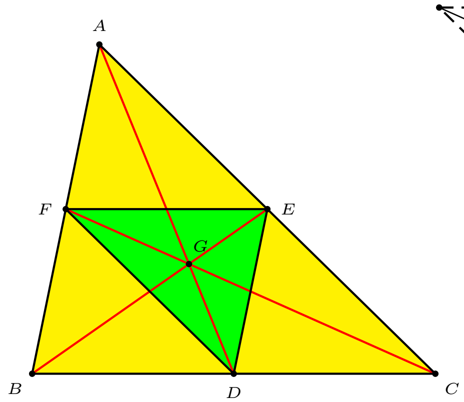
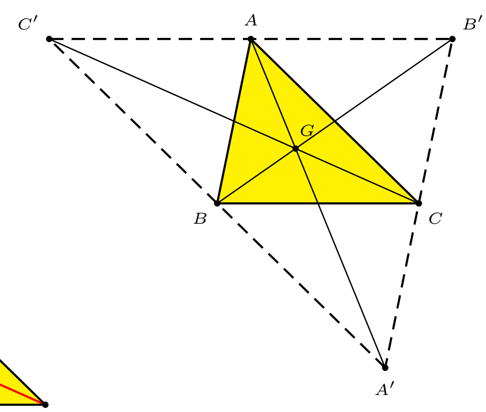
Chapter 4

The circumcircle and the incircle

4.1 The Euler line

4.1.1 Inferior and superior triangles





The *inferior triangle* of ABC is the triangle DEF whose vertices are the midpoints of the sides BC, CA, AB.

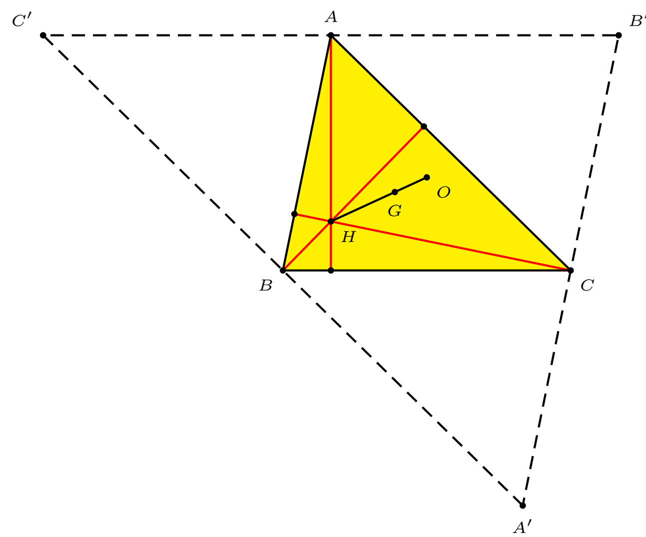
The two triangles share the same centroid G, and are homothetic at G with ratio $-1 : 2$.

The *superior triangle* of ABC is the triangle $A'B'C'$ bounded by the parallels of the sides through the opposite vertices.

The two triangles also share the same centroid G, and are homothetic at G with ratio $2 : -1$.

4.1.2 The orthocenter and the Euler line

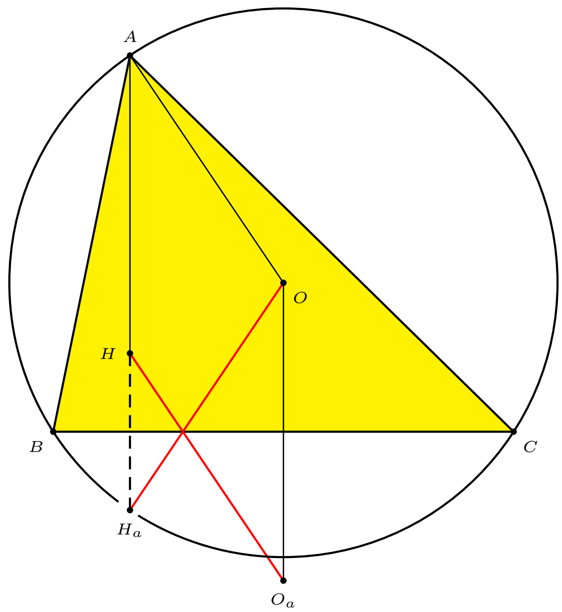
The three altitudes of a triangle are concurrent. This is because the line containing an altitude of triangle ABC is the perpendicular bisector of a side of its superior triangle. The three lines therefore intersect at the circumcenter of the superior triangle. This is the **orthocenter** of the given triangle.



The circumcenter, centroid, and orthocenter of a triangle are collinear. This is because the orthocenter, being the circumcenter of the superior triangle, is the image of the circumcenter under the homothety $\mathsf{h}(G, -2)$. The line containing them is called the **Euler line** of the reference triangle (provided it is non-equilateral).

The orthocenter of an acute (obtuse) triangle lies in the interior (exterior) of the triangle. The orthocenter of a right triangle is the right angle vertex.

Proposition 4.1. *The reflections of the orthocenter in the sidelines lie on the circumcircle.*

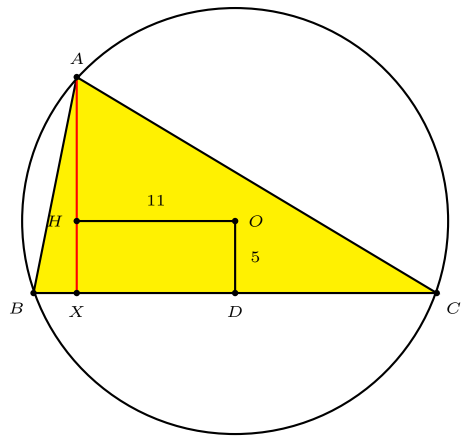


Proof. It is enough to show that the reflection H_a of H in BC lies on the circumcircle. Consider also the reflection O_a of O in BC. Since AH and OO_a are parallel and have the same length ($2R\cos\alpha$), AOO_aH is a parallelogram. On the other hand, HOO_aH_a is a isosceles trapezoid. It follows that $OH_a = HO_a = AO$, and H_a lies on the circumcircle. □

Exercise. **1.** A triangle is equilateral if and only if its circumcenter and centroid coincide.

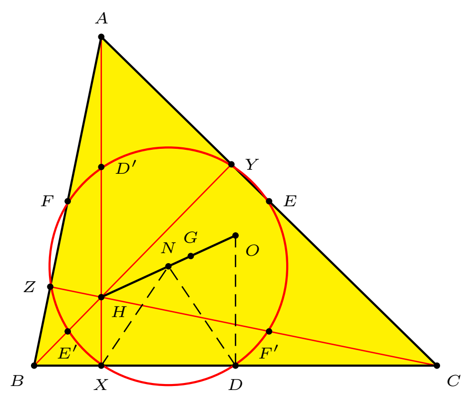
2. Let H be the orthocenter of triangle ABC. Show that
(i) A is the orthocenter of triangle HBC;
(ii) the triangles HAB, HBC, HCA and ABC have the same circumradius.

3. In triangle ABC with circumcenter O, orthocenter H, midpoint D of BC, and perpendicular foot X of A on BC, $OHXD$ is a rectangle of dimensions 11×5. Calculate the length of the side BC.



4.2 The nine-point circle

Theorem 4.1. *The following nine points associated with a triangle are on a circle whose center is the midpoint between the circumcenter and the orthocenter:*
(i) the midpoints of the three sides,
(ii) the pedals (orthogonal projections) of the three vertices on their opposite sides,
(iii) the midpoints between the orthocenter and the three vertices.



Proof. (1) Let N be the circumcenter of the inferior triangle DEF. Since DEF and ABC are homothetic at G in the ratio $1:2$, N, G, O are collinear, and $NG:GO=1:2$. Since $HG:GO=2:1$, the four are collinear, and

$$HN:NG:GO=3:1:2,$$

and N is the midpoint of OH.

(2) Let X be the pedal of H on BC. Since N is the midpoint of OH, the pedal of N is the midpoint of DX. Therefore, N lies on the perpendicular bisector of DX, and $NX=ND$. Similarly, $NE=NY$, and $NF=NZ$ for the pedals of H on CA and AB respectively. This means that the circumcircle of DEF also contains X, Y, Z.

