# Chapter 4 HW Solution

Review Questions.

1. Name the performance specification for first order systems. Time constant τ.

2. What does the performance specification for a first order system tell us? How fast the system responds.

5. The imaginary part of a pole generates what part of the response? The un-decaying sinusoidal part.

6. The real part of a pole generates what part of the response? The decay envelope.

8. If a pole is moved with a constant imaginary part, what will the responses have in common? Oscillation frequency.

9. If a pole is moved with a constant real part, what will the responses have in common? Decay envelope.

10. If a pole is moved along a radial line extending from the origin, what will the responses have in common? Damping ratio (and % overshoot).

13. What pole locations characterize (1) the underdamped system, (2) the overdamped system, and (3) the critically damped system?

1. Complex conjugate pole locations.

2. Real (and separate) pole locations.

3. Real identical pole locations.

14. Name two conditions under which the response generated by a pole can be neglected.

1. The pole is "far" to the left in the s-plane compared with the other poles.

2. There is a zero very near to the pole.

Problems.

Problem 2(a). This is a 1$^{\text{st}}$ order system with a time constant of 1/5 second (or 0.2 second). It also has a DC gain of 1 (just let $s = 0$ in the transfer function).
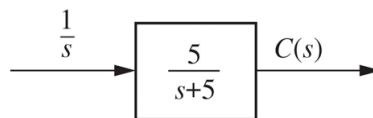
The input shown is a unit step; if we let the transfer function be called $G(s)$, the output is input $\times$ transfer function. The resulting response function $C(s)$ is #9 in my Laplace transform table, or you can expand the result in partial fractions,

$$C(s) = \frac{1}{s}\, G(s) = \frac{5}{s(s+5)} = \frac{1}{s} - \frac{1}{s+5} \tag{1}$$

Either way, the resulting response $c(t)$ is

$$\boxed{c(t) = 1 - e^{-5t}} \tag{2}$$

The time constant, rise time, and 2% settling time are:

$$\boxed{\begin{aligned} \tau &= 1/5 \text{ sec} \\ T_r &= 2.2\tau = 0.44 \text{ sec} \\ 2\% \ T_s &= 4\tau = 0.8 \text{ sec} \end{aligned}} \tag{3}$$

Problem 3(a). Same system as above, but use MATLAB `step` function to find step response. I did something like:

```
>> numG = [0 5]; % Define TF numerator
>> denG = [1 5]; % Define TF denominator
>> G = tf(num,den); % Define transfer function
>> [y,t] = step(G); % Find step response
>> plot(t,y); % Plot step response
```

The step response is shown in Figure 1 at right. At 0.2 seconds the response is 63% of the way to the final value. Hopefully the rise time and settling time are also about right.
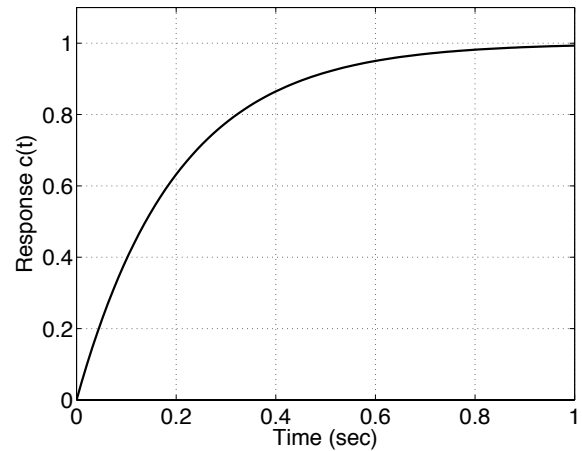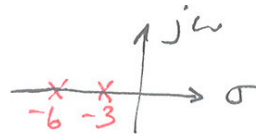
Figure 1: Step response of first-order system with $\tau = 0.2$ sec.

Problem 8. (b) The TF is

$$T(s) = \frac{5}{(s+3)(s+6)} \implies \text{poles at } s = -3, \ -6$$

and the poles are shown below.
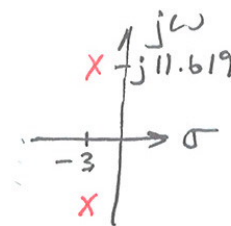
The general form of the step response will be

$$y(t) = A + Be^{-3t} + Ce^{-6t} \tag{4}$$

and it will be **OVERDAMPED**.

