A SERIES OF CLASS NOTES FOR 2005-2006 TO INTRODUCE LINEAR AND NONLINEAR
PROBLEMS TO ENGINEERS, SCIENTISTS, AND APPLIED MATHEMATICIANS

DE CLASS NOTES 1

A COLLECTION OF HANDOUTS ON
FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS (ODE's)

CHAPTER 5

Mathematical Modeling

Using First Order ODE's

1. Second Review of the Steps in Solving an Applied Math Problem

2. Applied Mathematics Problem #1: Radio Active Decay

3. Applied Mathematics Problem #2: Continuous Compounding

4. Applied Mathematics Problem #3: Mixing (Tank) Problems

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Handout #1 SECOND REVIEW OF THE STEPS IN SOLVING AN Professor Moseley
AN APPLIED MATH PROBLEM

The need to develop a mathematical model begins with specific questions that the solution of a mathematical model will answer. We review again the five basic steps used to solve any **applied math** or **application** problem. To answer specific questions in a particular application area we wish develop and solve a mathematical "find" problem which in this course will usually be an IVP that is well-posed in a set theoretic sense (i.e., has exactly one solution).

Step 1: UNDERSTAND THE CONCEPTS IN THE APPLICATION AREA. In order to answer specific questions, we wish to develop a mathematical model (or problem) whose solution will answer the specific questions of interest. Before we can build a mathematical model, we must first **understand** the concepts needed from the **application area** where answers to specific questions are desired. Solution of the model should provide answers to these questions. We start with a description of the phenomenon to be modeled, including the "laws" it must follow (e.g., that are imposed by nature, by an entrepreneurial environment or by the modeler). Recall that the need to answer questions about a ball being thrown up drove us to Newton's second law, F=MA.

Step 2: UNDERSTAND THE MATHEMATICAL CONCEPTS NEEDED. In order to develop and solve a mathematical model, we must first be sure we know the appropriate mathematics. For this course, you should have previously become reasonably proficient in **high school algebra** including how to solve **algebraic equations** and **calculus** including how to compute **derivatives** and **antiderivatives**. We are developing the required techniques and understanding of **differential equations**. Most of our models will be initial value problems. Additional required mathematics after first order ODE's (and solution of second order ODE's by first order techniques) is **linear algebra**. All of these must be mastered in order to understand the development and solution of mathematical models in science and engineering.

Step 3. DEVELOP THE MATHEMATICAL MODEL. The model must include those aspects of the application so that its solution will provide answers to the questions of interest. However, inclusion of too much complexity may make the model unsolvable and useless. To develop the mathematical model we use laws that must be followed, diagrams we have drawn to understand the process and notation and nomenclature we developed. Investigation of these laws results in a **mathematical model**. In this chapter our models are Initial Value Problems (IVP's) for a first order ODE that is a rate equation (dynamical system). This is indeed a "find" problem. Since the process evolves in time, we choose t as our independent variable and start it at $t = 0$. For our one state variable problem, we use y and hence use the general first order ODE with an initial condition as our model. For specific applications, finding $f(t,y)$ is a major part of the modeling process.

MATHEMATICAL MODEL: In mathematical language the general nonlinear model may be written as:

$$\text{IVP} \quad \begin{cases} \text{ODE} & \dfrac{dy}{dt} = f(t,y) \\ \text{IC} & y(0) = y_0. \end{cases}$$

For many (but not all) of the applications we investigate, the model is the simple linear autonomous model:

$$\text{ODE} \qquad \frac{dy}{dt} = k\,y + r_0 \tag{3}$$

$$\text{IVP} \qquad \text{IC} \qquad y(0) = y_0. \tag{4}$$

The parameters r_0, k and y_0 as well as the variable y and t are included in our nomenclature list.

Nomenclature

y = quantity of the state variable, t = time, r_0 = the rate of flow for the source or sink
k = constant of proportionality, y_0 = the initial amount of our state variable

The model is **general** in that we have not explicitly given the parameters r_0, k or y_0. These **parameters** are either given or found using specific (e.g., experimental) data. However, their values need not be known to solve the linear model.

Step 4: SOLVE THE MATHEMATICAL MODEL Once correctly formulated, the **solver** of the mathematical model can **rely completely on mathematics** and need not know where the model came from or what the Nomenclature stands for. Solution of the model requires both practical ("how to") skills and theoretical ("why") skills.

For the general linear autonomous model, we can obtain a general formula for its unique solution.

$$y = -\frac{r_0}{k} + \left(y_0 + \frac{r_0}{k} \right) e^{kt}.$$

Hence for this model we have a general solution (i.e., formula) for the model. If specific data is given, we can insert it into our formula.

Step 5: INTERPRETATION OF RESULTS. Although interpretation of results can involve lots of things, in the current context where the general model has been solved, it means insert the specific data given in the problem into the formula and **answer the questions** asked with regard to that specific data. This may require additional solution of **algebraic equations**, for example, the **formula** that you derived as the general solution of the IVP. However, some applications may involve other equations. The term general solution is used since arbitrary values of k, r_0, and y_0 are used. Recall that the term general solution is also used to indicate the (infinite) family of functions which are solutions to an ODE before a specific initial condition is imposed. We could argue that since the initial condition is arbitrary, we really have not imposed an initial condition, but again, general here means not only an arbitrary initial condition, but also an arbitrary value of k and r_0.

GENERAL AND SPECIFIC MODELS Once a general model has been formulated and solved, it can be applied using specific data. Alternately, the model can be written directly in terms of the specific data and then solved (again). If a general solution of the model has been obtained, this is redundant. However, writing a specific model and resolving provides much needed practice in the process of formulating and solving models and hence is useful in preparing for exams. Although it is sometimes useful to remember a general model, solutions of a general model should not normally be memorized and are usually not given on exams. Also specific data may simplify the process and the formulas obtained. Often it is better to solve a simple problem with specific data rather than try to apply a complicated formula resulting from a complicated model.

Repeating, it is acceptable (and indeed desirable since it gives practice in formulating and solving models) to formulate and solve a model using specific data.. The advantage of formulating and solving a model in a general context is that the solutions can be recorded in textbooks in physics, biology, etc. (and programed on personal computers) for those not interested in learning to solve differential equations. However, if the model assumptions change, a new model must be formulated and solved. Practice in formulating and solving specific models will help you to know when a different model is needed and in what generality a model can reasonably be developed. General models are useful when their results can be easily recorded (or can be programmed). On the other hand, trying to use the results of a complicated model can unduly complicate a simple problem.

Applied mathematics really begins with a desire to answer specific questions about "real world" problems. Hence in investigating applications, we will begin with specific questions that drive us to find answers. For our applications, answers require that we first develop and solve a **mathematical model** that is an **initial value problem**..

<u>APPLICATION</u> #1 **<u>RADIOACTIVE DECAY</u>**
Application Areas include Physics, Biology, and Nuclear Engineering.

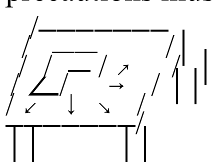
<u>QUESTION</u>: If the half-life of a particular radioactive substance is known to be 10 days and there are 25 milligrams initially, how much is present after 8 days?

To answer this question we use our **five step procedure**.

<u>Step 1: Understand the Concepts in the Application Area Where the Questions are Asked</u>. We first describe the phenomenon to be modeled, including the **laws** it must follow (e.g., that are imposed by nature, by an entrepreneurial environment or by the modeler). To understand radioactive decay, we consider the following **empirical physical law**.

<u>PHYSICAL LAW.</u> From physical experiments, it is found that radioactive substances **decay** at a **rate** that is **proportional** to the **amount present**.

