Chapter 5
Force Analysis

Static Force Analysis

► *Introduction*

A machine is a device that performs work and, as such, transmits energy by means mechanical force from a power source to a driven load. It is necessary in the design machine mechanisms to know the manner in which forces are transmitted from input to the output, so that the components of the machine can be properly size withstand the stresses that are developed. If the members are not designed to strong enough, then failure will occur during machine operation; if, on the other hand, the machine is over designed to have much more strength than required, then the machine may not be competitive with others in terms of cost, weight, size, power requirements, or other criteria. The bucket load and static weight loads may far exceed any dynamic loads due to accelerating masses, and a static-force analysis would be justified. An analysis that includes inertia effects is called a dynamic-force analysis and will be discussed in the next chapter. An example of an application where a dynamic-force analysis would be required is in the design of an automatic sewing machine, where, due to high operating speeds, the inertia forces may be greater than the external loads on the machine.

Another assumption deals with the rigidity of the machine components. No material is truly rigid, and all materials will experience significant deformation if the forces, either external or inertial in nature, are great enough. It will be assumed in this chapter and the next that deformations are so small as to be negligible and, therefore, the members will be treated as though they are rigid. The subject of mechanical vibrations, which is beyond the scope of this book, considers the flexibility of machine components and the resulting effects on machine behavior. A third major assumption that is often made is that friction effects are negligible. Friction is inherent in all devices, and its degree is dependent upon many factors, including types of bearings, lubrication, loads, environmental conditions, and so on. Friction will be neglected in the first few sections of this chapter, with an introduction to the subject presented. In addition to assumptions of the types discussed above, other assumptions may be necessary, and some of these will be addressed at various points throughout the chapter.

The first part of this chapter is a review of general force analysis principles and will also establish some of the convention and terminology to be used in succeeding sections. The remainder of the chapter will then present both graphical and analytical methods for static-force analysis of machines.

► *5.1.1 Free-Body Diagrams:*

Engineering experience has demonstrated the importance and usefulness of free-body diagrams in force analysis. A free-body diagram is a sketch or drawing of part or all of a system, isolated in order to determine the nature of forces acting on that body. Sometimes a free-body diagram may take the form of a mental picture; however, actual sketches are strongly recommended, especially for complex mechanical systems.

Generally, the first, and one of the most important, steps in a successful force analysis is the identification of the free bodies to be used. Figures 5.1B through 5.1E show examples of various free bodies that might be considered in the analysis of the four-bar linkage shown in Figure 5.1A. In Figure 5.1B, the free body consists of the three moving members isolated from the frame; here, the forces acting on the free body include a driving force or torque, external loads, and the forces transmitted:

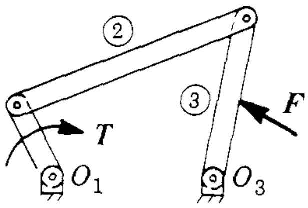


Figure 5.1(A) A four-bar linkage.

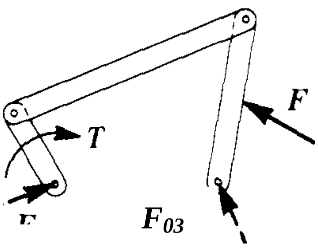


Figure 5.1(B) Free-body diagram of the three moving links

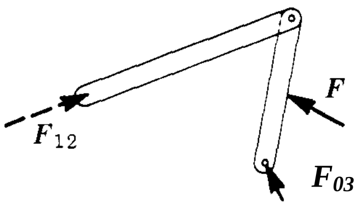


Figure 5.1(C) Free-body diagram of two connected links

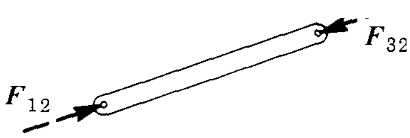


Figure 5.1(D) Free-body diagram of a single link

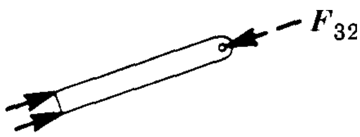


Figure 5.1(E) Free body diagram of part of a link.

► 5.1.2 Static Equilibrium:

For a free body in static equilibrium, the vector sum of all forces acting on the body must be zero and the vector sum of all moments about any arbitrary point must also be zero. These conditions can be expressed mathematically as follows:

$$\sum F = 0 \qquad (5.1A)$$

$$\sum T = 0 \qquad (5.1B)$$

Since each of these vector equations represents three scalar equations, there are a total of six independent scalar conditions that must be satisfied for the general case of equilibrium under three-dimensional loading.

There are many situations where the loading is essentially planar; in which case, forces can be described by two-dimensional vectors. If the *xy* plane designates the plane of loading, then the applicable form of Eqs. 5.1A and 5.1B is:-

$$\sum F_x = 0 \qquad (5.2A)$$

$$\sum F_y = 0 \qquad (5.2B)$$

$$\sum T_z = 0 \qquad (5.2C)$$

Eqs. 5.2A to 5.2C are three scalar equations that state that, for the case of two-dimensional *xy* loading, the summations of forces in the *x* and *y* directions must individually equal zero and the summation of moments about any arbitrary point in the plane must also equal zero. The remainder of this chapter deals with two-dimensional force analysis. A common example of three-dimensional forces is gear forces.

► 5.1.3 Superposition:

The principle of superposition of forces is an extremely useful concept, particularly in graphical force analysis. Basically, the principle states that, for linear systems, the net effect of multiple loads on a system is equal to the superposition (i.e., vector summation) of the effects of the individual loads considered one at a time. Physically, linearity refers to a direct proportionality between input force and output force. Its mathematical characteristics will be discussed in the section on analytical force analysis. Generally, in the absence of Coulomb or dry friction, most mechanisms are linear for force analysis purposes, despite the fact that many of these mechanisms exhibit very nonlinear motions. Examples and further discussion in later sections will demonstrate the application of this principle

► 5.1.4 Graphical Force Analysis:

Graphical force analysis employs scaled free-body diagrams and vector graphics in the determination of unknown machine forces. The graphical approach is best suited for planar force systems. Since forces are normally not constant during machine motion. analyses may be required for a number of mechanism positions; however, in many cases, critical maximum-force positions can be identified and graphical analyses performed for these positions only. An important advantage of the graphical approach is that it provides useful insight as to the nature of the forces in the physical system.

This approach suffers from disadvantages related to accuracy and time. As is true of any graphical procedure, the results are susceptible to drawing and measurement errors. Further, a great amount of graphics time and effort can be expended in the iterative design of a machine mechanism for which fairly thorough knowledge of force-time relationships is required. In recent years, the physical insight of the graphics approach and the speed and accuracy inherent in the computer-based analytical approach have been brought together through computer graphics systems, which have proven to be very effective engineering design tools. There are a few special types of member loadings that are repeatedly encountered in the force analysis of mechanisms, These include a member subjected to two forces, a member subjected to three forces, and a member subjected to two forces and a couple. These special cases will be considered in the following paragraphs, before proceeding to the graphical analysis of complete mechanisms.

► *5.2.1 Analysis of a Two-Force Member:*

A *member subjected to two forces is in equilibrium if and only if the two forces (1) have the same magnitude, (2) act along the same line, and (3) are opposite in sense*. Figure 5.2A shows a free-body diagram of a member acted upon by forces F_1 and F_2 where the points of application of these forces are points A and B. For equilibrium the directions of F_1 and F_2 must be along line *AB* and F_1 must equal $-F_2$ graphical vector addition of forces F_1 and F_2 is shown in Figure 5.2B, and, obviously, the resultant net force on the member is zero when $F_1 = -F_2$. The resultant moment about any point will also be zero.

Thus, if the load application points for a two-force member are known, the line of action of the forces is defined, and it the magnitude and sense of one of the forces are known, then the other force can immediately be determined. Such a member will either be in tension or compression.

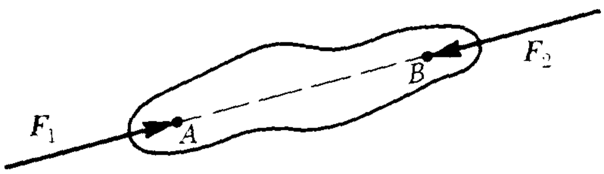


Figure 5.2(A) A two-force member. The resultant force and the resultant moment both equal Zero.

