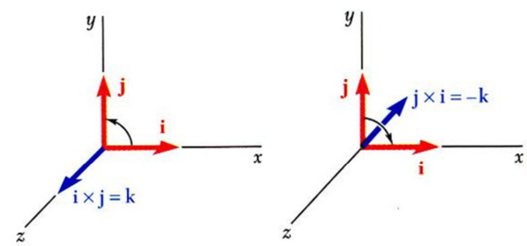
(a)

(b)

- Vector products:
  - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

- Vector products of Cartesian unit vectors,

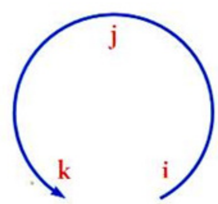
$$
\begin{array}{lll}
\vec{i}\times\vec{i}=0 & \vec{j}\times\vec{i}=-\vec{k} & \vec{k}\times\vec{i}=\vec{j} \\
\vec{i}\times\vec{j}=\vec{k} & \vec{j}\times\vec{j}=0 & \vec{k}\times\vec{j}=-\vec{i} \\
\vec{i}\times\vec{k}=-\vec{j} & \vec{j}\times\vec{k}=\vec{i} & \vec{k}\times\vec{k}=0
\end{array}
$$



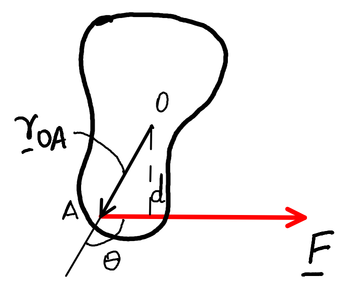
- Vector products in terms of rectangular coordinates

$$\vec{V} = \left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$

$$= \left(P_yQ_z - P_zQ_y\right)\vec{i} + \left(P_zQ_x - P_xQ_z\right)\vec{j} + \left(P_xQ_y - P_yQ_x\right)\vec{k}$$



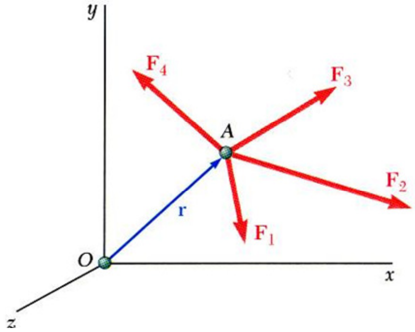
Thus Moment of F about O can be obtained as:



$$\underline{M}_O = \underline{r}_{OA} \times \underline{F}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

$$
\begin{array}{ccccc}
\underline{i} & \underline{j} & \underline{k} & \underline{i} & \underline{j} \\
r_x & r_y & r_z & r_x & r_y \\
F_x & F_y & F_z & F_x & F_y
\end{array}
$$



Varignon's Theorem

$$\vec{r}\times\left(\vec{F}_1+\vec{F}_2+\cdots\right)=\vec{r}\times\vec{F}_1+\vec{r}\times\vec{F}_2+\cdots$$

- Follows from the Distributive property of the vector cross product
- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

Equivalent forces

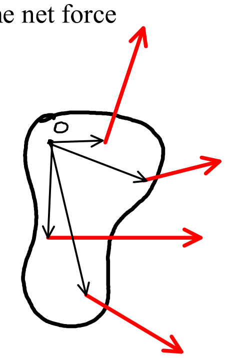
Two forces are said to be equivalent if they have the same magnitude and direction (i.e. they are equal) and produce the same moment about any point O (i.e. same line of action).

Note:

- Principle of transmissibility follows from this. Two forces that have the same line of action produce the same external effect (i.e.translation or rotation) on the body because they have the same net force and moment about any point.
- Later in Chapter 4 we will see that the equations for equilibrium for a rigid body are given by:

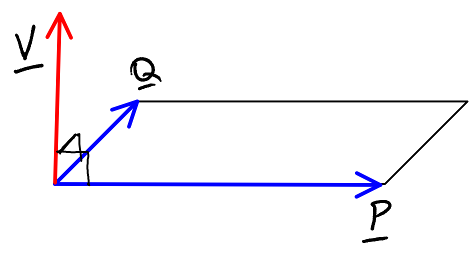
$$\Sigma \underline{F} = \underline{0}$$

$$\Sigma \underline{M}_o = \Sigma(\underline{r}_o \times \underline{F}) = \underline{0}$$



Applications of Cross product:

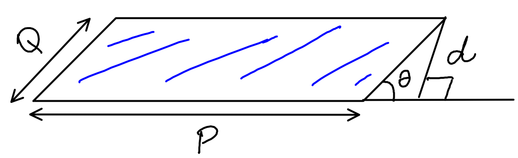
- Finding the direction perpendicular to two vectors.
- Calculation of Area of a Parallelogram.
- Finding the distance of a point from a line.



