# Chapter 4
# The Mathematics of Apportionment

Typical Problem

A school has one teacher available to teach all sections of Geometry, Precalculus and Calculus. She is able to teach 5 courses and no more.

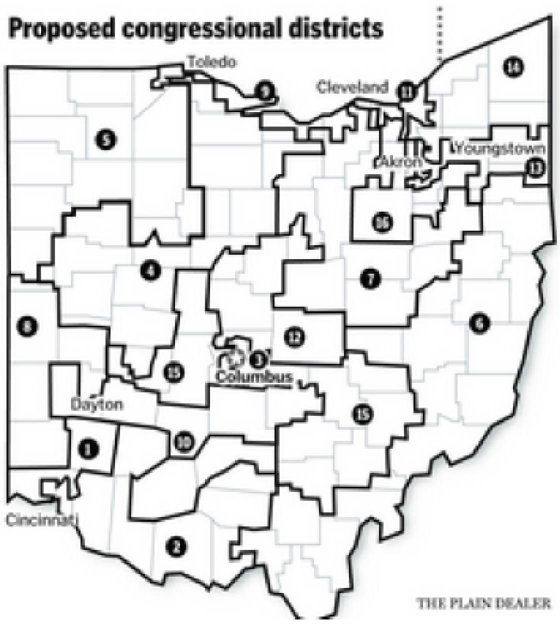
How do you decide how many of each course should be offered?
How do you apportion the five sections to the three courses?

1st. It depends on how many students will be in each course.

There are 100 students in all. They are as follows:
- Geometry 52
- Precalculus 33
- Calculus 15

How would you assign her sections?


18 Districts




16 Districts

We lost 2 districts in the last census. Why?

The first step is to find a good unit of measurement. The most natural unit of measurement is the ratio of students to sections. We call this ratio the **standard divisor** ***SD = P/M***

***SD*** **=** 100/*5 = 20 students per section*

For example, take Geometry. To find a section's standard quota, we divide the course's population by the standard divisor:
*Quota = population/SD* = 52/20 = 2.6

Geometry "should" have 2.6 sections...

Similarly, the quota for Precalcus is Pop/SD =
Finally, the quota for Calculus is Pop/SD =

Apportionment is the problem of rounding the quota to whole numbers in a way that is "fair" to everyone and satisfies the original problem. There are several ways to do this. None of which is perfect, but some are better than others.

First guess: Round each of the quotas to the nearest whole number. What happens in this case?

Geometry: Quota = Final Apportionment:

Precalculus: Quota = Final Apportionment:

Calculus: Quota= Final Apportionment:

What's wrong with that?

General Problem: Assign a number of "seats" to each of the "states" in proportion to the "population" of each state.

- **The "states."** This is the term we will use to describe the *players* involved in the apportionment.
- **The "seats."** This term describes the set of *M identical, indivisible objects* that are being divided among the *N* states.
- **The "populations."** This is a set of *N* positive numbers which are used as the basis for the apportionment of the seats to the states.

- **Upper quotas.** The quota rounded up and denoted by *U*..
- **Lower quotas.** The quota rounded down and is denoted by *L.*

In the unlikely event that the quota is a whole number, the lower and upper quotas are the same.

Another Example from the Book:

Table 4-3 Republic of Parador (Population by State)

Assign a number of seats in Congress to each of the following 6 states in proportion to their relative populations. There are 250 seats in the congress.

Find the Standard Quotient (Population per Seat)

Make a guess the apportionment. Does it work?

| State | A | B | C | D | E | F | Total |
|---|---|---|---|---|---|---|---|
| Population | 1,646,000 | 6,936,000 | 154,000 | 2,091,000 | 685,000 | 988,000 | 12,500,000 |

## Hamilton’s Method

- Step 1. Calculate each state’s standard quota.
- Step 2. Give to each state its *lower quota*.
- Step 3. Give the surplus seats to the state with the largest fractional parts until there are no more surplus seats.