It is useful to **<u>draw a sketch</u>** to help visualize the process being modeled. Try to visualize the radioactive substance on a table radiating out into the room. That is, the room is a **sink**. The amount of substance on the table is constantly decreasing. (Obviously, in a physics lab, safety precautions must be taken to protect against personal injury and pollution.) Now let us consider



the sentence "Radio active substances decay at a rate which is proportional to the amount present." **Rate** means **time rate of change** which implies **<u>derivative</u> with respect to time**. Thus our model will include a first order ODE that is a **rate equation**. (This is a special one-dimensional or **scalar** version of our quintessential model.) Always **make a list** of the **variables** and **parameters** you use. In an engineering research paper, this is called the **nomenclature** section. Begin with those stated in the problem. If you need a variable not given, choose one that is appropriate and helps you to remember what it stands for. We begin our list:

<u>Nomenclature</u>

Q = quantity of the radioactive substance (state variable)
t = time (independent variable)

To understand the concept of **half life**, we must first develop and solve the model.

<u>Step 2: Understand the Needed Concepts in Mathematics</u>. 1. High School Algebra, 2. Calculus, 3. Solution Techniques covered in this Part of the Notes.

<u>Step 3: Develop the Mathematical Model.</u> If the problem is not complicated, a **general model**

may be developed. By this we mean that arbitrary constants (**parameters**) are used instead of specific data. This general model may then be used for any specific problem where the **modeling assumptions** used to obtain the general model are satisfied. If the assumptions are changed, a new model must be formulated. If a general model can be developed and solved, the results can be recorded and used for any specific data. However, you may wish to redevelop the same model for different specific data in order to develop your modeling skills.

Let us more carefully analyze the sentence "Radio active substances decay at a rate which is proportional to the amount present." **Rate** means **time rate of change** which implies **derivative with respect to time**. **Decay** implies that the **derivative is negative**. **Proportional** means multiply the quantity by a **proportionality constant**, say k. Hence this sentence means the appropriate rate equation (first order ODE) to model radioactive decay is

$$\frac{dQ}{dt} = -kQ \qquad k > 0. \tag{1}$$

For this model we have followed the standard convention of putting in the minus sign explicitly since we know that the substance is always decaying (i.e., its time derivative is negative). This is not necessary, but forces the physical constant k to be positive. Physical constants are normally listed in reference books as positive quantities. You can and should check that the value you obtain for k in a specific problem is positive. If not, check your computations to find your mistake. Also, $k > 0$ makes the model more intuitive. We emphasize that the equation is a **rate equation** with **units** of mass per unit time (M/T e.g. grams per second, gm/sec). Thus it can be viewed as a **conservation law**. We only have a sink so that the rate of change is equal to the rate out. To determine the amount present at all times, we must also know the amount present initially (or at some time). Since no initial condition is given, we assume an arbitrary value, say Q_0 as a parameter. We add k and Q_0 to our nomenclature list.

Nomenclature

Q = quantity of the radioactive substance (state variable),
t = time (independent variable)
k = positive constant of proportionality (parameter),
Q_0 = initial amount of the radioactive substance (parameter)

The IVP that models radioactive decay is:

MATHEMATICAL MODEL: Radio active decay.

IVP

ODE $$\frac{dQ}{dt} = -kQ \tag{2}$$

IC $$Q(0) = Q_0 \tag{3}$$

Note that the model is "general" in that we have not explicitly given the proportionality constant k or the initial amount Q_0 of the substance. These **parameters** can be given or found using specific (e.g., experimental) data. However, we do not need to know the values of k and Q_0 to solve the model.

Step 4: Solve the Mathematical Model. Once the model is developed, it is not necessary that the solver of the model understand any of the **application concepts** in order to solve the model. What is required now is not an understanding of the physics, but an understanding of the mathematics.

To solve the ODE in this model, we note that it is both linear and separable. We choose to solve it as a separable problem, but recall that since it is linear, we can (and must) solve for Q explicitly. Separating variables we obtain the sequence of equivalent equations

$$\frac{dQ}{Q} = -k\ dt, \quad \int \frac{dQ}{Q} = -k \int dt, \quad \ell n\,|Q| = -kt + c, \quad |Q| = e^{-kt+c} = e^{c}e^{-kt}.$$

Letting $A = \pm e^c$ ($+e^c$ if $Q > 0$, $-e^c$ if $Q < 0$) we obtain $Q = Ae^{kt}$. Although the physics implies $Q \geq 0$, the mathematics does not require this in order for a unique solution to the IVP to exist. Applying the initial condition $Q(0) = Q_0$, we obtain $Q_0 = A$. Hence the unique solution to the IVP is

$$Q = Q_0 e^{-kt} \qquad (4)$$

It is the solution to the general model for radioactive decay for $Q_0 \geq 0$. Radioactive substances are said to experience **exponential decay**. The **formula** (4) is found in physics and biology texts. There are two constants (**parameters**) to be determined and we need further data to evaluate them. Known values of the constant k (with units 1/T e.g. 1/days) or its (multiplicative) inverse 1/k (which is referred to as a **time constant** since it has units of time) for specific substances could be given in reference books. (Usually half lives are given instead as explained below.) The existence and uniqueness theory says that exactly one solution exists for the IVP given by (2) and (3) and that the interval of validity is **R**. If we have any doubts that we have found it, we can check that it satisfies both the IC and the ODE for all $x \in \mathbf{R}$.

Step 5: Interpret the Results. Although interpretation of results can involve different things, in the context of this course it means "After you have solved the model (IVP) in whatever generality is appropriate, apply the specific data given to **answer the questions** that motivated our study". This may require additional solution of algebraic equations (e.g. the formula that you have derived for the general solution of the model). The term general solution is used since arbitrary values of k and Q_0 are used. (Recall that the term general solution is also used to indicate the family of functions which are solutions to an ODE before an initial condition is imposed. We could argue that since the initial condition is arbitrary, we really have not imposed an initial condition, but again, general here means not only an arbitrary initial condition, but also an arbitrary value of k.)

This brings us to the concept of **half life**. For an arbitrary value of Q_0, let t_{hl} be the time when only half of Q_0 is left. From (4) we obtain the sequence of equivalent scalar equations:

$(1/2)\ Q_0 = Q_0\, e^{-k\, t_h}$, $(1/2) = e^{-k\, t_h}$, $\ln(1/2) = -\, k\, t_h$, $t_h = -\dfrac{\ln(1/2)}{k}$.

First note that the half life depends only on the value of k and not on Q_0. In fact there is a **one-to-one correspondence** between values for k and values for t_{hl}. Thus we also have $k = -\dfrac{\ln(1/2)}{t_h}$. Note that although it may appear that k is negative, in fact ln(1/2) is negative and

$$k = \frac{\ln(2)}{t_h} \qquad (6)$$

Reference books generally give half lives. The value of k can then be computed using (6).

APPLICATION TO SPECIFIC DATA Once a general model has been formulated and solved, it can be applied to specific data. Alternately, the model can be written in terms of the specific data and then solved (again). If a general solution of the model has been obtained, this is redundant. However, resolving the model provides practice in the process of formulating and solving models and hence is useful in preparing for exams. Solutions of general models are not normally given on exams and are usually not memorized. Also specific data may simplify the process and the formulas obtained. Suppose that the following specific information is given:

SPECIFIC DATA. If the half-life (the time required for a given amount to decrease to half that amount) of a particular radioactive substance is known to be 10 days and there are 25 milligrams initially, then find the amount present after 8 days.

We develop a **data chart** so that the specific data and the questions to be answered are at our finger tips.

Data Chart:

t	$t_0 = 0$	$t_1 = 8$	$t_h = 10$
Q	$Q_0 = 25$	$Q_1 = ?$	$Q_h = ½\, Q_0$

All of the information in the sentence is now contained in the data chart for easy access. Recall that the "general" solution of the model (IVP) is given by $Q = Q_0e^{-kt}$. We need to apply the information in the data chart to obtain specific values for the constants (parameters) Q_0 and k, thus completing the model for this specific data.

