Chapter 5

Drawing a cube

Math 4520, Spring 2015

5.1 One and two-point perspective

In Chapter 5 we saw how to calculate the center of vision and the viewing distance for a square in one or two-point perspective. Suppose that we wish to extend our construction to a cube. In particular, what conditions must the projection of a cube satisfy?

For the projection of a cube, we say that it is in *one-point perspective* if only one of the three vanishing points, corresponding to three non-parallel sides, is finite. A one-point perspective drawing (or projection) of a cube is easy to handle. The front face of the cube projects onto a perfect square as in Figure 5.1. This is the extra condition.

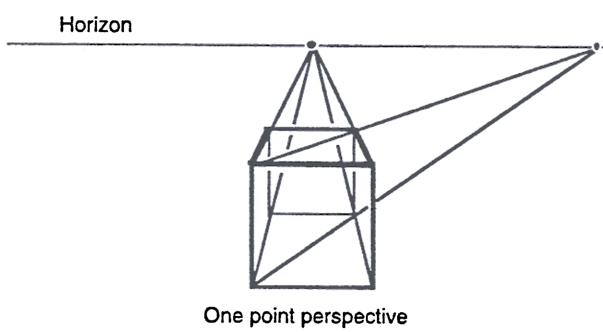

Figure 5.1

Two point perspective (where two of the three vanishing points correspond to finite points) is somewhat more complicated. This the same situation as in Chapter 5, except that the square is the bottom face of the cube that we are projecting. As in Chapter 5, we project orthogonally onto the base plane, which in Chapter 5 was called the object plane. This bird's-eye view is shown in Figure 5.2. If the cube has sides of length s, then the projection orthogonal to the picture plane of the two sides of the base of the cube, as in Figure 5.2, have lengths x and y, say. So we have

$$x^2 + y^2 = s^2.$$

If we draw the line in the base object plane parallel to the picture plane through the front corner of the cube, then we have the line L as in Figure 6.2. We can find the line segments of length x and y in line L.

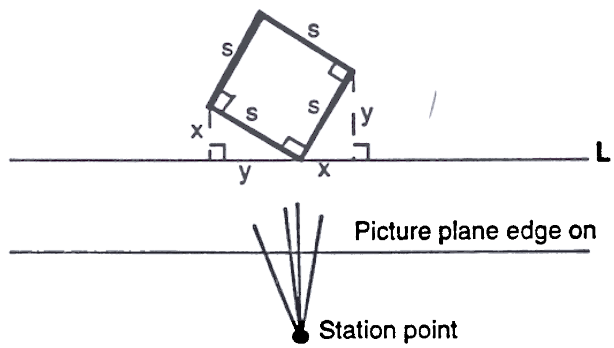

Figure 5.2

We next project this construction into the picture plane as in Figure 5.3. The projection of the lengths x and y in L and s in the vertical side of the cube are all proportional to their true lengths. So a simple Euclidean construction (constructing a right triangle with x and y as legs) allows us to find the projection of the vertical line segment in the picture plane.

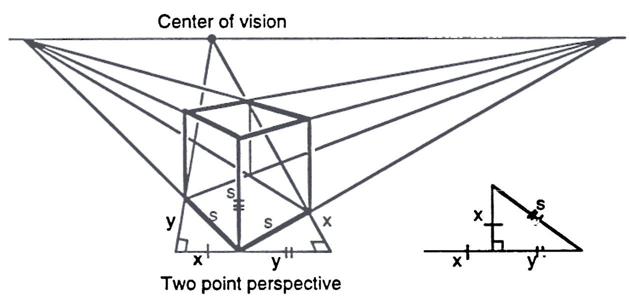

Figure 5.3

The moral of this story is that for two-point perspective, once the base of the cube is drawn, then the height and the rest of the cube is determined, as well as the unique station point in 3-space. If this height is not drawn correctly, then the drawing will probably not look like a cube. It will look like a box that is too high, too low, too short, or too long, etc. You may not be able to tell which dimension is the largest, but you can tell that the sides are not equal.

5.2 Three point perspective

If one draws a cube where none of its sides are parallel (or perpendicular) to the picture plane, we say the cube is in *three point perspective*. In other words, the vanishing point of each of the three sets of parallel sides project onto a finite point in the picture plane. Of course it could happen that these three points are not on the actual canvas, but nevertheless they are on the Euclidean plane through the canvas and not at infinity. Since the plane of any face of the cube is not perpendicular to the picture plane, our previous constructions for finding the center of vision and the viewing distance do not apply. However, there are other constructions that are relatively simple. Consider the tetrahedron determined by the station point and the three vanishing points in the picture plane as in Figure 5.4.

