Chapter 5

Dynamics

5.1 Introduction

The study of manipulator dynamics is essential for both the analysis of performance and the design of robot control. A manipulator is a multi-link, highly nonlinear and coupled mechanical system. In motion, this system is subjected to inertial, centrifugal, Coriolis, and gravity forces, which can greatly affect its performance in the execution of a task. If ignored, these dynamics may also lead to control instability, especially for tasks that involve contact interactions with the environment. Our goal here is to model the dynamics and establish the manipulator equation of motion in order to develop the appropriate control structures needed to achieve robot's stability and performance.

There are various formulations for modeling the dynamics of manipulators. We will discuss a recursive algorithm based on the Newton-Euler formulation, and present an approach for the explicit model, based on Lagrange's formulation. These two methodologies are similar to the recursive and explicit approaches we presented earlier for the kinematic model and the Jacobian matrix.

In the Newton-Euler method, the analysis is based on isolating each link and considering all the forces acting on it. This analysis is similar to the previous study of static forces, which lead to the relationship between

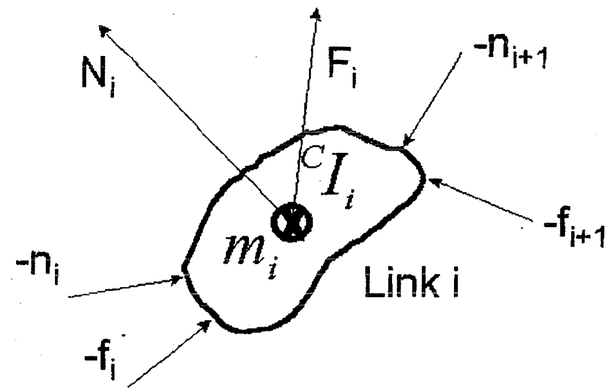

Figure 5.1: Link's Dynamics

end-effector forces and joint torques, i.e. $\tau = J^T \mathbf{F}$. The difference with the previous analysis is that now we must account for the inertial forces acting on the manipulator links.

For link i, we consider the forces $\mathbf{f}_i$ and $\mathbf{f}_{i+1}$, and the moments n_i and n_{i+1} acting at joints i and $i+1$. Because of the motion of the link, we must include the inertial forces associated with this motion. Let F_i and N_i be the inertial forces corresponding to the linear motion and angular motion respectively, expressed at the center of mass of the link. These dynamic forces are given by the equations of Newton (linear motion) and Euler (angular motion),

$$m_i \dot{\mathbf{v}}_{C_i} = F_i \tag{5.1}$$

$$^{C_i}I\dot{\omega}_i + \omega_i \times {^{C_i}I}\omega_i = N_i \tag{5.2}$$

where m_i and ^{C_i}I are the mass and link's tensor of inertia at the center of mass.

Similarly to the static analysis we have seen, recursive force and moment relationships can be developed, and internal forces and moments can be eliminated by projection on the joint axis,

$$\tau_i = \begin{cases} \mathbf{n}_i^T Z_i & \text{for a revolute joint} \\ \mathbf{f}_i^T Z_i & \text{for a prismatic joint} \end{cases} \tag{5.3}$$

The Newton-Euler algorithm consists of two propagation phases. A *forward propagation* of velocities, accelerations, and dynamic forces. *Backward propagation* then eliminates internal forces and moments. Internal forces are transmitted through the structure.

Lagrange's formulation relies on the concept of energy, the kinetic energy K and the potential energy U of the system. The kinetic energy is expressed in terms of the manipulator mass matrix M and the generalized velocities $\dot{\mathbf{q}}$ in the following quadratic form,

$$K = \frac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}} \tag{5.4}$$

Given the potential energy V, the Lagrangian is

$$L = K - V \tag{5.5}$$

and Lagrange's equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial q} = \tau \tag{5.6}$$

where τ is the vector of applied generalized torques. Both formalisms, Newton-Euler and Lagrange, lead to the same set of equations, which can be developed in the form

$$M\ddot{\mathbf{q}} + \mathbf{v} + \mathbf{g} = \tau \tag{5.7}$$

where $\mathbf{g}$ is the vector of gravity forces and $\mathbf{v}$ is the vector of centrifugal and Coriolis forces. These equations provide the relationship between torques applied to the manipulator and the resulting accelerations and velocities.

Analysis of Lagrange's equations shows that the coefficients involved in $\mathbf{v}$ can be obtained from M. This reduces the problem to finding M

and $\mathbf{g}$. The mass matrix M can be directly found from the total kinetic energy of the mechanism, and $\mathbf{g}$ can be determined simply from static analysis. This provides an *explicit* form of the equation of motion.

5.2 Newton-Euler Formulation

Newton's equation provides a description of the linear motion. Euler's equation, which describes the angular motion, involves the notion of angular momentum and the link's inertia tensor.

5.2.1 Linear and Angular Momentum

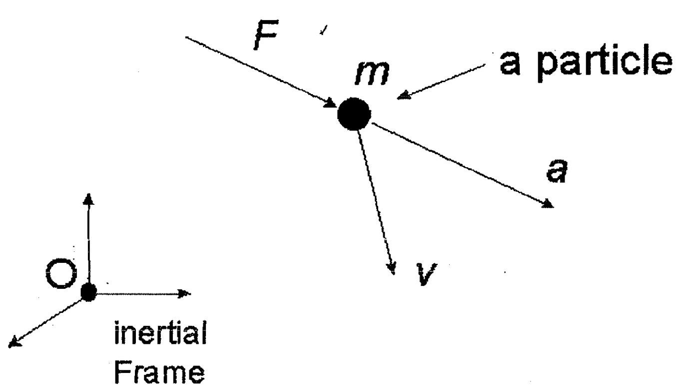

Figure 5.2: A particle's dynamics

Let us start with a simple particle. The kinetic energy of a particle with a velocity $\mathbf{v}$ is $1/2m\mathbf{v}^2$. Newton's law gives us the equation for the acceleration of the particle $\mathbf{a}$ with respect to an inertial frame, given an applied force $\mathbf{F}$