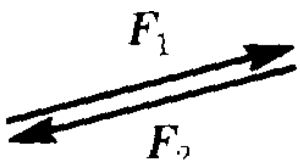


Figure 5.2(B) Force summation for a two-force member

► *5.2.2 Analysis of a Three-Force Member:*

A *member subjected to three forces is in equilibrium if and only if (1) the resultant of the three forces is zero, and (2) the lines of action of the forces all intersect at the same point*. The first condition guarantees equilibrium of forces, while the second condition guarantees equilibrium of moments. The second condition can be under-stood by considering the case when it is not satisfied. See Figure 5.3A. If moments are summed about point *P*, the intersection of forces F_1 and F_2, then the moments of these forces will be zero, but F_3 will produce a nonzero moment, resulting in a nonzero net moment on the member. On the other hand, if the line of action of force F_3 also passes through point *P* (Figure 5.3B), the net moment will be zero. This common point of intersection of the three forces is called the point of concurrency.

A typical situation encountered is that when one of the forces, F_1, is known completely, magnitude and direction, a second force, F_2, has known direction but unknown magnitude, and force F_3 has unknown magnitude and direction. The graphical solution of this case is depicted in Figures 5.4A through 5.4C. First, the free-body diagram is drawn to a convenient scale and the points of application of the three forces are identified. These are points *A, B,* and *C.* Next, the known force F_1 is drawn on the diagram with the proper direction and a suitable magnitude scale. The direction of force F_2 is then drawn, and the intersection of this line with an extension of the line of action of force F_1 is the concurrency point *P*. For equilibrium, the line of action of force F_3 must pass through points *C* and *P* and is therefore as shown in Figure 5.4A.

The force equilibrium condition states that

$$F_1 + F_2 + F_3 = 0$$

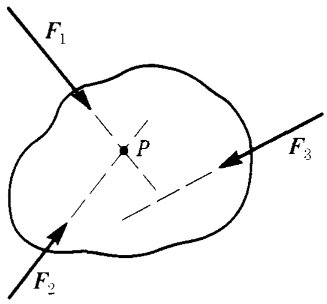


Figure 5.3(B) The three forces intersect at the same point *P*, called the *concurrency point,* and the net moment is zero.

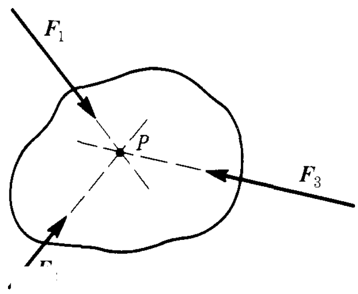


Figure 5.3(A) The three forces on the member do not intersect at a common point and there is a nonzero resultant moment.

Since the directions of all three forces are now known and the magnitude of F_1 were given, this equation can be solved for the remaining two magnitudes. A graphical

Solution follows from the fact that the three forces must form a closed vector loop, called a force polygon. The procedure is shown in Figure 5.4B. Vector F_1 is redrawn. From the head of this vector, a line is drawn in the direction of force F_2, and from the tail, a line is drawn parallel to F_3. The intersection of these lines closes the vector loop and determines the magnitudes of forces F_2 and F_3. Note that the same solution is obtained if, instead, a line parallel to F_3 is drawn from the head of F_1, and a line parallel to F_2 is drawn from the tail of F_1. See Figure 5.4C.

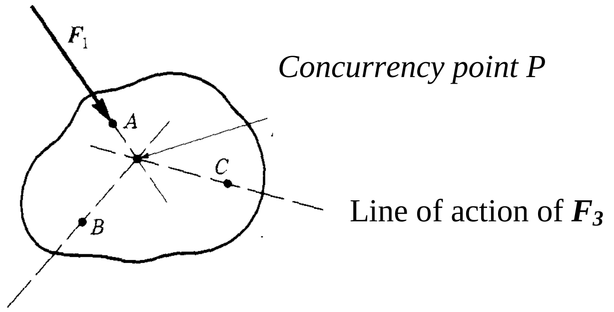


Figure 5.4(A) Graphical force analysis of a three- force member.

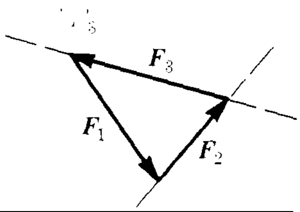


Figure 5.4(B) Force polygon for the three forces member.

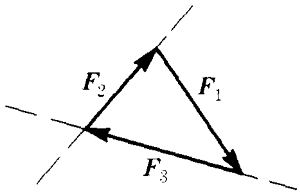


Figure 5.4(C) An equivalent force polygon for the three force member

This is so because vector addition is commutative, and, therefore, both force polygons are equivalent to the vector equation above. It is important to remember that, by the definition of vector addition, the force polygon corresponding to the general force equation

$$\sum F = 0$$

Will have adjacent vectors connected head to tail. This principle is used in identifying the sense of forces F_2 and F_3 in Figures 5.4B and 5.4C. Also, if the lines of action of F_1 and F_2 are parallel," then the point of concurrency is at infinity, and the third force F_3 must be parallel to the other two. In this case, the force polygon collapses to a straight line.

► *5.3.1 Graphical Force Analysis of the Slider Crank Mechanism:*

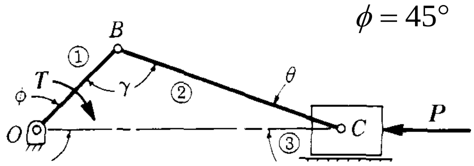
The slider crank mechanism finds extensive application in reciprocating compressors, piston engines, presses, toggle devices, and other machines where force characteristics are important. The force analysis of this mechanism employs most of the principles described in previous sections, as demonstrated by the following example.

▼ EXAMPLE 5.1

Static-force analysis of a slider crank mechanism is discussed. Consider the slider crank linkage shown in Figure 5.5A, representing a compressor, which is operating at so low a speed that inertia effects are negligible. It is also assumed that gravity forces are small compared with other forces and that all forces lie in the same plane. The dimensions are *OB* = 30 mm and *BC* == 70 mm, we wish to find the required crankshaft torque T and the bearing forces for a total gas pressure force *P* = 40N at the instant when the crank angle $\phi = 45°$.

OB = 30 mm
BC = 70 mm
$\phi = 45°$

Figure 5.5(A) Graphical force analysis of a slider crank mechanism, which is acted on by piston force ***P*** and crank torque ***T***



SOLUTION

The graphical analysis is shown in Figure 5.5B. First, consider connecting rod 2. In the absence of gravity and inertia forces, this link is acted on by two forces only, at pins *B* and C. These pins are assumed to be frictionless and, therefore, transmit no torque. Thus, link 2 is a two-force member loaded at each end as shown. The forces F_{12} and F_{32} lie along the link, producing zero net moment, and must be equal and opposite for equilibrium of the link. At this point, the magnitude and sense of these forces are unknown.

Next, examine piston 3, which is a three-force member. The pressure force *P is* completely known and is assumed to act through the center of the piston (i.e., the pressure distribution on the piston face is assumed to be symmetric). From Newton's third law, which states that for every action there is an equal and opposite reaction, it follows that $F_{23} = -F_{32}$, and the direction of F_{23} is therefore known. In the absence of friction, the force of the cylinder on the piston, F_{03}, is perpendicular to the cylinder wall, and it also must pass through the concurrency point, which is the piston pin *C*. Now, knowing the force directions, we can construct the force polygon for member 3 (Figure 5.5B). Scaling from this diagram, the contact force between the cylinder and piston is $F_{03} = 12.70N$ **,** acting upward, and the magnitude of the bearing force at *C* is $F_{23} = F_{32} = 42.0N$. This is also the bearing force at crankpin *B,* because $F_{12} = -F_{32}$. Further, the force directions for the connecting rod shown in the figure are correct, and the link is in compression.

Finally, crank 1 is subjected to two forces and a couple *T* (the shaft torque *T* is assumed to be a couple). The force at *B* is $F_{12} = -F_{21}$ and is now known. For force equilibrium, $F_{01} = -F_{21}$ as shown on the free-body diagram of link 1. However these forces are not collinear, and for equilibrium, the moment of this couple must be balanced by torque *T*. Thus, the required torque is clockwise and has magnitude

$$T = F_{21}h = (42.0N)(26.6mm) = 1120N.mm = 1.120N.m$$

It should be emphasized that this is the torque required for static equilibrium in the position shown in Figure 9.10A. If torque information is needed for a complete compression cycle, then the analysis must be repeated at other crank positions throughout the cycle. In general, the torque will vary with position.

