Chapter 5 — Polynomial and Rational Functions

We mostly study polynomial functions and their graphs in this chapter, with a brief diversion into rational functions, which are ratios (fractions) of polynomials. A lot of this course is about learning how to take what you know and use it to analyze new things, and polynomials are simple enough to provide a non-calculus experience in analysis. This gives students taking this course valuable experience at another aspect of analytical thinking.

Section 5.1 — Polynomial Functions and Models

Now we shall generalize the analytical techniques to polynomial functions of any degree, to the extent possible without using calculus.

Topic 1: Graphical features of polynomial functions.
Topic 2: Sketching polynomial functions with single roots.
Topic 3: Repeated roots.
Topic 4: Sketching polynomial functions in general.

Practice Problems: 37 – 43, 63 – 87 odd

Topic 1: Graphical features of polynomial functions.

A quadratic function is an example of a second degree polynomial function. Now we turn our attention to polynomial functions in general, which can have any integer degree greater than or equal to zero. I shall do this by introducing each degree polynomial up to degree 5 in a general way. In so doing, we shall build up the features of each type until all the patterns become apparent. This is a very pattern-driven topic. Unfortunately, I cannot justify all the patterns you will see through the mathematics of this course. Calculus is required in order to prove many things I'm going to show you.

Unlike the last section where better quality graphs were expected, in this section we are only interested in sketching the graphs in order to discern some of the basic behavioral characteristics. In the next section (of this book, not the textbook) we will see how this knowledge can be used for an analytical purpose.

In this topic, then, I will catalog each degree polynomial with its characteristics. Letters at the beginning of the alphabet, such as a, b and c, will represent constant numbers (numbers that can be fixed). For example, if $y = ax + b$ represents a polynomial function of degree 1, then $y = 2x - 3$ is an example of such a function, where $a = 2$ and $b = -3$.

First we need to talk about **roots**. There are at least three terms for this. When you set a function equal to zero and solve, such as $y = f(x) = 0$, then you are solving for the roots of the function. The roots of a function are also called the **zeros** of the function. Graphically, the y–axis is the line $y = 0$, so roots or zeros are also where the graph crosses the x–axis, i.e., the x–intercepts. As you might expect, the roots are an important feature of any function. In the context of polynomial functions it turns out that the maximum number of roots (including complex number roots) equal the degree of the polynomial. I will have more to say about that observation in a later section.

The following applies to the discussion of polynomial functions in this section.

> Except for the zero degree polynomial function, the leading coefficient $a \neq 0$.
>
> Any mention of roots only concerns real number roots, not complex number roots, unless otherwise stated.

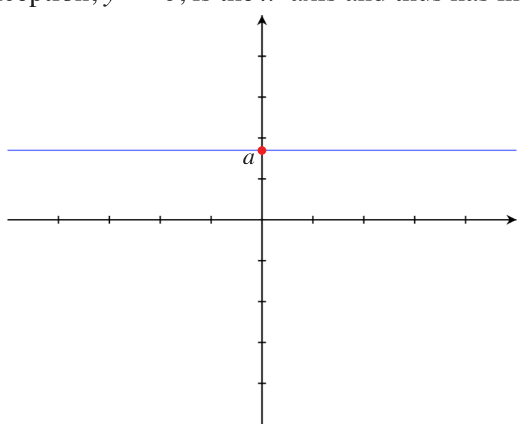
Degree 0 Polynomials

A zero degree polynomial has the form $y = a$.

It is called a **constant function** because the value of the function never changes.

The graph is a horizontal line.

There are no roots. (The exception, $y = 0$, is the x–axis and thus has infinitely many roots.)



You might say that zero degree polynomials are very uninteresting!

Degree 1 Polynomials

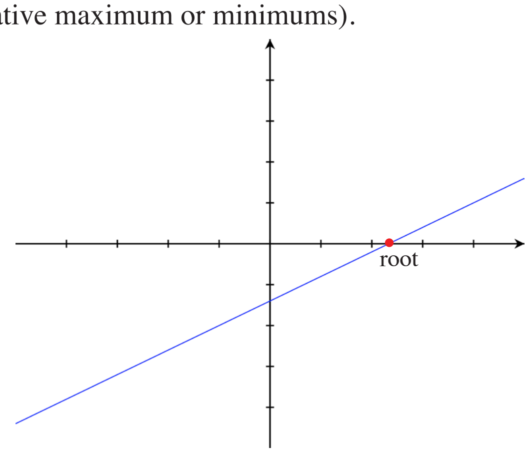
A first degree polynomial has the form $y = ax + b$.

It is called a **linear function**.

The graph is a slanted line (neither horizontal nor vertical).

There is one root because a slanted line must cross the x–axis once.

There are no **extrema** (relative maximum or minimums).



Of course, a slanted line may also slant downwards (negative slope) as you read left to right. The direction depends upon the leading coefficient, a.

Degree 2 Polynomials

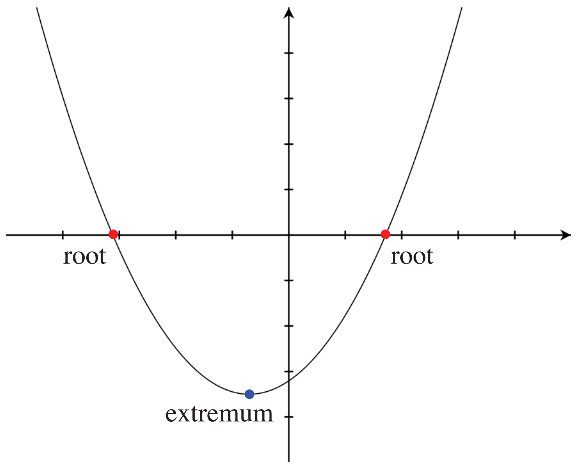
A second degree polynomial has the form $y = ax^2 + bx + c$.

It is called a quadratic function.

The graph is a parabola.

There are at most two roots.

There is one extremum, which is called the vertex.



Parabolas that are quadratic functions are line symmetric to the vertical line passing through the vertex. This line is called the axis of symmetry.

Degree 3 Polynomials

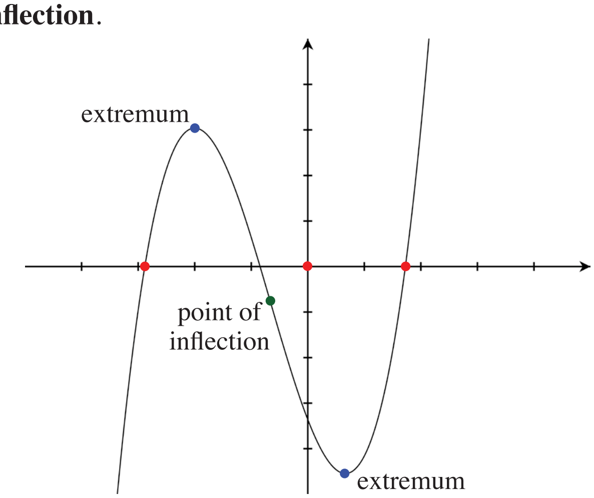
A third degree polynomial has the form $y = ax^3 + bx^2 + cx + d$.

It is called a **cubic** function.

There are one to three roots.

There are zero or two extremum.

There is one **point of inflection**.

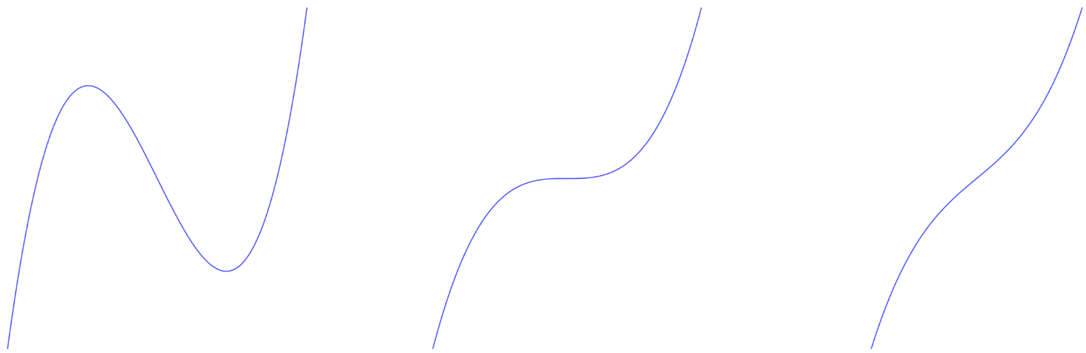


A point of inflection is where a graph changes **concavity**. Concavity is either up or down. If you look at the graph of the cubic above, the left side is **concave down** because the curve loops up and then down, sort of like an upside down "U". On the right side of the point of inflection the concavity changes to **concave up**, which means that the graph loops down and then up, a sort of "U" shape.

In general applications (not just polynomial functions) the point of inflection sometimes is interpreted as the point of diminishing returns.

Cubics have point symmetry. It can be shown that the point of inflection for a cubic is always the point of symmetry.

Beginning with third degree and higher polynomials there are an increasing number of varieties of shapes. Here are the three basic shapes for cubics.



If you account for reflections across the x–axis, then there are six basic shapes.

Notice that the cubic in the middle is a special case. The graph levels off at the point of inflection. Although it looks horizontal in that neighborhood, the only place such a cubic achieves true horizontal is at its point of inflection.

Degree 4 Polynomials

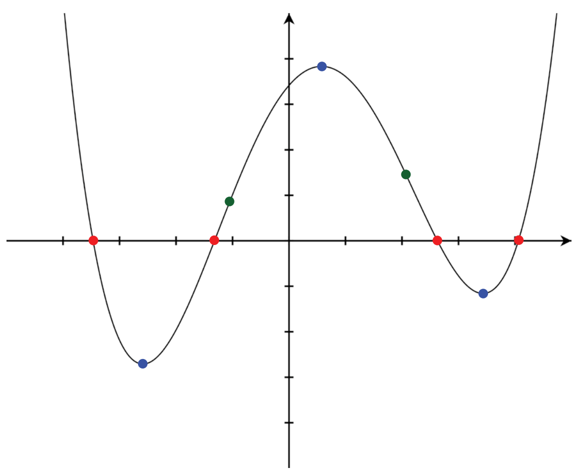
A fourth degree polynomial has the form $y = ax^4 + bx^3 + cx^2 + dx + e$.

It is called a **quartic** function.

There are zero to four roots.

There are one to three extremum.

There are zero to two points of inflection.



Polynomials of degree 4 and higher do not have any consistent symmetries, although certain ones can have symmetry under the right conditions.

I have identified seven types of quartic shapes by taking into account other factors. If you are interested, I have these catalogued on my web site:

http://www.sscc.edu/home/jdavidso/Math/Catalog/Polynomials/Fourth/Fourth.html

Degree 5 Polynomials

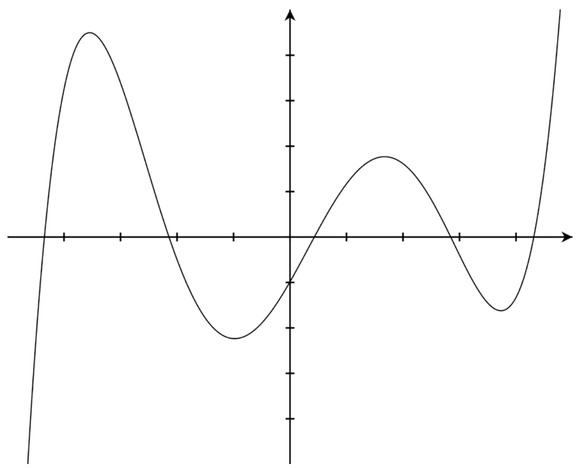
A fifth degree polynomial has the form $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.

It is called a **quintic** function.

There are one to five roots.

There are zero to four extremum.

There are one to three points of inflection.



How many different quintic shapes are there? I don't know, I haven't worked them all out. Let's say it's "many."

The patterns continue in a similar fashion for higher degree polynomials. We might summarize them a little more simply this way.

The graph of an nth degree polynomial function has the following properties

a) There are at most n roots.

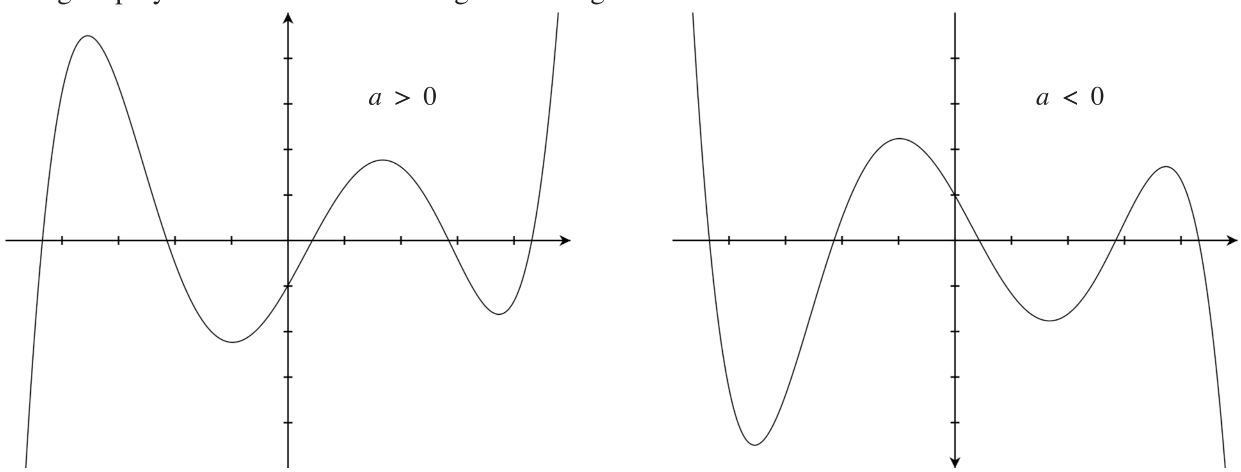
b) There are at most $n - 1$ extrema.

c) There are at most $n - 2$ points of inflection.