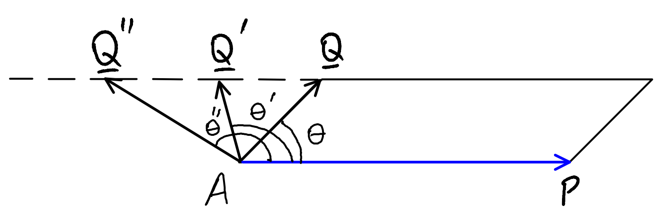
Examples:

$$\underline{V} = \underline{P} \times \underline{Q}$$

$$V = P\underbrace{Q \sin\theta}_{d}$$



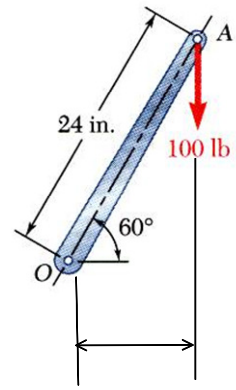
$$d = Q\sin\theta$$
$$= Q'\sin\theta'$$
$$= Q''\sin\theta''$$



Read Examples 3.2, 3.3, 3.4.

Example 3.1

A 100-lb vertical force is applied to the end of a lever which is attached to a shaft at *O*.



Determine:

a) moment about *O*,
b) horizontal force at *A* which creates the same moment,
c) smallest force at A which produces the same moment,
d) location for a 240-lb vertical force to produce the same moment,
e) whether any of the forces from b, c, and d is equivalent to the original force.

(a) $M_o = Fd = 100\text{lb} \times \underbrace{24\cos 60^\circ}_{12}$

$= 1200$ lb in (clockwise)

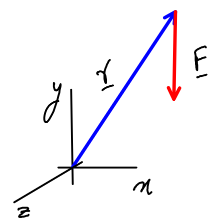
Alternatively

$\underline{M}_o = \underline{r} \times \underline{F}$

$\underline{r} = 12\,\underline{i} + 12\sqrt{3}\,\underline{j}$

$\underline{F} = -100\,\underline{j}$ lb

$\Rightarrow \boxed{\underline{M}_o = -1200\,\underline{k} \text{ lb in}}$

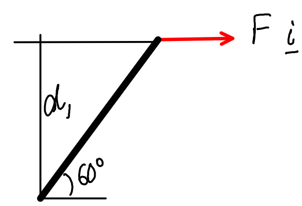


(b) $d_1 = 24\sin 60^\circ$

$= 12\sqrt{3} = 20.8$ in.

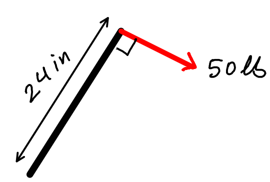
$F d_1 = M_o$

$\Rightarrow F = \dfrac{M_o}{d_1} = \boxed{57.7 \text{ lb}}$



(c) $d_2 = 24$ in

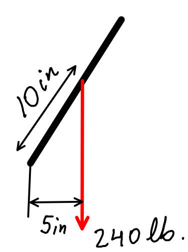
$F = \dfrac{1200}{24} = \boxed{50 \text{ lb}}$

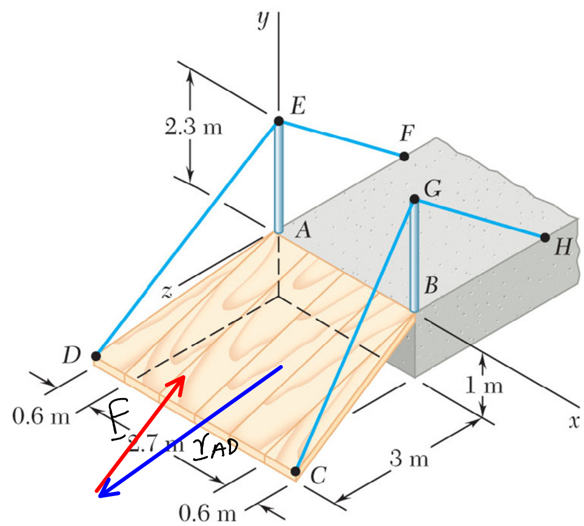


(d) $1200 = (240) \times d_3$

$\Rightarrow d_3 = 5$ in

$\Rightarrow l = \dfrac{5}{\cos 60^\circ} = \boxed{10 \text{ in}}$





Exercise 3.25 & 3.31

Given tension in each cable is 810N.

- Determine the moment at A due to the force at D.
- Find the distance of point A from the "line" DE.

Recall Moment of F about A can be obtained as:

$$\underline{M}_A = \underline{r}_{AD} \times \underline{F}$$

$$\underline{r}_{AD} = -0.6\,\underline{i} - 1\,\underline{j} + 3\,\underline{k} \quad m$$

$$\underline{F} = \frac{\overrightarrow{DE}}{|DE|}\,810 \quad N$$

$$\overrightarrow{DE} = 0.6\,\underline{i} + 3.3\,\underline{j} - 3\,\underline{k} \quad m$$

$$\Rightarrow |DE| = \sqrt{0.6^2 + 3.3^2 + 3^2} = \underline{4.5m}$$

$$\Rightarrow \underline{F} = 108\,\underline{i} + 594\,\underline{j} - 540\,\underline{k} \quad N$$

$$\Rightarrow \underline{M}_A = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -0.6 & -1 & 3 \\ 108 & 594 & -540 \end{vmatrix} = \begin{aligned} &\underline{i}\,(540 - 3\times594) \\ &+ \underline{j}\,(3\times108 - 0.6\times540) \\ &+ \underline{k}\,(-0.6\times594 + 108) \end{aligned}$$