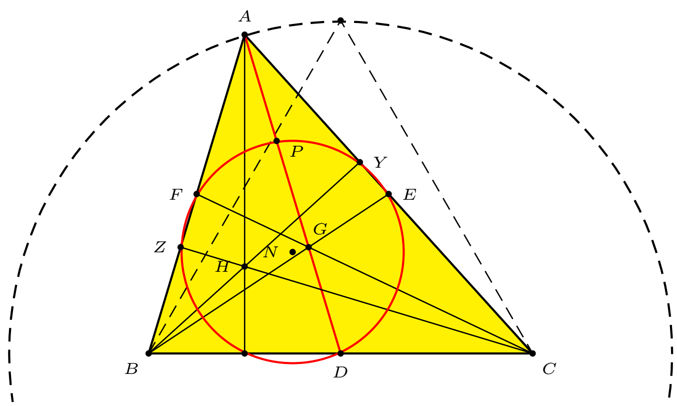
(3) Let D', E', F' be the midpoints of AH, BH, CH respectively. The triangle $D'E'F'$ is homothetic to ABC at H in the ratio $1:2$. Denote by N' its circumcenter. The points N', G, O are collinear, and $N'G:GO=1:2$. It follows that $N'=N$, and the circumcircle of DEF also contains D', E', F'. □

This circle is called the **nine-point circle** of triangle ABC. Its center N is called the nine-point center. Its radius is half of the circumradius of ABC.

Exercise. **1.** Let H be the orthocenter of triangle ABC. Show that the Euler lines of triangles ABC, HBC, HCA and HAB are concurrent. [1]

2. For what triangles is the Euler line parallel (respectively perpendicular) to an angle bisector? [2]

3. Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths are proportional to its medians lengths in some order.



Proof. $(\Rightarrow)$ $\frac{1}{3}m_a^2 = \frac{1}{4}(b^2+c^2-a^2)$; $4m_a^2 = 3(b^2+c^2-a^2)$; $2b^2+2c^2-a^2 = 3(b^2+c^2-a^2)$, $2a^2 = b^2+c^2$. Therefore, ABC is a root-mean-square triangle. $\square$

4. Let P be a point on the circumcircle. What is the locus of the midpoint of HP? Why?

5. If the midpoints of AP, BP, CP are all on the nine-point circle, must P be the orthocenter of triangle ABC? [3]

[1] Hint: find a point common to them all.

[2] The Euler line is parallel (respectively perpendicular) to the bisector of angle A if and only if $\alpha = 120^\circ$ (respectively 60°).

[3] P. Yiu and J. Young, Problem 2437 and solution, *Crux Math.* 25 (1999) 173; 26 (2000) 192.

Excursus: Triangles with nine-point center on the circumcircle

Begin with a circle, center O and a point N on it, and construct a family of triangles with (O) as circumcircle and N as nine-point center.

(1) Construct the nine-point circle, which has center N, and passes through the midpoint M of ON.

(2) Animate a point D on the minor arc of the nine-point circle *inside* the circumcircle.

(3) Construct the chord BC of the circumcircle with D as midpoint. (This is simply the perpendicular to OD at D).

(4) Let X be the point on the nine-point circle antipodal to D. Complete the parallelogram $ODXA$ (by translating the vector **DO** to X).

The point A lies on the circumcircle and the triangle ABC has nine-point center N on the circumcircle.

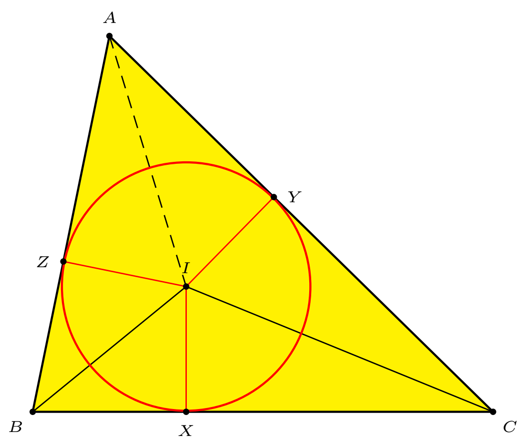
Here is a curious property of triangles constructed in this way: let A', B', C' be the reflections of A, B, C in their own opposite sides. The reflection triangle $A'B'C'$ degenerates, *i.e.*, the three points A', B', C' are collinear. [4]

[4] O. Bottema, *Hoofdstukken uit de Elementaire Meetkunde*, Chapter 16.

4.3 The incircle

The internal angle bisectors of a triangle are concurrent at the *incenter* of the triangle. This is the center of the *incircle*, the circle tangent to the three sides of the triangle.

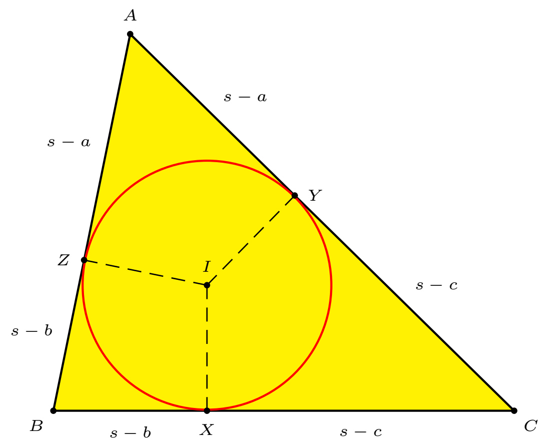
Let the bisectors of angles B and C intersect at I. Consider the pedals of I on the three sides. Since I is on the bisector of angle B, $IX = IZ$. Since I is also on the bisector of angle C, $IX = IY$. It follows $IX = IY = IZ$, and the circle, center I, constructed through X, also passes through Y and Z, and is tangent to the three sides of the triangle.



This is called the incircle of triangle ABC, and I the incenter.

Let s be the semiperimeter of triangle ABC. The incircle of triangle ABC touches its sides BC, CA, AB at X, Y, Z such that

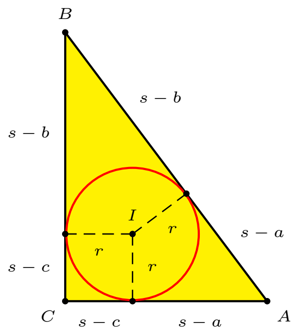
$$\begin{aligned} AY =& AZ = s-a, \\ BZ =& BX = s-b, \\ CX =& CY = s-c. \end{aligned}$$

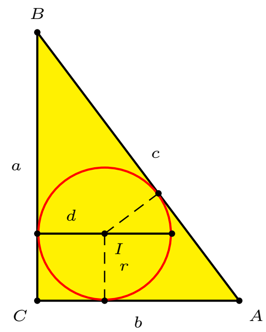


The inradius of triangle ABC is the radius of its incircle. It is given by

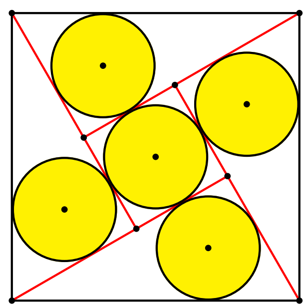
$$r = \frac{2\Delta}{a+b+c} = \frac{\Delta}{s}.$$

Exercise. **1.** Show that the inradius of a right triangle with hypotenuse c is $r = s - c$. Equivalently, if the remaining two sides have lengths a, b, and d is the diameter of the incircle, then $a + b = c + d$.

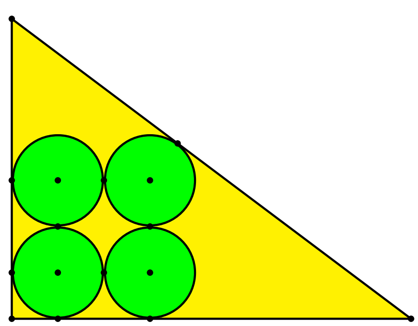




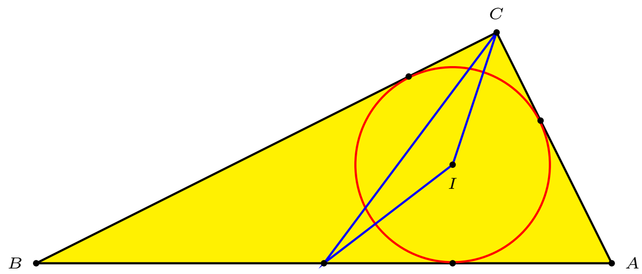
2. A square of side a is partitioned into 4 congruent right triangles and a small square, all with equal inradii r. Calculate r.