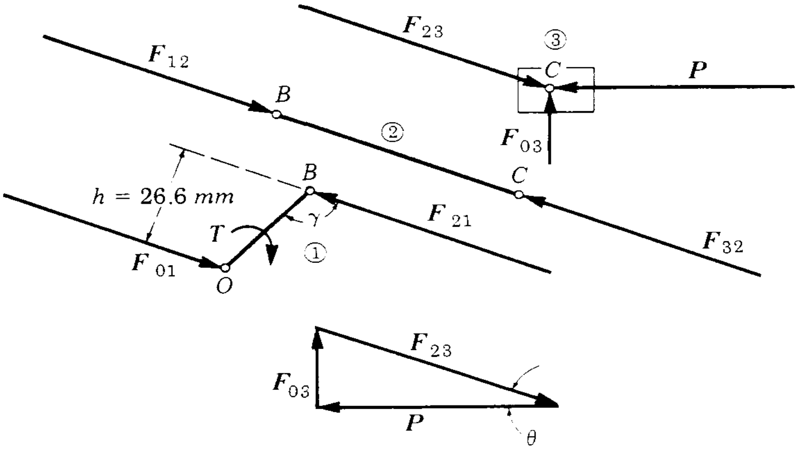


Figure 5.5(B) Static force balances for the three moving links, each considered as a free body

► *5.3.1 Graphical Force Analysis of the Four-Bar Linkage:*

The force analysis of the four-bar linkage proceeds in much the same manner as that of the slider crank mechanism. However, in the following example, we will consider the case of external forces on both the coupler and follower links and will utilize the principle of superposition.

▼ EXAMPLE 5.2

Static-force analysis of a four-bar linkage is considered. The link lengths for the four-bar linkage of Figure 5.6 A are given in the figure. In the position shown, coupler link 2 is subjected to force $\boldsymbol{F_2}$ of magnitude ***47 N,*** and follower link 3 is subjected to force $\boldsymbol{F_3}$, of magnitude ***30 N***. Determine the shaft torque ***Ti*** on input link1 and the bearing loads for static equilibrium.

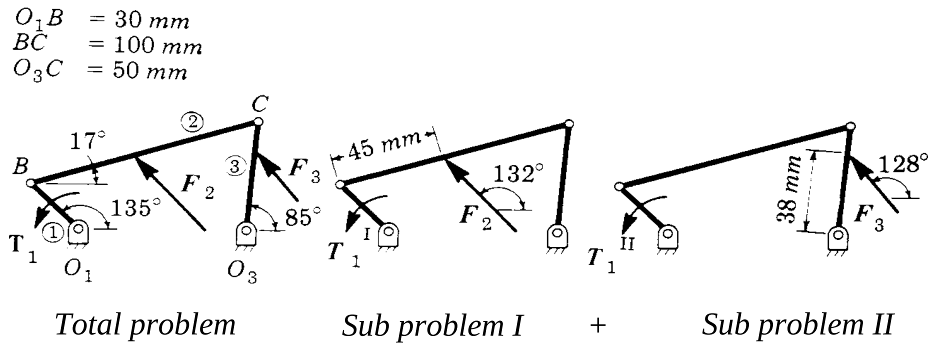


Figure 5.6(A) Graphical force analysis of a four-bar linkage, utilizing the principle of the superposition

SOLUTION

As shown in Figure 5.6A, the solution of the stated problem can be obtained by superposition of the solutions of sub problems *I* and *II*. In sub problem *I*, force $\boldsymbol{F_3}$ is neglected, and in sub problem *II*, force $\boldsymbol{F_2}$ is neglected. This process facilitates the solution by dividing a more difficult problem into two simpler ones.

The analysis of sub problem *I* is shown in Figure 5.6B, with quantities designated by superscript *I*. Here, member 3 is a two-force member because force $\boldsymbol{F_3}$ is neglected. The direction of forces F_{23}^{1} and F_{03}^{1} are as shown, and the forces are equal and opposite (note that the magnitude and sense of these forces are as yet unknown), This information allows the analysis of member 2, which is a three-force member with completely known force $\boldsymbol{F_2}$, known direction for F_{32}^{1}, and, using the concurrency point, known direction for F_{12}^{1}. Scaling from the force polygon, the following force magnitudes are determined (the force directions are shown in Figure (5.6B):

$$F_{32}^{1} = F_{23}^{1} = F_{03}^{1} = 21.0N \qquad F_{12}^{1} = F_{21}^{1} = 36N$$

Link 1 is subjected to two forces and couple T_1^{1}, and for equilibrium,

$$F_{03}^{11} = 29.0N \qquad F_{23}^{11} = F_{21}^{11} = F_{01}^{11}$$

And; $T_1^{1} = F_{21}^{1} h^{1} = (36N)(11mm) = 396N.mm \quad CW$

The analysis of sub problem *II* is very similar and is shown in Figure 5.6C, where superscript II is used. In this case, link 2 is a two-force member and link 3 is a three-force member, and the following results are obtained:

$$F_{03}^{11} = 29N \qquad F_{23}^{11} = F_{21}^{11} = F_{01}^{11} = 17N$$

And; $T_1^{11} = F_{21}^{11} h^{11} = (17N)(26mm) = 442N.mm \quad CW$

The superposition of the results of Figures 5.6B and 5.6C is shown in Figure 5.6D. The results must be added vectorially, as shown. By scaling from the free-body diagrams, the overall bearing force magnitudes are

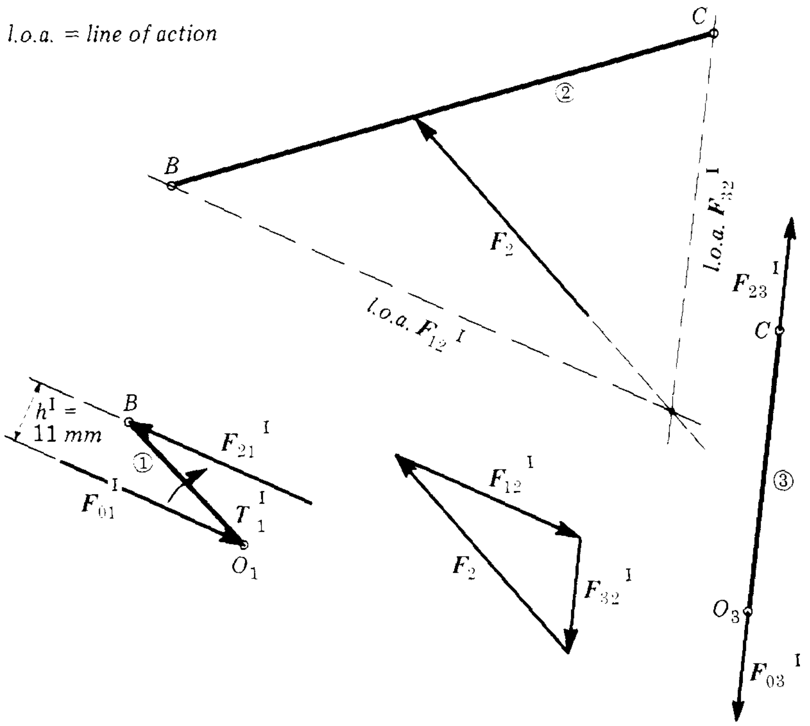


Figure 5.6B
The solution of *sub problem I*

$F_{01} = 50N$ $F_{23} = 31N$

$F_{12} = 50N$ $F_{03} = 49N$

And the net crankshaft torque is

$$T_1 = T_1^{1} + T_1^{11} = 396N.mm + 442N.mm = 838N.mm \quad CW$$

The directions of the bearing forces are as shown in the figure. These resultant quantities represent the actual forces experienced by the mechanism. It can be seen from the analysis that the effect of the superposition principle, in this example, was to create sub problems containing two-force members, from which the separate analyses could begin. In an attempt of a graphical analysis of the original problem without superposition, there is not enough intuitive force information to analyze three-force members 2 and 3, because none of the bearing force directions can be determined by inspection.