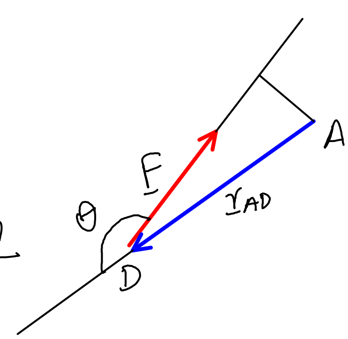
$$\boxed{\vec{M}_A = -1242\,\underline{i} + 0\,\underline{j} - 248.4\,\underline{k} \quad N.m}$$

Distance of A from line DE:-

Note

$$\vec{M}_A = \vec{r}_{AD} \times \vec{F}$$

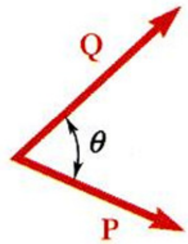
$$|\vec{M}_A| = |r_{AD}|\,|F| \sin\theta = \underline{1266.6 \text{ Nm}}$$



$$\text{i.e. } M_A = F \underbrace{r_{AD}\sin\theta}_{d}$$

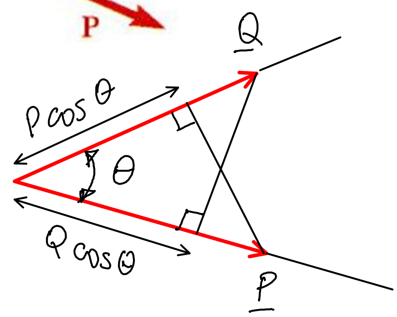
$$\Rightarrow d = \frac{M_A}{F} = \frac{1266.6}{810} = \boxed{1.56\ m}$$

3.9 Scalar Product of two Vectors



- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta \quad \text{(scalar result)}$$

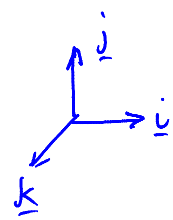


- Scalar products:
  - are commutative, $\vec{P} \bullet \vec{Q} = \vec{Q} \bullet \vec{P}$
  - are distributive, $\vec{P} \bullet \left(\vec{Q}_1 + \vec{Q}_2\right) = \vec{P} \bullet \vec{Q}_1 + \vec{P} \bullet \vec{Q}_2$
  - are not associative, $\left(\vec{P} \bullet \vec{Q}\right) \bullet \vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

$$\vec{P} \bullet \vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right) \bullet \left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

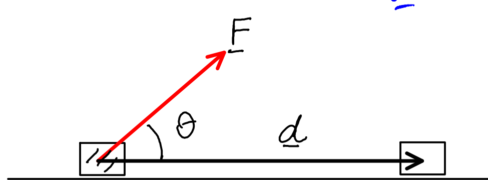
$$\vec{i} \bullet \vec{i} = 1 \quad \vec{j} \bullet \vec{j} = 1 \quad \vec{k} \bullet \vec{k} = 1 \quad \vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$



$$\vec{P} \bullet \vec{Q} = P_x Q_x + P_y Q_y + P_z Q_z$$
$$\vec{P} \bullet \vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

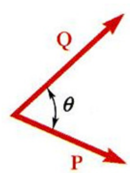
Applications of Dot Product

- Work done by a force



W = **F•d =** F d cosθ

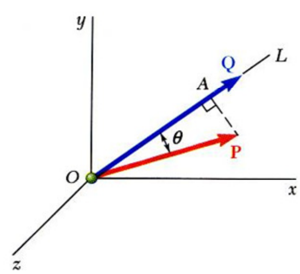
- Angle between two vectors:



$$\vec{P} \bullet \vec{Q} = PQ\cos\theta = P_x Q_x + P_y Q_y + P_z Q_z$$
$$\cos\theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ}$$

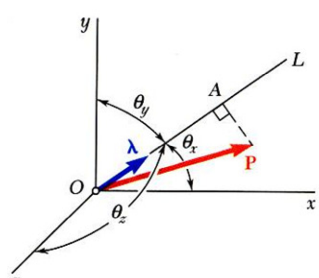
- Projection of a vector on a given axis:

$$P_{OL} = P\cos\theta = \text{projection of } P \text{ along } OL$$
$$\vec{P} \bullet \vec{Q} = PQ\cos\theta$$
$$\frac{\vec{P} \bullet \vec{Q}}{Q} = P\cos\theta = P_{OL}$$

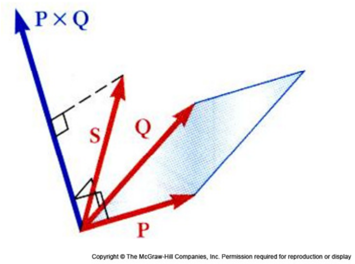


- Component in a given direction (unit vector):

$$P_{OL} = \vec{P} \bullet \vec{\lambda}$$
$$= P_x\cos\theta_x + P_y\cos\theta_y + P_z\cos\theta_z$$



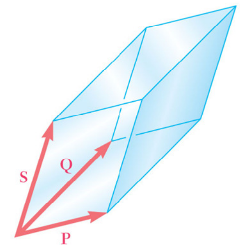
3.10 Mixed TRIPLE Product of 3 Vectors



- Mixed triple product of three vectors,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \text{scalar result}$$

