Chapter 5 - Structural Dynamics

Rev. 1

5.1 Introduction

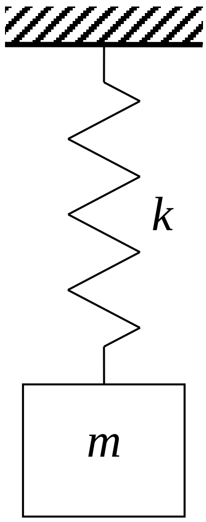
5.1.1 Outline of Structural Dynamics

Modern structures are increasingly slender and have reduced redundant strength due to improved analysis and design methods. Such structures are increasingly responsive to the manner in which loading is applied with respect to time and hence the dynamic behaviour of such structures must be allowed for in design; as well as the usual static considerations. In this context then, the word dynamic simply means “changes with time”; be it force, deflection or any other form of load effect.

Examples of dynamics in structures are:

- Soldiers breaking step as they cross a bridge to prevent harmonic excitation;
- The Tacoma Narrows Bridge footage, failure caused by vortex shedding;
- The London Millennium Footbridge: lateral synchronise excitation.

(a) (after Craig 1981)



(b)

Figure 1.1

The most basic dynamic system is the mass-spring system. An example is shown in Figure 1.1(a) along with the structural idealisation of it in Figure 1.1(b). This is known as a *Single Degree-of-Freedom* (SDOF) system as there is only one possible displacement: that of the mass in the vertical direction. SDOF systems are of great importance as they are relatively easily analysed mathematically, are easy to understand intuitively, and structures usually dealt with by Structural Engineers can be modelled approximately using an SDOF model (see Figure 1.2 for example).

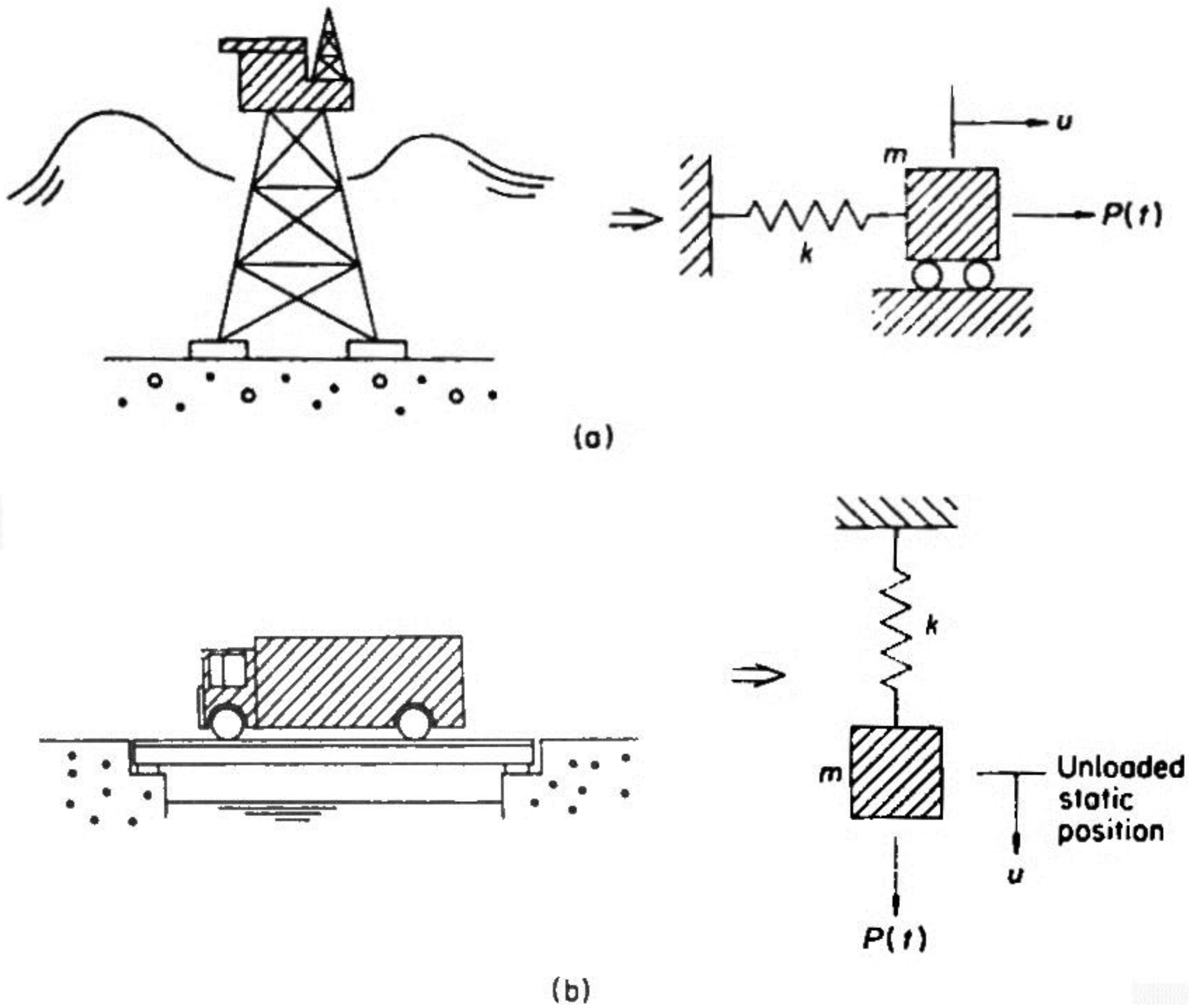


Figure 1.2 (after Craig 1981).

5.1.2 An Initial Numerical Example

If we consider a spring-mass system as shown in Figure 1.3 with the properties m = 10 kg and k = 100 N/m and if give the mass a deflection of 20 mm and then release it (i.e. set it in motion) we would observe the system oscillating as shown in Figure 1.3. From this figure we can identify that the time between the masses recurrence at a particular location is called the *period of motion or oscillation* or just the *period*, and we denote it T; it is the time taken for a single oscillation. The number of oscillations per second is called the *frequency*, denoted f, and is measured in Hertz (cycles per second). Thus we can say:

$$f = \frac{1}{T} \tag{5.1.1}$$

We will show (Section 2.b, equation (2.19)) for a spring-mass system that:

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{5.1.2}$$

In our system:

$$f = \frac{1}{2\pi}\sqrt{\frac{100}{10}} = 0.503 \text{ Hz}$$

And from equation (5.1.1):

$$T = \frac{1}{f} = \frac{1}{0.503} = 1.987 \text{ secs}$$

We can see from Figure 1.3 that this is indeed the period observed.

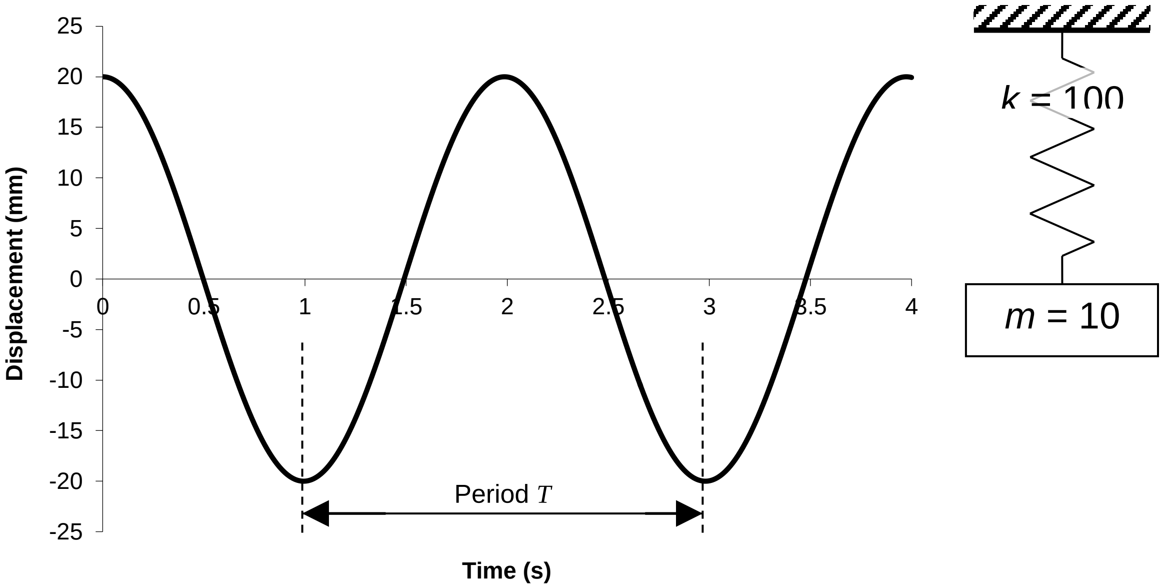


Figure 1.3

To reach the deflection of 20 mm just applied, we had to apply a force of 2 N, given that the spring stiffness is 100 N/m. As noted previously, the rate at which this load is applied will have an effect of the dynamics of the system. Would you expect the system to behave the same in the following cases?

- If a 2 N weight was dropped onto the mass from a very small height?
- If 2 N of sand was slowly added to a weightless bucket attached to the mass?

Assuming a linear increase of load, to the full 2 N load, over periods of 1, 3, 5 and 10 seconds, the deflections of the system are shown in Figure 1.4.

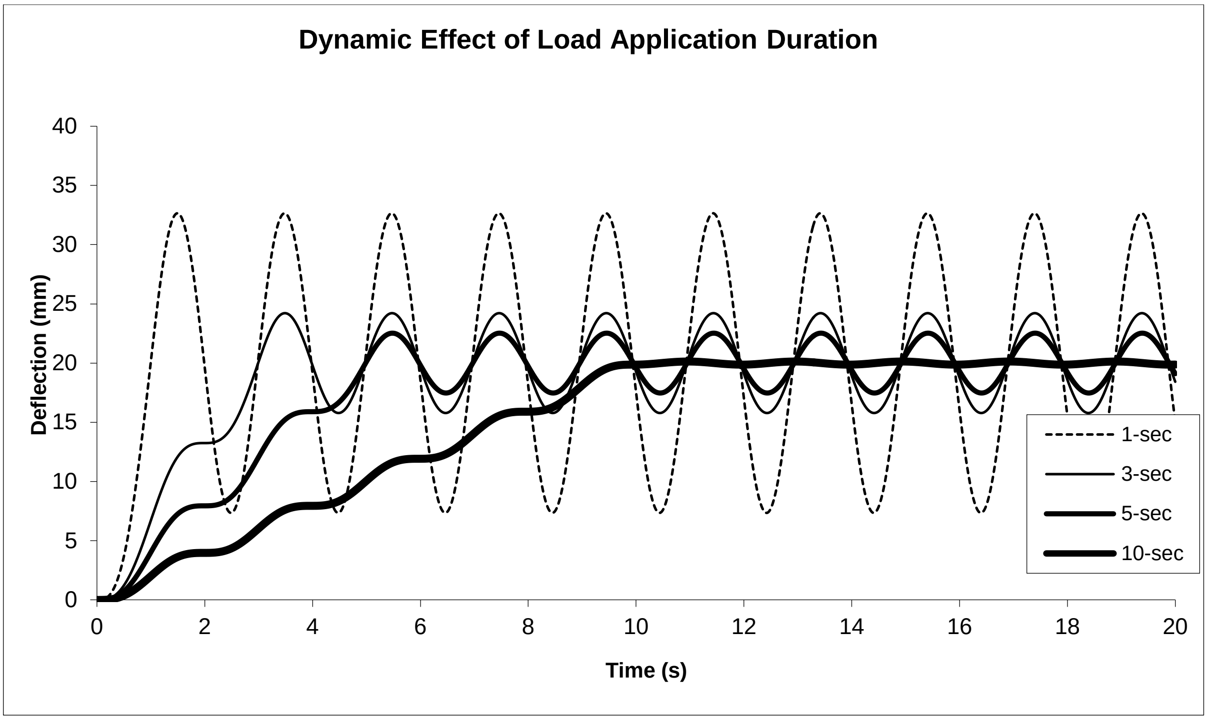


Figure 1.4

Remembering that the period of vibration of the system is about 2 seconds, we can see that when the load is applied faster than the period of the system, large dynamic effects occur. Stated another way, when the frequency of loading (1, 0.3, 0.2 and 0.1 Hz for our sample loading rates) is close to, or above the natural frequency of the system (0.5 Hz in our case), we can see that the dynamic effects are large. Conversely, when the frequency of loading is less than the natural frequency of the system little dynamic effects are noticed – most clearly seen via the 10 second ramp-up of the load, that is, a 0.1 Hz load.

5.1.3 Case Study – Aberfeldy Footbridge, Scotland

Aberfeldy footbridge is a glass fibre reinforced polymer (GFRP) cable-stayed bridge over the River Tay on Aberfeldy golf course in Aberfeldy, Scotland (Figure 1.5). Its main span is 63 m and its two side spans are 25 m, also, tests have shown that the natural frequency of this bridge is 1.52 Hz, giving a period of oscillation of 0.658 seconds.

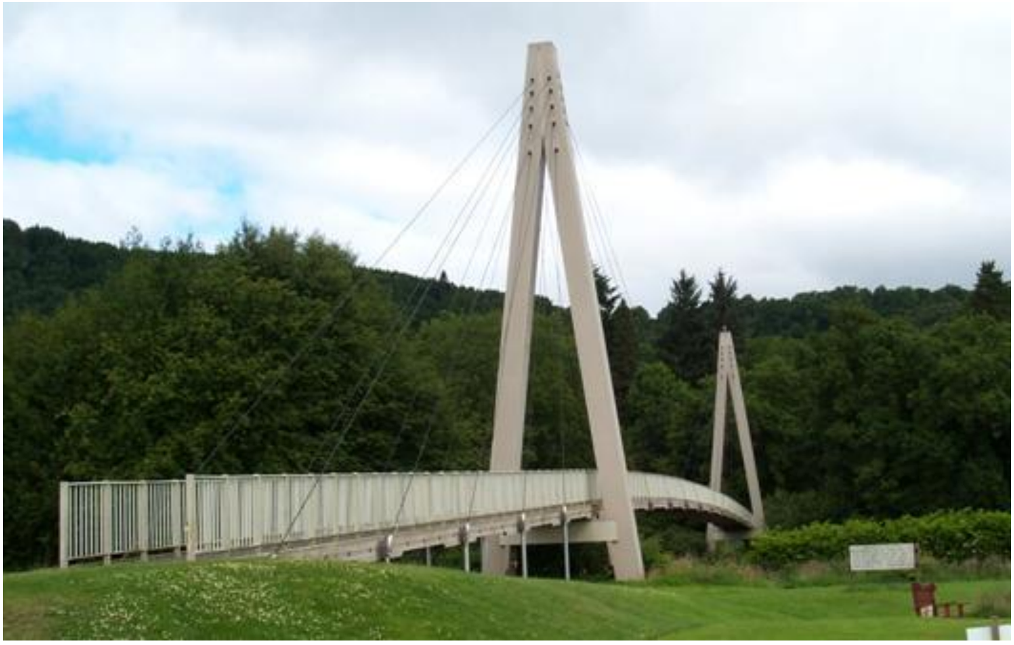


Figure 1.5: Aberfeldy Footbridge

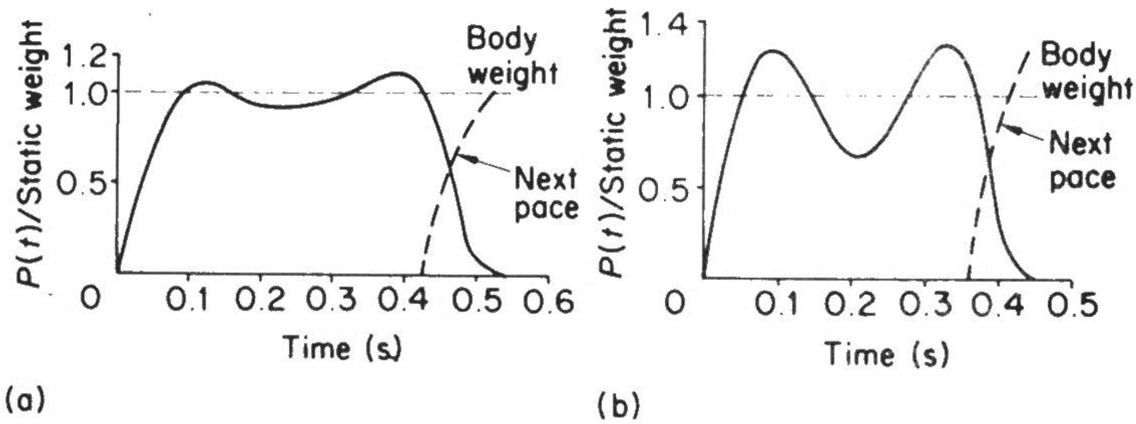


Figure 1.6: Force-time curves for walking: (a) Normal pacing. (b) Fast pacing

Footbridges are generally quite light structures as the loading consists of pedestrians; this often results in dynamically lively structures. Pedestrian loading varies as a person walks; from about 0.65 to 1.3 times the weight of the person over a period of about 0.35 seconds, that is, a loading frequency of about 2.86 Hz (Figure 1.6). When we compare this to the natural frequency of Aberfeldy footbridge we can see that pedestrian loading has a higher frequency than the natural frequency of the bridge – thus, from our previous discussion we would expect significant dynamic effects to results from this. Figure 1.7 shows the response of the bridge (at the mid-span) when a pedestrian crosses the bridge: significant dynamics are apparent.

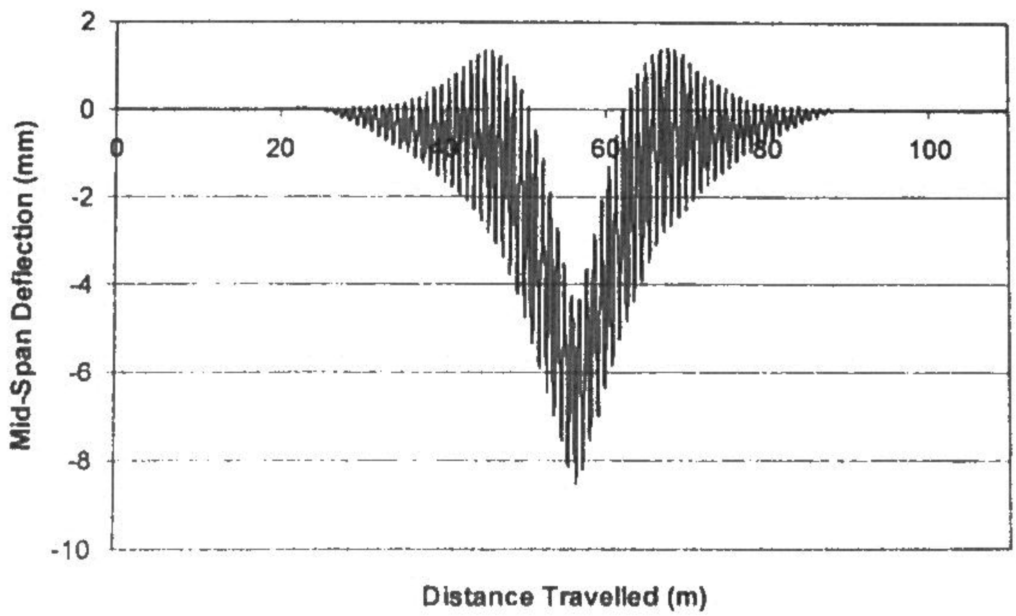


Figure 1.7: Mid-span deflection (mm) as a function of distance travelled (m).

Design codes generally require the natural frequency for footbridges and other pedestrian traversed structures to be greater than 5 Hz, that is, a period of 0.2 seconds. The reasons for this are apparent after our discussion: a 0.35 seconds load application (or 2.8 Hz) is slower than the natural period of vibration of 0.2 seconds (5 Hz) and hence there will not be much dynamic effect resulting; in other words the loading may be considered to be applied statically.

5.1.4 Structural Damping

Look again at the frog in Figure 1.1, according to the results obtained so far which are graphed in Figures 1.3 and 1.4, the frog should oscillate indefinitely. If you have ever cantilevered a ruler off the edge of a desk and flicked it you would have seen it vibrate for a time but certainly not indefinitely; buildings do not vibrate indefinitely after an earthquake; Figure 1.7 shows the vibrations dying down quite soon after the pedestrian has left the main span of Aberfeldy bridge - clearly there is another action opposing or "damping" the vibration of structures. Figure 1.8 shows the undamped response of our model along with the damped response; it can be seen that the oscillations die out quite rapidly – this depends on the level of damping.

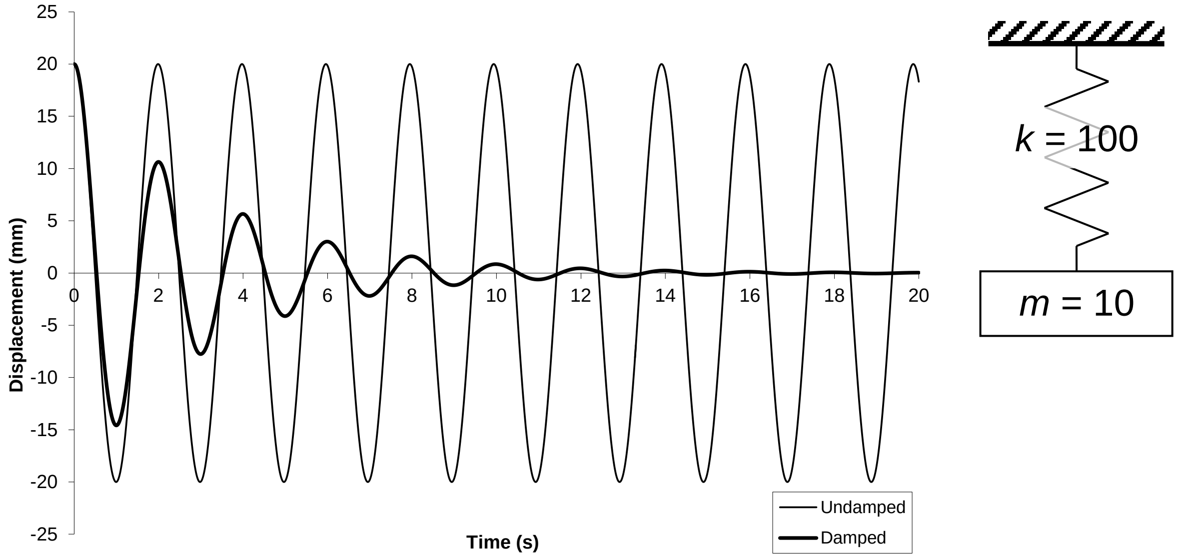


Figure 1.8

Damping occurs in structures due to energy loss mechanisms that exist in the system. Examples are friction losses at any connection to or in the system and internal energy losses of the materials due to thermo-elasticity, hysteresis and inter-granular bonds. The exact nature of damping is difficult to define; fortunately theoretical damping has been shown to match real structures quite well.

5.2 Single Degree-of-Freedom Systems

5.2.1 Fundamental Equation of Motion

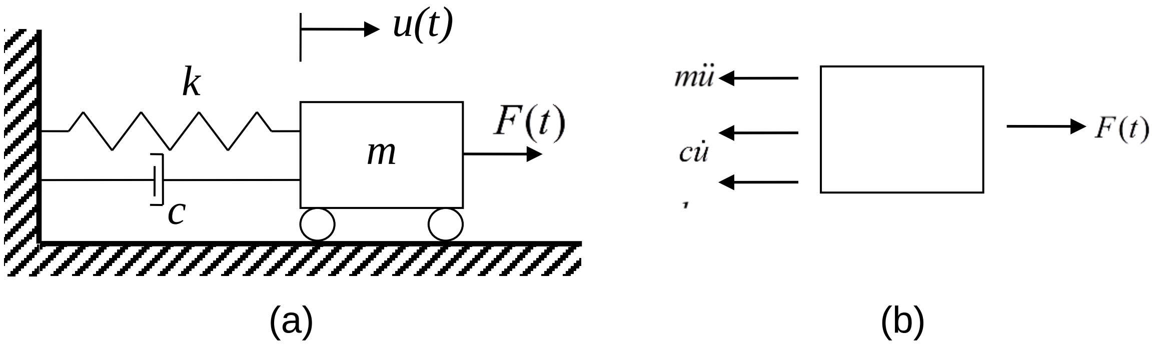


Figure 2.1: (a) SDOF system. (b) Free-body diagram of forces

Considering Figure 2.1, the forces resisting the applied loading are considered as:

- a force proportional to displacement (the usual static stiffness);
- a force proportional to velocity (the damping force);
- a force proportional to acceleration (D'Alambert's inertial force).

We can write the following symbolic equation:

$$\mathrm{F}_{\text{applied}} = \mathrm{F}_{\text{stiffness}} + \mathrm{F}_{\text{damping}} + \mathrm{F}_{\text{inertia}} \tag{5.2.1}$$

Noting that:

$$\left.\begin{aligned} \mathrm{F}_{\text{stiffness}} &= ku \\ \mathrm{F}_{\text{damping}} &= c\dot{u} \\ \mathrm{F}_{\text{inertia}} &= m\ddot{u} \end{aligned}\right\} \tag{5.2.2}$$

that is, stiffness × displacement, damping coefficient × velocity and mass × acceleration respectively. Note also that *u* represents displacement from the equilibrium position and that the dots over *u* represent the first and second derivatives with respect to time. Thus, noting that the displacement, velocity and acceleration are all functions of time, we have the Fundamental Equation of Motion:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F(t) \tag{5.2.3}$$

In the case of free vibration, there is no forcing function and so $F(t) = 0$ which gives equation (5.2.3) as:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = 0 \tag{5.2.4}$$

We note also that the system will have a state of *initial conditions*:

$$u_0 = u(0) \tag{5.2.5}$$

$$\dot{u}_0 = \dot{u}(0) \tag{5.2.6}$$

In equation (5.2.4), dividing across by *m* gives:

$$\ddot{u}(t) + \frac{c}{m}\dot{u}(t) + \frac{k}{m}u(t) = 0 \tag{5.2.7}$$

We introduce the following notation:

$$\xi = \frac{c}{c_{cr}} \tag{5.2.8}$$

$$\omega^2 = \frac{k}{m} \tag{5.2.9}$$

Or equally,

$$\omega = \sqrt{\frac{k}{m}} \qquad (5.2.10)$$

In which

- ω is called the *undamped circular natural frequency* and its units are radians per second (rad/s);
- ξ is the *damping ratio* which is the ratio of the *damping coefficient*, *c*, to the *critical value of the damping coefficient* c_{cr}.

We will see what these terms physically mean. Also, we will later see (equation (5.2.18)) that:

$$c_{cr} = 2m\omega = 2\sqrt{km} \qquad (5.2.11)$$

Equations (5.2.8) and (5.2.11) show us that:

$$2\xi\omega = \frac{c}{m} \qquad (5.2.12)$$

When equations (5.2.9) and (5.2.12) are introduced into equation (5.2.7), we get the *prototype SDOF equation of motion*:

$$\ddot{u}(t) + 2\xi\omega\dot{u}(t) + \omega^2 u(t) = 0 \qquad (5.2.13)$$

In considering free vibration only, the general solution to (5.2.13) is of a form

$$u = Ce^{\lambda t} \qquad (5.2.14)$$

When we substitute (5.2.14) and its derivates into (5.2.13) we get:

$$\left(\lambda^2 + 2\xi\omega\lambda + \omega^2\right)Ce^{\lambda t} = 0 \tag{5.2.15}$$

For this to be valid for all values of *t*, $Ce^{\lambda t}$ cannot be zero. Thus we get the characteristic equation:

$$\lambda^2 + 2\xi\omega\lambda + \omega^2 = 0 \tag{5.2.16}$$

the solutions to this equation are the two roots:

$$\begin{aligned} \lambda_{1,2} &= \frac{-2\omega\xi \pm \sqrt{4\omega^2\xi^2 - 4\omega^2}}{2} \\ &= -\omega\xi \pm \omega\sqrt{\xi^2 - 1} \end{aligned} \tag{5.2.17}$$

Therefore the solution depends on the magnitude of ξ relative to 1. We have:

- $\xi < 1$: Sub-critical damping or *under-damped*;

  Oscillatory response only occurs when this is the case – as it is for almost all structures.
- $\xi = 1$: *Critical damping*;

  No oscillatory response occurs.
- $\xi > 1$: Super-critical damping or *over-damped*;

  No oscillatory response occurs.

Therefore, when $\xi = 1$, the coefficient of $\dot{u}(t)$ in equation (5.2.13) is, by definition, the critical damping coefficient. Thus, from equation (5.2.12):

$$2\omega = \frac{c_{cr}}{m} \tag{5.2.18}$$

From which we get equation (5.2.11).

5.2.2 Free Vibration of Undamped Structures

We will examine the case when there is no damping on the SDOF system of Figure 2.1 so $\xi = 0$ in equations (5.2.13), (5.2.16) and (5.2.17) which then become:

$$\ddot{u}(t) + \omega^2 u(t) = 0 \tag{5.2.19}$$

respectively, where $i = \sqrt{-1}$. From the Appendix we see that the general solution to this equation is:

$$u(t) = A\cos\omega t + B\sin\omega t \tag{5.2.20}$$

where A and B are constants to be obtained from the initial conditions of the system, equations (5.2.5) and (5.2.6). Thus, at $t = 0$, from equation (5.2.20):

$$u(0) = A\cos\omega(0) + B\sin\omega(0) = u_0$$

$$A = u_0 \tag{5.2.21}$$

From equation (5.2.20):

$$\dot{u}(t) = -A\omega\sin\omega t + B\omega\cos\omega t \tag{5.2.22}$$

And so:

$$\dot{u}(0) = -A\omega\sin\omega(0) + B\omega\cos\omega(0) = \dot{u}_0$$

$$B\omega = \dot{u}_0$$

$$B = \frac{\dot{u}_0}{\omega} \tag{5.2.23}$$

Thus equation (5.2.20), after the introduction of equations (5.2.21) and (5.2.23), becomes:

$$u(t) = u_0 \cos \omega t + \left(\frac{\dot{u}_0}{\omega} \right) \sin \omega t \tag{5.2.24}$$

where u_0 and $\dot{u}_0$ are the initial displacement and velocity of the system respectively. Noting that cosine and sine are functions that repeat with period 2π, we see that $\omega(t_1 + T) = \omega t_1 + 2\pi$ (Figure 2.3) and so the undamped natural period of the SDOF system is:

$$T = \frac{2\pi}{\omega} \tag{5.2.25}$$

The natural frequency of the system is got from (1.1), (5.2.25) and (5.2.9):

$$f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{5.2.26}$$

and so we have proved (1.2). The importance of this equation is that it shows the natural frequency of structures to be proportional to $\sqrt{k/m}$. This knowledge can aid a designer in addressing problems with resonance in structures: by changing the stiffness or mass of the structure, problems with dynamic behaviour can be addressed.

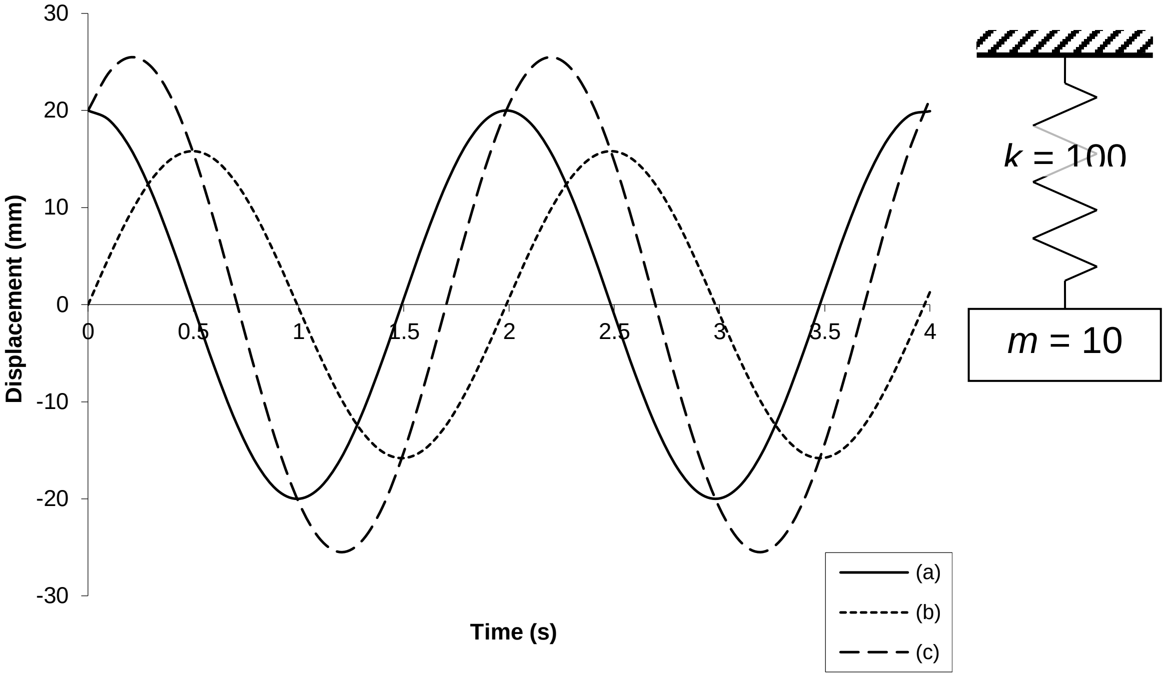


Figure 2.2: SDOF free vibration response for (a) $u_0 = 20\text{mm}$, $\dot{u}_0 = 0$, (b) $u_0 = 0$, $\dot{u}_0 = 50\text{mm/s}$, and (c) $u_0 = 20\text{mm}$, $\dot{u}_0 = 50\text{mm/s}$.