The first property about the number of roots can be justified through the Fundamental Theorem of Algebra, first proved around 1799 by Karl Friedrich Gauss (1777 – 1855), one of the top three mathematicians of all time. I will present an intuitive proof of this theorem in a later section. The other two properties are justified through calculus techniques, with the assist of the first property. In other words, the equation that gives us the extrema is itself a polynomial function, but of one degree less than the original. The equation that gives us the points of inflections is also a polynomial function, but of two degrees less than the original. The extrema and points of inflections are simply the roots of lower degree polynomial functions.

The sign of the leading coefficient, a, determines towards what direction the graph of a polynomial function points at the extreme ends (sides) of the graph. When a is positive, the graph always rises on the right side towards positive infinity, i.e., to the upper right. When a is negative, the graph will fall on the right side towards negative infinity. The reason that the leading coefficient is the determining factor in whether a graph rises or falls on one side is due to the dominating influence of the highest degree term. For example, consider a cubic polynomial function such as $y = x^3 - 10x^2 - 20x - 30$. When x is very close to zero then the function is approximately -30. The other terms contribute very little to the calculation. (For example, when $x = 0.01$ then $y = -30.200999$.) But as x grows large in either the positive or the negative direction then the leading term, x^3, begins to overwhelm all the other terms in size. (For example, when $x = 1000$, x^3 is positive one billion, whereas the remaining three terms together are around negative ten million.)

Of course, another consideration is that when the leading coefficient is negative, there has been a reflection of the basic power function $y = x^n$, in addition to other transformations. For example, below are graphs of two fifth degree polynomials with different signed leading coefficients.



What happens on the left side of the graph is more interesting. Have you seen the pattern? When the degree is odd then the graph goes in the opposite direction as the right side, as you can see in the two examples above. Why? When x is negative then x^n is going to be negative also since a negative raised to an odd power is a negative.

Even degree functions move in the same direction on both ends. Why? That is because any real number raised to an even integer power is still positive. Of course, a negative leading coefficient will change the leading term to a negative. Therefore, with the exception of constant functions, polynomial functions with even degrees will rise on both ends if $a > 0$, and fall on both ends if $a < 0$.

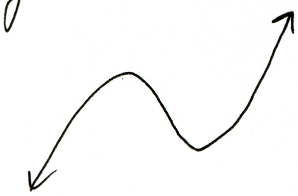
Topic 2: Sketching polynomial functions with single roots.

Sketch the graph of $y = (x - 4)(x + 3)(x - 1)$.

By "sketching" I mean draw a meaningful representation of the polynomial function indicating the roots and y–intercept. Other details, such as extrema and points of inflection, require methods of calculus. Thus our graphs in this section will lack some precision but will, nonetheless, give a good indication of function behavior.

What kind of polynomial is this? If you were to multiply everything out you would get this polynomial function: $y = x^3 - 2x^2 - 11x + 12$. There are only two coefficients that are useful for our purpose, the leading coefficient and the constant term. Rather than go to all that trouble of doing a lot of polynomial multiplications I prefer to find an easy way to figure out the two coefficients of interest so that I can ignore all that other stuff in between.

$$y = x^3 + \text{STUFF}$$



This indicates that if you multiply the x parts out then you will get x^3, meaning that we have a third degree polynomial with a positive leading coefficient, 1. Beneath it I sketched the basic shape for this kind of polynomial function.

Finding the y–intercept is a matter of plugging 0 in for x and simplifying.

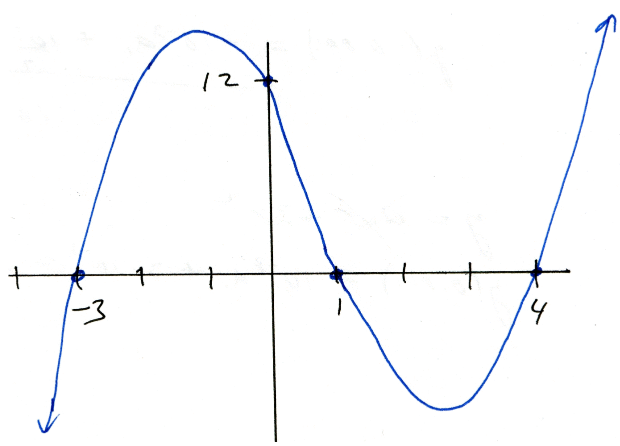
$$y(0) = (-4)(3)(-1) = 12$$

$$y\text{-INTERCEPT} = 12$$

Since the function is presented in factored form, then finding the roots is simply a matter of setting each factor equal to zero and solving. This you can do in your head.

$$\text{ROOTS ARE } x = 4, -3, 1$$

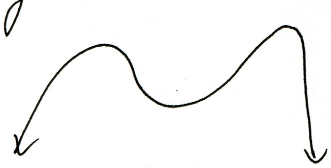
We now have enough information to draw a decent sketch.



Sketch the graph of $y = (x + 7)(2 - x)(x - 5)(x - 3)$.

This looks like a quartic. After multiplying the x parts out we get this.

$$y = -x^4 + \text{STUFF}$$



Some students will miss the negative sign inside the parentheses. In any event, this one's been reflected, and since it's an even degree polynomial, the graph points down at both ends.

$$y(0) = (7)(2)(-5)(-3) = 210$$

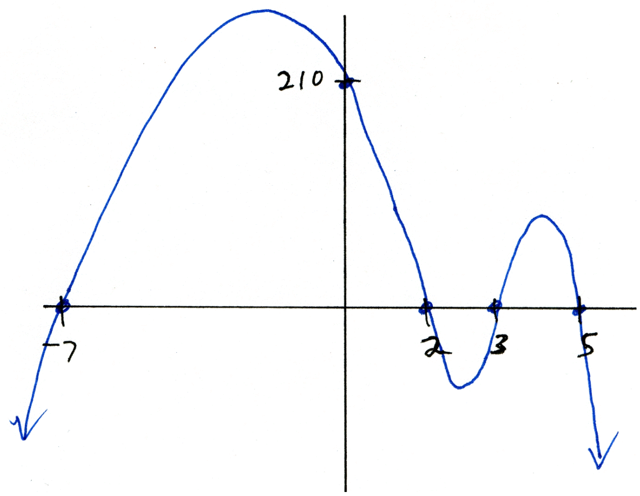
$$y\text{-INTERCEPT} = 210$$

The y–intercept is a large number compared to what we're used to dealing with in graphs, but we can scale the axes however we want.

Again, the roots are found by setting each factor equal to zero and solving.

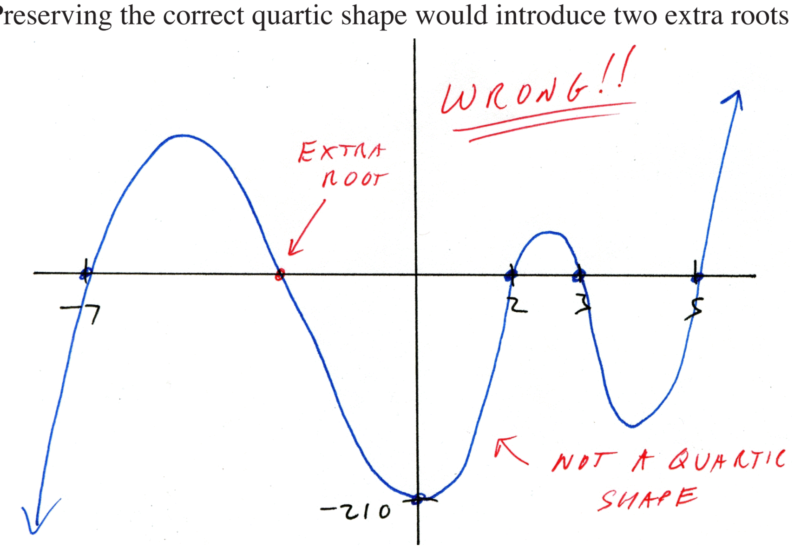
$$\text{ROOTS ARE } x = -7,\ 2,\ 5,\ 3$$

Now we can sketch the graph. The precise placement of the extrema are unknown without using calculus techniques, so as long as the graph meets the requirements of all the information we have uncovered then the sketch is enough.



If you are interested, the maximum on the left side is actually occurs around $x \approx -4.464$ and climbs up to nearly $y = 1{,}158$.

What happens if you get something wrong, how could you tell? Chances are if you get a wrong root or a wrong y–intercept then it will be impossible to graph it without violating part of the given information. For example, suppose that above you got $y = -210$ as the y–intercept by accident. That would be a common mistake. The only way to connect the dots in this case introduces an extra root that doesn't belong, and also changes the shape to a quintic. Preserving the correct quartic shape would introduce two extra roots.



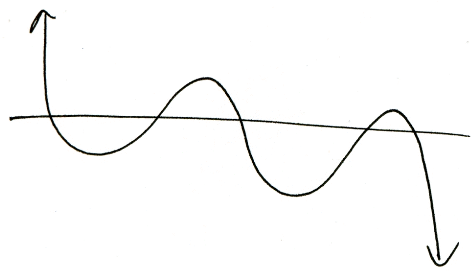
Sketch the graph of the polynomial function with the following information.

Fifth degree. y–intercept is -120.

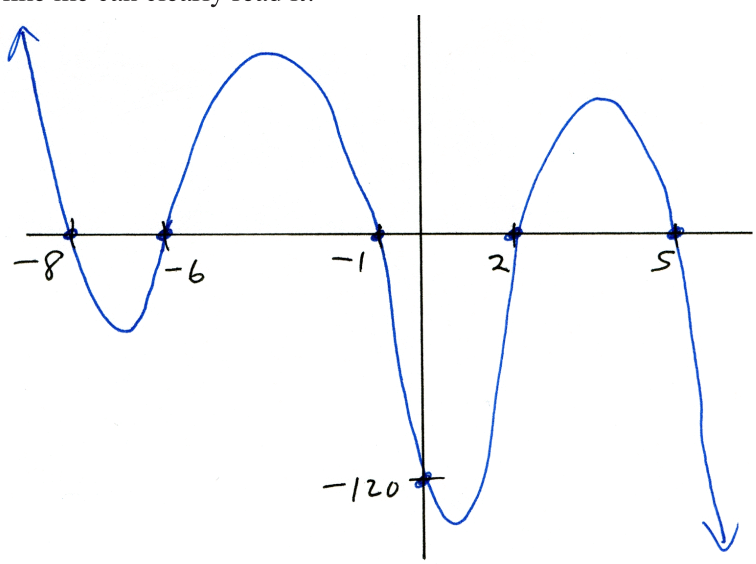
Leading coefficient is negative. Roots are $-8, -6, -1, 2, 5$.

This is one of the ways I get even with students who only want to push calculator buttons instead of learning how to think better!

All the information is here. I think it is a good idea to roughly sketch the shape before drawing a better graph. This is a quintic that is pointing down on the right hand side.



Now let's draw the sketch, labeling the pertinent information and drawing it large enough so that even someone with old eyes like me can clearly read it.



What is the polynomial function in the previous example?

This is a **curve fitting** question. Curve fitting means finding a function to fit given or observed criteria. It can be somewhat similar to mathematical modeling. Sometimes, statistical methods are used to fit data to a curve, especially when the data can only approximately fit a specific curve. That subject is known as regression analysis and is studied in statistics courses. In this problem, however, we are to find the precise function because the information will fit a quintic polynomial perfectly.

We can use the roots to construct the linear factors of our polynomial function. We don't yet know what the leading coefficient is, so I will write it as a factor, a, on the outside of the other factors.

$$y = a(x+8)(x+6)(x+1)(x-2)(x-5)$$

Now let's find out the *y*–intercept from the function we just wrote.

$$y(0) = a(8)(6)(1)(-2)(-5) = 480a$$

But the *y*–intercept is actually 120. This allows us to solve for *a*.

$$480a = -120$$

$$a = \frac{-120}{480} = -\frac{1}{4}$$

Finally, don't forget to write the answer.

$$y = \frac{-1}{4}(x+8)(x+6)(x+1)(x-2)(x-5)$$

Reinforcement Problems

1. Sketch the graph of $y = (x - 5)(x + 1)(x - 3)(x - 1)(x + 6)$.
2. Sketch the graph of $y = (x + 2)(x - 2)(4 - x)$.
3. Sketch the graph of $y = x(x + 1)(x - 4)(x - 2)$.
4. Sketch the graph of the polynomial function with the following information.

   Fourth degree. *y*–intercept is -360.

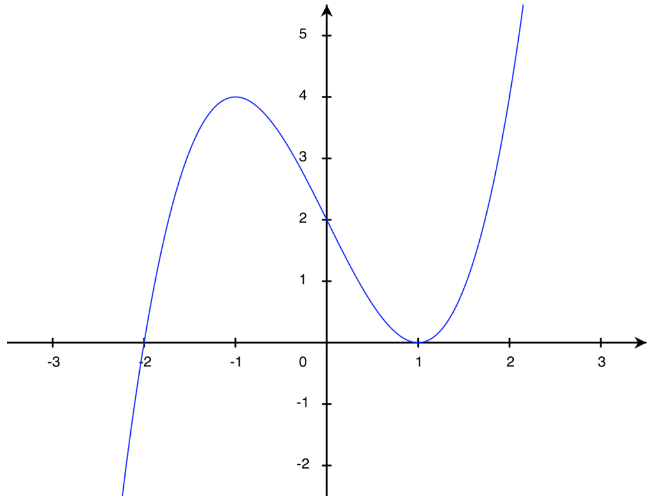
   Leading coefficient is negative. Roots are $2, -3, -4$ and 5.
5. Write the polynomial function described in problem 4.

Topic 3: Repeated roots.

Sketch the graph of $y = (x + 2)(x - 1)^2$.