- The six mixed triple products formed from ***S***, ***P***, and ***Q*** have equal magnitudes but not the same sign,

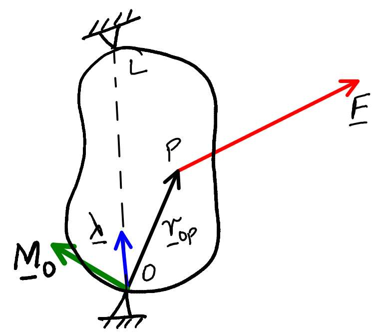


$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = \vec{P} \bullet \left(\vec{Q} \times \vec{S}\right) = \vec{Q} \bullet \left(\vec{S} \times \vec{P}\right)$$
$$= -\vec{S} \bullet \left(\vec{Q} \times P\right) = -\vec{P} \bullet \left(\vec{S} \times \vec{Q}\right) = -\vec{Q} \bullet \left(\vec{P} \times \vec{S}\right)$$

- Evaluating the mixed triple product,

$$\vec{S} \bullet \left(\vec{P} \times \vec{Q}\right) = S_x\left(P_y Q_z - P_z Q_y\right) + S_y\left(P_z Q_x - P_x Q_z\right) + S_z\left(P_x Q_y - P_y Q_x\right)$$
$$= \begin{vmatrix} S_x & S_y & S_z \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix}$$

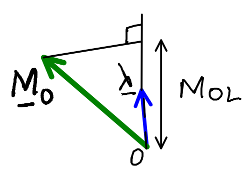
3.11 Moment of a force about an Axis



$$\underline{M}_O = \underline{r}_{OP} \times \underline{F}$$

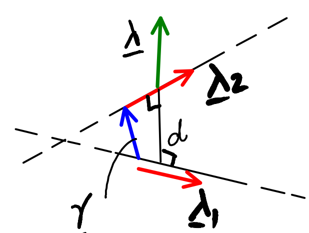
$$\boxed{M_{OL} = \underline{\lambda} \cdot \underline{M}_O = \underline{\lambda} \cdot \left(\underline{r}_{OP} \times \underline{F}\right)}$$

Note: M_{OL} is the projection of $\mathbf{M}_O$ along OL.



Application:

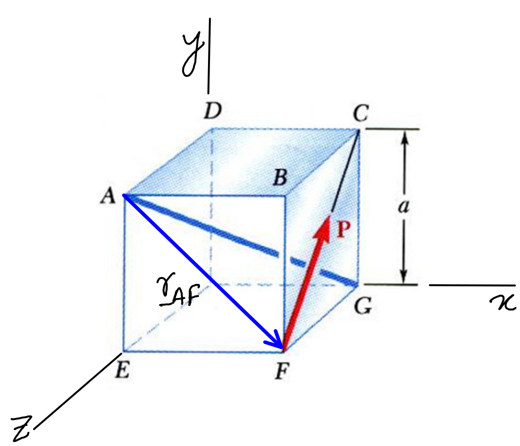
Finding the perpendicular distance between two non-intersecting lines in 3D space.



$$\underline{\lambda} = \underline{\lambda}_1 \times \underline{\lambda}_2$$

$$\boxed{d = \underline{r} \cdot \underline{\lambda} = \underline{r} \cdot \left(\underline{\lambda}_1 \times \underline{\lambda}_2\right)}$$

Read Example 3.5 in the book.



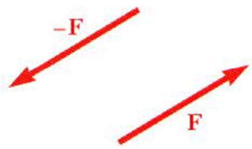
A cube is acted on by a force ***P*** as shown. Determine the moment of ***P***

a) about *A*
b) about the edge *AB* and
c) about the diagonal *AG* of the cube.
d) Determine the perpendicular distance between *AG* and *FC*.

$$\underline{M}_A = \underline{r}_{AF} \times \underline{P}$$

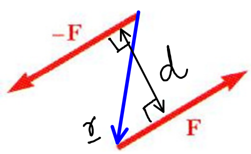
3.12-3.13 Force Couples

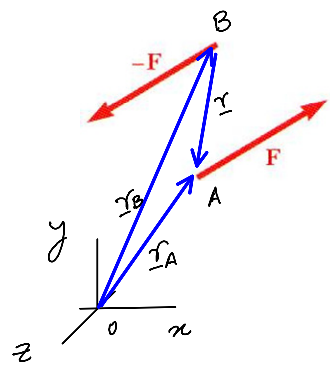
- Two forces ***F*** and ***-F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a **couple**.