Hamilton's Method of Apportionment

Section 2 of Constitution Apportionment of Representatives

Hamilton's Method worked out for our 6-state Congress Example

| State | Population | Step1 Quota | Step 2 Lower Quota | Fractional parts | Step 3 Surplus | Hamilton apportionment |
|---|---|---|---|---|---|---|
| A | 1,646,000 | 32.92 | 32 | 0.92 | First | 33 |
| B | 6,936,000 | 138.72 | 138 | 0.72 | Last | 139 |
| C | 154,000 | 3.08 | 3 | 0.08 | | 3 |
| D | 2,091,000 | 41.82 | 41 | 0.82 | Second | 42 |
| E | 685,000 | 13.70 | 13 | 0.70 | | 13 |
| F | 988,000 | 19.76 | 19 | 0.76 | Third | 20 |
| Total | 12,500,000 | 250.00 | 246 | 4.00 | 4 | 250 |

Rules that apportionments should follow:

**The Quota Rule**
No state should be apportioned a number of seats smaller than its lower quota or larger than its upper quota.

When a state is apportioned *a number smaller than its lower quota*, we call it a **lower-quota violation**;

when a state is apportioned a *number larger than its upper quota*, we call it an **upper-quota violation**.)

The most serious (in fact, the fatal) flaw of Hamilton's method is commonly know as the **Alabama paradox**.

In essence, the paradox occurs when an *increase in the total number of seats being apportioned, in and of itself, forces a state to lose one of its seats*.

After the 1880 census, C. W. Seaton, chief clerk of the United States Census Bureau, computed apportionments for all House sizes between 275 and 350, and discovered that Alabama would get 8 seats with a House size of 299 but only 7 with a House size of 300.

| | | With 10 seats | | With 11 seats | |
|---|---|---|---|---|---|
| **State** | **Size** | **Fair share** | **Seats** | **Fair share** | **Seats** |
| A | 6 | 4.286 | 4 | 4.714 | 5 |
| B | 6 | 4.286 | 4 | 4.714 | 5 |
| C | 2 | 1.429 | 2 | 1.571 | 1 |

Wikipedia Alabama Paradox

The Hamilton's method can fall victim to two other paradoxes called

**The population paradox-** when state A *loses* a seat to state B even though the population of A *grew at a higher rate* than the population of B.

**TABLE 4-10** Intergalactic Congress: Apportionment of 2525

| Planet | Population | Step 1 | Step 2 | Step 3 | Apportionment |
|---|---|---|---|---|---|
| Alanos | 150 | $8.\overline{3}$ | 8 | 0 | 8 |
| Betta | **78** | $4.\overline{3}$ | 4 | 0 | **4** |
| Conii | 173 | $9.6\overline{1}$ | 9 | 1 | 10 |
| Dugos | 204 | $11.\overline{3}$ | 11 | 0 | 11 |
| Ellisium | **295** | $16.3\overline{8}$ | 16 | 1 | **17** |
| Total | 900 | 50.00 | 48 | 2 | 50 |

**TABLE 4-11** Intergalactic Congress: Apportionment of 2535

| Planet | Population | Step 1 | Step 2 | Step 3 | Apportionment |
|---|---|---|---|---|---|
| Alanos | 150 | 8.25 | 8 | 0 | 8 |
| Betta | **78** | 4.29 | 4 | 1 | **5** |
| Conii | 181 | 9.96 | 9 | 1 | 10 |
| Dugos | 204 | 11.22 | 11 | 0 | 11 |
| Ellisium | **296** | 16.28 | 16 | 0 | **16** |
| Total | 909 | 50.00 | 48 | 2 | 50 |

**The new-states paradox-** that *the addition of a new state with its fair share of seats can, in and of itself, affect the apportionments of other states*.

**TABLE 4-12** Metro Garbage Truck Apportionments

| District | Homes serviced | Quota ($SD = 1000$) | Hamilton apportionment |
|---|---|---|---|
| Northtown | 10,450 | 10.45 | **10** |
| Southtown | 89,550 | 89.55 | **90** |
| Total | 100,000 | 100.00 | 100 |

**TABLE 4-13** Revised Metro Garbage Truck Apportionments

| District | Homes serviced | Quota ($SD \approx 1002.38$) | Hamilton apportionment |
|---|---|---|---|
| Northtown | 10,450 | 10.42 | **11** |
| Southtown | 89,550 | 89.34 | **89** |
| Newtown | 5,250 | 5.24 | 5 |
| Total | 105,250 | 105.00 | 105 |

Jefferson's Method

Step 1. Find a "suitable" divisor D.
A suitable or **modified divisor** is a divisor that produces and apportionment of exactly M seats when the quotas (populations divided by D) are *rounded down*.