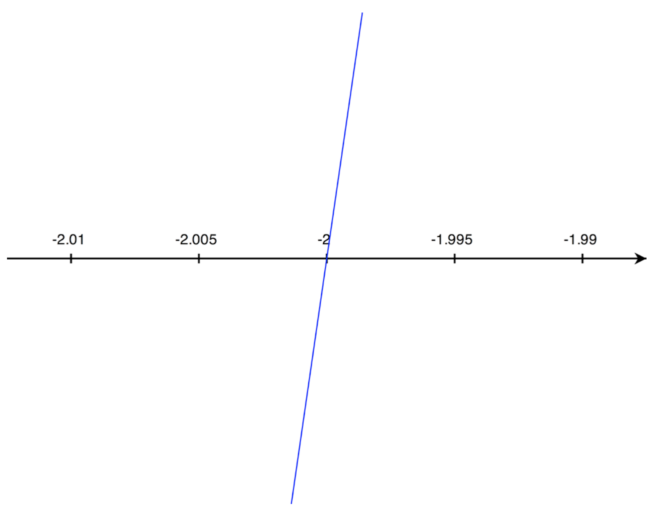
This is a cubic polynomial function with only two roots, $x = -2$ and 1. Because this function could also be written in this form, $y = (x + 2)(x - 1)(x - 1)$, you could also reasonably say that it has three roots, $x = -2, 1,$ and 1. Of course, this second way of looking at the problem has the root $x = 1$ repeated. In such situations we refer to these kinds of roots as **repeated roots**. To be more specific about how many times a root is repeated we identify it's **multiplicity**. In this example we say that $x = 1$ has "multiplicity 2."

Here is the graph of this function.



Notice how the graph just touches the x–axis at $x = 1$, but punches right through the axis at $x = -2$. The characteristic of just touching the axis means that the function graph is **tangent** at $x = 1$. You will see this for any root with an even multiplicity. Why is that?

Let's zoom in on our graph, which I have produced electronically this time in order to get believable precision. This is a close up in a neighborhood of the root $x = -2$. On this scale the graph looks like a straight line, i.e., linear.

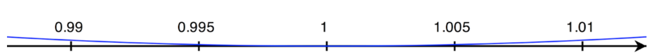


It's a steep line, at that, with a slope somewhat larger than 1. Calculus tells us that the slope at the root is exactly 9. We can approximate this slope without calculus. I deviated 0.001 on either side of -2 and calculated these points on the graph.

$$(-1.999, 0.008994001) \qquad (-2.001, -0.009006001)$$

The slope of the line connecting these two points is 9.000001, obviously a very close prediction to the actual slope of 9 at $x = -2$. The reason my calculation with the decimals did not exactly match the calculus result is due to the fact that although the graph looks like a straight line in this diagram, it really isn't. The curve is imperceptible to the eye, but it is there nonetheless.

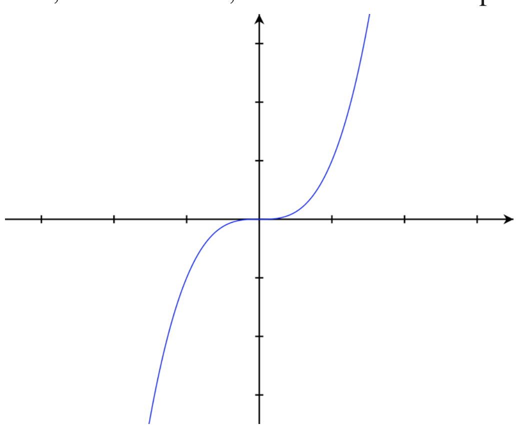
Now let's take a similar close up peek at our repeated root, $x = 1$.

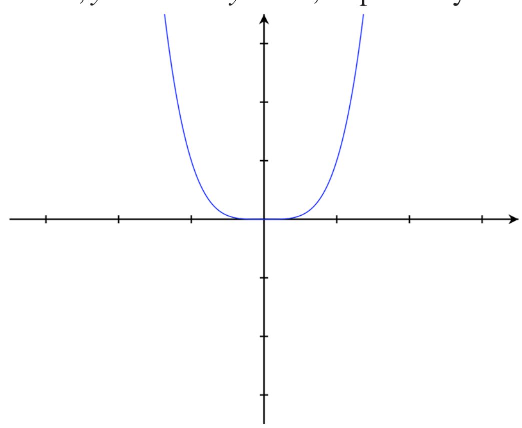


In both this and the previous diagram I rigged the scale so that both the x and y scales were about even, i.e., without distortion. If we could magnify a non-distorted graph of our polynomial function this is about what we would see. As you should expect from the main graph, the graph should curl down, touch at $x = 1$, then curl back up. Looks like a parabola here, doesn't it? It's very, very close, but technically, the graph is only a true parabola at the point $x = 1$. (That's paradoxical because a parabola is not a point, therefore the curve itself never attains parabola status along any length.) The approximate parabola in this neighborhood is $y = 3(x - 1)^2$, i.e., a basic parabola with a vertical stretch of 3. We can get that approximation via non-calculus means but it is very messy and I'm not inclined to write about it here. But if you are interested, e-mail me about it.

So what is happening with these repeated roots? Suppose you have a root r with multiplicity 1. This means that it comes from a linear term of the form $(x - r)$. The graph will resemble a line in the neighborhood of that root. If you have a root r of multiplicity 2 then it comes from a quadratic term of the form $(x - r)^2$. Thus the graph will resemble a parabola in the neighborhood of such a root. Similar patterns occur for multiplicities greater than 2, but you must keep in mind that the resemblance is to a power function of the simplest form, $y = x^n$.

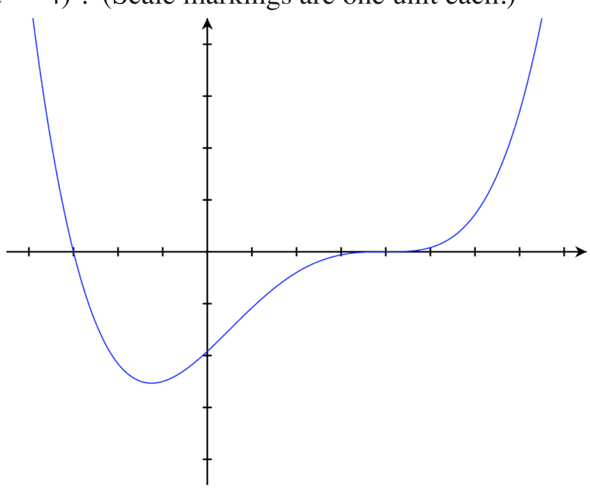
Here, as a reference, are the next two simple power functions, $y = x^3$ and $y = x^4$, respectively.





As you can see, the basic quartic resembles a parabola; the difference is that it is a little flatter near the origin and steeper as you move more than one unit away from the origin.

Here you can see how the cubic affects the behavior of a graph with a root of multiplicity 3. This is the graph of $y = 0.01(x + 3)(x - 4)^3$. (Scale markings are one unit each.)

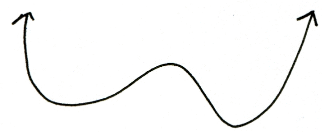


Topic 4: Sketching polynomial functions in general.

In this topic I will present a few more examples related to polynomial functions, but this time including repeated roots. There is nothing new to talk about. You should try these examples on your own before looking at the answers.

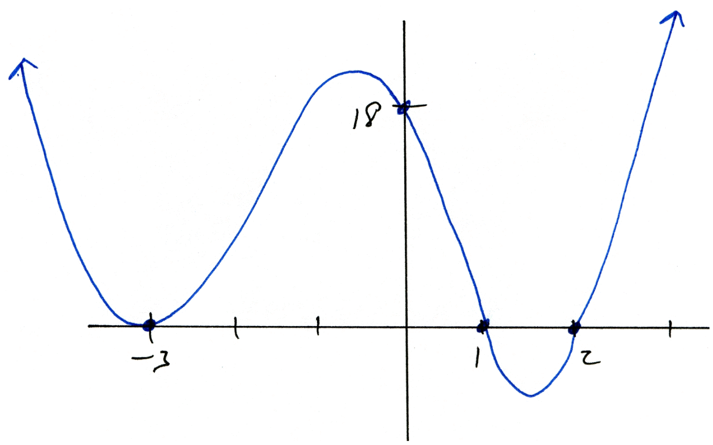
Sketch the graph of $y = (x - 1)(x + 3)^2(x - 2)$.

$y = x^4 + \text{STUFF}$



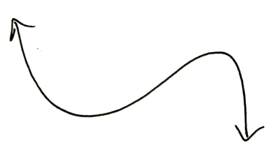
$y\text{-INTERCEPT} = y(0) = (-1)(3)^2(-2) = 18$

ROOTS ARE $x = 1, 2, -3, -3$.



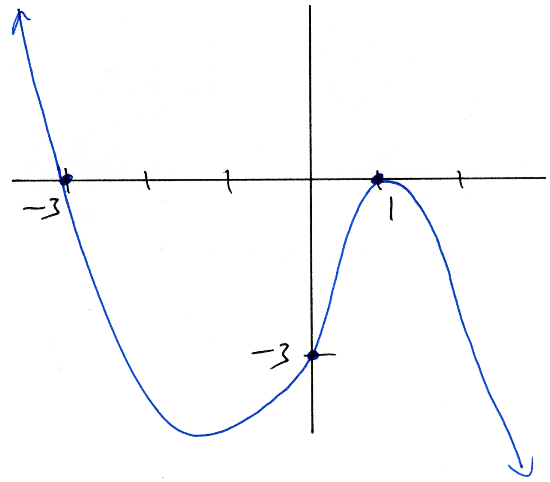
Sketch the graph of $y = (1 - x)^2(-3 - x)$.

$y = (-x)^2(-x) + \text{STUFF} = -x^3 + \text{STUFF}$



$y\text{-INTERCEPT} = y(0) = (1)^2(-3) = -3$

ROOTS ARE $x = 1, 1, -3$

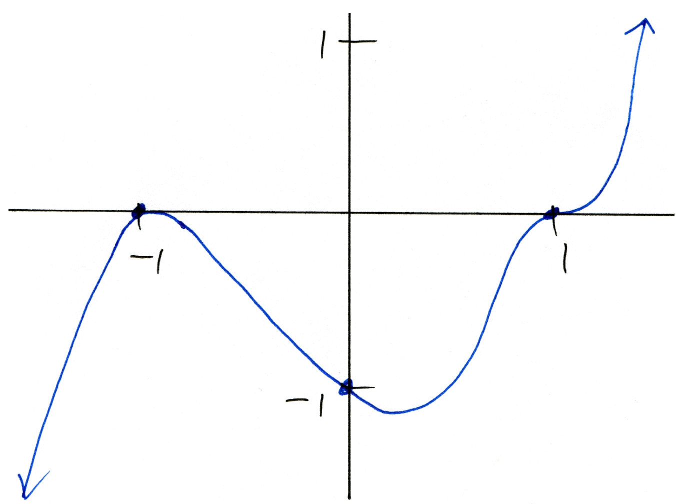


Sketch the graph of $y = (x - 1)^3 (x + 1)^2$.

$y = (x^3)(x^2) + \text{STUFF} = x^5 + \text{STUFF}$

$y(0) = (-1)^3(1)^2 = -1$ (y-INTERCEPT)

ROOTS ARE $x = 1, 1, 1, -1, -1$



Reinforcement Problems

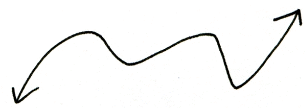
6. Sketch the graph of $y = (x - 2)(x + 4)^2 (x - 5)^2$.
7. Sketch the graph of $y = (x - 4)(x - 8)^2$.
8. Sketch the graph of $y = (5 - x)(x + 7)(x - 2)^2$.
9. Sketch the graph of the polynomial function with the following information.

   Fifth degree. y–intercept is -360.

   Leading coefficient is negative. Roots are $-1, 4, 2, -3$ and -3.
10. Write the polynomial function described in problem 9.