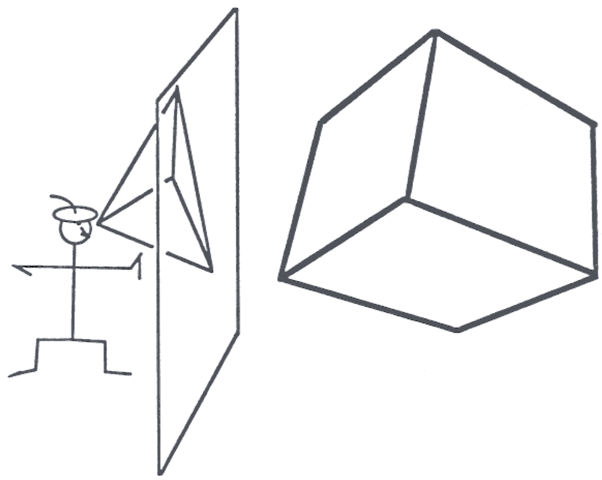

Figure 5.4

Note that the three edges of the tetrahedron incident to the station point are all perpendicular to each other, since they are parallel to the edges of the cube. The viewing distance is the length of the altitude of this tetrahedron, coming from the station point. The center of vision is the base of the altitude in the picture plane. The following Lemma tells us that the center of vision is just the altitude center of the three vanishing points in the picture plane.

Lemma 5.2.1. *Consider a tetrahedron T with one vertex O having its three incident edges mutually perpendicular. Then the foot of the altitude of T starting at O is the altitude center of the face opposite O.*

Proof. Call the vertices of the tetrahedron O, A, B, C, where OA, OB, OC are mutually perpendicular. Let X be the foot of the altitude to the tetrahedron in the plane of ABC. Then the plane of OXA is perpendicular to the plane of ABC. Then the plane of OXA is perpendicular to the plane of ABC, since it contains the line OX, which is perpendicular to the plane of ABC.

The plane through OXA is also perpendicular to the plane OBC, since it contains the line OA, which is perpendicular to the plane OBC. Thus the plane OXA is perpendicular to the intersection of the planes OBC and ABC, which is the line through BC. But the plane OXA and the plane ABC intersect in the line through AX. Thus the line AX and the line BC are perpendicular, since the line BC is perpendicular to the plane OXA and the line AX intersects the line BC (being in the plane of ABC). Thus X lies on the altitude of ABC from A. Similarly, X lies on the other two altitudes. Thus X is the altitude center of ABC. This is what we were to prove. See Figure 5.5. $\square$

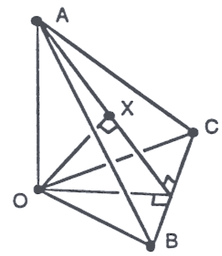

Figure 5.5

In the proof of the Lemma we have used the following three propositions from Euclidean geometry in three-space.

1. If a line L is perpendicular to a plane P, then any other plane through P is perpendicular to P.
2. Two intersecting lines in 3-space are perpendicular, if a plane through one is perpendicular to the other.
3. If one plane is perpendicular to two others, then it is perpendicular to their intersection (if the intersection is nonempty).

5.3 Constructing the center of vision in 3-space

If we "open up" the tetrahedron of the construction in Lemma 5.2.1 and lay the right angled sides flat in the picture plane next to the triangle of vanishing points, then we get Figure 5.6.

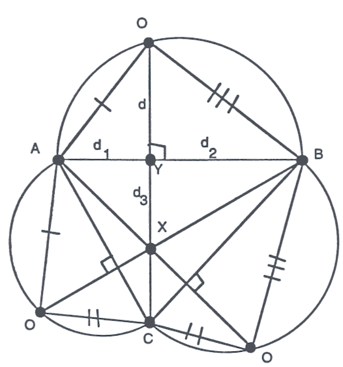

Figure 5.6

Lemma 5.3.1. *Let d_1, d_2, d_3, be the lengths indicated in Figure 6.6 determined by the center of vision X and the vanishing points A, B, C. Then the viewing distance (the length of $OABC$ from O) is*

$$d = \sqrt{d_1 d_2 - d_3^2}.$$

Proof. We use Figure 5.5. In the tetrahedron $OABC$ after folding it back into 3-space, OXY is a right triangle with OY as hypotenuse. Then OY has length $\sqrt{d_1 d_2}$ from Chapter 5. So the length of OX is $\sqrt{d_1 d_2 - d_3^2}$. □

Thus the expression of Lemma 5.3.1 gives a way of calculating the viewing distance d_3 if one knows the three vanishing points, ABC. Of course, the center of vision is the altitude center of ABC as in Figure 5.6.