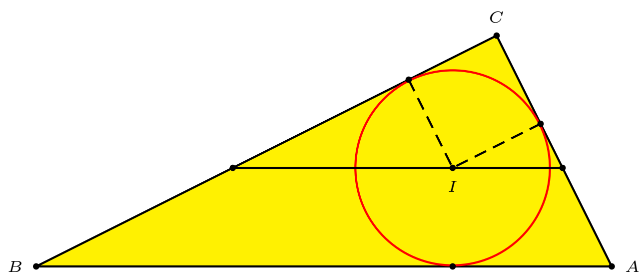
3. Calculate the radius of the congruent circles in terms of the sides of the right triangle.



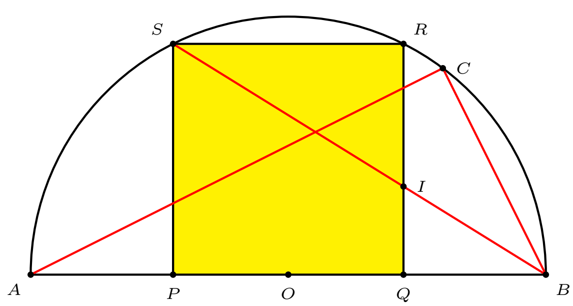
4. The incenter of a right triangle is equidistant from the midpoint of the hypotenuse and the vertex of the right angle. Show that the triangle contains a 30° angle.



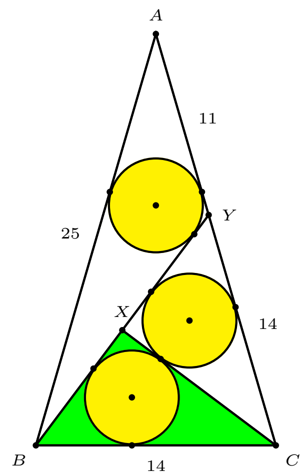
5. A line parallel to hypotenuse AB of a right triangle ABC passes through the incenter I. The segments included between I and the sides AC and BC have lengths 3 and 4. Calculate the area of the triangle.



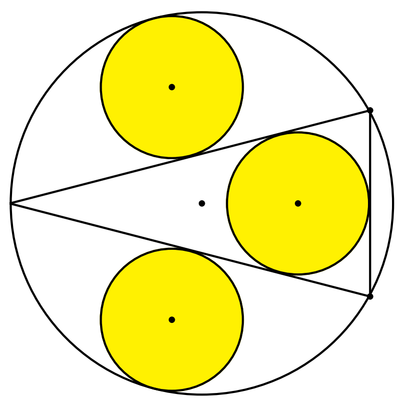
6. A square and a right triangle of equal areas are inscribed in a semicircle. Show that the lines BS and QR intersect at the incenter of triangle ABC.



7. ABC is an isosceles triangle with $AB = AC = 25$ and $BC = 14$. Y is a point on AC such that $CY = CB$, and X is the midpoint of BY. Calculate the inradii of the triangles XBC, XCY, and ABY.

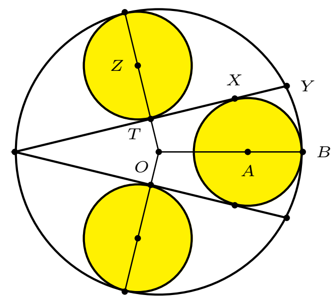


8. The triangle is isosceles and the three small circles have equal radii. Suppose the large circle has radius R. Find the radius of the small circles. [5]

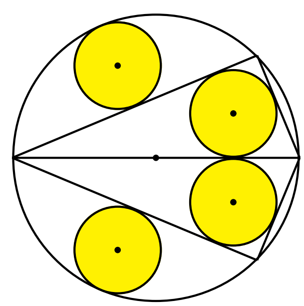


[5] Let θ be the semi-vertical angle of the isosceles triangle. The inradius of the triangle is $\frac{2R\sin\theta\cos^2\theta}{1+\sin\theta} = 2R\sin\theta(1-\sin\theta)$. If this is equal to $\frac{R}{2}(1-\sin\theta)$, then $\sin\theta = \frac{1}{4}$. From this, the inradius is $\frac{3}{8}R$.

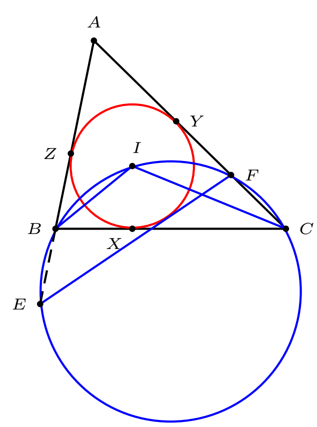
9. The three small circles are congruent. Show that each of the ratios $\frac{OA}{AB}$, $\frac{TX}{XY}$, $\frac{ZT}{TO}$. is equal to the golden ratio.



10. The large circle has radius R. The four small circles have equal radii. Calculate this common radius.

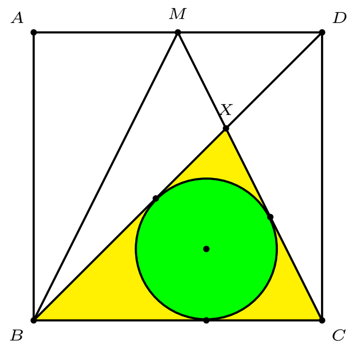


11. The circle BIC intersects the sides AC, AB at E and F respectively. Show that EF is tangent to the incircle of triangle ABC. [6]

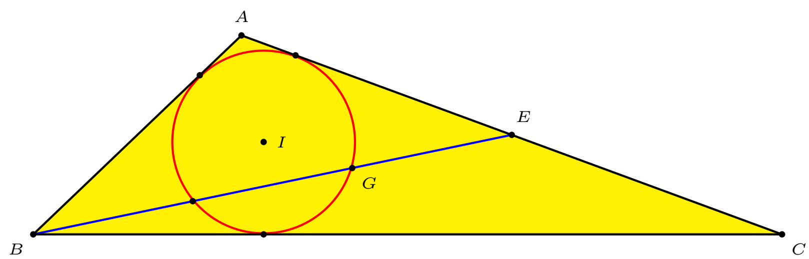


[6]Hint: Show that IF bisects angle AFE.

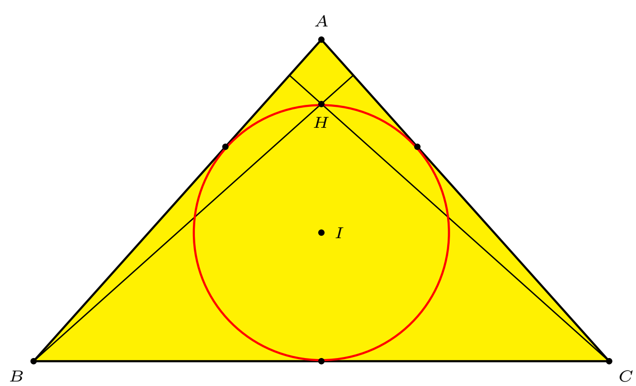
12. M is the midpoint of the side AD of square $ABCD$. The lines BD and CM intersect at X. Suppose each side of the square has length a. Calculate the inradius of triangle XBC.



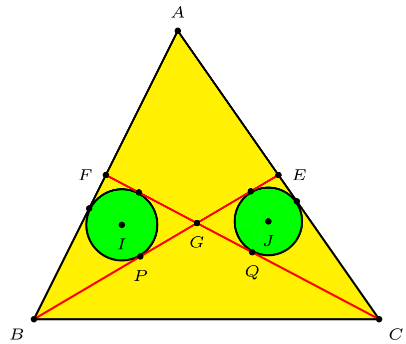
13. The median BE of triangle ABC is trisected by its incircle. Calculate $a : b : c$.



14. ABC is an isosceles triangle with $a : b : c = 4 : 3 : 3$. Show that its orthocenter lies on the incircle.



Example. The medians BE and CF of triangle ABC intersect at the centroid G. If the inradii of triangles BGF and CGE are equal, prove that the triangle is isosceles.