(d) This TF is

$$T(s) = \frac{20}{s^2 + 6s + 144} \implies \text{poles at } s = -3 \pm j11.619$$

and the poles are shown below.

The general form of the step response will be

$$y(t) = A + Be^{-3t}\cos(11.619t + \phi) \tag{5}$$

and it will be **UNDERDAMPED**.

Problem 9. To find the poles of

$$T(s) = \frac{s^2 + 2s + 2}{s^4 + 6s^3 + 4s^2 + 7s + 2} \tag{6}$$

one way is to use the MATLAB "`roots`" function:

```
>> roots([1 6 4 7 2])

ans = -5.4917
      -0.0955 + 1.0671i
      -0.0955 - 1.0671i
      -0.3173
```

So the poles of the given transfer function are:

$$\boxed{s = -0.0955 \pm j1.0671, \ -0.3173, \ -5.4917} \tag{7}$$

Note that poles (roots) always occur as real numbers or complex conjugates. This is why all systems are "made up" of first and second-order subsystems.

Problem 18. The standard form of a second-order transfer function denominator is

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

By equating coefficients and solving for damping ratio ζ and (undamped) natural frequency ω_n, we get:

> (b) $(s+3)(s+6) = s^2 + 9s + 18$ from which we find $\omega_n = \sqrt{18} = 4.24$ rad/s, $\zeta = 1.06$ (overdamped)

> (d) $s^2 + 6s + 144$ from which we find $\omega_n = \sqrt{144} = 12$ rad/s, $\zeta = 0.25$ (underdamped)

Problem 20(c). Similar approach to the previous problem. The transfer function now is

$$T(s) = \frac{1.05 \times 10^7}{s^2 + 1.6 \times 10^3 s + 1.05 \times 10^7} = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{8}$$

The natural frequency is

$$\boxed{\omega_n = \sqrt{1.05 \times 10^7} = 3,240 \text{ rad/s } (516 \text{ Hz})} \tag{9}$$

The damping ratio is

$$\boxed{2\zeta\omega_n = 1.6 \times 10^3 \implies \zeta = \frac{1.6 \times 10^3}{2\omega_n} = 0.247} \tag{10}$$

Settling time, peak time, rise time, and % overshoot: these are all functions of ζ and ω_n.

$$\text{Settling time } T_s = \frac{4}{\zeta\omega_n} = 0.005 \text{ sec} \tag{11}$$

$$\text{Peak time } T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.001 \text{ sec} \tag{12}$$

$$\text{Rise time } T_r \approx \frac{1.27}{\omega_n} \text{ (Fig. 4.16)} = 0.00039 \text{ sec} \tag{13}$$

$$\% \text{ Overshoot} = \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \times 100 = 45\% \tag{14}$$

