Chapter 6: Analysis of Structures

Some of the most common structures we see around us are buildings & bridges. In addition to these, one can also classify a lot of other objects as "structures."
For instance:

- The space station
- Chassis of your car
- Your chair, table, bookshelf etc. etc.

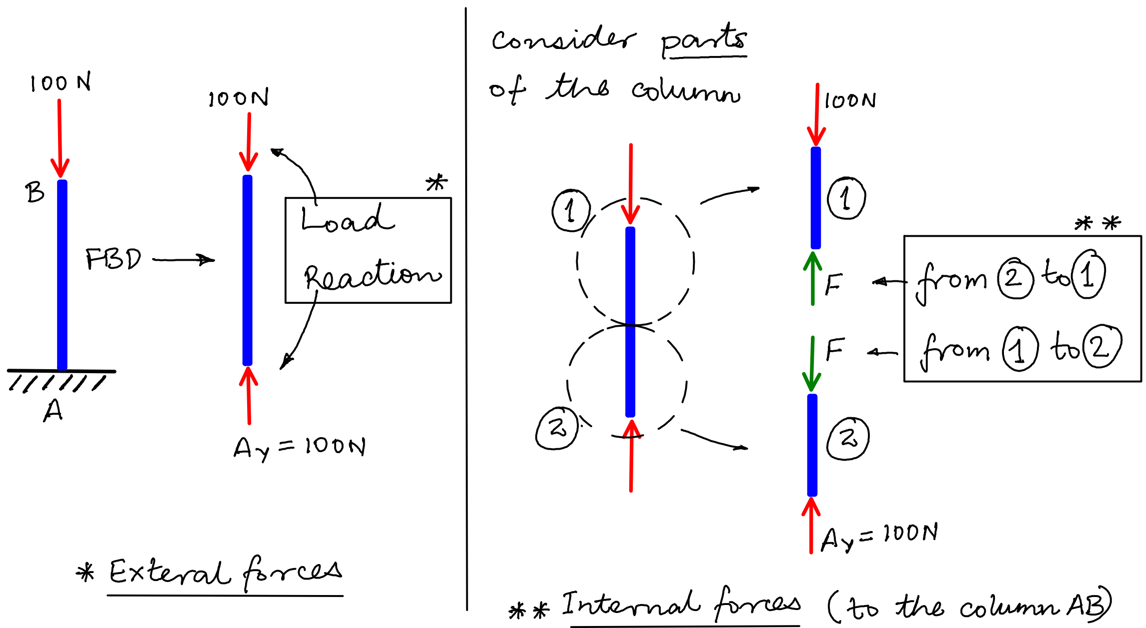
Almost everything has an internal structure and can be thought of as a "structure".

The objective of this chapter is to figure out the forces being carried by these structures so that as an engineer, you can decide whether the structure can sustain these forces or not.

Recall:

- External forces: "Loads" acting **on** your structure.
  Note: this includes "reaction" forces from the supports as well.

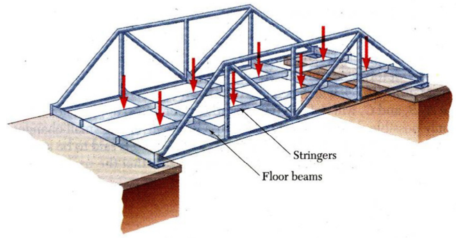
- Internal forces: Forces that develop within every structure that keep the different parts of the structure together.



In this chapter, we will find the internal forces in the following types of structures :

- Trusses
- Frames
- Machines

6.2-6.3 Trusses



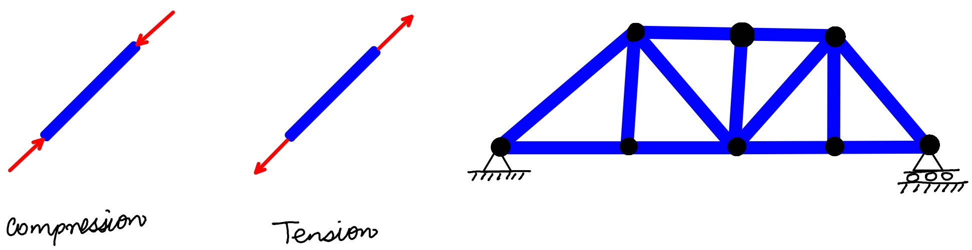
Trusses are used commonly in Steel buildings and bridges.

Definition: A truss is a structure that consists of

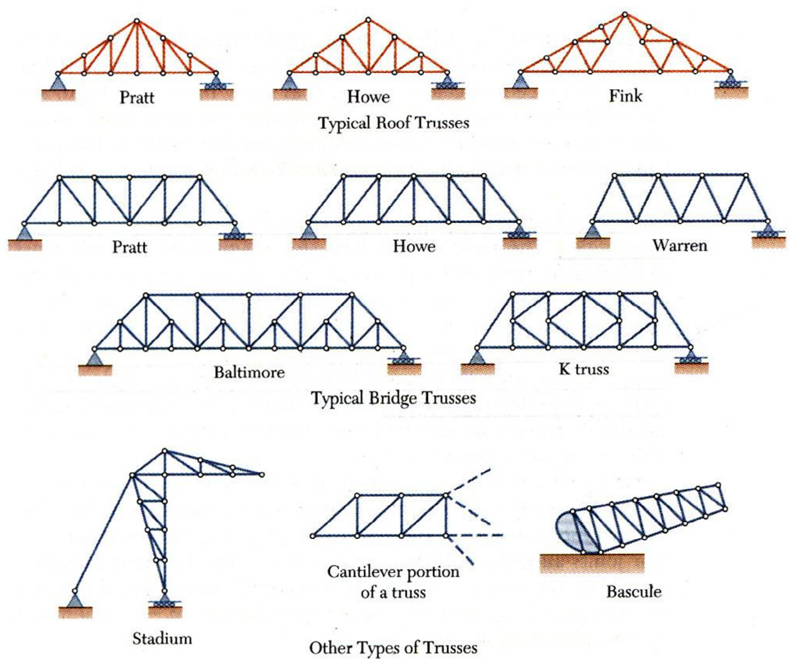
- All straight members
- connected together with pin joints
- connected only at the ends of the members
- and all external forces (loads & reactions) must be applied only at the joints.

Note:

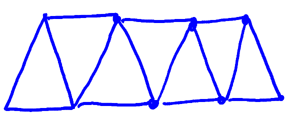
- Every member of a truss is a 2 force member.
- Trusses are assumed to be of negligible weight (compared to the loads they carry)



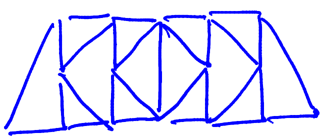
Types of Trusses



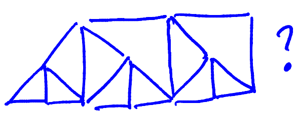
Simple Trusses: constructed from a "base" triangle by adding two members at a time.



simple



simple



? NOT simple

Note: For Simple Trusses (and in general statically determinate trusses)

$$\boxed{2n = m + r}$$

m: members, r: reactions } $(m+r)$ unknowns

n: joints } $2n$ equations

Note: This is a necessary condition for statical determinancy

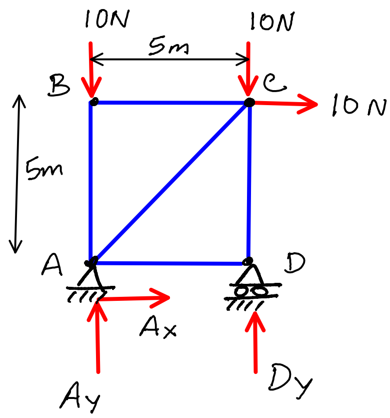
This is not sufficient condition. So even if a truss satisfies the above relation it may not be determinate.

But if it is determinate then it satisfies the above relation.

6.4 Analysis of Trusses: **Method of Joints**

Consider the truss shown. Truss analysis involves:

(i) Determining the EXTERNAL reactions.
(ii) Determining the INTERNAL forces in each of the members (tension or compression).



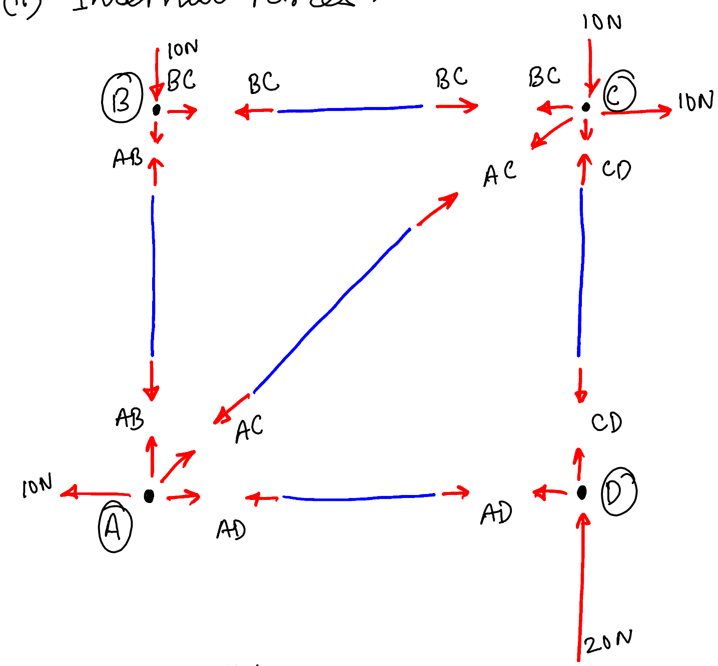
(i) External Reactions:

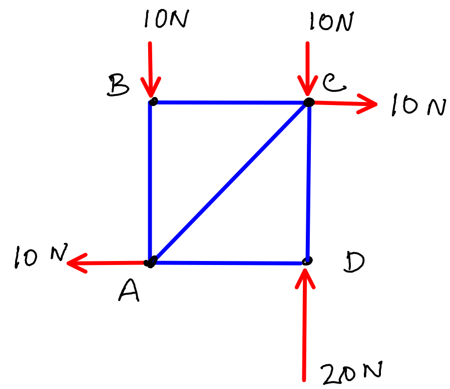
$\sum F_x = 0 \Rightarrow A_x + 10 = 0 \Rightarrow \boxed{A_x = -10N}$

$\sum F_y = 0 \Rightarrow A_y + D_y - 10 - 10 = 0 \Rightarrow \boxed{A_y = 0}$

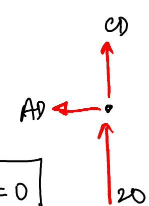
$\sum M_A = 0 \Rightarrow -(10 \times 5) - (10 \times 5) + D_y \times 5 = 0 \Rightarrow \boxed{D_y = 20N}$

(ii) Internal Forces:



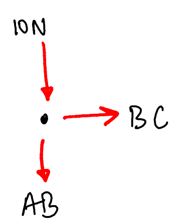


Joint D:



$\sum F_x = 0 \Rightarrow \boxed{AD = 0}$

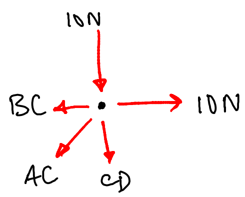
Joint B:



$\Rightarrow$ $\sum F_x = 0 \Rightarrow \boxed{BC = 0}$ (10 N Compressive)

$\sum F_y = 0 \Rightarrow -AB - 10 = 0 \Rightarrow \boxed{AB = -10}$

Joint C



$\sum F_x = 0 \Rightarrow -BC - AC(\cos 45^\circ) + 10 = 0$

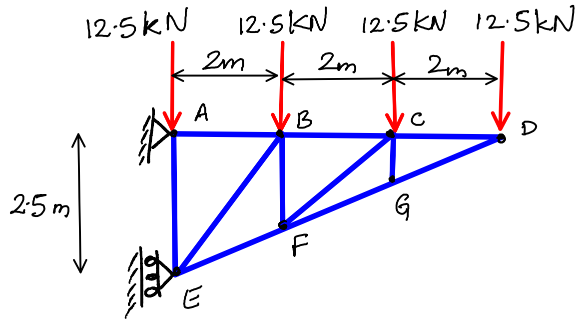
$\Rightarrow AC = 10\sqrt{2} = \boxed{14.14\,N}$ T

$\sum F_y = 0 \Rightarrow -10 - AC\sin 45^\circ - CD = 0$

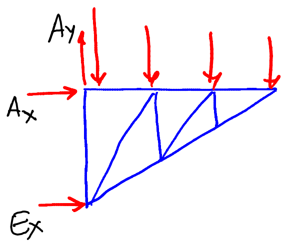
$\Rightarrow CD = -10N - 10 = \boxed{-20\,N}$ C

Read Example 6.1

Exercise 6.13



(i) External Reactions

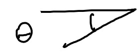


$\sum F_x = 0$

$\Rightarrow A_x + E_x = 0$

$\sum F_y = 0$

$\Rightarrow A_y - 4 \times 12.5 = 0 \Rightarrow \boxed{A_y = 50 \text{ kN}}$



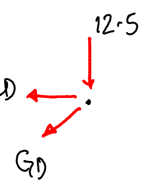
$\sin\theta = 5/13$

$\cos\theta = 12/13$

$\sum M_A = 0$

$\Rightarrow E_x(2.5) - 12.5(2+4+6) = 0 \Rightarrow \boxed{E_x = 60 \text{ kN}} \Rightarrow \boxed{A_x = -60 \text{ kN}}$

Joint D:

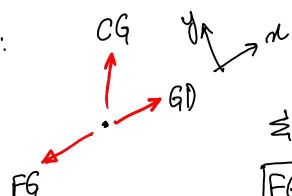


$\sum F_x = 0 \Rightarrow -CD - GD\cos\theta = 0 \Rightarrow CD = -32.5 \times \frac{12}{13} = \boxed{30 \text{ kN}}$ (T)

$\sum F_y = 0 \Rightarrow -12.5 - GD\sin\theta = 0 \Rightarrow GD = \frac{-12.5}{5} \times 13$

$= \boxed{-32.5 \text{ kN}}$ (comp)

Joint G:



$\sum F_y = 0 \Rightarrow \boxed{CG = 0}$

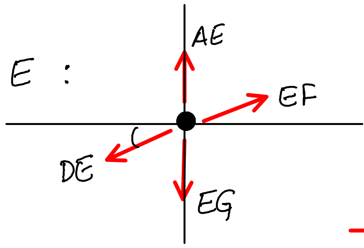
$\boxed{FG = GD = -32.5 \text{ kN}}$

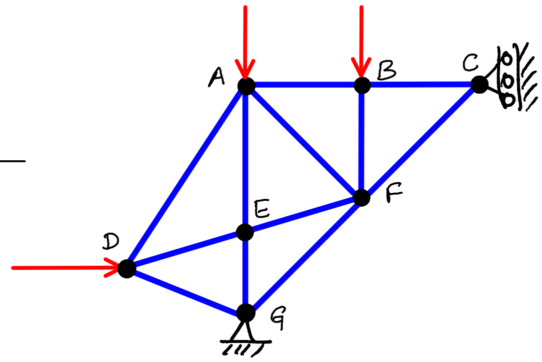
Similarly, solve joints C, F and B in that order and calculate the rest of the unknowns.

6.5 Joints under special loading conditions: Zero force members

Many times, in trusses, there may be joints that connect members that are "aligned" along the same line.

Consider joint E :





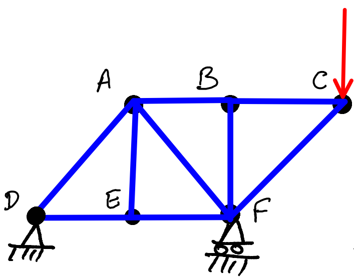
$\sum F_x = 0$

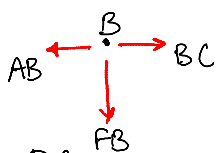
$\Rightarrow -DE\cos\theta + EF\cos\theta = 0$

$\Rightarrow \boxed{DE = EF}$

$\sum F_y = 0$

$\Rightarrow \boxed{AE = EG}$

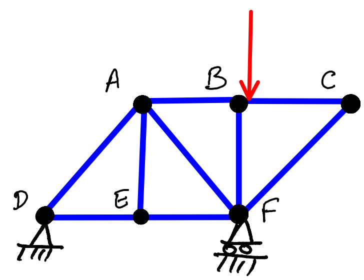


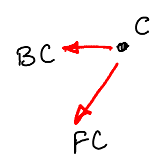


$\Rightarrow \underline{AB = BC}$

$\underline{FB = 0}$

Similarly, from joint E: DE=EF and AE=0





$\sum F_x = 0$

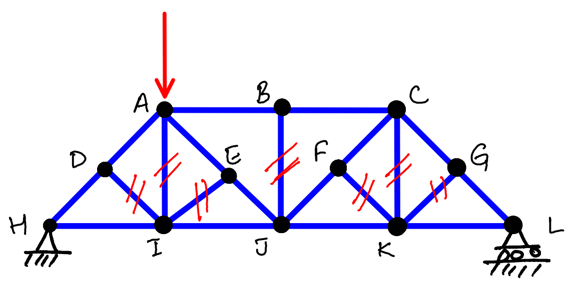
$\Rightarrow -BC - FC\cos\theta = 0$

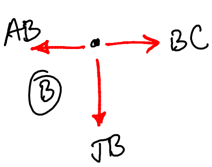
$\sum F_y = 0 \quad \Rightarrow \boxed{FC = 0}$

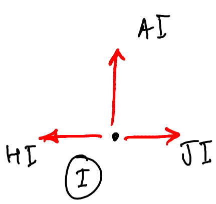
$\Rightarrow \boxed{BC = 0}$

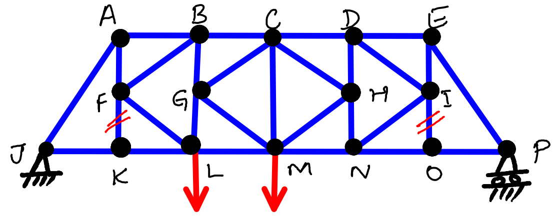
Exercise 6.32

Identify the zero-force members.



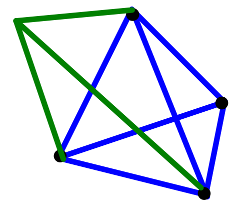






6.6 Space Trusses

Generalizing the structure of planar trusses to 3D results in space trusses.



The most elementary 3D space truss structure is the **tetrahedron**. The members are connected with ball-and-socket joints.

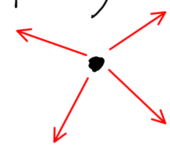
Simple space trusses can be obtained by adding 3 elements at a time to 3 existing joints and joining all the new members at a point.

Note: For a 3D determinate truss:

$$\boxed{3n = m + r}$$

n: joints] → $3n$ equilibrium equations $\left(\sum \vec{F} = 0\right)$

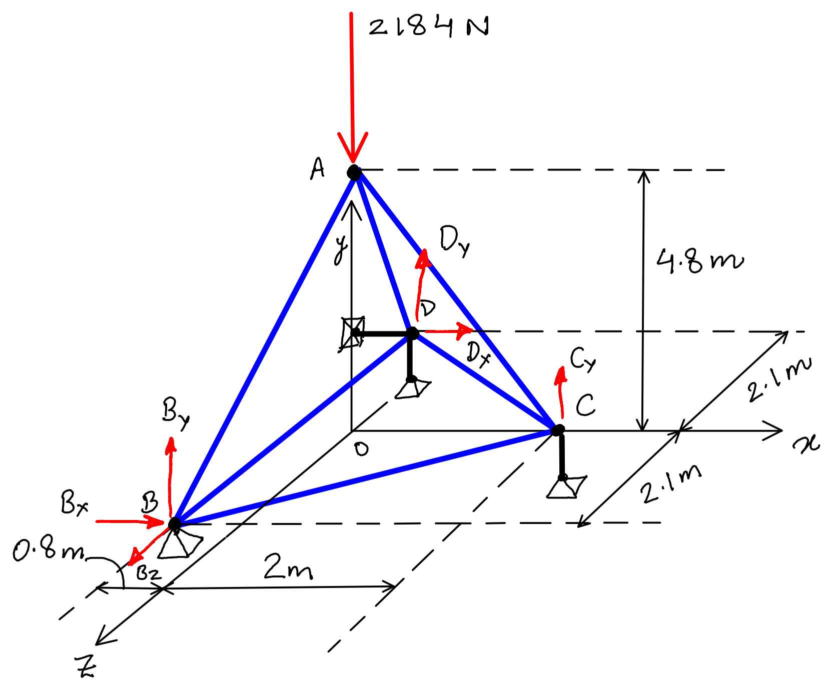
m: members
r : reactions] → $(m+r)$ unknowns



If the truss is "determinate" then this condition is satisfied.
However, even if this condition is satisfied, the truss may not be determinate.
Thus this is a Necessary condition (not sufficient) for solvability of a truss.

Exercise 6.36

Determine the forces in each member.



$\sum F_x = 0 \Rightarrow B_x + D_x = 0$

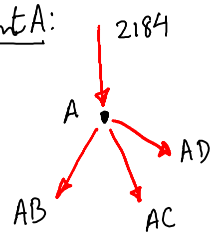
$\sum F_y = 0 \Rightarrow B_y + D_y + C_y - 2184 = 0$

$\sum F_z = 0 \Rightarrow \boxed{B_z = 0}$

Symmetry $\Rightarrow B_y = D_y$

$\boxed{B_x = D_x = 0}$

Joint A:



$$\sum \vec{F} = 0 \Rightarrow \vec{F}_{AB} + \vec{F}_{AC} + \vec{F}_{AD} - 2184\,\underline{j} = 0$$

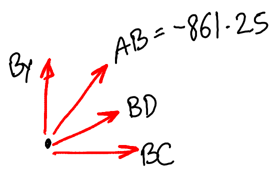
$$\vec{F}_{AB} = F_{AB}\frac{\vec{AB}}{|AB|} = F_{AB}\frac{(-0.8\underline{i} - 4.8\underline{j} + 2.1\underline{k})}{\sqrt{(0.8)^2 + (4.8)^2 + (2.1)^2}} \rightarrow 5.3$$

$$\vec{F}_{AC} = F_{AC}\frac{\vec{AC}}{|AC|} = F_{AC}\frac{(2\underline{i} - 4.8\underline{j} + 0\underline{k})}{\sqrt{2^2 + (4.8)^2}} \rightarrow 5.2$$

$$\vec{F}_{AD} = F_{AD}\frac{\vec{AD}}{|AD|} = F_{AD}\frac{(-0.8\underline{i} - 4.8\underline{j} - 2.1\underline{k})}{\sqrt{(0.8)^2 + (4.8)^2 + (2.1)^2}} \rightarrow 5.3$$

$$\left.\begin{aligned}
\Sigma F_x = 0 &\Rightarrow -\frac{0.8}{5.3} F_{AB} + \frac{2}{5.2} F_{AC} - \frac{0.8}{5.3} F_{AD} = 0 \\
\Sigma F_y = 0 &\Rightarrow -\frac{4.8}{5.3} F_{AB} - \frac{4.8}{5.2} F_{AC} - \frac{4.8}{5.3} F_{AD} - 2184 = 0 \\
\Sigma F_z = 0 &\Rightarrow \frac{2.1}{5.3} F_{AB} - \frac{2.1}{5.3} F_{AD} = 0
\end{aligned}\right] \Rightarrow \begin{aligned} &\boxed{F_{AB} = F_{AD} = -861.25\,N} \\ &\boxed{F_{AC} = -676\,N} \end{aligned}$$

Joint B



$$\Sigma \vec{F} = 0 \Rightarrow \vec{F}_{AB} + \vec{F}_{BD} + \vec{F}_{BC} + B_Y \underline{j} = 0$$

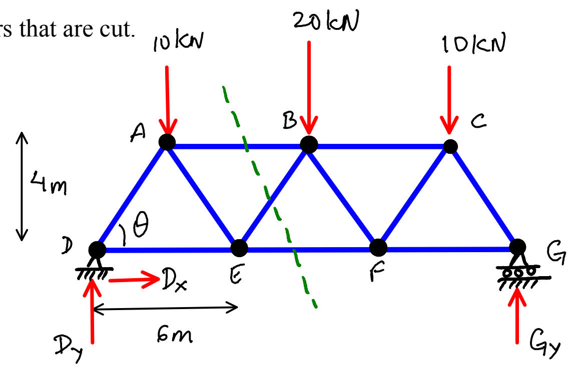
Similarly find the 3 unknowns F_{BD}, F_{BC} and B_Y at joint B.

6.7 Analysis of Trusses: **Method of Sections**

The method of joints is good if we have to find the internal forces in all the truss members.
In situations where we need to find the internal forces only in a few specific members of a truss, the method of sections is more appropriate.

Method of sections:

- Imagine a cut through the members of interest
- Try to cut the least number of members (preferably 3).
- Draw FBD of the 2 different parts of the truss
- Enforce Equilibrium to find the forces in the 3 members that are cut.



For example, find the force in member EF:

EXTERNAL (Entire Truss)

$\Sigma F_x = 0 \Rightarrow D_x = 0$

$\Sigma F_y = 0 \Rightarrow D_y + G_y = 0$

Symmetry $\Rightarrow D_y = G_y = 20$ kN

INTERNAL (Cut - - - -)

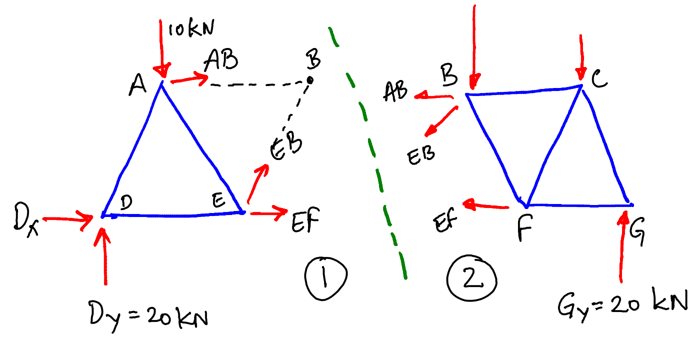
Body 1 :-

$\Sigma F_x = 0 \Rightarrow EF + EB\cos\theta + AB = 0$

$\Sigma F_y = 0 \Rightarrow EB\sin\theta + 20\text{kN} - 10\text{ kN} = 0$

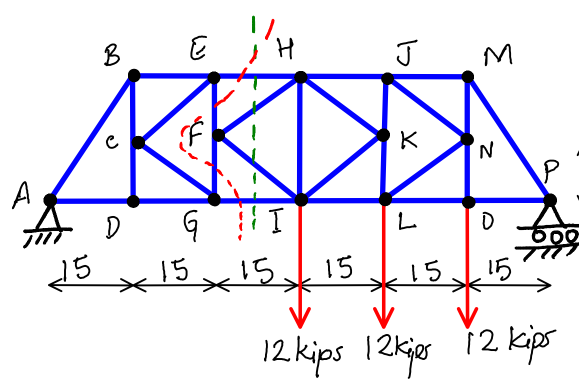
$\Sigma M_B = 0 \Rightarrow EF \times 4 - 20 \times 9 + 10 \times 6 = 0$

$\Rightarrow \boxed{EF = 30\text{kN}}$ (T)



Read Examples 6.2 and 6.3 from the book.

Exercise 6.63
Find forces in the members EH and GI.



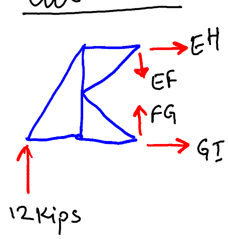
$\Sigma M_A = 0$ (for entire truss)

$\Rightarrow -36 \times 60 + P_y \times 90 = 0$

$\Rightarrow P_y = \frac{\cancel{36}^{4} \times 6\cancel{0}}{\cancel{90}} = \boxed{24\text{ kips}}$

$\Rightarrow \boxed{A_y = 12\text{ kips}}$

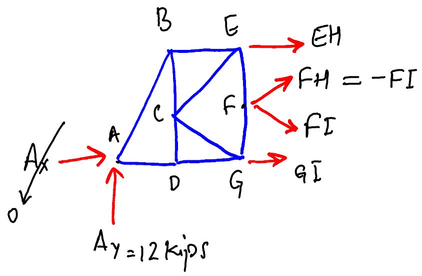
Cut - - - -



$\Sigma M_E = 0 \Rightarrow -12 \times 30 + GI \times 16 = 0 \Rightarrow GI = \frac{360}{16} = \boxed{22.5\text{ kips}}$ (T)

$\Sigma F_x = 0 \Rightarrow EH + GI = 0 \Rightarrow \boxed{EH = -22.5\text{ kips}}$ (C)

$\boxed{\text{OR}}$ Cut - - - - -

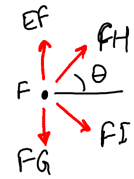


$$\Sigma F_x = 0 \Rightarrow EH + GI - FI\cancel{\cos\theta} + FI\cancel{\cos\theta} = 0$$

$$\Sigma F_y = 0 \Rightarrow 12 - FI\sin\theta - FI\sin\theta = 0$$

$$\Sigma M_F = 0 \Rightarrow -12\times 30 - EH\times 8 + GI\times 8 = 0$$

$$\Rightarrow GI = \frac{12\times 30}{16} = \boxed{22.5\ \text{kips}}\ (T) \Rightarrow \boxed{EH = -22.5\ \text{kips}}\ (C)$$



$$\Sigma F_x = 0 \Rightarrow \boxed{FH = -FI}$$

$$\Sigma F_y = 0$$

$$\Rightarrow EF - FG + FH\sin\theta - FI\sin\theta = 0$$

$$\Rightarrow EF - FG + 2FH\sin\theta = 0$$

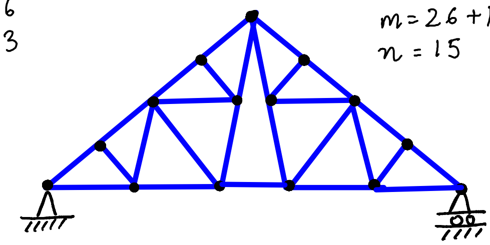
6.8 Compound Trusses; Determinate vs. Indeterminate Trusses.

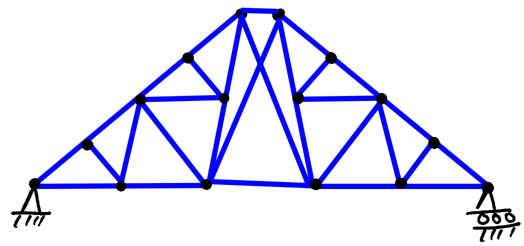
Trusses made by joining two or more simple trusses rigidly are called Compound Trusses.

$2n = 30$
$m + r = 26 + 3 = 29 + 1$

$2n = 16$
$m = 13$

$m = 26 + 1$
$n = 15$





$\boxed{2n > m + r}$ $\Longleftrightarrow$ Partially constrained

$\boxed{2n < m + r}$ $\Longleftrightarrow$ Overly constrained, Indeterminate

$\boxed{2n = m + r}$ $\Longleftarrow$ ✓ $\cancel{\Longrightarrow}$ Determinate

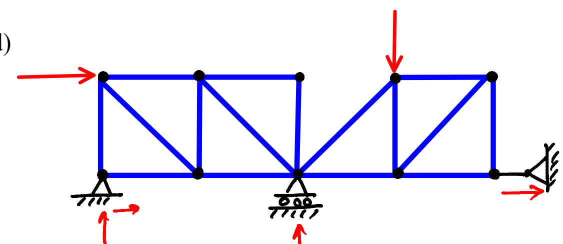
Exercise 6.69 Classify the trusses as:

Externally: Completely / Partially /Improperly constrained

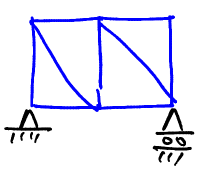
Internally: Determinate / Indeterminate. (if completely constrained)

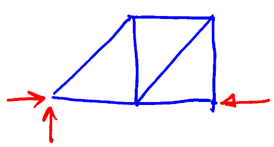
$n = 10 \Rightarrow 2n = 20$

$\left.\begin{matrix} m = 16 \\ r = 4 \end{matrix}\right\} \Rightarrow 20$



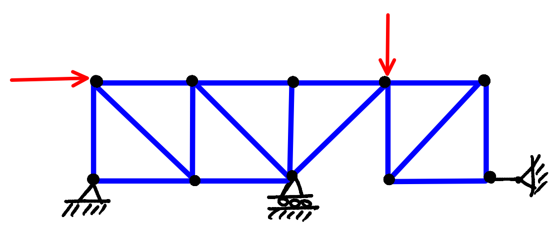
$\Rightarrow$ Partially restrained



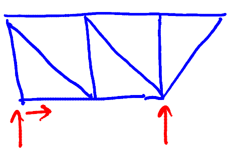


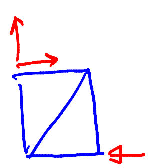
$n = 10 \Rightarrow 2n = 20$

$\left.\begin{matrix} m = 16 \\ r = 4 \end{matrix}\right] = 20$



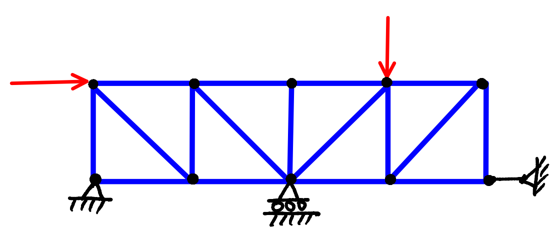
$\Rightarrow$ Determinate





$n = 10 \Rightarrow 2n = 20$ equations

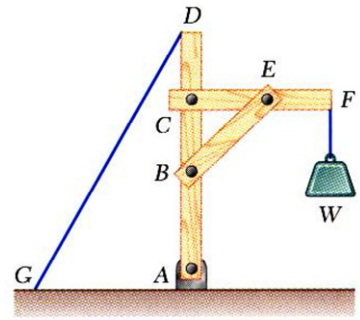
$\left.\begin{matrix} m = 17 \\ r = 4 \end{matrix}\right\} = 21$



$m + r > 2n$

$\Rightarrow$ Indeterminate

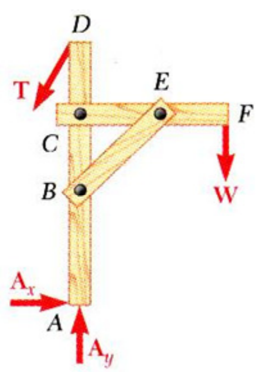
6.9 - 6.11 Frames



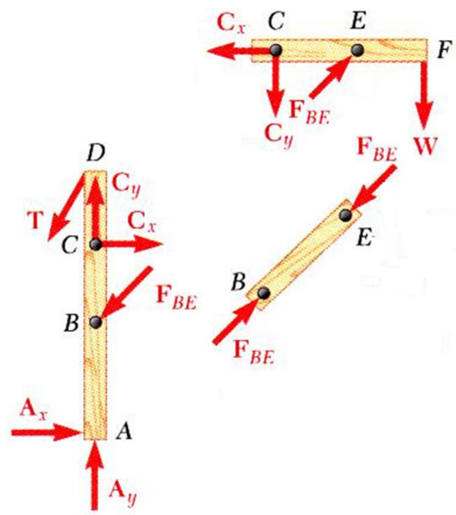
Frames are structures with at least one **multi-force** member,
i.e. atleast one member that has **3 or more** forces acting on it at different points.

Frame analysis involves determining:

(i) External Reactions



(ii) Internal forces at the joints



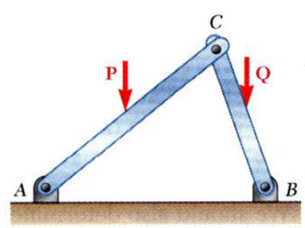
Note:
- Follow Newton's 3rd Law

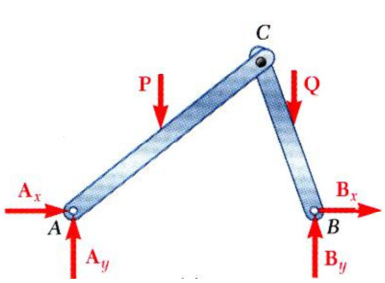
Frames that are not internally Rigid

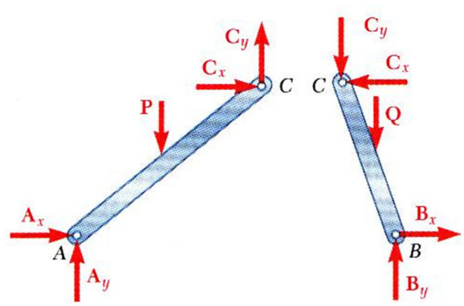
When a frame is not internally rigid, it has to be provided with additional external supports to make it rigid.

The support reactions for such frames cannot be simply determined by external equilibrium.

One has to draw the FBD of all the component parts to find out whether the frame is determinate or indeterminate.







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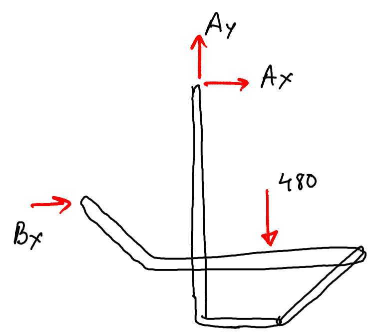
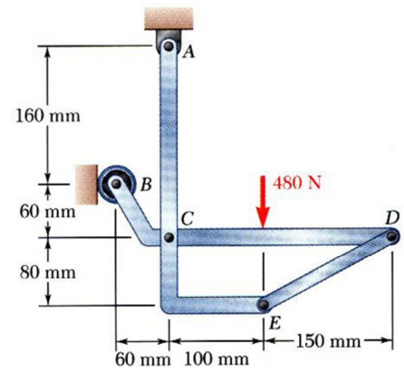
Example 6.4





$\sum F_x = 0$

$\Rightarrow A_x + B_x = 0$

$\sum F_y = 0$

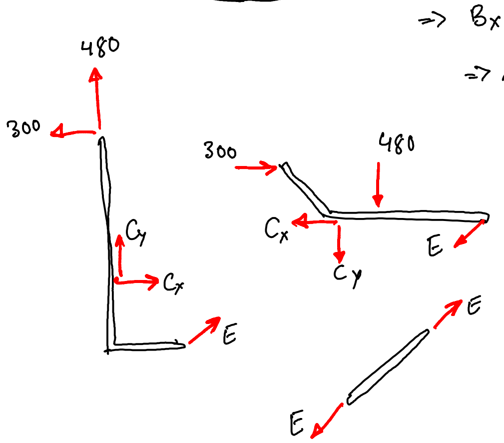
$A_y = 480$

$\sum M_A = 0$

$\Rightarrow -480 \times 0.1 + B_x \times 0.16 = 0$

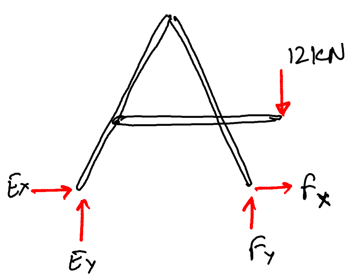
$\Rightarrow B_x = \frac{48}{0.16} = 300N$