It is interesting to note that, when the cube is on the opposite side of the picture plane from the station point, the triangle ABC is always acute, not matter how the cube is projected onto the picture plane. Also the center of vision X is always inside a circle with diameter AB, if A and B are vanishing points as above.

Another interesting point is that it is not necessary to use the fact that all three lengths of the sides of the cube are equal. All that was used was that the three sides are mutually perpendicular. In fact, the three vanishing points can be calculated from a picture (in three point perspective) of any "box" with arbitrary three values for length, width, and height. Knowing the three vanishing points allows one to compute the center of vision and the viewing distance. This is different from two-point perspective, where the center of vision cannot be determined unless, say, we know that the base is a square. We only know that the center of vision in two-point perspective lies on the horizon line.

On the other hand, if one only knows the three vanishing points of three point perspective, then one can construct a cube starting at any point. The key is to construct the vanishing points for the 45° lines of the base and sides by bisecting the O angles in the construction of Figure 5.6. This is shown in Figure 5.7.

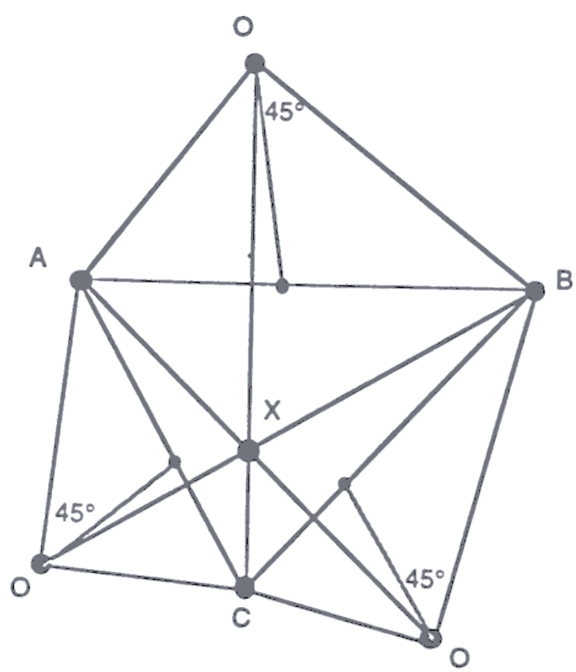

Figure 5.7

Once one has the 45° vanishing points determined, then it is an easy matter to draw a typical cube, as well (as a whole grid) as in Figure 5.8.

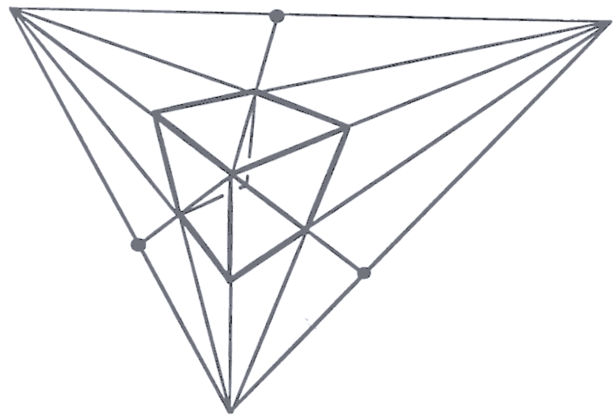

Figure 5.8

5.4 Exercises:

1. Why is it true that the vanishing points of three point perspective are so often off the paper?

2. Explain what a picture looks like when viewed from a distance greater that the proper viewing distance for two and three point perspective. Is it merely lengths or even relative lengths that appear distorted? Look at the pictures from Descargues' book where the vanishing points are on the paper.

3. Figure 5.9 is the projection of the edges of a cube in three point perspective. Assume that the cube is on the other side of the picture plane from the station point. Which is the point nearest to the picture plane, A or B? Explain your answer mathematically. Does your answer "look" right?