It is certainly acceptable (and indeed desirable since it gives practice in formulating and solving models) to formulate and solve the model using this specific data.. The advantage of formulating and solving a model in a general context is that the solutions can be recorded in textbooks in physics, biology, etc. (and programed on personal computers) for those not interested in learning to solve differential equations. However, if the model assumptions change, a new model **must** be formulated and solved. Practice in formulating and solving specific models will help you to know when a different model is needed and in what generality a model can reasonably be developed. General models are useful when their results can be easily recorded (or can be programmed). On the other hand, trying to use the results of a complicated model can unduly complicate a simple problem. Applying the data in the data chart we obtain:

At $t = 0$, $Q = 25 \Rightarrow Q_0 = 25$.
At $t = 10$, $Q = \frac{1}{2} Q_0 = \frac{1}{2}(25) = Q_0 e^{-k(10)} = 25e^{-k(10)}$

Hence $\ell n(\frac{1}{2}) = -k(10)$ (Note that this result is independent of the value of Q_0.) so that $k = \dfrac{\ln(2)}{10}$. Hence $Q = 25\ e^{-\frac{\ln 2}{10}t} = 25 \exp(-\dfrac{\ln 2}{10}\ t)$. Thus after 8 days

$Q(8) = 25\ e^{-\frac{\ln 2}{10}8} = 25 \exp(-\dfrac{\ln 2}{10} 8) = 25 \exp(\ell n(2^{-(4/5)}) = 25/(2^{4/5}) = 25/\sqrt[5]{16}$.

EXERCISES on Applied Math Problem #1: Radioactive Decay

EXERCISE #1. Use the solution (4) of the model (2), (3) to "solve" the following problems. Be sure to include a data chart.
(a) If the half life of a particular radioactive substance is known to be 10 years and there are 8 milligrams initially, find the amount after 8 years.
(b) If the half life of a particular radioactive substance is known to be 100 years and there are 8 milligrams initially, find the amount after 50 years.

EXERCISE #2. Suppose a radioactive material satisfies the model (2), (3) with decay rat r and half life τ. Determine τ in terms of r. By inverting this function, determine r in terms of τ. Copy down Table 1 below. Use the relations you have found to fill it in.

TABLE #1: DECAY RATES AND HALF LIVES FOR SOME RADIOACTIVE MATERIALS

Material	Units		Decay rate (r)	Half-Life (τ)
	Mass (Q)	Time (T)		
Plutonium - 241	milligrams	years	0.0525 1/years	
Einsteinium - 253				
Radium - 226	milligrams	years		1620 years
Thorium - 234				

EXERCISE #3. Suppose that a radioactive substance R has a decay constant r when the amount of the substance is measured in milligrams and time is in days. That is, if left alone, it obeys the model (2), (3). Now suppose that an additional amount of R is added at the constant rate of k mg/day. Develop a model (i.e., an IVP) for this experimental set up.

EXERCISE #4. Solve the model developed in Exercise #3.

Application #2 **CONTINUOUS COMPOUNDING**
Application Areas include Business and Economics.

QUESTIONS. If $1000 is invested at 6% annually compounded continuously how much will the investment be worth in 6 years. How long before the investment doubles?

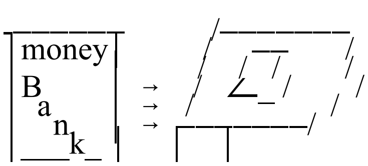
We again apply our four step procedure to solve this **applied math** or **application problem**:

Step 1: Understand the Concepts in the Application Area Where the Questions are Asked. This We describe the phenomenon to be modeled, including the **laws** it must follow. We consider the following "economic definition" for continuous compounding and develop a general model (IVP) that governs this process with no deposits or withdrawals after the initial investment.

ECONOMIC DEFINITION. **Continuous compounding** means that the time rate of change of the total investment (principle plus interest) is increasing in proportion to the amount present. (Money grows like rabbits. The more there is, the faster it grows.)

Unfortunately (in terms of understanding continuous compounding, but not in terms of understanding bank accounts) most people already have some understanding of **discrete compounding**. Example 2 (page 45) of Boyce and Diprima develops discrete as well as **continuous compounding** and compares them. As a first effort at trying to understand continuous compounding, it is probably better not to worry about discrete compounding (what banks do) and just focus on the model of continuous compounding as described in the above "economic definition". We note however that continuous compounding is in fact the limit of discrete compounding as the **interval of compounding** (i.e., Δt) is allowed to go to zero.

It is useful to **draw a sketch** to help visualize the process being modeled.



Your account

Try to visualize the amount of invested money on a table with the bank adding interest. The total amount of (principle plus interest) is increasing as the bank adds interest to the original principle invested. (Obviously, in a real life situation the money is in a vault or loaned to someone else at a higher interest rate.) Again our model will contain an ODE that is a **rate equation**.

Step 2: Understand the Needed Concepts in Mathematics. 1. High School Algebra, 2. Calculus, 3. Solution Techniques covered in this Part of the Notes.

Step 3: Develop the Mathematical Model. If the problem is not complicated, a **"general" model** may be developed and solved first. This "general" model may then be used for any specific problem where the modeling assumptions used to obtain the "general " model are satisfied. If they are not, a new model must be formulated and solved. To develop a model, let us analyze the sentence. "Continuous compounding means the time rate of change of the total investment (principle plus interest) is increasing in proportion to the amount present." This is just like radioactive decay, except that the rate is increasing, instead of decreasing. Hence we omit the minus sign and obtain

$$\frac{dS}{dt} = k\,S \qquad k > 0.$$

where

S = Total investment (Principle plus interest)
t = time
k = positive constant of proportionality (text uses r)

Always make a **list** of the **variables** and **parameters** you use. Begin with those stated in the problem. If you need a variable not given, choose one that is appropriate and helps you to remember what it stands for. Note that this **rate equation** has **units** of money per unit time (e.g. dollars per year, \$/yr). To determine the amount present at all times, we must also know the amount present initially (or at some time). Since no initial condition is given, we assume an arbitrary value, say S_0. Hence the IVP that models this phenomenon is given by:

MATHEMATICAL MODEL: Continuous Compounding.

IVP
- ODE $\quad \dfrac{dS}{dt} = k\,S$
- IC $\quad S(0) = S_0$

Note that the model is "general" in that we have not explicitly given the proportionality constant k or the initial investment S_0. These will have to be given or found using data. More needs to be said about the proportionality constant k. Unlike radioactive substances whose decay rates are set by nature, growth rates for money are set by bankers or the government). By looking at the definition of interest rates for discrete compounding and taking the limit as the time interval for compounding goes to zero (or simply by assuming this as a definition of continuous compounding) we agree that the constant k expressed as a fraction (e.g. 6% = 0.06) is the **rate of interest**. It is the (multiplicative) inverse of a time constant and has units of fractional portion (from the percentage rate)) per unit time (1/T, e.g. one over years, 1/yrs). Since it is a "rate of interest", we now replace k by r.

MATHEMATICAL MODEL: Continuous Compounding

IVP

ODE $$\frac{dS}{dt} = r\,S$$

IC $S(0) = S_0$.

Step 4: Solve the Mathematical Model. To solve the ODE, we note that it is essentially the same equation as for radioactive decay with $-k$ replaced by r. Hence the solution is given by

$$S = S_0 e^{rt}$$

There are two constants (parameters) to be determined and we need data to evaluate them. If the assumptions of the model are violated (e.g. if we add the additional assumption that we are adding to the original investment or withdrawing money on a continuous basis) the model must be reformulated and resolved.

Step 5: Interpret the Results. Although interpretation of results can involve a number of things, in the context of this course it usually means "After you have formulated and solved the "general" model (IVP) for the conditions presented, use your results and the specific data given to answer the specific questions asked". This may require additional solution of algebraic equations obtained in solving the model (IVP), for example, the equation obtained as the "general" solution of the model. The term "general" solution is used here since arbitrary values of r and S_0 are used. (Recall that the term general solution is also used to indicate the family of functions which are solutions to an ODE before an initial condition is imposed. We could argue that since the initial condition is arbitrary, we really have not imposed an initial condition, but again, "general" here means not only an arbitrary initial condition, but also an arbitrary value of r.)