Figure 2.2 shows the free-vibration response of a spring-mass system for various initial states of the system. It can be seen from (b) and (c) that when $\dot{u}_0 \neq 0$ the amplitude of displacement is not that of the initial displacement; this is obviously an important characteristic to calculate. The cosine addition rule may also be used to show that equation (5.2.20) can be written in the form:

$$u(t) = C\cos\left(\omega t - \theta\right) \tag{5.2.27}$$

where $C = \sqrt{A^2 + B^2}$ and $\tan\theta = {-B}/{A}$. Using A and B as calculated earlier for the initial conditions, we then have:

$$u(t) = \rho\cos\left(\omega t - \theta\right) \tag{5.2.28}$$

where ρ is the amplitude of displacement and θ is the phase angle, both given by:

$$\rho = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega}\right)^2} \tag{5.2.29}$$

$$\tan\theta = \frac{\dot{u}_0}{u_0\omega} \tag{5.2.30}$$

The phase angle determines the amount by which $u(t)$ lags behind the function $\cos\omega t$. Figure 2.3 shows the general case.

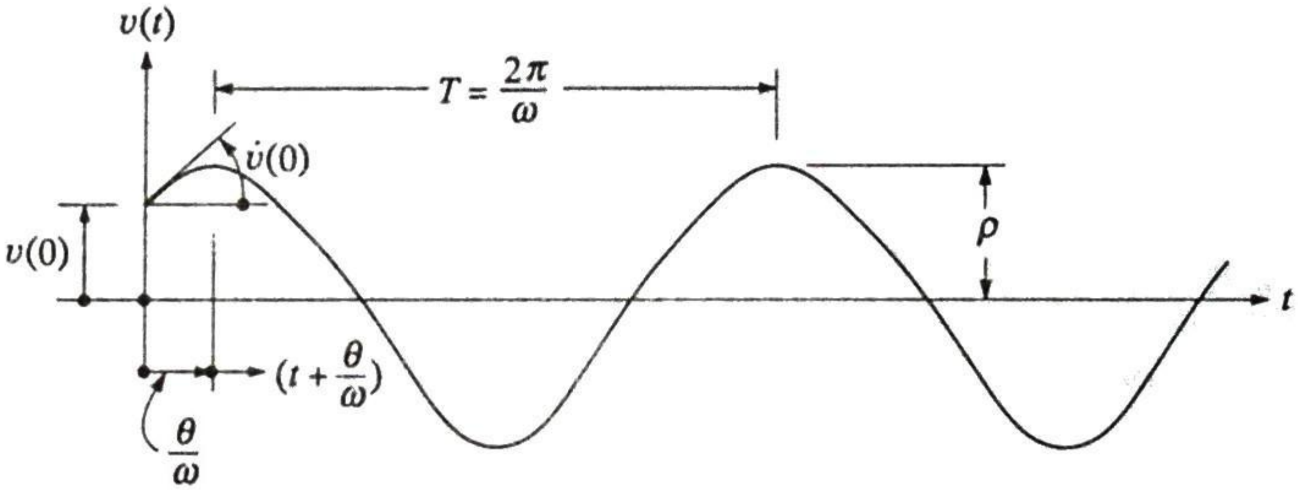
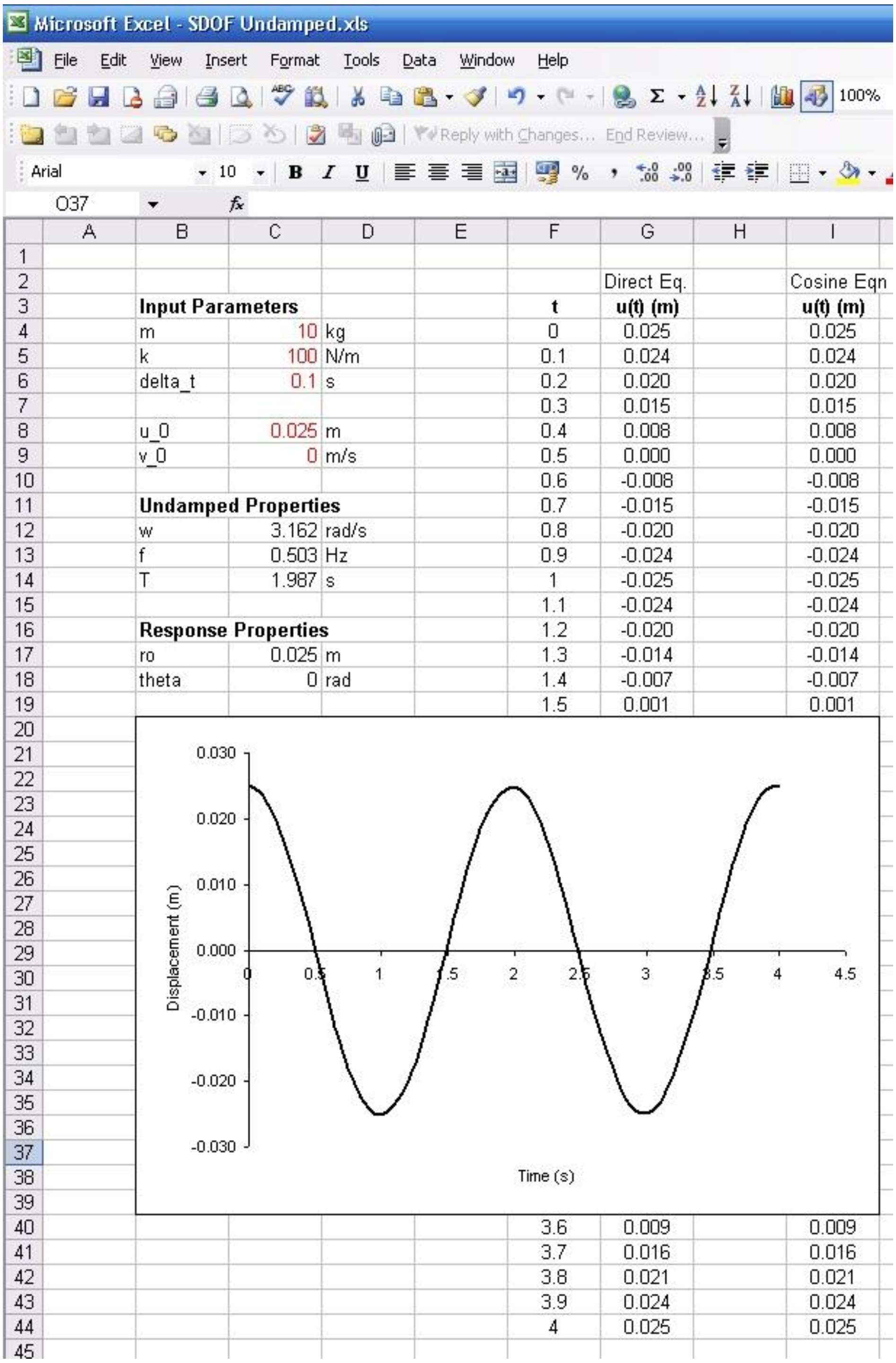


Figure 2.3 Undamped free-vibration response.

5.2.3 *Computer Implementation & Examples*

Using MS Excel

To illustrate an application we give the spreadsheet used to generate Figure 1.3. This can be downloaded from the course website.



	A	B	C	D	E	F	G	H	I
1									
2							Direct Eq.		Cosine Eqn
3		**Input Parameters**				**t**	**u(t) (m)**		**u(t) (m)**
4		m	10	kg		0	0.025		0.025
5		k	100	N/m		0.1	0.024		0.024
6		delta_t	0.1	s		0.2	0.020		0.020
7						0.3	0.015		0.015
8		u_0	0.025	m		0.4	0.008		0.008
9		v_0	0	m/s		0.5	0.000		0.000
10						0.6	-0.008		-0.008
11		**Undamped Properties**				0.7	-0.015		-0.015
12		w	3.162	rad/s		0.8	-0.020		-0.020
13		f	0.503	Hz		0.9	-0.024		-0.024
14		T	1.987	s		1	-0.025		-0.025
15						1.1	-0.024		-0.024
16		**Response Properties**				1.2	-0.020		-0.020
17		ro	0.025	m		1.3	-0.014		-0.014
18		theta	0	rad		1.4	-0.007		-0.007
19						1.5	0.001		0.001
40						3.6	0.009		0.009
41						3.7	0.016		0.016
42						3.8	0.021		0.021
43						3.9	0.024		0.024
44						4	0.025		0.025
45									

The input parameters (shown in red) are:

- *m* – the mass;
- *k* – the stiffness;
- *delta_t* – the time step used in the response plot;
- *u_0* – the initial displacement, u_0;
- *v_0* – the initial velocity, $\dot{u}_0$.

The properties of the system are then found:

- *w*, using equation (5.2.10);
- *f*, using equation (5.2.26);
- *T*, using equation (5.2.26);
- ρ, using equation (5.2.29);
- θ, using equation (5.2.30).

A column vector of times is dragged down, adding *delta_t* to each previous time value, and equation (5.2.24) ("Direct Eqn"), and equation (5.2.28) ("Cosine Eqn") is used to calculate the response, $u(t)$, at each time value. Then the column of *u*-values is plotted against the column of *t*-values to get the plot.

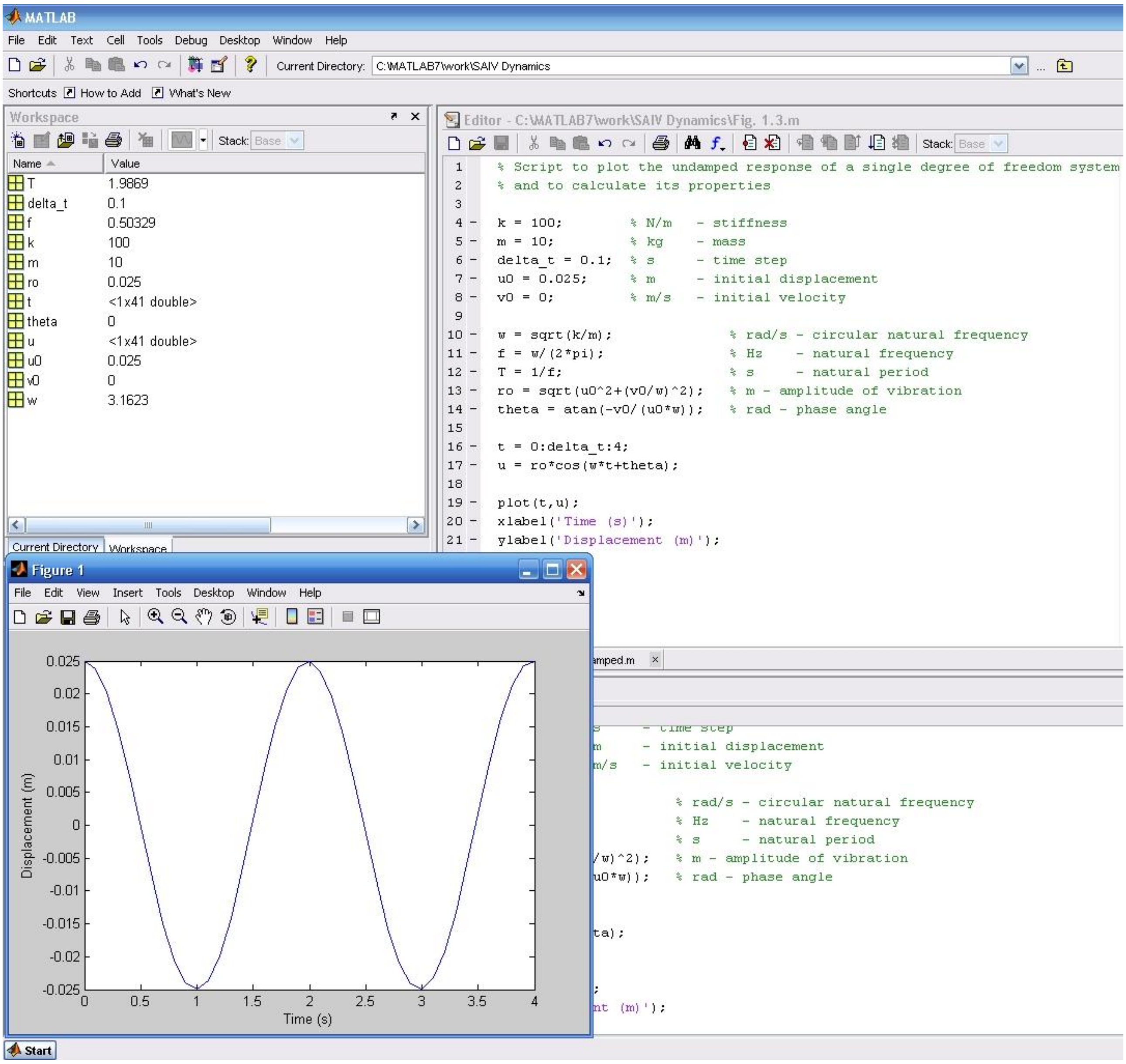
Using Matlab

Although MS Excel is very helpful since it provides direct access to the numbers in each equation, as more concepts are introduced, we will need to use loops and create regularly-used functions. Matlab is ideally suited to these tasks, and so we will begin to use it also on the simple problems as a means to its introduction.

A script to directly generate Figure 1.3, and calculate the system properties is given below:

```
% Script to plot the undamped response of a single degree of freedom system
% and to calculate its properties

k = 100;          % N/m   - stiffness
m = 10;           % kg    - mass
delta_t = 0.1;    % s     - time step
u0 = 0.025;       % m     - initial displacement
v0 = 0;           % m/s   - initial velocity

w = sqrt(k/m);                % rad/s - circular natural frequency
f = w/(2*pi);                 % Hz    - natural frequency
T = 1/f;                      % s     - natural period
ro = sqrt(u0^2+(v0/w)^2);     % m     - amplitude of vibration
theta = atan(v0/(u0*w));      % rad   - phase angle

t = 0:delta_t:4;
u = ro*cos(w*t-theta);
plot(t,u);
xlabel('Time (s)');
ylabel('Displacement (m)');
```

The results of this script are the system properties are displayed in the workspace window, and the plot is generated, as shown below:



Whilst this is quite useful, this script is limited to calculating the particular system of Figure 1.3. Instead, if we create a function that we can pass particular system properties to, then we can create this plot for any system we need to. The following function does this.

Note that we do not calculate f or T since they are not needed to plot the response. Also note that we have commented the code very well, so it is easier to follow and understand when we come back to it at a later date.

```
function [t u] = sdof_undamped(m,k,u0,v0,duration,plotflag)
% This function returns the displacement of an undamped SDOF system with
% parameters:
% m - mass, kg
% k - stiffness, N/m
% u0 - initial displacement, m
% v0 - initial velocity, m/s
% duration - length of time of required response
% plotflag - 1 or 0: whether or not to plot the response
% This function returns:
% t - the time vector at which the response was found
% u - the displacement vector of response

Npts = 1000;     % compute the response at 1000 points
delta_t = duration/(Npts-1);

w = sqrt(k/m);              % rad/s - circular natural frequency
ro = sqrt(u0^2+(v0/w)^2);   % m - amplitude of vibration
theta = atan(v0/(u0*w));    % rad - phase angle

t = 0:delta_t:duration;
u = ro*cos(w*t-theta);

if(plotflag == 1)
    plot(t,u);
    xlabel('Time (s)');
    ylabel('Displacement (m)');
end
```

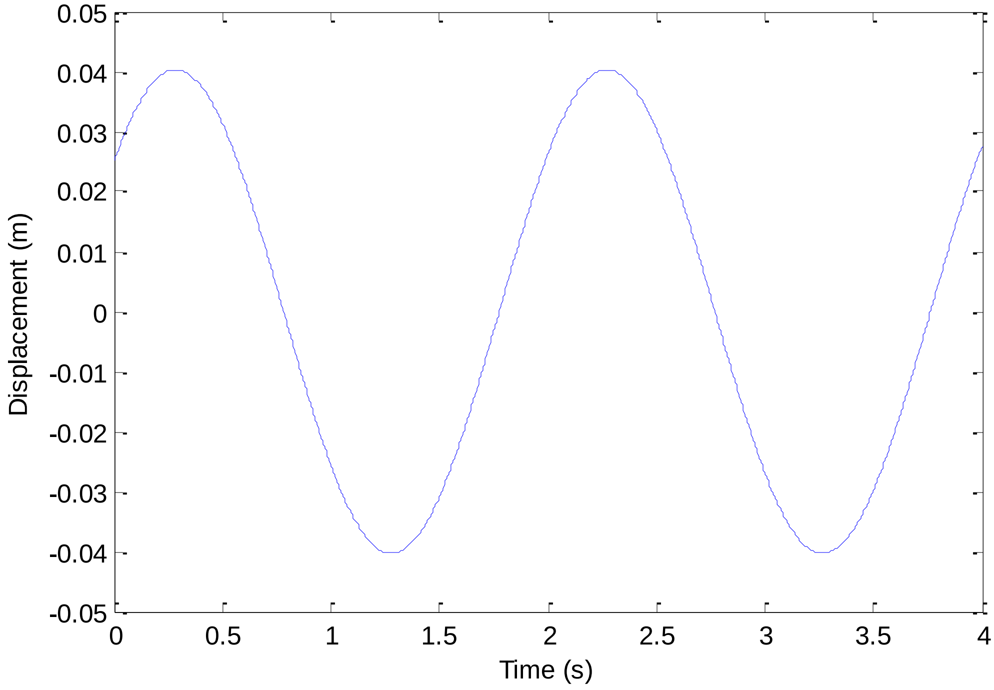
To execute this function and replicate Figure 1.3, we call the following:

```
[t u] = sdof_undamped(10,100,0.025,0,4,1);
```

And get the same plot as before. Now though, we can really benefit from the function. Let's see the effect of an initial velocity on the response, try +0.1 m/s:

```
[t u] = sdof_undamped(10,100,0.025,0.1,4,1);
```

Note the argument to the function in bold – this is the +0.1 m/s initial velocity. And from this call we get the following plot:



From which we can see that the maximum response is now about 40 mm, rather than the original 25.

Download the function from the course website and try some other values.

5.2.4 Free Vibration of Damped Structures

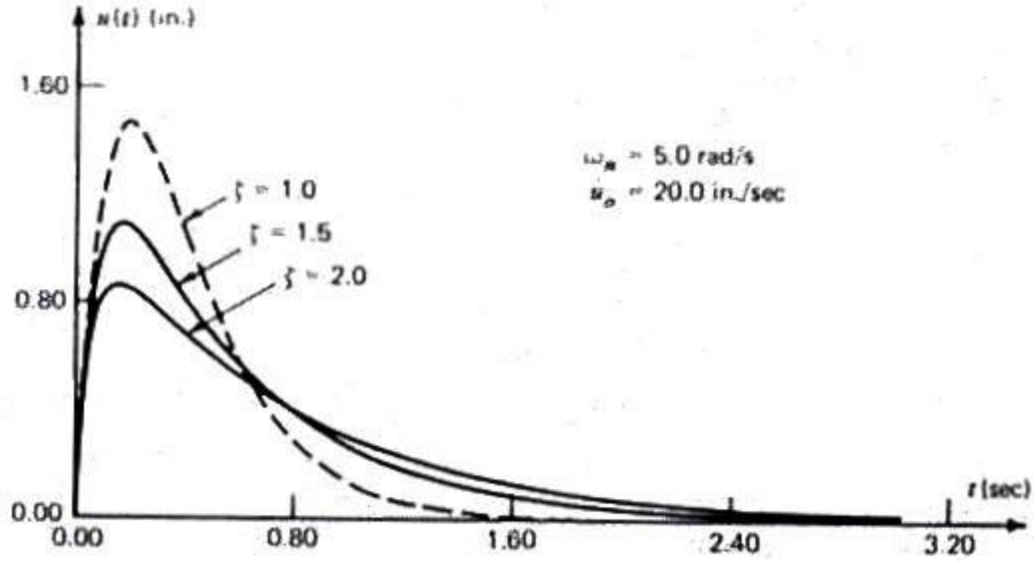


Figure 2.4: Response with critical or super-critical damping

When taking account of damping, we noted previously that there are 3, cases but only when $\xi < 1$ does an oscillatory response ensue. We will not examine the critical or super-critical cases. Examples are shown in Figure 2.4.

To begin, when $\xi < 1$ (5.2.17) becomes:

$$\lambda_{1,2} = -\omega\xi \pm i\omega_d \tag{5.2.31}$$

where ω_d is the *damped circular natural frequency* given by:

$$\omega_d = \omega\sqrt{1-\xi^2} \tag{5.2.32}$$

which has a corresponding *damped period and frequency* of:

$$T_d = \frac{2\pi}{\omega_d} \tag{5.2.33}$$

$$f_d = \frac{\omega_d}{2\pi} \tag{5.2.34}$$

The general solution to equation (5.2.14), using Euler's formula again, becomes:

$$u(t) = e^{-\xi\omega t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \tag{5.2.35}$$

and again using the initial conditions we get:

$$u(t) = e^{-\xi\omega t}\left[u_0\cos\omega_d t + \left(\frac{\dot{u}_0 + \xi\omega_d u_0}{\omega_d}\right)\sin\omega_d t\right] \tag{5.2.36}$$

Using the cosine addition rule again we also have:

$$u(t) = \rho e^{-\xi\omega t}\cos\left(\omega_d t - \theta\right) \tag{5.2.37}$$

In which

$$\rho = \sqrt{u_0^2 + \left(\frac{\dot{u}_0 + \xi\omega u_0}{\omega_d}\right)^2} \tag{5.2.38}$$

$$\tan\theta = \frac{\dot{u}_0 + u_0\xi\omega}{u_0\omega_d} \tag{5.2.39}$$

Equations (5.2.35) to (5.2.39) correspond to those of the undamped case looked at previously when $\xi = 0$.

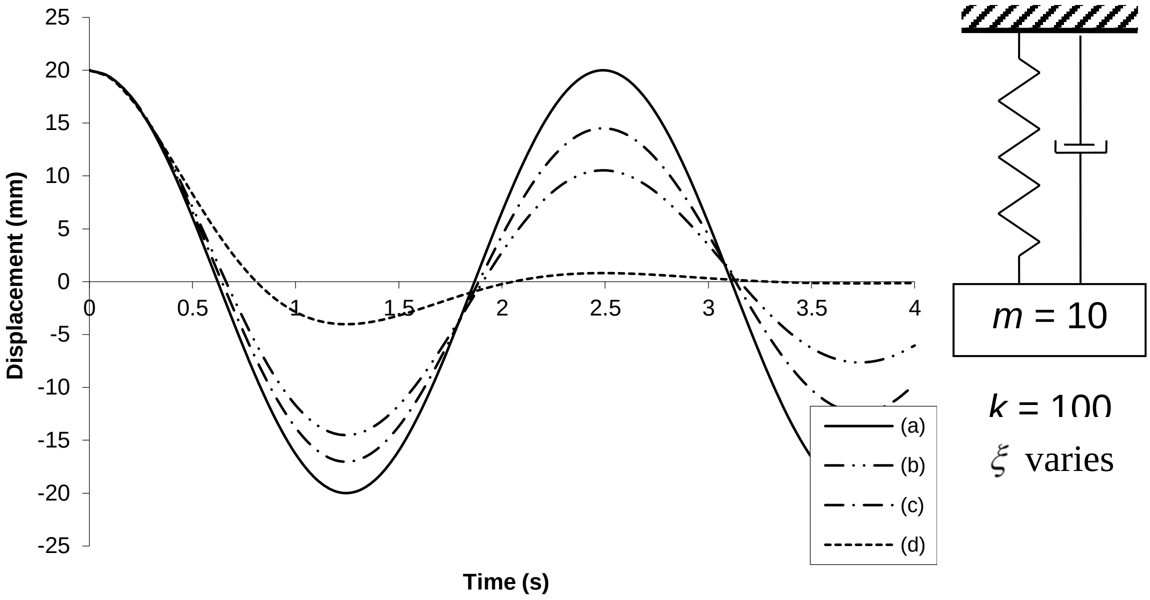


Figure 2.5: SDOF free vibration response for:
(a) $\xi = 0$; (b) $\xi = 0.05$; (c) $\xi = 0.1$; and (d) $\xi = 0.5$.

Figure 2.5 shows the dynamic response of the SDOF model shown. It may be clearly seen that damping has a large effect on the dynamic response of the system – even for small values of ξ. We will discuss damping in structures later but damping ratios for structures are usually in the range 0.5 to 5%. Thus, the damped and undamped properties of the systems are very similar for these structures.

Figure 2.6 shows the general case of an under-critically damped system.

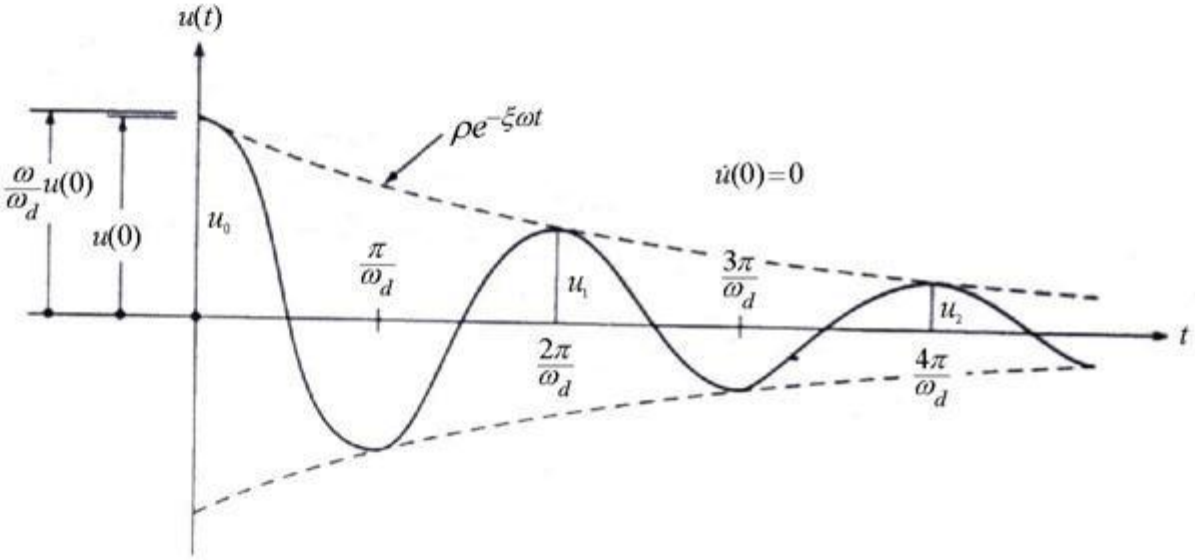
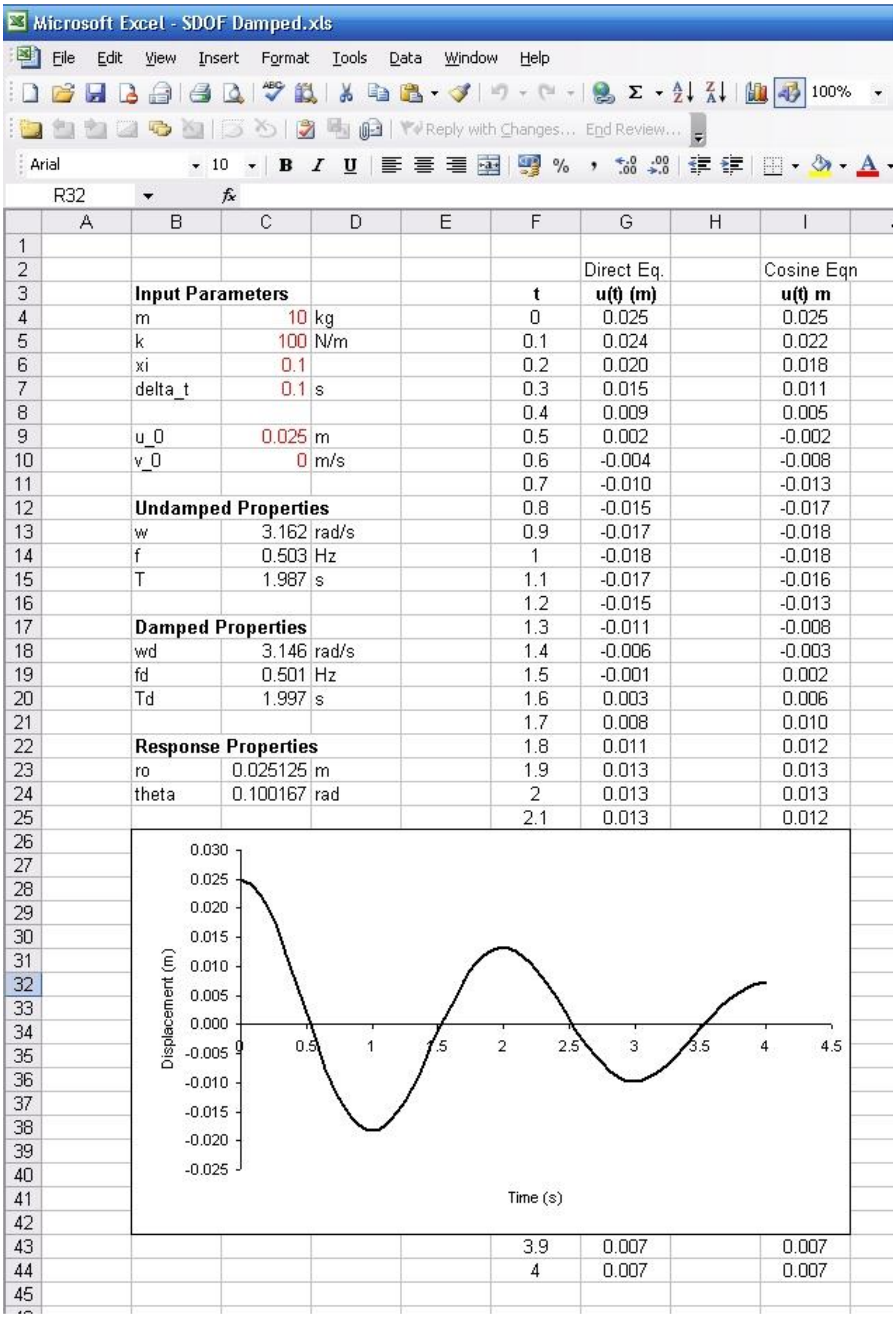


Figure 2.6: General case of an under-critically damped system.