$$m\mathbf{a} = \mathbf{F}$$

This equation can be also written in terms of the linear momentum, $m\mathbf{v}$ of this particle. The rate of change of the linear momentum is equal to the applied forces,

$$\frac{d}{dt}(m\mathbf{v}) = \mathbf{F} \tag{5.8}$$

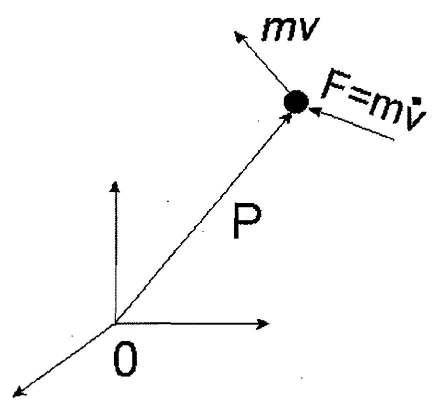

Figure 5.3: Angular Momentum Computation

To introduce the angular momentum, we take the moment of the forces that appear on both sides of the above equation. The moment N of $\mathbf{F}$ with respect to some point O is the cross product of the vector $\mathbf{p}$ locating the particle and the vector F. Taking the moment with respect to the same point of the left hand side of the equation yields

$$\mathbf{p} \times m\dot{\mathbf{v}} = \mathbf{p} \times \mathbf{F} = N \tag{5.9}$$

Let us consider the rate of change of the quantity $\mathbf{p} \times m\mathbf{v}$,

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{p} \times m\dot{\mathbf{v}} + \mathbf{v} \times m\mathbf{v} = \mathbf{p} \times m\dot{\mathbf{v}} \tag{5.10}$$

This yields

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = N \tag{5.11}$$

The quantity $\mathbf{p} \times m\mathbf{v}$ is the angular momentum with respect to O of the particle. Thus the rate of change of the angular momentum is equal to the applied moment. This equation complements the one above for the rate of change of the linear momentum and the applied forces.

5.2.2 Euler Equation

To develop Euler's equations, we must extend our previous result to the rigid body case. A rigid body can be treated as a large set of particles, and the previous analysis can be extended to the sum over this set.

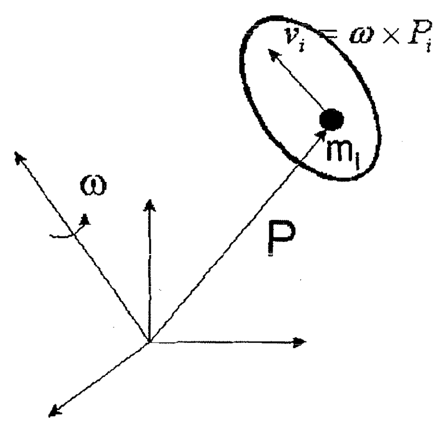

Figure 5.4: Rigid body rotational motion

Let us consider the angular motion of a rigid body rotating with respect to some fixed point O at an angular velocity ω. The linear velocity, $\mathbf{v}_i$, of a particle i of this rigid body is $\omega \times \mathbf{p}_i$. where $\mathbf{p}_i$ is the position vector for the particle with respect to O. The angular momentum, Φ, of the rigid body – the sum of the angular momentums of all particles – is

$$\Phi = \sum_i m_i \mathbf{p}_i \times (\omega \times \mathbf{p}_i) \tag{5.12}$$

Let us assume that the mass density of the rigid body is ρ. The mass m_i can be approximated by the product of the density of the rigid body ρ by a small volume dv occupied by a particle. Integrating over the rigid body's volume, V, we obtain

$$\Phi = \int_V \mathbf{p} \times (\omega \times \mathbf{p}) \rho dv \tag{5.13}$$

Observing that ω is independent of the variable in this integral, and replacing $\mathbf{p}\times$ by the cross product operator $\hat{\mathbf{p}}$, yields

$$\Phi = [\int_V -\hat{\mathbf{p}}\hat{\mathbf{p}} \rho dv]\omega \tag{5.14}$$

The quantity in brackets is called the inertia tensor of the rigid body, I, hence

$$I = [\int_V -\hat{\mathbf{p}}\hat{\mathbf{p}} \rho dv]$$

Finally, the angular momentum of this rigid body is

$$\Phi = I\omega \tag{5.15}$$

Euler's equations for the rotational motion with respect to some point O state that the rate of change of the angular momentum of the rigid body is equal to the applied moments

$$\dot{\Phi} = I\dot{\omega} + \omega \times I\omega = N \tag{5.16}$$

Together with Newton's law, these equations provide the description of the linear and angular motions for a manipulator, subjected to external forces.