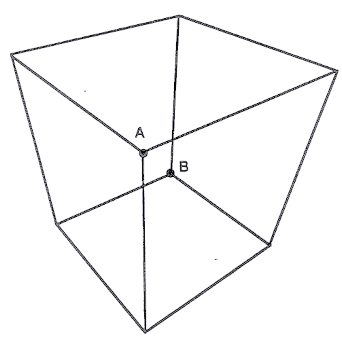

Figure 5.9

4. Suppose that one has a picture of a rectangle in two-point perspective as in Figure 5.10 (Assume that the plane of the rectangle is perpendicular to the picture plane.) At what points for the center of vision will it appear that $x \leq y$?

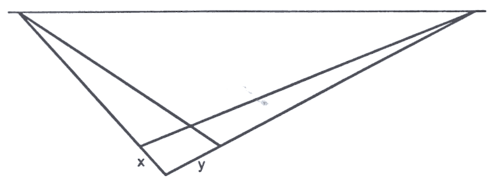

Figure 5.10

5. Suppose that Figure 5.10 is a picture of a square, but in three point perspective. That is, the plane of the square is not perpendicular to the picture plane. What is the locus of the possible centers of vision?

6. Locate the center of vision and the viewing distance for the Escher drawing of a lattice. Are the blocks cubes? You can make a transparancy and use the sketches in Geometer's Sketchpad.

7. Explain the "mistakes" in the handout of odd drawings.

8. In technical drawing the term *cone of vision* is sometimes used. This is the cone from the station along the line of sight to the center of vision. See Figure 5.11.

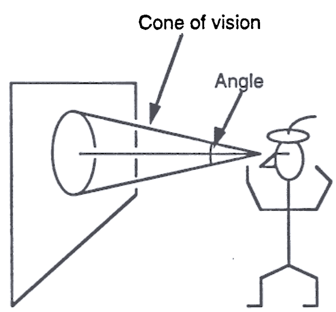

Figure 5.11

   (a) If the cone of vision is restricted to 30° (as is often done), how many vanishing points for a cube can there be inside the cone?

   (b) If all three vanishing points for a cube are in the cone of vision, what is the smallest that the angle can be? What is the position of the vanishing points at this minimum angle?

9. The following exercise provides a formula for the viewing distance in two-point perspective when the distance between the side vanishing points and one of the 45° vanishing points is known.

   (a) Suppose we have a right angled triangle with legs of length x and y, while the hypotenuse has length $a + b$, where a and b are determined as in Figure 5.12 by

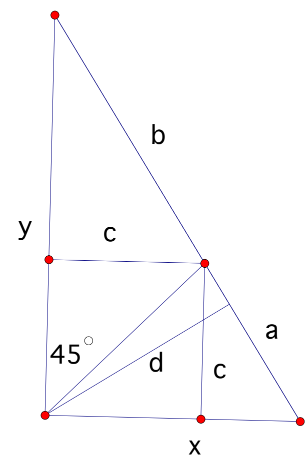

Figure 5.12

the inscribed square with side length c. Show that $\frac{a}{c} = \frac{a+b}{y}$ and $\frac{b}{c} = \frac{a+b}{x}$ and conclude that $\frac{x}{y} = \frac{a}{b}$.

(b) The altitude of triangle in Figure 5.12 has length d. By calculating the area of the right triangle two ways, show that $d(a+b) = xy$.

(c) Use the calculations of Part (a) and Part (b) to show that

$$d = \frac{ab(a+b)}{a^2+b^2}.$$

(d) Show that the formula of Part (c) can be used to calculate the viewing distance for two-point perspective as in Figure 5.13.

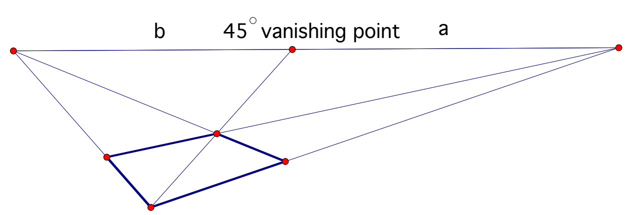

Figure 5.13

10. (Challenging) The following exercise provides a formula for the viewing distance in three-point perspective in terms of the distances between the three vanishing points. This calculation is easy with a simple calculator.