Problem 21(c). The unit step response for the system of Problem 20(c) is shown in Figure 2 on the next page, with the response characteristics indicated. I got them all from the response data rather than the expressions of Problem 20(c).
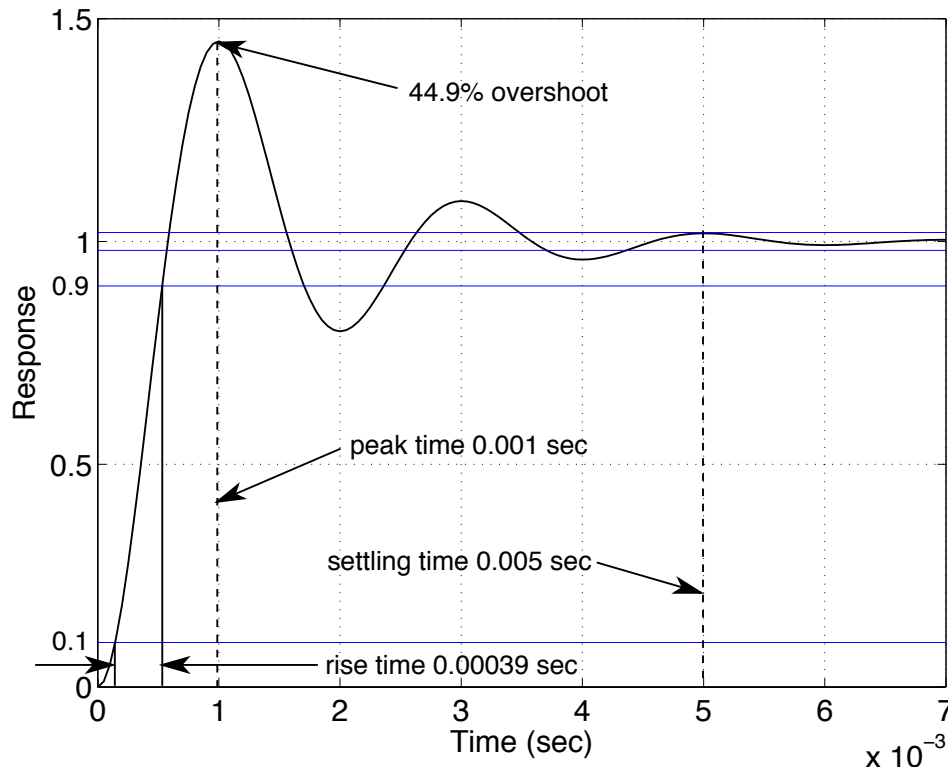
Figure 2: Unit step response of Problem 21(c) using MATLAB. All response characteristics obtained from data, not analytical expressions. They seem to agree quite well.

Problem 23. For the following second-order response specs, find the corresponding pole locations.

(a) Overshoot of 12% means $\zeta = 0.55$, and $T_s = \dfrac{4}{\zeta\omega_n} = 0.6$ sec means $\zeta\omega_n = 6.67$, so $\omega_n = 12.1$ rad/s. So the pole location is

$$\boxed{s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \approx -6.65 + j10.1}$$

(b) Overshoot of 10% means $\zeta = 0.6$, and $T_p = \dfrac{\pi}{\omega_d} = 5$ sec means $\omega_d = 0.628$ sec, and $\omega_n = \omega_d/\sqrt{1-\zeta^2} = 0.78$. So the pole location is

$$\boxed{s = -\zeta\omega_n \pm j\omega_d \approx -0.47 \pm j0.628} \tag{15}$$

(c) Settling time $T_s = \dfrac{4}{\zeta\omega_n} = 7$ sec means $\zeta\omega_n = 0.57$. Peak time $T_p = \dfrac{\pi}{\omega_d} = 3$ sec means $\omega_d \approx 1.05$ rad/s. So like part (b), the pole location is

$$\boxed{s = -\zeta\omega_n \pm j\omega_d \approx -0.57 \pm j1.05} \tag{16}$$

Problem 29(c). From Figure P4.9(c), the step response has 40% overshoot, hence the damping ratio $\zeta \approx 0.3$. The peak time T_p is about 4 sec, so the damped frequency $\omega_d \approx 0.78$ rad/s. Then the *undamped* natural frequency $\omega_n = \omega_d/\sqrt{1-\zeta^2} \approx 0.82$ rad/s. Finally, the DC gain is 1. So the transfer function is

$$\boxed{\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{0.67}{s^2 + 0.49s + 0.67}}$$

Problem 30. This is on **pole-zero cancellation.** As I indicated, this problem was **poorly posed**!! To change the "response functions" into "transfer functions," simply remove the "s" from the denominator of each function (this is removing the unit step input). Then the $C(s)$ becomes $G(s)$ (a better letter to use for a transfer function).

I decided I just wanted for you to plot the unit step response of each original system plus that of the "cancelled" system on the same plot (four plots: two responses on each plot). Adjust the numerator coefficient of the "cancelled" system so the "DC Gain" of the "cancelled" system is the same as the original. The four TFs (original on left; cancelled on right) are:

$$(a) \qquad G(s) = \frac{s+3}{(s+2)(s^2+3s+10)} = \frac{s+3}{(s+2)(s+1.5 \pm j2.8)}, \qquad G_c(s) = \frac{1.5}{s^2+3s+10}$$