5.2.3 Inertia Tensor

The inertia tensor I is defined by

$$I = \int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv \tag{5.17}$$

The quantity $-\hat{\mathbf{p}}\hat{\mathbf{p}}$ can be computed as

$$(-\hat{\mathbf{p}}\hat{\mathbf{p}}) = (\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T \tag{5.18}$$

The inertia tensor is therefore

$$I = \int_V [(\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T]\rho dv \tag{5.19}$$

Let us consider a Cartesian representation for the position vector $\mathbf{p}$,

$$\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{5.20}$$

The term in the integral is

$$[(\mathbf{p}^T\mathbf{p})I_3 - \mathbf{p}\mathbf{p}^T] = \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & z^2+x^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix} \tag{5.21}$$

The inertia tensor I is represented by the matrix

$$I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \tag{5.22}$$

where

$$I_{xx} = \iiint (y^2 + z^2)\rho dxdydz \tag{5.23}$$

$$I_{yy} = \iiint (z^2 + x^2)\rho dxdydz \tag{5.24}$$

$$I_{zz} = \iiint (x^2 + y^2)\rho dxdydz \tag{5.25}$$

$$I_{xy} = \iiint xy\rho dxdydz \tag{5.26}$$

$$I_{xz} = \iiint xz\rho dxdydz \tag{5.27}$$

$$I_{yz} = \iiint yz\rho dxdydz \tag{5.28}$$

I_{xx}, I_{yy}, and I_{zz} are called the moments of inertia and I_{xy}, I_{yz} and I_{zx} are called products of inertia. When the matrix I is diagonal, the diagonal moments of inertia are called the *principal moments of inertia.*

Parallel Axis Theorem

Because of the symmetries generally found in rigid bodies, it is more efficient to compute the body's inertia tensor with respect to its center of mass. If needed with respect to another point and axes, the inertia tensor can be obtained from the tensor computed at the center of mass through a translation and rotation transformation, determined by the parallel axis theorem.

Assuming the the inertia tensor has been computed with respect to the frame $\{C\}$ (at the body's center of mass), to find the inertia tensor with respect to another frame $\{A\}$, whose axes are parallel to those of $\{C\}$, we can proceed as follows.

Let $\mathbf{p}_C$ be the vector locating point C in frame $\{A\}$. The parallel axis theorem states:

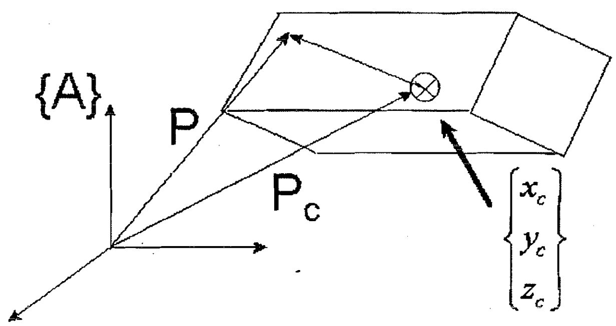

Figure 5.5: Parallel Axis Theorem

$$^{A}I = {}^{C}I + m[(\mathbf{p}_C^T \mathbf{p}_C) I_3 - \mathbf{p}_C \mathbf{p}_C^T] \tag{5.29}$$

If (x_c, y_c, z_c) are the Cartesian coordinates of point C in frame $\{A\}$, the relationships between the two tensors are

$$^{A}I_{zz} = {}^{C}I_{zz} + m(x_C^2 + y_C^2) \tag{5.30}$$

$$^{A}I_{xy} = {}^{C}I_{xy} + m x_C y_C \tag{5.31}$$

Rotation Transformation Let us consider the case where we wish to express the inertia tensor with respect to another frame rotated with respect to the rigid body frame. The angular momentum expressed in frame $\{A\}$ is

$$^{A}\Phi = {}^{A}I\,^{A}\omega$$

Let's express this quantity in a frame $\{B\}$, having the same origin as $\{A\}$ and obtained by a rotation $^{B}_{A}R$,

$$^{B}\Phi = {}^{B}_{A}R\,^{A}\Phi = {}^{B}_{A}R\,^{A}I\,^{A}\omega$$

where

$$^{A}\omega = {}^{A}_{B}R^{B}\omega = {}^{B}_{A}R^{TB}\omega$$

thus

$$^{B}\Phi = {}^{B}_{A}R^{A}I({}^{B}_{A}R^{T}\ {}^{B}\omega)$$

also

$$^{B}\Phi = {}^{B}I\ {}^{B}\omega$$

finally

$$^{B}I = [{}^{B}_{A}R^{A}I^{B}_{A}R^{T}]$$

The relationship described above is a similarity transformation. For a general frame transformation involving both translation and rotation. We first proceed with a translation using the parallel axis theorem, and then apply the similarity transformation for the rotation.

5.3 Lagrange Formulation

Given a set of *generalized coordinates,* **q**, describing the configuration of a mechanism, there is a set of corresponding *generalized forces,* τ, acting along (or about) each of these coordinates. If the coordinate q_i represents the rotation of a revolute joint, the corresponding force τ_i would be a torque acting about the joint axis. For a prismatic joint, τ_i is a force acting along the axis of the joint.

Lagrange's equations involve a scalar quantity L, the Lagrangian, which represents the difference between the two scalars corresponding to the kinetic energy K and the potential energy V of the mechanism,

$$L = K - V \tag{5.32}$$

The Lagrangian, L is a function, of the generalized coordinates $\mathbf{q}$ and the generalized velocities, $\dot{\mathbf{q}}$.

$$L = L(\mathbf{q}, \dot{\mathbf{q}})$$

For an n DOF mechanism, the Lagrange formulation provides the n equations of motion in the following from

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\mathbf{q}}}) - \frac{\partial L}{\partial \mathbf{q}} = \tau \tag{5.33}$$

Since the potential energy (due to the gravity) is only dependent on the configuration, these equations can be written as

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{\mathbf{q}}}) - \frac{\partial K}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \tau \tag{5.34}$$

The first two terms define the inertial forces associated with the motion of the mechanism, and the third term represents the gradient of the gravity potential acting on it. This gradient is the gravity force vector.