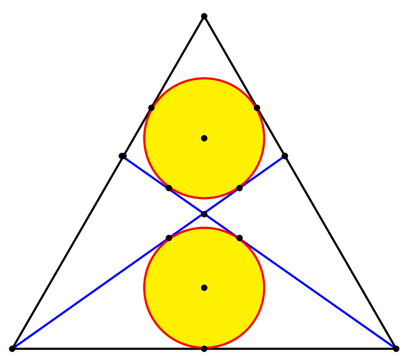


Proof. The triangles BGF and CGE have equal areas. If they have equal inradii, then they have the same semiperimeter s.

Now, if I and J are the incenters of the triangles, and P, Q points their points of tangency with the medians BE and CF, then triangles GIP and GJQ are congruent. It follows that $GP = GQ$.

Since $GP = s - \frac{c}{2}$ and $GQ = s - \frac{b}{2}$, it is clear that $b = c$, and triangle ABC is isosceles. ☐

Example. An equilateral triangle of side $2a$ is partitioned symmetrically into a quadrilateral, an isosceles triangle, and two other congruent triangles. If the inradii of the quadrilateral and the isosceles triangle are equal, find this inradius. [7]



[7] $(\sqrt{3} - \sqrt{2})a$.

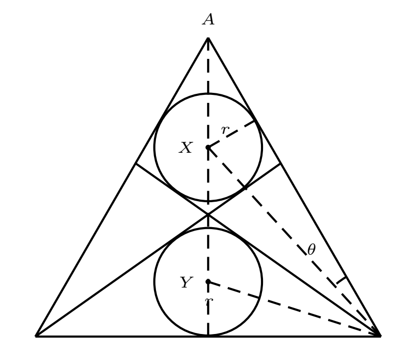
Suppose each side of the equilateral triangle has length 2, each of the congruent circles has radius r, and $\angle ACX = \theta$. See Figure GC8A.

(i) From triangle AXC, $r = \frac{2}{\cot 30^\circ + \cot\theta}$.

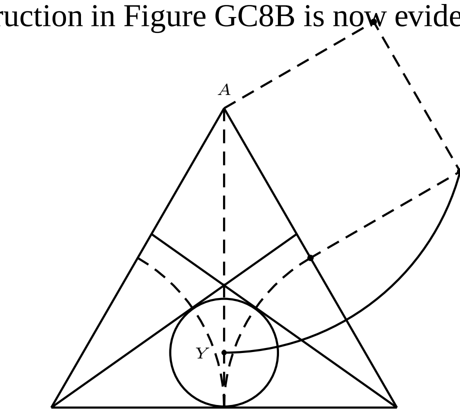
(ii) Note that $\angle BCY = \frac{1}{2}(60^\circ - 2\theta) = 30^\circ - \theta$. It follows that $r = \tan(30^\circ - \theta) = \frac{1}{\cot(30^\circ - \theta)} = \frac{\cot\theta - \cot 30^\circ}{\cot 30^\circ \cot\theta + 1}$.

By putting $\cot\theta = x$, we have $\frac{2}{\sqrt{3}+x} = \frac{x-\sqrt{3}}{\sqrt{3}x+1}$; $x^2 - 3 = 2\sqrt{3}x + 2$; $x^2 - 2\sqrt{3}x - 5 = 0$, and $x = \sqrt{3} + 2\sqrt{2}$. (The negative root is rejected). From this, $r = \frac{2}{\sqrt{3}+x} = \frac{1}{\sqrt{3}+\sqrt{2}} = \sqrt{3} - \sqrt{2}$.

To construct diagram GC8, it is enough to mark Y on the altitude through A such that $AY = \sqrt{3} - r = \sqrt{2}$. The construction in Figure GC8B is now evident.



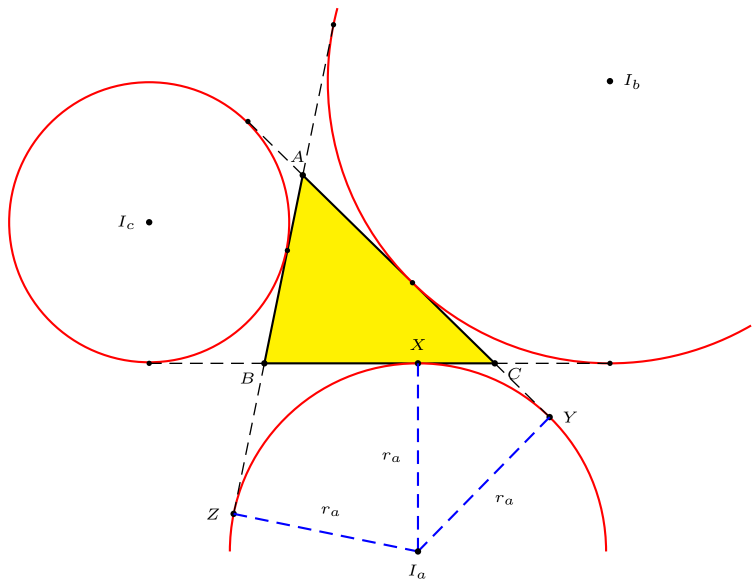
(y) GC8A



(z) GC8B

4.4 The excircles

The internal bisector of each angle and the *external* bisectors of the remaining two angles are concurrent at an *excenter* of the triangle. An *excircle* can be constructed with this as center, tangent to the lines containing the three sides of the triangle.



The exradii of a triangle with sides a, b, c are given by

$$r_a = \frac{\Delta}{s-a}, \qquad r_b = \frac{\Delta}{s-b}, \qquad r_c = \frac{\Delta}{s-c}.$$

The areas of the triangles I_aBC, I_aCA, and I_aAB are $\frac{1}{2}ar_a$, $\frac{1}{2}br_a$, and $\frac{1}{2}cr_a$ respectively. Since

$$\Delta = -\Delta I_aBC + \Delta I_aCA + \Delta I_aAB,$$

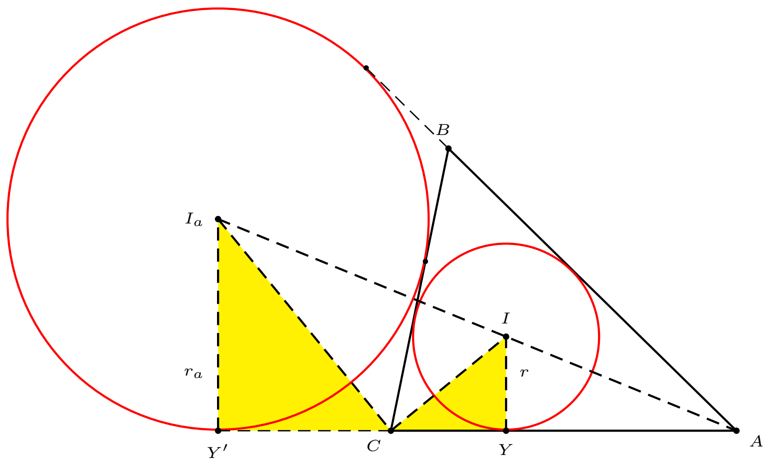
we have

$$\Delta = \frac{1}{2}r_a(-a+b+c) = r_a(s-a),$$

from which $r_a = \frac{\Delta}{s-a}$.

4.4.1 Heron's formula for the area of a triangle

Consider a triangle ABC with area Δ. Denote by r the inradius, and r_a the radius of the *excircle* on the side BC of triangle ABC. It is convenient to introduce the *semiperimeter* $s = \frac{1}{2}(a+b+c)$.



(1) From the similarity of triangles AIZ and $AI'Z'$,

$$\frac{r}{r_a} = \frac{s-a}{s}.$$

(2) From the similarity of triangles CIY and $I'CY'$,

$$r \cdot r_a = (s-b)(s-c).$$

(3) From these,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}},$$
$$r_a = \sqrt{\frac{s(s-b)(s-c)}{s-a}}.$$

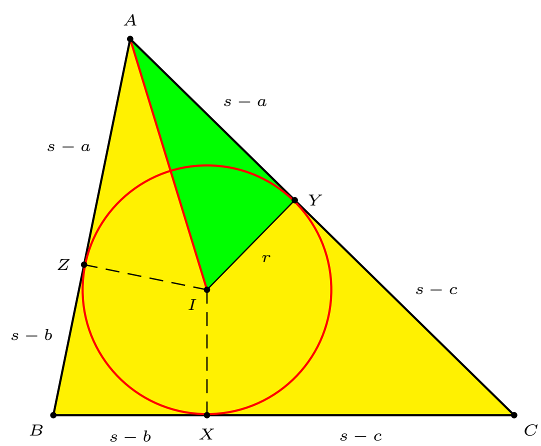
Theorem 4.2 (Heron's formula).