$$(b) \qquad G(s) = \frac{s+2.5}{(s+2)(s^2+4s+20)} = \frac{s+3}{(s+2)(s+2 \pm j4)}, \qquad G_c(s) = \frac{1.25}{s^2+4s+20}$$

$$(c) \qquad G(s) = \frac{s+2.1}{(s+2)(s^2+2s+5)} = \frac{s+3}{(s+2)(s+0.5 \pm j2.2)}, \qquad G_c(s) = \frac{1.05}{s^2+2s+5}$$

$$(d) \qquad G(s) = \frac{s+2.01}{(s+2)(s^2+5s+20)} = \frac{s+3}{(s+2)(s+2.5 \pm j3.7)}, \qquad G_c(s) = \frac{1.005}{s^2+5s+20}$$

Note that I showed the complex pole locations for the quadratic polynomial in the "second" version of the original transfer function.
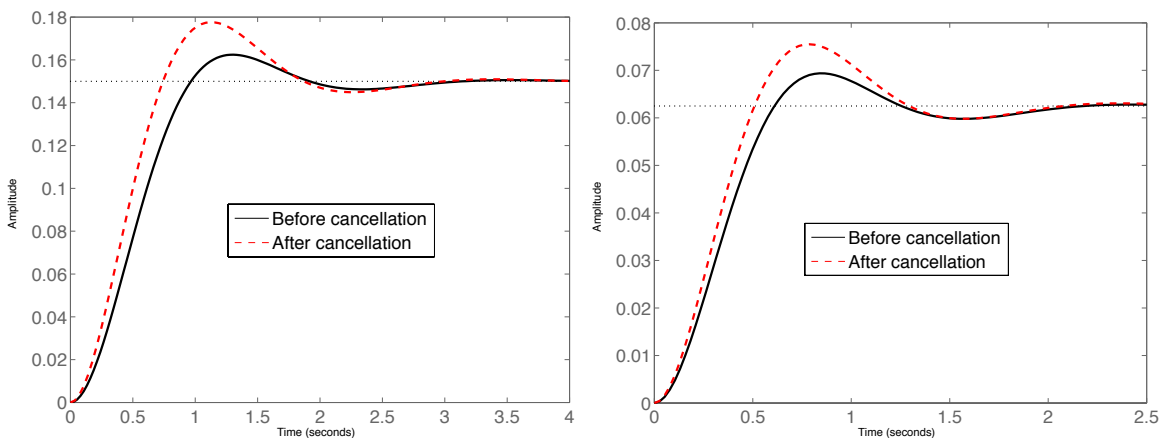
(a) Difference in response; probably shouldn't cancel.

(b) A little less difference, but still some.
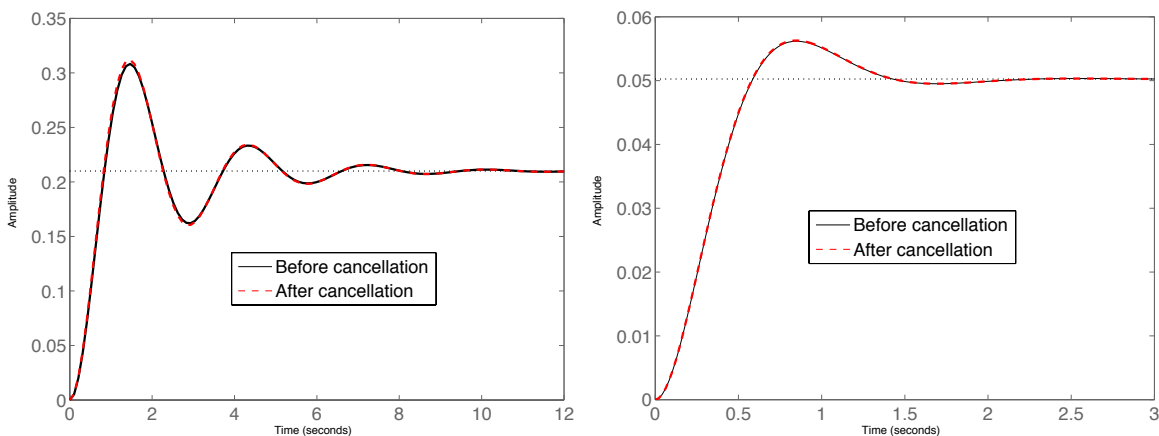
(c) Can definitely cancel here; almost no difference.