For a single mass m with a velocity v, the kinetic energy is $1/2(v^T m v)$. In the case of a multi-link manipulator with a mass matrix M and generalized velocities, $\dot{\mathbf{q}}$, the kinetic energy is the scalar given by the quadratic form

$$K = \frac{1}{2}\dot{\mathbf{q}}^T M \dot{\mathbf{q}} \tag{5.35}$$

Using this expression of K we can write

$$\frac{\partial K}{\partial \dot{\mathbf{q}}} = \frac{\partial}{\partial \dot{\mathbf{q}}}(\frac{1}{2}\dot{\mathbf{q}}^T M(q)\dot{\mathbf{q}}) = M\dot{\mathbf{q}} \tag{5.36}$$

Differentiating with respect to time we obtain:

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{\mathbf{q}}}) = M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}} \tag{5.37}$$

The inertial forces in the equation of Lagrange can be expressed as

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{\mathbf{q}}}) - \frac{\partial K}{\partial \mathbf{q}} = M\ddot{\mathbf{q}} + \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix} \tag{5.38}$$

This equation can be developed in the form

$$\frac{d}{dt}(\frac{\partial K}{\partial \dot{\mathbf{q}}}) - \frac{\partial K}{\partial \mathbf{q}} = M\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) \tag{5.39}$$

where $\mathbf{v}$ is the vector of centrifugal and Coriolis forces given by

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix} \tag{5.40}$$

Finally adding the inertial and gravity terms in the Lagrange equations, yields

$$M(q)\ddot{\mathbf{q}} + \mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{g}(\mathbf{q}) = \tau \tag{5.41}$$

The vector of centrifugal and Coriolis forces can be expressed as

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] \tag{5.42}$$

5.3.1 Explicit Form of the Mass Matrix

The mass matrix M plays a central role in the dynamics of manipulator. In particular, the elements of the matrices B and C can be completely determined from this matrix.

Because of its additive property, the kinetic energy of the total system is the sum of the kinetic energies associated with its links.

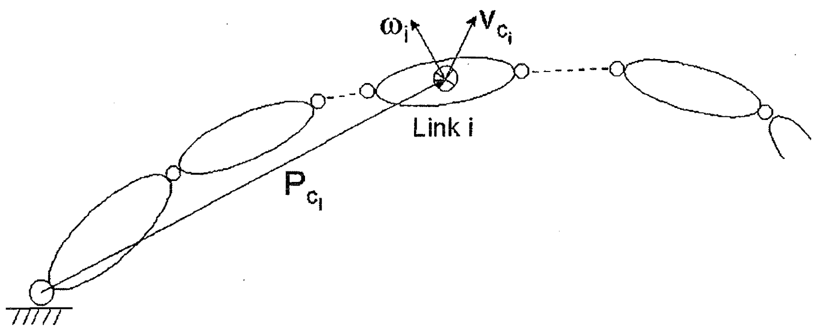

Figure 5.6: Explicit Form

$$K = \sum_{i=1}^{n} K_i \tag{5.43}$$

The kinetic energy of a link has two components: one that is due to its linear motion, and the second due to its rotational motion. If the linear velocity of the center of mass of a link is $\mathbf{v}_{C_i}$, and if the angular velocity of the link is ω_i, the kinetic energy, K_i of this link is

$$K_i = \frac{1}{2}(m_i \mathbf{v}_{C_i}^T \mathbf{v}_{C_i} + \omega_i^T I_{C_i} \omega_i) \tag{5.44}$$

where I_{C_i} is the inertia tensor of link i computed with respect to the link's center of mass, C_i. The linear velocity at the center v_{C_i} can be expressed as a linear combination of the joint velocities, $\dot{\mathbf{q}}$. Introducing a Jacobian matrix, J_{v_i}, corresponding to the linear motion of the center-of-mass of link i, the velocity vector v_{C_i} can be written as

$$\mathbf{v}_{C_i} = J_{v_i} \dot{\mathbf{q}} \tag{5.45}$$

where

$$J_{v_i} = [\, \tfrac{\partial \mathbf{p}_{C_i}}{\partial q_1} \quad \tfrac{\partial \mathbf{p}_{C_i}}{\partial q_2} \quad \cdots \quad \tfrac{\partial \mathbf{p}_{C_i}}{\partial q_i} \quad 0 \quad 0 \quad \cdots \quad 0\,] \tag{5.46}$$

In this matrix, the columns $i+1$ to n in J_{v_i} are zero columns, since the velocity $\mathbf{v}_{C_i}$ at the center of mass of link i is independent of the velocities of joint $i+1$ to joint n. Similarly the angular velocity can be expressed as

$$\omega_i = J_{\omega_i}\dot{\mathbf{q}} \tag{5.47}$$

where

$$J_{\omega_i} = [\bar{\varepsilon}_1 Z_1 \quad \bar{\varepsilon}_2 Z_2 \quad \cdots \quad \bar{\varepsilon}_i Z_i \quad 0 \quad 0 \quad \cdots \quad 0] \tag{5.48}$$

Using these expressions, the kinetic energy becomes

$$K = \frac{1}{2}\sum_{i=1}^{n}(m_i\dot{\mathbf{q}}^T J_{v_i}^T J_{v_i}\dot{\mathbf{q}} + \dot{\mathbf{q}}^T J_{\omega_i}^T I_{C_i} J_{\omega_i}\dot{\mathbf{q}}) \tag{5.49}$$

Factoring out the generalized velocities, the kinetic energy can be expressed as

$$K = \frac{1}{2}\dot{\mathbf{q}}^T[\sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i})]\dot{\mathbf{q}} \tag{5.50}$$

Equating this expression to the quadratic form of the kinetic energy leads to the following explicit form of the mass matrix$M$,

$$M = \sum_{i=1}^{n}(m_i J_{v_i}^T J_{v_i} + J_{\omega_i}^T I_{C_i} J_{\omega_i}) \tag{5.51}$$