- Moment of the couple,

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times (-\vec{F})$$
$$= (\vec{r}_A - \vec{r}_B) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF\sin\theta = Fd$$

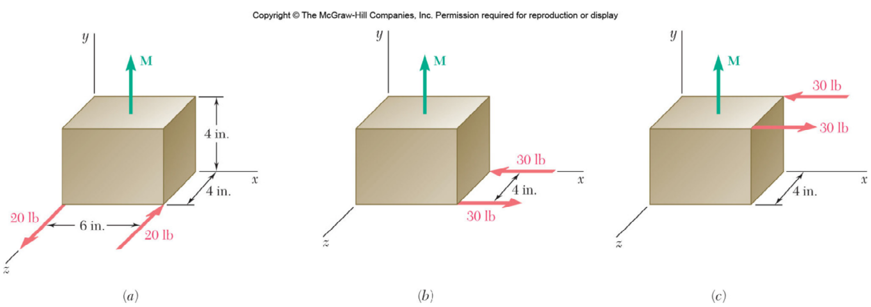


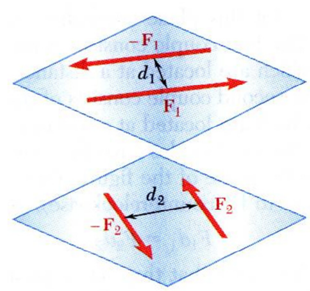


$r = (r_A - r_B)$

Note: Moment of a "couple" is always the same about any point.

Equivalent Couples

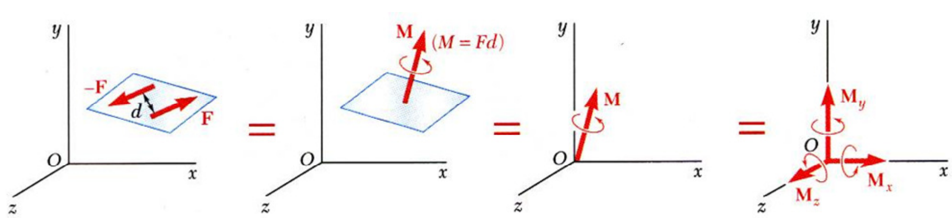




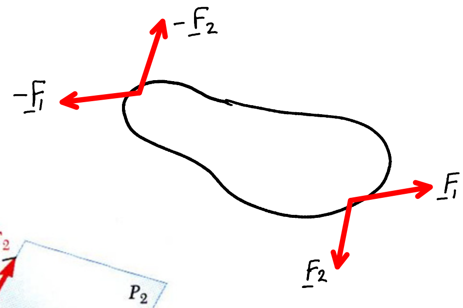
Two or more couples are equivalent iff they produce the same moment.

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.

3.14-3.15 Couples (and Moments) are Vectors



Addition of Couples



Moment due to the <u>resultant</u> of the forces:

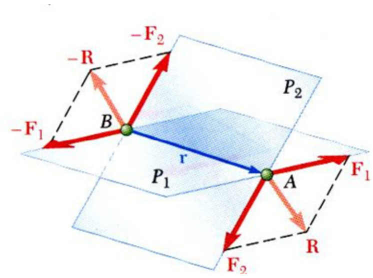
$$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

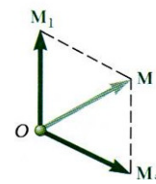
Moment due to the individual couples:

$$\vec{M}_1 = \vec{r} \times \vec{F}_1 \text{ in plane } P_1$$
$$\vec{M}_2 = \vec{r} \times \vec{F}_2 \text{ in plane } P_2$$

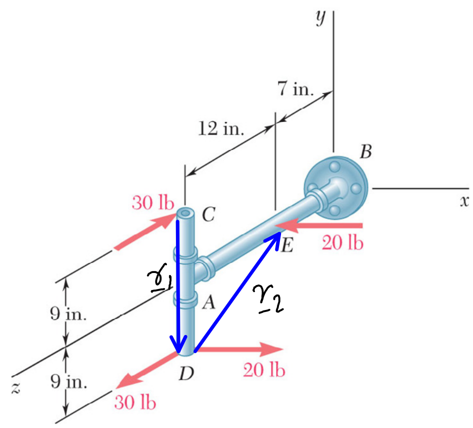
VECTOR sum of the two moments:

$$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$
$$= \vec{M}_1 + \vec{M}_2$$





Example 3.6



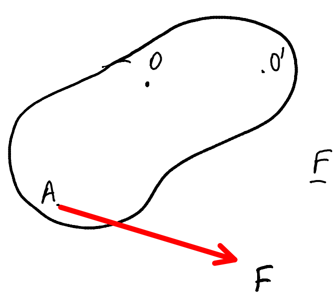
$$\underline{M} = (30 \times 18)(-\underline{i}) + (20 \times 12)(\underline{j}) + (20 \times 9)(\underline{k})$$

$$\underline{M} = \underbrace{-540\,\underline{i} + 240\,\underline{j} + 180\,\underline{k}}\ \text{lb in}$$

Alternatively.

$$\underline{M} = \underline{r}_1 \times (30\,\underline{k}) + \underline{r}_2 \times (-20\,\underline{i})$$
$$= (-18\underline{j}) \times (30\underline{k}) + (9\underline{j} - 12\underline{k}) \times (-20\underline{k})$$
$$= \underbrace{-540\,\underline{i} + 240\,\underline{j} + 180\,\underline{k}}\ \text{lb in}$$