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

Proof. $\Delta = rs$. $\square$

Proposition 4.2.

1. $\tan\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$,
2. $\cos\frac{\alpha}{2}=\sqrt{\frac{s(s-a)}{bc}}$,
3. $\tan\frac{\alpha}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$.



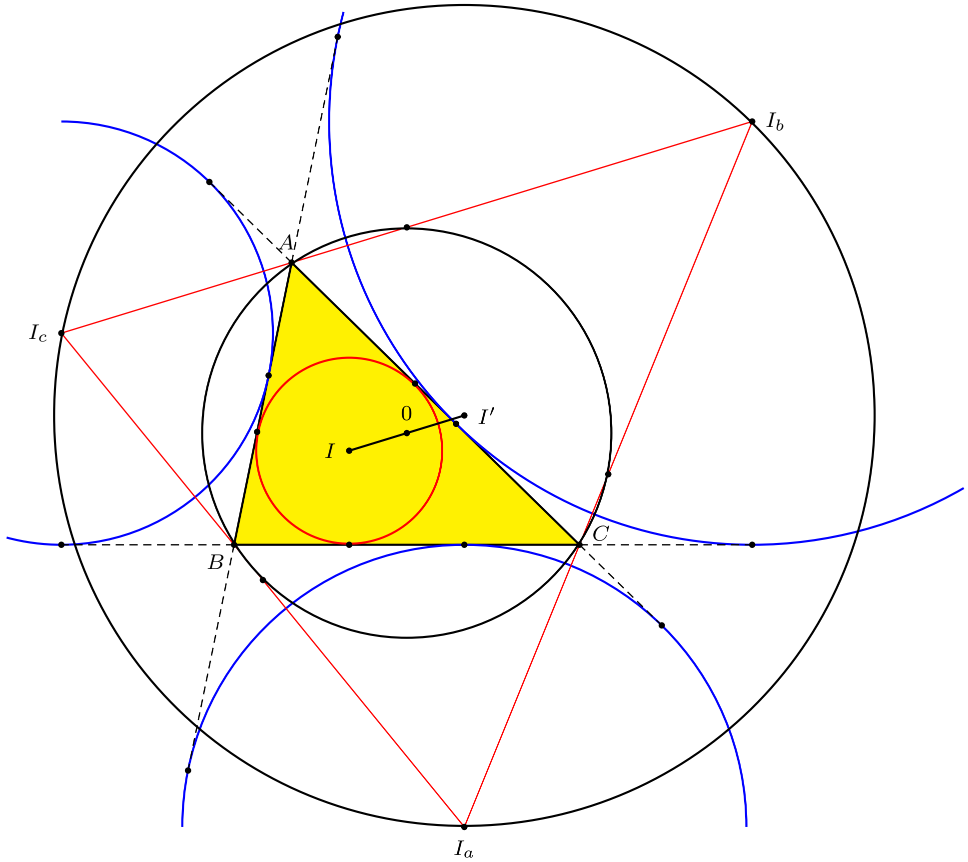
Proof. (1) $\tan\frac{\alpha}{2}=\frac{r}{s-a}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

(2) and (3) follow from (1) and

$$(s-b)(s-c)+s(s-a)=2s^2-(b+c-a)s+bc=bc.$$

$\square$

Exercise. **1.** Show that (i) the incenter is the orthocenter of the excentral triangle
(ii) the circumcircle is the nine-point circle of the excentral triangle, (iii) the circumcenter of the excentral triangle is the reflection of I in O.



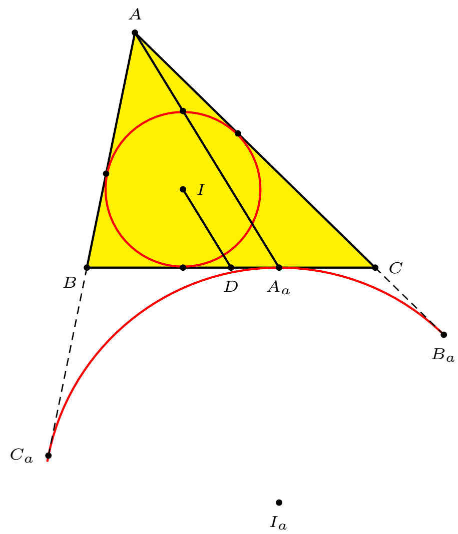
2. If the incenter is equidistant from the three excenters, show that the triangle is equilateral.

3. The *altitudes* a triangle are 12, 15 and 20. What is the area of the triangle ?[8]

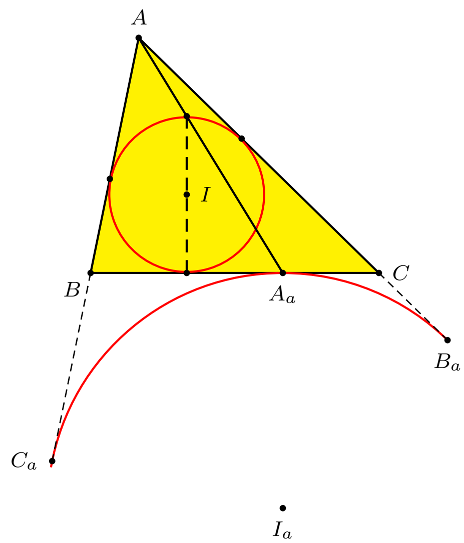
4. Find the inradius and the exradii of the (13,14,15) triangle.

[8]triangle $= 150$. The lengths of the sides are 25, 20 and 15.

5. Show that the line joining the incenter to the midpoint of a side is parallel to the line joining the point of the tangency of the excircle on the side to its opposite vertex.

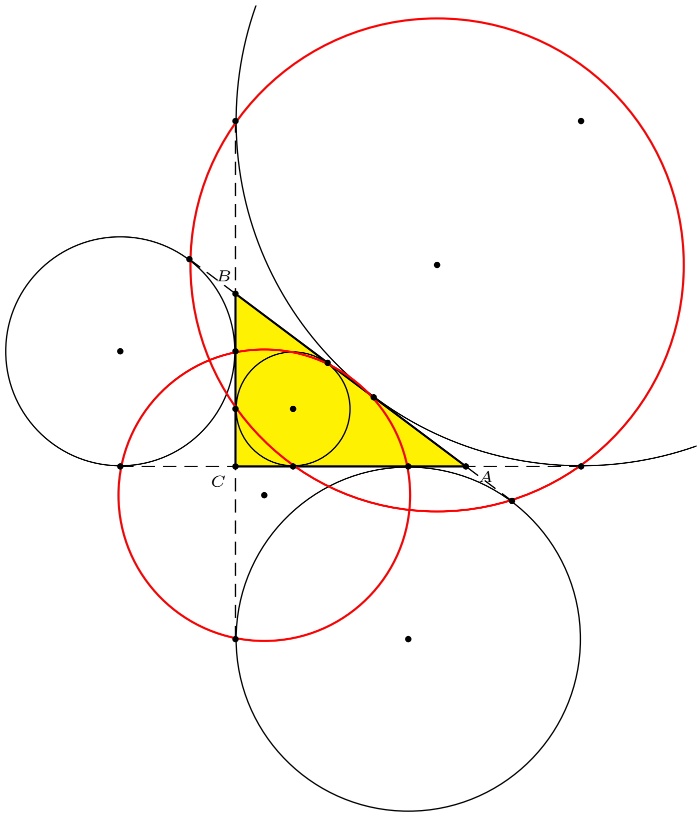


6. Show that the line joining vertex A to the point of tangency of BC with the A-excircle intersects the incircle at the antipode of its point of tangency with BC.