The mass matrix M is a symmetric positive definite matrix, i.e. $m_{ij} = m_{ji}$ and $\dot{\mathbf{q}}^T M\dot{\mathbf{q}} > 0$ for $\dot{\mathbf{q}} \neq 0$

5.3.2 Centrifugal and Coriolis Forces

We now consider the relationships between the matrices B and C with the matrix M. These relationships can be obtained from the development of equation 5.40 defining the vector of centrifugal and Coriolis forces

$$\mathbf{v}(\mathbf{q}, \dot{\mathbf{q}}) = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_1}\dot{\mathbf{q}} \\ \vdots \\ \dot{\mathbf{q}}^T \frac{\partial M}{\partial q_n}\dot{\mathbf{q}} \end{bmatrix}$$

This equation involves time derivatives and partial derivatives of the elements m_{ij} of the matrix M. We denote by m_{ijk} the partial derivatives

$$m_{ijk} \equiv \frac{\partial m_{ij}}{\partial q_k} \tag{5.52}$$

The time derivative of an element m_{ij} is

$$\frac{dm_{ij}}{dt} = \sum_{k=1}^{n} m_{ijk}\dot{q}_k$$

To simplify the development, let us consider the case of a 2 DOF manipulator. The mass matrix is

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \tag{5.53}$$

The vector $\mathbf{v}$ of centrifugal and Coriolis forces is

$$\mathbf{v} = \dot{M}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T M_{q_1}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T M_{q_2}\dot{\mathbf{q}} \end{bmatrix} = \begin{pmatrix} \dot{m}_{11} & \dot{m}_{12} \\ \dot{m}_{12} & \dot{m}_{22} \end{pmatrix}\dot{\mathbf{q}} - \frac{1}{2}\begin{bmatrix} \dot{\mathbf{q}}^T \begin{pmatrix} m_{111} & m_{121} \\ m_{121} & m_{221} \end{pmatrix}\dot{\mathbf{q}} \\ \dot{\mathbf{q}}^T \begin{pmatrix} m_{112} & m_{122} \\ m_{122} & m_{222} \end{pmatrix}\dot{\mathbf{q}} \end{bmatrix}$$

These expressions can be developed in the form

$$\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) = \begin{bmatrix} \frac{1}{2}(m_{111}+m_{111}-m_{111}) & \frac{1}{2}(m_{122}+m_{122}-m_{221}) \\ \frac{1}{2}(m_{211}+m_{211}-m_{112}) & \frac{1}{2}(m_{222}+m_{222}-m_{222}) \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} m_{112}+m_{121}-m_{121} \\ m_{212}+m_{221}-m_{122} \end{bmatrix} [\dot{q}_1\dot{q}_2] \quad (5.54)$$

Expansion in this form shows a pattern of grouping of coefficients that leads to the following representation of Christoffel symbols,

$$b_{ijk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki}) \quad (5.55)$$

Using these symbols the equation above can be written as:

$$\mathbf{v} = \begin{bmatrix} b_{111} & b_{122} \\ b_{211} & b_{222} \end{bmatrix} \begin{bmatrix} \dot{q}_1^2 \\ \dot{q}_2^2 \end{bmatrix} + \begin{bmatrix} 2b_{112} \\ 2b_{212} \end{bmatrix} [\dot{q}_1\dot{q}_2] \quad (5.56)$$

In this equation, the first matrix corresponds to the matrix C of the coefficients associated with centrifugal forces, and the second matrix represents the matrix B corresponding to the the coefficients of Coriolis forces. In this case of 2 DOF, the matrix C is of dimension (2×2) and B is (2×1).

In the general case of n DOF, C is an $(n \times n)$ matrix, while B is of dimensions $n \times (n \times \frac{(n-1)n}{2})$. Using these matrices, the vector $\mathbf{v}$ is

$$\mathbf{v}(\mathbf{q},\dot{\mathbf{q}}) = C(\mathbf{q})[\dot{\mathbf{q}}^2] + B(\mathbf{q})[\dot{\mathbf{q}}\dot{\mathbf{q}}] \quad (5.57)$$

$[\dot{\mathbf{q}}^2]$ is the symbolic representation of the $n \times 1$ vector of components $\dot{q}_i^2$ (square joint velocities),

$$[\dot{\mathbf{q}}^2]^T = [\dot{q}_1^2 \; \dot{q}_2^2 \; \dot{q}_3^2 \ldots \dot{q}_n^2]^T$$

$[\dot{\mathbf{q}}\dot{\mathbf{q}}]$ is the $(\frac{(n-1)n}{2} \times 1)$ vector of product of joint velocities

$$[\dot{\mathbf{q}}\dot{\mathbf{q}}]^T = [\dot{q}_1\dot{q}_2 \; \dot{q}_1\dot{q}_3 \ldots \dot{q}_1\dot{q}_n \; \dot{q}_2\dot{q}_3 \; \dot{q}_2\dot{q}_4 \ldots \dot{q}_2\dot{q}_n \ldots \dot{q}_{(n-1)} \; \dot{q}_n]^T$$

The general forms of the matrices B and C are

$$C(\mathbf{q}) = \begin{bmatrix} b_{1,11} & b_{1,22} & \cdots & b_{1,nn} \\ b_{2,11} & b_{2,22} & \cdots & b_{2,nn} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n,11} & b_{n,22} & \cdots & b_{n,nn} \end{bmatrix} \tag{5.58}$$

and

$$B(\mathbf{q}) = \begin{bmatrix} 2b_{1,12} & 2b_{1,13} & \cdots & 2b_{1,1n} & 2b_{1,23} & \cdots & 2b_{1,2n} & \cdots & 2b_{1,(n-1)n} \\ 2b_{2,12} & 2b_{2,13} & \cdots & 2b_{2,1n} & 2b_{2,23} & \cdots & 2b_{2,2n} & \cdots & 2b_{2,(n-1)n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 2b_{n,12} & 2b_{n,13} & \cdots & 2b_{n,1n} & 2b_{n,23} & \cdots & 2b_{n,2n} & \cdots & 2b_{n,(n-1)n} \end{bmatrix} \tag{5.59}$$