5.2.5 Computer Implementation & Examples

Using MS Excel

We can just modify our previous spreadsheet to take account of the revised equations for the amplitude (equation (5.2.38)), phase angle (equation (5.2.39)) and response (equation (5.2.37)), as well as the damped properties, to get:



	B	C	D	F	G	I
1						
2					Direct Eq.	Cosine Eqn
3	**Input Parameters**			**t**	**u(t) (m)**	**u(t) m**
4	m	10	kg	0	0.025	0.025
5	k	100	N/m	0.1	0.024	0.022
6	xi	0.1		0.2	0.020	0.018
7	delta_t	0.1	s	0.3	0.015	0.011
8				0.4	0.009	0.005
9	u_0	0.025	m	0.5	0.002	-0.002
10	v_0	0	m/s	0.6	-0.004	-0.008
11				0.7	-0.010	-0.013
12	**Undamped Properties**			0.8	-0.015	-0.017
13	w	3.162	rad/s	0.9	-0.017	-0.018
14	f	0.503	Hz	1	-0.018	-0.018
15	T	1.987	s	1.1	-0.017	-0.016
16				1.2	-0.015	-0.013
17	**Damped Properties**			1.3	-0.011	-0.008
18	wd	3.146	rad/s	1.4	-0.006	-0.003
19	fd	0.501	Hz	1.5	-0.001	0.002
20	Td	1.997	s	1.6	0.003	0.006
21				1.7	0.008	0.010
22	**Response Properties**			1.8	0.011	0.012
23	ro	0.025125	m	1.9	0.013	0.013
24	theta	0.100167	rad	2	0.013	0.013
25				2.1	0.013	0.012
43				3.9	0.007	0.007
44				4	0.007	0.007
45						

Using Matlab

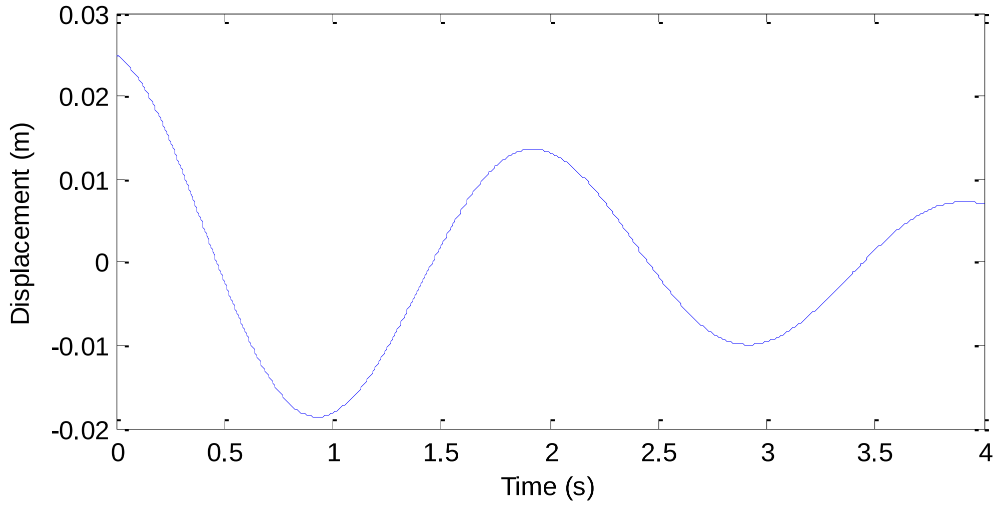
Now can just alter our previous function and take account of the revised equations for the amplitude (equation (5.2.38)), phase angle (equation (5.2.39)) and response (equation (5.2.37)) to get the following function. This function will (of course) also work for undamped systems where $\xi = 0$.

```
function [t u] = sdof_damped(m,k,xi,u0,v0,duration,plotflag)
% This function returns the displacement of a damped SDOF system with
% parameters:
% m - mass, kg
% k - stiffness, N/m
% xi - damping ratio
% u0 - initial displacement, m
% v0 - initial velocity, m/s
% duration - length of time of required response
% plotflag - 1 or 0: whether or not to plot the response
% This function returns:
% t - the time vector at which the response was found
% u - the displacement vector of response

Npts = 1000;     % compute the response at 1000 points
delta_t = duration/(Npts-1);

w = sqrt(k/m);                % rad/s - circular natural frequency
wd = w*sqrt(1-xi^2);          % rad/s - damped circular frequency
ro = sqrt(u0^2+((v0+xi*w*u0)/wd)^2);     % m - amplitude of vibration
theta = atan((v0+u0*xi*w)/(u0*w));       % rad - phase angle

t = 0:delta_t:duration;
u = ro*exp(-xi*w.*t).*cos(w*t-theta);

if(plotflag == 1)
    plot(t,u);
    xlabel('Time (s)');
    ylabel('Displacement (m)');
end
```

Let's apply this to our simple example again, for $\xi = 0.1$:

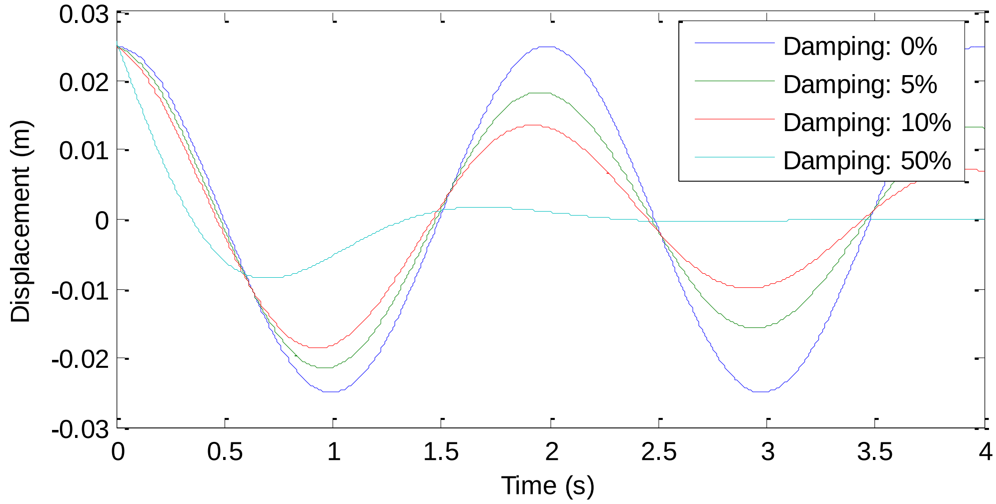
```
[t u] = sdof_damped(10,100,0.1,0.025,0,4,1);
```

To get:



To plot Figure 2.5, we just call out function several times (without plotting it each time), save the response results and then plot all together:

```
xi = [0,0.05,0.1,0.5];
for i = 1:length(xi)
    [t u(i,:)] = sdof_damped(10,100,xi(i),0.025,0,4,0);
end
plot(t,u);
xlabel('Time (s)');
ylabel('Displacement (m)');
legend('Damping: 0%','Damping: 5%','Damping: 10%','Damping: 50%');
```



5.2.6 Estimating Damping in Structures

Examining Figure 2.6, we see that two successive peaks, u_n and u_{n+m}, m cycles apart, occur at times nT and $(n+m)T$ respectively. Using equation (5.2.37) we can get the ratio of these two peaks as:

$$\frac{u_n}{u_{n+m}} = \exp\left(\frac{2m\pi\xi\omega}{\omega_d}\right) \tag{5.2.40}$$

where $\exp(x) \equiv e^x$. Taking the natural log of both sides we get the *logarithmic decrement of damping*, δ, defined as:

$$\delta = \ln\frac{u_n}{u_{n+m}} = 2m\pi\xi\frac{\omega}{\omega_d} \tag{5.2.41}$$

for low values of damping, normal in structural engineering, we can approximate this:

$$\delta \cong 2m\pi\xi \tag{5.2.42}$$

thus,

$$\frac{u_n}{u_{n+m}} = e^{\delta} \cong \exp(2m\pi\xi) \cong 1 + 2m\pi\xi \tag{5.2.43}$$

and so,

$$\xi \cong \frac{u_n - u_{n+m}}{2m\pi u_{n+m}} \tag{5.2.44}$$

This equation can be used to estimate damping in structures with light damping ($\xi < 0.2$) when the amplitudes of peaks m cycles apart is known. A quick way of doing this, known as the *Half-Amplitude Method*, is to count the number of peaks it takes to halve the amplitude, that is $u_{n+m} = 0.5u_n$. Then, using (5.2.44) we get:

$$\xi \cong \frac{0.11}{m} \text{ when } u_{n+m} = 0.5u_n \tag{5.2.45}$$

Further, if we know the amplitudes of two successive cycles (and so $m = 1$), we can find the amplitude after p cycles from two instances of equation (5.2.43):

$$u_{n+p} = \left(\frac{u_{n+1}}{u_n}\right)^p u_n \tag{5.2.46}$$

5.2.7 Response of an SDOF System Subject to Harmonic Force

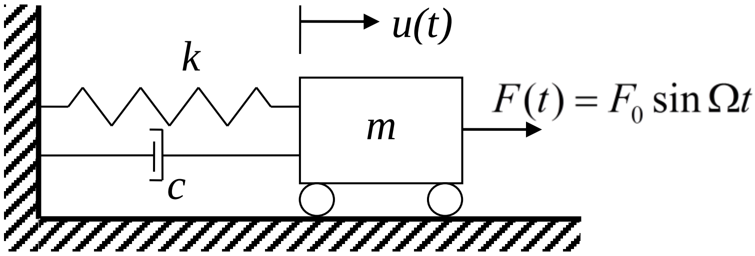


Figure 2.7: SDOF undamped system subjected to harmonic excitation

So far we have only considered free vibration; the structure has been set vibrating by an initial displacement for example. We will now consider the case when a time varying load is applied to the system. We will confine ourselves to the case of harmonic or sinusoidal loading though there are obviously infinitely many forms that a time-varying load may take – refer to the references (Appendix) for more.

To begin, we note that the forcing function $F(t)$ has excitation amplitude of F_0 and an excitation circular frequency of Ω and so from the fundamental equation of motion (5.2.3) we have:

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F_0 \sin \Omega t \qquad (5.2.47)$$

The solution to equation (5.2.47) has two parts:

- The *complementary solution*, similar to (5.2.35), which represents the *transient response* of the system which damps out by $\exp(-\xi\omega t)$. The transient response may be thought of as the vibrations caused by the initial application of the load.
- The *particular solution*, $u_p(t)$, representing the *steady-state harmonic response* of the system to the applied load. This is the response we will be interested in as it will account for any resonance between the forcing function and the system.

The complementary solution to equation (5.2.47) is simply that of the damped free vibration case studied previously. The particular solution to equation (5.2.47) is developed in the Appendix and shown to be:

$$u_p(t) = \rho \sin(\Omega t - \theta) \qquad (5.2.48)$$

In which

$$\rho = \frac{F_0}{k}\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]^{-1/2} \qquad (5.2.49)$$

$$\tan\theta = \frac{2\xi\beta}{1-\beta^2} \qquad (5.2.50)$$

where the phase angle is limited to $0 < \theta < \pi$ and the ratio of the applied load frequency to the natural undamped frequency is:

$$\beta = \frac{\Omega}{\omega} \qquad (5.2.51)$$

the maximum response of the system will come at $\sin(\Omega t - \theta) = 1$ and dividing (5.2.48) by the static deflection F_0/k we can get the *dynamic amplification factor* (DAF) of the system as:

$$\text{DAF} \equiv D = \left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]^{-1/2} \qquad (5.2.52)$$

At resonance, when $\Omega = \omega$, we then have:

$$D_{\beta=1} = \frac{1}{2\xi} \qquad (5.2.53)$$

Figure 2.8 shows the effect of the frequency ratio β on the DAF. Resonance is the phenomenon that occurs when the forcing frequency coincides with that of the natural frequency, $\beta = 1$. It can also be seen that for low values of damping, normal in structures, very high DAFs occur; for example if $\xi = 0.02$ then the dynamic amplification factor will be 25. For the case of no damping, the DAF goes to infinity - theoretically at least; equation (5.2.53).

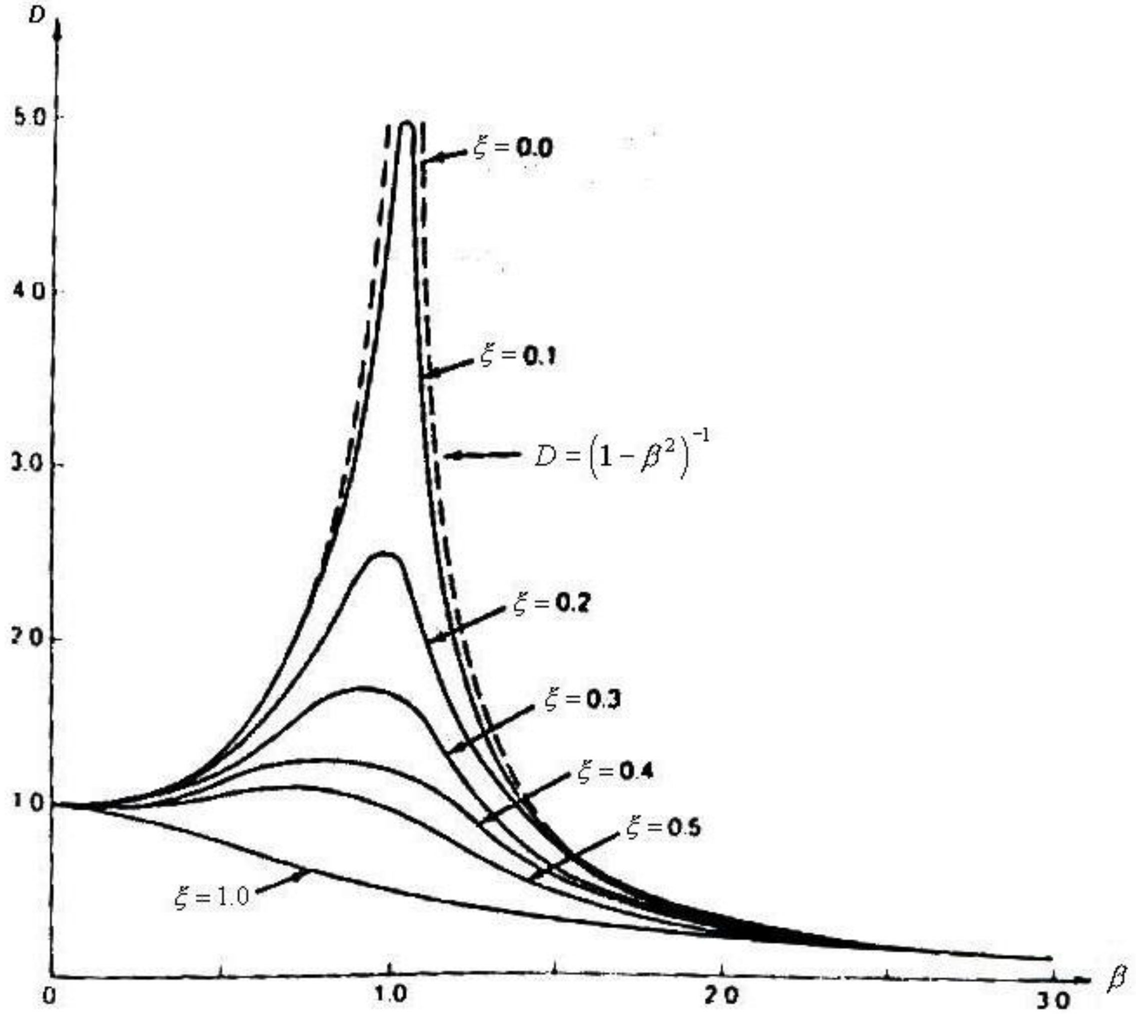


Figure 2.8: Variation of DAF with damping and frequency ratios.

The phase angle also helps us understand what is occurring. Plotting equation (5.2.50) against β for a range of damping ratios shows:

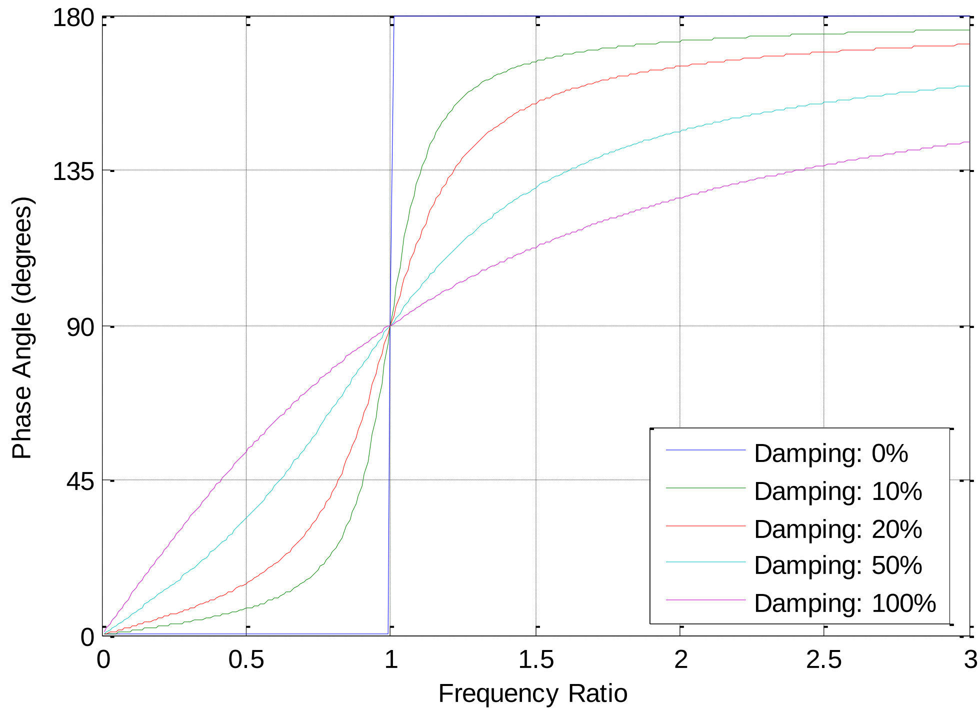


Figure 2.9: Variation of phase angle with damping and frequency ratios.

Looking at this then we can see three regions:

- $\beta << 1$: the force is slowly varying and θ is close to zero. This means that the response (i.e. displacement) is in phase with the force: for example, when the force acts to the right, the system displaces to the right.
- $\beta >> 1$: the force is rapidly varying and θ is close to 180°. This means that the force is out of phase with the system: for example, when the force acts to the right, the system is displacing to the left.
- $\beta = 1$: the forcing frequency is equal to the natural frequency, we have resonance and $\theta = 90$. Thus the displacement attains its peak as the force is zero.

We can see these phenomena by plotting the response and forcing fun(5.2.54)ction together (though with normalized displacements for ease of interpretation), for different values of β. In this example we have used $\xi = 0.2$. Also, the three phase angles are $\theta/2\pi = 0.04$, 0.25, 0.46 respectively.

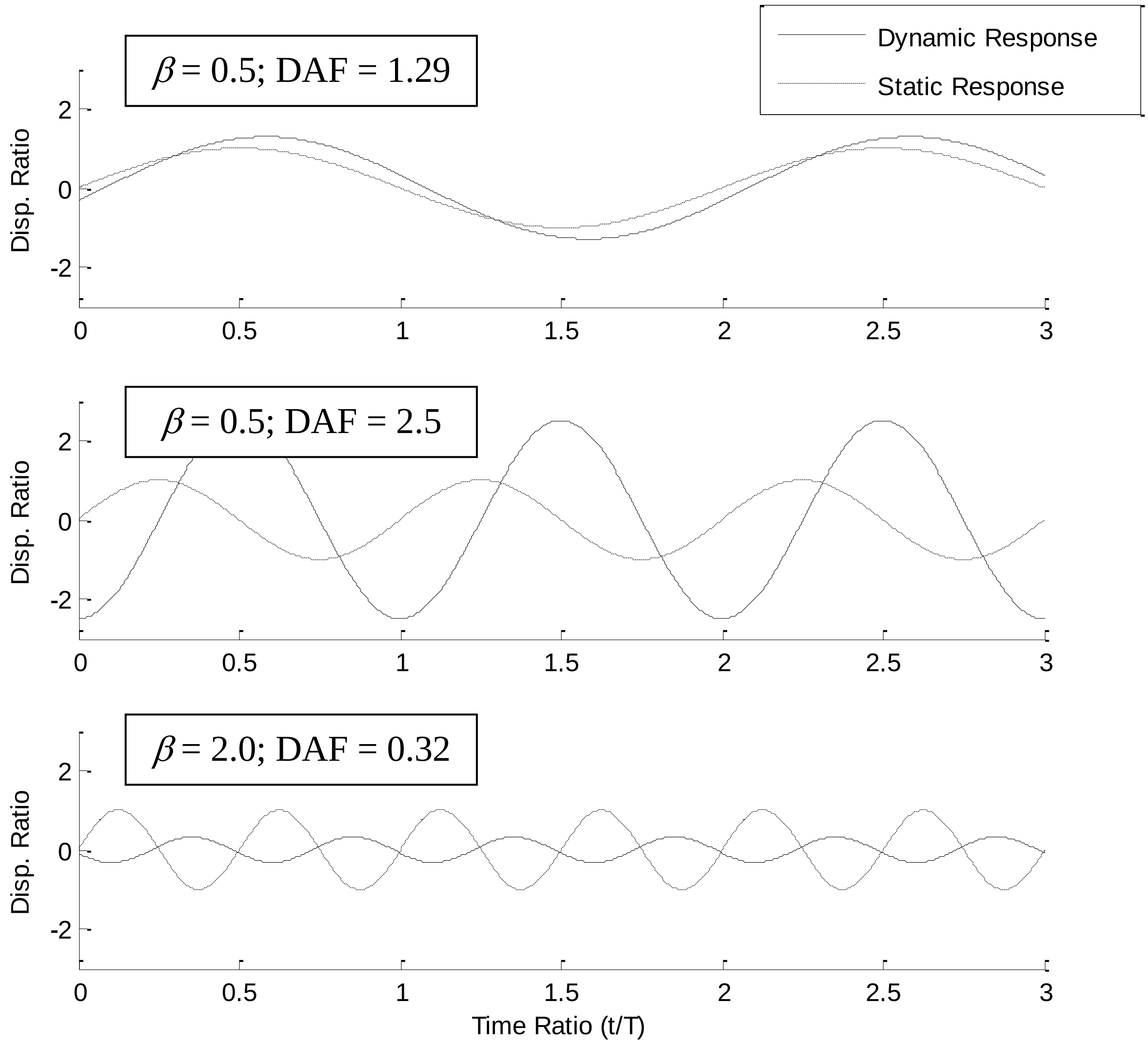


Figure 2.10: Steady-state responses to illustrate phase angle.

Note how the force and response are firstly "in sync" ($\theta \sim 0$), then "halfway out of sync" ($\theta = 90^\circ$) at resonance; and finally, "fully out of sync" ($\theta \sim 180^\circ$) at high frequency ratio.

Maximum Steady-State Displacement

The maximum steady-state displacement occurs when the DAF is a maximum. This occurs when the denominator of equation (5.2.52) is a minimum:

$$\frac{dD}{d\beta} = 0$$

$$\frac{d}{d\beta}\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]^{1/2} = 0$$

$$\frac{1}{2}\left[\frac{-4\beta\left(1-\beta^2\right) + 4\beta\left(2\xi^2\right)}{\sqrt{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2}}\right] = 0$$

$$\beta\left(-1+\beta^2+2\xi^2\right) = 0$$

The trivial solution to this equation of $\beta = 0$ corresponds to an applied forcing function that has zero frequency –the static loading effect of the forcing function. The other solution is:

$$\beta = \sqrt{1-2\xi^2} \tag{5.2.54}$$

Which for low values of damping, $\xi \leq 0.1$ approximately, is very close to unity. The corresponding maximum DAF is then given by substituting (5.2.54) into equation (5.2.52) to get:

$$D_{\max} = \frac{1}{2\xi\sqrt{1-\xi^2}} \tag{5.2.55}$$

Which reduces to equation (5.2.53) for $\beta = 1$, as it should.

Measurement of Natural Frequencies

It may be seen from equation (5.2.50) that when $\beta = 1$, $\theta = \pi/2$; this phase relationship allows the accurate measurements of the natural frequencies of structures. That is, we change the input frequency Ω in small increments until we can identify a peak response: the value of Ω at the peak response is then the natural frequency of the system. Example 2.1 gave the natural frequency based on this type of test.

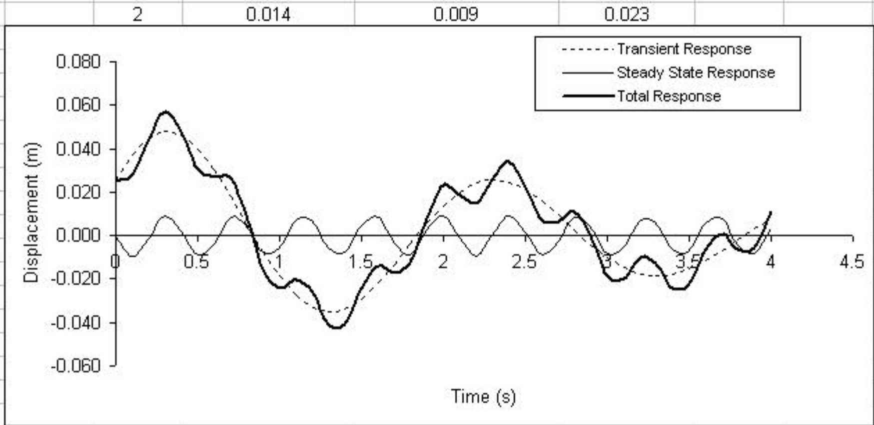
5.2.8 Computer Implementation & Examples

Using MS Excel

Again we modify our previous spreadsheet and include the extra parameters related to forced response. We've also used some of the equations from the Appendix to show the transient, steady-sate and total response. Normally however, we are only interested in the steady-state response, which the total response approaches over time.

Microsoft Excel - SDOF Forced.xls

	B	C	D	F	G	H	I
1							
2					Transient Response	Steady State Response	Total Response
3	**Input Parameters**			**t**	**u(t) (m)**	**u(t) m**	**u(t) m**
4	F0	20	N	0	0.025	0.000	0.025
5	Omega	15	rad/s	0.1	0.037	-0.009	0.028
6	m	10	kg	0.2	0.045	-0.001	0.044
7	k	100	N/m	0.3	0.048	0.009	0.057
8	xi	0.1		0.4	0.046	0.002	0.048
9	delta_t	0.1	s	0.5	0.040	-0.009	0.031
10				0.6	0.030	-0.003	0.027
11	u_0	0.025	m	0.7	0.018	0.008	0.027
12	v_0	0	m/s	0.8	0.005	0.005	0.010
13				0.9	-0.007	-0.008	-0.015
14	**Undamped Properties**			1	-0.019	-0.006	-0.024
15	w	3.162	rad/s	1.1	-0.027	0.007	-0.021
16	f	0.503	Hz	1.2	-0.033	0.007	-0.026
17	T	1.987	s	1.3	-0.035	-0.006	-0.041
18				1.4	-0.034	-0.008	-0.041
19	**Damped Properties**			1.5	-0.029	0.005	-0.024
20	wd	3.146	rad/s	1.6	-0.022	0.008	-0.014
21	fd	0.501	Hz	1.7	-0.013	-0.004	-0.017
22	Td	1.997	s	1.8	-0.004	-0.009	-0.012
23				1.9	0.006	0.002	0.008
24	**Response Properties**			2	0.014	0.009	0.023
25	Beta	4.743416					
26	DAF	0.046466					
27	ro	0.009293	m				
28	theta	3.097496	rad				
29	Aconst	0.02541					
30	Bconst	0.046815					
42				3.8	-0.002	-0.004	-0.006
43				3.9	0.003	-0.008	-0.005
44				4	0.007	0.003	0.011



Using Matlab

First let's write a little function to return the DAF, since we will use it often:

```
function D = DAF(beta,xi)
% This function returns the DAF, D, associated with the parameters:
% beta - the frequency ratio
% xi - the damping ratio

D = 1./sqrt((1-beta.^2).^2+(2*xi.*beta).^2);
```

And another to return the phase angle (always in the region $0 < \theta < \pi$):

```
function theta = phase(beta,xi)
% This function returns the pahse angle, theta, associated with the
% parameters:
% beta - the frequency ratio
% xi - the damping ratio

theta = atan2((2*xi.*beta),(1-beta.^2)); % refers to complex plane
```

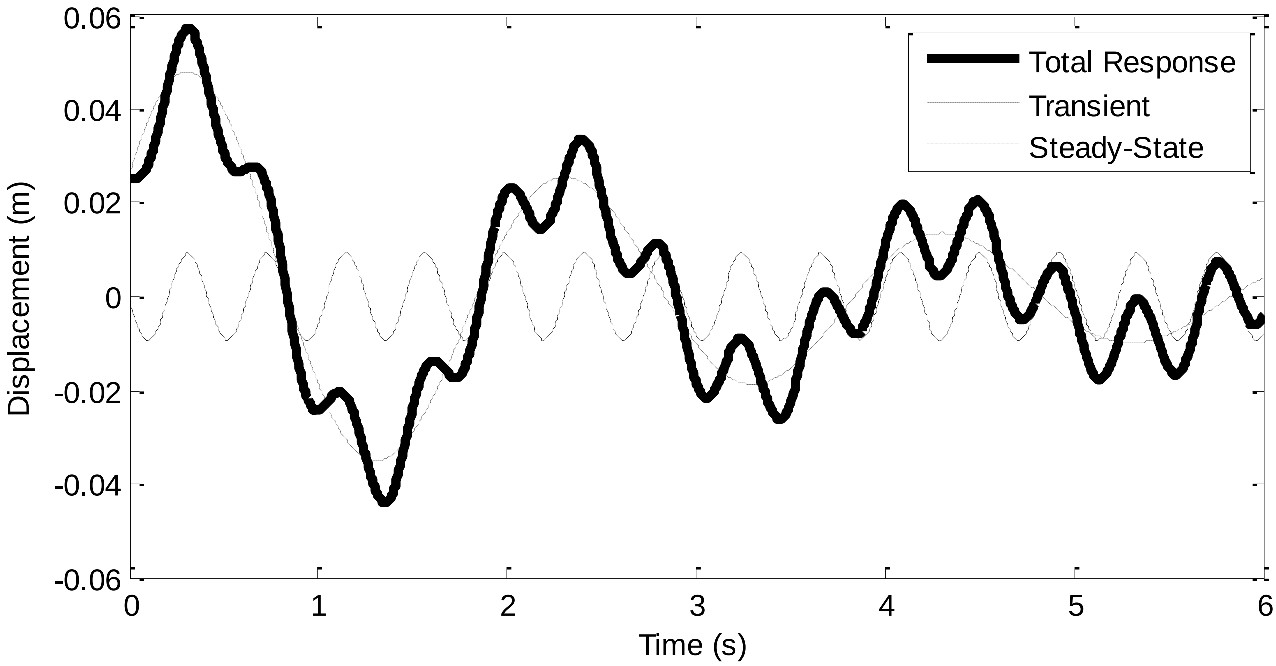
With these functions, and modifying our previous damped response script, we have:

```
function [t u] = sdof_forced(m,k,xi,u0,v0,F,Omega,duration,plotflag)
% This function returns the displacement of a damped SDOF system with
% parameters:
% m - mass, kg
% k - stiffness, N/m
% xi - damping ratio
% u0 - initial displacement, m
% v0 - initial velocity, m/s
% F - amplitude of forcing function, N
% Omega - frequency of forcing function, rad/s
% duration - length of time of required response
% plotflag - 1 or 0: whether or not to plot the response
% This function returns:
% t - the time vector at which the response was found
% u - the displacement vector of response

Npts = 1000;     % compute the response at 1000 points
delta_t = duration/(Npts-1);

w = sqrt(k/m);           % rad/s - circular natural frequency
wd = w*sqrt(1-xi^2);     % rad/s - damped circular frequency

beta = Omega/w;          % frequency ratio
D = DAF(beta,xi);        % dynamic amplification factor
ro = F/k*D;              % m - amplitude of vibration
theta = phase(beta,xi); % rad - phase angle
```

```
% Constants for the transient response
Aconst = u0+ro*sin(theta);
Bconst = (v0+u0*xi*w-ro*(Omega*cos(theta)-xi*w*sin(theta)))/wd;

t = 0:delta_t:duration;
u_transient = exp(-xi*w.*t).*(Aconst*cos(wd*t)+Bconst*sin(wd*t));
u_steady = ro*sin(Omega*t-theta);
u = u_transient + u_steady;

if(plotflag == 1)
    plot(t,u,'k');
    hold on;
    plot(t,u_transient,'k:');
    plot(t,u_steady,'k--');
    hold off;
    xlabel('Time (s)');
    ylabel('Displacement (m)');
    legend('Total Response','Transient','Steady-State');
end
```

Running this for the same problem as before with $F_0 = 10\text{ N}$ and $\Omega = 15\text{ rad/s}$ gives:

```
[t u] = sdof_forced(10,100,0.1,0.025,0,20,15,6,1);
```

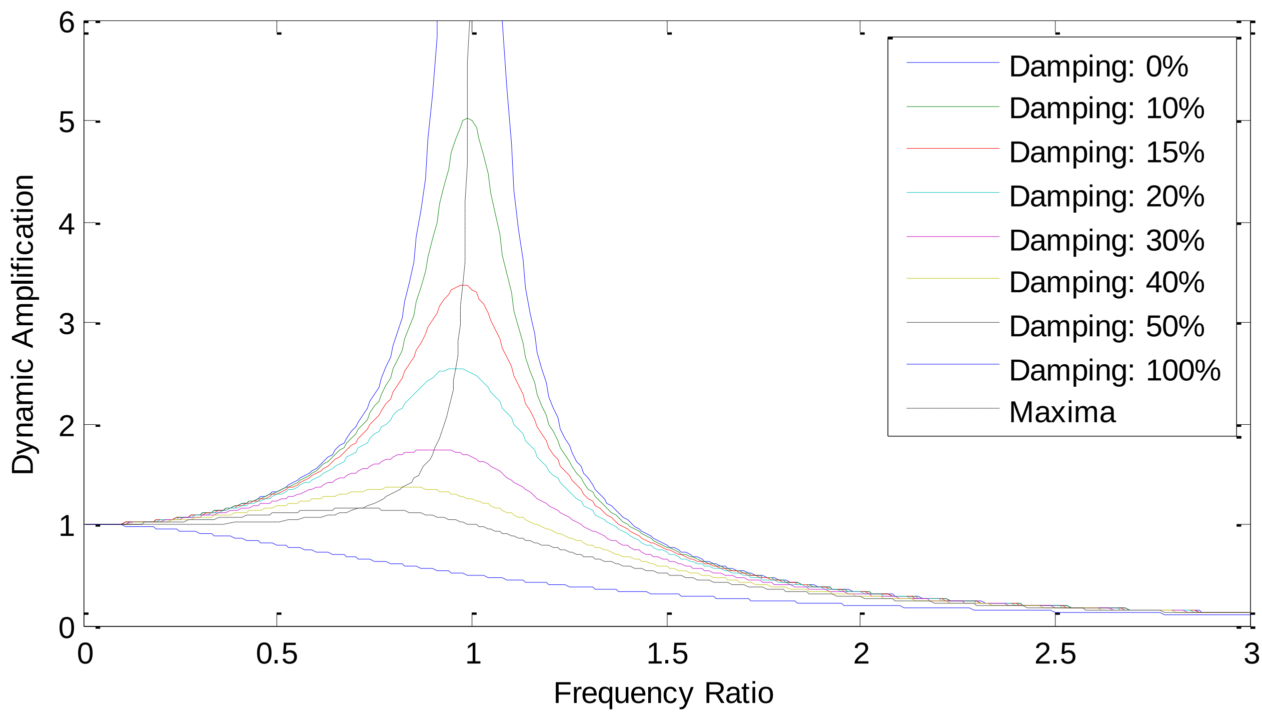


As can be seen, the total response quickly approaches the steady-state response.