APPLICATION OF SPECIFIC DATA Once a general model has been formulated and solved, it can be applied using specific data. Alternately, the model can be written in terms of the specific data and resolved. Although redundant, this resolving of the model provides much needed practice in the process of formulating and solving models. This is useful in preparation for exams since solutions of general models are not normally given on exams and are usually not memorized. Also specific data may simplify the process and the formulas obtained. Suppose that the following specific information is given:

SPECIFIC DATA. If \$1000 is invested at 6% annually compounded continuously how much will the investment be worth in 6 years. How long before the investment doubles?

We develop a **data chart** for: $r = 6\% = 0.06$

t	t0 = 0	$t_1 = 6$	t_d=?
S	$S_0 = 1000$	$S_1 = ?$	$S_d = 2\ S_0$

All of the information in the sentence except r = 6% = 0.06 is now contained in the data chart for easy access. Recall that the "general" solution of the model (IVP) is given by $S = S_0e^{-rt}$. We need to apply the information in the data chart to obtain values for the ?'s in the chart. It is certainly acceptable to include computation of S_0 (i.e. writing the formula with the value given) as part of the solution process with Step 2, but normally Step 2 involves the solution of the model (IVP) in the most general form that is reasonable.) Letting r = 6% = 0.06 and applying the data in the data chart we obtain:

At t = 0, S = 1000 which implies $S_0 = 1000$. Hence $S = 1000\ e^{0.06\ t}$

Hence at t = 6 years, $S = 1000e^{0.06(6)} = 1000e^{0.36}$ (≈ $ 1433.33 using a calculator)

At $t = t_d$, $S = 2\ S_0 = 2000 = S_0\ e^{0.06\ td} = 1000\ e^{0.06\ td}$ so that $2 = e^{0.06\ td}$ and hence $0.06\ t_d = \ell n(2)$.

Thus $t_d = (100/6)\ell n(2)$ (≈ 11.55 years using a calculator)

Similar to half life, the **doubling time** for the initial investment (or the doubling time for rabbits) is not dependent on value of the initial investment (or the initial number of rabbits). It is important to emphasize that if the modeling assumptions are changed, the result (i.e. formula for the solution) derived for the above model in Steps 1 and 2 and applied in Step 3, is not valid. The model must be reformulated and re-solved.

EXERCISES on Applied Math Problem #1: Continuous Compounding

Application #3 **MIXING (TANK) PROBLEMS**.
Application Areas include Civil and Chemical Engineering.

QUESTIONS. Suppose a tank initially has 10 pounds of salt dissolved in 100 gallons of water. If brine at a concentration of 1/4 pound of salt per gallon is entering the tank at the rate of 3 gallons per minute and the well stirred mixture leaves the tank at the same rate, how much salt is left in the tank after 30 minutes? What is the maximum amount of salt which accumulates in the tank?

Again we use our four step procedure to solve this **applied math** or **application problem**:

Step 1: Understand the Concepts in the Application Area Where the Questions are Asked. This process begins with a description of the phenomenon to be modeled, including the **laws** it must follow. We consider the following conservation (of stuff) law for a fixed location (container) in space.

PHYSICAL LAW. **Conservation of mass**. If we are not dealing with nuclear energy, then mass can neither be created or destroyed. When dealing with a fixed **control volume** (container), this becomes the **rate equation**:

Rate of change of amount in the control volume = (Rate in) – (Rate out).

We consider the following physical setup and develop a general model that governs the water and salt flow in a tank using appropriate modeling assumptions. That is, we apply the physical law of conservation of mass to a specific control volume (tank) containing two substances, salt and water.

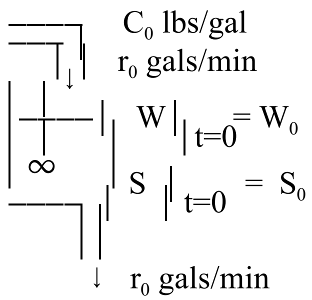
SET UP OF PHYSICAL SYSTEM. Let T be a tank which initially has S_0 lbs of salt dissolved in W_0 gallons of water. Suppose brine at a concentration of C_0 lbs of salt per gallon is entering the tank at the rate of r_0 gal./min. and the well stirred mixture leaves the tank at the same rate.

Step 2: Understand the Needed Concepts in Mathematics. 1. High School Algebra, 2. Calculus, 3. Solution Techniques covered in this Part of the Notes.

Step 3: Develop the Mathematical Model. If the problem is not complicated, a **general model** may be developed and solved first. Results for this general model may then be used for any specific data where the modeling assumptions used to obtain the general model are satisfied. If different assumptions are desired, a new model must be formulated and solved.

We could give numerical values for S_0, W_0, C_0, and r_0 since these are assumed to be constants (**parameters**). This might make the problem appear less abstract. However, working the problem with "symbols" instead of numbers will yield **formulas** instead of numbers as answers. These formulas can then be programed on a computer, PC, or programmable calculator. These formulas can then be used for any values of S_0, W_0, C_0, and r_0 desired. Thus we only have to solve the model once. Of course, if we change the model assumptions and make it more complicated (e.g. if the flow rate in is not the same as the flow rate out), then we must reformulate the model and re-solve it. On the other hand, having specific values for S_0, G_0, C_0, and r_0 may make the problem seem more "real world" or "applied". You are encourage to solve models in as general form as you can handle. On the other hand resolving the model for different specific data provides much needed practice in te formulation and solution of models.

To develop a model, let us analyze the sentence. "Suppose brine at a concentration of C_0 lbs. of salt per gallon is entering the tank at the rate of r_0 gallons per minute and the well stirred mixture leaves the tank at the same rate." It is useful to **draw a sketch** to help visualize the process being modeled. Try to visualize the brine (salt water) flowing into the tank and the **well stirred mixture** flowing out **at the same rate**.



We apply the general physical law of conservation of "Stuff" (i.e. **conservation of mass**, but it can be measured in force units e.g., lbs., since weight is proportional to mass and for liquids, in volume e.g., gallons, since the **density** is assumed to be constant). The general conservation equation is

Rate of change of amount in the tank = (Rate in) – (Rate out)

Since there are two ingredients (in a chemical reactor, there are may be many chemical species), there are <u>two</u> conservation equations. That is, we have **two state variables** and hence a **system** of ODE's. However, the coupling of these is only one way and we can easily treat the system as two scalar equations, one of which is trivial. Always **make a list** of the **variables** you use. Use those stated in the problem. If you need a variable not given, choose one that is appropriate and helps you to remember what it stands for.

Nomenclature

W = Amount of water (brine, gals.) in the tank at time t (state variable #1)
S = Amount of salt (lbs.) in the tank at time t (state variable #2)
t = time
W_0 = Initial amount of water (brine, gals.) in the tank.
S_0 = Initial amount of salt (lbs.) in the tank.

We write the conservation equation for the water first.

ASSUMPTION #1: The volume and flow of the water is not affected by the salt it contains.

That is, we assume that the concentrations are small enough so that the amount of salt in solution (lbs.) does not affect the space occupied by the water (gals.)). Hence we only have one-way (forward) coupling. We may compute the amount of water in the tank first and this provides definite information for computing the amount of salt. Hence we develop a **submodel** for the water flow.

ASSUMPTION #2: The rate of flow of brine into the tank is the same as the rate of flow out. This is a very important assumption since it leads to much simplification. It means that the amount of brine in the tank remains constant.

$$\frac{dW}{dt} = \text{rate of flow in minus rate of flow out} = r_0 - r_0 = 0.$$

Note that the units of this rate equation are gals/min. We need an initial condition. We assume

$W(0) = W\Big|_{t=0} = W_0$. Hence we have the IVP

MATHEMATICAL SUBMODEL #1: Water in tank.