Because of the properties of the mass matrix, many of the elements b_{ijk} are zero. This symmetric, positive definite matrix represents the inertial properties of the manipulator with respect to joint motion. For instance, if joint 1 was revolute, m_{11} would represent the inertia (mass if it were prismatic) of the whole manipulator as it rotates about the joint axis 1. m_{11} is independent of the first joint, but varies with the configuration of the links following in the chain $(q_2, q_3, \ldots, q_n)$. Similarly m_{22} depends only on $q_3, \ldots, q_n$, and $m_{(n-1)(n-1)}$ depends only on q_n. Finally m_{nn} is a constant element. These properties result in a number of zero partial derivatives of the elements of the mass matrix, and leads to significant simplification of the elements involved in B and C.

5.3.3 Gravity Forces

The gravity forces are the gradient of the potential energy of the mechanism. The potential energy of link i increases with the elevation of its center of mass. This energy is proportional to the mass, the gravity constant, and to the height of the center of mass.

$$V_i = m_i g_0 h_i + V_0 \tag{5.60}$$

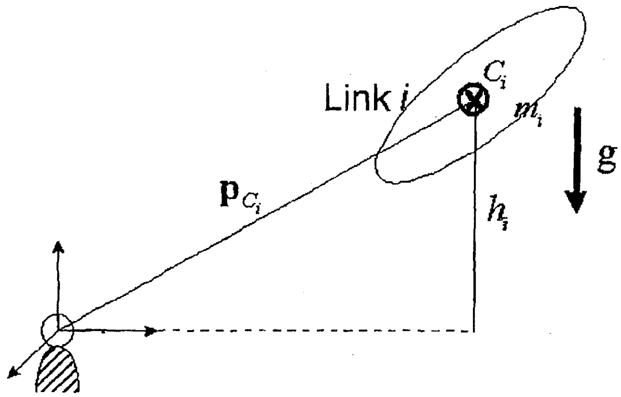

Figure 5.7: Potential Energy

Where V_0 represents the potential energy at some reference level. The height is given as the projection of the position vector $\mathbf{p}_{C_i}$ along the gravity direction,

$$V_i = m_i(-g^T p_{C_i}) \tag{5.61}$$

The potential energy of the whole manipulator is

$$V = \sum_i V_i \tag{5.62}$$

Using the matrix J_{v_i}, the gradient of the potential energy is

$$\mathbf{g} = -\begin{pmatrix} J_{v_1}^T & J_{v_2}^T & \cdots & J_{v_n}^T \end{pmatrix} \begin{pmatrix} m_1 g \\ m_2 g \\ \vdots \\ m_n g \end{pmatrix} \tag{5.63}$$

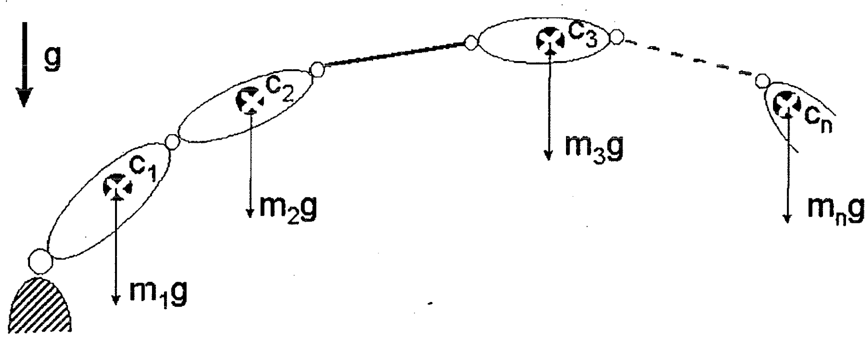

Figure 5.8: Gravity Vector

Direct Computation of g The gravity forces can be directly also by considering the gravity at the link's as weights acting at each link's center of mass. The gravity forces can then be directly computed as the torques needed to compensate for these weights. This leads

$$\mathbf{g} = -(J_{v_1}^T(m_1 g) + J_{v_2}^T(m_2 g) + \cdots + J_{v_n}^T(m_n g)) \tag{5.64}$$

5.3.4 Example: 2-DOF RP Manipulator

The links of the RP manipulator shown in Figure 5.9 have total masses of m_1 and m_2. The center of mass of link 1 is located at a distance l_1 from the joint axis 1, and the center of mass of link 2 is located at the distance d_2 from the joint axis 1. The inertia tensors of these links are

$${}^1I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix}; \text{ and } {}^2I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}.$$

The Mass Matrix M The mass matrix M can be obtained by applying equation 5.51 to this 2 DOF manipulator:

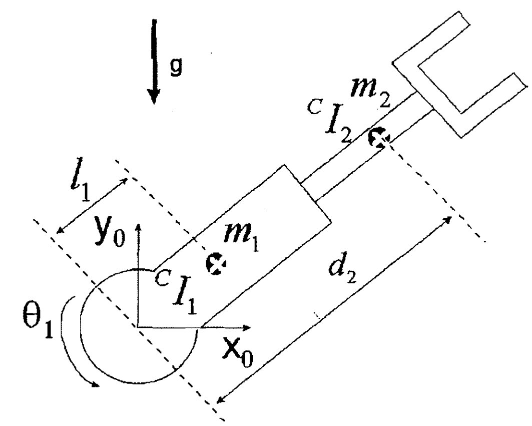

Figure 5.9: 2 DOF RP Manipulator

$$M = m_1 J_{v1}^T J_{v1} + J_{\omega 1}^T I_1 J_{\omega 1} + m_2 J_{v2}^T J_{v2} + J_{\omega 2}^T I_2 J_{\omega 2}.$$