Step 2. Each state is apportioned its *lower quota*.

*Bad News*- Jefferson's method can produce **upper-quota violations**!
To make matters worse, the upper-quota violations tend to consistently favor the larger states.

The apportionment method suggested by Alexander Hamilton was approved by Congress in 1791, but was subsequently vetoed by president Washington - in the very first exercise of the veto power by President of the United States. Hamilton's method was adopted by the US Congress in 1852 and was in use through 1911 when it was replaced by Webster's method.

Hamilton's Method (Round Down) on 6-State Congress
Decrease Divisor until Correct number of seats

**TABLE 4-14** Rounding Down the Quotas: $SD = 50{,}000$ and $D = 49{,}500$

| State | Population | Standard quota ($SD = 50{,}000$) | Lower quota | Modified quota ($D = 49{,}500$) | Jefferson apportionment |
|---|---|---|---|---|---|
| *A* | 1,646,000 | 32.92 | 32 | 33.25 | 33 |
| *B* | 6,936,000 | **138.72** | 138 | 140.12 | **140** |
| *C* | 154,000 | 3.08 | 3 | 3.11 | 3 |
| *D* | 2,091,000 | 41.82 | 41 | 42.24 | 42 |
| *E* | 685,000 | 13.70 | 13 | 13.84 | 13 |
| *F* | 988,000 | 19.76 | 19 | 19.96 | 19 |
| Total | 12,500,000 | 250.00 | 246 | | 250 |

Adams's Method

Step 1. Find a "suitable" divisor D.
A suitable or **modified divisor** is a divisor that produces and apportionment of exactly M seats when the quotas (populations divided by D) are *rounded up*.

Step 2. Each state is apportioned its *upper quota*.

*Bad News*- Adam's method can produce **lower-quota violations**!
We can reasonably conclude that Adam's method is no better (or worse) than Jefferson's method– just different.

Adams's Method (Round Up) on 6-State Congress
Increase Divisor until Correct number of seats

**TABLE 4-15 Rounding Up the Quotas: $D = 50{,}500$, and $D = 50{,}700$**

| State | Population | Quota ($D = 50{,}500$) | Upper quota ($D = 50{,}500$) | Quota ($D = 50{,}700$) | Adams's apportionment |
|---|---|---|---|---|---|
| *A* | 1,646,000 | 32.59 | 33 | 32.47 | 33 |
| *B* | 6,936,000 | 137.35 | 138 | 136.80 | 137 |
| *C* | 154,000 | 3.05 | 4 | 3.04 | 4 |
| *D* | 2,091,000 | 41.41 | 42 | 41.24 | 42 |
| *E* | 685,000 | 13.56 | 14 | 13.51 | 14 |
| *F* | 988,000 | 19.56 | 20 | 19.49 | 20 |
| Total | 12,500,000 | | 251 | | 250 |

Webster’s Method

Step 1. Find a “suitable” divisor *D*.

Here a suitable divisor means a divisor that produces an apportionment of exactly *M* seats when the quotas (populations divided by *D*) are *rounded the conventional way*.

Step 2. Find the apportionment of each state by rounding its quota the conventional way.

Webster's Method Finding Suitable Divisor

**TABLE 4-16 Conventional Rounding: $D = 50{,}100$**

| State | Population | Standard quota ($D = 50{,}000$) | Nearest integer | Quota ($D = 50{,}100$) | Webster's apportionment |
|---|---|---|---|---|---|
| *A* | 1,646,000 | 32.92 | 33 | 32.85 | 33 |
| *B* | 6,936,000 | 138.72 | 139 | 138.44 | 138 |
| *C* | 154,000 | 3.08 | 3 | 3.07 | 3 |
| *D* | 2,091,000 | 41.82 | 42 | 41.74 | 42 |
| *E* | 685,000 | 13.70 | 14 | 13.67 | 14 |
| *F* | 988,000 | 19.76 | 20 | 19.72 | 20 |
| Total | 12,500,000 | 250.00 | 251 | | 250 |

Daniel Webster proposed his apportionment method in 1832.It was adopted by the Congress in 1842, and then replaced by Alexander Hamilton's in 1852. It was again adopted in 1901 and reconfirmed in 1911. Finally, it was replaced by Huntington-Hill's method in 1941.