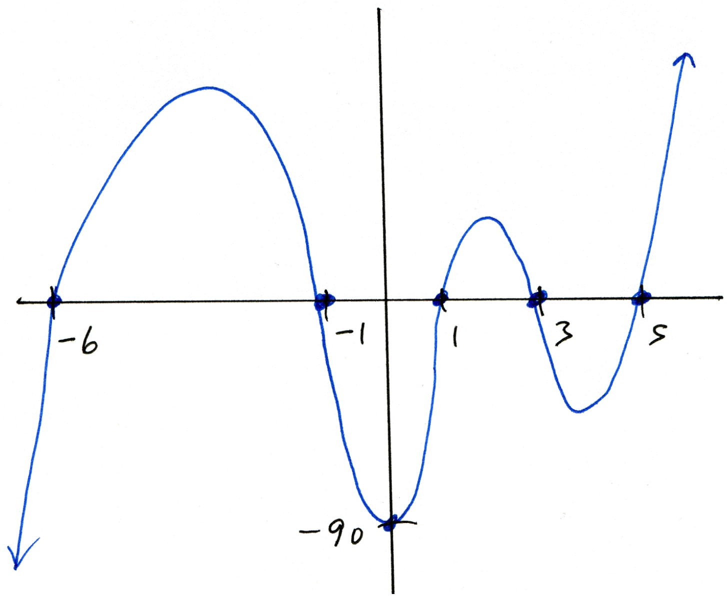
Solutions to Reinforcement Problems

1. $y = x^5 + \text{STUFF}$

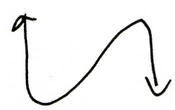


$y(0) = (-5)(1)(-3)(-1)(6) = -90$

ROOTS: $x = 5, -1, 3, 1, -6$

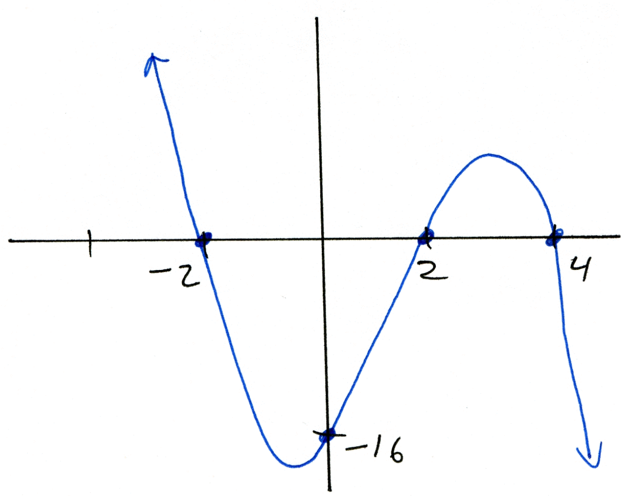


2. $y = -x^3 + \text{STUFF}$



$y(0) = (2)(-2)(4) = -16$

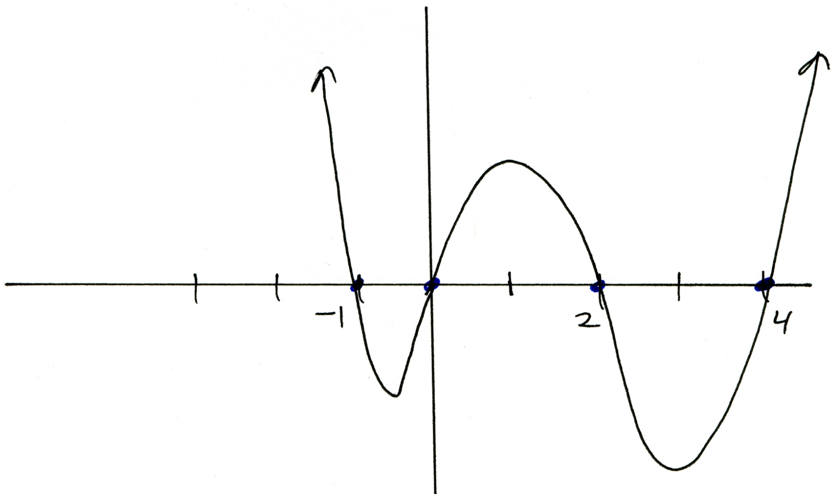
ROOTS ARE $x = -2, 2, 4.$



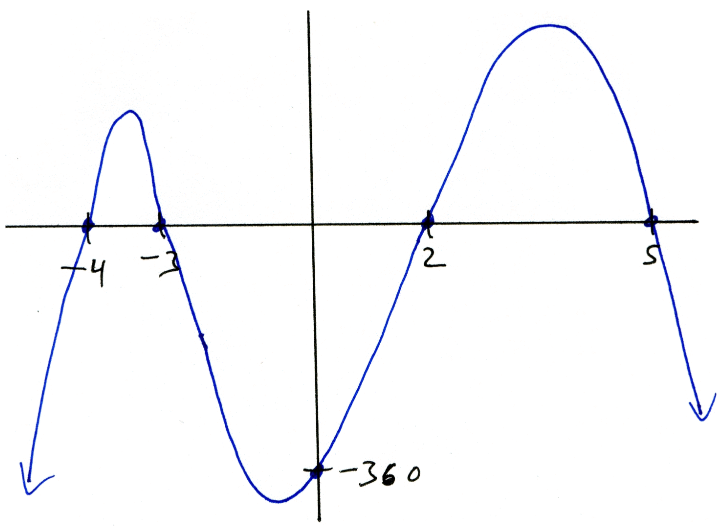
3. $y = x^4 + \text{STUFF}$

$y(0) = (0)(1)(-4)(-2) = 0$

ROOTS ARE $x = 0, -1, 4, 2$



4.



5. $y = a(x-2)(x+3)(x+4)(x-5)$

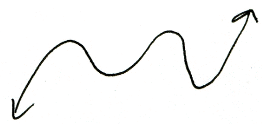
$y(0) = a(-2)(3)(4)(-5) = 120a = -360$

$$a = \frac{-360}{120} = -\frac{1}{3}$$

$$y = -\frac{1}{3}(x-2)(x+3)(x+4)(x-5)$$

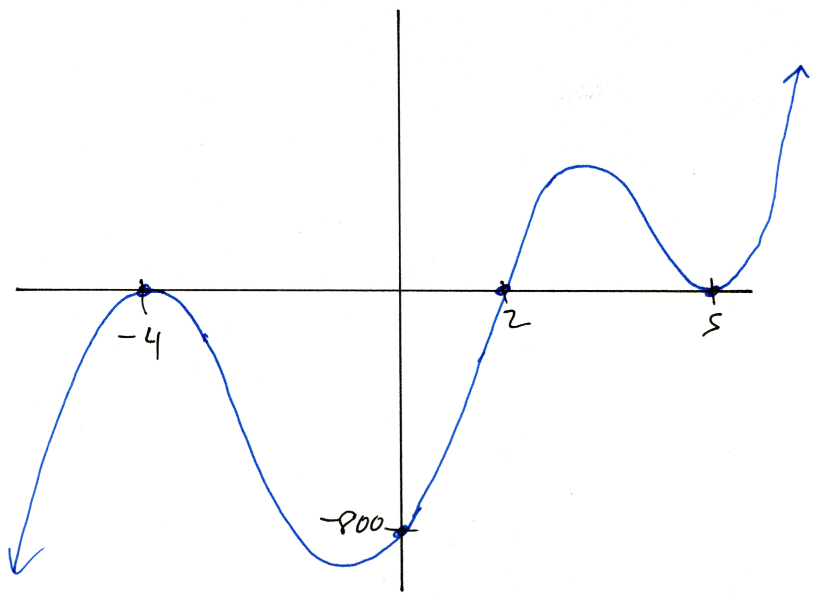
Solutions to Reinforcement Problems

6. $y = x^5 + \text{STUFF}$



$y(0) = (-2)(4)^2(-5)^2 = -800$

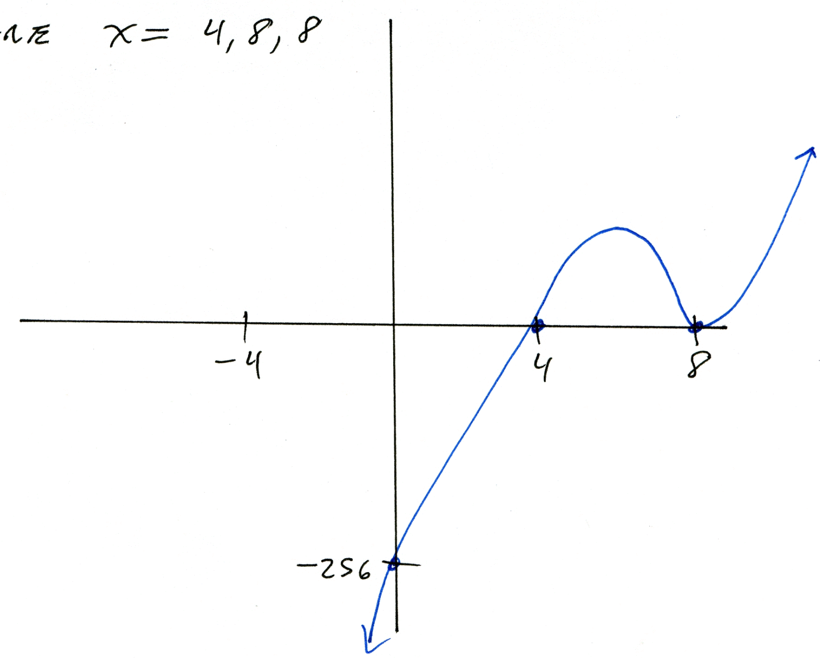
ROOTS: $x = 2, -4, -4, 5, 5$



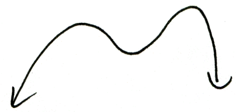
7. $y = x^3 + \text{STUFF}$

$y(0) = (-4)(-8)^2 = -256$

ROOTS ARE $x = 4, 8, 8$

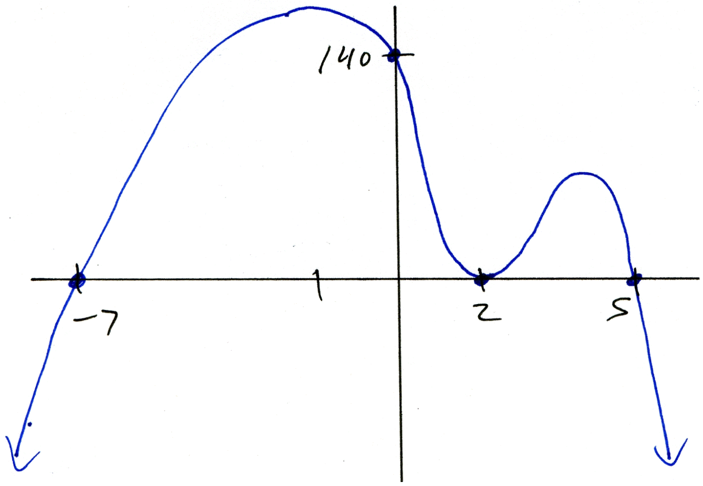


8. $y = -x^4 + \text{STUFF}$

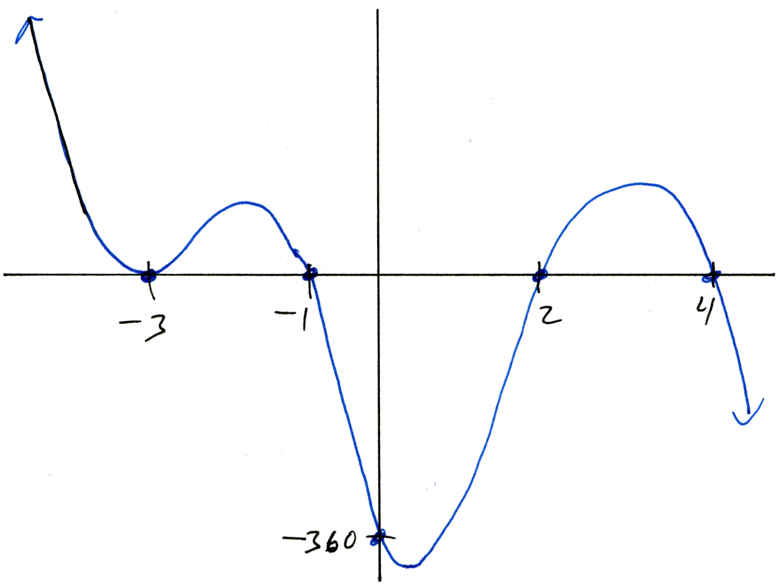


$y(0) = (5)(7)(-2)^2 = 140$

ROOTS: $x = 5, -7, 2, 2$



9.



10. $y = a(x+1)(x-4)(x-2)(x+3)^2$

$y(0) = a(1)(-4)(-2)(3)^2 = 72a = -360$

$a = \dfrac{-360}{72} = -5$

$y = -5(x+1)(x-4)(x-2)(x+3)^2$

Section 5.4 — Polynomial and Rational Inequalities

I like to skip ahead to this section in order to take advantage of the analysis you learned about polynomial functions. In a regular class I simply teach both sections at the same time. Your textbook author, for some reason, has chosen not to take advantage of this easy way to solve polynomial inequalities. Instead, the textbook shows you an alternative that can be more broadly applied to any inequalities. If you take calculus then you will learn this textbook's technique.

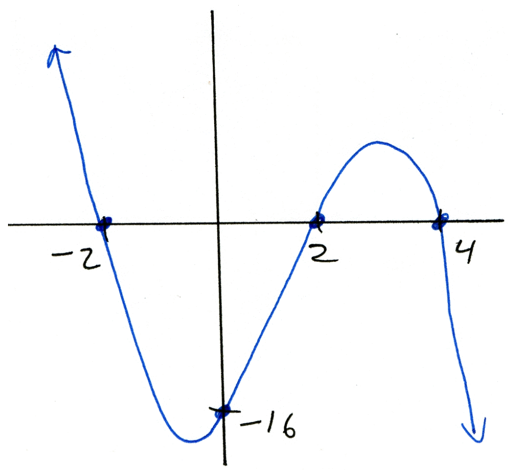
Topic 1: Polynomial inequalities.

Practice Problems: 3 – 19 odd; also, practice Section 4.5, problems 7 – 21 odd

Topic 1: Polynomial inequalities.

Solve: $(x + 2)(x - 2)(4 - x) \geq 0$.

I am mostly going to adapt the reinforcement problems from the last section. This problem comes from number 2. That problem asked you to sketch the graph of $y = (x + 2)(x - 2)(4 - x)$. Here is that sketch.

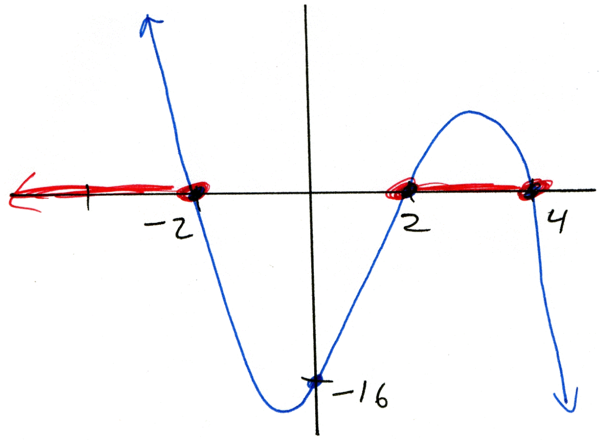


Suppose we consider our inequality problem a little differently, as solving this problem.

$$\text{Solve: } y = (x + 2)(x - 2)(4 - x) \geq 0.$$

Instead of being a problem on a number line with just x, we have restated the problem in two dimensions, the x–y plane. With the sketch of the polynomial function in hand, our problem becomes identifying where $y \geq 0$.

So where is y greater than or equal to zero? Very simple: above the x–axis (also including the x–axis.) So to work this inequality we only need to figure out which intervals in the domain, i.e., which intervals of x's, cause $y \geq 0$. I have marked these on the graph below.



If you are my student then I will prefer that you write the solution in interval notation.