SUBMODEL#1	IVP	ODE	$\frac{dW}{dt} = 0$
		IC	$W(0) = W_0$

This submodel is easily solved to obtain $W = W_0$ = constant. Since by Assumption 2, the flow rate out is the same as the flow rate in, the amount of water in the tank remains constant.

We now write the conservation equation for the salt. We must first note that the rate of flow of a solid in solution is given by its **concentration** times the **flow rate** of the liquid. Thus

$$\frac{dS}{dt} = \text{(conc. in)(flow rate of water in)} - \text{(conc. out)(flow rate of water out)}.$$

Note that the units of the **rate equation** are lbs/min. on the left and (lbs/gal)(gals/min) = lbs/min on the right. The concentration of the solution flowing in is given. However, the concentration of the solution flowing out is not. The phrase "**well stirred mixture**" is a clue. If the salt is evenly distributed (by the clapper in the sketch) through out the tank, then the concentration in the tank is the same at all points and is equal to S/W ($=S/W_0$).

ASSUMPTION #3: The concentration of the salt is uniform in the entire tank ("reactor"). That is, it si independent of the point in the tank where we might choose to measure it.
Hence the conservation equation for salt is

$$\frac{dS}{dt} = C_0\, r_0 - S/W.$$

To determine the amount present at all times, we need an initial condition. To make the model as general as possible, we assume an arbitrary initial condition:

$S(0) = S\Big|_{t=0} = S_0$. Hence we have the IVP

MATHEMATICAL SUBMODEL #2: Salt in tank.

SUBMODEL#2 IVP

ODE $$\frac{dS}{dt} = C_0\, r_0 - (S/W)r_0.$$

IC $S(0) = S_0$

Note that this **rate equation** has **units** of mass (or weight) per unit time (e.g. pounds per minute, lbs/min).

The mathematical model consists of the two submodels taken together. We could write this as a **system** of two ODE's. We will learn to solve general systems of ODE's later. However this one is simple to solve without the general machinery and hence provides a good introduction to the concept.)

MODEL IVP

ODE System
$$\frac{dW}{dt} = 0$$
$$\frac{dS}{dt} = C_0\, r_0 - (S/W)r_0.$$

IC
$$W(0) = W_0$$
$$S(0) = S_0$$

The reason that this system is easy to solve is that it is uncoupled (or at least only one-way coupled). We can solve for the amount of water in the tank first and then find the amount of salt in the tank. Since the solution of the first equation was trivial resulting in $W = W_0$ = constant, we can argue that we should be allowed to substitute this value in the second submodel to obtain the **scalar model**.

SCALAR MODEL: Salt in tank.

IVP

ODE $$\frac{dS}{dt} = C_0\, r_0 - (S/W_0)r_0.$$

IC $S(0) = S_0$

Step 4: Solve the Mathematical Model. Since Submodel#1 was trivial, we solved it immediately to obtain $W = W_0$ = constant. We then put this constant into Submodel#2 to obtain the scalar model given above. To solve the ODE in the scalar model, we note that it is both linear and separable. We choose to view it as a separable problem, but recall that since it is linear, we can (and must) solve explicitly for Q. Also from the theory, we have that $p(t) = r_0/W_0$ and $g(t) = C_0\, r_0$ are in $A(\mathbf{R})$ so that all solutions are in $A(\mathbf{R})$. From our previous experience, we expect exponential growth or decay. As previously demonstrated, we separate variables in a way to make it easy to obtain the time constant and avoid integration errors.

$$\frac{dS}{dt} = -\frac{r_0}{W_0} (S - W_0 C_0) \ dt$$

$$\int \frac{dS}{S - W_0 C_0} = -\frac{r_0}{W_0} \int dt$$

$$\ell n | S - W_0 C_0 | = -(r_0 / W_0) t + c$$

$$| S - W_0 C_0 | = \exp(-(r_0 / W_0)t + c) = e^{-(r_0 / W_0) t + c} = e^{-(r_0 / W_0) t} e^c$$

Letting $A = \pm e^c$, we may rewrite this as

$$S - W_0 C_0 = A \exp(-(r_0/W_0)t) = A e^{-(r_0 / W_0) t}$$

or

$$S = W_0 C_0 + A \exp(-(r_0/W_0)t) = W_0 C_0 + A e^{-(r_0 / W_0) t}$$

Applying the initial condition we obtain

$$S_0 = W_0 C_0 + A \exp(-(r_0 /W_0) 0) = W_0 C_0 + A e^{-(r_0 / W_0) 0}$$

$$S_0 = W_0 C_0 + A \Rightarrow A = S_0 - W_0 C_0$$

$$S = W_0 C_0 + (S_0 - W_0 C_0) \exp(-(r_0 /W_0) t) = W_0 C_0 + (S_0 - W_0 C_0) e^{-(r_0 / W_0) t}$$

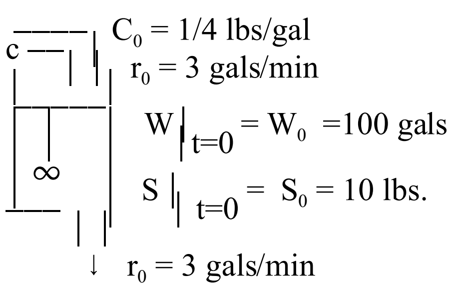
Note there are <u>four</u> constants (parameters) to be determined and we need further data to evaluate these. Also note that if the assumptions of the model do not hold (e.g. the flow of liquid in is not the same as the flow out) the model <u>must</u> be reformulated and resolved.

<u>Step 5: Interpret the Results</u>. Although interpretation of results can involve a number of things, in the context of this course it usually means " After you have formulated and solved the (general) model (IVP) for the general modeling assumptions, apply the specific data given to answer the questions asked". This may require additional solution of algebraic formulas (functions or equations) obtained in solving the model (IVP), for example, the equation obtained as the "general" solution of the model. The term "general" solution is used here since arbitrary values of C_0 and r_0 as well as W_0 and S_0 are used.

<u>APPLICATION OF SPECIFIC DATA</u> Once a general model has been formulated and solved, it can be applied using specific data. Alternately, the model can be written in terms of the specific data and resolved. Although redundant, this resolving of the model provides much needed practice in the process of formulating and solving models. This is useful in preparation for exams since solutions of general models are not normally given on exams and are usually not memorized. Also specific data may simplify the process and the formulas obtained. Suppose that the following specific information is given:

<u>SPECIFIC DATA</u>. Let T be a tank which initially has 10 lbs of salt dissolved in 100 gals of water. If brine at a concentration of 1/4 lb of salt per gallon is entering the tank at the rate of 3 gals/min, and the well stirred mixture leaves the tank at the same rate. Determine the amount of salt in the tank after 30 min. What is the maximum amount of salt which accumulates in the tank.

Since the assumption of the <u>general model</u> are satisfied, we may use the "general" solution. However, first we redraw our sketch giving the specific data and develop a data chart for <u>both</u> IVPs.



Water

t	$t_0 = 0$
W	$W_0 = 1000$

Salt

t	$t_0 = 0$	$t_1 = 30$	$t_m = ?$
S	$S_0 = 10$	$S_1 = ?$	$S_{max} = ?$

All of the information in the problem is now contained in the sketch and the data chart for easy access. Since these values are now given in the problem statement, it is certainly acceptable (and in fact desirable) to include them in your own formulation of the IVP and repeat Steps 1 and 2 using these specific values. We leave this as an exercise.

<u>EXERCISE.</u> Formulate and solve (repeat steps 2 and 3) the model for the specific data given above.