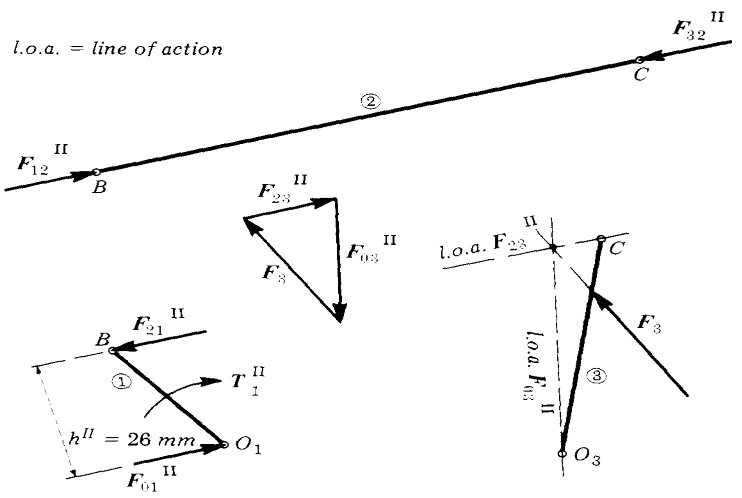


Figure 5.6C
The solution of *sub problem II*

► *PROBLEMS* ◄

Perform a graphical static-force analysis of the given mechanism. Construct the complete force polygon for determining bearing forces and the required input force or torque. Mechanism dimensions are given in the accompanying figures.

1- The applied piston load ***P*** on the offset slider crank mechanism of Figure1 remains constant as angle ϕ varies and has a magnitude of ***100 Ib***. Determine the required input torque T_1 for static equilibrium at the following crank positions:

a. $\phi = 45°$

b. $\phi = 135°$

c. $\phi = 270°$

d. $\phi = 315°$

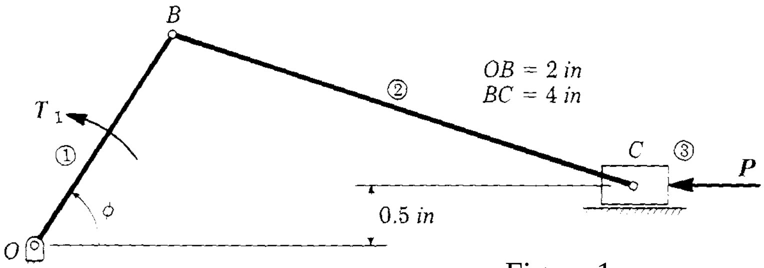


Figure 1

2- Determine the required input torque Ti for static equilibrium of the mechanism shown in Figure2. Forces F_2 and F_3, have magnitudes of ***20 Ib*** and ***10 Ib.*** respectively. Force F_a acts in the horizontal direction.

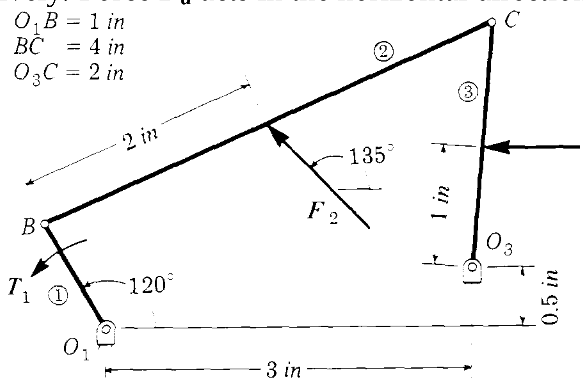


Figure 2

3- Determine the required input torque T_1 for static equilibrium of the mechanism shown in Figure3. Torques T_2 and T_3 are pure torques, having magnitudes of ***10N.m*** • m and ***7 N.m***, respectively.

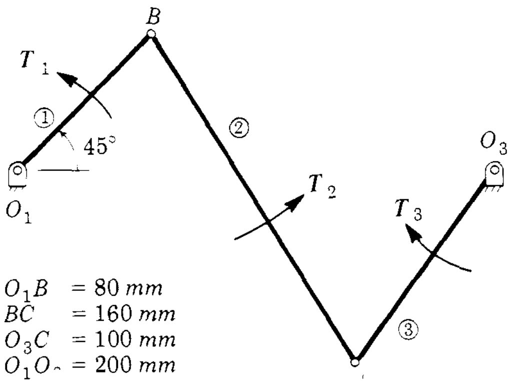


Figure 3

Dynamic Force Analysis

► *5.4.1 D'Alembert's Principle and Inertia Forces:*

An important principle, known as d'Alembert's principle, can be derived from Newton's second law. In words, d'Alembert's principle states that *the reverse-effective forces and torques and the external forces and torques on a body together give statical equilibrium*.

$$F + (-ma_G) = 0 \quad (5.3A)$$

$$T_{eG} + (-I_G \alpha) = 0 \quad (5.3B)$$

The terms in parentheses in Eqs. 5.3A and 5.3B are called the reverse-effective force and the reverse-effective torque, respectively. These quantities are also referred to as inertia force and inertia torque. Thus, we define the inertia force ***F***, as

$$\boldsymbol{F_i = -ma_G} \quad (5.4A)$$

This reflects the fact that a body resists any change in its velocity by an inertia force proportional to the mass of the body and its acceleration. The inertia force acts through the center of mass ***G*** of the body. The inertia torque or inertia couple ***C***, is given by:

$$C_i = -I_G \alpha \quad (5.4B)$$

As indicated, the inertia torque is a pure torque or couple. From Eqs. 5.4A and 5.4B, their directions are opposite to that of the accelerations. Substitution of Eqs. 5.4A and 5.4B into Eqs, 5.3A and 5.3B leads to equations that are similar to those used for static-force analysis:

$$\sum F = \sum F_e + F_i = 0 \quad (5.5A)$$

$$\sum T_G = \sum T_{eG} + C_i = 0 \quad (5.5B)$$

Where $\sum F$ refers here to the summation of external forces and, therefore, is the resultant external force, and $\sum T_{eG}$ is the summation of external moments, or resultant external moment, about the center of mass ***G***. Thus, the dynamic analysis problem is reduced in form to a static force and moment balance where inertia effects are treated in the same manner as external forces and torques. In particular for the case of assumed mechanism motion, the inertia forces and couples can be determined completely and thereafter treated as known mechanism loads.

Furthermore, d'Alembert's principle facilitates moment summation about any arbitrary point *P* in the body, if we remember that the moment due to inertia force ***F*,** must be included in the summation. Hence,

$$\sum T_P = \sum T_{eP} + C_i + R_{PG} \times F_t = 0 \quad (5.5C)$$

Where; $\sum T_P$ is the summation of moments, including inertia moments, about point ***P***. $\sum T_{eP}$ is the summation of external moments about ***P*, *C***, is the inertia couple defined by Eq. 5.4B, ***F*,** is the inertia force defined by Eq. 5.4A, and $\boldsymbol{R_{PG}}$ is a vector from point P to point ***C***. It is clear that Eq. 5.5B is the special case of Eq.5.5C, where point ***P*** is taken as the center of mass G (i.e., $\boldsymbol{R_{PG}} = 0$).

For a body in plane motion in the ***xy*** plane with all external forces in that plane. Eqs. 5.5A and 5.5B become:

$$\sum F_x = \sum F_{ex} + F_{ix} = \sum F_{ex} + (-ma_{Gx}) = 0 \quad (5.6A)$$

$$\sum F_y = \sum F_{ey} + F_{iy} = \sum F_{ey} + (-ma_{Gy}) = 0 \quad (5.6B)$$

$$\sum T_G = \sum T_{eG} + C_i = \sum T_{eG} + (-I_G \alpha) = 0 \quad (5.6C)$$

Where $\boldsymbol{a_{Gx}}$ and $\boldsymbol{a_{Gy}}$ are the ***x*** and ***y*** components of $\boldsymbol{a_G}$. These are three scalar equations, where the sign convention for torques and angular accelerations is based on a right-hand ***xyz*** coordinate system; that is. Counterclockwise is positive and clockwise is negative. The general moment summation about arbitrary point ***P***, Eq. 5.5C, becomes:

$$\begin{aligned}\sum T_P &= \sum T_{eP} + C_i + R_{PGx}.F_{iy} - R_{PGy}.F_{ix} \\ &= \sum T_{eP} + (-I_G \alpha) + R_{PGx}(-ma_{Gy}) - R_{PGy}(-ma_{Gx}) = 0\end{aligned} \quad (5.6D)$$

Where $\boldsymbol{R_{PGx}}$ and $\boldsymbol{R_{PGy}}$ are the ***x*** and ***y*** components of position vector $\boldsymbol{R_{PG}}$. This expression for dynamic moment equilibrium will be useful in the analyses to be presented in the following sections of this chapter.

