Chapter 3 - Vectors

I. Definition

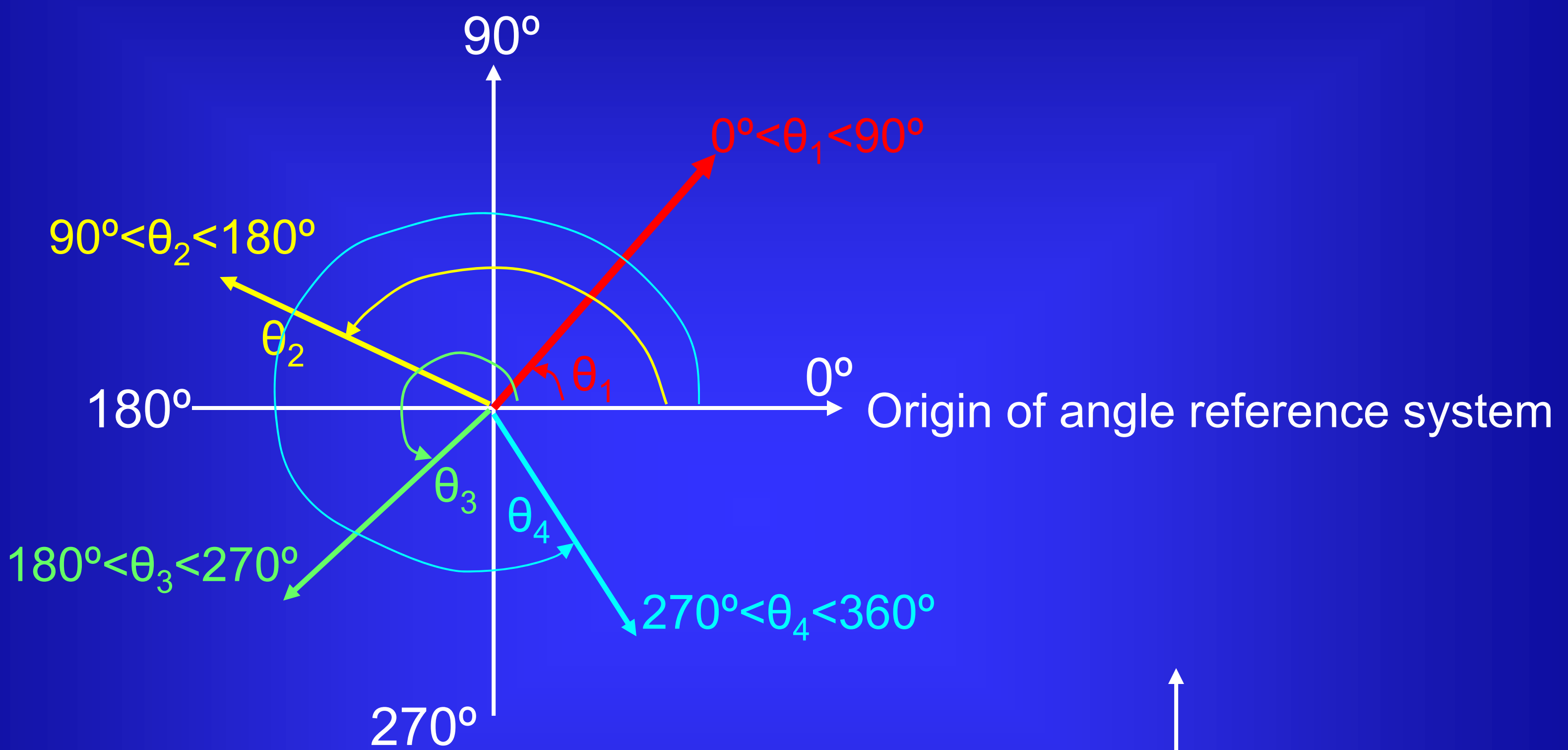
II. Arithmetic operations involving vectors