J_{v1} and J_{v2} are obtained by direct differentiation of the vectors:

$$^0\mathbf{p}_{C1} = \begin{bmatrix} l_1 C1 \\ l_1 S1 \\ 0 \end{bmatrix}; \text{ and } ^0\mathbf{p}_{C2} = \begin{bmatrix} d_2 C1 \\ d_2 S1 \\ 0 \end{bmatrix}.$$

In frame $\{0\}$, these matrices are:

$$^0J_{v1} = \begin{bmatrix} -l_1 S1 & 0 \\ l_1 C1 & 0 \\ 0 & 0 \end{bmatrix}; \quad ^0J_{v2} = \begin{bmatrix} -d_2 S1 & C1 \\ d_2 C1 & S1 \\ 0 & 0 \end{bmatrix}.$$

This yields

$$m_1({}^0J_{v1}^T\,{}^0J_{v1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \quad (m_2\,{}^0J_{v2}^T\,{}^0J_{v2}) = \begin{bmatrix} m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}.$$

The matrices $J_{\omega 1}$ and $J_{\omega 2}$ are given by

$$J_{\omega 1} = \begin{bmatrix} \bar{\epsilon}_1 \mathbf{z}_1 & \mathbf{0} \end{bmatrix} = \text{ and } J_{\omega 2} = \begin{bmatrix} \bar{\epsilon}_1 \mathbf{z}_1 & \bar{\epsilon}_2 \mathbf{z}_2 \end{bmatrix}.$$

Joint 1 is revolute and joint 2 is prismatic. In frame $\{0\}$, these matrices are:

$$^0J_{\omega 1} = {}^0J_{\omega 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Noting that for this planar mechanism $^1J_{\omega 1} = {}^0J_{\omega 1}$ and $^1J_{\omega 2} = {}^0J_{\omega 2}$ yields

$$({}^1J_{\omega 1}^T\, {}^1I_1\, {}^1J_{\omega 1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \quad ({}^2J_{\omega 2}^T\, {}^2I_2\, {}^2J_{\omega 2}) = \begin{bmatrix} I_{zz2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, the matrix M is

$$M = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix}.$$

Centrifugal and Coriolis Vector v The Christoffel Symbols are defined as

$$b_{i,jk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki}); \text{ where } m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}; \text{ with } b_{iii} = b_{iji} = 0.$$

For this manipulator, only m_{11} (see matrix M) is configuration dependent – a function of d_2. This implies that only m_{112} is non-zero,

$$m_{112} = 2m_2 d_2.$$

<u>Matrix B</u>

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix} = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix}.$$

<u>Matrix C</u>

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix}.$$

<u>Vector V</u>

$$V = \begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot\theta_1 \dot d_2] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot\theta_1^2 \\ \dot d_2^2 \end{bmatrix}.$$

The Gravity Vector g

$$\mathbf{G} = -[J_{v1}^T m_1 \mathbf{g} + J_{v2}^T m_2 \mathbf{g}].$$

In frame $\{0\}$, the gravity vector is

$$^0G = -\begin{bmatrix} -l_1 S1 & l_1 C1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -d_2 S1 & d_2 C1 & 0 \\ C1 & S1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix};$$

and

$$^0G = \begin{bmatrix} (m_1 l_1 + m_2 d_2) g C1 \\ m_2 g S1 \end{bmatrix}.$$

Equations of Motion

$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2 d_2^2 + I_{zz2} & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{d}_2 \end{bmatrix} +$$
$$\begin{bmatrix} 2m_2 d_2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{d}_2] + \begin{bmatrix} 0 & 0 \\ -m_2 d_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{d}_2^2 \end{bmatrix} + \begin{bmatrix} (m_1 l_1 + m_2 d_2) g C1 \\ m_2 g S1 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$$

5.3.5 Example: 2-DOF RR Equations of Motion

The masses of the links are m_1 and m_2. The center of mass for the first link is located on the second joint axis at a distance l_1 from the fixed origin. The distance from the second joint axis to the center of mass of link 2 is denoted by l_2. The inertia tensors of the links are I_1 and I_2.

$$^1I_1 = \begin{pmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{pmatrix}; \text{ and } ^2I_2 = \begin{pmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{pmatrix}.$$

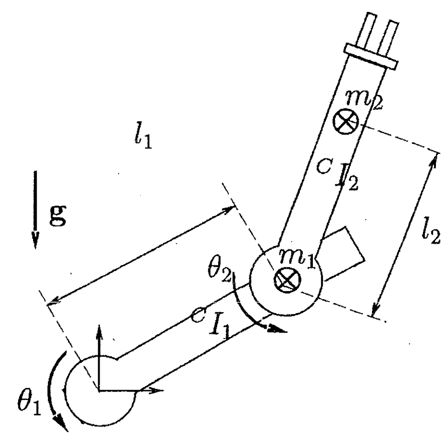

Figure 5.10: 2-DOF RR Equations of Motion

Matrix M The mass matrix M is obtained by applying equation 5.51.

$$M = m_1 J_{v1}^T J_{v1} + J_{\omega 1}^T I_1 J_{\omega 1} + m_2 J_{v2}^T J_{v2} + J_{\omega 2}^T I_2 J_{\omega 2}.$$

We compute J_{v_1} and J_{v_2} by direct differentiation of P_{C_1} and P_{C_2}.