7. If one of the ex-radii of a triangle is equal to its semiperimeter, then the triangle contains a right angle.

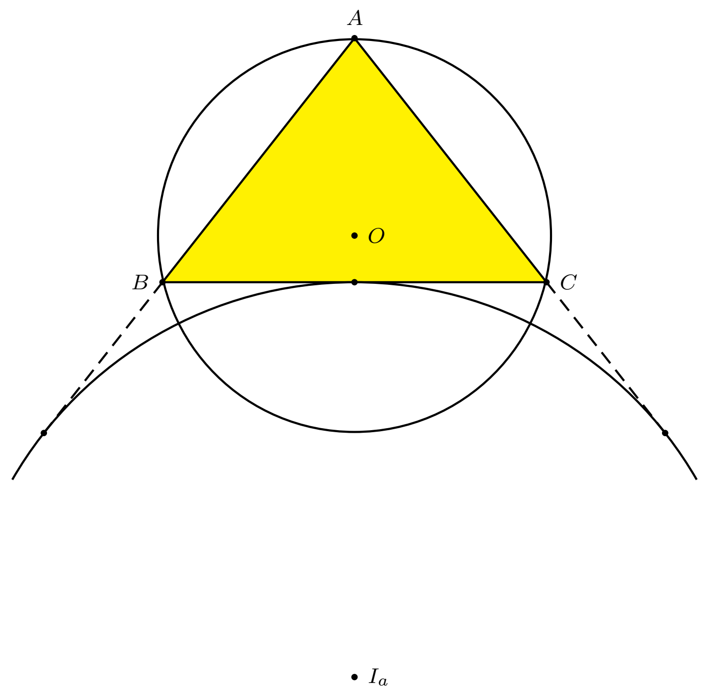
8. Show that in a right triangle the twelve points of contact of the inscribed and escribed circles form two groups of six points situated on two circles which cut each other orthogonally at the points of intersection of the cirucmcircle with the line joining the midpoints of the legs of the triangle.

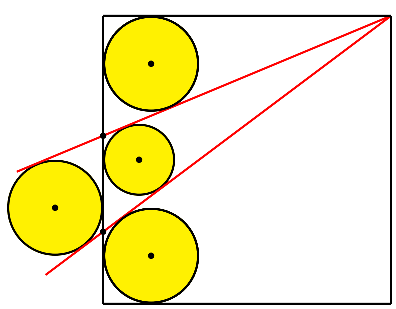


9. ABC is an isosceles triangle with $AB = AC$ and $a : b : c = 2 : \varphi : \varphi$. Show that the circumcircle and the A-excircle are orthogonal to each other, and find the ratio $r_a : R$.

10. The length of each side of the square is $6a$, and the radius of each of the top and bottom circles is a. Calculate the radii of the other two circles. [9]

[9] a and $\frac{3}{4}a$.

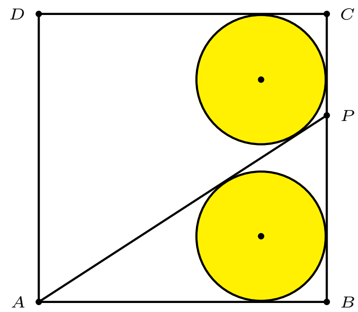




11. $ABCD$ is a square of unit side. P is a point on BC so that the incircle of triangle ABP and the circle tangent to the lines AP, PC and CD have equal radii. Show that the length of BP satisfies the equation

$$2x^3 - 2x^2 + 2x - 1 = 0.$$

12. $ABCD$ is a square of unit side. Q is a point on BC so that the incircle of triangle ABQ and the circle tangent to AQ, QC, CD touch each other at a

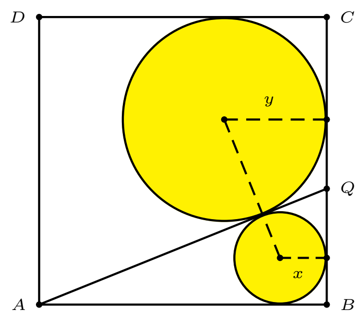


point on AQ. Show that the radii x and y of the circles satisfy the equations

$$y = \frac{x(3-6x+2x^2)}{1-2x^2}, \qquad \sqrt{x}+\sqrt{y} = 1.$$

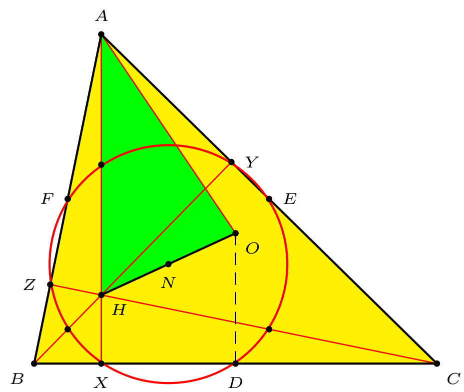
Deduce that x is the root of

$$4x^3 - 12x^2 + 8x - 1 = 0.$$



4.5 Feuerbach's theorem

Excursus: Distance between the circumcenter and orthocenter



Proposition 4.3. $OH^2 = R^2(1 - 8\cos\alpha\cos\beta\cos\gamma)$.

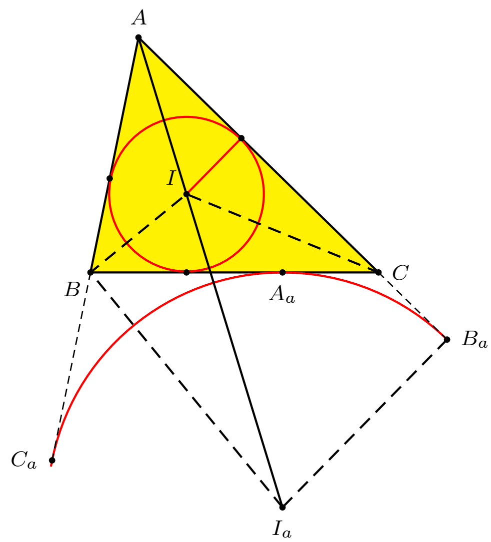
Proof. In triangle AOH, $AO = R$, $AH = 2R\cos\alpha$, and $\angle OAH = |\beta - \gamma|$. By the law of cosines,

$$\begin{aligned} OH^2 &= R^2(1 + 4\cos^2\alpha - 4\cos\alpha\cos(\beta-\gamma)) \\ &= R^2(1 - 4\cos\alpha(\cos(\beta+\gamma) + \cos(\beta-\gamma)) \\ &= R^2(1 - 8\cos\alpha\cos\beta\cos\gamma). \end{aligned}$$

□

Proposition 4.4.

$$r = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2},$$
$$s = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$



Proof. (1) In triangle IAB, $AB = 2R\sin\gamma$, $\angle ABI = \frac{\beta}{2}$, and $\angle AIB = 180° - \frac{\alpha+\beta}{2}$. Applying the law of sines, we have

$$IA = AB\cdot\frac{\sin\frac{\beta}{2}}{\sin\frac{\alpha+\beta}{2}} = 2R\sin\gamma\cdot\frac{\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}} = 4R\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\cdot\frac{\sin\frac{\beta}{2}}{\cos\frac{\gamma}{2}} = 4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

From this,

$$r = IA\cdot\sin\frac{\alpha}{2} = 4R\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$$

(2) Similarly, in triangle I_aAB, $\angle ABI_a = 90° + \frac{\beta}{2}$, we have

$$I_aA = 4R\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

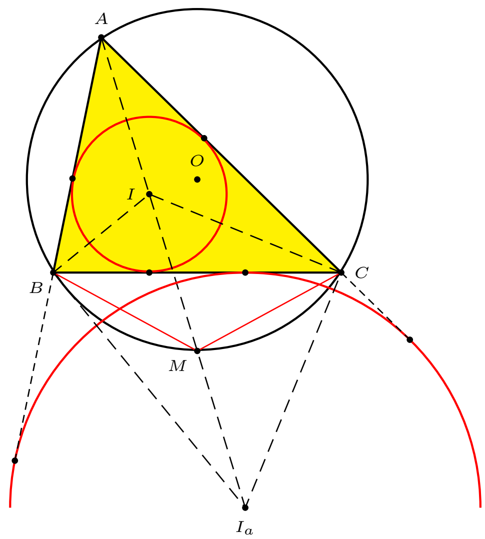
It follows that

$$s = I_aA\cdot\cos\frac{\alpha}{2} = 4R\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$$

$\square$

4.5.1 Distance between circumcenter and tritangent centers

Lemma 4.3. *If the bisector of angle A intersects the circumcircle at M, then M is the center of the circle through B, I, C, and I_a.*



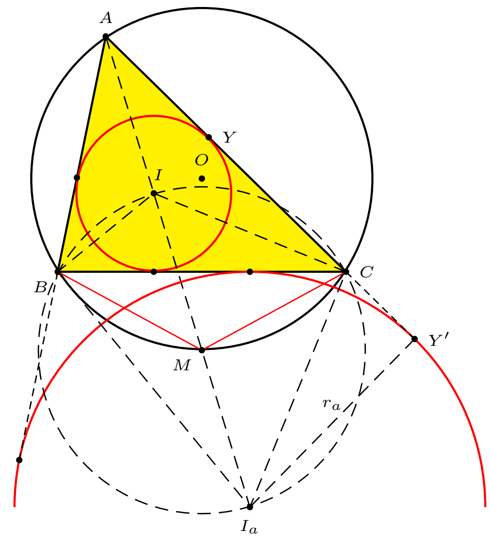
Proof. (1) Since M is the midpoint of the arc BC, $\angle MBC = \angle MCB = \angle MAB$. Therefore,

$$\angle MBI = \angle MBC + \angle CBI = \angle MAB + \angle IBA = \angle MIB,$$

and $MB = MI$. Similarly, $MC = MI$.