(a) In the usual orthogonal Cartesian coordinate system let a, b, c be the distances along the x-axis, y-axis, z-axis, respectively, for three separate points A, B, C, respectively. What is the equation of the plane through A, B, C?

(b) For the same points as in part a, what is the distance d of the plane from the origin?

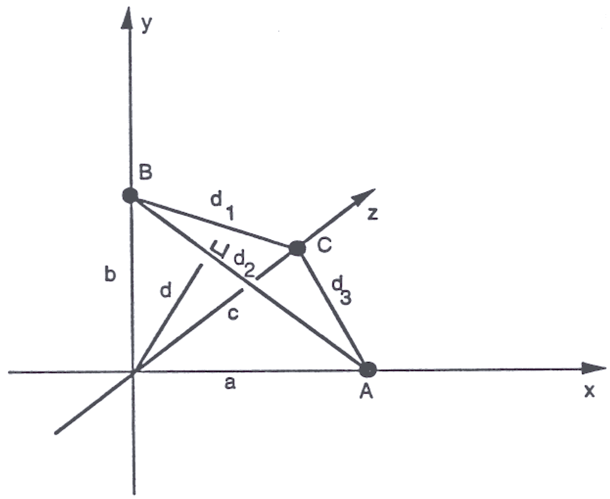

Figure 5.14

(c) Let d_1, d_2, d_3, be the distances between the three pairs of points of Part (a). Find an expression for a, b, c in terms of these distances.

(d) Given the distances between the three pairs of vanishing points in three point perspective, find an expression for the viewing distance. (Hint: The answer is the following, where d is the viewing distance.)

$$d = \sqrt{\frac{1}{\frac{2}{-d_1^2+d_2^2+d_3^2} + \frac{2}{d_1^2-d_2^2+d_3^2} + \frac{2}{d_1^2+d_2^2-d_3^2}}}$$

11. Use the formula in Problem 10d to calculate the viewing distance in the following Escher picture Figure 5.16, OR find your own example of a drawing or picture in 3-point perspective. Then find the correct viewing distance for Figure 5.16

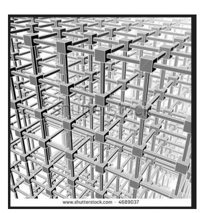

Figure 5.15

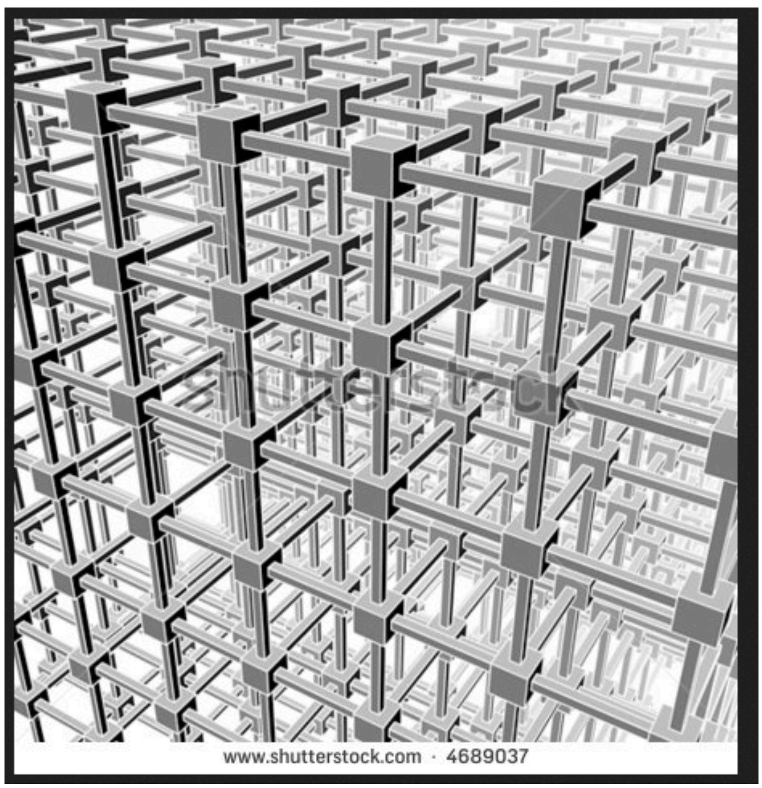

Figure 5.16: This is Figure 5.16 rescaled by a factor of 4.