Instead of reworking the problem for this specific data we simply recall that the "general" solution for this model (IVP) is given by

$$S = W_0 C_0 + (S_0 - W_0 C_0) \exp(-(r_0/W_0)\, t) = W_0 C_0 + (S_0 - W_0 C_0)\, e^{-(r_0/W_0)\, t}$$

where $C_0 = 1/4$ lbs/gal, $r_0 = 3$ gals/min, $W_0 = 100$ gals, $S_0 = 10$ lbs.
Hence we obtain

$$S = (100)(1/4) + (10 - (100)(1/4)) \exp(-(3/100)\, t) = 25 + (10 - 25)\, e^{-(3/100)\, t}$$
$$= 25 - 15\, e^{-(3/100)\, t}$$

We use this result to obtain values for the " ?'s " in the chart. The first question is easy.

$$S(30) = S(t_1) = S\Big|_{t = t_1 = 30} = S_1 = 25 - 15\, e^{-(3/100)(30)} = 25 - 15\, e^{-9/10} \approx 18.00$$

The second is a little more tricky since it might at first appear to be a straight forward max-min problem. However, recall that e^{-x} decreases to zero as $x \to \infty$. Also
$dS/dt = -(15)(-3/100)\, e^{-(3/100)\,t} > 0 \quad \forall\, t \in \mathbf{R}$. Hence the amount of salt is always increasing as the term $15\, e^{-(3/100)\,t}$ decreases to zero. Hence the maximum amount of salt in the tank is

$$S(t=\infty) \;=\; S(t_m) \;=\; S\Big|_{t = t_1 = 30} \;=\; S_{max} = \lim_{t\to\infty} \left(25 - 15\, e^{-(3/100)(t)}\right) = 25.$$

As $t \to \infty$ the amount of salt in the tank approaches 25 lbs. This is called the <u>steady</u>-<u>state</u>.

EXERCISES on Applied Math Problem #3: Mixing Problems

Handout #5 A GENERIC FIRST ORDER LINEAR MATHEMATICAL MODEL WITH ONE STATE VARIABLE Professor Moseley

Review the three models developed and note their commonality. (The mixing problem was originally a nonlinear vector problem with two state variables, but, even with nonequal flow rates, because of the one-way coupling, it could be reduced to a scalar linear model.) We wish to step back and get an overview of how to use first order (linear) ODE initial value problems as mathematical models. We first review our five step procedure for solving any **applied math** or **application problem**:

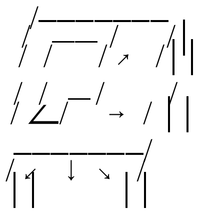
Step 1. **Understand** the concepts in the application area where a mathematical **model** is desired.
Step 2: **Understand** the Needed Concepts in **Mathematics**.
Step 3. **Develop** the mathematical **model**.
Step 4. **Solve** the mathematical **model**.
Step 5. **Interpret** your **results** This includes **answer**ing the **questions asked**.

QUESTIONS. Suppose a container (e.g., a tank, test tube, Petri dish, bank account, human body, or ocean) initially contains y_0 units of stuff (e.g., salt, chemical, culture, money, heat energy, or fish). Suppose, with no source term, stuff accumulates (or drains) by a linear law. If in addition, stuff is added at the constant rate of r_0 units of stuff per unit of time. how much stuff will be left after t_1 units of time?

Step 1: Understand the Concepts in the Application Area Where the Questions are Asked.. This process begins with a description of the phenomenon to be modeled, including the **laws** it must follow (e.g., that are imposed by nature, by an entrepreneurial environment or by the modeler). For our generic model this is a law which gives the rate of change of the **state variable** based on its current value. it is a **rate equation** that does not depend on history. We assume a linear law if there is no source term.

SOMEBODY'S LINEAR LAW. From data or theory, it is known that if there is no source term, the **rate** at which the amount of "stuff" we have changes (increases or decreases, but going who knows where) is **proportional** to the **amount present**.

We will see that this leads to a first order **linear** ODE. It is useful to **draw a sketch** to help visualize the process being modeled. Try to visualize the amount of stuff in a container on a table. It may be increasing or decreasing. Most physical laws assume decay (radioactive decay, discharge of a capacitor, cooling of a body) when there is no source term. Our law does not say which. If we assume it is decreasing, then as in the sketch below, we draw arrows away from the container (and put in a minus sign explicitly in the model). If it is increasing, we draw the arrows toward the table. Now consider the sentence "the **rate** at which "stuff" changes is **proportional**



to the **amount present.”** **Rate** means **time rate of change** which implies **derivative with respect to time**. Thus our model will include an ODE that is a **rate equation**. This is the one-dimensional or scalar case of our quintessential model. Now **make a list** of the **variables** you use. In an engineering research paper, this is in the **nomenclature** section. Begin with those stated in the problem. If you need a variable not given, choose one that is appropriate and helps you to remember what it stands for. We begin our list with y as our only state variable that is a quantitative measure of “stuff” and t as time.

Nomenclature

y = our only state variable
t = time

Step 2: Understand the Needed Concepts in Mathematics. 1. High School Algebra, 2. Calculus, 3. Solution Techniques covered in this Part of the Notes.

Step 3: DEVELOPMENT OF THE MODEL. If the problem is not complicated, a **general model** may be developed and solved first. This general model may then be used for any **specific problem** where the **modeling assumptions** used to obtain the general model are satisfied. If they are not, a new model that governs this new phenomenon must be formulated and solved. Let us more carefully analyze the sentence "the **rate** at which our state variable changes is **proportional** to the **amount present**." **Rate** means **time rate of change** which implies **derivative with respect to time**. **Proportional** means multiply the amount of “stuff” currently present by a **proportionality constant**, say k. Hence this sentence means the appropriate ODE to model this behavior if there is no source term is

$$\frac{dy}{dt} = k\,y. \qquad (1)$$

If we know that y is decreasing, we put in the minus sign explicitly so that $-k<0$ and $k>0$. Since we assume a source term, we modify (1) to obtain

$$\frac{dy}{dt} = f(y;k,r_0) = k\,y + r_0. \qquad (2)$$

We emphasize that the ODE is a **rate equation** with **units** of “units of stuff” per unit time. Here r_0 can be positive or negative; we could have a source or sink. Also. the constant k could be positive or negative, depending on the phenomenon being modeled. To determine the amount present at all times, we must also know the amount present initially (or at some time). Since no **initial condition** is given, we assume an arbitrary value, say y_0. Hence the IVP that models the phenomenon described by “Somebody’s Law” with a source (or sink) term is:

MATHEMATICAL MODEL:

$$\text{IVP}\quad \begin{cases} \text{ODE} & \dfrac{dy}{dt} = f(y;k,r_0) = k\,y + r_0 & (3) \\ \text{IC} & y(0) = y_0 & (4) \end{cases}$$

We add the parameters r_0, k and y_0 to our nomenclature list.

Nomenclature
y = quantity of the state variable, t = time, r_0 = the rate of flow for the source or sink
k = constant of proportionality, y_0 = the initial amount of our state variable

The model is **general** in that we have not explicitly given the parameters r_0, k or y_0. These **parameters** are either given or found using specific (e.g., experimental) data. However, they are not necessary in order to solve the model. We do note that $f(y;k,r_0)$ is independent of time t. Such systems are called **autonomous**. They have certain special properties. One is that equilibrium (or constant) solutions are $y = y_e$ where y_e is any solution of $f(y) = 0$. For our problem $f(y;k,r_0) = k\,y + r_0 = 0$ implies that if $k\neq 0$, then $y_e = -r_0/k$ is the only equilibrium solution.

Step 4: Solve the Mathematical Model Even though we have not specified values for the parameters k and y_0, this does not impair our ability to solve the general model. To solve the ODE, we note that it is both linear and separable. We choose to view it as a separable problem, but recall that since it is linear, we can (and must) solve for y explicitly. Also, since $p(t) = k$ is continuous, indeed analytic, for all t ($p\in \mathcal{A}(\mathbf{R})\subseteq C(\mathbf{R})$), we know a priori that all solutions to the ODE are in $\mathcal{A}(\mathbf{R})$. Hence the solution to the IVP will be in $\mathcal{A}(\mathbf{R})$.