(2) On the other hand, since $\angle IBI_a$ and ICI_a are both right angles, the four points B, I, C, I_aM are concyclic, with center at the midpoint of II_A. This is the point M. $\square$

Theorem 4.4 (Euler). (a) $OI^2 = R^2 - 2Rr$.
(b) $OI_a^2 = R^2 + 2Rr_a$.



Proof. (a) Considering the power of I in the circumcircle, we have

$$R^2 - OI^2 = AI \cdot IM = AI \cdot MB = \frac{r}{\sin\frac{\alpha}{2}} \cdot 2R \cdot \sin\frac{\alpha}{2} = 2Rr.$$

(b) Consider the power of I_a in the circumcircle.
Note that $I_aA = \frac{r_a}{\sin\frac{\alpha}{2}}$. Also, $I_aM = MB = 2R\sin\frac{\alpha}{2}$.

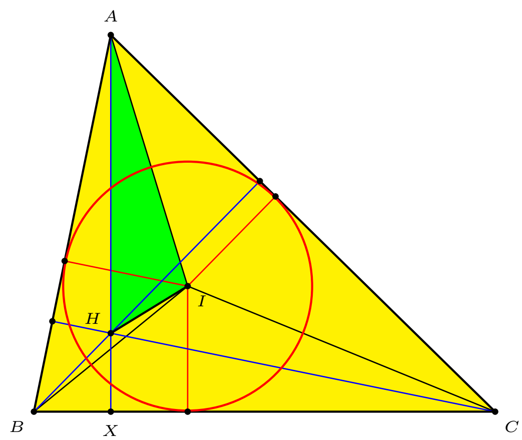
$$\begin{aligned} OI_a^2 &= R^2 + I_aA \cdot I_aM \\ &= R^2 + \frac{r_a}{\sin\frac{\alpha}{2}} \cdot 2R\sin\frac{\alpha}{2} \\ &= R^2 + 2Rr_a. \end{aligned}$$

□

4.5.2 Distance between orthocenter and tritangent centers

Proposition 4.5.

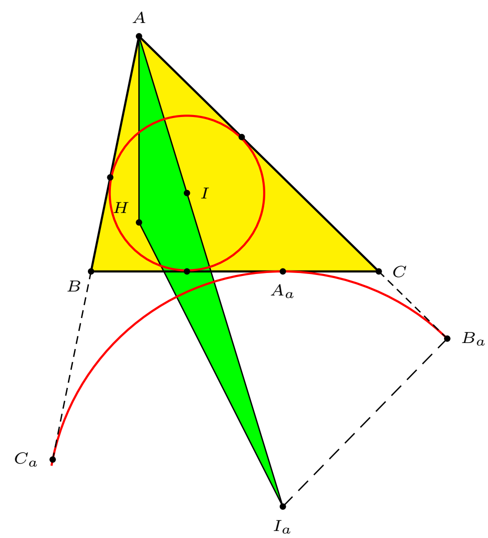
$$HI^2 = 2r^2 - 4R^2 \cos\alpha \cos\beta \cos\gamma,$$
$$HI_a^2 = 2r_a^2 - 4R^2 \cos\alpha \cos\beta \cos\gamma.$$



Proof. In triangle AIH, we have $AH = 2R\cos\alpha$, $AI = 4R\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$ and $\angle HAI = \frac{|\beta-\gamma|}{2}$. By the law of cosines,

$$\begin{aligned}
HI^2 &= AH^2 + AI^2 - 2AI \cdot AH \cdot \cos\frac{\beta-\gamma}{2}\\
&= 4R^2\left(\cos^2\alpha + 4\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2} - 4\cos\alpha\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\beta-\gamma}{2}\right)\\
&= 4R^2\left(\cos^2\alpha + 4\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2} - 4\cos\alpha\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - 4\cos\alpha\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}\right)\\
&= 4R^2\left(\cos^2\alpha + 4\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2} - \cos\alpha\sin\beta\sin\gamma - 4\left(1 - 2\sin^2\frac{\alpha}{2}\right)\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}\right)\\
&= 4R^2\left(\cos\alpha(\cos\alpha - \sin\beta\sin\gamma) + 8\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}\right)\\
&= 4R^2\left(-\cos\alpha\cos\beta\cos\gamma + 8\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}\right)\\
&= 2r^2 - 4R^2\cos\alpha\cos\beta\cos\gamma.
\end{aligned}$$

(2) In triangle AHI_a, $AI_a = 4R\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

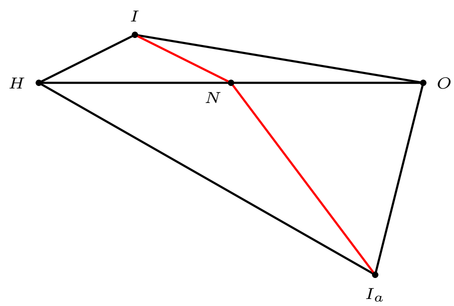


By the law of cosines, we have

$$\begin{aligned}
HI_a^2 &= AH^2 + AI_a^2 - 2AI_a\cdot AH\cdot\cos\frac{\beta-\gamma}{2}\\
&= 4R^2\left(\cos^2\alpha + 4\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} - 4\cos\alpha\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\cos\frac{\beta-\gamma}{2}\right)\\
&= 4R^2\left(\cos^2\alpha + 4\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} - 4\cos\alpha\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} - 4\cos\alpha\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)\\
&= 4R^2\left(\cos^2\alpha + 4\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} - 4\left(1-2\sin^2\frac{\alpha}{2}\right)\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2} - \cos\alpha\sin\beta\sin\gamma\right)\\
&= 4R^2\left(\cos\alpha(\cos\alpha-\sin\beta\sin\gamma) + 8\sin^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}\right)\\
&= 4R^2\left(-\cos\alpha\cos\beta\cos\gamma + 8\sin^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}\right)\\
&= 2r_a^2 - 4R^2\cos\alpha\cos\beta\cos\gamma.
\end{aligned}$$

$\square$

Theorem 4.5 (Feuerbach). *The nine-point circle is tangent internally to the incircle and externally to each of the excircles.*



Proof. (1) Since N is the midpoint of OH, IN is a median of triangle IOH. By Apollonius' theorem,