Next let's use our little DAF function to plot something similar to Figure 2.8, but this time showing the frequency ratio and maximum response from equation (5.2.54):

```
% Script to plot DAF against Beta for different damping ratios
xi = [0.0001,0.1,0.15,0.2,0.3,0.4,0.5,1.0];
beta = 0.01:0.01:3;
for i = 1:length(xi)
    D(i,:) = DAF(beta,xi(i));
end
% A new xi vector for the maxima line
xi = 0:0.01:1.0;
xi(end) = 0.99999;    % very close to unity
xi(1)   = 0.00001;    % very close to zero
for i = 1:length(xi)
    betamax(i) = sqrt(1-2*xi(i)^2);
    Dmax(i) = DAF(betamax(i),xi(i));
end
plot(beta,D); hold on;
plot(betamax,Dmax,'k--');
xlabel('Frequency Ratio');
ylabel('Dynamic Amplification');
ylim([0 6]); % set y-axis limits since DAF at xi = 0 is enormous
legend( 'Damping: 0%','Damping: 10%','Damping: 15%',...
        'Damping: 20%','Damping: 30%','Damping: 40%',...
        'Damping: 50%', 'Damping: 100%', 'Maxima');
```

This gives:



Lastly then, using the phase function we wrote, we can generate Figure 2.9:

```
% Script to plot phase against Beta for different damping ratios
xi = [0.0001,0.1,0.2,0.5,1.0];
beta = 0.01:0.01:3;
for i = 1:length(xi)
    T(i,:) = phase(beta,xi(i))*(180/pi); % in degrees
end
plot(beta,T);
xlabel('Frequency Ratio');
ylabel('Phase Angle (degrees)');
ylim([0 180]);
set(gca,'ytick',[0 45 90 135 180]);
grid on;
legend('Damping: 0%','Damping: 10%','Damping: 20%','Damping: 50%',...
    'Damping: 100%','Location','SE');
```

5.2.9 Numerical Integration – Newmark's Method

Introduction

The loading that can be applied to a structure is infinitely variable and closed-form mathematical solutions can only be achieved for a small number of cases. For arbitrary excitation we must resort to computational methods, which aim to solve the basic structural dynamics equation, at the next time-step:

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + ku_{i+1} = F_{i+1} \tag{5.2.56}$$

There are three basic time-stepping approaches to the solution of the structural dynamics equations:

1. Interpolation of the excitation function;
2. Use of finite differences of velocity and acceleration;
3. An assumed variation of acceleration.

We will examine one method from the third category only. However, it is an important method and is extensible to non-linear systems, as well as multi degree-of-freedom systems (MDOF).

Development of Newmark's Method

In 1959 Newmark proposed a general assumed variation of acceleration method:

$$\dot{u}_{i+1} = \dot{u}_i + \left[(1-\gamma)\Delta t\right]\ddot{u}_i + (\gamma\,\Delta t)\ddot{u}_{i+1} \tag{5.2.57}$$

$$u_{i+1} = u_i + (\Delta t)\dot{u}_i + \left[(0.5-\beta)(\Delta t)^2\right]\ddot{u}_i + \left[\beta(\Delta t)^2\right]\ddot{u}_{i+1} \tag{5.2.58}$$

The parameters β and γ define how the acceleration is assumed over the time step, Δt. Usual values are $\gamma = \frac{1}{2}$ and $\frac{1}{6} \le \beta \le \frac{1}{4}$. For example:

- Constant (average) acceleration is given by: $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$;
- Linear variation of acceleration is given by: $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$.

The three equations presented thus far (equations (5.2.56), (5.2.57) and (5.2.58)) are sufficient to solve for the three unknown responses at each time step. However to avoid iteration, we introduce the incremental form of the equations:

$$\Delta u_i \equiv u_{i+1} - u_i \tag{5.2.59}$$

$$\Delta \dot{u}_i \equiv \dot{u}_{i+1} - \dot{u}_i \tag{5.2.60}$$

$$\Delta \ddot{u}_i \equiv \ddot{u}_{i+1} - \ddot{u}_i \tag{5.2.61}$$

$$\Delta F_i \equiv F_{i+1} - F_i \tag{5.2.62}$$

Thus, Newmark's equations can now be written as:

$$\Delta \dot{u}_i = (\Delta t)\ddot{u}_i + (\gamma\,\Delta t)\Delta\ddot{u}_i \tag{5.2.63}$$

$$\Delta u_i = (\Delta t)\dot{u}_i + \frac{(\Delta t)^2}{2}\ddot{u}_i + \beta(\Delta t)^2 \Delta\ddot{u}_i \quad (5.2.64)$$

Solving equation (5.2.64) for the unknown change in acceleration gives:

$$\Delta\ddot{u}_i = \frac{1}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta(\Delta t)}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \quad (5.2.65)$$

Substituting this into equation (5.2.63) and solving for the unknown increment in velocity gives:

$$\Delta\dot{u}_i = \frac{\gamma}{\beta(\Delta t)}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i \quad (5.2.66)$$

Next we use the incremental equation of motion, derived from equation (5.2.56):

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + k\Delta u_i = \Delta F_i \quad (5.2.67)$$

And introduce equations (5.2.65) and (5.2.66) to get:

$$\begin{aligned} & m\left[\frac{1}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta(\Delta t)}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i\right] \\ & \quad + c\left[\frac{\gamma}{\beta(\Delta t)}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i\right] + k\Delta u_i = \Delta F_i \end{aligned} \quad (5.2.68)$$

Collecting terms gives:

$$\left[m\frac{1}{\beta(\Delta t)^2}+c\frac{\gamma}{\beta(\Delta t)}+k\right]\Delta u_i = \Delta F_i + \left[\frac{1}{\beta(\Delta t)}m+\frac{\gamma}{\beta}c\right]\dot{u}_i + \left[\frac{1}{2\beta}m+\Delta t\left(\frac{\gamma}{2\beta}-1\right)c\right]\ddot{u}_i \tag{5.2.69}$$

Let's introduce the following for ease of representation:

$$\hat{k} = m\frac{1}{\beta(\Delta t)^2}+c\frac{\gamma}{\beta(\Delta t)}+k \tag{5.2.70}$$

$$\Delta\hat{F}_i = \Delta F_i + \left[\frac{1}{\beta(\Delta t)}m+\frac{\gamma}{\beta}c\right]\dot{u}_i + \left[\frac{1}{2\beta}m+\Delta t\left(\frac{\gamma}{2\beta}-1\right)c\right]\ddot{u}_i \tag{5.2.71}$$

Which are an effective stiffness and effective force at time *i*. Thus equation (5.2.69) becomes:

$$\hat{k}\Delta u_i = \Delta\hat{F}_i \tag{5.2.72}$$

Since $\hat{k}$ and $\Delta\hat{F}_i$ are known from the system properties (*m*, *c*, *k*); the algorithm properties (γ, β, Δt); and the previous time-step ($\dot{u}_i$, $\ddot{u}_i$), we can solve equation (5.2.72) for the displacement increment:

$$\Delta u_i = \frac{\Delta\hat{F}_i}{\hat{k}} \tag{5.2.73}$$

Once the displacement increment is known, we can solve for the velocity and acceleration increments from equations (5.2.66) and (5.2.65) respectively. And once all the increments are known we can compute the properties at the current time-step by just adding to the values at the previous time-step, equations (5.2.59) to (5.2.61).

Newmark's method is stable if the time-steps is about $\Delta t = 0.1T$ of the system.

The coefficients in equation (5.2.71) are constant (once Δt is), so we can calculate these at the start as:

$$A = \frac{1}{\beta(\Delta t)} m + \frac{\gamma}{\beta} c \tag{5.2.74}$$

$$B = \frac{1}{2\beta} m + \Delta t \left(\frac{\gamma}{2\beta} - 1 \right) c \tag{5.2.75}$$

Making equation (5.2.71) become:

$$\Delta \hat{F}_i = \Delta F_i + A\dot{u}_i + B\ddot{u}_i \tag{5.2.76}$$

Newmark's Algorithm

1. Select algorithm parameters, γ, β and Δt;
2. Initial calculations:
    a. Find the initial acceleration:

$$\ddot{u}_0 = \frac{1}{m}\left(F_0 - c\dot{u}_0 - ku_0\right) \tag{5.2.77}$$

    b. Calculate the effective stiffness, $\hat{k}$ from equation (5.2.70);
    c. Calculate the coefficients for equation (5.2.71) from equations (5.2.74) and (5.2.75).
3. For each time step, *i*, calculate:

$$\Delta\hat{F}_i = \Delta F_i + A\dot{u}_i + B\ddot{u}_i \tag{5.2.78}$$

$$\Delta u_i = \frac{\Delta\hat{F}_i}{\hat{k}} \tag{5.2.79}$$

$$\Delta\dot{u}_i = \frac{\gamma}{\beta(\Delta t)}\Delta u_i - \frac{\gamma}{\beta}\dot{u}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\ddot{u}_i \tag{5.2.80}$$

$$\Delta\ddot{u}_i = \frac{1}{\beta(\Delta t)^2}\Delta u_i - \frac{1}{\beta(\Delta t)}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \tag{5.2.81}$$

$$u_i = u_{i-1} + \Delta u_i \tag{5.2.82}$$

$$\dot{u}_i = \dot{u}_{i-1} + \Delta\dot{u}_i \tag{5.2.83}$$

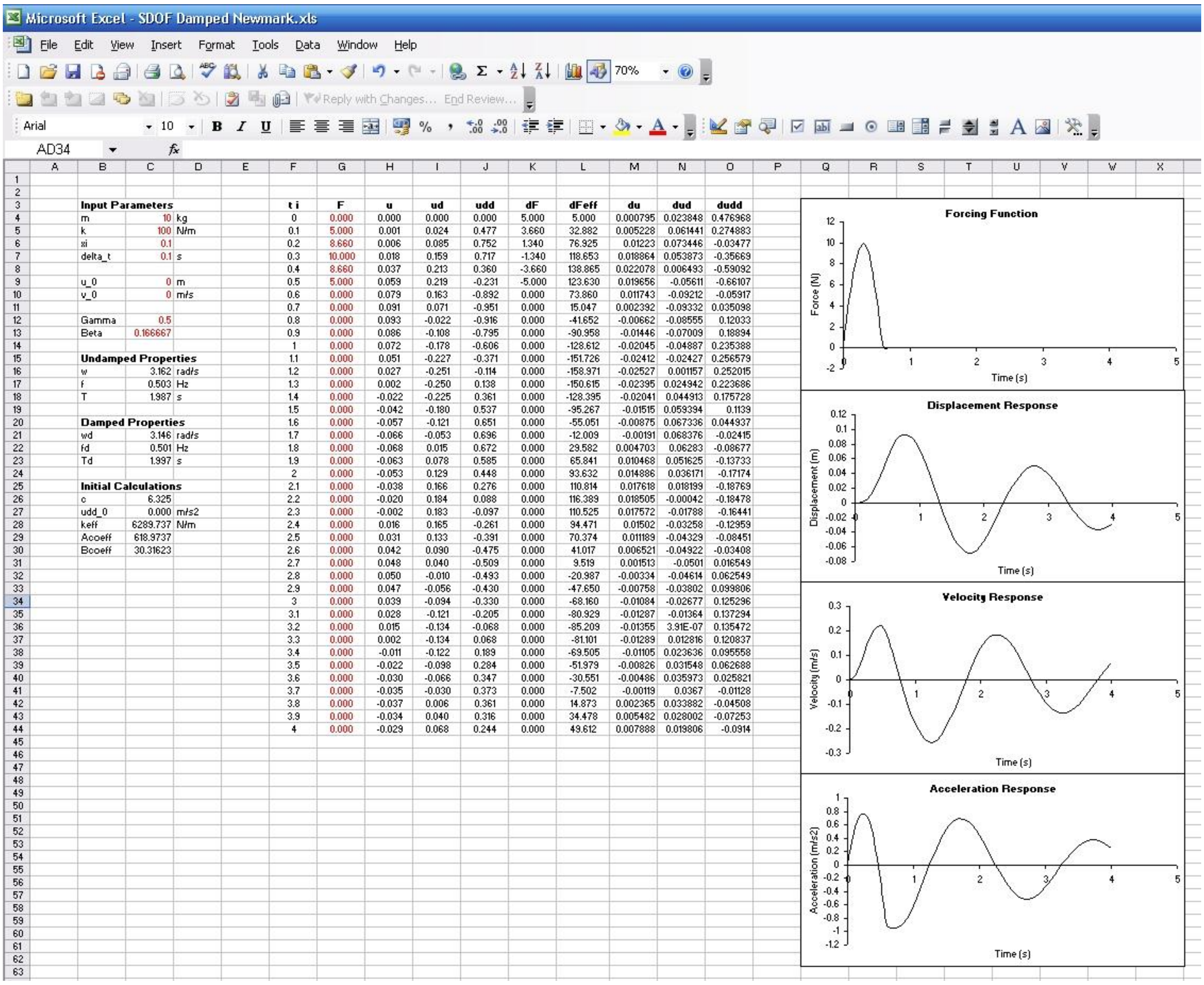
$$\ddot{u}_i = \ddot{u}_{i-1} + \Delta\ddot{u}_i \tag{5.2.84}$$

5.2.10 Computer Implementation & Examples

Using MS Excel

Based on our previous spreadsheet, we implement Newmark Integration. Download it from the course website, and see how the equations and algorithm are implemented.

In the example shown, we've applied a sinusoidal load of 10 N for 0.6 secs to the system we've been using so far:



Input Parameters		
m	10	kg
k	100	N/m
xi	0.1	
delta_t	0.1	s
u_0	0	m
v_0	0	m/s
Gamma	0.5	
Beta	0.166667	
Undamped Properties		
w	3.162	rad/s
f	0.503	Hz
T	1.987	s
Damped Properties		
wd	3.146	rad/s
fd	0.501	Hz
Td	1.997	s
Initial Calculations		
c	6.325	
udd_0	0.000	m/s2
keff	6289.737	N/m
Acoeff	618.9737	
Bcoeff	30.31623	

t i	F	u	ud	udd	dF	dFeff	du	dud	dudd
0	0.000	0.000	0.000	0.000	5.000	5.000	0.000795	0.023848	0.476968
0.1	5.000	0.001	0.024	0.477	3.660	32.882	0.005228	0.061441	0.274883
0.2	8.660	0.006	0.085	0.752	1.340	76.925	0.01223	0.073446	-0.03477
0.3	10.000	0.018	0.159	0.717	-1.340	118.653	0.018864	0.053873	-0.35669
0.4	8.660	0.037	0.213	0.360	-3.660	138.865	0.022078	0.006493	-0.59092
0.5	5.000	0.059	0.219	-0.231	-5.000	123.630	0.019656	-0.05611	-0.66107
0.6	0.000	0.079	0.163	-0.892	0.000	73.860	0.011743	-0.09212	-0.05917
0.7	0.000	0.091	0.071	-0.951	0.000	15.047	0.002392	-0.09332	0.035098
0.8	0.000	0.093	-0.022	-0.916	0.000	-41.652	-0.00662	-0.08555	0.12033
0.9	0.000	0.086	-0.108	-0.795	0.000	-90.958	-0.01446	-0.07009	0.18894
1	0.000	0.072	-0.178	-0.606	0.000	-128.612	-0.02045	-0.04887	0.235388
1.1	0.000	0.051	-0.227	-0.371	0.000	-151.726	-0.02412	-0.02427	0.256579
1.2	0.000	0.027	-0.251	-0.114	0.000	-158.971	-0.02527	0.001157	0.252015
1.3	0.000	0.002	-0.250	0.138	0.000	-150.615	-0.02395	0.024942	0.223686
1.4	0.000	-0.022	-0.225	0.361	0.000	-128.395	-0.02041	0.044913	0.175728
1.5	0.000	-0.042	-0.180	0.537	0.000	-95.267	-0.01515	0.059394	0.1139
1.6	0.000	-0.057	-0.121	0.651	0.000	-55.051	-0.00875	0.067336	0.044937
1.7	0.000	-0.066	-0.053	0.696	0.000	-12.009	-0.00191	0.068376	-0.02415
1.8	0.000	-0.068	0.015	0.672	0.000	29.582	0.004703	0.06283	-0.08677
1.9	0.000	-0.063	0.078	0.585	0.000	65.841	0.010468	0.051625	-0.13733
2	0.000	-0.053	0.129	0.448	0.000	93.632	0.014886	0.036171	-0.17174
2.1	0.000	-0.038	0.166	0.276	0.000	110.814	0.017618	0.018199	-0.18769
2.2	0.000	-0.020	0.184	0.088	0.000	116.389	0.018505	-0.00042	-0.18478
2.3	0.000	-0.002	0.183	-0.097	0.000	110.525	0.017572	-0.01788	-0.16441
2.4	0.000	0.016	0.165	-0.261	0.000	94.471	0.01502	-0.03258	-0.12959
2.5	0.000	0.031	0.133	-0.391	0.000	70.374	0.011189	-0.04329	-0.08451
2.6	0.000	0.042	0.090	-0.475	0.000	41.017	0.006521	-0.04922	-0.03408
2.7	0.000	0.048	0.040	-0.509	0.000	9.519	0.001513	-0.0501	0.016549
2.8	0.000	0.050	-0.010	-0.493	0.000	-20.987	-0.00334	-0.04614	0.062549
2.9	0.000	0.047	-0.056	-0.430	0.000	-47.650	-0.00758	-0.03802	0.099806
3	0.000	0.039	-0.094	-0.330	0.000	-68.160	-0.01084	-0.02677	0.125296
3.1	0.000	0.028	-0.121	-0.205	0.000	-80.929	-0.01287	-0.01364	0.137294
3.2	0.000	0.015	-0.134	-0.068	0.000	-85.209	-0.01355	3.91E-07	0.135472
3.3	0.000	0.002	-0.134	0.068	0.000	-81.101	-0.01289	0.012816	0.120837
3.4	0.000	-0.011	-0.122	0.189	0.000	-69.505	-0.01105	0.023636	0.095558
3.5	0.000	-0.022	-0.098	0.284	0.000	-51.979	-0.00826	0.031548	0.062688
3.6	0.000	-0.030	-0.066	0.347	0.000	-30.551	-0.00486	0.035973	0.025821
3.7	0.000	-0.035	-0.030	0.373	0.000	-7.502	-0.00119	0.0367	-0.01128
3.8	0.000	-0.037	0.006	0.361	0.000	14.873	0.002365	0.033882	-0.04508
3.9	0.000	-0.034	0.040	0.316	0.000	34.478	0.005482	0.028002	-0.07253
4	0.000	-0.029	0.068	0.244	0.000	49.612	0.007888	0.019806	-0.0914

Using Matlab

There are no shortcuts to this one. We must write a completely new function that implements the Newmark Integration algorithm as we've described it:

```
function [u ud udd] = newmark_sdof(m, k, xi, t, F, u0, ud0, plotflag)
% This function computes the response of a linear damped SDOF system
% subject to an arbitrary excitation. The input parameters are:
% m         - scalar, mass, kg
% k         - scalar, stiffness, N/m
% xi        - scalar, damping ratio
% t         - vector of length N, in equal time steps, s
% F         - vector of length N, force at each time step, N
% u0        - scalar, initial displacement, m
% v0        - scalar, initial velocity, m/s
% plotflag  - 1 or 0: whether or not to plot the response
% The output is:
% u         - vector of length N, displacement response, m
% ud        - vector of length N, velocity response, m/s
% udd       - vector of length N, acceleration response, m/s2

% Set the Newmark Integration parameters
% gamma = 1/2 always
% beta = 1/6 linear acceleration
% beta = 1/4 average acceleration
gamma = 1/2;
beta = 1/6;

N = length(t);  % the number of integration steps
dt = t(2)-t(1); % the time step
w = sqrt(k/m);  % rad/s - circular natural frequency
c = 2*xi*k/w;   % the damping coefficient

% Calulate the effective stiffness
keff = k + (gamma/(beta*dt))*c+(1/(beta*dt^2))*m;
% Calulate the coefficients A and B
Acoeff = (1/(beta*dt))*m+(gamma/beta)*c;
Bcoeff = (1/(2*beta))*m + dt*(gamma/(2*beta)-1)*c;

% calulate the change in force at each time step
dF = diff(F);

% Set initial state
u(1) = u0;
ud(1) = ud0;
udd(1) = (F(1)-c*ud0-k*u0)/m; % the initial acceleration

for i = 1:(N-1) % N-1 since we already know solution at i = 1
    dFeff = dF(i) + Acoeff*ud(i) + Bcoeff*udd(i);
    dui = dFeff/keff;
    dudi = (gamma/(beta*dt))*dui-(gamma/beta)*ud(i)+dt*(1-
gamma/(2*beta))*udd(i);
    duddi = (1/(beta*dt^2))*dui-(1/(beta*dt))*ud(i)-(1/(2*beta))*udd(i);
    u(i+1) = u(i) + dui;
    ud(i+1) = ud(i) + dudi;
```

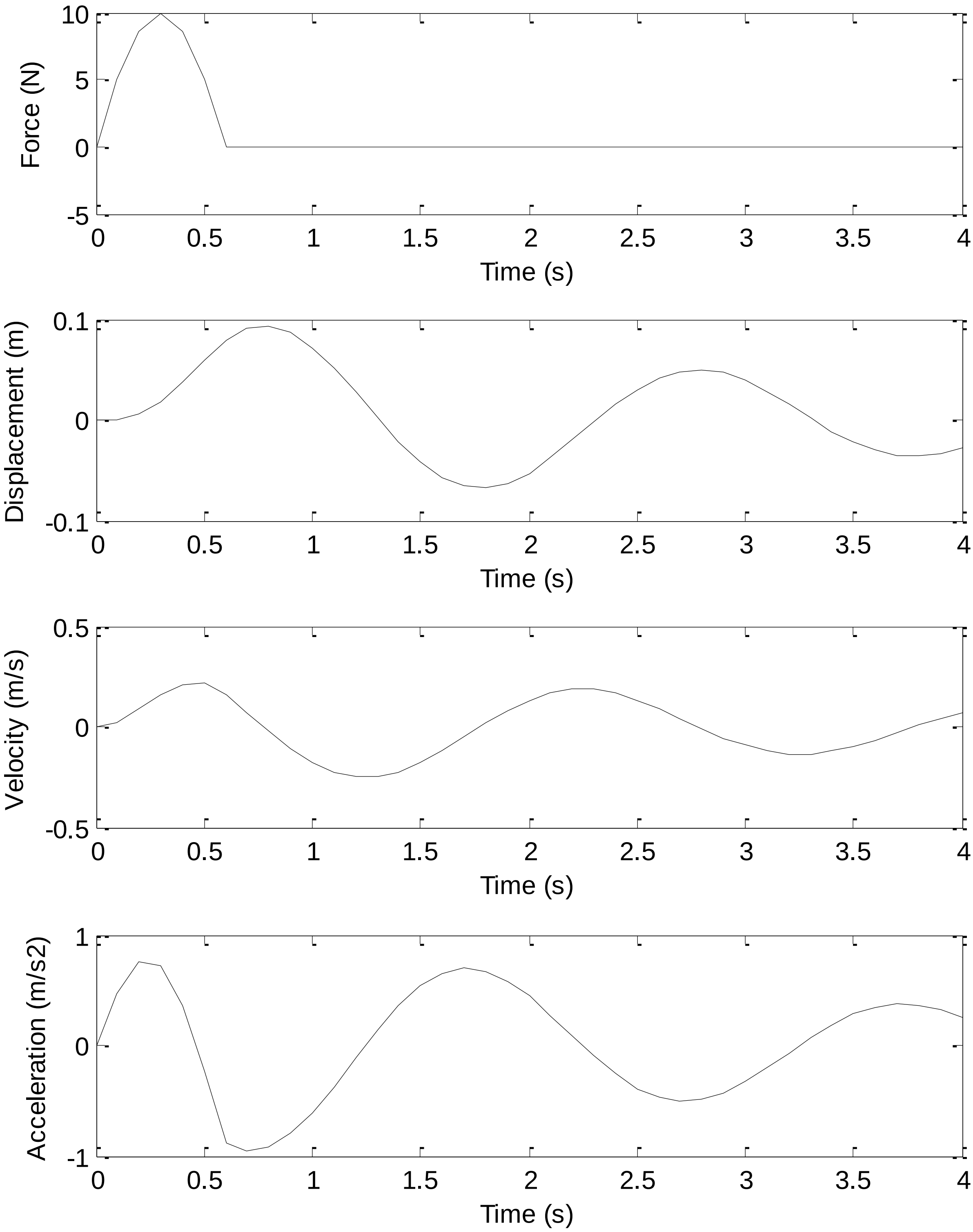
```
    udd(i+1) = udd(i) + duddi;
end

if(plotflag == 1)
    subplot(4,1,1)
    plot(t,F,'k');
    xlabel('Time (s)');
    ylabel('Force (N)');
    subplot(4,1,2)
    plot(t,u,'k');
    xlabel('Time (s)');
    ylabel('Displacement (m)');
    subplot(4,1,3)
    plot(t,ud,'k');
    xlabel('Time (s)');
    ylabel('Velocity (m/s)');
    subplot(4,1,4)
    plot(t,udd,'k');
    xlabel('Time (s)');
    ylabel('Acceleration (m/s2)');
end
```

Bear in mind that most of this script is either comments or plotting commands – Newmark Integration is a fast and small algorithm, with a huge range of applications.

In order to use this function, we must write a small script that sets the problem up and then calls the `newmark_sdof` function. The main difficulty is in generating the forcing function, but it is not that hard:

```
% script that calls Newmark Integration for sample problem
m = 10;
k = 100;
xi = 0.1;
u0 = 0;
ud0 = 0;
t = 0:0.1:4.0;              % set the time vector
F = zeros(1,length(t));     % empty F vector
% set sinusoidal force of 10 over 0.6 s
Famp = 10;
Tend = 0.6;
i = 1;
while t(i) < Tend
    F(i) = Famp*sin(pi*t(i)/Tend);
    i = i+1;
end

[u ud udd] = newmark_sdof(m, k, xi, t, F, u0, ud0, 1);
```

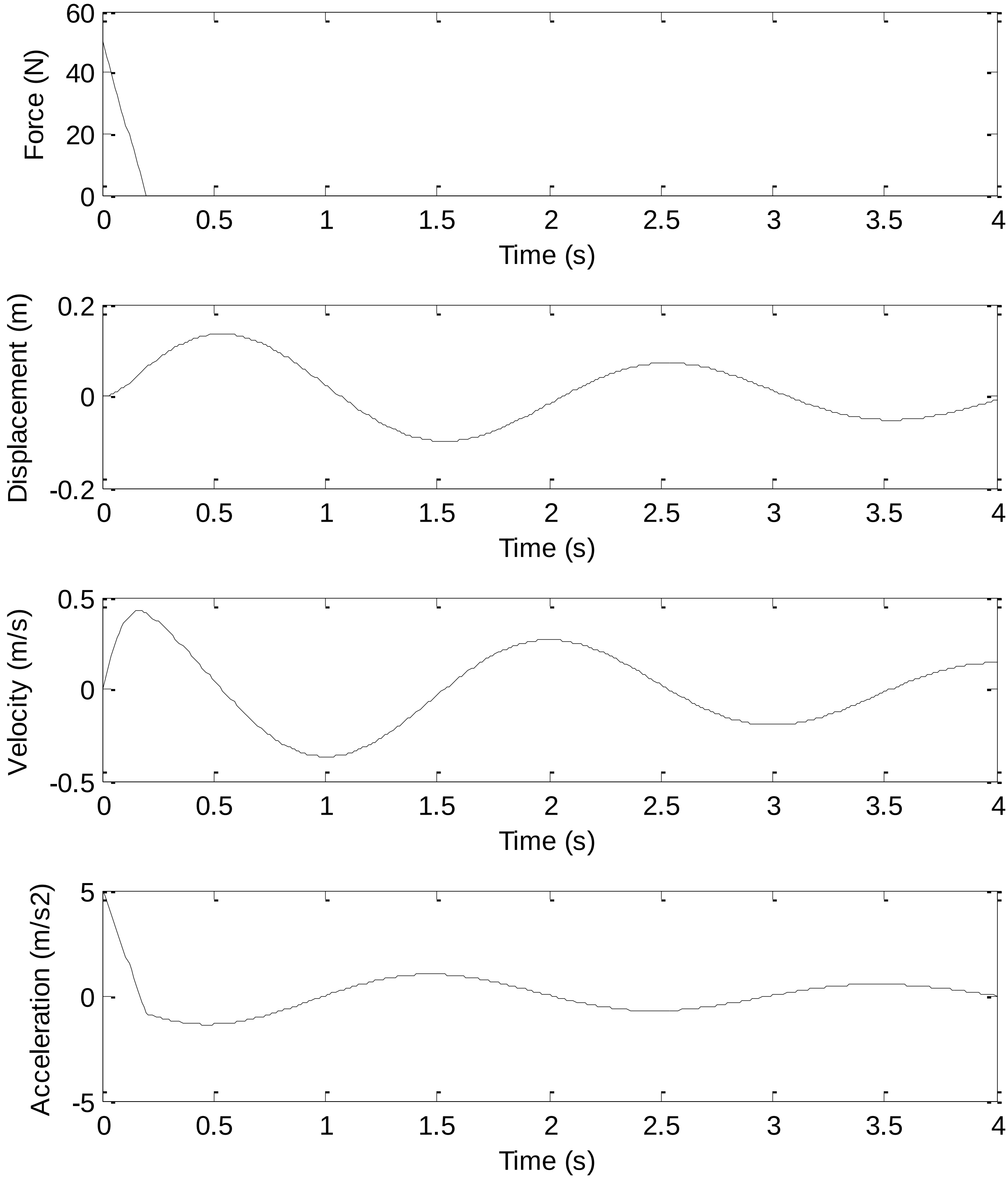
This produces the following plot:



Explosions are often modelled as triangular loadings. Let's implement this for our system:

```
% script that finds explosion response
m = 10;
k = 100;
xi = 0.1;
u0 = 0;
ud0 = 0;
Fmax = 50;      % N
Tend = 0.2;     % s
t = 0:0.01:2.0;             % set the time vector
F = zeros(1,length(t));     % empty F vector
% set reducing triangular force
i = 1;
while t(i) < Tend
    F(i) = Fmax*(1-t(i)/Tend);
    i = i+1;
end

[u ud udd] = newmark_sdof(m, k, xi, t, F, u0, ud0, 1);
```

As can be seen from the following plot, even though the explosion only lasts for a brief period of time, the vibrations will take several periods to dampen out. Also notice that the acceleration response is the most sensitive – this is the most damaging to the building, as force is mass times acceleration: the structure thus undergoes massive forces, possibly leading to damage or failure.