Separating variables we obtain $\dfrac{dy}{y + r_0/k} = k\,dt$ so that $\displaystyle\int \frac{dy}{y + r_0/k} = k\int dt$ and hence $\ell n\,|\,y + r_0/k\,| = kt + c$. Raising both sides to the e power, we obtain

$$|y + r_0/k| = e^{kt+c} = e^c\, e^{kt}$$

Letting $A = \pm e^c$ ($+\,e^c$ if $y > 0$, $-\,e^c$ if $y < 0$), we obtain $y + r_0/k = Ae^{kt}$. Applying the initial condition $y(0) = y_0$, we obtain $y_0 + r_0/k = A$. Hence we have the solution

$$y = (y_0 + r_0/k)\, e^{kt} - r_0/k \qquad (4)$$

to the general model. The existence and uniqueness theory told us that exactly one solution exists for the IVP given by (3) and (4) and that it is in $\mathcal{A}(\mathbf{R})$. "Clearly" y is in $\mathcal{A}(\mathbf{R})$. If we have any doubts that it is what we seek, we can check that it satisfies both the IC and the ODE for all t in **R**. Note that $y_0 = -\,r_0/k$ (so that $y = y_e = -\,r_0/k$) yields the equilibrium solution and that if $k<0$, then all solutions approach $y_e = -\,r_0/k$ so that it is stable.

Step 5: INTERPRETATION OF RESULTS. Although interpretation of results can involve a number of things, in the context of this course it generally means the following: Apply the specific data given in the problem and **answer the questions** asked with regard to that specific data. This may require additional solution of **algebraic equations**, for example, the **formula** (4) that we derived as the general solution of the IVP. However, some applications may involve other equations. The term general solution is used since arbitrary values of k, r_0, and y_0 are used. Recall that the term **general solution** is also used to indicate the (infinite) family of functions which are solutions to an ODE before a specific initial condition is imposed. We could argue that since the initial condition is arbitrary, we really have not imposed an initial condition, but again, general here means not only an arbitrary initial condition, but also an arbitrary value of k and r_0.

APPLICATION OF SPECIFIC DATA Once a general model has been formulated and solved, it can be applied using specific data. Alternately, the model can be written directly in terms of the specific data and then solved (again). If a general solution of the model has been obtained, this is redundant. However, resolving the model provides much needed practice in the process of formulating and solving models and hence is useful in preparing for exams. Although it is sometimes useful to remember a general model, solutions of a general model should not normally be memorized and are usually not given on exams. Also specific data may simplify the process and the formulas obtained. It is better to solve a simple model than to try to apply a complicated formula resulting from a complicated model.

Repeating, it is acceptable (and indeed desirable since it gives practice in formulating and solving models) to formulate and solve a model using specific data.. The advantage of formulating and solving a model in a general context is that the solutions can be recorded in textbooks in physics, biology, etc. (and programed on personal computers) for those not interested in learning to solve differential equations. However, if the model assumptions change, a new model must be formulated and solved. Practice in formulating and solving specific models will help you to know when a different model is needed and in what generality a model can reasonably be developed. General models are useful when their results can be easily recorded (or can be programmed). On the other hand, trying to use the results of a complicated model can unduly complicate a simple problem..

Application #4 **ONE DIMENSIONAL MOTION OF A POINT MASS**
Application Areas include Physics, Mechanics, and Mechanical Engineering.

QUESTIONS. A body with mass 5 grams falls from rest in a medium offering resistance proportional to the square of the velocity. If the limiting velocity is 2 centimeters per second, find the velocity v as a function of time t.

Step 1: Understand the Concepts in the Application Area Where the Questions Are Asked. This process begins with a description of the phenomenon to be modeled, including the "laws" it must follow. We consider the following physical law that was reviewed in Chapter 1.

THEORETICAL (AND EMPIRICAL) PHYSICAL LAW. (Newton's Second Law of Motion, Conservation of Momentum,) This law states that at any given time, the net **force** on a particle (point mass) is equal to its **mass** times its **acceleration** (F = MA).

Now consider the following **empirical physical law**.

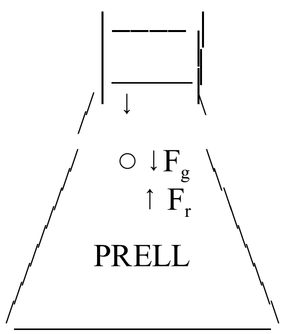
EMPIRICAL PHYSICAL LAW. When a particular body falls from rest in a particular medium, the force of resistance to movement that is to the square of the velocity.

We call this the Prell problem since in a commercial for Prell shampoo, a pearl was dropped into a bottle of Prell shampoo. Since Prell was a very viscous liquid, the pearl dropped very slowly. We could give numerical values for mass and other relevant parameters, but if we can solve the resulting model in some generality, the resulting formula(s) can then be programed on a computer, PC, or programmable calculator. These formula(s) can then be used for any values of the parameters desired. Thus we only have to solve the model once. Of course, if we change the model assumptions and make the problem more complicated (e.g. if the medium offers resistance proportional to the velocity or to the cube of the velocity), then we must reformulate the model and re-solve it. On the other hand, having specific values for the parameters may make the problem seem more "real world" or "applied". You are encourage to solve models in as general form as you can handle, but formulating and solving specific models gives much needed experience in the process.

Step 2: Understand the Needed Concepts in Mathematics. 1. High School Algebra, 2. Calculus, 3. Solution Techniques covered in this Part of the Notes.

Step 3: DEVELOPMENT OF THE MODEL. If the problem is not complicated, a general model may be developed and solved first. This general model may then be used for any specific data where the modeling assumptions used to obtain the general model are satisfied. If they are not, a new model must be formulated and solved

To develop a model, we begin with Newton's law. It is usually useful to **draw a sketch** to help visualize the process being modeled. Try to visualize the pearl falling slowly in the bottle of Prell. Applying Newton's law, **Force = mass times acceleration, F = ma** .(i.e. conservation of momentum but under certain circumstances it can be used to derive conservation of energy. Actually, we prefer to write it as **ma = F** where a =dv/dt = the derivative of the velocity v. Thus we obtain the differential equation $m\,\frac{dv}{dt} = F$. We now consider the forces acting on the pearl. We are told that the "pearl" **falls** from rest. Hence we choose our coordinate system to point down. Our initial velocity is v(0) = 0. We could choose to have an arbitrary initial velocity. However, then the pearl could initially go up (like throwing a ball up). It would eventually come down. However, the change in direction would cause a problem for our model. Hence we start with the simpler model using v(0) = 0 with some intuitive assurance that the motion will always be down.



There are two forces acting on the "pearl". The force of gravity and the resistance offered by the (viscous) fluid. Since we are taking positive x as down, F_g = mg where g is the acceleration due to gravity (32 ft/s^2). The resistance offered by the fluid is a little more tricky. Its magnitude is $k\,|v^2|$ where k is the (positive) proportionality constant. However, the direction of the resistance offered by the fluid (force is really a vector quantity even though we are assuming one dimensional motion.) is always opposite that of the motion. Thus $F_r = -\,k\,v^2$ provided the motion is down (recall down is positive and up is negative). If the motion were allowed to be up, we would have to stop the process and replace $-\,k\,v^2$ with $k\,v^2$ ($k > 0$) when the "pearl" was going up ($v < 0$). Since we do not expect this to happen we have:

F_g = mg = force of gravity (positive since positive x is down)
$F_r = -\,k\,v^2$ = resistance offered by the fluid (negative since it opposes the downward, positive, motion
where
m = mass of the particle
g = acceleration due to gravity (32ft/s^2)
k = proportionality constant determined empirically (i.e. experimentally and not from theory)
v = velocity of particle.
t = time.