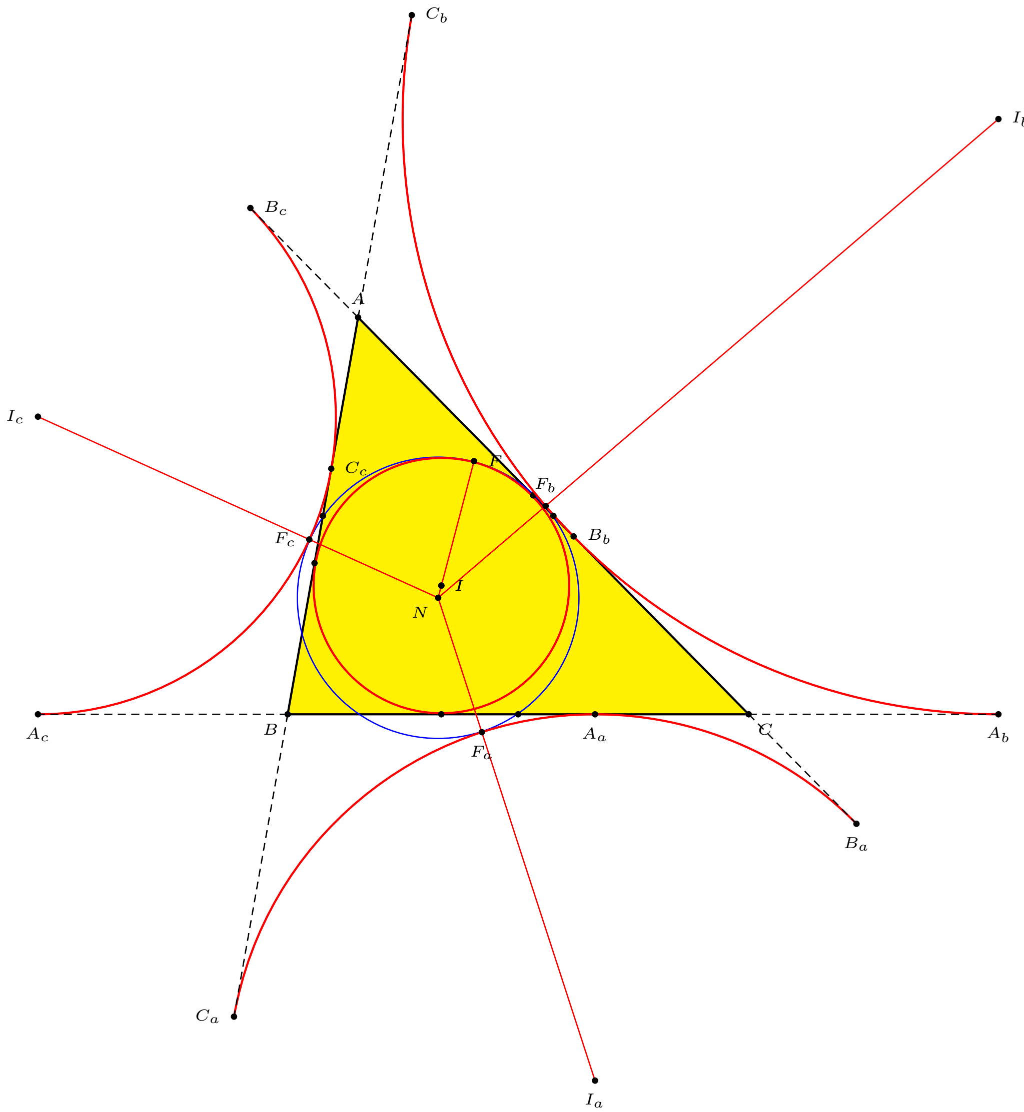
$$\begin{aligned} NI^2 &= \frac{1}{2}(IH^2 + OI^2) - \frac{1}{4}OH^2 \\ &= \frac{1}{4}R^2 - Rr + r^2 \\ &= \left(\frac{R}{2} - r\right)^2. \end{aligned}$$

Therefore, NI is the difference between the radii of the nine-point circle and the incircle. This shows that the two circles are tangent to each other internally.

(2) Similarly, in triangle I_aOH,

$$\begin{aligned} NI_a^2 =& \frac{1}{2}(HI_a^2 + OI_a^2) - \frac{1}{4}OH^2 \\ &= \frac{1}{4}R^2 + Rr_a + r_a^2 \\ &= \left(\frac{R}{2} + r_a\right)^2. \end{aligned}$$

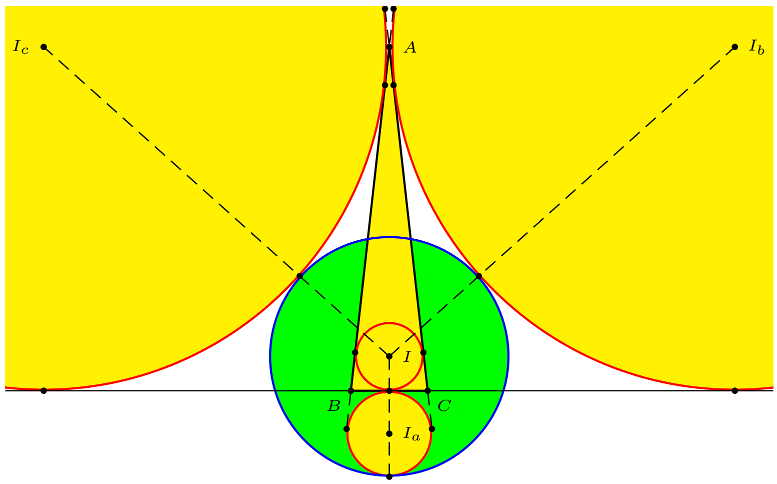
This shows that the distance between the centers of the nine-point and an excircle is the sum of their radii. The two circles are tangent externally. ☐



C_b
I_b
B_c
A
I_c
C_c
F
F_b
B_b
F_c
I
N
A_c
B
A_a
C
A_b
F_a
B_a
C_a
I_a

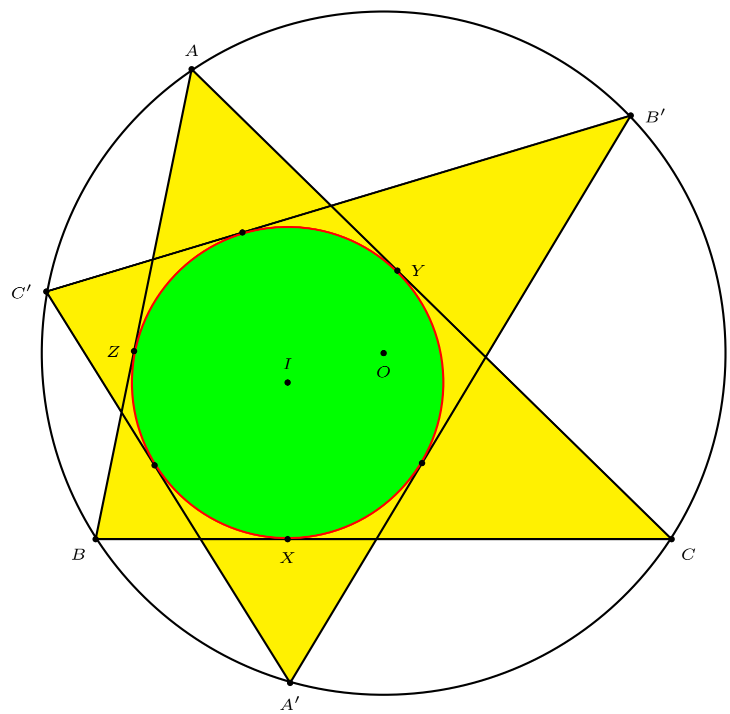
Exercise. **1.** Suppose there is a circle, center I, tangent externally to all three excircles. Show that triangle ABC is equilateral.

2. Find the dimensions of an isosceles (but non-equilateral) triangle for which there is a circle, center I, tangent to all three excircles.



Excursus: Steiner's porism

Construct the circumcircle (O) and the incircle (I) of triangle ABC. Animate a point A' on the circumcircle, and construct the tangents from A' to the incircle (I). Extend these tangents to intersect the circumcircle again at B' and C'. The lines $B'C'$ is always tangent to the incircle. This is the famous theorem on Steiner porism: if two given circles are the circumcircle and incircle of one triangle, then they are the circumcircle and incircle of a continuous family of poristic triangles.



Exercise. **1.** $r \leq \frac{1}{2}R$. When does equality hold?

2. Suppose $OI = d$. Show that there is a right-angled triangle whose sides are d, r and $R - r$. Which one of these is the hypotenuse?

3. Given a point I inside a circle $O(R)$, construct a circle $I(r)$ so that $O(R)$ and $I(r)$ are the circumcircle and incircle of a (family of poristic) triangle(s).

4. Given the circumcenter, incenter, and one vertex of a triangle, construct the triangle.

5. Construct an animation picture of a triangle whose circumcenter lies on the incircle. [10]

6. What is the locus of the centroids of the poristic triangles with the same circumcircle and incircle of triangle ABC? How about the orthocenter?

7. Let $A'B'C'$ be a poristic triangle with the same circumcircle and incircle of triangle ABC, and let the sides of $B'C'$, $C'A'$, $A'B'$ touch the incircle at X, Y, Z.

(i) What is the locus of the centroid of XYZ?

(ii) What is the locus of the orthocenter of XYZ?

(iii) What can you say about the Euler line of the triangle XYZ?

[10]Hint: $OI = r$.