► *5.4.2 Equivalent Offset Inertia Force:*

For purposes of graphical plane force analysis, it is convenient to define what is known as the equivalent offset inertia force. This is a single force that accounts for both translational inertia and rotational inertia corresponding to the plane motion of a rigid body. Its derivation will follow, with reference to Figures 5.7A through 5.7D.

Figure 5.7A shows a rigid body with planar motion represented by center of mass acceleration $\boldsymbol{a_C}$ and angular acceleration α. The inertia force and inertia torque associated with this motion are also shown. The inertia torque $-I_G \alpha$ can be expressed as a couple consisting of forces ***Q*** and (- ***Q***) separated by perpendicular

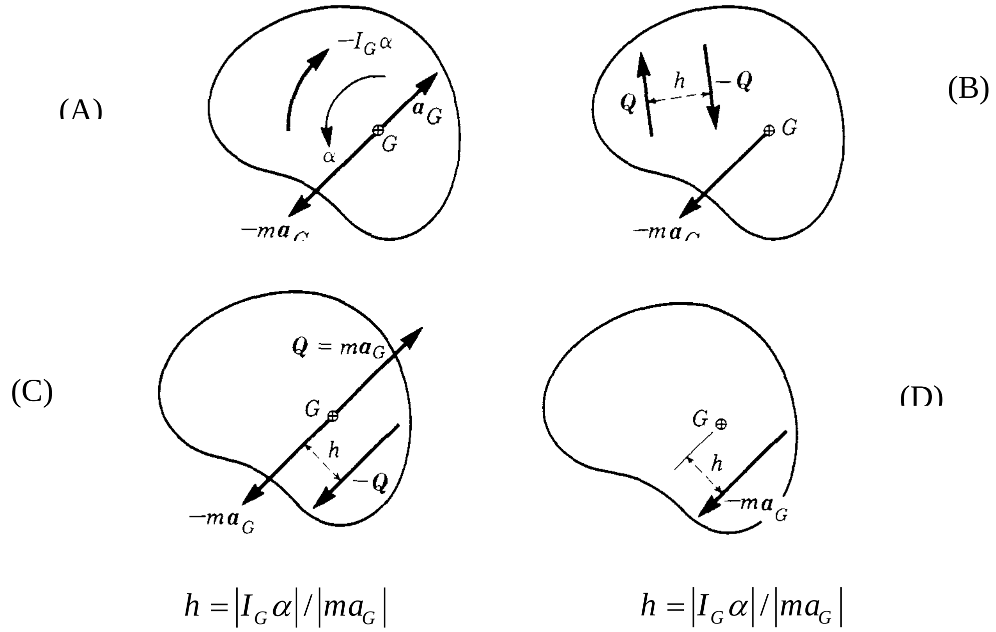


Figure 5.7 (A) Derivation of the equivalent offset inertia force associated with planer motion of a rigid body. (B) Replacement of the inertia torque by a couple. (C) The strategic choice of a couple. (D) The single force is equivalent to the combination of a force and a torque in figure 5.7(A)

Distance ***h***, as shown in Figure 5.7B. The necessary conditions for the couple to be equivalent to the inertia torque are that the sense and magnitude be the same. Therefore, in this case, the sense of the couple must be clockwise and the magnitudes of ***Q*** and ***h*** must satisfy the relationship

$$|\ Q.h| = |\ I_G.\alpha|$$

Otherwise, the couple is arbitrary and there are an infinite number of possibilities that will work. Furthermore, the couple can be placed anywhere in the plane.

Figure 5.7C shows a special case of the couple, where force vector ***Q*** is equal to ma_G and acts through the center of mass. Force ***(- Q)*** must then be placed as shown to produce a clockwise sense and at a distance;

$$h = \frac{|\ I_G\alpha|}{|\ Q|} = \frac{|\ I_G\alpha|}{|ma_G|} \qquad (5.7)$$

Force ***Q*** will cancel with the inertia force $F_i = -ma_G$, leaving the single equivalent offset force shown in Figure 5.7D, which has the following characteristics:

1. The magnitude of the force is $|\ ma_G\ |$.
2. The direction of the force is opposite to that of acceleration α .
3. The perpendicular offset distance from the center of mass to the line of action of the force is given by Eq. 5.7.
4. The force is offset from the center of mass so as to produce a moment about the center of mass that is opposite in sense to acceleration *a*.

The usefulness of this approach for graphical force analysis will be demonstrated in the following section. It should be emphasized, however, that this approach is usually unnecessary in analytical solutions, where Eqs. 5.6A to 5.6D. Including the original inertia force and inertia torque, can be applied directly.

► *5.4.3 Dynamic Analysis of the Four-Bar Linkage:*

The analysis of a four-bar linkage will effectively illustrate most of the ideas that have been presented; furthermore, the extension to other mechanism types should become clear from the analysis of this mechanism.

▼ EXAMPLE 5.3

The four-bar linkage shown in Figure 5.8A has the dimensions shown in the figure where ***G*** refers to center of mass, and the mechanism has the following mass properties:

$$m_1 = 0.10kg \qquad I_{G1} = 20kg.mm^2$$

$$m_2 = 0.20kg \qquad I_{G2} = 400kg.mm^2$$

$$m_3 = 0.30kg \qquad I_{G3} = 20kg.mm^2$$

Determine the instantaneous value of drive torque ***T*** required to produce an assumed motion given by input angular velocity $\omega = 95rad/s$ counterclockwise and input angular acceleration $a_1 = 0$ for the position shown in the figure. Neglect gravity and friction effects.

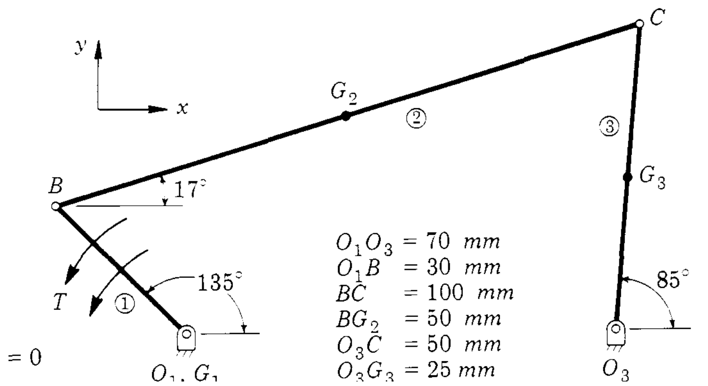


Figure 5.8(A) The four-bar linkage of Example 5.3

SOLUTION

This problem falls in the first analysis category that is given the mechanism motion, determine the resulting bearing forces and the necessary input torque. Therefore, the first step in the solution process is to determine the inertia forces and inertia torques. Thereafter, the problem can be treated as though it were a static-force analysis problem.

Kinematics analysis of the mechanism can be accomplished by using any of the methods presented in earlier chapters. Figure 5.8B shows a graphical analysis employing velocity and acceleration polygons. From the analysis, the following accelerations are determined:

$$a_{C1} = 0(Stationary\ Center\ of\ mass) \qquad \alpha_1 = 0(given)$$
$$a_{C2} = 235{,}000\angle 312° mm/Sec^2 \qquad \alpha_2 = 520 rad/s^2 \quad ccw$$
$$a_{C3} = 235{,}000\angle 308° mm/Sec^2 \qquad \alpha_3 = 2740 rad/s^2 \quad cw$$

Where the angles of the acceleration vectors are measured counterclockwise from the positive *x* direction shown in Figure 5.8A. From Eqs. 5.4A and 5.4B, the inertia forces and inertia torques are;

$$F_{i1} = 0$$
$$F_{i2} = -m_2 a_{G2} = 47{,}000\angle 132° kg.mm/s^2 = 47\angle 132° N$$
$$F_{i3} = -m_3 a_{G3} = 30{,}000\angle 128° kg.mm/s^2 = 30\angle 132° N$$
$$C_{i1} = 0$$
$$C_{i2} = -I_{G2}\alpha_2 = 208{,}000 kg.mm^2/s^2 cw = 208 N.mm \quad cw$$
$$C_{i3} = -I_{G3}\alpha_3 = 274{,}000 kg.mm^2/s^2 ccw = 274 N.mm \quad ccw$$

The inertia forces have lines of action through the respective centers of mass, and the inertia torqueses are pure couples.