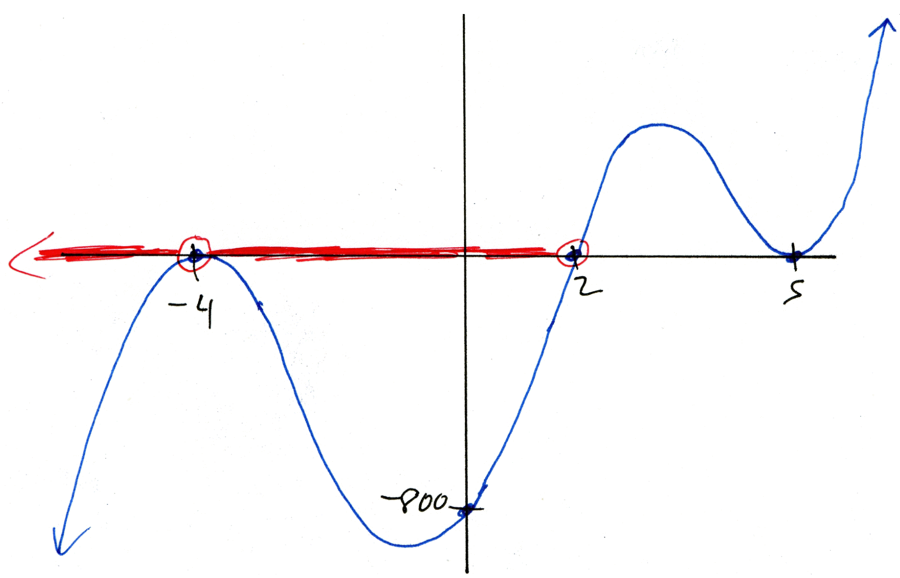
$$x \in (-\infty, -2] \text{ or } [2, 4]$$

Solve: $(x - 2)(x + 4)^2 (x - 5)^2 < 0$

This one is adapted from reinforcement problem number 6. As before, the first order of business is to sketch the related graph of $y = (x - 2)(x + 4)^2 (x - 5)^2$. It is not necessary to find the y–intercept for these inequality problems, though I think it is a good idea to do so as a partial check that you have graphed the function correctly.

This one is a little unusual. The function has a repeated root of multiplicity 2 at $x = -4$, so the graph just touches the x–axis there. Since the x–axis is where $y = 0$, then so would $(x - 2)(x + 4)^2 (x - 5)^2 = 0$ at this point. Thus we must exclude $x = -4$ from the solution to the inequality.

Again, I will mark the intervals on the x–axis of the graph, this time where the graph is below the axis, where $y < 0$.



And here is the answer.

$$x \in (-\infty, -4) \text{ or } (-4, 2)$$

There really is nothing much more to say about solving polynomial inequalities. The main purpose in analyzing polynomial functions was to give you experience at analytical thinking. There are no jobs out there graphing polynomial functions or solving polynomial inequalities, of course, but there are great jobs for people who have developed their capacity for analytical thinking, so that is why a course such as College Algebra is a staple of the core curriculum in higher education.

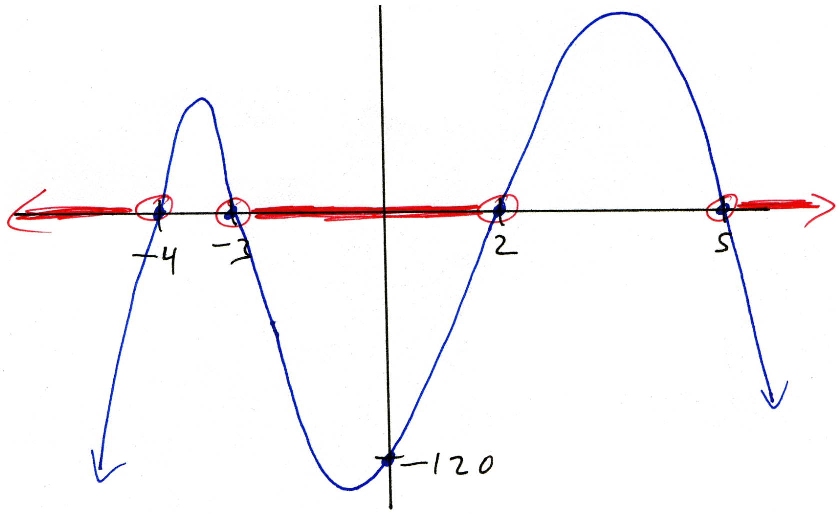
As for the polynomial inequalities, we have found a use for the analysis we did on polynomial functions. Thus you can sketch the inequality expression as if it was the meat of a polynomial function, then simply look at the graph in order to flush out the proper intervals.

Reinforcement Problems

1. Solve: $(x + 4)(2 - x)(x - 5)(x + 3) < 0$
2. Solve: $(x + 1)(x - 2)(x + 3)^2(4 - x) \geq 0$

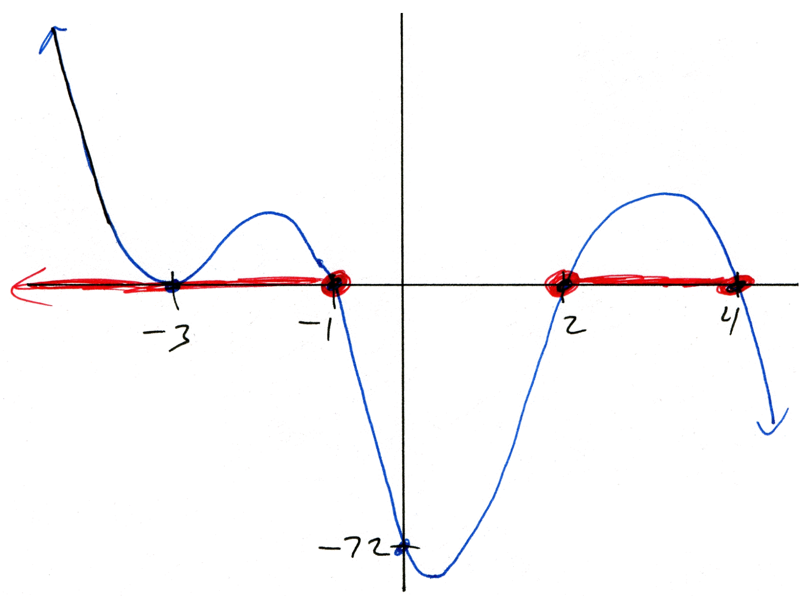
Solutions to Reinforcement Problems

1.



$x \in (-\infty, -4)$ OR $(-3, 2)$ OR $(5, \infty)$

2.



$x \in (-\infty, -1]$ OR $[2, 4]$

Section 5.5 — The Real Zeros of a Polynomial Function

Up to now our study of polynomial functions has only included those that were already factored. What happens if you have one that is not factored? We shall explore a technique for factoring polynomials that lend themselves to factoring in this section. In a sense, being able to factor means being able to find roots, but it also depends on what you mean by factoring. In algebra we normally only factor polynomials into linear terms of the form $(x - r)$ where r is an integer, or into the form $(ax - b)$ where both a and b are integers. Considered together, mathematicians say that this is factoring over the field of rational numbers (fractions). If you extend the idea of factoring to where r could be any real number then we are at an impasse. I will discuss that situation at the end of this section. Our primary purpose is to explore ways to find roots or factor polynomials when those roots are rational numbers.

Topic 1: Synthetic division.
Topic 2: Using synthetic division to evaluate and factor polynomial functions.
Topic 3: The Rational Roots Theorem.
Topic 4: Applying the Rational Roots Theorem.
Topic 5: Solving polynomial equations in general.
Topic 6: The Fundamental Theorem of Algebra

Practice Problems: 11 – 19, 33 – 55 all odd

Topic 1: Synthetic division.

In an earlier algebra you may have learned the process of long division of polynomials. It is a tedious process. If your divisor has the form $(x - r)$ then there is a streamlined process called **synthetic division**. It resembles long division but is much faster. Note that synthetic division does not work when your divisor has the form $(ax - b)$ where $a \neq 1$. We can work around that stricture later.

Divide: $\left(x^3 - 2x^2 - 5x + 6\right) \div (x + 2)$

Here is how the synthetic division is set up.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & & & \\ \hline \end{array}$$

The four numbers inside the box obviously come from the coefficients of the dividend polynomial (the cubic), but why -2 on the outside? The root of $(x + 2)$ is -2. If you want to think of it as the opposite signed number of the number in the divisor that is fine, but I encourage you to think of it as the root of the divisor since that terminology will persist as we apply synthetic division in this section.

Now we shall begin the process by bringing the first number inside the box down.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & & & \\ \hline & 1 & & & \end{array}$$

Now we're going to multiply the root, -2, by the number we just brought down, 1, and write that result beneath the second number in the row inside, also -2. We then add the second column to get -4.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & & \\ \hline & 1 & -4 & & \end{array}$$

From here we continuie to repeat these steps. This time we multiply the new number, -4, by the root, -2 and then write that result in the third column beneath the -5. We then add the third column.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & \\ \hline & 1 & -4 & 3 & \end{array}$$

One more repetition of this process completes the calculations. Sometimes when I write this on the board I clutter the diagram with arrows in order to show the flow of the work.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

The last number, 0, indicates that there is no remainder after doing this division. The first three numbers in the row represent the coefficients of the quotient polynomial, which is one degree less than the dividend. In other words, we divided a third degree polynomial by a first degree polynomial and got a second degree polynomial. This is consistent with division by exponent numbers (subtracting exponents).

So here is the outcome of this division.

$$\frac{x^3-2x^2-5x+6}{x+2} = x^2-4x+3$$

This result can be verified by multiplying it back. By this, I mean that we can show that this multiplication works out.

$$\left(x^2 - 4x + 3\right)\left(x + 2\right) = x^3 - 2x^2 - 5x + 6$$

Let's do another example, this time one with a remainder, and then we'll discuss this synthetic division business.

Divide: $\left(x^4 + 3x^2 - 6x - 7\right) \div (x - 3)$

This time we are "missing" the x^3 term. It's really there in the dividend, but as $0x^3$. In order for this process to work properly we must fill in the gaps with placeholder zeros, just as we do with ordinary numbers such as 5,024. Thus we should regard this problem as $\left(x^4 + 0x^3 + 3x^2 - 6x - 7\right) \div (x - 3)$ for purposes of synthetic division. Here is the synthetic division set up.

$$\begin{array}{c|ccccc} 3 & 1 & 0 & 3 & -6 & -7 \\ & & & & & \\ \hline & & & & & \end{array}$$

I'm going to describe this similar to how I described the first problem. First of all, we are dividing by $(x - 3)$, so the root of $(x - 3)$ is 3. That is why 3 is written on the left. Once again, we must account for the missing power in the polynomial dividend with a placeholder 0, otherwise you will get worthless junk for an answer.

Now let's bring the first number inside the box down, 1, and multiply it by the root and write that result in the second column. We add the two numbers in the second column and write that sum at the bottom.

$$\begin{array}{c|ccccc} 3 & 1 & 0 & 3 & -6 & -7 \\ & & 3 & & & \\ \hline & 1 & 3 & & & \end{array}$$

Next we multiply the number we just wrote, 3, by the root, 3, write that in the next column and add down.

$$\begin{array}{c|ccccc} 3 & 1 & 0 & 3 & -6 & -7 \\ & & 3 & 9 & & \\ \hline & 1 & 3 & 12 & & \end{array}$$

Same process as before. We mulitply the last number written, 9, by the root, 3, write 27 in the next column and add down.

$$\begin{array}{c|ccccc} 3 & 1 & 0 & 3 & -6 & -7 \\ & & 3 & 9 & 36 & \\ \hline & 1 & 3 & 12 & 30 & \end{array}$$

We repeat the same process one more time by shifting over a column, ending with this.

$$\begin{array}{c|ccccc} 3 & 1 & 0 & 3 & -6 & -7 \\ & & 3 & 9 & 36 & 90 \\ \hline & 1 & 3 & 12 & 30 & 83 \end{array}$$

This time we ended up with a remainder after the division, 83. Here is how to write the quotient so that it makes sense in an algebraic way.

$$\frac{x^4+3x^2-6x-7}{x-3} = x^3+3x^2+12x+30+\frac{83}{x-3}$$

The following would be a wrong way to write this answer.

WRONG! $$\frac{x^4+3x^2-6x-7}{x-3} = x^3+3x^2+12x+30\frac{83}{x-3}$$

Literally, the last term is just what it reads, 30 times the fraction $83/(x-3)$. Some students will write the remainder of the division this way because that's how it is done in arithemetic; e.g., $7 \div 3 = 2\,{}^1/_3$. Arithmetic is the only place in mathematics where one number next to another without anything in between means to add, not multiply. In algebra it is a mistake to write it this way.

Why does synthetic division work? I don't exactly know because I haven't attempted a proof; it's looks messy and not terribly interesting from my point of view. Nevertheless, in playing around with it I do think this process is strongly related to how we divide by long division in arithmetic, provided that you regard polynomials as a "base x" number system. We ordinarily work in a base 10 system where every number is expressed as a string of digits that are coefficients of powers of 10. A polynomial is the same thing, except that you are working with powers of x instead of 10.

Reinforcement Problems

1. Divide: $\left(x^3 + 6x^2 - 4x + 9\right) \div (x-2)$
2. Divide: $\left(x^5 - 2x^4 - 4x^2 + 3x - 5\right) \div (x+3)$

Topic 2: Using synthetic division to evaluate and factor polynomial functions.

One advantage of synthetic division is that it is a convenient tool for rapidly evaluating a polynomial function. I will introduce this with examples, and then we'll look into why this works.

Let $p(x) = 2x^4 - 3x^3 + 5x^2 + 7x - 10$. Find $p(-2)$.

It is fairly customary in algebra to represent polynomial functions using p and q, instead of f and g, but that's not a binding rule by any means.