Force (N)
60
40
20
0
Time (s)
Displacement (m)
0.2
0
-0.2
Time (s)
Velocity (m/s)
0.5
0
-0.5
Time (s)
Acceleration (m/s2)
5
0
-5
0 0.5 1 1.5 2 2.5 3 3.5 4
Time (s)

5.2.11 Problems

Problem 1

A harmonic oscillation test gave the natural frequency of a water tower to be 0.41 Hz. Given that the mass of the tank is 150 tonnes, what deflection will result if a 50 kN horizontal load is applied? You may neglect the mass of the tower.

Ans: 50.2 mm

Problem 2

A 3 m high, 8 m wide single-bay single-storey frame is rigidly jointed with a beam of mass 5,000 kg and columns of negligible mass and stiffness of $EI_c = 4.5\times10^3$ kNm^2. Calculate the natural frequency in lateral vibration and its period. Find the force required to deflect the frame 25 mm laterally.

Ans: 4.502 Hz; 0.222 sec; 100 kN

Problem 3

An SDOF system (m = 20 kg, k = 350 N/m) is given an initial displacement of 10 mm and initial velocity of 100 mm/s. (a) Find the natural frequency; (b) the period of vibration; (c) the amplitude of vibration; and (d) the time at which the third maximum peak occurs.

Ans: 0.666 Hz; 1.502 sec; 25.91 mm; 3.285 sec.

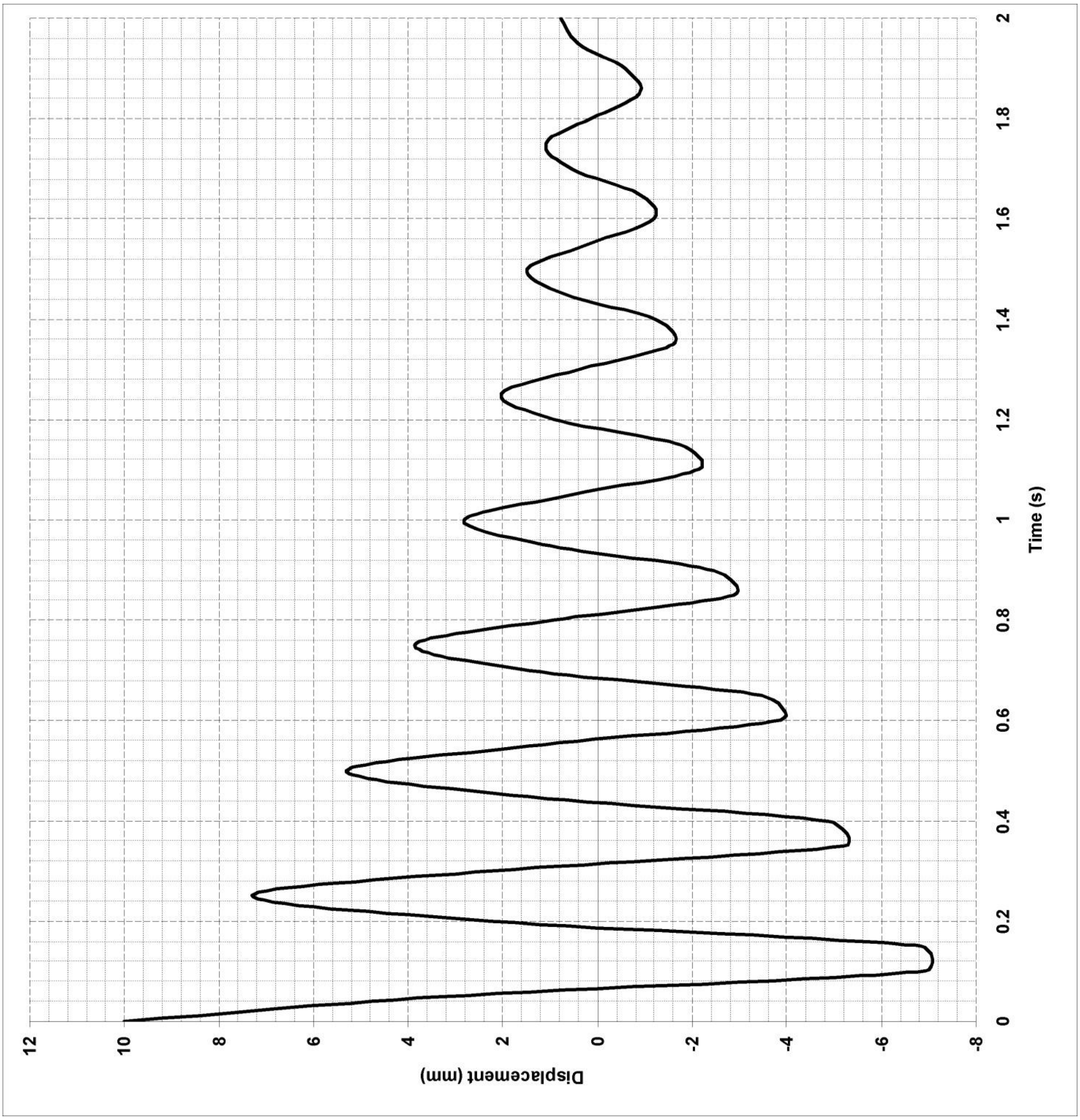
Problem 4

For the frame of Problem 2, a jack applied a load of 100 kN and then instantaneously released. On the first return swing a deflection of 19.44 mm was noted. The period of motion was measured at 0.223 sec. Assuming that the stiffness of the columns cannot change, find (a) the damping ratio; (b) the coefficient of damping; (c) the undamped frequency and period; and (d) the amplitude after 5 cycles.

Ans: 0.04; 11,367 kg·s/m; 4.488 Hz; 0.2228 sec; 7.11 mm.

Problem 5

From the response time-history of an SDOF system given:



(a) estimate the damped natural frequency; (b) use the half amplitude method to calculate the damping ratio; and (c) calculate the undamped natural frequency and period.

Ans: 4.021 Hz; 0.05; 4.026 Hz; 0.248 sec.

Problem 6

Workers' movements on a platform (8 × 6 m high, m = 200 kN) are causing large dynamic motions. An engineer investigated and found the natural period in sway to be 0.9 sec. Diagonal remedial ties (E = 200 kN/mm^2) are to be installed to reduce the natural period to 0.3 sec. What tie diameter is required?

Ans: 28.1 mm.

Problem 7

The frame of examples 2.2 and 2.4 has a reciprocating machine put on it. The mass of this machine is 4 tonnes and is in addition to the mass of the beam. The machine exerts a periodic force of 8.5 kN at a frequency of 1.75 Hz. (a) What is the steady-state amplitude of vibration if the damping ratio is 4%? (b) What would the steady-state amplitude be if the forcing frequency was in resonance with the structure?

Ans: 2.92 mm; 26.56 mm.

Problem 8

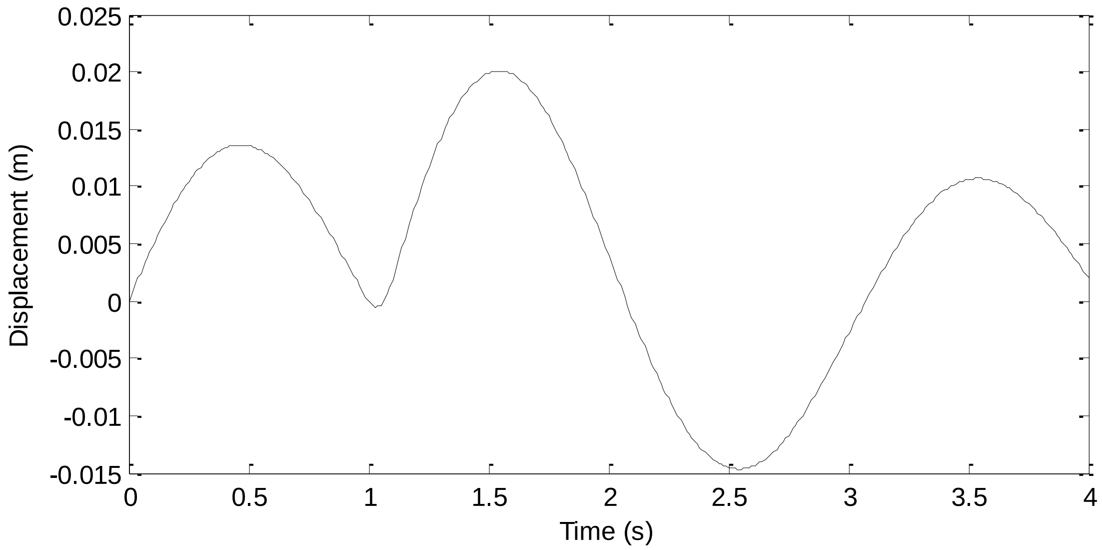
An air conditioning unit of mass 1,600 kg is place in the middle (point C) of an 8 m long simply supported beam (EI = 8×10^3 kNm^2) of negligible mass. The motor runs at 300 rpm and produces an unbalanced load of 120 kg. Assuming a damping ratio of 5%, determine the steady-state amplitude and deflection at C. What rpm will result in resonance and what is the associated deflection?

Ans: 1.41 mm; 22.34 mm; 206.7 rpm; 36.66 mm.

Problem 9

Determine the response of our example system, with initial velocity of 0.05 m/s, when acted upon by an impulse of 0.1 s duration and magnitude 10 N at time 1.0 s. Do this up for a duration of 4 s.

Ans. below



Problem 10

Determine the maximum responses of a water tower which is subjected to a sinusoidal force of amplitude 445 kN and frequency 30 rad/s over 0.3 secs. The tower has properties, mass 17.5 t, stiffness 17.5 MN/m and no damping.

Ans. 120 mm, 3.8 m/s, 120.7 m/s²

Problem 11

Determine the maximum response of a system (m = 1.75 t, k = 1.75 MN/m, ξ = 10%) when subjected to an increasing triangular load which reaches 22.2 kN after 0.1 s.

Ans. 14.6 mm, 0.39 m/s, 15.0 m/s²

5.3 Multi-Degree-of-Freedom Systems

5.3.1 General Case (based on 2DOF)

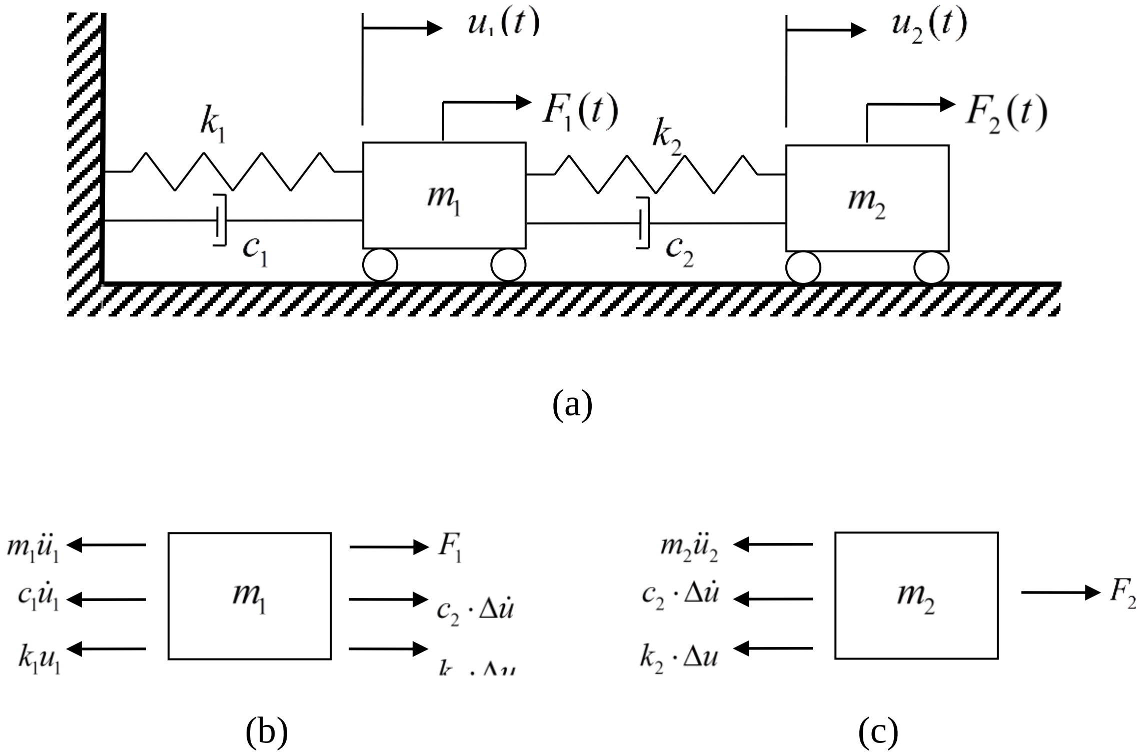


Figure 3.1: (a) 2DOF system. (b) and (c) Free-body diagrams of forces

Considering Figure 3.1, we can see that the forces that act on the masses are similar to those of the SDOF system but for the fact that the springs, dashpots, masses, forces and deflections may all differ in properties. Also, from the same figure, we can see the interaction forces between the masses will result from the relative deflection between the masses; the change in distance between them.

For each mass, $\sum F_x = 0$, hence:

$$m_1\ddot{u}_1 + c_1\dot{u}_1 + k_1u_1 + c_2\left(\dot{u}_1 - \dot{u}_2\right) + k_2\left(u_1 - u_2\right) = F_1 \qquad (5.3.1)$$

$$m_2\ddot{u}_2 + c_2\left(\dot{u}_2 - \dot{u}_1\right) + k_2\left(u_2 - u_1\right) = F_2 \tag{5.3.2}$$

In which we have dropped the time function indicators and allowed Δu and $\Delta \dot{u}$ to absorb the directions of the interaction forces. Re-arranging we get:

$$\begin{aligned} \ddot{u}_1 m_1 \quad &+\dot{u}_1\left(c_1 + c_2\right) &+\dot{u}_2\left(-c_2\right) \quad &+u_1\left(k_1 + k_2\right) &+u_2\left(-k_2\right) \quad &= F_1 \\ \ddot{u}_2 m_2 \quad &+\dot{u}_1\left(-c_2\right) &+\dot{u}_2\left(c_2\right) \quad &+u_1\left(-k_2\right) &+u_2\left(k_2\right) \quad &= F_2 \end{aligned} \tag{5.3.3}$$

This can be written in matrix form:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{u}_1 \\ \dot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \tag{5.3.4}$$

Or another way:

$$\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{F} \tag{5.3.5}$$

where:

M is the mass matrix (diagonal matrix);

ü is the vector of the accelerations for each DOF;

C is the damping matrix (symmetrical matrix);

u̇ is the vector of velocity for each DOF;

K is the stiffness matrix (symmetrical matrix);

u is the vector of displacements for each DOF;

F is the load vector.

Equation (5.3.5) is quite general and reduces to many forms of analysis:

Free vibration:

$$\mathbf{M\ddot{u} + C\dot{u} + Ku = 0} \tag{5.3.6}$$

Undamped free vibration:

$$\mathbf{M\ddot{u} + Ku = 0} \tag{5.3.7}$$

Undamped forced vibration:

$$\mathbf{M\ddot{u} + Ku = F} \tag{5.3.8}$$

Static analysis:

$$\mathbf{Ku = F} \tag{5.3.9}$$

We will restrict our attention to the case of undamped free-vibration – equation (5.3.7) - as the inclusion of damping requires an increase in mathematical complexity which would distract from our purpose.

5.3.2 Free-Undamped Vibration of 2DOF Systems

The solution to (5.3.7) follows the same methodology as for the SDOF case; so following that method (equation (2.42)), we propose a solution of the form:

$$\mathbf{u} = \mathbf{a}\sin\left(\omega t + \phi\right) \tag{5.3.10}$$

where $\mathbf{a}$ is the *vector of amplitudes* corresponding to each degree of freedom. From this we get:

$$\ddot{\mathbf{u}} = -\omega^2\mathbf{a}\sin\left(\omega t + \phi\right) = -\omega^2\mathbf{u} \tag{5.3.11}$$

Then, substitution of (5.3.10) and (5.3.11) into (5.3.7) yields:

$$-\omega^2\mathbf{M}\mathbf{a}\sin\left(\omega t + \phi\right) + \mathbf{K}\mathbf{a}\sin\left(\omega t + \phi\right) = \mathbf{0} \tag{5.3.12}$$

Since the sine term is constant for each term:

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{a} = \mathbf{0} \tag{5.3.13}$$

We note that in a dynamics problem the amplitudes of each DOF will be non-zero, hence, $\mathbf{a} \neq \mathbf{0}$ in general. In addition we see that the problem is a standard eigenvalues problem. Hence, by Cramer's rule, in order for (5.3.13) to hold the determinant of $\mathbf{K} - \omega^2\mathbf{M}$ must then be zero:

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0 \tag{5.3.14}$$

For the 2DOF system, we have:

$$\left|\mathbf{K}-\omega^2\mathbf{M}\right| = \left[\left(k_2+k_1\right)-\omega^2 m_1\right]\left[k_2-\omega^2 m_2\right]-k_2^2=0 \qquad (5.3.15)$$

Expansion of (5.3.15) leads to an equation in ω^2 called the *characteristic polynomial* of the system. The solutions of ω^2 to this equation are the eigenvalues of $\left[\mathbf{K}-\omega^2\mathbf{M}\right]$. There will be two solutions or roots of the characteristic polynomial in this case and an *n*-DOF system has *n* solutions to its characteristic polynomial. In our case, this means there are two values of ω^2 (ω_1^2 and ω_2^2) that will satisfy the relationship; thus there are two frequencies for this system (the lowest will be called the fundamental frequency). For each ω_n^2 substituted back into (5.3.13), we will get a certain amplitude vector $\mathbf{a}_n$. This means that each frequency will have its own characteristic displaced shape of the degrees of freedoms called the mode shape. However, we will not know the absolute values of the amplitudes as it is a free-vibration problem; hence we express the mode shapes as a *vector of relative amplitudes,* $\boldsymbol{\varphi}_n$, relative to, normally, the first value in $\mathbf{a}_n$.

As we will see in the following example, the implication of the above is that MDOF systems vibrate, not just in the fundamental mode, but also in higher harmonics. From our analysis of SDOF systems it's apparent that should any loading coincide with any of these harmonics, large DAF's will result (Section 2.d). Thus, some modes may be critical design cases depending on the type of harmonic loading as will be seen later.

5.3.3 Example of a 2DOF System

The two-storey building shown (Figure 3.2) has very stiff floor slabs relative to the supporting columns. Calculate the natural frequencies and mode shapes.

$$EI_c = 4.5 \times 10^3 \text{ kNm}^2$$

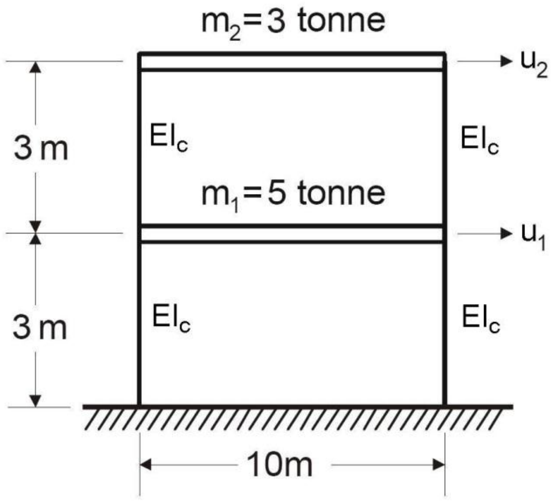


Figure 3.2: Shear frame problem.

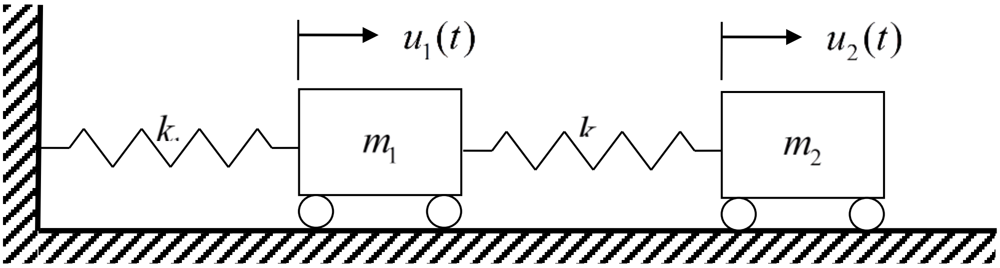


Figure 3.3: 2DOF model of the shear frame.

We will consider the free lateral vibrations of the two-storey shear frame idealised as in Figure 3.3. The lateral, or shear stiffness of the columns is:

$$k_1 = k_2 = k = 2\left[\frac{12EI_c}{h^3}\right]$$

$$\therefore k = \frac{2 \times 12 \times 4.5 \times 10^6}{3^3}$$

$$= 4 \times 10^6 \text{ N/m}$$

The characteristic polynomial is as given in (5.3.15) so we have:

$$\left[8\times10^6-\omega^2 5000\right]\left[4\times10^6-\omega^2 3000\right]-16\times10^{12}=0$$
$$\therefore 15\times10^6\omega^4-4.4\times10^{10}\omega^2+16\times10^{12}=0$$

This is a quadratic equation in ω^2 and so can be solved using $a=15\times10^6$, $b=-4.4\times10^{10}$ and $c=16\times10^{12}$ in the usual expression

$$\omega^2=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Hence we get $\omega_1^2=425.3$ and $\omega_2^2=2508$. This may be written:

$$\boldsymbol{\omega}_n^2=\begin{Bmatrix}425.3\\2508\end{Bmatrix}\text{ hence }\boldsymbol{\omega}_n=\begin{Bmatrix}20.6\\50.1\end{Bmatrix}\text{ rad/s and }\mathbf{f}=\frac{\boldsymbol{\omega}_n}{2\pi}=\begin{Bmatrix}3.28\\7.97\end{Bmatrix}\text{ Hz}$$

To solve for the mode shapes, we will use the appropriate form of the equation of motion, equation (5.3.13): $\left[\mathbf{K}-\omega^2\mathbf{M}\right]\mathbf{a}=\mathbf{0}$. First solve for the $\mathbf{E}=\left[\mathbf{K}-\omega^2\mathbf{M}\right]$ matrix and then solve $\mathbf{Ea}=\mathbf{0}$ for the amplitudes $\mathbf{a}_n$. Then, form $\boldsymbol{\varphi}_n$.

In general, for a 2DOF system, we have:

$$\mathbf{E}_n=\begin{bmatrix}k_1+k_2 & -k_2\\-k_2 & k_2\end{bmatrix}-\omega_n^2\begin{bmatrix}m_1 & 0\\0 & m_2\end{bmatrix}=\begin{bmatrix}k_1+k_2-\omega_n^2 m_1 & -k_2\\-k_2 & k_2-\omega_n^2 m_2\end{bmatrix}$$

For $\omega_1^2=425.3$:

$$\mathbf{E}_1=\begin{bmatrix}5.8735 & -4\\-4 & 2.7241\end{bmatrix}\times10^6$$

Hence

$$\mathbf{E}_1\mathbf{a}_1 = 10^6 \begin{bmatrix} 5.8735 & -4 \\ -4 & 2.7241 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Taking either equation, we calculate:

$$\left.\begin{array}{ll} 5.8735a_1 - 4a_2 = 0 & \Rightarrow a_1 = 0.681a_2 \\ -4a_1 + 2.7241a_2 = 0 & \Rightarrow a_1 = 0.681a_2 \end{array}\right\} \therefore \boldsymbol{\varphi}_1 = \begin{Bmatrix} 1 \\ 0.681^{-1} \end{Bmatrix}$$

Similarly for $\omega_2^2 = 2508$:

$$\mathbf{E}_2 = \begin{bmatrix} -4.54 & -4 \\ -4 & -3.524 \end{bmatrix} \times 10^6$$

Hence, again taking either equation, we calculate:

$$\left.\begin{array}{ll} -4.54a_1 - 4a_2 = 0 & \Rightarrow a_1 = -0.881a_2 \\ -4a_1 - 3.524a_2 = 0 & \Rightarrow a_1 = -0.881a_2 \end{array}\right\} \therefore \boldsymbol{\varphi}_2 = \begin{Bmatrix} 1 \\ -0.881^{-1} \end{Bmatrix}$$

The complete solution may be given by the following two matrices which are used in further analysis for more complicated systems.

$$\boldsymbol{\omega}_n^2 = \begin{Bmatrix} 425.3 \\ 2508 \end{Bmatrix} \text{ and } \boldsymbol{\Phi} = \begin{bmatrix} 1 & 1 \\ 1.468 & -1.135 \end{bmatrix}$$

For our frame, we can sketch these two frequencies and associated mode shapes: Figure 3.4.

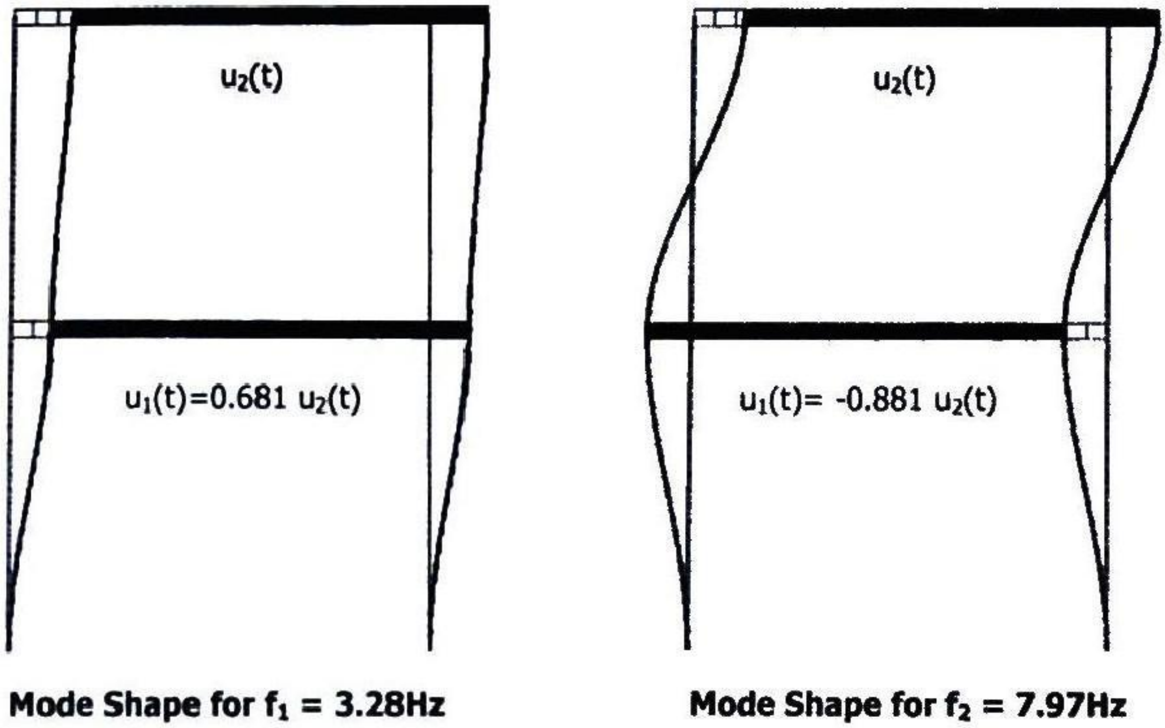


Figure 3.4: Mode shapes and frequencies of the example frame.

Larger and more complex structures will have many degrees of freedom and hence many natural frequencies and mode shapes. There are different mode shapes for different forms of deformation; torsional, lateral and vertical for example. Periodic loads acting in these directions need to be checked against the fundamental frequency for the type of deformation; higher harmonics may also be important.

As an example; consider a 2DOF idealisation of a cantilever which assumes stiffness proportional to the static deflection at $0.5L$ and L as well as half the cantilever mass 'lumped' at the midpoint and one quarter of it lumped at the tip. The mode shapes are shown in Figure 3.5. In Section 4(a) we will see the exact mode shape for this – it is clear that the approximation is rough; but, with more DOFs it will approach a better solution.

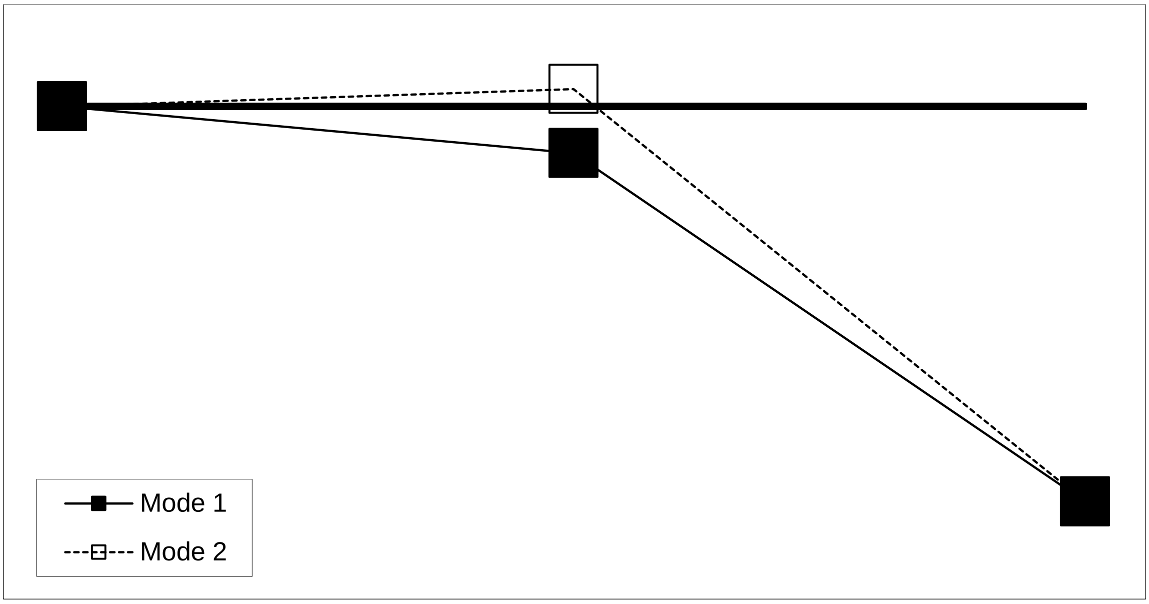


Figure 3.5: Lumped mass, 2DOF idealisation of a cantilever.

5.3.4 Case Study – Aberfeldy Footbridge, Scotland

Returning to the case study in Section 1, we will look at the results of some research conducted into the behaviour of this bridge which forms part of the current research into lateral synchronise excitation discovered on the London Millennium footbridge. This is taken from a paper by Dr. Paul Archbold, formerly of University College Dublin.