Velocity polygon

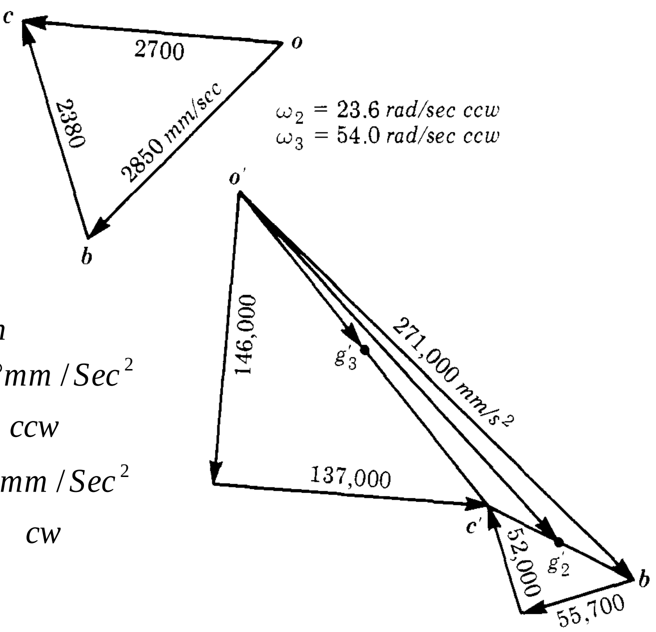


Figure 5.8(B) the velocity and acceleration analysis necessary for determination of inertia forces and inertia

Acceleration polygon

$a_{G2} = 235,000\angle 312° mm/Sec^2$

$\alpha_2 = 520 rad/Sec \quad ccw$

$a_{G3} = 100,000\angle 308° mm/Sec^2$

$\alpha = 2740 rad/Sec \quad cw$

GRAPHICAL SOLUTION

In order to simplify the graphical force analysis, we will account for the inertia torques by introducing equivalent offset inertia forces. These forces are shown in Figure 2.8C, and their placement is determined according to the previous section. For link 2, the offset force $\boldsymbol{F}_2$ is equal and parallel to inertia force $\boldsymbol{F}_{12}$.Therefore,

$$F_2 = 47\angle 132° N$$

It is offset from the center of mass $\boldsymbol{G}_2$ by a perpendicular amount equal to

$$h_2 = \frac{|\ I_{G2}\alpha_2|}{|\ m_2 a_{G2}|} = \frac{208}{47} = 4.43mm$$

And this offset is measured to the left as shown to produce the required clockwise direction for the inertia moment about point $\boldsymbol{G}_2$. In a similar manner, the equivalent offset inertia force for link 3 is

$$F_3 = 30\angle 128° N \quad \text{at an offset distance} \quad h_3 = \frac{|\ I_{G3}\alpha_3|}{|\ m_3 a_{G3}|} = \frac{274}{30} = 9.13mm$$

Where this offset is measured to the right from $\boldsymbol{G}_3$ to produce the necessary counterclockwise inertia moment about $\boldsymbol{G}_3$. From the values of $\boldsymbol{h}_2$ and $\boldsymbol{h}_3$ and the angular relationships, the force positions $\boldsymbol{r_2}$ and $\boldsymbol{r_3}$ in Figure 5.8C are computed to be

$$r_2 = BG_2 - \frac{h_2}{\cos(132° - 17° - 90°)} = 45.10mm$$

$$r_3 = O_3G_3 + \frac{h_3}{\cos(90° + 85° - 128°)} = 38.40mm$$

Now, we wish to perform a graphical force analysis for known forces $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$. This has been done in Example Problem 9.2, and the reader is referred to that

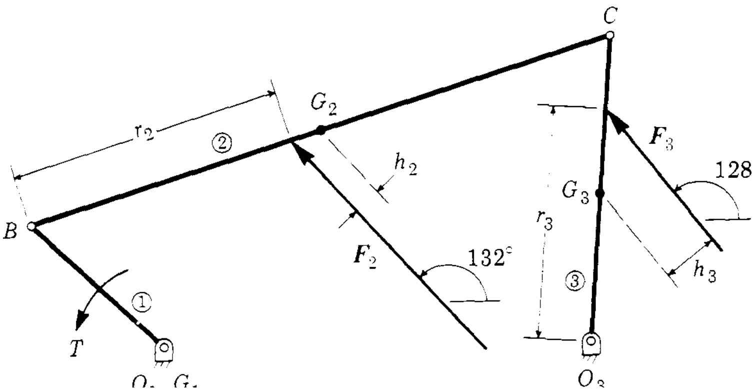


Figure 5.8(C) Equivalent offset inertia forces for members 2 and 3

Analysis. The required input torque was found to be ***T = 383N.mm cw***

ANALYTICAL SOLUTION

Having determined the equivalent offset inertia forces $\boldsymbol{F_2}$ and $\boldsymbol{F_3}$ the analytical solution could proceed according to Example Problem 9, 6, which examined the same problem. However, it is not necessary to convert to the offset force, and here we will carry out the analytical solution in terms of the original inertia forces and inertia couples.

Figure 5.8D shows the linkage with the inertia torques and the inertia forces in ***xy*** coordinate form. Consistent with Figure 9.15A, we define the following quantities:

$$\ell_1 = 30mm \qquad \ell_2 = 100mm \qquad \ell_3 = 50mm$$

$$\phi_1 = 135° \qquad \phi_2 = 17° \qquad \phi_3 = 85°$$

$$r_1 = 0 \qquad r_2 = 50mm \qquad r_3 = 25mm$$

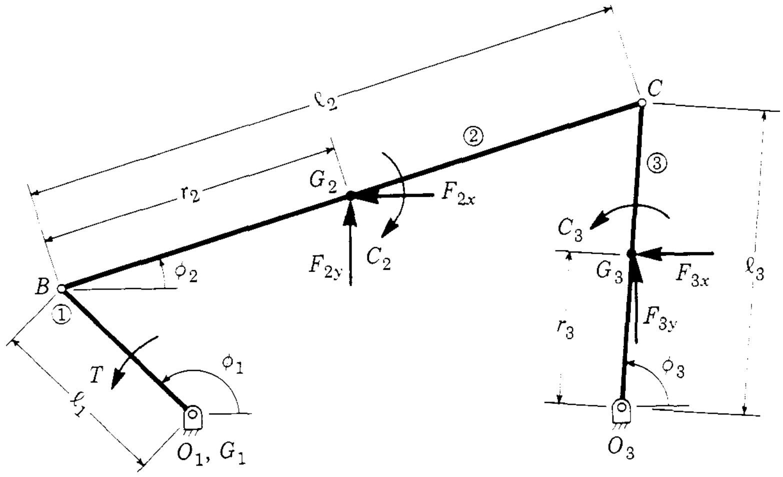
$$F_{2x} = 47\cos(132°) = -31.40N \qquad F_{2y} = 47\sin(132°) = 34.90N$$

$$F_{3x} = 30\cos(128°) = -18.50N \qquad F_{3y} = 30\sin(128°) = 23.60N$$

$$C_2 = -208N.mm \qquad C_3 = 274N.mm$$

$$F_{1x} = F_{1y} = C_1 = 0$$

Figure 5.8(D) Combinations of inertia forces and inertia torques for members 2 and 3



Where the differences are due to round off:

$a_{11} = -49.8 \qquad a_{21} = 29.2 \qquad b_1 = -786$

$a_{12} = 4.36 \qquad a_{22} - 95.6 \qquad b_2 = -1920$

Then, $F_{23} = 31.30N \qquad F_{12} = 50.30N$

$F_{03} = 49.20N \qquad F_{01} = 50.30N$

And $T = -851N.mm$

Thus, it can be seen that the general analytical solution of the four-bar linkage presented in this Chapter for static-force analysis is equally well suited for dynamic-force analysis. Before leaving this example, a couple of general comments should be made.