First I will work this out the conventional way so that we'll know what answer we're supposed to get.

$$\begin{aligned} p(-2) &= 2(-2)^4 - 3(-2)^3 + 5(-2)^2 + 7(-2) - 10 \\ &= 2(16) - 3(-8) + 5(4) - 14 - 10 \\ &= 32 - 24 + 20 - 14 - 10 = 52 \\ p(-2) &= 52 \end{aligned}$$

Now I will show you the alternative calculation. Let's consider -2 as the root of the linear expression $(x - (-2)) = x + 2$. Look what happens when we divide the polynomial function $p(x)$ by $(x + 2)$.

$$\begin{array}{r|rrrrr} -2 & 2 & -3 & 5 & 7 & -10 \\ & & -4 & 14 & -38 & 62 \\ \hline & 2 & -7 & 19 & -31 & 52 \end{array}$$

The remainder is 52, as you can see, which equals $p(-2)$. This works all the time, in fact. The remainder under synthetic division is always the value of the dividend polynomial function evaluated at the root of the divisor.

Let's try it again.

Let $p(x) = x^5 - 3x^4 + 7x^2 - 12x - 28$. Find $p(4)$.

Since we will regard 4 as the root of the divisor under synthetic division, we do not change its sign.

$$\begin{array}{r|rrrrrr} 4 & 1 & -3 & 0 & 7 & -12 & -28 \\ & & 4 & 4 & 16 & 92 & 320 \\ \hline & 1 & 1 & 4 & 23 & 80 & 292 \end{array}$$

This ends up being a convenient tool for factoring. Consider the implications of when the remainder is zero. A zero remainder in arithmetic occurs when one number divides into the other evenly with no remainder. In a sense, it is no different with polynomials. We will be looking for numbers that give us zero remainders.

Factor: $x^3 - 2x^2 - 5x + 6$

There are no general methods for factoring polynomials of degree 3 or higher other than what you will see here, or something closely related involving division. We did see a specialized method of factoring called grouping earlier in this book, but that is only possible under very limited circumstances.

Suppose that you knew that -2 is a root of this polynomial. We can verify that we get a zero remainder under synthetic division. In fact, this is the first synthetic division I showed you, at the beginning of this section.

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -5 & 6 \\ & & -2 & 8 & -6 \\ \hline & 1 & -4 & 3 & 0 \end{array}$$

But using -2 in this synthetic division is equivalent to dividing by $(x - (-2)) = (x + 2)$. In a sense, we have divided $(x + 2)$ out of $(x^3 - 2x^2 - 5x + 6)$ giving us $(x^2 - 4x + 3)$. Thus we have the first step in factoring this cubic polynomial.

$$x^3 - 2x^2 - 5x + 6 = (x+2)(x^2 - 4x + 3)$$

The remaining quadratic could also be factored if we knew one of its roots, but it is easy enough to factor in the conventional way. Thus we can complete the problem.

$$= (x+2)(x-1)(x-3)$$

Now I can show you why all this works the way it does. All this is related to an old theorem, misnamed the Division Algorithm. (It's a theorem, something that can be proved, not an algorithm, which is simply a set of rules for performing a task.) I will demonstrate it with the first example in this topic.

Let $p(x) = 2x^4 - 3x^3 + 5x^2 + 7x - 10$. Find $p(-2)$.

Suppose that we divide this polynomial by $x + 2$. Here is the synthetic division shown earlier.

$$\begin{array}{r|rrrrr} -2 & 2 & -3 & 5 & 7 & -10 \\ & & -4 & 14 & -38 & 62 \\ \hline & 2 & -7 & 19 & -31 & 52 \end{array}$$

This means that the quotient polynomial is $2x^3 - 7x^2 + 19x - 31$, and the remainder is 52. The following can be verified.

$$p(x) = 2x^4 - 3x^3 + 5x^2 + 7x - 10 = (2x^3 - 7x^2 + 19x - 31)(x + 2) + 52$$

Look what happens when you consider $p(-2)$ in the above equation. It creates a zero in this multiplication, $(2x^3 - 7x^2 + 19x - 31)(x + 2)$, because $(x + 2)$ is zero when $x = -2$. Thus the right side equals to 52, which means that $p(-2) = 52$, the remainder after doing the synthetic division by $(x + 2)$.

A little more formally, the Division Algorithm tells us that when you take a polynomial $p(x)$ and divide it by $(x - a)$, you will get a quotient polynomial $q(x)$, and a remainder R such that the following is true.

$$p(x) = q(x)(x - a) + R.$$

If now we consider $p(a)$, then we'll have the following.

$$p(a) = q(a)(a - a) + R = q(a) \bullet 0 + R = R.$$

That's why the remainder after synthetic division is the value of the polynomial with the root of the divisor plugged in.

Where are we headed with all of this?

Our ultimate goal in this section is to use synthetic division to help us find the roots of a polynomial. Thus far we have seen that synthetic division is a fast way of dividing polynomials when the divisor is $(x - r)$, that synthetic division is a tool for quickly evaluating a number substituted into a polynomial, and that if you knew a root to a polynomial then you could start to factor the polynomial and find other roots.

Our problem in finding the roots of a polynomial, then, amounts to knowing where to start in factoring it. That's what the Rational Roots Theorem will settle for us in the next topic. After learning that then we will be able to fully use the factoring capabilities available to us through synthetic division.

Reinforcement Problems

3. Suppose that $p(x) = x^3 + 3x^2 + 6x - 9$. Use synthetic division to find $p(-2)$.

4. Suppose that $p(x) = -2x^4 + 5x^3 - 4x^2 + 7x - 6$. Use synthetic division to find $p(3)$.

Topic 3: The Rational Roots Theorem.

The Rational Roots Theorem is an amazing theorem that tells us that if a polynomial with integer coefficients has any rational roots (roots that are either integers or fractions), what form those roots can possibly take. This limits the search for possible roots.

I will first introduce this theorem with an example.

List the possible rational roots of $-2x^4 + 5x^3 - 4x^2 + 7x - 6$.

The Rational Roots Theorem states that any rational roots, if there are any, must take the following form.

$$\frac{\text{FACTORS OF } -6}{\text{FACTORS OF } -2}$$

In words, this means that if there is a root in the form of an integer or a fraction, then it can only take the form of a factor of the constant term divided by a factor of the leading coefficient. In this example, I will list the possible factors, using the ± symbol in order to indicate both negatives and positives.

$$\frac{\text{FACTORS OF } -6}{\text{FACTORS OF } -2} = \frac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2}$$

This makes for a lot of possibilities, but not as many as may appear because there are some of duplicates, such as 3/1 and 6/2. Here is the complete list.

$$1, -1, 2, -2, 3, -3, 6, -6, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, -\frac{3}{2}$$

By amazing coincidence, this polynomial has two rational roots. I say amazing coincidence because I made that polynomial up at random, and usually you don't get rational roots when you make up polynomials at random. Let's find them, shall we?

I will work through the above list of possibilities. First I will try $x = 1$ to see if it is a root.

$$\begin{array}{r|rrrrr} 1 & -2 & 5 & -4 & 7 & -6 \\ & & -2 & 3 & -1 & 6 \\ \hline & -2 & 3 & -1 & 6 & 0 \end{array}$$

It is a root because the remainder is 0. And here is the nice part. Everytime you uncover a root you can reduce the size of the polynomial you are using, and often the list of possibilities. In this case, the fact that $x = 1$ is a root of the polynomial, then we can factor $(x - 1)$ out.

$$\begin{aligned} &-2x^4 + 5x^3 - 4x^2 + 7x - 6 \\ &= (x-1)(-2x^3 + 3x^2 - x + 6) \end{aligned}$$

Now we can focus on the quotient polynomial, $-2x^3 + 3x^2 - x + 6$, which we got through the synthetic division process. Unfortunately, its list of possible rational roots is just as long as the original, factors of 6 divided by factors of -2. So I will continue to work off the original list of possible rational roots. We need to see if $x = 1$ is a root again because it could be a repeated root.

$$\begin{array}{r|rrrr} 1 & -2 & 3 & -1 & 6 \\ & & -2 & 1 & 0 \\ \hline & -2 & 1 & 0 & 6 \end{array}$$

Well, it isn't a repeated root because the remainder was not zero. So we move on to the next number in the list, -1, and see if it is a root.

$$\begin{array}{r|rrrr} -1 & -2 & 3 & -1 & 6 \\ & & 2 & -5 & 6 \\ \hline & -2 & 5 & -6 & 12 \end{array}$$

And so -1 is not a root. On to the next number in our list of possible roots, 2.

$$\begin{array}{r|rrrr} 2 & -2 & 3 & -1 & 6 \\ & & -4 & -2 & -6 \\ \hline & -2 & -1 & -3 & 0 \end{array}$$

Bingo! We see that $x = 2$ is also a root. That allows us to rewrite the original polynomial with both factors we have uncovered.

$$\begin{aligned} &-2x^4 + 5x^3 - 4x^2 + 7x - 6 \\ &= (x-1)(-2x^3 + 3x^2 - x + 6) \\ &= (x-1)(x-2)(-2x^2 - x - 3) \end{aligned}$$

I'm inclined to factor a negative out of the remaining term.

$$= -(x-1)(x-2)(2x^2 + x + 3)$$

At this point it would make sense to try to factor or otherwise solve for the roots of the remaining quadratic term, $2x^2 + x + 3$, rather than continue with using synthetic division as a trail and error method. It will not factor. The discriminant of this quadratic, $b^2 - 4ac$, equals a negative number, -23, so the roots are complex conjugates. If we wanted to, we could find them and then list all the roots of the original polynomial.

THE ROOTS ARE $x = 1,\ 2,\ \dfrac{-1 \pm i\sqrt{23}}{4}$.

List the possible rational roots of $6x^7 - 9x^5 + 4x^4 - x^2 + 9$.

As before, the Rational Roots Theorem tells us that any possible rational roots must have the form of a factor of the constant term, 9, divided by a factor of the leading coefficient, 6.

$$\text{POSSIBLE ROOTS: } \frac{\text{FACTORS OF } 9}{\text{FACTORS OF } 6} = \frac{\pm 1, \pm 3, \pm 9}{\pm 1, \pm 2, \pm 3, \pm 6}$$

I will list all of these individually, as I expect you would, reducing fractions as needed and eliminating duplicates.

$$= 1, -1, 3, -3, 9, -9, \frac{1}{2}, \frac{-1}{2}, \frac{1}{3}, \frac{-1}{3}, \frac{1}{6}, \frac{-1}{6}, \frac{3}{2}, \frac{-3}{2}, \frac{9}{2}, \frac{-9}{2}$$

I checked, by the way. None of these are the actual roots of the original seventh degree polynomial, which means that all seven roots are either irrational or complex numbered. In fact, there is one irrational root, approximately -1.459, and three pairs of complex conjugates.

I am going to present a proof of the Rational Roots Theorem for cubic polynomials that you should be able to follow without much trouble. The full theorem is merely a more abstract looking extension of what I will show you.

Suppose that we want to solve this cubic equation in which all the coefficients (the subscripted numbers, a_3, a_2, a_1 and a_0) are all integers.

$$a_3 x^3 + a_2 x^2 + a_1 x + a_0 = 0$$

Suppose further that there may exist a rational root, what must it look like in terms of the coefficients? Rational numbers are integers or fractions of integers, so we could write it as a reduced fraction in terms of integers p and q.

$$x = \frac{p}{q}$$

If this number is a root, then it solves the equation. So, let's substitute it in.

$$\frac{a_3 p^3}{q^3} + \frac{a_2 p^2}{q^2} + \frac{a_1 p}{q} + a_0 = 0$$

Finally, let's clear the fractions by multiplying everything by q^3, and then we'll take stock of the situation.

$$a_3 p^3 + a_2 p^2 q + a_1 p q^2 + a_0 q^3 = 0$$

I don't know if history records who first proved the Rational Roots Theorem. People probably noticed in the Middle Ages, maybe even further back, that there were consistent patterns involving rational roots of polynomials and their coefficients. At some point someone must have proceeded along the lines of what I've shown you and, after some thought, hit upon the key to this. In hindsight, it is easier to explain the breakthrough, which is this. Each term in the above equation is an integer. The first three terms can be evenly divided by the integer p

and the last three terms can be divided by q. Let's start with p first.

$$\frac{a_3p^3 + a_2p^2q + a_1pq^2 + a_0q^3}{p} = \frac{0}{p}$$

$$a_3p^2 + a_2pq + a_1q^2 + \frac{a_0q^3}{p} = 0$$

The first three terms are still integers, and they add together with the last term to give 0, then the last term, although appearing to be a fraction, must also be an integer. Remember that we decided that p/q is a reduced fraction (since you can always reduce a fraction of integers), so p and q have no common factors, which means that p must be able to divide evenly into a_0. This means that p must be a factor of a_0, the constant term.

By similar means we can show that q must be a factor of a_3. This time we divide that equation by q instead.

$$\frac{a_3p^3 + a_2p^2q + a_1pq^2 + a_0q^3}{q} = \frac{0}{q}$$

$$\frac{a_3p^3}{q} + a_2p^2 + a_1pq + a_0q^2 = 0$$

This time the last three terms are integers, so the first term must also be an integer, and since p and q have no common factors, then q must divide evenly into a_3, the leading coefficient.

This proves the Rational Roots Theorem for cubic polynomials. The general proof is merely a generalization of what was shown above.

Reinforcement Problems

5. Suppose that $p(x) = 4x^3 + 3x^2 + 5x - 6$. List all the possible rational roots.

6. Suppose that $p(x) = -5x^4 + 2x^3 - 4x^2 + 7x - 15$. List all the possible rational roots.

Topic 4: Applying the Rational Roots Theorem.

Find all the roots of $x^4 - 2x^3 - 21x^2 + 22x + 40$.

Let's list all the possible rational roots, which in this case must be just integers.

$$\text{POSSIBLE ROOTS} = \frac{\pm 1, \pm 2, \pm 4, \pm 5, \pm 8, \pm 10, \pm 20, \pm 40}{\pm 1}$$

$$= 1, -1, 2, -2, 4, -4, 5, -5, 8, -8, 10, -10, 20, -20, 40, -40$$

The good news is that it would be very unusual for the polynomial to have roots much larger than any of the coefficients. So we are most likely to find our roots on the left side of this list. From here there is a lot of trial an error as we try each number to see if it is a root. The number 1 is not a root, but -1 is.