Mode	Mode Type	Measured Frequency (Hz)	Predicted Frequency (Hz)	
1	L1	0.98	1.14	+16%
2	V1	1.52	1.63	+7%
3	V2	1.86	1.94	+4%
4	V3	2.49	2.62	+5%
5	L2	2.73	3.04	+11%
6	V4	3.01	3.11	+3%
7	V5	3.50	3.63	+4%
8	V6	3.91	4.00	+2%
9	T1	3.48	4.17	20%
10	V7	4.40	4.45	+1%
11	V8	4.93	4.90	-1%
12	T2	4.29	5.20	+21%
13	L3	5.72	5.72	+0%
14	T3	5.72	6.07	+19%

Table 1: Modal frequencies

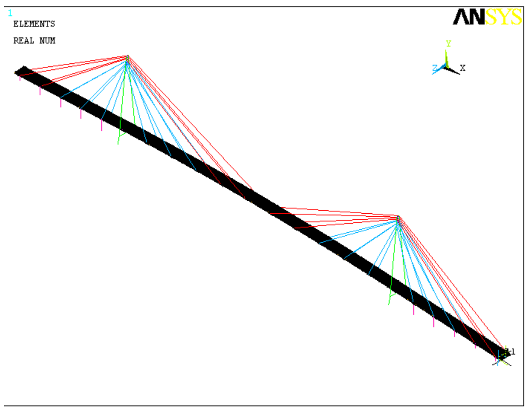
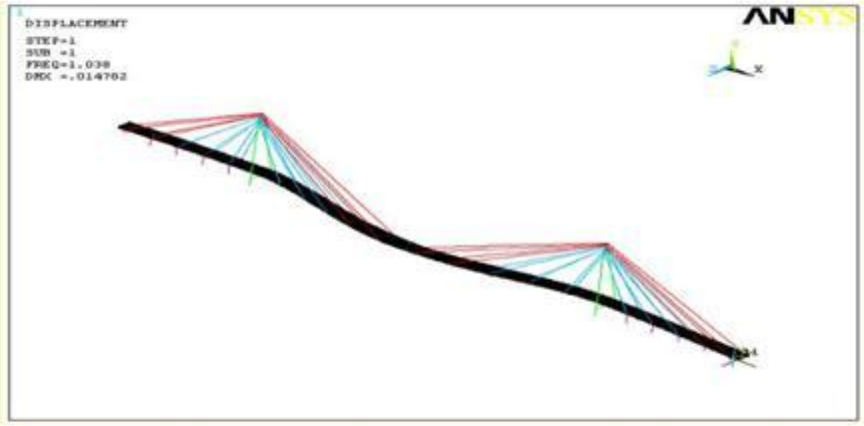


Figure 3.6: Undeformed shape

Table 1 gives the first 14 mode and associated frequencies from both direct measurements of the bridge and from finite-element modelling of it. The type of mode is also listed; L is lateral, V is vertical and T is torsional. It can be seen that the predicted frequencies differ slightly from the measured; however, the modes have been estimated in the correct sequence and there may be some measurement error.

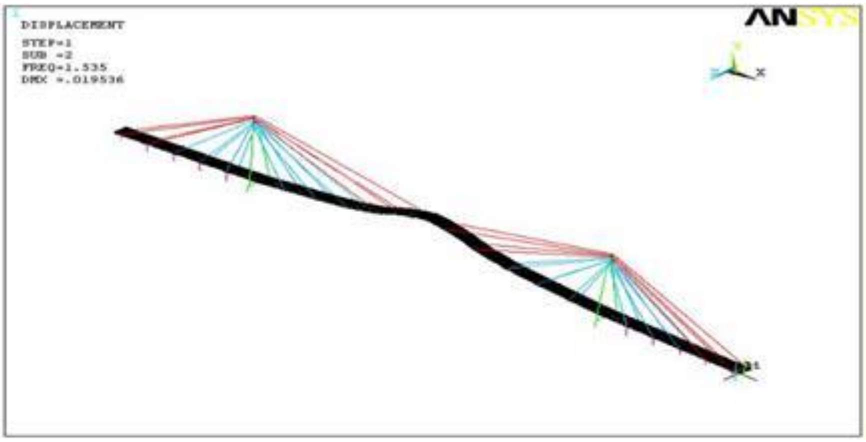
We can see now that (from Section 1) as a person walks at about 2.8 Hz, there are a lot of modes that may be excited by this loading. Also note that the overall fundamental mode is lateral – this was the reason that this bridge has been analysed – it is similar to the Millennium footbridge in this respect. Figure 1.7 illustrates the dynamic motion due to a person walking on this bridge – this is probably caused by the third or fourth mode. Several pertinent mode shapes are given in Figure 3.7.



Mode 1:

1st Lateral mode

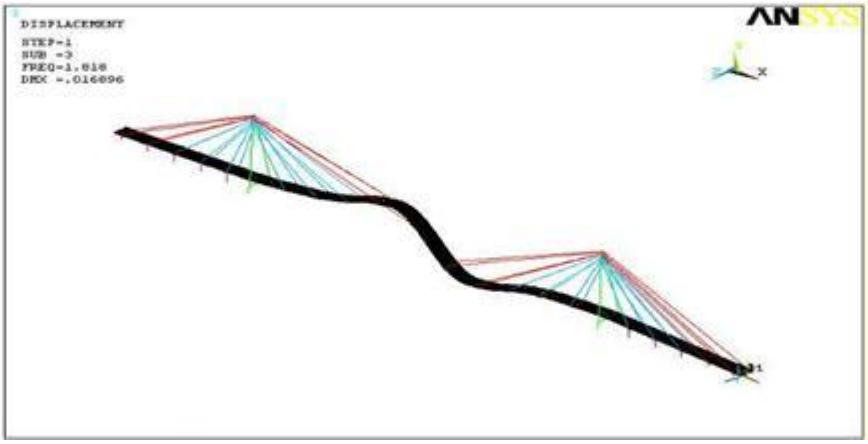
1.14 Hz



Mode 2:

1st Vertical mode

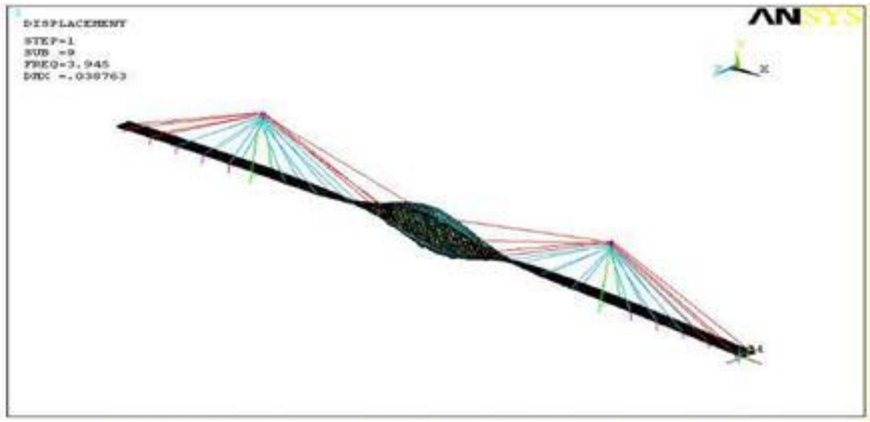
1.63 Hz



Mode 3:

2nd Vertical mode

1.94 Hz



Mode 9:

1st Torsional mode

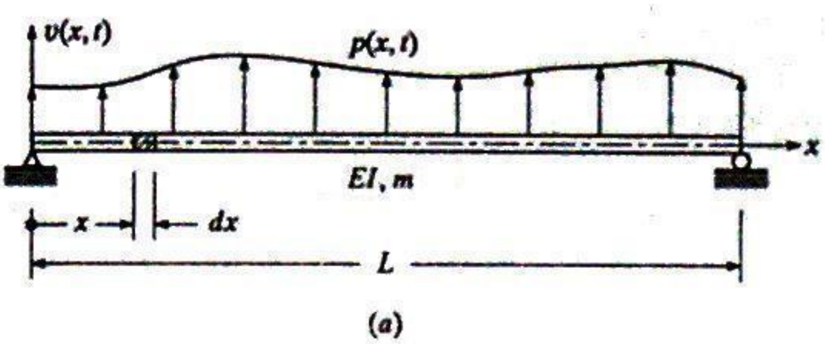
4.17 Hz

Figure 3.7: Various Modes of Aberfeldy footbridge.

5.4 Continuous Structures

5.4.1 Exact Analysis for Beams

General Equation of Motion



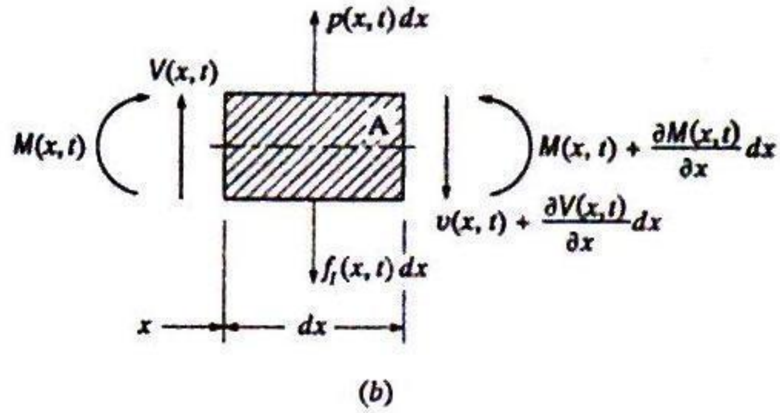


Figure 4.1: Basic beam subjected to dynamic loading: (a) beam properties and coordinates; (b) resultant forces acting on the differential element.

In examining Figure 4.1, as with any continuous structure, it may be seen that any differential element will have an associated stiffness and deflection – which changes with time – and hence a different acceleration. Thus, any continuous structure has an infinite number of degrees of freedom. Discretization into an MDOF structure is certainly an option and is the basis for finite-element dynamic analyses; the more DOF's used the more accurate the model (Section 3.b). For some basic structures though, the exact behaviour can be explicitly calculated. We will limit ourselves to free-undamped vibration of beams that are thin in comparison to their length. A general expression can be derived and from this, several usual cases may be established.

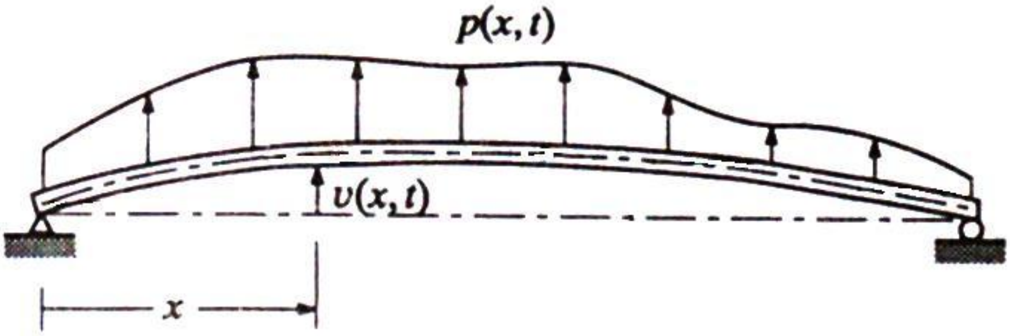


Figure 4.2: Instantaneous dynamic deflected position.

Consider the element A of Figure 4.1(b); $\sum F_y = 0$, hence:

$$p(x,t)dx - \frac{\partial V(x,t)}{\partial x}dx - f_I(x,t)dx = 0 \tag{5.4.1}$$

after having cancelled the common $V(x,t)$ shear term. The resultant transverse inertial force is (mass × acceleration; assuming constant mass):

$$f_I(x,t)dx = mdx\frac{\partial^2 v(x,t)}{\partial t^2} \tag{5.4.2}$$

Thus we have, after dividing by the common dx term:

$$\frac{\partial V(x,t)}{\partial x} = p(x,t) - m\frac{\partial^2 v(x,t)}{\partial t^2} \tag{5.4.3}$$

which, with no acceleration, is the usual static relationship between shear force and applied load. By taking moments about the point A on the element, and dropping second order and common terms, we get the usual expression:

$$V(x,t) = \frac{\partial M(x,t)}{\partial x} \tag{5.4.4}$$

Differentiating this with respect to x and substituting into (5.4.3), in addition to the relationship $M = EI \, \partial^2 v / \partial x^2$ (which assumes that the beam is of constant stiffness):

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} + m \frac{\partial^2 v(x,t)}{\partial t^2} = p(x,t) \tag{5.4.5}$$

With free vibration this is:

$$EI \frac{\partial^4 v(x,t)}{\partial x^4} + m \frac{\partial^2 v(x,t)}{\partial t^2} = 0 \tag{5.4.6}$$

General Solution for Free-Undamped Vibration

Examination of equation (5.4.6) yields several aspects:

- It is separated into spatial (x) and temporal (t) terms and we may assume that the solution is also;
- It is a fourth-order differential in x; hence we will need four spatial boundary conditions to solve – these will come from the support conditions at each end;
- It is a second order differential in t and so we will need two temporal initial conditions to solve – initial deflection and velocity at a point for example.

To begin, assume the solution is of a form of separated variables:

$$v(x,t) = \phi(x) Y(t) \tag{5.4.7}$$

where $\phi(x)$ will define the deformed shape of the beam and $Y(t)$ the amplitude of vibration. Inserting the assumed solution into (5.4.6) and collecting terms we have:

$$\frac{EI}{m} \frac{1}{\phi(x)} \frac{\partial^4 \phi(x)}{\partial x^4} = -\frac{1}{Y(t)} \frac{\partial^2 Y(t)}{\partial t^2} = \text{constant} = \omega^2 \tag{5.4.8}$$

This follows as the terms each side of the equals are functions of x and t separately and so must be constant. Hence, each function type (spatial or temporal) is equal to ω^2 and so we have:

$$EI \frac{\partial^4 \phi(x)}{\partial x^4} = \omega^2 m \phi(x) \tag{5.4.9}$$

$$\ddot{Y}(t) + \omega^2 Y(t) = 0 \tag{5.4.10}$$

Equation (5.4.10) is the same as for an SDOF system (equation (2.4)) and so the solution must be of the same form (equation (2.17)):

$$Y(t) = Y_0 \cos \omega t + \left(\frac{\dot{Y}_0}{\omega} \right) \sin \omega t \tag{5.4.11}$$

In order to evaluate ω we will use equation (5.4.9) and we introduce:

$$\alpha^4 = \frac{\omega^2 m}{EI} \tag{5.4.12}$$

And assuming a solution of the form $\phi(x) = G \exp(sx)$, substitution into (5.4.9) gives:

$$\left(s^4 - \alpha^4\right) G \exp(sx) = 0 \tag{5.4.13}$$

There are then four roots for s and when each is put into (5.4.13) and added we get:

$$\phi(x) = G_1 \exp(i\alpha x) + G_2 \exp(-i\alpha x) + G_3 \exp(\alpha x) + G_4 \exp(-\alpha x) \tag{5.4.14}$$

In which the G's may be complex constant numbers, but, by using Euler's expressions for cos, sin, sinh and cosh we get:

$$\phi(x) = A_1 \sin(\alpha x) + A_2 \cos(\alpha x) + A_3 \sinh(\alpha x) + A_4 \cosh(-\alpha x) \tag{5.4.15}$$

where the A's are now real constants; three of which may be evaluated through the boundary conditions; the fourth however is arbitrary and will depend on ω.

Simply-supported Beam

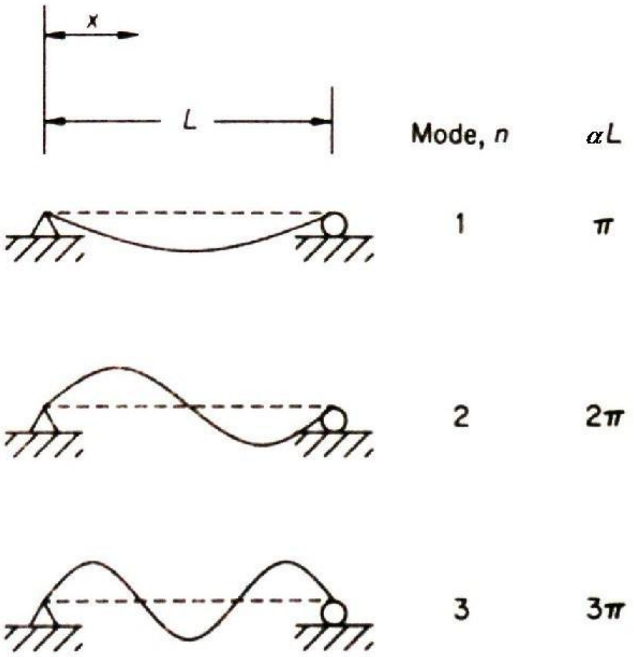


Figure 4.3: First three mode shapes and frequency parameters for an s-s beam.

The boundary conditions consist of zero deflection and bending moment at each end:

$$v(0,t)=0 \text{ and } EI\frac{\partial^2 v}{\partial x^2}(0,t)=0 \tag{5.4.16}$$

$$v(L,t)=0 \text{ and } EI\frac{\partial^2 v}{\partial x^2}(L,t)=0 \tag{5.4.17}$$

Substituting (5.4.16) into equation (5.4.14) we find $A_2 = A_4 = 0$. Similarly, (5.4.17) gives:

$$\begin{aligned}\phi(L) &= A_1\sin(\alpha L)+A_3\sinh(\alpha L)=0\\ \phi''(L) &= -\alpha^2 A_1\sin(\lambda L)+\alpha^2 A_3\sinh(\alpha L)=0\end{aligned} \tag{5.4.18}$$

from which, we get two possibilities:

$$\begin{aligned} 0 &= 2A_3 \sinh(\alpha L) \\ 0 &= A_1 \sin(\alpha L) \end{aligned} \tag{5.4.19}$$

however, since $\sinh(\lambda x)$ is never zero, A_3 must be, and so the non-trivial solution $A_1 \neq 0$ must give us:

$$\sin(\alpha L) = 0 \tag{5.4.20}$$

which is the *frequency equation* and is only satisfied when $\lambda L = n\pi$. Hence, from (5.4.12) we get:

$$\omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{m}} \tag{5.4.21}$$

and the corresponding modes shapes are therefore:

$$\phi_n(x) = A_1 \sin\left(\frac{n\pi x}{L}\right) \tag{5.4.22}$$

where A_1 is arbitrary and normally taken to be unity. We can see that there are an infinite number of frequencies and mode shapes ($n \to \infty$) as we would expect from an infinite number of DOFs. The first three mode shapes and frequencies are shown in Figure 4.3.

Cantilever Beam

This example is important as it describes the sway behaviour of tall buildings. The boundary conditions consist of:

$$v(0,t)=0 \text{ and } \frac{\partial v}{\partial x}(0,t)=0 \tag{5.4.23}$$

$$EI\frac{\partial^2 v}{\partial x^2}(L,t)=0 \text{ and } EI\frac{\partial^3 v}{\partial x^3}(L,t)=0 \tag{5.4.24}$$

Which represent zero displacement and slope at the support and zero bending moment and shear at the tip. Substituting (5.4.23) into equation (5.4.14) we get $A_4=-A_2$ and $A_3=-A_1$. Similarly, (5.4.24) gives:

$$\begin{aligned}\phi''(L)&=-\alpha^2A_1\sin(\alpha L)-\alpha^2A_2\cos(\alpha L)+\alpha^2A_3\sinh(\alpha L)+\alpha^2A_4\cosh(\alpha L)=0\\ \phi'''(L)&=-\alpha^3A_1\cos(\alpha L)+\alpha^3A_2\sin(\alpha L)+\alpha^3A_3\cosh(\alpha L)+\alpha^3A_4\sinh(\alpha L)=0\end{aligned} \tag{5.4.25}$$

where a prime indicates a derivate of x, and so we find:

$$\begin{aligned}&A_1\left(\sin(\alpha L)+\sinh(\alpha L)\right)+A_2\left(\cos(\alpha L)+\cosh(\alpha L)\right)=0\\ &A_1\left(\cos(\alpha L)+\cosh(\alpha L)\right)+A_2\left(-\sin(\alpha L)+\sinh(\alpha L)\right)=0\end{aligned} \tag{5.4.26}$$

Solving for A_1 and A_2 we find:

$$\begin{aligned}&A_1\begin{bmatrix}\left(\cos(\alpha L)+\cosh(\alpha L)\right)^2\\ \qquad -\left(\sin(\alpha L)+\sinh(\alpha L)\right)\left(-\sin(\alpha L)+\sinh(\alpha L)\right)\end{bmatrix}=0\\ &A_2\begin{bmatrix}\left(\cos(\alpha L)+\cosh(\alpha L)\right)^2\\ \qquad -\left(\sin(\alpha L)+\sinh(\alpha L)\right)\left(-\sin(\alpha L)+\sinh(\alpha L)\right)\end{bmatrix}=0\end{aligned} \tag{5.4.27}$$

In order that neither A_1 and A_2 are zero, the expression in the brackets must be zero and we are left with the *frequency equation*:

$$\cos(\alpha L)\cosh(\alpha L)+1=0 \tag{5.4.28}$$

The mode shape is got by expressing A_2 in terms of A_1:

$$A_2=-\frac{\sin(\alpha L)+\sinh(\alpha L)}{\cos(\alpha L)+\cosh(\alpha L)}A_1 \tag{5.4.29}$$

and the modes shapes are therefore:

$$\phi_n(x)=A_1\left[\begin{array}{l}\sin(\alpha x)-\sinh(\alpha x)\\ \qquad +\dfrac{\sin(\alpha L)+\sinh(\alpha L)}{\cos(\alpha L)+\cosh(\alpha L)}\left(\cosh(\alpha x)-\cos(\alpha x)\right)\end{array}\right] \tag{5.4.30}$$

where again A_1 is arbitrary and normally taken to be unity. We can see from (5.4.28) that it must be solved numerically for the corresponding values of αL The natural frequencies are then got from (5.4.21) with the substitution of αL for $n\pi$. The first three mode shapes and frequencies are shown in Figure 4.4.

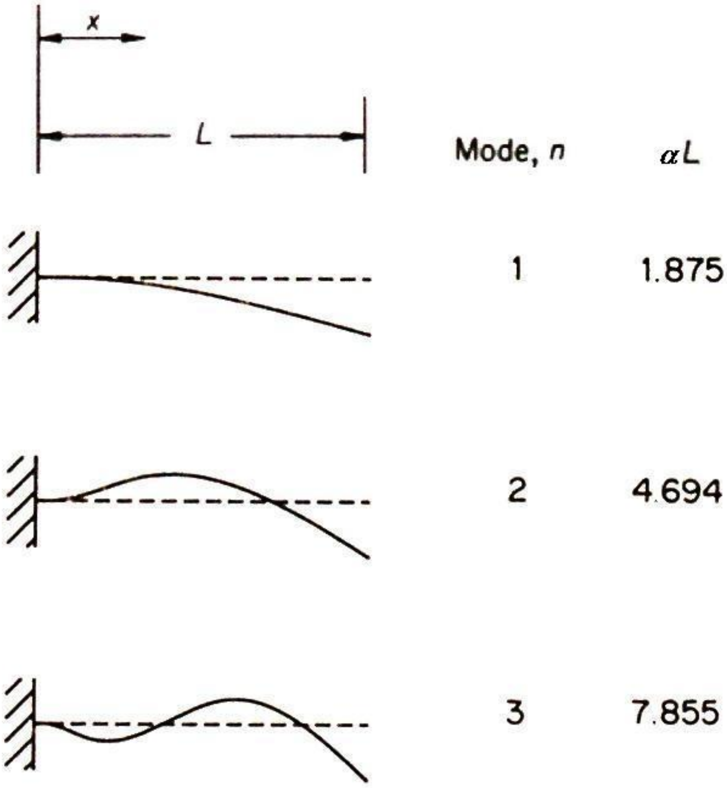


Figure 4.4: First three mode shapes and frequency parameters for a cantilever.

5.4.2 Approximate Analysis – Bolton's Method

We will now look at a simplified method that requires an understanding of dynamic behaviour but is very easy to implement. The idea is to represent, through various manipulations of mass and stiffness, any complex structure as a single SDOF system which is easily solved via an implementation of equation (1.2):

$$f = \frac{1}{2\pi}\sqrt{\frac{K_E}{M_E}} \tag{5.4.31}$$

in which we have equivalent SDOF stiffness and mass terms.

Consider a mass-less cantilever which carries two different masses, Figure 4.5:

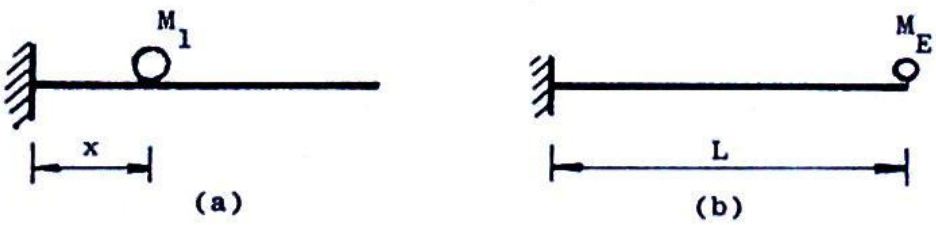


Figure 4.5: Equivalent dynamic mass distribution for a cantilever.

The end deflection of a cantilever loaded at its end by a force P is well known to be $PL^3/3EI$ and hence the stiffness is $3EI/L^3$. Therefore, the frequencies of the two cantilevers of Figure 4.5 are:

$$f_1 = \frac{1}{2\pi}\sqrt{\frac{3EI}{M_1 x^3}} \tag{5.4.32}$$

$$f_E = \frac{1}{2\pi}\sqrt{\frac{3EI}{M_E L^3}}; \tag{5.4.33}$$

And so, if the two frequencies are to be equal, and considering M_1 as the mass of a small element dx when the mass per metre is m, the corresponding part of M_E is:

$$dM_E = \left(\frac{x}{L}\right)^3 mdx \tag{5.4.34}$$

and integrating:

$$\begin{aligned} M_E &= \int_0^L \left(\frac{x}{L}\right)^3 mdx \\ &= 0.25mL \end{aligned} \tag{5.4.35}$$

Therefore the cantilever with self-mass uniformly distributed along its length vibrates at the same frequency as would the mass-less cantilever loaded with a mass one quarter its actual mass. This answer is not quite correct but is within 5%; it ignores the fact that every element affects the deflection (and hence vibration) of every other element. The answer is reasonable for design though.

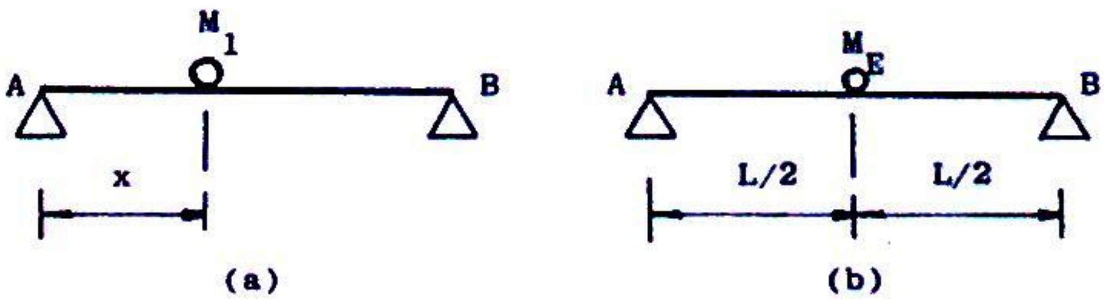


Figure 4.6: Equivalent dynamic mass distribution for an s-s beam

Similarly for a simply supported beam, we have an expression for the deflection at a point:

$$\delta_x = \frac{Px^2\left(L-x\right)^2}{3EIL} \tag{5.4.36}$$

and so its stiffness is:

$$K_x = \frac{3EIL}{x^2\left(L-x\right)^2} \tag{5.4.37}$$

Considering Figure 4.6, we see that, from (5.4.31):

$$\frac{3EIL}{x^2\left(L-x\right)^2 M_1} = \frac{48EI}{L^3 M_E} \tag{5.4.38}$$

and as the two frequencies are to be equal:

$$\begin{aligned} M_E &= \int_0^L 16x^2 \frac{\left(L-x\right)^2}{L^4} m dx \\ &= 8/15 mL \end{aligned} \tag{5.4.39}$$

which is about half of the self-mass as we might have guessed.

Proceeding in a similar way we can find equivalent spring stiffnesses and masses for usual forms of beams as given in Table 1. Table 4.1 however, also includes a refinement of the equivalent masses based on the known dynamic deflected shape rather than the static deflected shape.

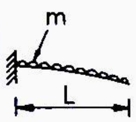
Member		Known frequency $\sqrt{EI/mL^4}$ ×	Equivalent spring stiffness EI/L^3 ×	Equivalent mass, M_E mL ×	Amplitude $\frac{\text{Mass}}{\text{stiffness}}$ mL^4/EI ×	Relative amplitude mL^4/EI ×
1		0·560	3	0·2422	0·08073	7·861
2		$\pi/2$	48	0·4928	0·01027	1
3		2·45	101·9	0·4299	0·004219	0·4108
4		3·56	192	0·3837	0·001998	0·1945
5		2π	768	0·4928	0·0006417	0·0625

Table 4.1: Bolton's table for equivalent mass, stiffnesses and relative amplitudes.

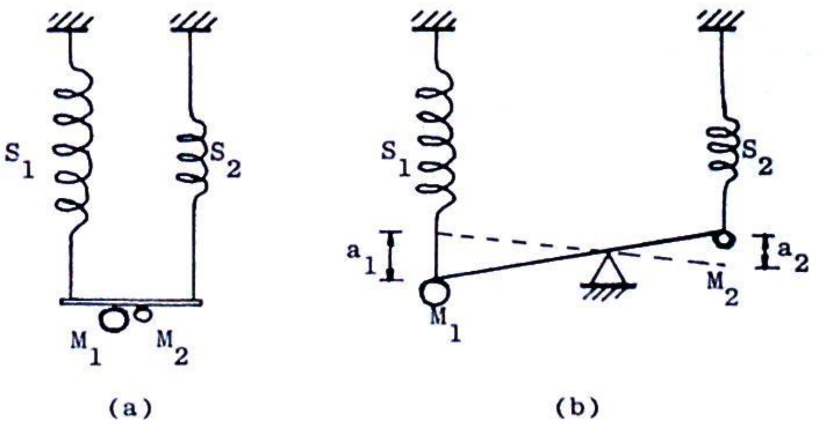


Figure 4.7: Effective SDOFs: (a) neglecting relative amplitude; (b) including relative amplitude.

In considering continuous beams, the continuity over the supports requires all the spans to vibrate at the same frequency for each of its modes. Thus we may consider summing the equivalent masses and stiffnesses for each span and this is not a bad approximation. It is equivalent to the SDOF model of Figure 4.7(a). But, if we allowed for the relative amplitude between the different spans, we would have the model of Figure 4.7(b) which would be more accurate – especially when there is a significant difference in the member stiffnesses and masses: long heavy members will have larger amplitudes than short stiff light members due to the amount of kinetic energy stored. Thus, the stiffness and mass of each span must be weighted by its relative amplitude before summing. Consider the following examples of the beam shown in Figure 4.8; the exact multipliers are known to be 10.30, 13.32, 17.72, 21.67, 40.45, 46.10, 53.89 and 60.53 for the first eight modes.

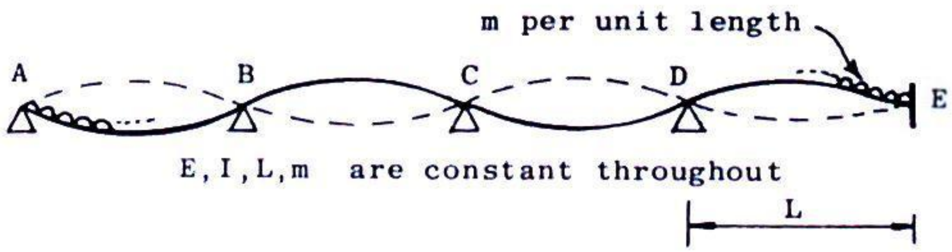


Figure 4.8: Continuous beam of Examples 1 to 3.

Example 1: Ignoring relative amplitude and refined M_E

From Table 4.1, and the previous discussion:

$$\sum K_E = \frac{EI}{L^3}\left(48\times 3+101.9\right);\text{ and } \sum M_E = mL\left(3\times\frac{8}{15}+\frac{1}{2}\right),$$

and applying (5.4.31) we have: $f = \left(10.82\right)\frac{1}{2\pi}\sqrt{\frac{EI}{mL^4}}$

The multiplier in the exact answer is 10.30: an error of 5%.