**TABLE 4-18 Summary of the Four Methods**

| | Hamilton | Jefferson | Adams | Webster |
|---|---|---|---|---|
| **Quota rule** | No violations | Upper-quota violations possible | Lower-quota violations possible | Upper- and lower-quota violations possible |
| **Alabama paradox** | Possible | Not possible | Not possible | Not possible |
| **Population paradox** | Possible | Not possible | Not possible | Not possible |
| **New-states paradox** | Possible | Not possible | Not possible | Not possible |
| **Bias in favor of** | Large states | Large states | Small states | Neutral |

**Appendix** 2000–2010 Apportionments of the House of Representatives

**TABLE 1-9** Current Huntington-Hill Apportionments

| State | Population | Seats | State | Population | Seats |
|---|---|---|---|---|---|
| Alabama | 4,461,130 | 7 | Montana | 905,316 | 1 |
| Alaska | 628,933 | 1 | Nebraska | 1,715,369 | 3 |
| Arizona | 5,140,683 | 8 | Nevada | 2,002,032 | 3 |
| Arkansas | 2,679,733 | 4 | New Hampshire | 1,238,415 | 2 |
| California | 33,930,798 | 53 | New Jersey | 8,424,354 | 13 |
| Colorado | 4,311,882 | 7 | New Mexico | 1,823,821 | 3 |
| Connecticut | 3,409,535 | 5 | New York | 19,004,973 | 29 |
| Delaware | 785,068 | 1 | North Carolina | 8,067,673 | 13 |
| Florida | 16,028,890 | 25 | North Dakota | 643,756 | 1 |
| Georgia | 8,206,975 | 13 | Ohio | 11,374,540 | 18 |
| Hawaii | 1,216,642 | 2 | Oklahoma | 3,458,819 | 5 |
| Idaho | 1,297,274 | 2 | Oregon | 3,428,543 | 5 |
| Illinois | 12,439,042 | 19 | Pennsylvania | 12,300,670 | 19 |
| Indiana | 6,090,782 | 9 | Rhode Island | 1,049,662 | 2 |
| Iowa | 2,931,923 | 5 | South Carolina | 4,025,061 | 6 |
| Kansas | 2,693,824 | 4 | South Dakota | 756,874 | 1 |
| Kentucky | 4,049,431 | 6 | Tennessee | 5,700,037 | 9 |
| Louisiana | 4,480,271 | 7 | Texas | 20,903,994 | 32 |
| Maine | 1,277,731 | 2 | Utah | 2,236,714 | 3 |
| Maryland | 5,307,886 | 8 | Vermont | 609,890 | 1 |
| Massachusetts | 6,355,568 | 10 | Virginia | 7,100,702 | 11 |
| Michigan | 9,955,829 | 15 | Washington | 5,908,684 | 9 |
| Minnesota | 4,925,670 | 8 | West Virginia | 1,813,077 | 3 |
| Mississippi | 2,852,927 | 4 | Wisconsin | 5,371,210 | 8 |
| Missouri | 5,606,260 | 9 | Wyoming | 495,304 | 1 |
| | | | **Total** | **281,424,177** | **435** |

Projected Changes in Representatives 2010 Census

| Gain more than one | Gain one | | Lose one | | Lose more than one |
|---|---|---|---|---|---|
| Texas +4<br>Arizona +2 | Florida +1<br>Georgia +1<br>Nevada +1 | Oregon +1<br>South Carolina +1<br>Utah +1 | Illinois -1<br>Iowa -1<br>Louisiana -1<br>Massachusetts -1<br>Michigan -1 | Minnesota -1<br>Missouri -1<br>New Jersey -1<br>New York -1<br>Pennsylvania -1 | Ohio -2 |

State Populations as of 2008

Attachments

- Hamilton's Method of Apportionment
- Section 2 of Constitution: Apportionment of Representatives
- Wikipedia Alabama Paradox
- Webster's Method Finding Suitable Divisor
- Projected Changes in Representatives 2010 Census
- State Populations as of 2008