Always **make a list** of the **variables** you use. Use those stated in the problem. If you need a variable not given, choose one that is appropriate and helps you to remember what it stands for Recalling that v(0) = 0, we obtain the model(IVP):

ODE $$m\,\frac{dv}{dt} = F_g + F_r = mg - k\,v^2$$

MODEL IVP

IC $v(0) = v_0 = 0.$

Note that the ODE has units of force (e.g. Newtons = kg m/s^2 , pounds force = slug ft/s^2 , dynes = gm cm/s^{2,} etc.). Hence k must have units of force / (velocity)^2 = (ML/T^2)/(L/T)^2 = M/L = (gm/cm).

<u>Step 4: SOLUTION OF THE MATH MODEL</u> To solve the ODE, we note that it is <u>not</u> linear. Why? However, it is separable. We solve it in the usual way. However, we have no guarantee that we can solve for the velocity explicitly. From our physical intuition, we expect the velocity to in crease. However, it is not obvious that a <u>terminal</u> velocity will be obtained (like the maximum amount of salt in the tank). We must wait and see.

$$\frac{dv}{dt} = \frac{mg - kv^2}{m} = -\frac{k}{m}\left(v^2 - \frac{mg}{k}\right)$$

$$\int \frac{dv}{v^2 - (mg/k)} = -\frac{k}{m}\int dt$$

Interestingly the units of this equation are
LHS = velocity/(velocity)^2 = T/M = (sec/cm)
RHS = ((M/L) / M) T = (sec/cm)

The integral on the left requires partial fractions. We do it separately. Since

$$\frac{1}{v^2 - (mg/k)} = \frac{1}{(v - \sqrt{mg/k})(v + \sqrt{mg/k})} = \frac{A}{v - \sqrt{mg/k}} + \frac{B}{v + \sqrt{mg/k}}$$

we have (using the "cover up" method with $v = \sqrt{mg/k}$ and $v = -\sqrt{mg/k}$),

$$\frac{1}{\sqrt{mg/k} + \sqrt{mg/k}} = \frac{A}{1} + \frac{B(0)}{\sqrt{mg/k} + \sqrt{mg/k}}\ .$$

$$\frac{1}{-\sqrt{mg/k} - \sqrt{mg/k}} = \frac{A(0)}{-\sqrt{mg/k} - \sqrt{mg/k}} + \frac{B}{1}$$

So $A = 1/(2\sqrt{mg/k}\ )$, $B = -1/(2\sqrt{mg/k}\ )$.

$$\int \frac{dv}{v^2 - (mg/k)} = \int \frac{A}{v - \sqrt{mg/k}} + \int \frac{B}{v + \sqrt{mg/k}}$$

$$= A\,\ell n \,|\, v - \sqrt{mg/k} \,| + B\,\ell n \,|\, v + \sqrt{mg/k} \,|$$

$$= 1/(2\sqrt{mg/k}\,)\,(\,\ell n \,|\, v - \sqrt{mg/k} \,| - \ell n \,|\, v + \sqrt{mg/k} \,|\,)$$

$$= \frac{1}{2\sqrt{mg/k}} \ln\left|\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}}\right|$$

Substituting back into the solution of the ODE, we obtain:

$$\frac{1}{2\sqrt{mg/k}} \ln\left|\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}}\right| = -\frac{k}{m}\,t + c$$

Recall that the ODE was nonlinear. Hence an implicit solution may be the best we can do. However, you may be able to tell that, indeed, you can solve for v. But this could get messy if an arbitrary IC is used. Recall, however, that to keep the model simple, we <u>dropped</u> the particle. Thus v(0) =0 and we can apply it in the implicit form.

$$\frac{1}{2\sqrt{mg/k}} \ln\left|\frac{0 - \sqrt{mg/k}}{0 + \sqrt{mg/k}}\right| = -\frac{k}{m}\,(0) + c \quad \Rightarrow \quad c = 0.\ \text{Hence}$$

$$\frac{1}{2\sqrt{mg/k}} \ln\left|\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}}\right| = -\frac{k}{m}\,t,$$

$$\ln\left|\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}}\right| = -\,(2\sqrt{mg/k}\,)\,\frac{k}{m}\,t = -2\sqrt{kg/m}\;t.$$

Although this may be an acceptable solution under some circumstances, an explicit solution is desirable. Raising both sides to the e power, we obtain

$$\left|\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}}\right| = \exp\{\,-2\sqrt{kg/m}\;t\,\} = e^{-2\sqrt{kg/m}\,t}$$

To get rid of the absolute value, we must know the sign of the terms inside. From the physics, we expect v to start a zero and increase. However, does it increase without limit? Note that the right hand side goes to zero as $t \to \infty$. Hence we obtain the terminal velocity from

$$\left|\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}}\right| = 0$$

as $v_{max} = \sqrt{mg/k}$. Note that the units of mg/k are

$$\text{force}/(M/L) = (ML/T^2)/(M/L) = (L/T)^2$$

as required. Thus the velocity increases from zero but never gets above this terminal value. Hence, $v - \sqrt{mg/k} < 0$, so that we have

$$-\frac{v - \sqrt{mg/k}}{v + \sqrt{mg/k}} = \exp\{-2\sqrt{kg/m}\;t\} = e^{-2\sqrt{kg/m}t}$$

$$-(v - \sqrt{mg/k}) = (v + \sqrt{mg/k})\exp\{-2\sqrt{kg/m}\;t\} = (v + \sqrt{mg/k})\; e^{-2\sqrt{kg/m}t}$$

$$-v + \sqrt{mg/k} = v\exp\{-2\sqrt{kg/m}\;t\} + \sqrt{mg/k}\exp\{-2\sqrt{kg/m}\;t\}$$

$$\sqrt{mg/k}\,(1 - \exp\{-2\sqrt{kg/m}\;t\}) = v(1 + \exp\{-2\sqrt{kg/m}\;t\})$$

$$v = \sqrt{mg/k}\,\frac{1 - \exp\{-2\sqrt{kg/m}\;t}{1 + \exp\{-2\sqrt{kg/m}\;t}$$

The results of solving the model are this equation and $v_{max} = \sqrt{mg/k}$.
Note there are <u>two</u> constants to be determined, m and k (g is the acceleration due to gravity and is a constant, but not a parameter), and we need further data to evaluate these. Also note that if the assumptions of the model do not hold (e.g. the initial velocity is not zero) the model <u>must</u> be reformulated and re-solved.

<u>Step 4: INTERPRETATION OF RESULTS.</u> Although interpretation of results can involve a number of things, in the context of this course it usually means " After you have formulated and solved the model (IVP) in as much generality as is appropriate, (it is important to record the modeling assumptions clearly e.g. v(0) =0 in the statement of the IVP), answer the questions asked for the specific data given". This may require additional solution of algebraic formulas (functions or equations) obtained in solving the model (IVP).

Suppose that the more specific problem is given:

A body with mass 5 grams falls from rest in a medium offering resistance proportional to the square of the velocity. If the limiting velocity is 2 centimeters per second, find the velocity v as a function of time t.

Recall

$$v = \sqrt{mg/k}\,\frac{1 - \exp\{-2\sqrt{kg/m}\;\; t}{1 + \exp\{-2\sqrt{kg/m}\;\; t} \;, \qquad v_{max} = \sqrt{mg/k}$$

The mass m is given as 5 grams. the constant g = 980 cm/s^2. The parameter k can now be obtain from the limiting velocity equation.

$$2 = v_{max} = \sqrt{mg/k} = \sqrt{5(980)/k} = \sqrt{5^2(198)/k} = \sqrt{5^2(198)}\;/\sqrt{k}$$

$$\sqrt{k} = \sqrt{mg/2} = \sqrt{5^2(198)/2} = \sqrt{5^2(198)/2} = 5\sqrt{99} = 5\sqrt{9(11)} = 15\sqrt{11}$$

$k \approx 34.1$ gr/cm

EXERCISES on Applied Math Problem #4: Newton's Second Law of Motion