First, the torque determined is the instantaneous value required for the prescribed motion, and the value will vary with position. Furthermore, for the position considered, the torque is opposite in direction to the angular velocity of the crank. This can be explained by the fact that the inertia of the mechanism in this position is tending to accelerate the crank in the counterclockwise direction, and, therefore, the required torque must be clockwise to maintain a constant angular speed. If a constant speed is to be maintained throughout the mechanism cycle, then there will be other positions of the mechanism for which the required torque will be counterclockwise. The second comment is that it may be impossible to find a mechanism actuator, such as an electric motor, that will supply the required torque versus position behavior. This problem can be alleviated, however, in the case of a "constant" rotational speed mechanism through the use of a device called a flywheel, which is mounted on the input shaft and produces a relatively large mass moment of inertia for crank 1. The flywheel can absorb mechanism torque and energy- variations with minima] speed fluctuation and. thus, maintains an essentially constant input speed. In such a case. The assumed-motion approach to dynamic-force analysis is appropriate.

► *5.4.3 Dynamic Analysis of the Slider-Crank Mechanism:*

Dynamic forces are a very important consideration in the design of slider crank mechanisms for use in machines such as internal combustion engines and reciprocating compressors. Dynamic-force analysis of this mechanism can be carried out in exactly the same manner as for the four-bar linkage in the previous section. Following such a process a kinematics analysis is first performed from which expressions are developed for the inertia force and inertia torque for each of the moving members, These quantities may then be converted to equivalent offset inertia forces for graphical analysis or they may be retained in the form of forces and torques for analytical solution, utilizing, in either case, the methods presented in this chapter. In fact, the analysis of the slider crank mechanism is somewhat easier than that of the four-bar linkage because there is no rotational motion and, in turn, no inertia torque for the piston or slider, which has translating motion only. The following paragraphs will describe an analytical approach in detail.

Figure 5.9A is a schematic diagram of a slider crank mechanism, showing the crank 1, the connecting rod 2, and the piston 3, all of which are assumed to be rigid. The center of mass locations are designated by letter **G,** and the members have masses ***m***, and moments of inertia I_{Gi}, i = 1, 2, 3. The following analysis will consider the relationships of the inertia forces and torques to the bearing reactions and the drive torque on the crank, at an arbitrary mechanism position given by crank angle ϕ Friction will be neglected.

Figure 5.9B shows free-body diagrams of the three moving members of the linkage. Applying the dynamic equilibrium conditions. Eqs. 5.6A to 5.6D, to each member yields the following set of equations. For the piston (moment equation not included):

$$F_{23x} + (-m_3 a_{G3}) = 0 \qquad (5.8A)$$

$$F_{03y} + F_{23y} = 0 \qquad (5.8B)$$

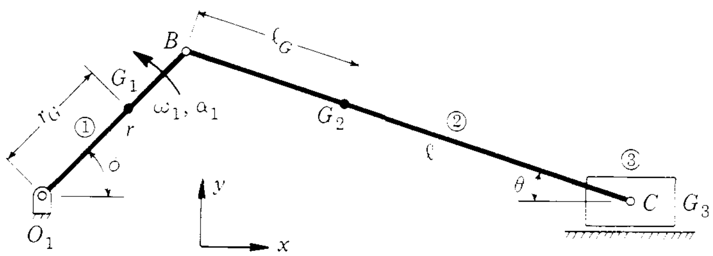


Figure 5.9(A) Dynamic-force analysis of a slider crank mechanism

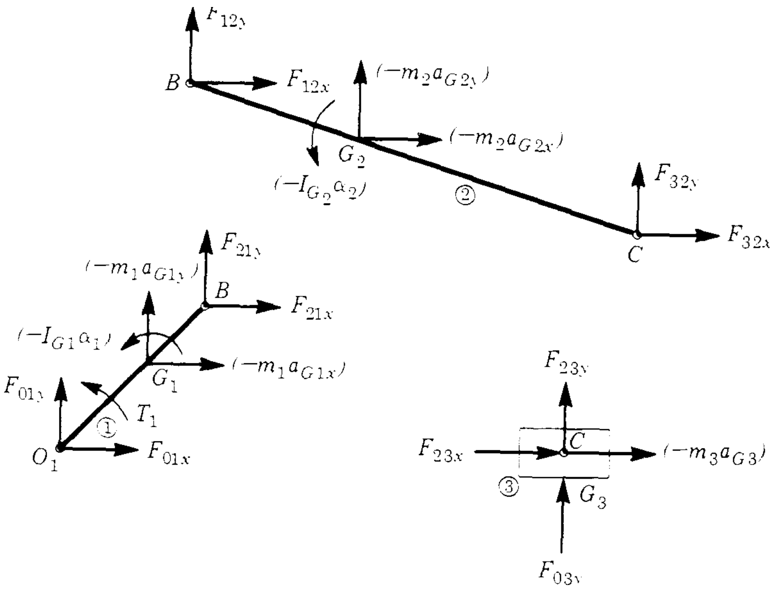


Figure 5.9(B) Free-body diagrams of the moving members

For the connecting rod (moments about point ***B***):

$$F_{12x} + F_{32x} + (-m_2a_{G2x}) = 0 \qquad (5.8C)$$

$$F_{12y} + F_{32y} + (-m_2a_{G2y}) = 0 \qquad (5.8D)$$

$$F_{32x}\ell\sin\theta + F_{32y}\ell\cos\theta + (-m_2a_{G2x})\ell_G\sin\theta + (-m_2a_{G2y})\ell_G\cos\theta + (-I_{G2}\alpha_2) = 0 \qquad (5.8E)$$

For the crank (moments about point $\boldsymbol{O_1}$**):**

$$F_{01x} + F_{21x} + (-m_1a_{G1x}) = 0 \qquad (5.8F)$$

$$F_{01y} + F_{21y} + (-m_1a_{G1y}) = 0 \qquad (5.8G)$$

$$T_1 - F_{21x}r\sin\phi + F_{21y}r\cos\phi + (-m_1a_{G1x})r_G\sin\phi + (-m_1a_{G1y})r_G\cos\phi + (-I_{G1}\alpha_1) = 0 \qquad (5.8H)$$

Where ***T*** is the input torque on the crank. This set of equations embodies both of the dynamic-force analysis approaches described in Newton's Laws. However, its form is best suited for the case of known mechanism motion, as illustrated by the following example.

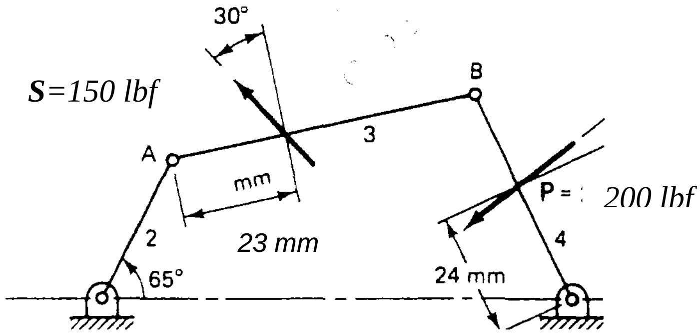
MEG 373 Kinematics and Dynamics of Machines
Force Analysis (Graphical Method)

Student Name: Student No.:

Question 1:

The four-bar mechanism of Figure has one external force ***P* = 200 Ibf** and one inertia force ***S* = 150 Ibf** acting on it. The system is in dynamic equilibrium as a result of torque $\boldsymbol{T_2}$ applied to link 2. *Find* $\boldsymbol{T_2}$ and *the pin forces*.

(a) Use the graphical method based on free-body diagrams.