3.16 Equivalent Force and Moment



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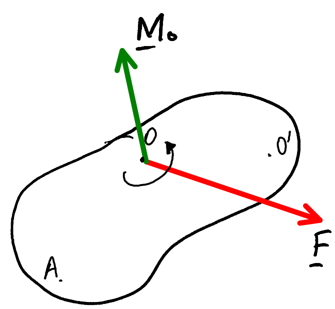
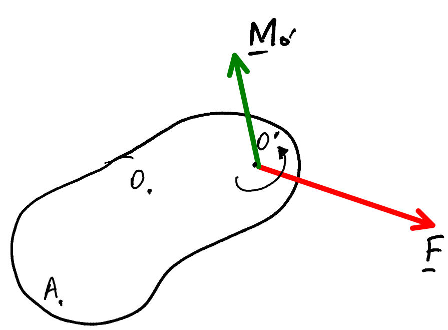


$$\underline{M}_o = \underline{r}_{OA} \times \underline{F}$$

$$\underline{M}_{o'} = \underline{M}_o + \underline{r}_{o'o} \times \underline{F}$$

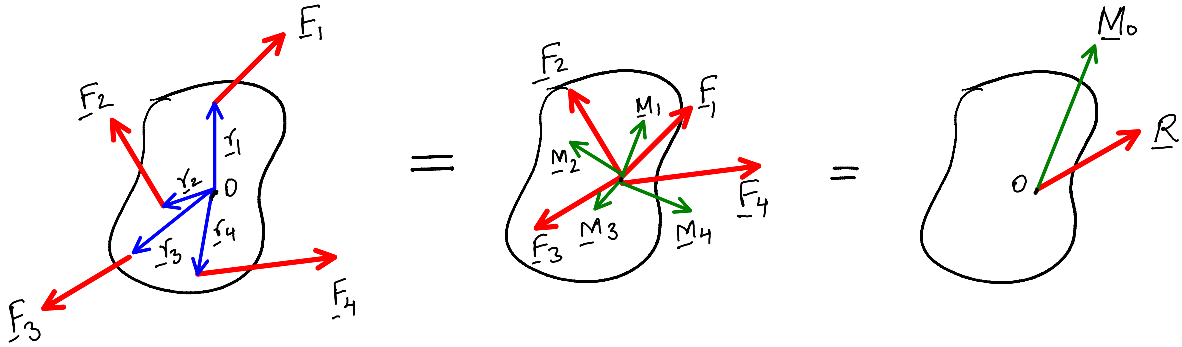
$$\underline{M}_{o'} = \underbrace{(\underline{r}_{OA} + \underline{r}_{o'o})}_{\underline{r}_{o'A}} \times \underline{F}$$

=



3.17-3.18 Equivalent Systems of Forces & Moments

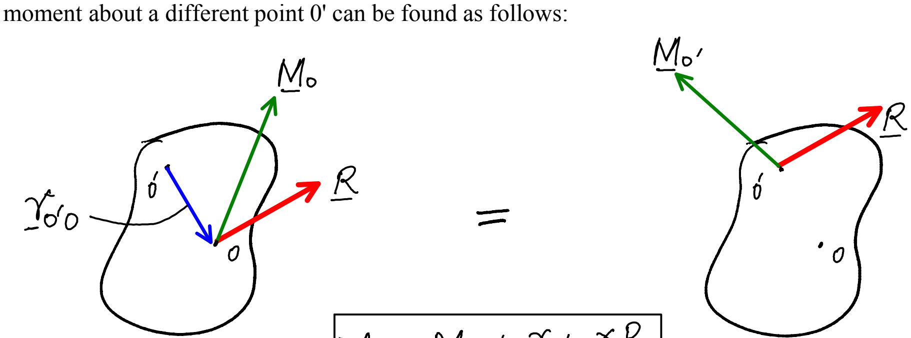
- Any system of forces can be reduced to ONE resultant force and ONE resultant moment.



$$\underline{M}_1 = \underline{r}_1 \times \underline{F}_1$$
$$\underline{M}_2 = \underline{r}_2 \times \underline{F}_2$$
$$\underline{M}_3 = \underline{r}_3 \times \underline{F}_3$$
$$\vdots$$

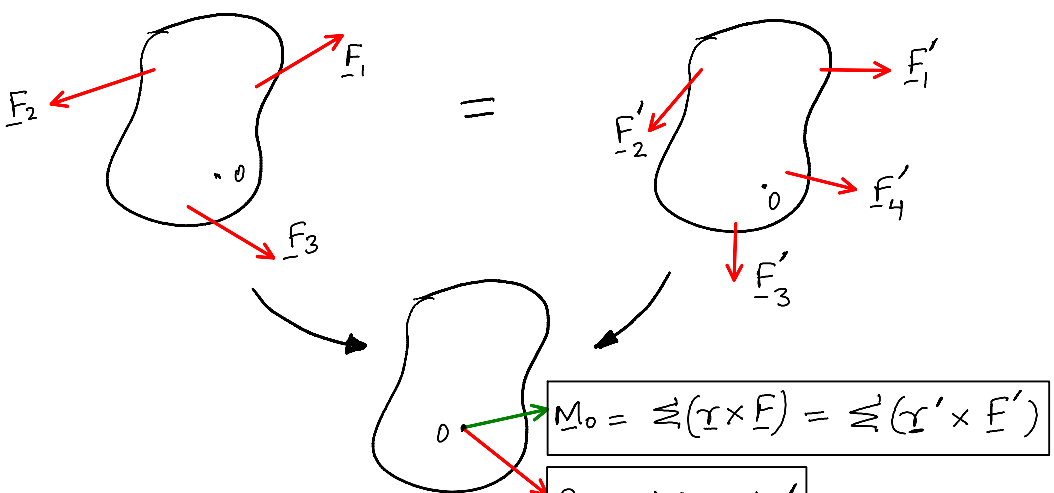
$$\boxed{\begin{aligned} \underline{R} &= \underline{F}_1 + \underline{F}_2 + \underline{F}_3 + \cdots = \sum \underline{F} \\ \underline{M}_0 &= \underline{M}_1 + \underline{M}_2 + \underline{M}_3 + \cdots = \sum (\underline{r} \times \underline{F}) \end{aligned}}$$