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & -21 & 22 & 40 \\ & & 1 & -1 & -22 & 0 \\ \hline & 1 & -1 & -22 & 0 & 40 \end{array} \qquad \begin{array}{r|rrrrr} -1 & 1 & -2 & -21 & 22 & 40 \\ & & -1 & 3 & 18 & -40 \\ \hline & 1 & -3 & -18 & 40 & 0 \end{array}$$

Each time you find a root this reduces the degree of the polynomial you need to find roots for.

$$x^4 - 2x^3 - 21x^2 + 22x + 40$$

$$= (x+1)(x^3 - 3x^2 - 18x + 40)$$

We still need to check -1 again just in case it is a repeated root. It is not, but 2 is a root.

$$\begin{array}{r|rrrr} -1 & 1 & -3 & -18 & 40 \\ & & -1 & 4 & 14 \\ \hline & 1 & -4 & -14 & 54 \end{array} \qquad \begin{array}{r|rrrr} 2 & 1 & -3 & -18 & 40 \\ & & 2 & -2 & -40 \\ \hline & 1 & -1 & -20 & 0 \end{array}$$

This allows us to refactor the original polynomial again.

$$= (x+1)(x-2)(x^2 - x - 20)$$

Once we get down to a quadratic term then we can abandon the trail and error search using synthetic division and just solve the quadratic. This one factors.

$$= (x+1)(x-2)(x+4)(x-5)$$

So we can now write the answer to the problem.

$$\text{THE ROOTS ARE } x = -1, 2, -4, 5.$$

Find all the roots of $24x^3 - 26x^2 - 13x + 10$.

It would be unusual to have to use fractions in this process. The only time it would be required is if you have three or more roots that are fractions. So normally you can focus only on the possible integer roots and reduce the polynomial you need to factor down to a quadratic through synthetic division. Just so that you would see one, I rigged this problem to have three fractional roots. One of them is $-2/3$. Here is how the synthetic division looks.

$$\begin{array}{r|rrrr} -\frac{2}{3} & 24 & -26 & -13 & 10 \\ & & -16 & 28 & -10 \\ \hline & 24 & -42 & 15 & 0 \end{array}$$

This means that we can factor this root out of the original polynomial, of course.

$$24x^3 - 26x^2 - 13x + 10$$
$$= \left(x + \tfrac{2}{3}\right)\left(24x^2 - 42x + 15\right)$$

Notice that we can factor 3 out of the quadratic. That I shall do, but then I'll multiply it inside the other factor. The advantage is that this makes the quadratic term is more manageable.

$$= \left(x + \tfrac{2}{3}\right)(3)\left(8x^2 - 14x + 5\right)$$
$$= (3x + 2)\left(8x^2 - 14x + 5\right)$$

Now let's factor the quadratic and finish the problem.

$$= (3x + 2)(2x - 1)(4x - 5)$$

THE ROOTS ARE $x = -\frac{2}{3}, \frac{1}{2}, \frac{5}{4}$.

Reinforcement Problems

7. Find all the roots of $3x^4 + 7x^3 - 11x^2 - 11x + 12$.

8. Find all the roots of $x^3 + x^2 - 6x - 8$.

Topic 5: Solving polynomial equations in general.

Polynomials fascinated mathematicians for many centuries. One of the most vexing questions went unanswered for nearly three centuries: How do you solve a fifth degree polynomial equation? The answer was finally worked out in the early 19th century and was very short: You can't!

I hope you also have some fascination with this topic. I will briefly discuss the history of these problems and show you not only how to solve a particular cubic equation, but also how this particular problem might have settled an even older, more famous geometric problem. Along the way you will see why mathematicians had to start taking complex numbers seriously in the 16th century.

As the quest evolved in medieval times the question became how to solve certain types of polynomial equations. Linear equations can reduce to the form $ax + b = 0$, which is very easy to solve. Quadratic equations can reduce to the form $ax^2 + bx + c$, and of course, you have seen how the quadratic formula was derived and solves all of those. But is there a similar cubic formula to solve $ax^3 + bx^2 + cx + d = 0$? There is, but it is somewhat large and very complicated to use.

Mathematics, as practiced in the 15th and 16th century, was partly court entertainment. Mathematicians would engage in contests where they were given problems to solve, or perhaps they gave each other problems to solve. Some of the best were focused on trying to solve cubic equations and it was through this trial and error—research 16th century style—that a general method was finally discovered in the 1530's by several Italians, notably Cardano and Tartaglia. Girolamo Cardano (1501 – 76) is also famous for inventing the universal joint (u-joint), a mechanical piece found in modern day cars and trucks, for inventing the combination lock, and for being the first to describe typhoid fever.

The method of solving cubic equations begins with first dividing by the leading coefficient in order to produce a cubic of the form $x^3 + ax^2 + bx + c = 0$. Then a substitution is made, letting $y = x - a/3$, which ends up eliminating the second degree term. Thus, through a series of transformations any cubic equation can be put into this format: $x^3 + px + q = 0$. Those Italian mathematicians had gradually discovered ways to solve equations of this form, so it was a matter of finding a way to turn any cubic equation into this form.

I'm going to solve a specific equation, $8x^3 - 6x - 1 = 0$, by way of giving you an indication of how solving a cubic works. You should make an effort to follow my steps, realizing that the reason for the steps is best described as "because it works." It took a number of years of trial and error to come up with this method, and if you saw what algebra looked like in the 16th century you would wonder how they ever managed to succeed. Algebra as we know it with all the variables and operation symbols did not come about until about a hundred years later.

This happens to be a very special equation, which I will explain afterwards.

Our first step is to make a substitution for x.

$$8x^3 - 6x - 1 = 0$$

$$\text{LET} \quad x = u - v$$

I may skip a step here or there because my purpose is to show you some great math of historical importance, not to teach you anything of practical value. No one solves cubic equations this way anymore, they simply approximate solutions through fast calculus techniques, such as Newton's Method, mentioned in an earlier chapter.

$$8(u-v)^3 - 6(u-v) - 1 = 0$$

$$8(u^3 - 3u^2v + 3uv^2 - v^3) - 6u + 6v - 1 = 0$$

$$8u^3 - 24u^2v + 24uv^2 - 8v^3 - 6u + 6v - 1 = 0$$

This looks much worse than what we started with, but there is a clever way to eliminate four of these terms with the right substitution. First we will factor uv out of two of the terms.

$$8u^3 - 24(uv)u + 24(uv)v - 8v^3 - 6u + 6v - 1 = 0$$

Our next substitution allows us to eliminate those four terms I mentioned.

$$\text{LET } uv = \frac{-1}{4}.$$

$$8u^3 - 24\left(-\frac{1}{4}\right)u + 24\left(-\frac{1}{4}\right)v - 8v^3 - 6u + 6v - 1 = 0$$

$$8u^3 + 6u - 6v - 8v^3 - 6u + 6v - 1 = 0$$

$$8u^3 - 8v^3 - 1 = 0$$

After taking into account the last substitution, this result ends up being a quadratic form. So we solve for v and substitute.

$$uv = -\frac{1}{4}$$

$$v = \frac{-1}{4u}$$

$$8u^3 - 8\left(\frac{-1}{4u}\right)^3 - 1 = 0$$

$$8u^3 + \frac{1}{8u^3} - 1 = 0$$

Once we clear the fractions and rearrange the terms then you will see the quadratic form appear.

$$\left[8u^3 + \frac{1}{8u^3} - 1 = 0\right]8u^3$$

$$64u^6 + 1 - 8u^3 = 0$$

$$64u^6 - 8u^3 + 1 = 0$$

$$64(u^3)^2 - 8(u^3) + 1 = 0$$

So this whole technique hinges on being able to turn the original cubic equation into a quadratic form. But we're not out of the woods yet because this quadratic gives us complex numbered roots, yet any cubic equation must have at least one real numbered root since the graph of any cubic crosses the x–axis at least once. This must have caused some consternation in the 1500's because mathematicians dismissed complex numbers as irrelevant since they did not appearently exist in nature.

Let's go ahead and solve for u^3. We will need the quadratic formula.

$$u^3 = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(64)(1)}}{2(64)}$$

$$= \frac{8 \pm i\sqrt{192}}{128} = \frac{8 \pm 8i\sqrt{3}}{128}$$

$$u^3 = \frac{1 \pm i\sqrt{3}}{16}$$

Here is where things get very sticky. Just as you get two different answers when you solve by taking the square root of both sides of an equation, so you will get three roots when solving by taking the cube root of both sides. This leads to potentially six different values for u. But recall that $x = u - v$. There could be six values for v as well, but somehow there ends up being only three different values for x. It's a mess. Some books in advanced (abstract) algebra walk you through all the nuances. Here I will concentrate on one value for u. I will use the plus sign above and the principal cube root, as follows.

$$u = \sqrt[3]{\frac{1 + i\sqrt{3}}{16}}$$

To solve for x we can bypass v by using the earlier substitutions.

$$uv = -\frac{1}{4} \Rightarrow v = \frac{-1}{4u}$$

$$x = u - v = u - \left(\frac{-1}{4u}\right)$$

$$x = u + \frac{1}{4u} = \frac{4u^2 + 1}{4u}$$

Thus we have a calculation for one of the roots of this particular cubic equation. Writing it out, here's the calculation we face.

$$x = \frac{4\sqrt[3]{\frac{1+i\sqrt{3}}{16}}^{\,2} + 1}{4\sqrt[3]{\frac{1+i\sqrt{3}}{16}}}$$

This may be typical of what faced mathematicians in the 16th century, except they would have written the square root of -3 in this instance since the imaginary number i had not yet been defined. This calculation can be rearranged, perhaps even simplified a little. Since it is a real number then the imaginary parts of it must numerically cancel each other in the expression. I can think of no way possible, however, to simplify this such that the imaginary parts cancel leaving only an exact symbolic calculation with real numbers only.

Imagine how confounded mathematicians must have been when Cardano published his results in 1545. Without calculators and without any knowledge of complex numbers how would they have computed an approximation of such a result? Such outcomes did force them to start thinking seriously about complex numbers, however.

There are trigonometric techniques for calculating cube roots of complex numbers which were uncovered about a century or two after Cardano and friends. This root, above, works out to approximately 0.93969.

I mentioned that this particular problem has special significance, so please indulge me in a quick detour. The ancient Greeks, as part of their geometry, worked at constructing certain figures and angles with only a compass and unmarked straightedge. If you took a geometry class in high school perhaps you tinkered with this. There were three construction problems from ancient Greece that were never solved. The most famous was the problem of trisecting a given angle. In other words, if you draw an arbitrary angle for me, can I find a way to split that angle into three exactly even parts by using only the tools of a compass and an umarked straightedge? This problem was finally settled in 1836, although Gauss claimed, with much credibility, to have solved it in 1800. I will give you an intutive argument.

The answer is no, you cannot trisect an angle using those tools. To prove that this is not possible in general you only need to find one example, called a counter example, of an angle that could never be divided equally into three parts using the given tools. This angle is 60˚. The problem becomes finding a way to prove that no one can construct a 20˚ angle. It turns out that the equation I solved above, $8x^3 - 6x - 1 = 0$, is associated with 20˚ angles. The root I calculated is the cosine of 20˚, written cos 20˚. This forms the length of a side of a right triangle with a hypotenuse of length 1 and an angle of 20˚. This root, of course, consists of cube roots, not to mention the imaginary numbers. It is not possible to construct an irrational cube root using compass and unmarked straightedge. Why is this? A compass can only draw circles or arcs of circles, which are second degree polynomials in two variables. You can only draw line segments with a straightedge, parts of lines, which are first degree equations. The intersections of lines and circles can, at best, only produce even roots such as square roots and fourth roots, never odd roots such as cube roots. Therefore it would be impossible to trisect this particular angle.

Cardano also published a way to solve quartic equations. As you might expect, fourth degree polynomial equations involve even more work to solve than the cubic, but the method is fairly similar. But that's where it stopped. No one could find a formula for solving a general quintic, or fifth degree polynomial equation. The search proceeded for nearly three hundred years. In 1799 Paolo Ruffini (1765 – 1822) nearly settled the issue, but it was left to the Norwegian, Neils Abel (1802 – 1829), to complete Ruffini's proof in 1824 that it is impossible to find a formula in terms of roots that would solve a general fifth degree equation.

A young Frenchman, Evariste Galois (1811 – 1832), laid much of the groundwork for a new and important branch of mathematics, called group theory, in proving that no polynomial of degree 5 or greater can be solved by such a formula involving roots. His accomplishment, at age 21 no less, managed to connect two branches of mathematics, the algebra of fields (the more general, sometimes abstract form of algebra you have been studying) to the algebra of groups, which are related to more general ideas of symmetry. His accomplishment was one of the more amazing mathematical discoveries of the 19th century, and it was a tragedy that he allowed himself to accept a challenge to a duel, and lost.

Thus the story of finding roots of polynomial equations is a rich one. I have only provided a thumbnail sketch.

Topic 6: The Fundamental Theorem of Algebra

It seems fitting to me to bring this section to a close with a look at the Fundamental Theorem of Algebra. This states that any polynomial equation (except the constant function) with real or complex numbered coefficients has at least one root. This root may be a complex number, or could be a real number. Since we have been studying roots of polynomials it seems natural to show that every polynomial has at leats one root.

This theorem was first proposed in the 17rth century but a complete proof was not furnished until 1816 by, again, Karl Friedrich Gauss. His was a very difficult proof, but later in the century a far more easier proof was given in the new branch of complex analysis, the study of complex numbers and functions. It is the latter method of proof which I will present in an intuitive way. The requires that we work in the complex plane. Recall that this plane uses real numbers as the horizontal axis and imaginary numbers as the vertical axis.