O_2A = 30 mm
AB = 60 mm
O_4B = 45 mm
O_2O_4 = 90 mm

MEG 373 Kinematics and Dynamics of Machines
Force Analysis (Graphical Method)

Student Name: Student No.:

Question 2:

The input crank of the four-bar linkage of Figure rotates at a constant speed of ***w_2 = 500 rad/Sec (C.W)***. Each link has significant inertia. The velocity and acceleration diagrams are provided in the figure. Calculate *the values of all velocities and accelerations in these diagrams.*

Then;

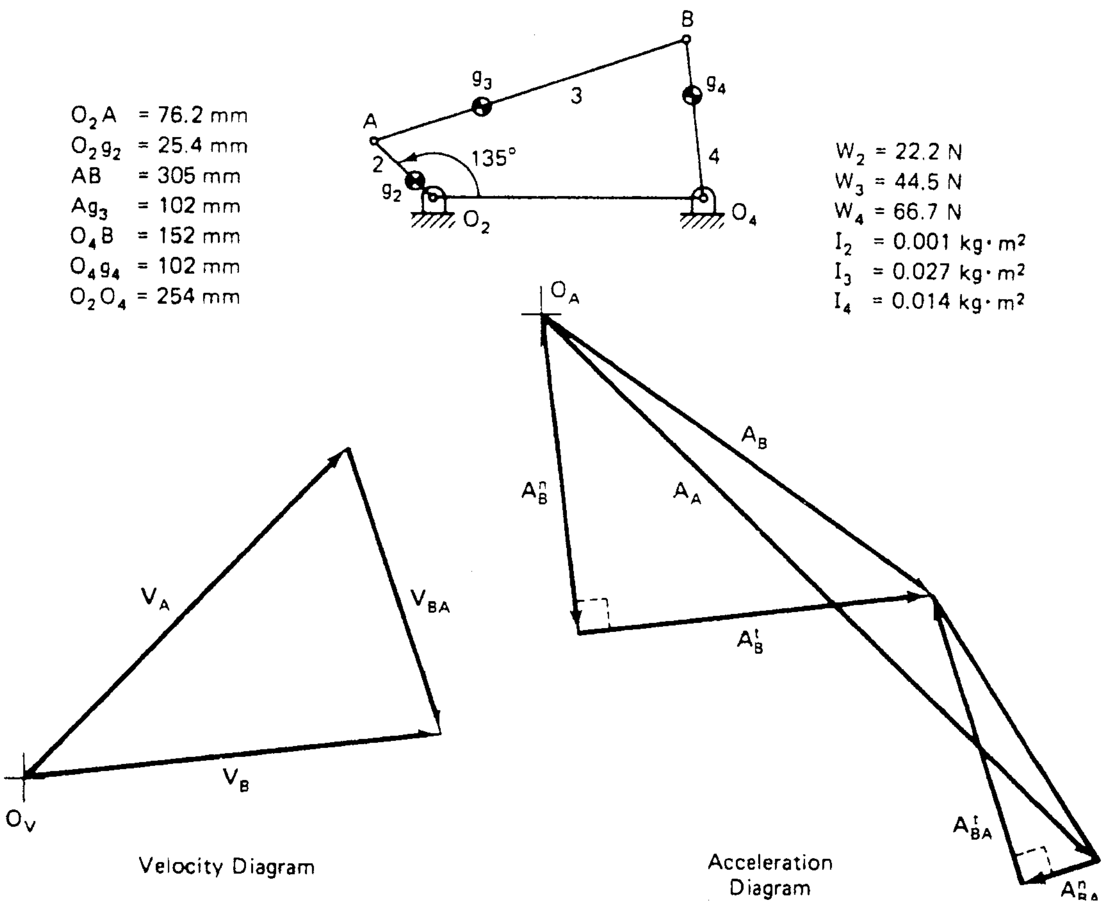
(a) Determine the linear accelerations of each center of gravity and angular accelerations α_2, α_3 and α_4.

(b) Find the inertia forces F_{02}, F_{03} and F_{04}.

(c) Find the offsets $\varepsilon_2, \varepsilon_3$ and ε_4 of the inertia forces.

(d) Sketch the inertia forces in their correct positions on the linkage.

(e) Find the directions and magnitudes of the pin forces at *A* and *B*.



MEG 373 Kinematics and Dynamics of Machines
Force Analysis (Graphical Method)

Student Name: Student No.:

Question 3:

The slider-crank mechanism of Figure is to be analyzed to determine the effect of the inertia of the connecting rod (link 3). The velocity diagram is shown in the figure and the magnitude of $\boldsymbol{V_A}$ is given. *Calculate the crank vector* $\mathbf{O_2A}$ and *the input angular velocity* w_2, and proceed to *calculate the values of all vectors in the velocity diagram*. Then;

(a) Determine the linear acceleration of the center of gravity of link 3 and the angular acceleration α_3.

(b) Find the inertia force $\boldsymbol{F_{03}}$ of the coupler link.

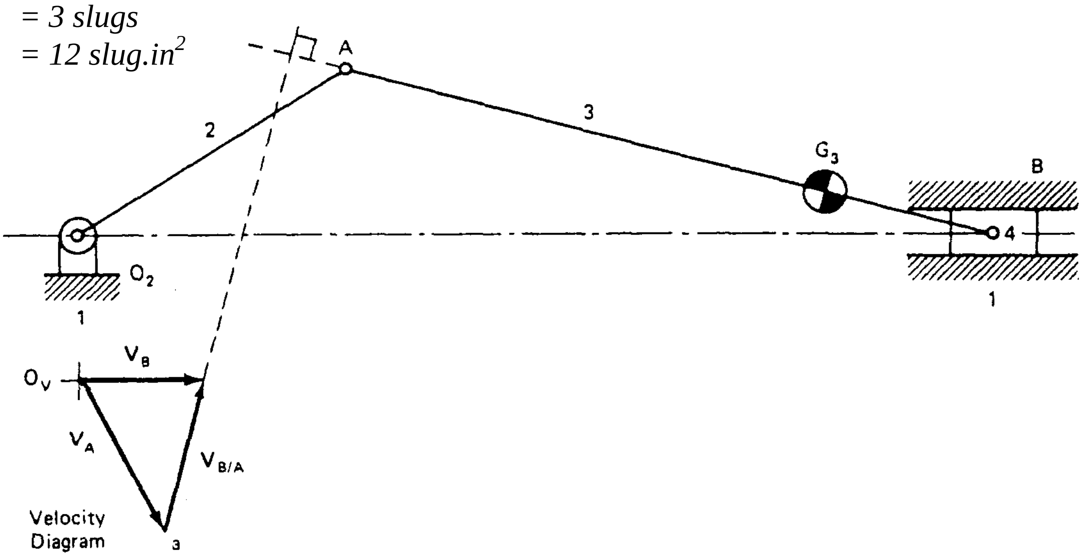
(c) Find the offset ε_3 of the inertia force $\boldsymbol{F_{03}}$.

(d) *Sketch* the inertia force in its correct position on the linkage.

(e) Find the directions and magnitudes of the pin forces at *A* and *B*.

(f) Determine the required input torque to drive this mechanism in this position under the conditions described in this problem.

$AB = 4$ *in*
$AG_3 = 3$ *in*
$O_2B = 5.5$ *in*
$M_3 = 3$ *slugs*
$I_3 = 12$ *slug.in*^{2}



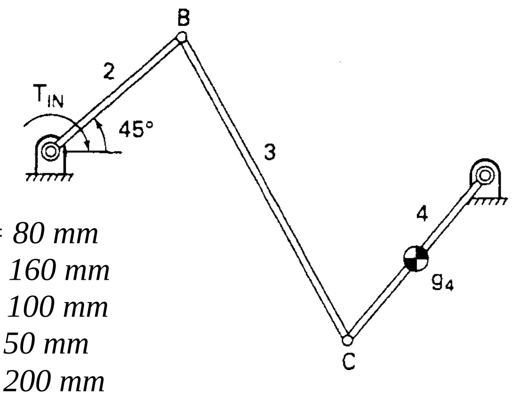
$\boldsymbol{V_A = 20}$ ***in/Sec***

MEG 373 Kinematics and Dynamics of Machines
Force Analysis (Graphical Method)

Student Name: Student No.:

Question 4:

(a) Find the magnitude A_{g4}.
(b) Find the angular accelerator α_4 .
(c) What is the magnitude of the inertia force F_{04}?
(d) What is the magnitude of the offset ε_4 ?
(e) *Draw* the vector F_{04} in the correct location on the mechanism.
(f) Given that the mechanism is driven by an input torque, T_{IN}, applied to link 2. *Determine* the following: magnitudes of all pin forces, and magnitude and direction of the input torque.



O_2B = 80 mm
BC = 160 mm
O_4C = 100 mm
O_4G_4 = 50 mm
O_2G_4 = 200 mm