- Once a resultant force & moment has been found about O, a new resultant force & moment about a different point 0' can be found as follows:



$$\boxed{\underline{M}_{o'} = \underline{M}_o + \underline{r}_{o'o} \times \underline{R}}$$

- Two or more systems of forces & Moments are said to be equivalent iff they have the same resultant force and the resultant moment about any (and all) points O.



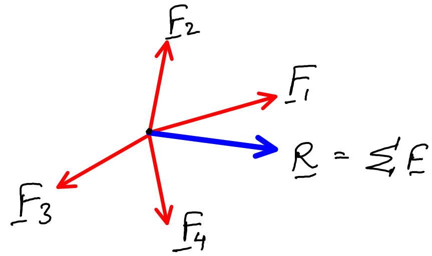
$$\boxed{\underline{M}_o = \sum (\underline{r} \times \underline{F}) = \sum (\underline{r}' \times \underline{F}')}$$
$$\boxed{\underline{R} = \sum \underline{F} = \sum \underline{F}'}$$

3.20 Special Case: Reduction to a SINGLE force

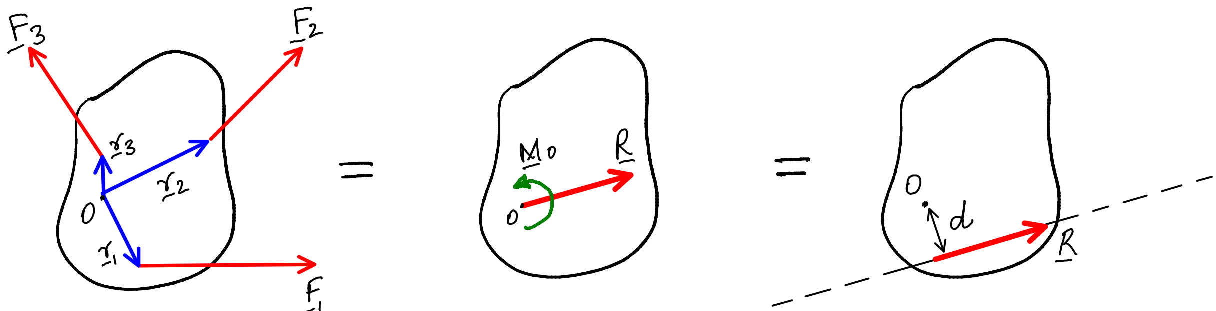
In general, a system of forces and moments cannot be reduced to just a single force. However, if the resultant moment is **perpendicular to** the resultant force, one can reduce the system to just ONE force and NO moment.

Particular cases:

- Concurrent forces: Forces acting at the same point.



- Coplanar forces: Forces contained in the same plane (with non-concurrent lines of action)

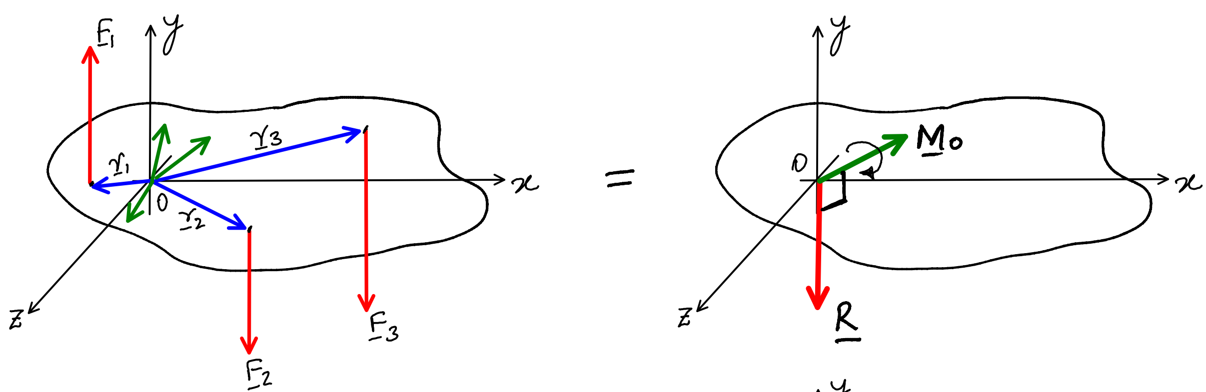


$$\boxed{\begin{aligned}\underline{R} &= \sum \underline{F} \\ \underline{M}_O &= \sum (\underline{r} \times \underline{F})\end{aligned}}$$

$$M_O = R\,d$$

$$\Rightarrow \boxed{d = \frac{M_O}{R}}$$

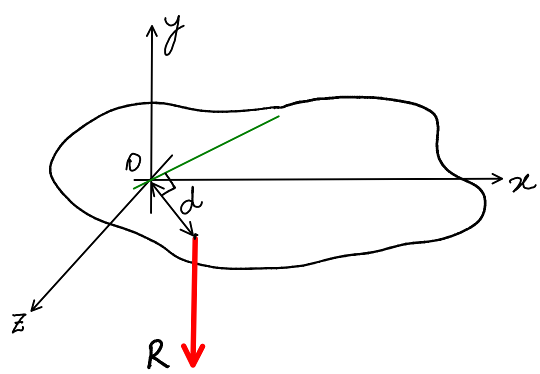
- Parallel forces in 3D space.