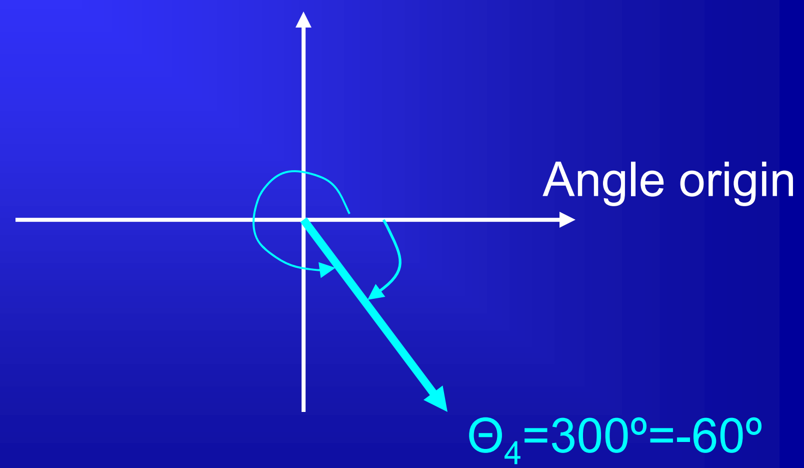
A) Addition and subtraction

- Graphical method
- Analytical method → Vector components

B) Multiplication

Review of angle reference system

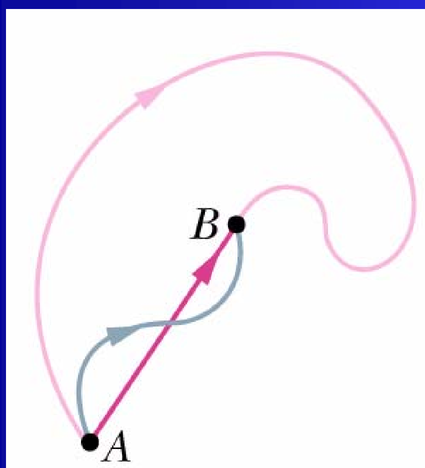


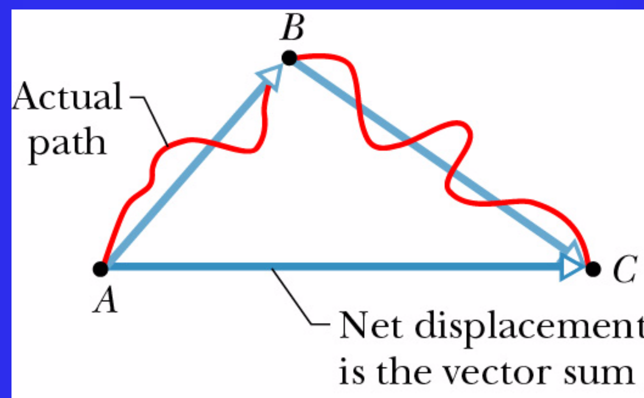


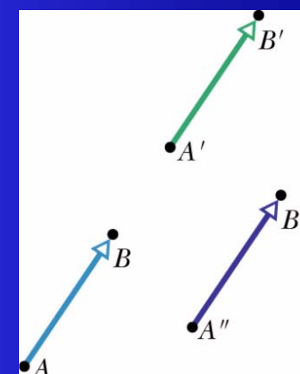
I. Definition

Vector quantity: quantity with a magnitude and a direction. It can be represented by a vector.

Examples: displacement, velocity, acceleration.







Same displacement

Displacement → does not describe the object's path.

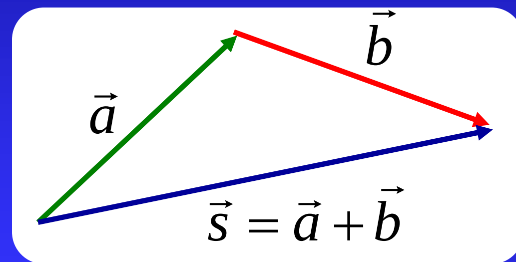
Scalar quantity: quantity with magnitude, no direction.

Examples: temperature, pressure

II. Arithmetic operations involving vectors

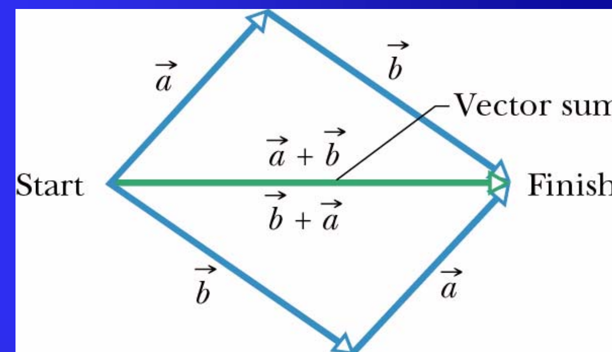
Vector addition: $\vec{s} = \vec{a} + \vec{b}$

- Geometrical method

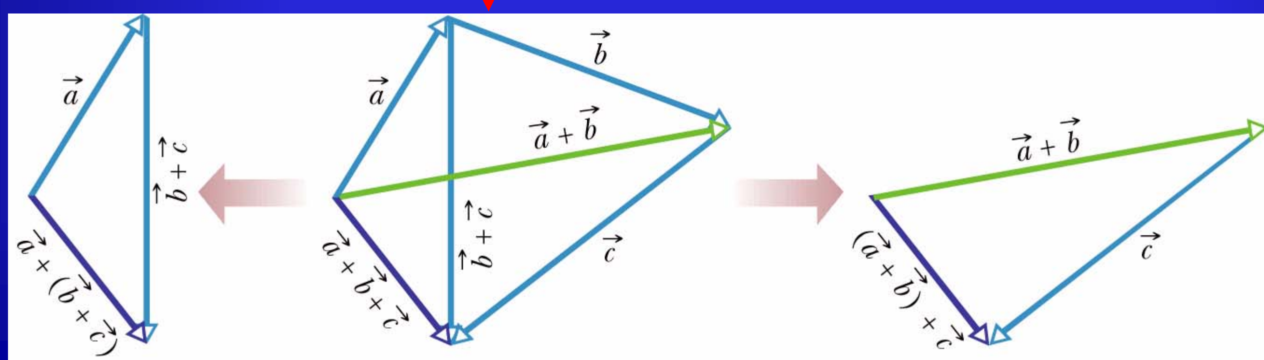


Rules:

$$\vec{a} + \vec{b} = \vec{b} + \vec{a} \qquad \text{(commutative law)} \qquad (3.1)$$

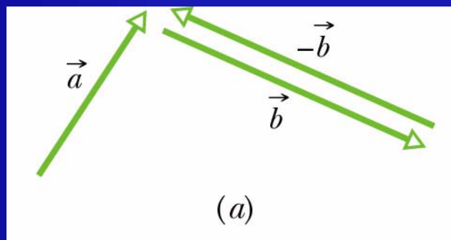


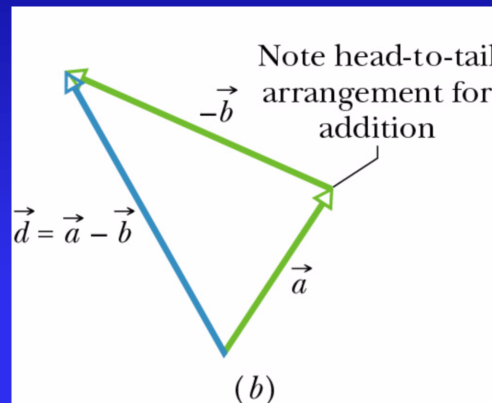
$$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c}) \qquad \text{(associative law)} \qquad (3.2)$$



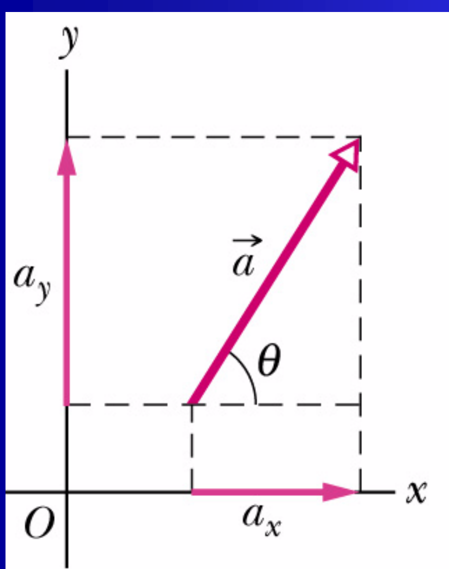
Vector subtraction:

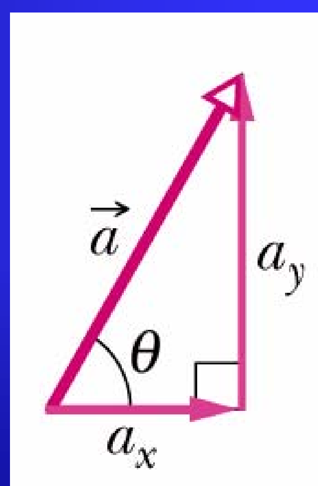
$$\vec{d} = \vec{a} - \vec{b} = \vec{a} + (-\vec{b}) \qquad (3.3)$$





Vector component: projection of the vector on an axis.





$$a_x = a\cos\theta$$

$$a_y = a\sin\theta \qquad (3.4)$$

Scalar components of $\vec{a}$

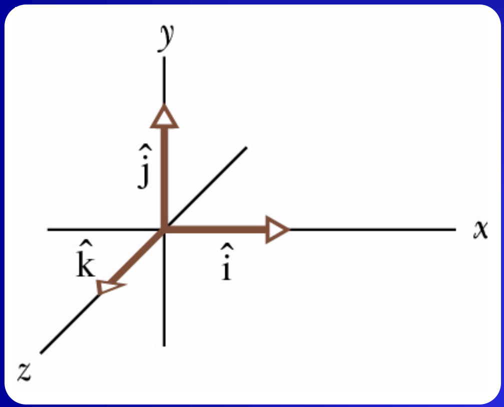
$$a = \sqrt{a_x^2 + a_y^2}$$

$$\tan\theta = \frac{a_y}{a_x} \qquad (3.5)$$

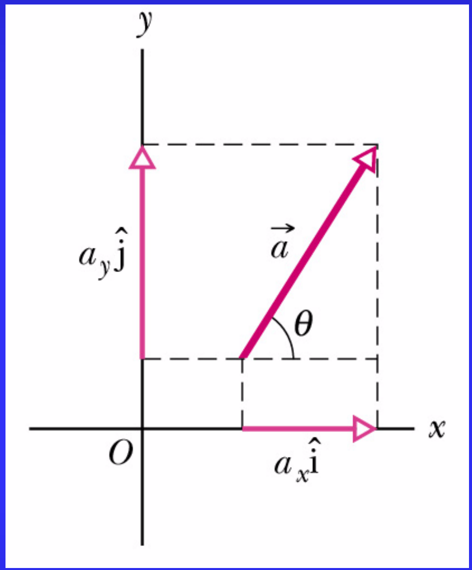
Vector magnitude

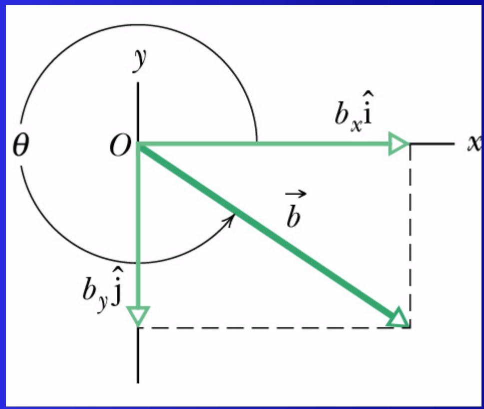
Vector direction

Unit vector: Vector with magnitude 1.
No dimensions, no units.



$$\hat{i}, \hat{j}, \hat{k} \rightarrow \textit{unit vectors in positive direction of } x, y, z \textit{ axes}$$





$$\vec{a} = a_x\hat{i} + a_y\hat{j} \qquad (3.6)$$

Vector component

Vector addition:

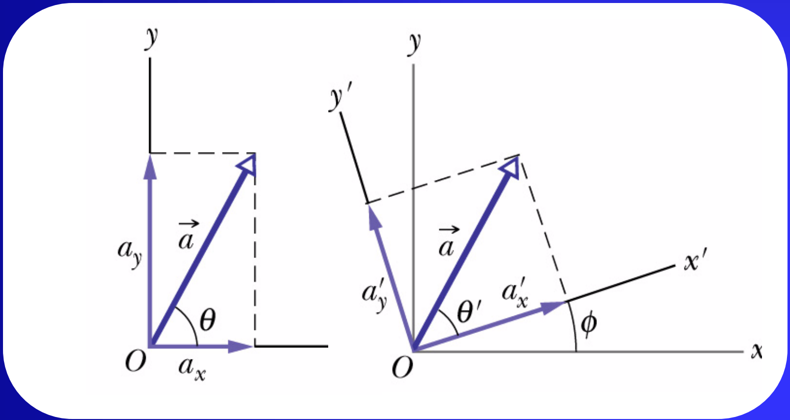
- **Analytical method:** adding vectors by components.

$$\vec{r} = \vec{a} + \vec{b} = (a_x + b_x)\hat{i} + (a_y + b_y)\hat{j} \qquad (3.7)$$

Vectors & Physics:

-The relationships among vectors do not depend on the location of the origin of the coordinate system or on the orientation of the axes.

- The laws of physics are independent of the choice of coordinate system.



$$a = \sqrt{a_x^2 + a_y^2} = \sqrt{a'^2_x + a'^2_y} \qquad (3.8)$$

$$\theta = \theta' + \phi$$

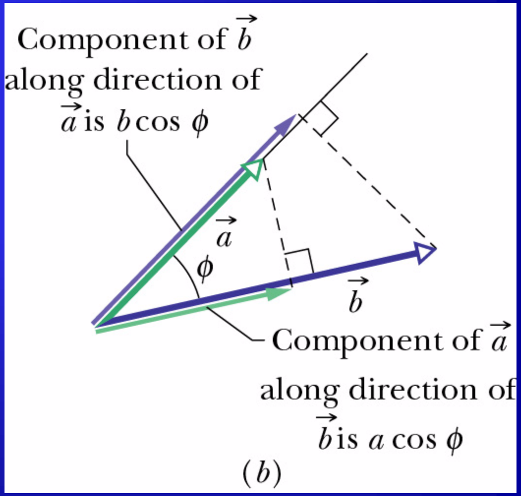
Multiplying vectors:

- **Vector by a scalar:** $\vec{f} = s \cdot \vec{a}$

- **Vector by a vector:**

Scalar product = scalar quantity

(dot product)



$$\vec{a} \cdot \vec{b} = ab\cos\phi = a_x b_x + a_y b_y + a_z b_z \qquad (3.9)$$

Rule: $\vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$ (3.10)

$$\vec{a}\cdot\vec{b}=ab \quad \leftarrow \cos\phi=1 \ \ (\phi=0^\circ)$$

$$\vec{a}\cdot\vec{b}=0 \quad \leftarrow \cos\phi=0 \ \ (\phi=90^\circ)$$

$$\vec{i}\cdot\vec{i}=\vec{j}\cdot\vec{j}=\vec{k}\cdot\vec{k}=1\cdot 1\cdot\cos 0^\circ=1$$

$$\vec{i}\cdot\vec{j}=\vec{j}\cdot\vec{i}=\vec{i}\cdot\vec{k}=\vec{k}\cdot\vec{i}=\vec{j}\cdot\vec{k}=\vec{k}\cdot\vec{j}=1\cdot 1\cdot\cos 90^\circ=0$$

Angle between two vectors:

$$\cos\varphi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}$$

Multiplying vectors:

- **Vector by a vector**

<u>**Vector product**</u> **= vector** (cross product)

$$\vec{a}\times\vec{b}=\vec{c}=(a_y b_z-b_y a_z)\hat{i}-(b_z a_x-a_z b_x)\hat{j}+(a_x b_y-b_x a_y)\hat{k}$$

$$c=ab\sin\phi$$

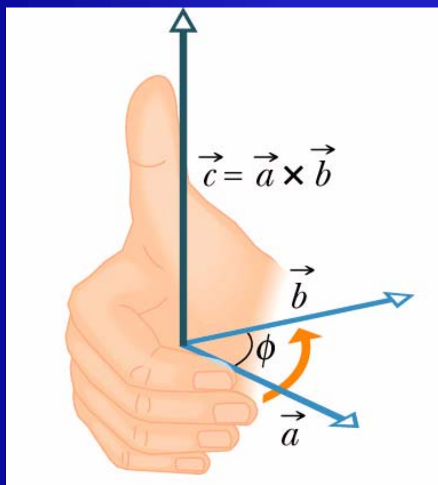
Magnitude

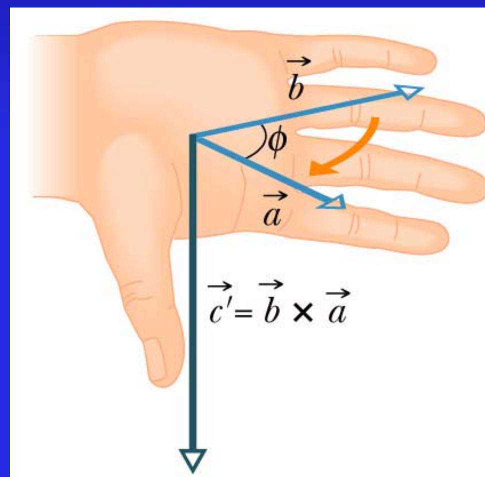
$$\vec{a}\times\vec{b}=0 \;\leftarrow \sin\phi=0\;(\phi=0^\circ)$$

$$\left|\vec{a}\times\vec{b}\right|=ab \;\leftarrow \sin\phi=1\;(\phi=90^\circ)$$

Vector product

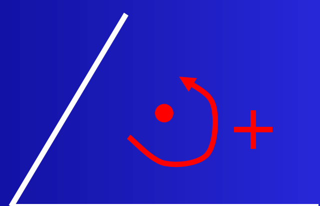
Direction → right hand rule

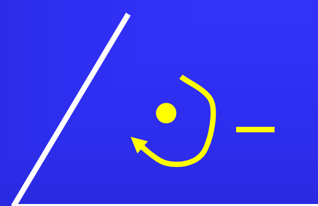




Rule:

$$\vec{b}\times\vec{a}=-(\vec{a}\times\vec{b}) \qquad (3.12)$$

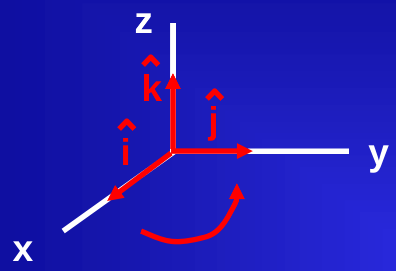




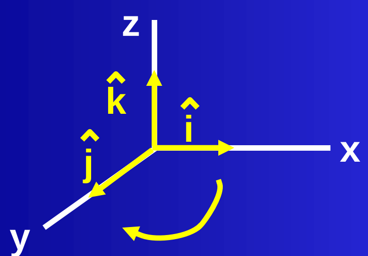
$\vec{c}$ *perpendicular to plane containing* $\vec{a},\vec{b}$

1) Place $\vec{a}$ and $\vec{b}$ tail to tail without altering their orientations.
2) $\vec{c}$ will be along a line perpendicular to the plane that contains $\vec{a}$ and $\vec{b}$ where they meet.
3) Sweep $\vec{a}$ into $\vec{b}$ through the smallest angle between them.

Right-handed coordinate system



Left-handed coordinate system

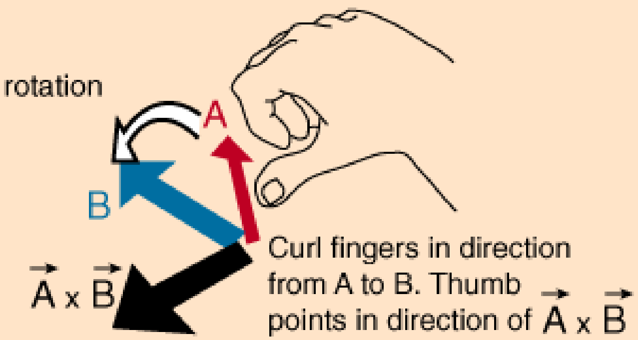


Right Hand Rule, Vector Product

The direction of the vector product can be visualized with the right-hand rule. If you curl the fingers of your right hand so that they follow a rotation from vector A to vector B, then the thumb will point in the direction of the vector product.

Note that the direction of rotation is significant and that

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}$$



The vector product of A and B is always perpendicular to both A and B. Another way of stating that is to say that the vector product is perpendicular to the plane formed by vectors A and B. This right-hand rule direction is produced mathematically by the vector product expression.

Vector Product, Determinant Form

The vector product is compactly stated in the form of a determinant which for the 3x3 case has a convenient evaluation procedure:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Calculation

Once the scheme for determinant evaluation is familiar, this is a convenient way to reconstruct the expanded form:

$$\vec{A} \times \vec{B} = \vec{i}(A_y B_z - A_z B_y) - \vec{j}(A_x B_z - A_z B_x) + \vec{k}(A_x B_y - A_y B_x)$$

$$\left|\vec{i}\times\vec{i}\right| = \left|\vec{j}\times\vec{j}\right| = \left|\vec{k}\times\vec{k}\right| = 1\cdot 1\cdot \sin 0^\circ = 0$$

$$\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = \vec{0}$$

$$\vec{i}\times\vec{j} = -(\vec{j}\times\vec{i}) = \vec{k}$$

$$\vec{j}\times\vec{k} = -(\vec{k}\times\vec{j}) = \vec{i}$$

$$\vec{k}\times\vec{i} = -(\vec{i}\times\vec{k}) = \vec{j}$$

Determinant Evaluation Example

For a determinant of order three the evaluation rule is

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \underbrace{a_{11}}_{\text{First row element}}\underbrace{(a_{22}\, a_{33} - a_{23}\, a_{32})}_{\text{Cofactor}} \underbrace{-}_{\text{Alternating signs}} \underbrace{a_{12}}_{\text{First row element}}\underbrace{(a_{21}\, a_{33} - a_{23}\, a_{31})}_{\text{Cofactor}} + a_{13}(a_{21}\, a_{32} - a_{22}\, a_{31})$$

etc.

Take the elements of the top row and multiply them times the determinant of their cofactors. The cofactor is the array left when the row and column of the given top row element is eliminated. The evaluation of the determinant of the cofactor follows the same pattern until the cofactor has dimension two. At that point, it's value is the difference of the diagonal products.

$$\underbrace{a_{11}}_{\text{First row element}}(\underbrace{a_{22}\, a_{33}}_{\text{Diagonal product}} - \underbrace{a_{23}\, a_{32}}_{\text{Diagonal product}}) = \begin{vmatrix} a_{11} & & \\ & a_{22} & a_{23} \\ & a_{32} & a_{33} \end{vmatrix}$$

P1: If $\vec{B}$ is added to $\vec{C} = 3\hat{i} + 4\hat{j}$, the result is a vector in the positive direction of the y axis, with a magnitude equal to that of $\vec{C}$. What is the magnitude of $\vec{B}$?

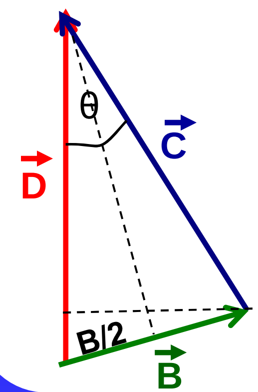
Method 1

$$\vec{B}+\vec{C}=\vec{B}+(3\hat{i}+4\hat{j})=\vec{D}=D\hat{j}$$

$$|C|=|D|=\sqrt{3^2+4^2}=5$$

$$\vec{B}+(3\hat{i}+4\hat{j})=5\hat{j}\rightarrow\vec{B}=-3\hat{i}+\hat{j}\rightarrow|B|=\sqrt{9+1}=3.2$$

Method 2

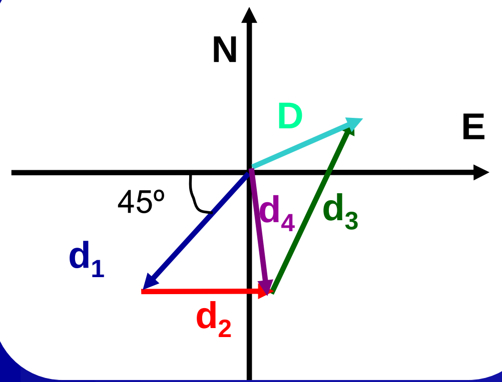


Isosceles triangle

$$\tan\theta=(3/4)\rightarrow\theta=36.9^{\circ}$$

$$\sin\left(\frac{\theta}{2}\right)=\frac{B/2}{D}\rightarrow B=2D\sin\left(\frac{\theta}{2}\right)=3.2$$

P2: A fire ant goes through three displacements along level ground: $\vec{d}_1$ for 0.4m SW, $\vec{d}_2$ 0.5m E, $\vec{d}_3$=0.6m at 60º North of East. Let the positive x direction be East and the positive y direction be North. (a) What are the x and y components of $\vec{d}_1$, $\vec{d}_2$ and $\vec{d}_3$? (b) What are the x and the y components, the magnitude and the direction of the ant's net displacement? (c) If the ant is to return directly to the starting point, how far and in what direction should it move?



(a)

$$d_{1x}=-0.4\cos45^{\circ}=-0.28m$$

$$d_{1y}=-0.4\sin45^{\circ}=-0.28m$$

$$d_{2x}=0.5m$$

$$d_{2y}=0$$

$$d_{3x}=0.6\cos60^{\circ}=0.30m$$

$$d_{3y}=0.6\sin60^{\circ}=0.52m$$

(b)

$$\vec{d}_4=\vec{d}_1+\vec{d}_2=(-0.28\hat{i}-0.28\hat{j})+0.5\hat{i}=(0.22\hat{i}-0.28\hat{j})m$$

$$\vec{D}=\vec{d}_4+\vec{d}_3=(0.22\hat{i}-0.28\hat{j})+(0.3\hat{i}+0.52\hat{j})=(0.52\hat{i}+0.24\hat{j})m$$

$$|D|=\sqrt{0.52^2+0.24^2}=0.57m$$

$$\theta=\tan^{-1}\left(\frac{0.24}{0.52}\right)=24.8^{\circ}\quad North\ of\ East$$

(c) Return vector → negative of net displacement, D=0.57m, directed 25º South of West

P2

$\vec{d}_1 = 4\hat{i} + 5\hat{j} - 6\hat{k}$

$\vec{d}_2 = -\hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{d}_3 = 4\hat{i} + 3\hat{j} + 2\hat{k}$

$(a)\ \vec{r} = \vec{d}_1 - \vec{d}_2 + \vec{d}_3\,?$

(b) *Angle between* $\vec{r}$ *and* $+z$?