Let's suppose that we are given a polynomial function of degree n, as follows.

$$p(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$$

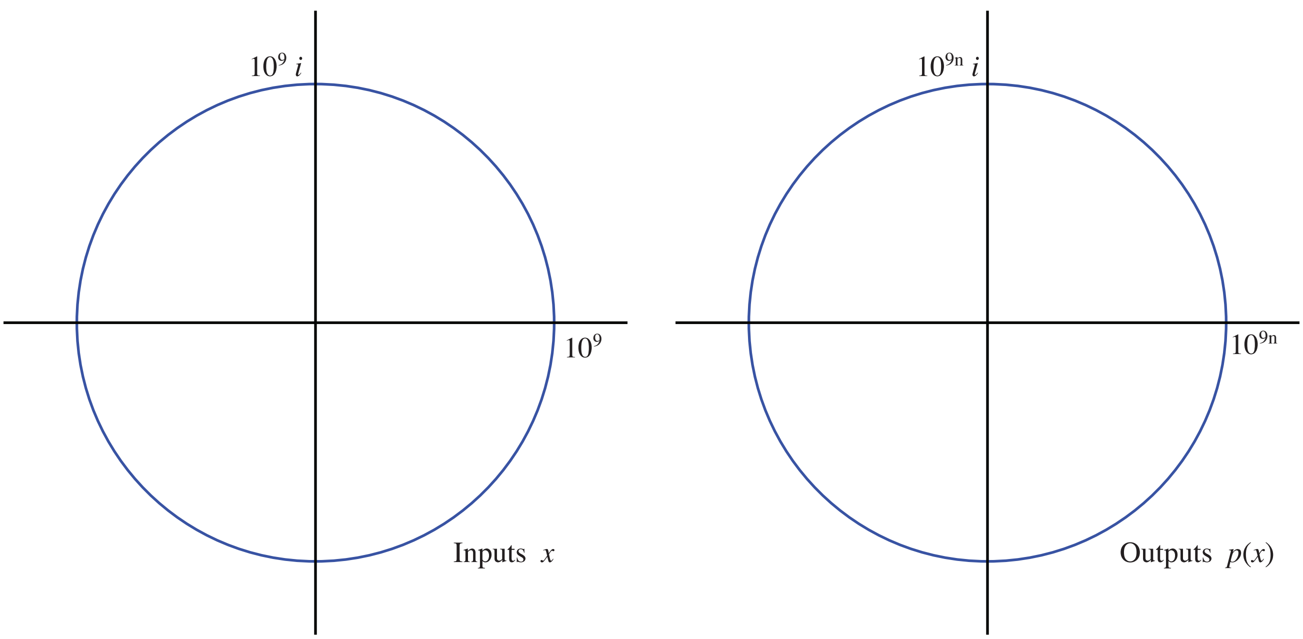
Notice that this polynomial has a leading coefficient of 1. It ultimately doesn't matter whether a polynomial does or not, since one could always factor the leading coefficient out of the polynomial and focus on the polynomial inside the parentheses with a leading coefficient of 1.

One other assumption I shall make is that our other coefficients are relatively small, say, single digits. Ultimately it doesn't matter. If the function had one or more very large coefficients then we could adjust the argument likewise for size.

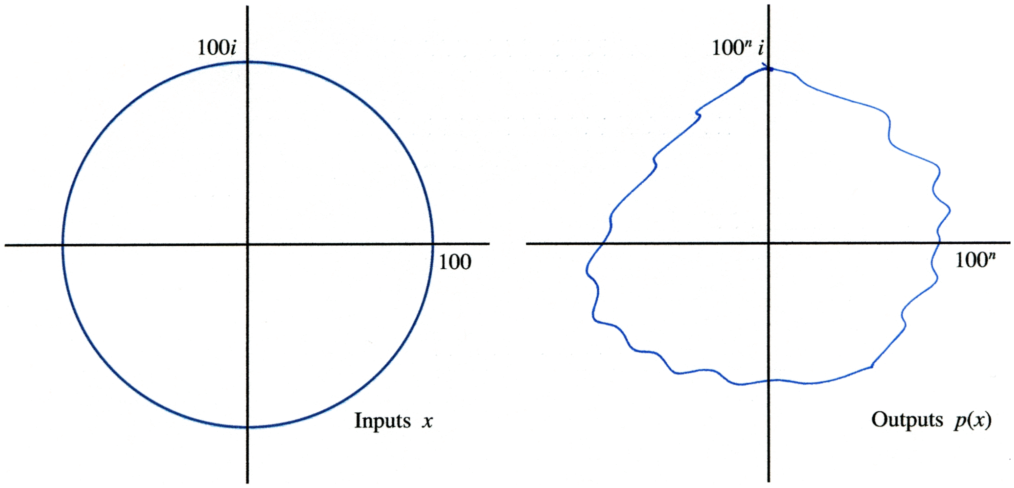
Our argument will involve a comparison of two graphs in the complex plane, one of which is a collection of inputs, and the other the collection of outputs of the polynomial function given the inputs.

On the left I have formed a very large circle with a radius of 10^9, or one billion. These are all the points in the complex plane that are a distance of one billion from the origin, which is the number 0.

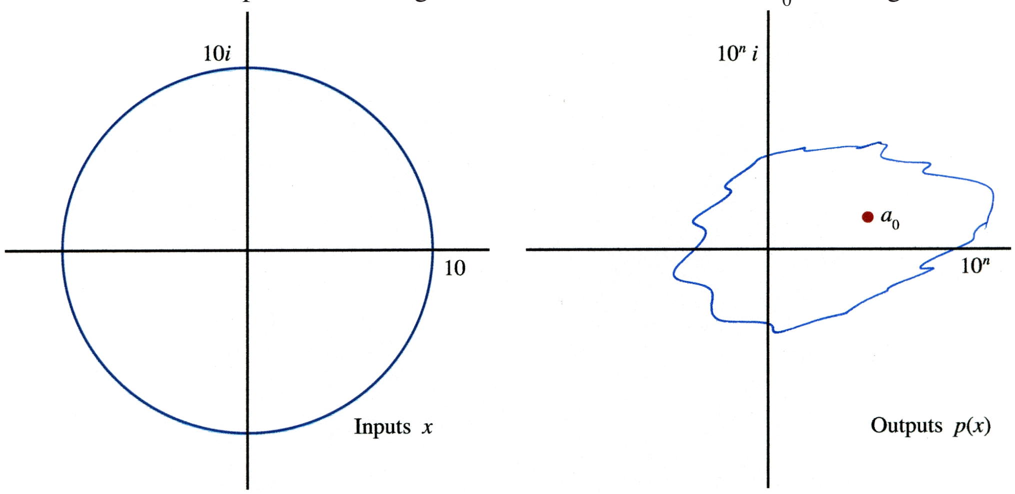
On the right I have formed an even larger circle (the scale of the graph is different) of outputs of our function $p(x)$. Well, it's not quite a circle, but almost. The leading term, x^n, will easily dominate all the smaller degree terms. Thus when you plug in a complex number on the left circle into the function which is a billion away from the origin, the output will be, for all practical purposes, on or very close to a circel with a radius of one billion raised to the nth power, or 10^{9n}.



We proceed by smoothly shrinking the circle of inputs on the left. Suppose that we shrink them down to a radius of 100, far smaller than one billion. The right hand circle will start to deform somewhat due to the increased influence of the terms less than degree n. Maybe it could look something like this.

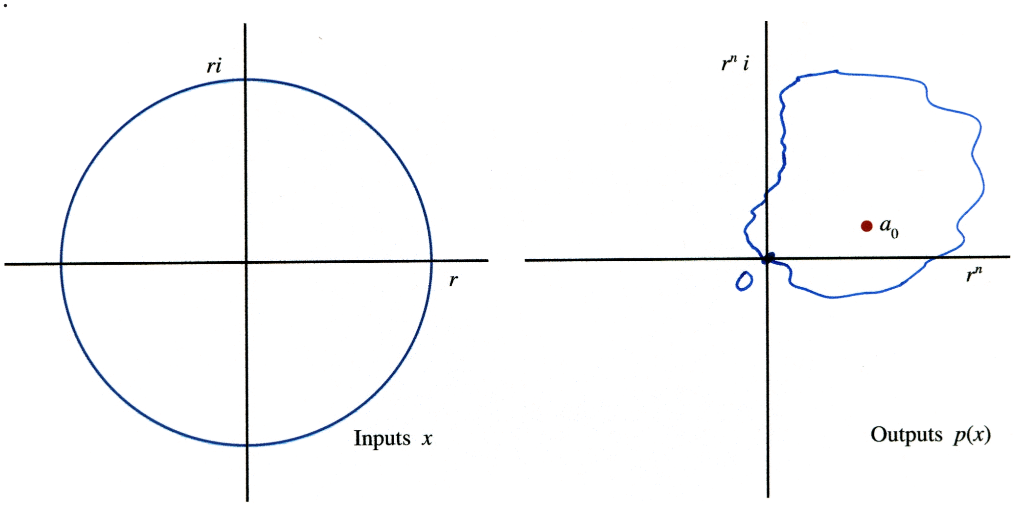


As we shrink the circle on the left even more then the graph of outputs to our function on the right could take on even more deformed shapes, but as we get smaller the constant term, a_0, will begin to come into view.



What happens as the circle on the left shrinks down to almost zero? The object on the right will also shrink down until it is a very small circle like object surrounding the constant term a_0. Why? $p(0) = a_0$. If we shrink the circle on the left down to the center point at the origin, which is the number 0, then on the right the surrounding object must shrink to the constant term.

So as the circle of inputs on the left continuously shrink from a very large radius at some point there will be at least one radius which causes the graph on the right to pass through the origin, or 0. In my picture I called this radius r.



That means that there is at least one number on this size circle of inputs that gives an output of 0, which means that it is a root of the polynomial.

Since this method of argument can be used on any polynomial, then all polynomials (except the constant function) have at least one root, which may or may not be a complex number.

Now let's wrap this up by extending the Fundamental Theorem of Algebra to show that any *n*th degree polynomial has n roots, accounting for repeated roots. Since one root is guaranteed to exist, we could use synthetic division to factor that root out of the original polynomial, which leaves a different polynomial of one degree lower. This new polynomial must also have a root, so if we factor it out then we are left with a polynomial yet another degree lower, which also has a root. We can continue this process in theory until we have found *n* roots. As I mentioned, it is possible that some of these roots will be repeated. And as we saw earlier, if our original polynomial function has degree 5 of greater, then we cannot expect to be able to find these roots in exact form. We can, however, use one of several techniques in order to estimate these roots to any degree of accuracy we are willing to work for.

Solutions to Reinforcement Problems

1.

$$\begin{array}{r|rrrr} 2 & 1 & 6 & -4 & 9 \\ & & 2 & 16 & 24 \\ \hline & 1 & 8 & 12 & 33 \end{array}$$

$$\frac{x^3+6x^2-4x+9}{x-2} = x^2+8x+12+\frac{33}{x-2}$$

2.

$$\begin{array}{r|rrrrrr} -3 & 1 & -2 & 0 & -4 & 3 & -5 \\ & & -3 & 15 & -45 & 147 & -450 \\ \hline & 1 & -5 & 15 & -49 & 150 & -455 \end{array}$$

$$\frac{x^5-2x^4-4x^2+3x-5}{x+3} = x^4-5x^3+15x^2-49x+150-\frac{455}{x+3}$$

3.

$$\begin{array}{r|rrrr} -2 & 1 & 3 & 6 & -9 \\ & & -2 & -2 & -8 \\ \hline & 1 & 1 & 4 & -17 \end{array}$$

$$p(-2) = -17$$

4.

$$\begin{array}{r|rrrrr} 3 & -2 & 5 & -4 & 7 & -6 \\ & & -6 & -3 & -21 & -42 \\ \hline & -2 & -1 & -7 & -14 & -48 \end{array}$$

5. POSSIBLE ROOTS ARE $\dfrac{\text{FACTORS OF } -6}{\text{FACTORS OF } 4} = \dfrac{\pm 1, \pm 2, \pm 3, \pm 6}{\pm 1, \pm 2, \pm 4}$

$= 1, -1, 2, -2, 3, -3, 6, -6, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \frac{3}{2}, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{4}$

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6. POSSIBLE ROOTS ARE $\dfrac{\text{FACTORS OF } -15}{\text{FACTORS OF } -5} = \dfrac{\pm 1, \pm 3, \pm 5, \pm 15}{\pm 1, \pm 5}$

$= 1, -1, 3, -3, 5, -5, 15, -15, \frac{3}{5}, -\frac{3}{5}$

---

7. POSSIBLE ROOTS ARE $= 1, -1, 2, -2, 3, -3, 4, -4, 6, -6, 12, -12,$

$\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}, \frac{4}{3}, -\frac{4}{3}$

$x = 1$ IS A ROOT:

$$\begin{array}{r|rrrrr} 1 & 3 & 7 & -11 & -11 & 12 \\ & & 3 & 10 & -1 & -12 \\ \hline & 3 & 10 & -1 & -12 & 0 \end{array}$$

$(x-1)(3x^3 + 10x^2 - x - 12)$

$x = 1$ IS ANOTHER ROOT:

$$\begin{array}{r|rrrr} 1 & 3 & 10 & -1 & -12 \\ & & 3 & 13 & 12 \\ \hline & 3 & 13 & 12 & 0 \end{array}$$

$(x-1)^2(3x^2 + 13x + 12)$

$= (x-1)^2(3x+4)(x+3)$

THE ROOTS ARE $x = 1, 1, -3, -\frac{4}{3}$.

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8. POSSIBLE RATIONAL ROOTS ARE $\dfrac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1}$

$= 1, -1, 2, -2, 4, -4, 8, -8$

$x = -2$ IS A ROOT:

$$\begin{array}{r|rrrr} -2 & 1 & 1 & -6 & -8 \\ & & -2 & 2 & 8 \\ \hline & 1 & -1 & -4 & 0 \end{array}$$

$(x+2)(x^2 - x - 4)$

$\hookrightarrow a = 1, b = -1, c = -4$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-4)}}{2(1)} = \frac{1 \pm \sqrt{17}}{2}$$

THE ROOTS ARE $x = -2, \frac{1+\sqrt{17}}{2}, \frac{1-\sqrt{17}}{2}$.