(d) Pole & zero completely cancel (almost).

Figure 3: The four step responses for Problem 30, showing situations where the pole and zero don't cancel, and when they do.

DC Motor Problem. Use the system of part (c) in the Chapter 3 HW assignment, and find transfer function $G(s)$, where

$$G(s) = \frac{\theta_L(s)}{E_a(s)} \quad \frac{\text{rad}}{\text{V}}$$

Plot the response of θ_L (rad) to a 10V step input in motor voltage e_a. Use MATLAB, and plot for 0.1 second.

Solution. From my notes, the transfer function from motor armature voltage $e_a(t)$ to load angular position $\theta_L(t)$ is:

$$\frac{\theta_L(s)}{E_a(s)} = \frac{\dfrac{K_t}{nR_aJ_t}}{s\left(s + \dfrac{b_t}{J_t}\right)} \tag{17}$$

where

$$n = \text{ gear ratio; in denominator of numerator to convert to } \theta_L$$

$$J_t = J_m + \frac{J_L}{n^2} \quad \text{(total inertia "seen" by motor)}$$

$$b_t = b_{eq} + \frac{b_L}{n^2} \quad \text{(and} \quad b_{eq} = b_m + \frac{K_tK_b}{R_a})$$

The response to a 10V step input is shown below in Figure 3. Note that in 0.1 second the load moves about 0.8 rad ($45°$), which is the same as we saw in the Chapter 3 HW state-space model.
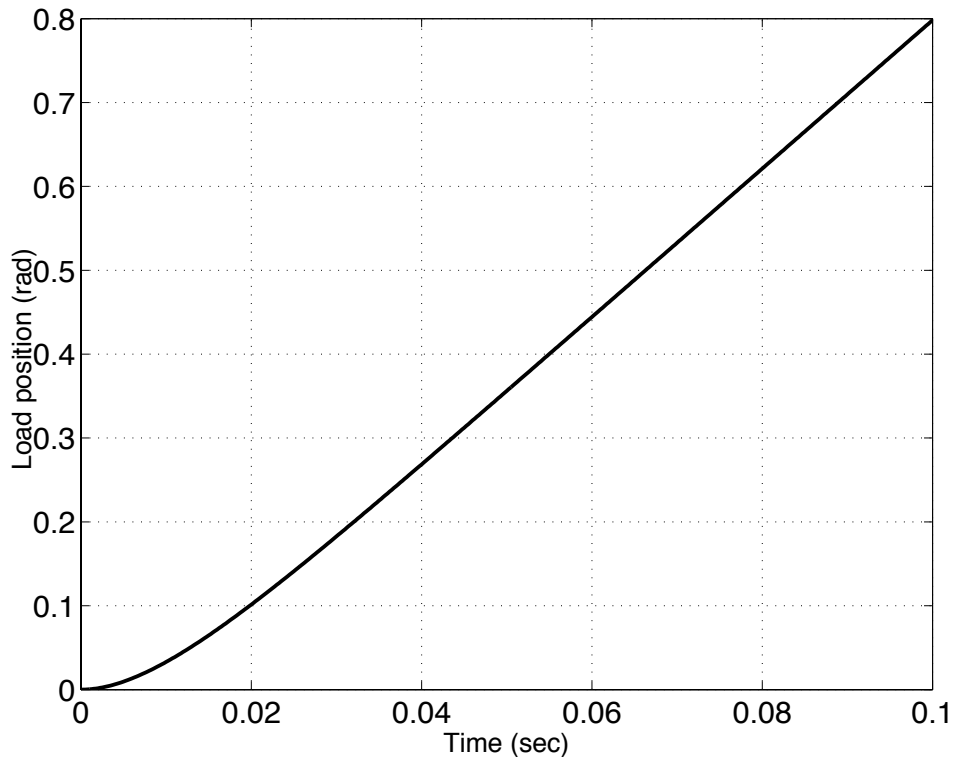

Figure 4: Response of motor to 10V step input. Note that in 0.1
second the load moves about 0.8 rad ($45°$), the same as in the
state-space model of Chapter 3.