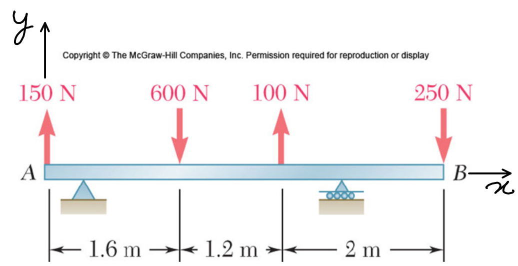
$$\boxed{\begin{aligned}\underline{R} &= \sum \underline{F} \\ \underline{M}_O &= \sum (\underline{r} \times \underline{F})\end{aligned}}$$

$$M_O = R\,d \quad \Rightarrow \boxed{d = \frac{M_O}{R}}$$

=



Example 3.8



Reduce the forces to an equivalent force & moment

- at A
- at B

Reduce them to SINGLE force and find where it acts.

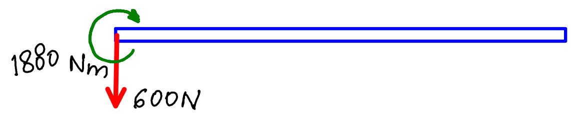
(a) at A:

$$\underline{R} = \sum \underline{F} = (150 - 600 + 100 - 250)\underline{j} \text{ N}$$

$$= \boxed{-600\underline{j}\text{ N}}$$

$$\underline{M}_A = (150 \times 0 - 600 \times 1.6 + 100 \times 2.8 - 250 \times 4.8)\ \underline{k}\ \text{Nm}$$

$$= \boxed{-1880\ \underline{k}\ \text{Nm}}$$

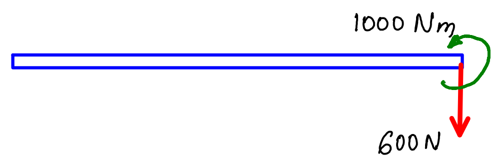


(b) at B:

$$\underline{R} = \boxed{-600\,\underline{j}\text{ N}}$$

$$\underline{M} = -1880\underline{k} + 600 \times 4.8\underline{k}\ \text{Nm}$$

$$= \boxed{1000\ \underline{k}\ \text{Nm}}$$



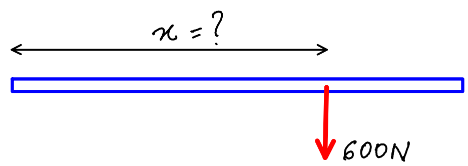
(c) Single Force:

$$M = 0 = -600 \times (4.8 - x) + 1000$$

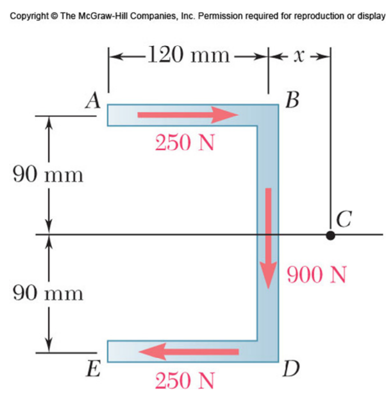
or

$$M = 0 = 600 \times x - 1880$$

$$\Rightarrow \boxed{x = 3.13\text{ m}}$$



Exercise 3.88



The shearing forces exerted on the cross section of a steel channel can be represented as 900N vertical and two 250 N horizontal forces.

- Replace these forces with a SINGLE force at C.
  (C is called the shear center)
- Determine *x*.

$$\underline{R} = \sum \underline{F} = 250\underline{i} - 250\underline{i} - 900\underline{j}\ \text{N}$$

$$= -900\,\underline{j}\ \text{N}$$

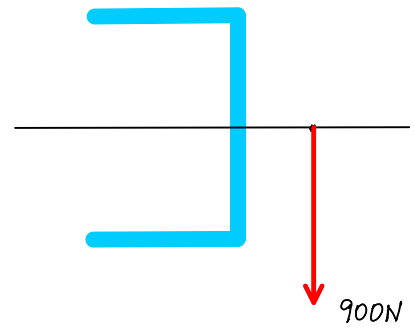
$$\underline{M}_C = -(250 \times 180) + 900 \times x\ \underline{k}\ \text{Nmm}$$

(= 0 for C to be the shear center)

$$\Rightarrow x = \frac{250 \times 180}{900} = \boxed{50\text{ mm}}$$

Note:

The resultant force is not an actual applied force. Therefore it is OK that it acts at a point which is not even in the body.



Analogy: