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Geometric Series

Motivation (I hope)

Geometric series are a basic artifact of algebra that everyone should know.[1] I am teaching them here because they come up remarkably often with Markov chains. The finite geometric series formula is at the heart of many of the fundamental formulas of financial mathematics. All students of the mathematical sciences should be intimately familiar with this topic and have all the formulas memorized. Geometric series can be characterized by the following properties:

> A geometric series is a sum of either a finite or an infinite number of terms. Each term after the first term of a geometric series is a multiple of the previous term by some fixed constant, x.

Example 25 + 50 + 100 + 200 + 400 is a geometric series because each term is twice the previous term.

Example 4 + 2 + 1 + .5 + .25 + .125 + .625 + ... is an (infinite) geometric series because each term is 1/2 the previous term.

> Multiplication of a geometric series by a constant does not affect its nature. It is still a geometric series. Whether it converges (actually adds up to anything) is unaffected. If $x + x^2 + x^3 + x^4 + ... = L$, then $C \cdot x + C \cdot x^2 + C \cdot x^3 + C \cdot x^4 + ... = C \cdot L$.

[1]Japanese children are thoroughly trained in geometric series before they enter pre-school.

The Finite Geometric Series

The most basic geometric series is $1 + x + x^2 + x^3 + x^4 + ... + x^n$. This is the *finite geometric series* because it has exactly n + 1 terms. It has a simple formula:

$$1 + x + x^2 + x^3 + x^4 + x^5 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$$

Formula 1 The Finite Geometric Series

This formula is easy to prove: just multiply both sides by 1 - x. All but two terms on the left will cancel. It can be proven just as easily by induction (proving it is an exercise in Section 6).

Example There is a simple fairy tale known to many people that I cannot tell here because this is a college text and it would be improper. However, if I were to tell it, it goes something like this: some *ordinary bloke* saves the king's life. The king, being at heart a regular guy, is grateful. He offers *ordinary bloke* ⅓ of his kingdom. But all *ordinary bloke* wants is that a chessboard be brought and on its first square be a grain of wheat, and on the second square two grains of wheat; then four on the next square and so on. The king thinks this is nothing. He offers *ordinary bloke* one of his daughters to go with ⅓ of his kingdom. He offers both of his daughters (this is actually a very sneaky trick, but that is another fairy tale). However, all *ordinary bloke* wants is a chessboard, and on its first square he wants a single grain of wheat. On the second square he wants 2 wheat grains. On the third square he wants 4 grains, and so on. The king tries to get him to go for something else, but *ordinary bloke* won't budge. Finally, the king says *let it be done*. The Chancellor of the Wheatery comes back and says there is not enough wheat! It turns out that the wheat was all eaten by rats. For embarrassing the monarchy the king has *ordinary bloke's* head cut off with a rusty axe. The moral of this story is quite simple: if a King offers you one of his daughters, take her; you can always find some way of dumping her later.

However, we being serious academics can go ahead and ask how much wheat did that guy want anyway? Well, on the first square he wanted $1 = 2^0$ grains. On the next square he wanted $2 = 2^1$ grains. On the next square he wanted $4 = 2^2$ grains, and so on. By the time he gets to the 64'th square he wants 2^{63} grains which is over nine-quintillion grains. But that is just the last square; the total he wanted is:

$$1 + 2 + 2^2 + 2^3 + ... + 2^{63} = \frac{1 - 2^{64}}{1 - 2} = 2^{64} - 1$$

In other words, he didn't just want over nine-quintillion grains, he want over eighteen-quintillion and that of course changes everything.

☐ **Exercise 1** Solve $1 + 10 + 10^2 + 10^3 + ... + 10^{10}$.

☐ **Exercise 2** Solve $1 - 3 + 9 - 27 + ... + (-3)^{10}$. Ordinarily a first problem like this requires a hint. In this case the hint is given in the last term.

☐ **Exercise 3** Solve 1/4 + 1/2 + 1 + 2 + 4 + ... + 1024. (In this case, the hint is to factor out 1/4.)

☐ **Exercise 4** Solve 1/6 - 1 + 6 - 36 + ... + 7776.

☐ **Exercise 5** Solve - 1/2 + 3 - 18 + 108 - ... - 23328.

Infinite Geometric Series

In **some** cases we can sum infinite geometric series. A simple case is 1/2 + 1/4 + 1/8 + 1/16 + ... This series can be seen to sum to 1. If you add it up by hand, you will see that it gets very close to 1 and it gets closer and closer and it gets arbitrarily close.[1] We know from above that the first n terms of the infinite series,

$1 + x + x^2 + x^3 + x^4 + \ldots$, is $\dfrac{1 - x^{n+1}}{1 - x}$. This sum will be finite if and only if the

term x^{n+1} goes to 0. That happens if and only if $-1 < x < 1$ (or more succinctly, $|x| < 1$). To see this, use your calculator and examine high powers of numbers between 1 and -1. Notice, that if x = 1 then, in the series, we are simply adding up an infinite number of 1's and of course the sum goes to infinity. Likewise, if x = −1, then we have the series 1 − 1 + 1 − 1 + 1 − 1 + ... It oscillates between 1 and 0. If x is less than −1, the series oscillates towards ± ∞, (take a look at what happens when x = −2). We have the law:

$$1 + x + x^2 + x^3 + x^4 + \ldots = \frac{1}{1 - x}; \quad -1 < x < 1$$

Formula 2 The Infinite Geometric Series

The restriction $-1 < x < 1$ is not a restriction as far as probability is concerned because the case where x is a probability and x = 1 is always trivial.

<u>Example</u> Find the sum of: 10 + 1 + .1 + .01 + .001 + .0001 + .00001 + ... We know that this is a geometric series since each term is .1 times the previous term. The series is infinite in form. However, since .1 is between −1 and 1, we know that the series has a finite sum. To get the series into the form of **Formula 2** , we

[1]Closer and closer does **not** imply arbitrarily close. In truth we have ventured into the realm of calculus. Do not panic, people were doing calculus long before it was invented. We are only skirting the edges and there is no law that says you have to have had the course.

factor 10 out of each term to get: 10(1 + .1 + .01 + .001 + .0001 + .00001 + ...).

According to the formula this is $10\left(\frac{1}{1-.1}\right) = 11\frac{1}{9}$.

Example Find the sum of: − .5 + .25 − .125 + .625 − .3125 + ... In this case, we factor −.5 out of each term to get: −.5(1 − .5 + .25 − .125 + .625 −...). This is just the infinite geometric series with x = −.5. By the formula, the sum is:

$$-.5\left(\frac{1}{1 - (-.5)}\right) = -.5\left(\frac{2}{3}\right) = -\frac{1}{3}$$

The High School Derivation of the Infinite Series Formula

You may recall an easier derivation of the infinite series formula from high school. It goes like this. We want to sum $1 + x + x^2 + x^3 + x^4 + ...$ We set it equal to S; that is $S = 1 + x + x^2 + x^3 + x^4 + ...$ Multiplying both sides by x we get $xS = x + x^2 + x^3 + x^4 + ...$ In other words: $S = 1 + xS$. Solving for S, we get $S = 1/(1 - x)$ which is precisely the formula derived above. The only problem with this solution technique is that when we set $S = 1 + x + x^2 + x^3 + x^4 + ...$, we assumed that the sum S exists. However, we know from above that the sum S exists if and only if $-1 < x < 1$.

☐ **Exercise 6** Find the sum of $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots$.

☐ **Exercise 7** Using Exercise 6, find the rational equivalent of 1.11111... (that is, put it in the form of a fraction).

☐ **Exercise 8** Find the sum of the infinite geometric series 1 + 2 + 4 + 8 + 16 + ...

☐ **Exercise 9** Find the sum of $12 + 4 + \frac{4}{3} + \frac{4}{9} + \frac{4}{27} + \cdots$.

☐ **Exercise 10** Find the sum of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \cdots$.

Finite Geometric Series Again

Above, we derived the formula for the sum of the infinite geometric series from the formula for the sum of the finite geometric series. (We also looked at an easier way of deriving the infinite case that always assumes that the limit exists.) Since it is easier to remember the infinite formula than the finite formula, it is worth looking at a way of getting the finite formula from the infinite formula. This is made especially useful since it uses an otherwise useful trick.

We have (for $-1 < x < 1$) that $1 + x + x^2 + x^3 + x^4 + \ldots = 1/(1 - x)$. Hence, if we multiply both sides by x^{n+1}, we get $x^{n+1} + x^{n+2} + x^{n+3} + \ldots = x^{n+1}/(1-x)$. Subtracting the second expression from the first we get:

$$1 + x + x^2 + x^3 + \ldots - (x^{n+1} + x^{n+2} + x^{n+3} + \ldots) =$$

$$\frac{1}{1 - x} - \frac{x^{n+1}}{1 - x} = \frac{1 - x^{n+1}}{1 - x}$$

Variations of Geometric Series

There are two variations of infinite geometric series that appear a lot in probability problems. The first can be derived in several ways. First, if you know calculus, you can derive this series by differentiating both sides of **Formula 2** . If you do not know calculus, you can derive it simply by squaring both sides of **Formula 2** . With the second approach, you sum the infinite sequence of equations:

$$1 + x + x^2 + x^3 + x^4 + x^5 + \dots = \frac{1}{1 - x}$$
$$x + x^2 + x^3 + x^4 + x^5 + \dots = \frac{x}{1 - x}$$
$$x^2 + x^3 + x^4 + x^5 + \dots = \frac{x^2}{1 - x}$$
$$\vdots$$

to get:

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \dots = \frac{1}{1 - x}(1 + x + x^2 + x^3 + x^4 + x^5 + \dots)$$

This can be stated more succinctly:

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = \frac{1}{(1 - x)^2} \quad ; \quad -1 < x < 1$$

Formula 3 A Variation of the Infinite Geometric Series

A second variation is obtained simply by multiplying both sides of **Formula 3** by x. This yields:

$$x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + \ldots = \frac{x}{(1-x)^2}\ ; \qquad -1 < x < 1$$

Formula 4 A Second Variation of the Infinite Geometric Series

The Return of the Geometric Distribution

You may recall from Section 26 that in the geometric distribution, that we are doing a Bernoulli experiment (that is an independent sequence of binary experiments) until we get a success. If the probability of a success is p and the probability of failure is q (= 1 − p) then the probability that the first success occurs on the nth experiment is $q^{n-1}p$. This experiment is represented in **Figure 1**. The only reason that there is a transition from the second state to itself is so that this graph is a Markov chain.

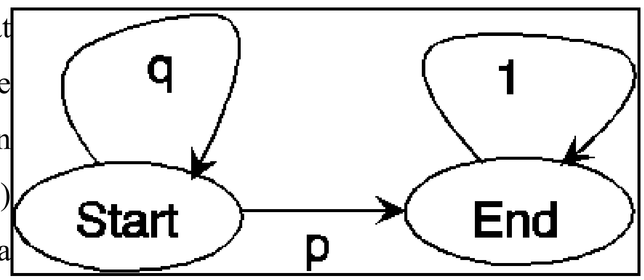

Figure 1 The Geometric Distribution

Again, **Figure 1** represents an experiment that we repeat until we get a success. We already know the probability that success occurs on a particular experiment. However, we could ask: *How many experiments do we have to perform on average before we have a success?* More precisely, what is the expected number of experiments until we have a success? Remember (again from Section 26) the expected outcome of an experiment is the sum, $\sum x \cdot p(x)$, of all the outcomes (x) times their respective probabilities. In this case, the outcome we are interested in is the number experiments until we have a success. Hence, the possible outcomes are the positive integers: 1, 2, 3, ... We have that the probability of a specific outcome n is $q^{n-1}p$.

Therefore the expected number of experiments is the sum $\sum nq^{n-1}p$ as n takes on all the values of the positive integers 1, 2, 3, ... Writing this out we have:

$$\sum_{n=1}^{\infty} n\cdot q^{n-1}p = p + 2qp + 3q^2p + 4q^3p + \dots = p\sum_{n=1}^{\infty} n\cdot q^{n-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

The summation was reduced using **Formula 3** . The next simplification used the fact that $q = 1 - p$, or $1 - q = p$. Note that this answer is entirely intuitive (and note also that intuition in probability is to be highly suspected!).[1] If the probability of a successful experiment is .1 then the expected number of experiments until a success is $1/.1 = 10$.

The Probability That A Occurs Before B[2]

Consider a collection of **independent** events, including two events A and B. We are interested in the probability that the event A occurs before the event B. Let us denote the new event *not A nor B* as C so that $P(C) = 1 - P(B) - P(C)$. The probability that A occurs before B is the probability that we have a sequence of events of the form C^kA where k is equal to 0, 1, 2, ... For example that case that A occurs immediately is just C^0A (since a non-zero entity to

[1]Another faster way of getting this result is to use the following fact. If the expected number of experiments to success is μ then $\mu = 1\cdot p + (1-p)\cdot(1+\mu)$. Solving for μ we get the above result.

[2]This is a charming elementary application of geometric series. However, I would not have thought of it, if I had not seen it in *The Art of Probability For Scientists and Engineers*, by R. W. Hamming (Addison-Wesley) 1991, p. 156. Professor Hamming uses this formula to easily calculate the probability of winning at craps. Later, we will do a more extensive analysis of craps.

the zero'th power is one). The case that three events neither A nor B occur before A occurs is C^3A. **The probability that A occurs before B is:**

$$\mathrm{P}\left(\sum_{k=0}^{k=\infty} C^k A\right) = \sum_{k=0}^{k=\infty} \mathrm{P}\left(C^k A\right) = \mathrm{P}(A)\sum_{k=0}^{k=\infty} \mathrm{P}(C^k) = \mathrm{P}(A)\sum_{k=0}^{k=\infty} \left(\mathrm{P}(C)\right)^k$$

$$= \frac{\mathrm{P}(A)}{1 - \mathrm{P}(C)} = \frac{\mathrm{P}(A)}{\mathrm{P}(A) + \mathrm{P}(B)}$$

Since $0 < \mathrm{P}(C) < 1$, we are able to use the formula for the infinite geometric series. Again, the result is both elegant and intuitive.

1. 11,111,111,111

2. $1 - 3 + .. + (-3)^{10} = 44{,}287$

3. $\frac{1}{4}(1 + .. + 2^{12}) = 2{,}047.75$

4. $(1/6)(1 - 6 + .. + (-6)^6) = 6{,}665.166666$

5. $-\frac{1}{2}(1 - 6 + .. + (-6)^6) = -19{,}995.5$

6. 10/9

7. 1.11111... is precisely the same thing that the previous problem asks for: 10/9

8. This is a infinite geometric series with $|x| > 1$. It goes to infinity. We say that it is an illegal problem or that it is has no solution.

9. $12(1 + \frac{1}{3} + (\frac{1}{3})^2 + ...) = 18$

10. $1 + (-\frac{1}{2}) + (-\frac{1}{2})^2 + ... = \frac{2}{3}$