Example 2: Including relative amplitude and refined M_E

From Table 4.1 and the previous discussion, we have:

$$\sum K_E = 3\times\frac{48EI}{L^3}\times 1+\frac{101.9EI}{L^3}\times 0.4108 = 185.9\frac{EI}{L^3}$$

$$\sum M_E = 3\times 0.4928mL\times 1+0.4299mL\times 0.4108 = 1.655mL$$

and applying (5.4.31) we have:

$$f = \left(10.60\right)\frac{1}{2\pi}\sqrt{\frac{EI}{mL^4}}$$

The multiplier in the exact answer is 10.30: a reduced error of 2.9%.

Example 3: Calculating the frequency of a higher mode

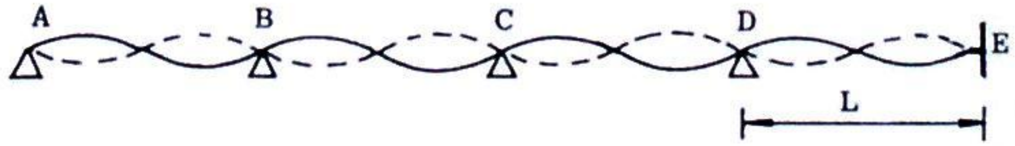


Figure 4.9: Assumed mode shape for which the frequency will be found.

The mode shape for calculation is shown in Figure 4.7. We can assume supports at the midpoints of each span as they do not displace in this mode shape. Hence we have seven simply supported half-spans and one cantilever half-span, so from Table 4.1 we have:

$$\sum K_E = 7\times\frac{48EI}{(0.5L)^3}\times 1+\frac{101.9EI}{(0.5L)^3}\times 0.4108$$
$$=3022.9\frac{EI}{L^3}$$
$$\sum M_E = 7\times 0.4928m(0.5L)\times 1+0.4299m(0.5L)\times 0.4108$$
$$=1.813mL$$

again, applying (5.4.31), we have:

$$f=(40.8)\frac{1}{2\pi}\sqrt{\frac{EI}{mL^4}}$$

The multiplier in the exact answer is 40.45: and error of 0.9%.

Mode Shapes and Frequencies

Section 2.d described how the DAF is very large when a force is applied at the natural frequency of the structure; so for any structure we can say that when it is vibrating at its natural frequency it has very low stiffness – and in the case of no damping: zero stiffness. Higher modes will have higher stiffnesses but stiffness may also be recognised in one form as

$$\frac{M}{EI} = \frac{1}{R} \tag{5.4.40}$$

where R is the radius of curvature and M is bending moment. Therefore, smaller stiffnesses have a larger R and larger stiffnesses have a smaller R. Similarly then, lower modes have a larger R and higher modes have a smaller R. This enables us to distinguish between modes by their frequencies. Noting that a member in single curvature (i.e. no point of contraflexure) has a larger R than a member in double curvature (1 point of contraflexure) which in turn has a larger R than a member in triple curvature (2 points of contraflexure), we can distinguish modes by deflected shapes. Figures 4.3 and 4.4 illustrate this clearly.

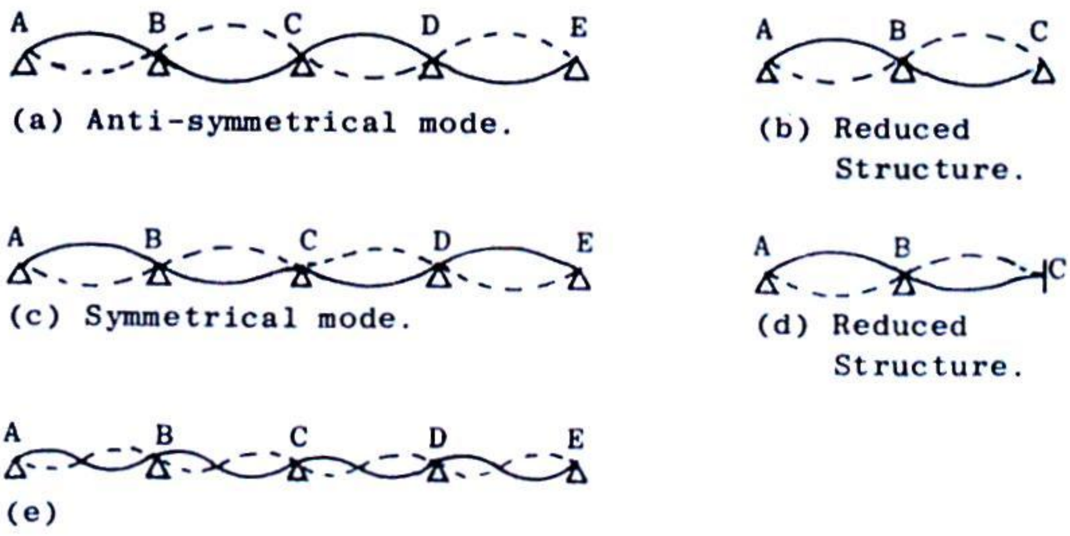


Figure 4.10: Typical modes and reduced structures.

An important fact may be deduced from Figure 4.10 and the preceding arguments: a continuous beam of any number of identical spans has the same fundamental frequency as that of one simply supported span: symmetrical frequencies are similarly linked. Also, for non-identical spans, symmetry may exist about a support and so reduced structures may be used to estimate the frequencies of the total structure; reductions are shown in Figure 4.10(b) and (d) for symmetrical and anti-symmetrical modes.

5.4.3 Problems

Problem 1

Calculate the first natural frequency of a simply supported bridge of mass 7 tonnes with a 3 tonne lorry at its quarter point. It is known that a load of 10 kN causes a 3 mm deflection.

Ans.: 3.95Hz.

Problem 2

Calculate the first natural frequency of a 4 m long cantilever (EI = 4,320 kNm^2) which carries a mass of 500 kg at its centre and has self weight of 1200 kg.

Ans.: 3.76 Hz.

Problem 3

What is the fundamental frequency of a 3-span continuous beam of spans 4, 8 and 5 m with constant EI and m? What is the frequency when $EI = 6\times10^3$ kNm^2 and m = 150 kg/m?

Ans.: 6.74 Hz.

Problem 4

Calculate the first and second natural frequencies of a two-span continuous beam; fixed at *A* and on rollers at *B* and *C*. Span *AB* is 8 m with flexural stiffness of $2EI$ and a mass of $1.5m$. Span *BC* is 6 m with flexural stiffness EI and mass m per metre. What are the frequencies when $EI = 4.5\times10^3$ kNm^2 and m = 100 kg/m?

Ans.: 9.3 Hz; ? Hz.

Problem 5

Calculate the first and second natural frequencies of a 4-span continuous beam of spans 4, 5, 4 and 5 m with constant EI and m? What are the frequencies when $EI = 4\times10^3$ kNm^2 and $m = 120$ kg/m? What are the new frequencies when support A is fixed? Does this make it more or less susceptible to human-induced vibration?

Ans.: ? Hz; ? Hz.

5.5 Practical Design Considerations

5.5.1 Human Response to Dynamic Excitation

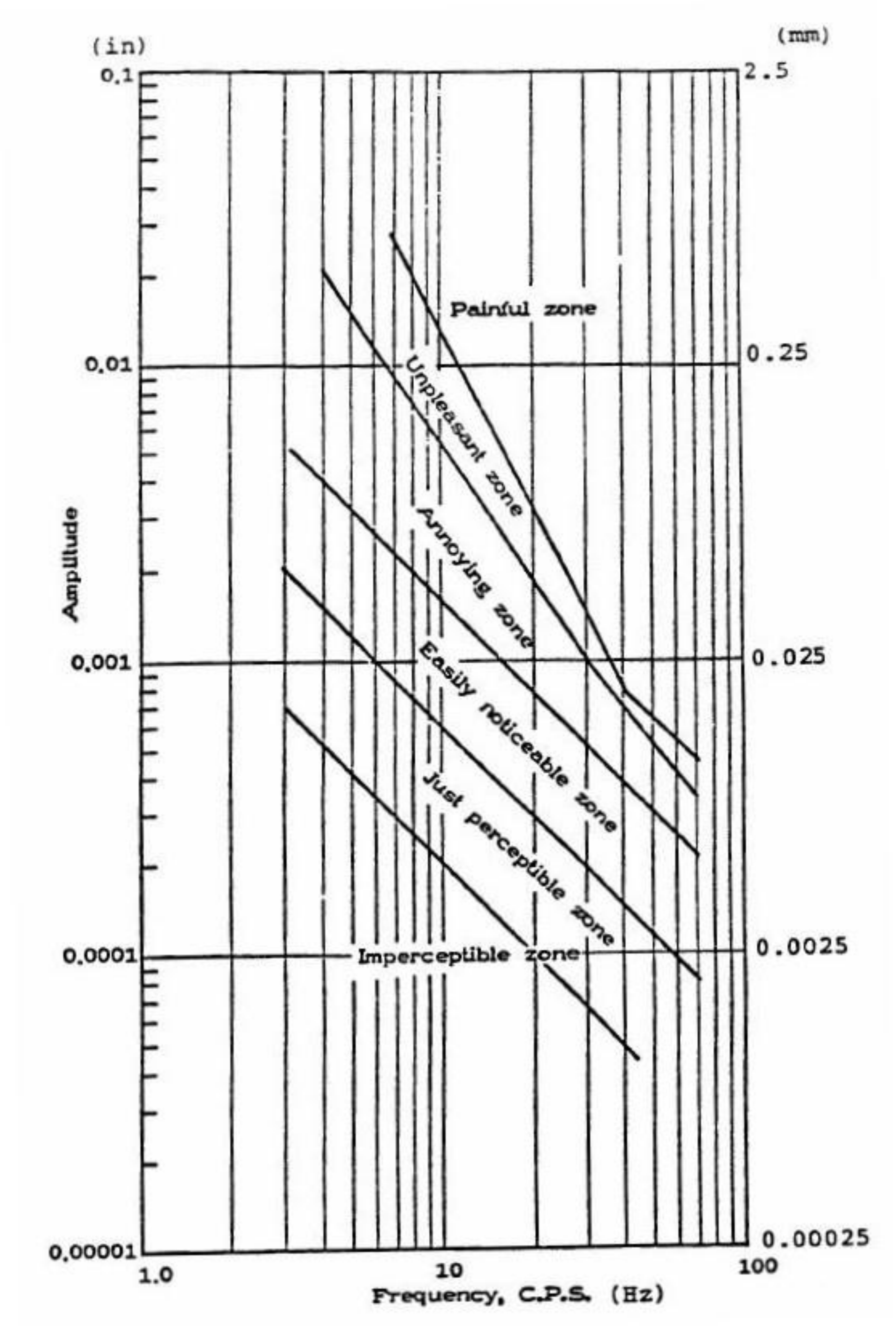
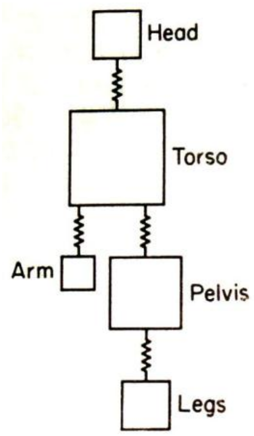


Figure 5.1: Equal sensation contours for vertical vibration

The response of humans to vibrations is a complex phenomenon involving the variables of the vibrations being experienced as well as the perception of it. It has been found that the frequency range between 2 and 30 Hz is particularly uncomfortable because of resonance with major body parts (Figure 5.2). Sensation contours for vertical vibrations are shown in Figure 5.1. This graph shows that for a given frequency, as the amplitude gets larger it becomes more uncomfortable; thus it is acceleration that is governing the comfort. This is important in the design of tall buildings which sway due to wind loading: it is the acceleration that causes discomfort. This may also be realised from car-travel: at constant velocity nothing is perceptible, but, upon rapid acceleration the motion if perceived ($F = ma$).



Frequency (Hz)	*Nature of motion*
2	No resonance. Body acts as a unit mass
3–3.5	Oscillation of abdominal viscera coupled with air oscillation of the mouth and chest
5–7	In-phase motion of torso and pelvis
10–14	Upper torso with a forward nodding motion of the upper spine
20–30	Relative movement between head and shoulders
60–90	Eyeball resonance

Figure 5.2: Human body response to vibration

Response graphs like Figure 5.1 have been obtained for each direction of vibration but vertical motion is more uncomfortable for standing subjects; for the transverse and longitudinal cases, the difference has the effect of moving the illustrated bands up a level. Other factors are also important: the duration of exposure; waveform (which is again linked to acceleration); type of activity; and, psychological factors. An example is that low frequency exposure can result in motion sickness.

5.5.2 Crowd/Pedestrian Dynamic Loading

Lightweight Floors

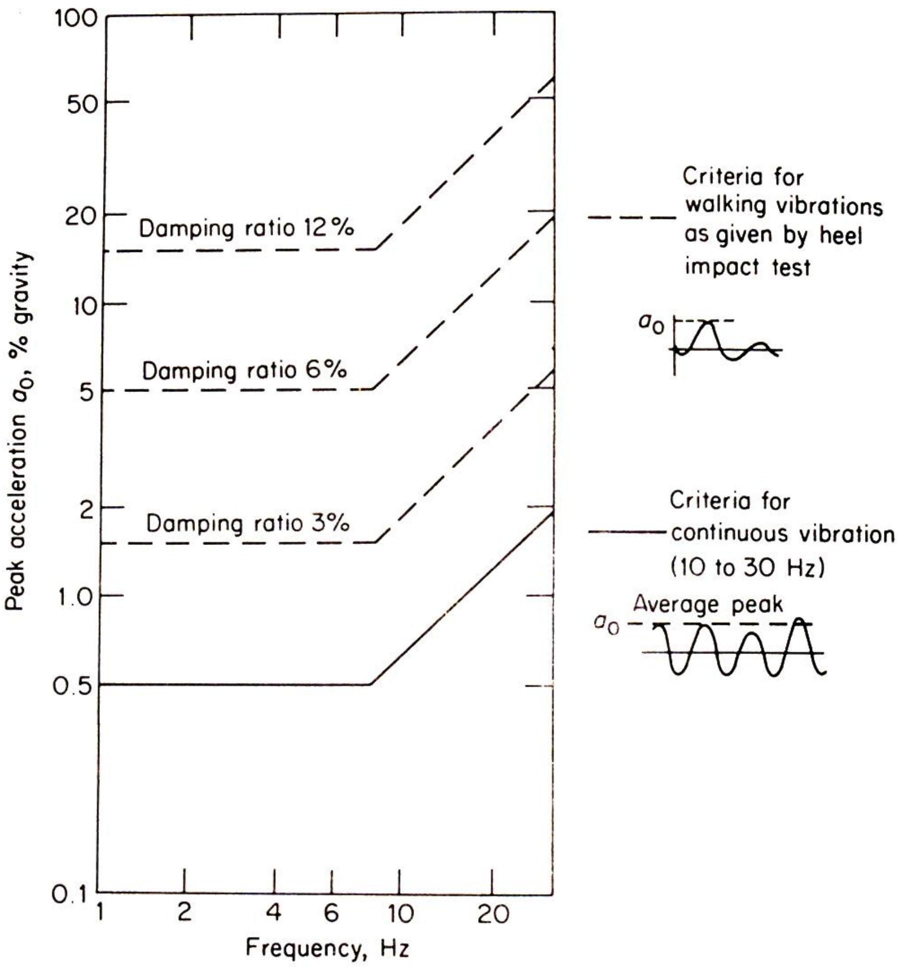


Figure 5.3: Recommended vibration limits for light floors.

Vibration limits for light floors from the 1984 Canadian Standard is shown in Figure 5.2; the peak acceleration is got from:

$$a_0 = (0.9)2\pi f \frac{I}{M} \quad (5.5.1)$$

where I is the impulse (the area under the force time graph) and is about 70 Ns and M is the equivalent mass of the floor which is about 40% of the distributed mass.

This form of approach is to be complemented by a simple analysis of an equivalent SDOF system. Also, as seen in Section 1, by keeping the fundamental frequency above 5 Hz, human loading should not be problematic.

Crowd Loading

This form of loading occurs in grandstands and similar structures where a large number of people are densely packed and will be responding to the same stimulus. Coordinated jumping to the beat of music, for example, can cause a DAF of about 1.97 at about 2.5 Hz. Dancing, however, normally generates frequencies of 2 – 3 Hz. Once again, by keeping the natural frequency of the structure above about 5 Hz no undue dynamic effects should be noticed.

In the transverse or longitudinal directions, allowance should also be made due to the crowd-sway that may accompany some events a value of about 0.3 kN per metre of seating parallel and 0.15 kN perpendicular to the seating is an approximate method for design.

Staircases can be subject to considerable dynamic forces as running up or down such may cause peak loads of up to 4-5 times the persons bodyweight over a period of about 0.3 seconds – the method for lightweight floors can be applied to this scenario.

Footbridges

As may be gathered from the Case Studies of the Aberfeldy Bridge, the problem is complex, however some rough guidelines are possible. Once again controlling the fundamental frequency is important; the lessons of the London Millennium and the Tacoma Narrows bridges need to be heeded though: dynamic effects may occur in any direction or mode that can be excited by any form of loading.

An approximate method for checking foot bridges is the following:

$$u_{\max} = u_{st} K \psi \tag{5.5.2}$$

where u_{st} is the static deflection under the weight of a pedestrian at the point of maximum deflection; K is a configuration factor for the type of structure (given in Table 5.1); and ψ is the dynamic response factor got again from Figure 5.4. The maximum acceleration is then got as $\ddot{u}_{\max} = \omega^2 u_{\max}$ (see equations (2.30) and (3.11) for example, note: $\omega^2 = 2\pi f$). This is then compared to a rather simple rule that the maximum acceleration of footbridge decks should not exceed $\pm 0.5\sqrt{f}$.

Alternatively, BD 37/01 states:

"For superstructures for which the fundamental natural frequency of vibration exceeds 5Hz for the unloaded bridge in the vertical direction and 1.5 Hz for the loaded bridge in the horizontal direction, the vibration serviceability requirement is deemed to be satisfied." – Appendix B.1 General.

Adhering to this clause (which is based on the discussion of Section 1's Case Study) is clearly the easiest option.

Also, note from Figure 5.4 the conservative nature of the damping assumed, which, from equation (2.35) can be seen to be so based on usual values of damping in structures.

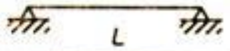
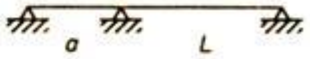
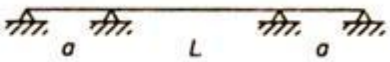
Configuration	a/L	K
	-	1.0
	1.0	0.7
	0.8	0.9
	<0.6	1.0
	1.0	0.6
	0.8	0.8
	<0.6	0.9

Table 5.1: Configuration factors for footbridges.

Bridge superstructure	δ
Steel with asphalt or epoxy surfacing	0.03
Composite steel/concrete	0.04
Prestressed and reinforced concrete	0.05

Table 5.2: Values of the logarithmic decrement for different bridge types.

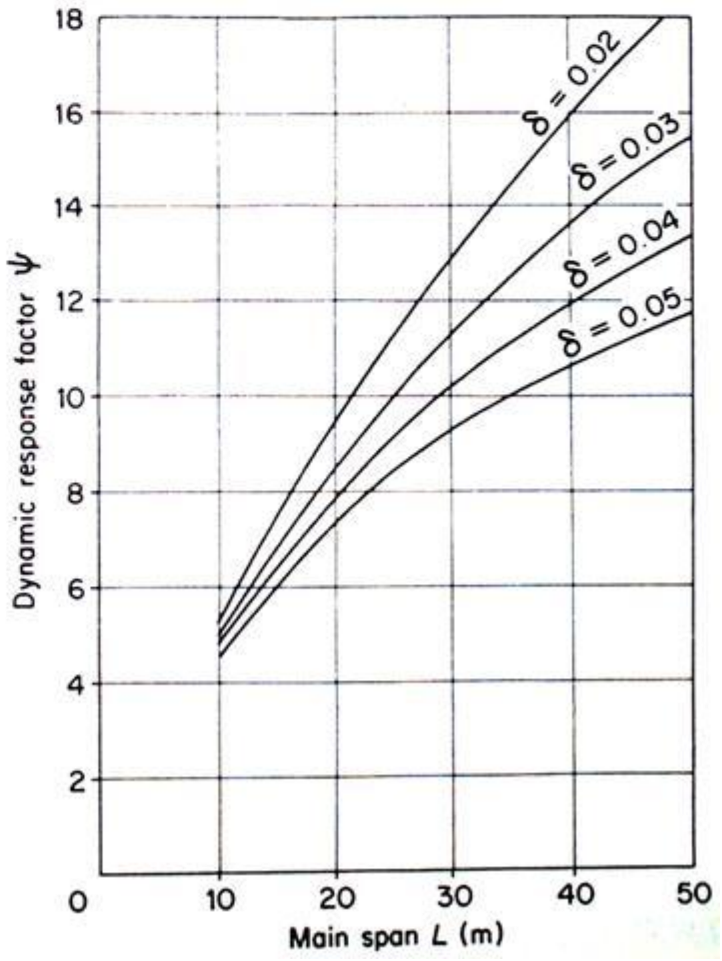


Figure 5.4: Dynamic response factor for footbridges

Design Example

A simply-supported footbridge of 18 m span has a total mass of 12.6 tonnes and flexural stiffness of 3×10^5 kNm^2. Determine the maximum amplitude of vibration and vertical acceleration caused by a 0.7 kN pedestrian walking in frequency with the bridge: the pedestrian has a stride of 0.9 m and produces an effective pulsating force of 180 N. Assume the damping to be related to $\delta = 0.05$. Is this a comfortable bridge for the pedestrian (Figure 5.1)?

The natural frequency of the bridge is, from equations (2.19) and (4.21):

$$f = \frac{\pi}{2\times18^2}\sqrt{\frac{3\times10^8}{12600/18}} = 3.17 \text{ Hz}$$

The static deflection is:

$$u_{st} = \frac{700\times18^3}{48\times3\times10^8} = 0.2835 \text{ mm}$$

Table 5.1 gives $K=1$ and Figure 5.4 gives $\psi = 6.8$ and so, by (5.5.2) we have:

$$u_{\max} = 0.2835\times1.0\times6.8 = 1.93 \text{ mm}$$

and so the maximum acceleration is:

$$\ddot{u}_{\max} = \omega^2 u_{\max} = \left(2\pi\times3.17\right)^2\times1.93\times10^{-3} = 0.78 \text{ m/s}^2$$

We compare this to the requirement that:

$$\left|\ddot{u}_{\max}\right| \le \left|0.5\sqrt{f}\right|$$
$$\le 0.5\sqrt{f}$$
$$0.78 \le 0.89 \text{ m/s}^2$$

And so we deem the bridge acceptable. From Figure 5.1, with the amplitude 1.93 mm and 3.17 Hz frequency, we can see that this pedestrian will feel decidedly uncomfortable and will probably change pace to avoid this frequency of loading.

The above discussion, in conjunction with Section 2.d reveals why, historically, soldiers were told to break step when crossing a slender bridge – unfortunately for some, it is more probable that this knowledge did not come from any detailed dynamic analysis; rather, bitter experience.

5.5.3 Damping in Structures

The importance of damping should be obvious by this stage; a slight increase may significantly reduce the DAF at resonance, equation (2.47). It was alluded to in Section 1 that the exact nature of damping is not really understood but that it has been shown that our assumption of linear viscous damping applies to the majority of structures – a notable exception is soil-structure interaction in which alternative damping models must be assumed. Table 5.3 gives some typical damping values in practice. It is notable that the materials themselves have very low damping and thus most of the damping observed comes from the joints and so can it depend on:

- The materials in contact and their surface preparation;
- The normal force across the interface;
- Any plastic deformation in the joint;
- Rubbing or fretting of the joint when it is not tightened.

Type of structure	*Damping* ξ (%)
Material damping – steel	0.03–0.15
– concrete	0.15–1.0
– wire rope	0.4 –2.0
Bridges – all steel	0.2 –1.0
– composite construction	0.3 –1.6
– reinforced concrete or prestressed concrete	0.3 –1.6
Chimneys – steel	0.3 –0.8
– concrete	0.5 –1.0
Steel masts and towers	0.3 –2.9
Multi-storey buildings	0.7 –2.9
One- or two-storey houses	1.0 –5.0

Table 5.4: Recommended values of damping.

When the vibrations or DAF is unacceptable it is not generally acceptable to detail joints that will have higher damping than otherwise normal – there are simply too many variables to consider. Depending on the amount of extra damping needed, one could wait for the structure to be built and then measure the damping, retro-fitting vibration isolation devices as required. Or, if the extra damping required is significant, the design of a vibration isolation device may be integral to the structure.

The devices that may be installed vary; some are:

- Tuned mass dampers (TMDs): a relatively small mass is attached to the primary system and is 'tuned' to vibrate at the same frequency but to oppose the primary system;
- Sloshing dampers: A large water tank is used – the sloshing motion opposes the primary system motion due to inertial effects;
- Liquid column dampers: Two columns of liquid, connected at their bases but at opposite sides of the primary system slosh, in a more controlled manner to oppose the primary system motion.

These are the approaches taken in many modern buildings, particularly in Japan and other earthquake zones. The Citicorp building in New York (which is famous for other reasons also) and the John Hancock building in Boston were among the first to use TMDs. In the John Hancock building a concrete block of about 300 tonnes located on the 54th storey sits on a thin film of oil. When the building sways the inertial effects of the block mean that it moves in the opposite direction to that of the sway and so opposes the motion (relying heavily on a lack of friction). This is quite a rudimentary system compared to modern systems which have computer controlled actuators that take input from accelerometers in the building and move the block an appropriate amount.

5.5.4 Design Rules of Thumb

General

The structure should not have any modal frequency close to the frequency of any form of periodic loading, irrespective of magnitude. This is based upon the large DAFs that may occur (Section 2.d).

For normal floors of span/depth ratio less than 25 vibration is not generally a problem. Problematic floors are lightweight with spans of over about 7 m.

Human loading

Most forms of human loading occur at frequencies < 5 Hz (Sections 1 and 5.a) and so any structure of natural frequency greater than this should not be subject to undue dynamic excitation.

Machine Loading

By avoiding any of the frequencies that the machine operates at, vibrations may be minimised. The addition of either more stiffness or mass will change the frequencies the structure responds to. If the response is still not acceptable vibration isolation devices may need to be considered (Section 5.c).

Approximate Frequencies

The Bolton Method of Section 4.b is probably the best for those structures outside the standard cases of Section 4.a. Careful thought on reducing the size of the problem to an SDOF system usually enables good approximate analysis.

Other methods are:

Structures with concentrated mass: $f = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

Simplified rule for most structures: $f = \frac{18}{\sqrt{\delta}}$

where δ is the static deflection and g is the acceleration under gravity.

Rayleigh Approximation

A method developed by Lord Rayleigh (which is always an upper bound), based on energy methods, for estimating the lowest natural frequency of transverse beam vibration is:

$$\omega_1^2 = \frac{\int_0^L EI\left(\frac{d^2y}{dx^2}\right)^2 dx}{\int_0^L y^2 dm} \tag{5.5.3}$$

This method can be used to estimate the fundamental frequency of MDOF systems. Considering the frame of Figure 5.5, the fundamental frequency in each direction is given by:

$$\omega_1^2 = g\frac{\sum_i Q_i u_i}{\sum_i Q_i u_i^2} = g\frac{\sum_i m_i u_i}{\sum_i m_i u_i^2} \tag{5.5.4}$$

where u_i is the static deflection under the dead load of the structure Q_i, acting in the direction of motion, and g is the acceleration due to gravity. Thus, the first mode is approximated in shape by the static deflection under dead load. For a building, this can be applied to each of the X and Y directions to obtain the estimates of the fundamental sway modes.

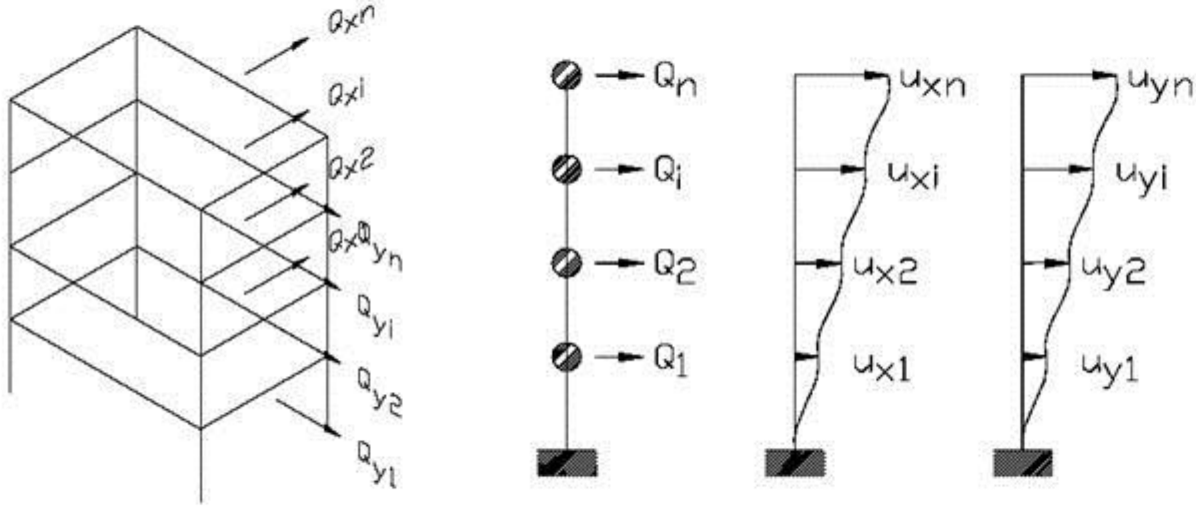


Figure 5.5: Rayleigh approximation for the fundamental sway frequencies of a building.

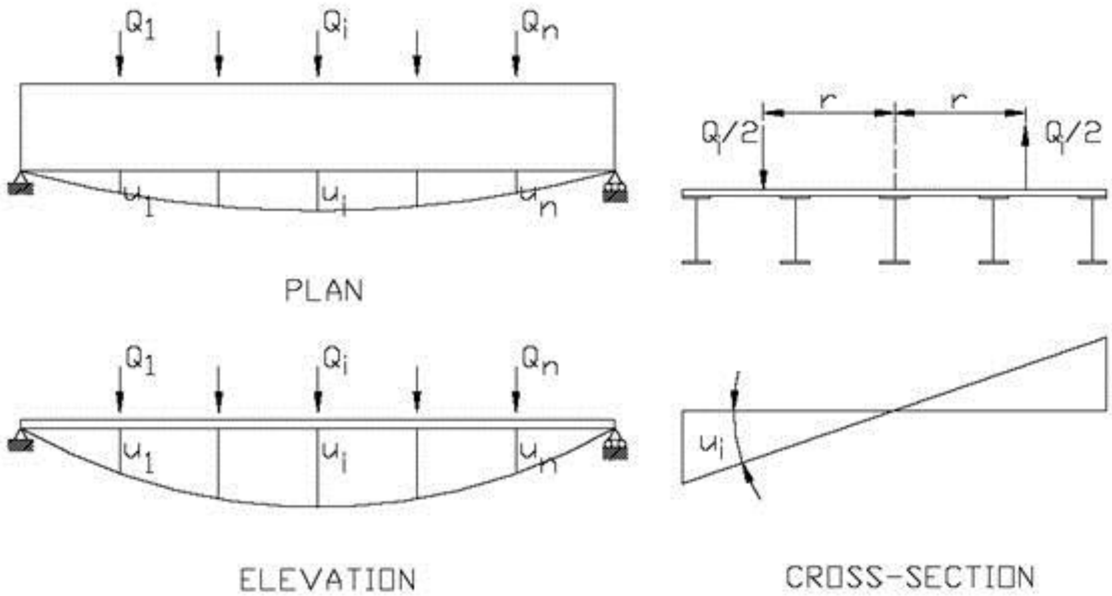


Figure 5.6: Rayleigh method for approximating bridge fundamental frequencies.

Likewise for a bridge, by applying the dead load in each of the vertical and horizontal directions, the fundamental lift and drag modes can be obtained. The torsional mode can also be approximated by applying the dead load at the appropriate radius of gyration and determining the resulting rotation angle, Figure 5.6.

This method is particularly useful when considering the results of a detailed analysis, such as finite-element. It provides a reasonable approximate check on the output.

5.6 Appendix

5.6.1 Past Exam Questions

Summer 2005

Question 5

(a) The system shown in Figure Q.5(a) is known to have a static deflection of 32.7 mm for an unknown mass.

1) Find the natural frequency of the system.