$$^0\mathbf{p}_{C1} = \begin{bmatrix} l_1 C1 \\ l_1 S1 \\ 0 \end{bmatrix}; \quad \text{and} \quad ^0\mathbf{p}_{C2} = \begin{bmatrix} l_1 C1 + l_2 C12 \\ l_1 S1 + l_2 S12 \\ 0 \end{bmatrix}.$$

In frame $\{0\}$, these matrices are:

$$^0J_{v1} = \begin{bmatrix} -l_1 S1 & 0 \\ l_1 C1 & 0 \\ 0 & 0 \end{bmatrix}; \quad ^0J_{v2} = \begin{bmatrix} -l_1 S1 - l_2 S12 & -l_2 S12 \\ l_1 C1 + l_2 C12 & l_2 C12 \\ 0 & 0 \end{bmatrix}.$$

This yields

$$m_1({}^0J_{v1}^T\,{}^0J_{v1}) = \begin{bmatrix} m_1 l_1^2 & 0 \\ 0 & 0 \end{bmatrix}; \quad (m_2\,{}^0J_{v2}^T\,{}^0J_{v2}) = \begin{bmatrix} m_2(l_1^2 + l_2^2 + 2l_1l_2C2 & m_2(l_2^2 + l_1l_2C2) \\ m_2(l_2^2 + l_1l_2C2) & m_2l_2^2 \end{bmatrix}.$$

The matrices $J_{\omega 1}$ and $J_{\omega 2}$ are given by

$$J_{\omega 1} = [\,\bar{\epsilon}_1\mathbf{z}_1 \quad \mathbf{0}\,] = \text{ and } J_{\omega 2} = [\,\bar{\epsilon}_1\mathbf{z}_1 \quad \bar{\epsilon}_2\mathbf{z}_2\,].$$

Both joints are revolute. In frame $\{0\}$, these matrices are:

$${}^0J_{\omega 1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad {}^0J_{\omega 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix};$$

and since ${}^1J_{\omega 1} = {}^0J_{\omega 1}$ and ${}^1J_{\omega 2} = {}^0J_{\omega 2}$, we have

$$({}^1J_{\omega 1}^T\,{}^1I_1\,{}^1J_{\omega 1}) = \begin{bmatrix} I_{zz1} & 0 \\ 0 & 0 \end{bmatrix}; \quad ({}^2J_{\omega 2}^T\,{}^2I_2\,{}^2J_{\omega 2}) = \begin{bmatrix} I_{zz2} & I_{zz2} \\ I_{zz2} & I_{zz2} \end{bmatrix}.$$

Finally, the matrix M is

$$M = \begin{bmatrix} m_1l_1^2 + I_{zz1} + m_2(l_1^2 + l_2^2 + 2l_1l_2C2) + I_{zz2} & m_2(l_2^2 + l_1l_2C2) + I_{zz2} \\ m_2(l_2^2 + l_1l_2C2) + I_{zz2} & l_2^2m_2 + I_{zz2} \end{bmatrix}.$$

Centrifugal and Coriolis Vector v The Christoffel Symbols are defined as

$$b_{i,jk} = \frac{1}{2}(m_{ijk} + m_{ikj} - m_{jki}); \text{ where } m_{ijk} = \frac{\partial m_{ij}}{\partial q_k}; \text{ with } b_{iii} = b_{iji} = 0.$$

<u>**Matrix B**</u>

$$B = \begin{bmatrix} 2b_{112} \\ 0 \end{bmatrix}; \quad b_{112} = \frac{1}{2}m_{112};$$

$$B = \begin{bmatrix} -2l_1l_2m_2S2 \\ 0 \end{bmatrix}.$$

Matrix C

$$C = \begin{bmatrix} 0 & b_{122} \\ b_{211} & 0 \end{bmatrix}; \; b_{211} = \frac{1}{2}(-m_{112}); \quad \text{and} b_{122} = m_{122};$$

$$C = \begin{bmatrix} 0 & -l_1 l_2 m_2 S2 \\ l_1 l_2 m_2 S2 & 0 \end{bmatrix}.$$

Vector v

$$V = \begin{bmatrix} -2l_1 l_2 m_2 S2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{\theta}_2] + \begin{bmatrix} 0 & -l_1 l_2 m_2 S2 \\ l_1 l_2 m_2 S2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix}.$$

The Gravity Vector g

$$\mathbf{G} = -[J_{v1}^T m_1 \mathbf{g} + J_{v2}^T m_2 \mathbf{g}].$$

In frame $\{0\}$, the gravity vector is

$${}^0G = -\begin{bmatrix} -l_1 S1 & l_1 C1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_1 g \\ 0 \end{bmatrix} - \begin{bmatrix} -l_1 S1 - l_2 S12 & l_1 C1 + l_2 C12 & 0 \\ -l_2 S12 & l_2 C12 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -m_2 g \\ 0 \end{bmatrix};$$

and

$${}^0G = \begin{bmatrix} [(m_1 + m_2) l_1 C1 + m_2 l_2 C12] g \\ m_2 l_2 C12 g \end{bmatrix}.$$

Equations of Motion

$$\begin{bmatrix} m_1 l_1^2 + I_{zz1} + m_2(l_1^2 + l_2^2 + 2l_1 l_2 C2) + I_{zz2} & m_2(l_2^2 + l_1 l_2 C2) + I_{zz2} \\ m_2(l_2^2 + l_1 l_2 C2) + I_{zz2} & l_2^2 m_2 + I_{zz2} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -2l_1 l_2 m_2 S2 \\ 0 \end{bmatrix} [\dot{\theta}_1 \dot{\theta}_2] + \begin{bmatrix} 0 & -l_1 l_2 m_2 S2 \\ l_1 l_2 m_2 S2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} +$$

$$\begin{bmatrix} [(m_1 + m_2) l_1 C1 + m_2 l_2 C12] g \\ m_2 l_2 C12 g \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$