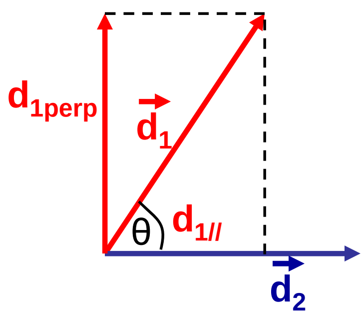
(c) *Component of* $\vec{d}_1$ *along* $\vec{d}_2$?

(d) *Component of* $\vec{d}_1$ *perpendicular to* $\vec{d}_2$ *and in plane of* $\vec{d}_1, \vec{d}_2$?

$$(a)\ \vec{r} = \vec{d}_1 - \vec{d}_2 + \vec{d}_3 = (4\hat{i} + 5\hat{j} - 6\hat{k}) - (-\hat{i} + 2\hat{j} + 3\hat{k}) + (4\hat{i} + 3\hat{j} + 2\hat{k}) = 9\hat{i} + 6\hat{j} - 7\hat{k}$$

$$(b)\ \vec{r}\cdot\hat{k} = r\cdot 1\cdot\cos\theta = -7 \rightarrow \theta = \cos^{-1}\left(\frac{-7}{12.88}\right) = 123^\circ$$

$$r = \sqrt{9^2 + 6^2 + 7^2} = 12.88m$$



$$(c)\ \vec{d}_1\cdot\vec{d}_2 = -4 + 10 - 18 = -12 = d_1 d_2 \cos\theta \rightarrow \cos\theta = \frac{\vec{d}_1\cdot\vec{d}_2}{d_1 d_2}$$

$$d_{1//} = d_1\cos\theta = d_1\frac{\vec{d}_1\cdot\vec{d}_2}{d_1 d_2} = \frac{-12}{3.74} = -3.2m$$

$$d_2 = \sqrt{1^2 + 2^2 + 3^2} = 3.74m$$

$$(d)\ d_1 = \sqrt{d_{1//}^2 + d_{1perp}^2} \rightarrow d_{1perp} = \sqrt{8.77^2 - 3.2^2} = 8.16m$$

$$d_1 = \sqrt{4^2 + 5^2 + 6^2} = 8.77m$$

P3

If $\vec{d}_1 = 3\hat{i} - 2\hat{j} + 4\hat{k}$

$\vec{d}_2 = -5\hat{i} + 2\hat{j} - \hat{k}$

$(\vec{d}_1 + \vec{d}_2)\cdot(\vec{d}_1 \times 4\vec{d}_2)$?

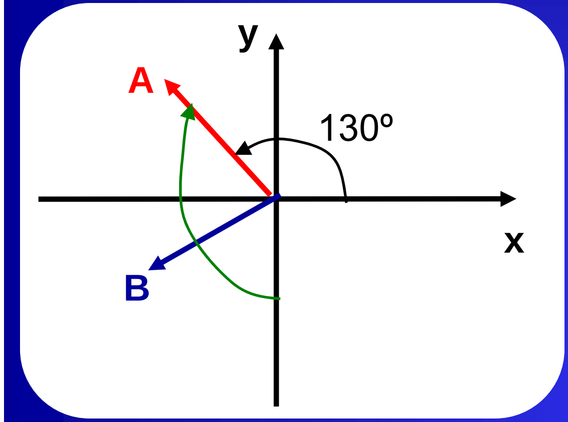
$(\vec{d}_1 + \vec{d}_2) = \vec{a} \rightarrow$ *contained in* $(\vec{d}_1, \vec{d}_2)$ *plane*

$(\vec{d}_1 \times 4\vec{d}_2) = 4(\vec{d}_1 \times \vec{d}_2) = 4\vec{b} \rightarrow$ *perpendicular to* $(\vec{d}_1, \vec{d}_2)$ *plane*

$\vec{a}$ *perpendicular to* $\vec{b} \rightarrow \cos 90^\circ = 0 \rightarrow 4\vec{a}\cdot\vec{b} = 0$

Tip: Think before calculate !!!

P4: Vectors $\vec{A}$ and $\vec{B}$ lie in an xy plane. $\vec{A}$ has a magnitude 8.00 and angle 130º; $\vec{B}$ has components B_x= -7.72, B_y= -9.20. What are the angles between the negative direction of the y axis and (a) the direction of $\vec{A}$, (b) the direction of $\vec{A}\times\vec{B}$, (c) the direction of $\vec{A}\times(\vec{B}+3\hat{k})$?