(10%)

2) Given that the mass is 10 kg, find the peak displacement when this mass is given an initial velocity of 500 mm/s and an initial displacement of 25 mm.

(10%)

3) What time does the first positive peak occur?

(10%)

4) What value of damping coefficient is required such that the amplitude after 5 oscillations is 10% of the first peak?

(10%)

5) What is the peak force in the spring?

(20%)

(b) A cantilever riverside boardwalk has been opened to the public as shown in Figure Q.5(b); however, it was found that the structure experiences significant human- and traffic-induced vibrations. An harmonic oscillation test found the natural frequency of the structure to be 2.25 Hz. It is proposed to retro-fit braced struts at 5m spacings so that the natural period of vibration will be 9 Hz – given E = 200 kN/mm^2 and ignoring buckling effects, what area of strut is required?

(40%)

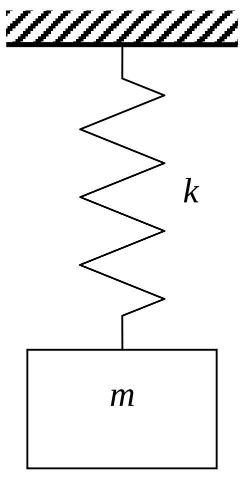


FIG. Q.5(a)

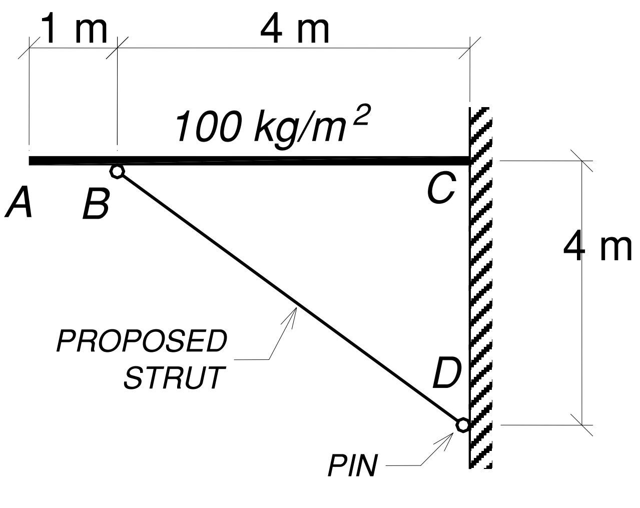


FIG. Q.5(b)

Ans. (a) 2.756 Hz; 38.2 mm; 0.05 s; 99 kg.s/m;114.5 N; (b) 67.5 mm^2.

Sample Paper Semester 1 2006/7

5. (*a*) The single-degree-of-freedom system shown in Fig. Q5(a) is known to have a static deflection of 32.7 mm for an unknown mass.

(i) Find the natural frequency of the system;

(*2 marks*)

(ii) Given that the mass is 10 kg, find the peak displacement when the mass is given an initial velocity of 500 mm/s and an initial displacement of 25 mm;

(*2 marks*)

(iii) At what time does the first positive peak occur?

(*2 marks*)

(iv) What damping ratio is required such that the amplitude after 5 oscillations is 10% of the first peak?

(*2 marks*)

(v) What is the peak force in the spring?

(*6 marks*)

FIG. Q5(a)

(*b*) The beam shown in Fig. Q5(b) is loaded with an air conditioning (AC) unit at its tip. The AC unit produces an unbalanced force of 100 kg which varies sinusoidally. When the speed of the AC unit is varied, it is found that the maximum steady-state deflection is 20.91 mm. Determine:

(i) The damping ratio;

(*4 marks*)

(ii) The maximum deflection when the unit's speed is 250 rpm;

(*7 marks*)

Take the following values:

- $EI = 1\times10^6$ kNm^2;
- Mass of the unit is 500 kg.

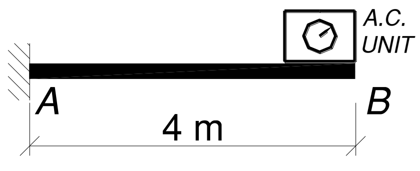


FIG. Q5(a)

Ans. (a) 2.756 Hz; 38.2 mm; 0.05 s; 99 kg.s/m;114.5 N; (b) ??.

Semester 1 2006/7

5. (*a*) A simply-supported reinforced concrete beam, 300 mm wide × 600 mm deep spans 8 m. Its fundamental natural frequency is measured to be 6.5 Hz. In your opinion, is the beam cracked or uncracked?

Use a single degree-of-freedom (SDOF) system to represent the deflection at the centre of the beam. Assume that 8/15 of the total mass of the beam contributes to the SDOF model. Take the density of reinforced concrete to be 24 kN/m^3 and E = 30 kN/mm^2.

(*10 marks*)

(*b*) The beam shown in Fig. Q5(b) is loaded with an air conditioning (AC) unit at its tip. The AC unit produces an unbalanced force of 200 kg which varies sinusoidally. When the speed of the AC unit is varied, it is found that the maximum steady-state amplitude of vibration is 34.6 mm. Determine:

(i) The damping ratio;

(*5 marks*)

(ii) The maximum deflection when the unit's speed is 100 rpm;

(*10 marks*)

Take the following values:

- $EI = 40\times10^3$ kNm^2;
- Mass of the unit is 2000 kg;
- Ignore the mass of the beam.

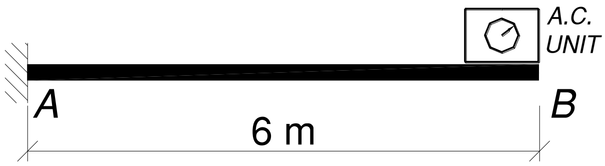


FIG. Q5(b)

Ans. (a) Cracked; (b) 5.1%; 41.1 mm.

Semester 1 2007/8

QUESTION 5

(a) For the frame shown in Fig. Q5, using a single-degree-of-freedom model, determine:
- (i) The natural frequency and period in free vibration;
- (ii) An expression for the displacement at time t if member BC is displaced 20 mm and suddenly released at time t = 1 sec.

(8 marks)

(b) The frame is found to have 5% damping. Using appropriate approximations, what is the percentage change in deflection, 4 cycles after the frame is released, of the damped behaviour compared to the undamped behaviour?

(10 marks)

(c) A machine is placed on member BC which has an unbalanced force of 500 kg which varies sinusoidally. Neglecting the mass of the machine, determine:
- (i) the maximum displacement when the unit's speed is 150 rpm;
- (ii) the speed of the machine at resonance;
- (iii) the displacement at resonance.

(7 marks)

Note:
Take the following values:
- $EI = 20\times10^3$ kNm^2;
- M = 20 tonnes;
- Consider BC as infinitely rigid.

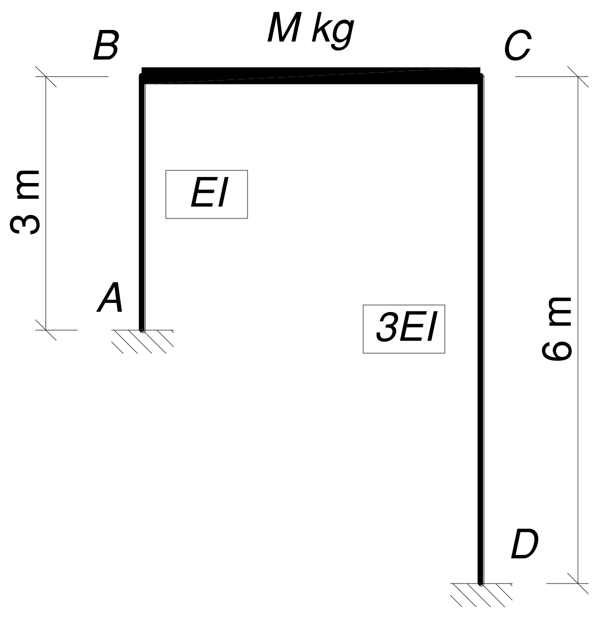


FIG. Q5

Ans.(a) 3.93 Hz; 0.254 s; 20cos[24.72(t-1)], t>1; (b) Ratio: 28.4%, change: 71.6%; (c) 0.67 mm; 236 rpm; 4.01 mm.

Semester 1 2008/9

QUESTION 5

The structure shown in Fig. Q5 supports a scoreboard at a sports centre. The claxton (of total mass *M*) which sounds the end of playing periods includes a motor which has an unbalanced mass of 100 kg which varies sinusoidally when sounded. Using a single-degree-of-freedom model for vibrations in the vertical direction, and neglecting the mass of the truss members, determine:

(i) the natural frequency and period in free vibration;
(ii) the damping, given that a test showed 5 cycles after a 10 mm initial displacement was imposed, the amplitude was 5.30 mm;
(iii) the maximum displacement when the unit's speed is 1500 rpm;
(iv) the speed of the machine at resonance;
(v) the displacement at resonance.

(25 marks)

Note:
Take the following values:

- For all truss members: $EA = 20 \times 10^3$ kN;
- M = 5 tonnes;
- Ignore the stiffness and mass of member *EF*.

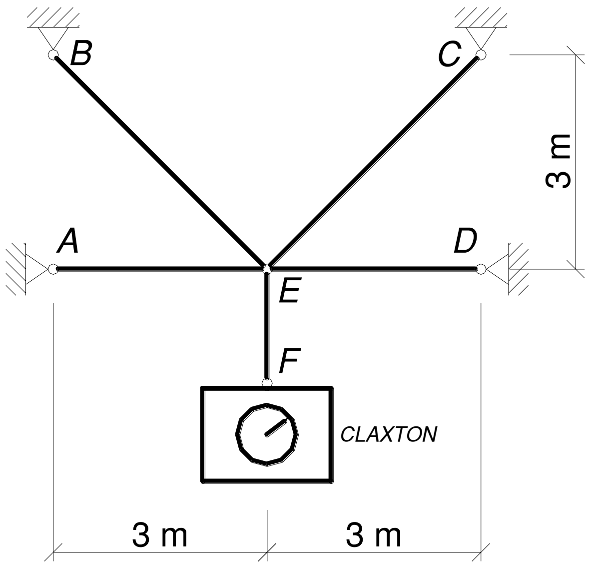


FIG. Q5

Ans. 4.9 Hz; 0.205 s; 2%; 0.008 mm; 293.2 rpm; 5.2 mm.

Semester 1 2009/10

QUESTION 5

(a) A 3 m high, 6 m wide single-bay single-storey frame is rigidly jointed with a beam of mass 2,000 kg and columns of negligible mass and stiffness of $EI = 2.7 \times 10^3$ kNm^2. Assuming the beam to be infinitely rigid, calculate the natural frequency in lateral vibration and its period. Find the force required to deflect the frame 20 mm laterally.

(10 marks)

(b) A spring-mass-damper SDOF system is subject to a harmonically varying force. At resonance, the amplitude of vibration is found to be 10 mm, and at 0.80 of the resonant frequency, the amplitude is found to be 5.07 mm. Determine the damping of the system.

(15 marks)

Ans. 5.51 Hz, 48 kN.; 0.1.

Semester 1 2010/11

QUESTION 5

(a) For the shear frame shown in Fig. Q5(a), ignoring the mass of the columns:
- (i) How many modes will this structure have?
- (ii) Sketch the mode shapes;
- (iii) Indicate the order of the natural frequencies associated with each mode shape (i.e. lowest to highest).

(10 marks)

(b) For the frame shown in Fig. Q5(b), using a single-degree-of-freedom model, determine the natural frequency and period in free vibration given that EI = 27×10^3 kNm^2 and M = 24 tonnes. If a machine is placed on member BC which has an unbalanced force of 500 kg varying sinusoidally, neglecting the mass of the machine, determine:
- (i) the maximum displacement when the unit's speed is 360 rpm;
- (ii) the speed of the machine at resonance;
- (iii) the displacement at resonance.

(15 marks)

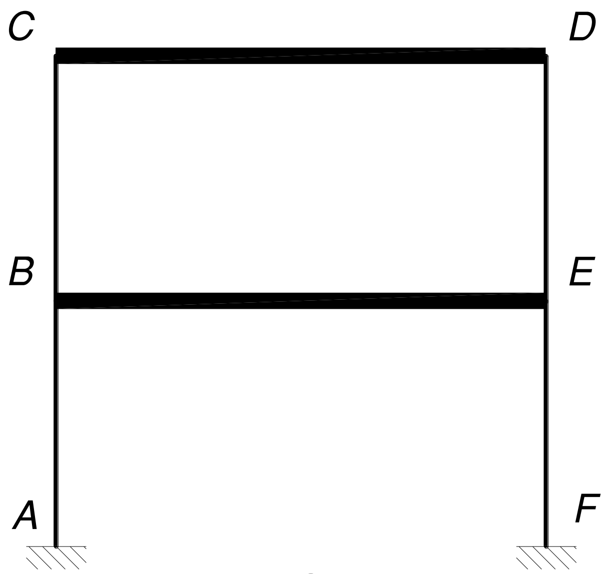


FIG. Q5(a)

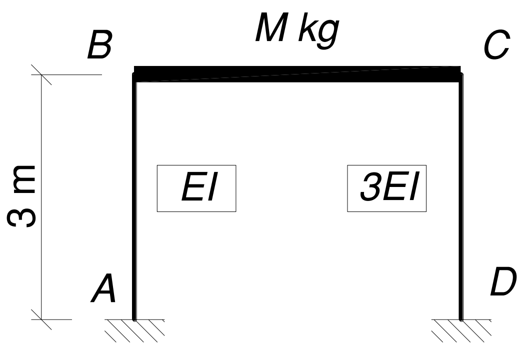


FIG. Q5(b)

Ans. 0.34 mm, 426.6 rpm, 1.02 mm.

5.6.2 References

The following books/articles were referred to in the writing of these notes; particularly Clough & Penzien (1993), Smith (1988) and Bolton (1978) - these should be referred to first for more information. There is also a lot of information and software available online; the software can especially help intuitive understanding. The class notes of Mr. R. Mahony (D.I.T.) and Dr. P. Fanning (U.C.D.) were also used.

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5.6.3 Amplitude Solution to Equation of Motion

The solution to the equation of motion is found to be in the form:

$$u(t) = A\cos\omega t + B\sin\omega t \tag{5.5.5}$$

However, we regularly wish to express it in one of the following forms:

$$u(t) = C\cos(\omega t + \alpha) \tag{5.5.6}$$

$$u(t) = C\cos(\omega t - \beta) \tag{5.5.7}$$

Where

$$C = \sqrt{A^2 + B^2} \tag{5.5.8}$$

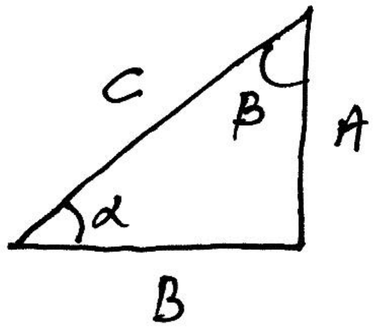
$$\tan\alpha = \frac{A}{B} \tag{5.5.9}$$

$$\tan\beta = \frac{B}{A} \tag{5.5.10}$$

To arrive at this result, re-write equation (5.5.5) as:

$$u(t) = C\left[\frac{A}{C}\cos\omega t + \frac{B}{C}\sin\omega t\right] \tag{5.5.11}$$

If we consider that *A*, *B* and *C* represent a right-angled triangle with angles α and β, then we can draw the following:



Thus:

$$\sin\alpha = \cos\beta = \frac{A}{C} \tag{5.5.12}$$

$$\cos\alpha = \sin\beta = \frac{B}{C} \tag{5.5.13}$$

Introducing these into equation (5.5.11) gives two relationships:

$$u(t) = C\left[\sin\alpha\cos\omega t + \cos\alpha\sin\omega t\right] \tag{5.5.14}$$

$$u(t) = C\left[\cos\beta\cos\omega t + \sin\beta\sin\omega t\right] \tag{5.5.15}$$

And using the well-known trigonometric identities:

$$\sin(X+Y) = \sin X\cos Y + \cos X\sin Y \tag{5.5.16}$$

$$\cos(X-Y) = \cos X\cos Y + \sin X\sin Y \tag{5.5.17}$$

Gives the two possible representations, the last of which is the one we adopt:

$$u(t) = C\sin(\omega t + \alpha) \tag{5.5.18}$$

$$u(t) = C\cos(\omega t - \beta) \tag{5.5.19}$$

5.6.4 Solutions to Differential Equations

The Homogenous Equation

To find the solution of:

$$\frac{d^2y}{dx^2} + k^2 y = 0 \tag{5.5.20}$$

we try $y = e^{\lambda x}$ (note that this *k* has nothing to do with stiffness but is the conventional mathematical notation for this problem). Thus we have:

$$\frac{dy}{dx} = \lambda e^{\lambda x}; \qquad \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$$

Substituting this into (5.5.20) gives:

$$\lambda^2 e^{\lambda x} + k^2 e^{\lambda x} = 0$$

And so we get the characteristic equation by dividing out $e^{\lambda x}$:

$$\lambda^2 + k^2 = 0$$

From which:

$$\lambda = \pm\sqrt{-k^2}$$

Or,

$$\lambda_1 = +ik; \qquad \lambda_2 = -ik$$

Where $i = \sqrt{-1}$. Since these are both solutions, they are both valid and the expression for y becomes:

$$y = A_1 e^{ikx} + A_2 e^{-ikx} \tag{5.5.21}$$

In which A_1 and A_2 are constants to be determined from the initial conditions of the problem. Introducing Euler's equations:

$$\begin{aligned} e^{ikx} &= \cos kx + i \sin kx \\ e^{-ikx} &= \cos kx - i \sin kx \end{aligned} \tag{5.5.22}$$

into (5.5.21) gives us:

$$y = A_1 \left(\cos kx + i \sin kx\right) + A_2 \left(\cos kx - i \sin kx\right)$$

Collecting terms:

$$y = \left(A_1 + A_2\right) \cos kx + \left(iA_1 - iA_2\right) \sin kx$$

Since the coefficients of the trigonometric functions are constants we can just write:

$$y = A \cos kx + B \sin kx \tag{5.5.23}$$

The Non-homogenous Equation

Starting with equation (5.2.47) (repeated here for convenience):

$$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=F_0\sin\Omega t \tag{5.5.24}$$

We divide by *m* and introduce equations (5.2.10) and (5.2.12) to get:

$$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2u(t)=\frac{F_0}{m}\sin\Omega t \tag{5.5.25}$$

At this point, recall that the solution to non-homogenous differential equations is made up of two parts:

- The **complimentary solution** ($u_C(t)$): this is the solution to the corresponding homogenous equation, which we already have (equation (5.5.23));
- The **particular solution** ($u_P(t)$): particular to the function on the right hand side of equation (5.5.24), which we must now find.

The final solution is the sum of the complimentary and particular solutions:

$$u(t)=u_C(t)+u_P(t) \tag{5.5.26}$$

For the particular solution we try the following:

$$u_P(t)=C\sin\Omega t+D\cos\Omega t \tag{5.5.27}$$

Then we have:

$$\dot{u}_P(t)=\Omega C\cos\Omega t-\Omega D\sin\Omega t \tag{5.5.28}$$

And

$$\ddot{u}_P(t) = -\Omega^2 C \sin \Omega t - \Omega^2 D \cos \Omega t \tag{5.5.29}$$

Substituting equations (5.5.27), (5.5.28) and (5.5.29) into equation (5.5.25) gives:

$$\begin{aligned} &\left[-\Omega^2 C \sin \Omega t - \Omega^2 D \cos \Omega t\right] \\ &\quad + 2\xi\omega\left[\Omega C \cos \Omega t - \Omega D \sin \Omega t\right] \\ &\quad + \omega^2\left[C \sin \Omega t + D \cos \Omega t\right] = \frac{F_0}{m} \sin \Omega t \end{aligned} \tag{5.5.30}$$

Collecting sine and cosine terms:

$$\begin{aligned} &\left[\left(\omega^2 - \Omega^2\right) C - 2\xi\omega\Omega D\right] \sin \Omega t \\ &\quad + \left[2\xi\omega\Omega C + \left(\omega^2 - \Omega^2\right) D\right] \cos \Omega t = \frac{F_0}{m} \sin \Omega t \end{aligned} \tag{5.5.31}$$

For this to be valid for all *t*, the sine and cosine terms on both sides of the equation must be equal. Thus:

$$\left(\omega^2 - \Omega^2\right) C - 2\xi\omega\Omega D = \frac{F_0}{m} \tag{5.5.32}$$

$$2\xi\omega\Omega C + \left(\omega^2 - \Omega^2\right) D = 0 \tag{5.5.33}$$

Next, divide both sides by ω^2:

$$\left(1-\frac{\Omega^2}{\omega^2}\right)C-2\xi\frac{\Omega}{\omega}D=\frac{F_0}{\omega^2 m} \tag{5.5.34}$$

$$2\xi\frac{\Omega}{\omega}C+\left(1-\frac{\Omega^2}{\omega^2}\right)D=0 \tag{5.5.35}$$

Introduce the frequency ratio, equation (5.2.51), $\beta=\Omega/\omega$, and $k=\omega^2 m$ from equation (5.2.9) to get:

$$\left(1-\beta^2\right)C-2\xi\beta D=\frac{F_0}{k} \tag{5.5.36}$$

$$2\xi\beta C+\left(1-\beta^2\right)D=0 \tag{5.5.37}$$

From equation (5.5.37), we have:

$$C=-\frac{\left(1-\beta^2\right)}{2\xi\beta}D \tag{5.5.38}$$

And using this in equation (5.5.36) gives:

$$\left[-\frac{\left(1-\beta^2\right)^2}{2\xi\beta}-2\xi\beta\right]D=\frac{F_0}{k} \tag{5.5.39}$$

To get:

$$-\left[\frac{\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2}{2\xi\beta}\right]D=\frac{F_0}{k} \tag{5.5.40}$$

And rearrange to get, finally:

$$D = \frac{F_0}{k} \frac{-2\xi\beta}{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2} \tag{5.5.41}$$

Now using this with equation (5.5.38), we have:

$$C = -\frac{\left(1-\beta^2\right)}{2\xi\beta}\left[\frac{F_0}{k} \frac{-2\xi\beta}{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2}\right] \tag{5.5.42}$$

To get, finally:

$$C = \frac{F_0}{k} \frac{\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2} \tag{5.5.43}$$

Again we use the cosine addition rule:

$$\rho = \sqrt{C^2 + D^2} \tag{5.5.44}$$

$$\tan\theta = -\frac{D}{C} \tag{5.5.45}$$

To express the solution as:

$$u_P(t) = \rho \sin\left(\Omega t - \theta\right) \tag{5.5.46}$$

So we have, from equations (5.5.44), (5.5.43) and (5.5.41):

$$\rho = \frac{F_0}{k}\sqrt{\left[\frac{\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2}\right]^2+\left[\frac{-2\xi\beta}{\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2}\right]^2} \tag{5.5.47}$$

This simplifies to:

$$\rho = \frac{F_0}{k}\sqrt{\frac{\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2}{\left[\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2\right]^2}} \tag{5.5.48}$$

And finally we have the amplitude of displacement:

$$\rho = \frac{F_0}{k}\left[\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2\right]^{-\frac{1}{2}} \tag{5.5.49}$$

To obtain the phase angle, we use equation (5.5.45) with equations (5.5.43) and (5.5.41) again to get:

$$\tan\theta = -\frac{\dfrac{F_0}{k}\dfrac{-2\xi\beta}{\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2}}{\dfrac{F_0}{k}\dfrac{\left(1-\beta^2\right)}{\left(1-\beta^2\right)^2+\left(2\xi\beta\right)^2}} \tag{5.5.50}$$

Immediately we see that several terms (and the minus signs) cancel to give:

$$\tan\theta = \frac{2\xi\beta}{1-\beta^2} \tag{5.5.51}$$

Thus we have the final particular solution of equation (5.5.46) in conjunction with equations (5.5.49) and (5.5.51).

As referred to previously, the total solution is the sum of the particular and complimentary solutions, which for us now becomes:

$$u(t) = u_C(t) + u_P(t) \tag{5.5.52}$$

$$u(t) = e^{-\xi\omega t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + \rho\sin\left(\Omega t - \theta\right) \tag{5.5.53}$$

Notice here that we used equation (5.2.35) since we have redefined the amplitude and phase in terms of the forcing function. To determine the unknown constants from the initial parameters, u_0 and $\dot{u}_0$ we differentiate equation (5.5.53) to get:

$$\dot{u}(t) = e^{-\xi\omega t}\left[\left(\omega_d B - A\xi\omega\right)\cos\omega_d t - \left(A\omega_d + B\xi\omega\right)\sin\omega_d t\right] + \Omega\rho\cos\left(\Omega t - \theta\right) \tag{5.5.54}$$

Now at $t = 0$, we have from equations (5.5.53) and (5.5.54):

$$u(0) = u_0 = A - \rho\sin\theta \tag{5.5.55}$$

And:

$$\dot{u}(0) = \dot{u}_0 = \left(\omega_d B - A\xi\omega\right) + \Omega\rho\cos\theta \tag{5.5.56}$$

Solving for *A* first from equation (5.5.55) gives:

$$A = u_0 + \rho\sin\theta \tag{5.5.57}$$

And introducing this into equation (5.5.56) gives:

$$\dot{u}_0 = \omega_d B - \left(u_0 + \rho \sin\theta\right)\xi\omega + \Omega\rho\cos\theta \tag{5.5.58}$$

Multiplying out and rearranging gives:

$$\dot{u}_0 = \omega_d B - u_0\xi\omega + \rho\left(\Omega\cos\theta - \xi\omega\sin\theta\right) \tag{5.5.59}$$

From which we get:

$$B = \frac{\dot{u}_0 + u_0\xi\omega - \rho\left(\Omega\cos\theta - \xi\omega\sin\theta\right)}{\omega_d} \tag{5.5.60}$$

And now we have completely defined the time history of the problem in terms of its initial parameters.

Remember that:

- The complimentary solution ($u_C\left(t\right)$): represents the transient state of the system which dampens out after a period of time, as may be realized when it is seen that it is only the complementary response that is affected by the initial state (u_0 and $\dot{u}_0$) of the system, in addition to the exponentially reducing term in equation (5.5.53);
- The particular solution ($u_P\left(t\right)$): represents the steady state of the system which persists as long as the harmonic force is applied, as again may be seen from equation (5.5.53).

5.6.5 Important Formulae

SDOF Systems

Fundamental equation of motion	$m\ddot{u}(t)+c\dot{u}(t)+ku(t)=F(t)$
Equation of motion for free vibration	$\ddot{u}(t)+2\xi\omega\dot{u}(t)+\omega^2u(t)=0$
Relationship between frequency, circular frequency, period, stiffness and mass: Fundamental frequency for an SDOF system.	$f=\frac{1}{T}=\frac{\omega}{2\pi}=\frac{1}{2\pi}\sqrt{\frac{k}{m}}$
Coefficient of damping	$2\xi\omega=\frac{c}{m}$
Circular frequency	$\omega^2=\frac{k}{m}$
Damping ratio	$\xi=\frac{c}{c_{cr}}$
Critical value of damping	$c_{cr}=2m\omega=2\sqrt{km}$
General solution for free-undamped vibration	$u(t)=\rho\cos\left(\omega t-\theta\right)$ $\rho=\sqrt{u_0^2+\left(\frac{\dot{u}_0}{\omega}\right)^2}$; $\tan\theta=\frac{\dot{u}_0}{u_0\omega}$
Damped circular frequency, period and frequency	$\omega_d=\omega\sqrt{1-\xi^2}$ $T_d=\frac{2\pi}{\omega_d}$; $f_d=\frac{\omega_d}{2\pi}$
General solution for free-damped vibrations	$u(t)=\rho e^{-\xi\omega t}\cos\left(\omega_d t-\theta\right)$ $\rho=\sqrt{u_0^2+\left(\frac{\dot{u}_0+\xi\omega u_0}{\omega_d}\right)^2}$; $\tan\theta=\frac{\dot{u}_0+\xi\omega u_0}{u_0\omega_d}$

Logarithmic decrement of damping	$\delta = \ln\frac{u_n}{u_{n+m}} = 2m\pi\xi\frac{\omega}{\omega_d}$
Half-amplitude method	$\xi \cong \frac{0.11}{m}$ when $u_{n+m} = 0.5u_n$
Amplitude after p-cycles	$u_{n+p} = \left(\frac{u_{n+1}}{u_n}\right)^p u_n$
Equation of motion for forced response (sinusoidal)	$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = F_0 \sin\Omega t$
General solution for forced-damped vibration response and frequency ratio	$u_p(t) = \rho\sin(\Omega t - \theta)$ $\rho = \frac{F_0}{k}\left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]^{-1/2}$; $\tan\theta = \frac{2\xi\beta}{1-\beta^2}$ $\beta = \frac{\Omega}{\omega}$
Dynamic amplification factor (DAF)	$\text{DAF} \equiv D = \left[\left(1-\beta^2\right)^2 + \left(2\xi\beta\right)^2\right]^{-1/2}$

MDOF Systems

Fundamental equation of motion	$\mathbf{M\ddot{u}} + \mathbf{C\dot{u}} + \mathbf{Ku} = \mathbf{F}$
Equation of motion for undamped-free vibration	$\mathbf{M\ddot{u}} + \mathbf{Ku} = \mathbf{0}$
General solution and derivates for free-undamped vibration	$\mathbf{u} = \mathbf{a}\sin(\omega t + \phi)$ $\mathbf{\ddot{u}} = -\omega^2\mathbf{a}\sin(\omega t + \phi) = -\omega^2\mathbf{u}$
Frequency equation	$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{a} = \mathbf{0}$
General solution for 2DOF system	$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$
Determinant of 2DOF system from Cramer's rule	$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = \left[(k_2 + k_1) - \omega^2 m_1\right]\left[k_2 - \omega^2 m_2\right] - k_2^2 = 0$
Composite matrix	$\mathbf{E} = \left[\mathbf{K} - \omega^2\mathbf{M}\right]$
Amplitude equation	$\mathbf{Ea} = \mathbf{0}$

Continuous Structures

Equation of motion	$EI\frac{\partial^4 v(x,t)}{\partial x^4}+m\frac{\partial^2 v(x,t)}{\partial t^2}=p(x,t)$
Assumed solution for free-undamped vibrations	$v(x,t)=\phi(x)Y(t)$
General solution	$\phi(x)=A_1\sin(\alpha x)+A_2\cos(\alpha x)+A_3\sinh(\alpha x)+A_4\cosh(\alpha x)$
Boundary conditions for a simply supported beam	$v(0,t)=0$ and $EI\frac{\partial^2 v}{\partial x^2}(0,t)=0$ $v(L,t)=0$ and $EI\frac{\partial^2 v}{\partial x^2}(L,t)=0$
Frequencies of a simply supported beam	$\omega_n=\left(\frac{n\pi}{L}\right)^2\sqrt{\frac{EI}{m}}$
Mode shape or mode n: (A_1 is normally unity)	$\phi_n(x)=A_1\sin\left(\frac{n\pi x}{L}\right)$
Cantilever beam boundary conditions	$v(0,t)=0$ and $\frac{\partial v}{\partial x}(0,t)=0$ $EI\frac{\partial^2 v}{\partial x^2}(L,t)=0$ and $EI\frac{\partial^3 v}{\partial x^3}(L,t)=0$
Frequency equation for a cantilever	$\cos(\alpha L)\cosh(\alpha L)+1=0$
Cantilever mode shapes	$\phi_n(x)=A_1\left[\sin(\alpha x)-\sinh(\alpha x)+\frac{\sin(\alpha L)+\sinh(\alpha L)}{\cos(\alpha L)+\cosh(\alpha L)}\times\left(\cosh(\alpha x)-\cos(\alpha x)\right)\right]$
Bolton method general equation	$f=\frac{1}{2\pi}\sqrt{\frac{K_E}{M_E}}$

Practical Design

Peak acceleration under foot-loading	$a_0 = (0.9)2\pi f \dfrac{I}{M}$ $I \approx 70$ Ns $M \approx 40\%$ mass per unit area
Maximum dynamic deflection	$u_{\max} = u_{st} K \psi$
Maximum vertical acceleration	$\ddot{u}_{\max} = \omega^2 u_{\max}$
BD37/01 requirement for vertical acceleration	$\pm 0.5\sqrt{f}$

5.6.6 Glossary

Structural dynamics introduces many new terms and concepts so it's beneficial to keep track of them in one place. Fill this out as you progress through the notes.

Symbol	**Name**	**Units**