$$(a)\ \ Angle\ between\ -y\ and\ \vec{A} = 90^\circ + 50^\circ = 140^\circ$$

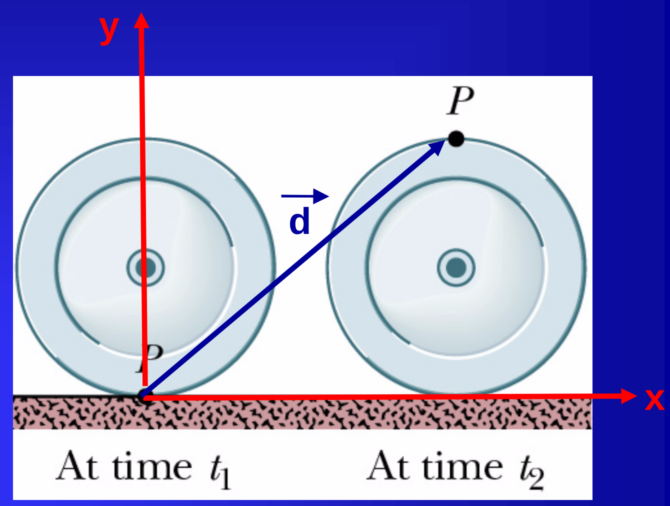
$$(b)\ \ Angle\ -y\ ,(\vec{A}\times\vec{B}) = \vec{C} \to angle\ -\hat{j}\ ,\hat{k}\ because\ \vec{C}\ perpendicular$$
$$plane\ (\vec{A},\vec{B}) = (xy) \to 90^\circ$$

$$(c)\ \ Direction\ \vec{A}\times(\vec{B}+3\hat{k}) = \vec{D}$$
$$\vec{E} = \vec{B}+3\hat{k} = -7.72\hat{i} - 9.2\hat{j} + 3\hat{k}$$

$$\vec{D} = \vec{A}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5.14 & 6.13 & 0 \\ -7.72 & -9.20 & 3 \end{vmatrix} = 18.39\hat{i} + 15.42\hat{j} + 94.61\hat{k}$$

$$|D| = \sqrt{18.39^2 + 15.42^2 + 94.61^2} = 97.61$$
$$-\hat{j}\cdot\vec{D} = -\hat{j}\cdot(18.39\hat{i} + 15.42\hat{j} + 94.61\hat{k}) = -15.42$$
$$\cos\theta = \left(\frac{-\hat{j}\cdot\vec{D}}{1\cdot|D|}\right) = \left(\frac{-15.42}{97.61}\right) \to \theta = 99^\circ$$

P5: A wheel with a radius of 45 cm rolls without sleeping along a horizontal floor. At time t_1 the dot P painted on the rim of the wheel is at the point of contact between the wheel and the floor. At a later time t_2, the wheel has rolled through one-half of a revolution. What are (a) the magnitude and (b) the angle (relative to the floor) of the displacement P during this interval?



Vertical displacement: $2R = 0.9m$

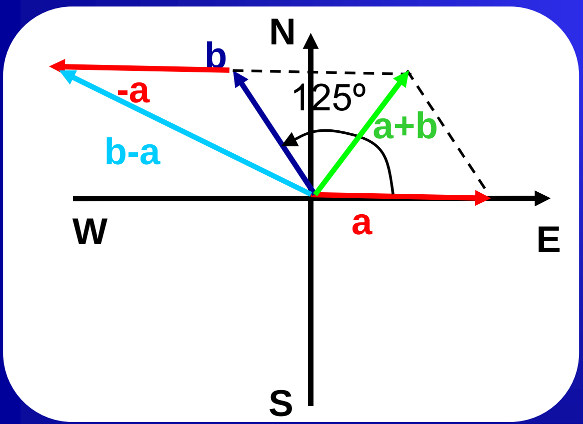
Horizontal displacement: $\frac{1}{2}(2\pi R) = 1.41m$

$$\vec{r} = (1.41m)\hat{i} + (0.9m)\hat{j}$$

$$|\vec{r}| = \sqrt{1.41^2 + 0.9^2} = 1.68m$$

$$\tan\theta = \left(\frac{2R}{\pi R}\right) \rightarrow \theta = 32.5^\circ$$

P6: Vector $\vec{a}$ has a magnitude of 5.0 m and is directed East. Vector $\vec{b}$ has a magnitude of 4.0 m and is directed 35º West of North. What are (a) the magnitude and direction of $(\vec{a}+\vec{b})$?. (b) What are the magnitude and direction of $(\vec{b}-\vec{a})$?. (c) Draw a vector diagram for each combination.



$$\vec{a} = 5\hat{i}$$

$$\vec{b} = -4\sin 35^\circ\,\hat{i} + 4\cos 35^\circ\,\hat{j} = -2.29\hat{i} + 3.28\hat{j}$$

(a) $\vec{a}+\vec{b} = 2.71\hat{i} + 3.28\hat{j}$

$$|\vec{a}+\vec{b}| = \sqrt{2.71^2 + 3.28^2} = 4.25m$$

$$\tan\theta = \left(\frac{3.28}{2.71}\right) \rightarrow \theta = 50.43^\circ$$

(b) $\vec{b}-\vec{a} = \vec{b} + (-\vec{a}) = -7.29\hat{i} + 3.28\hat{j}$

$$|\vec{b}-\vec{a}| = \sqrt{7.29^2 + 3.28^2} = 8m$$

$$\tan\theta = \left(-\frac{3.28}{7.29}\right) \rightarrow \theta = -24.2^\circ$$

$$or\ 180^\circ + (-24.2^\circ) = \boxed{155.8^\circ}$$

$$180^\circ - 155.8^\circ = 24.2^\circ\ \textit{North of West}$$