$\Rightarrow A_x = -300N$



Read examples 6.5 and 6.6

Exercise 6.101



$\sum F_x = 0$

$\Rightarrow E_x + F_x = 0$

$\sum F_y = 0$

$\Rightarrow E_y + F_y = 12$

$\sum M_F = 0 \Rightarrow -E_y \times 1.8 - 12 \times 0.3 = 0$ $\Rightarrow \boxed{E_y = -2\ kN}$ $\Rightarrow \boxed{F_y = 14kN}$

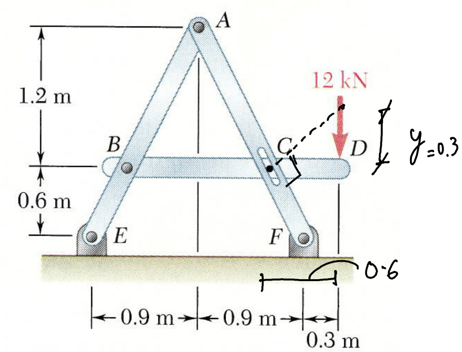
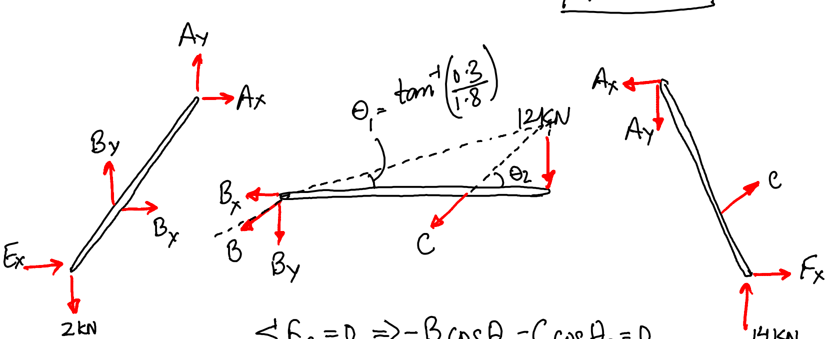


Fig. P6.101

x

$\frac{x}{0.9} = \frac{1.2}{1.8} \Rightarrow x = 0.6$

$\frac{y}{0.6} = \frac{0.9}{1.8} \Rightarrow y = 0.3$



$\sum F_x = 0 \Rightarrow -B\cos\theta_1 - C\cos\theta_2 = 0$

$\sum F_y = 0 \Rightarrow -B\sin\theta_1 - C\sin\theta_2 = 0$

Exercise 6.120

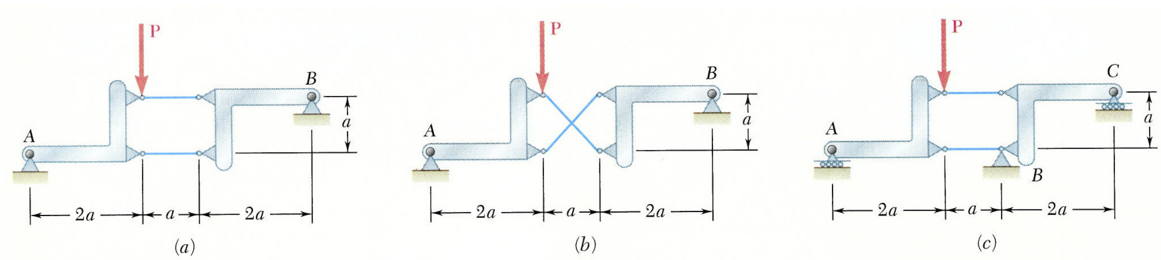
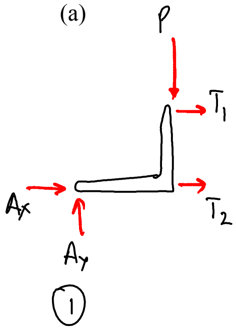


Fig. P6.120

(a)

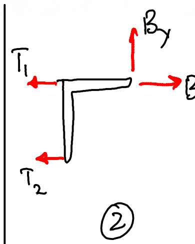


$\sum F_x = 0 \Rightarrow A_x + T_1 + T_2 = 0$

$\sum F_y = 0 \Rightarrow \boxed{A_y = P}$

$\sum M_A = 0 \Rightarrow -P \times 2a - T_1 a = 0$

$\Rightarrow \boxed{T_1 = -\frac{P}{2}}$



$\sum F_x = 0$

$\Rightarrow -T_1 + B_x - T_2 = 0$

$\sum F_y = 0$

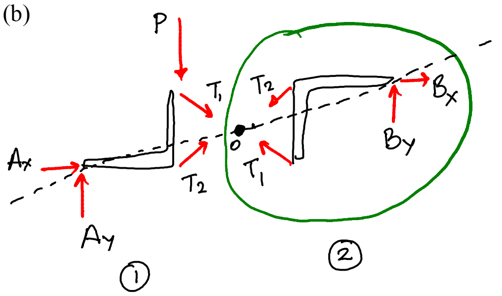
$\Rightarrow \boxed{B_y = 0}$

$\sum M_B = 0 \Rightarrow \boxed{T_2 = 0}$

$\Rightarrow \boxed{A_x = \frac{P}{2}} \Rightarrow \boxed{B_x = \frac{P}{2}}$

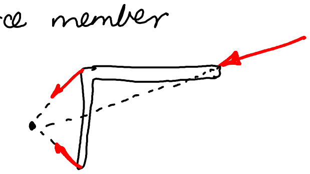
$\Rightarrow$ Completely constrained & Determinate.

(b)

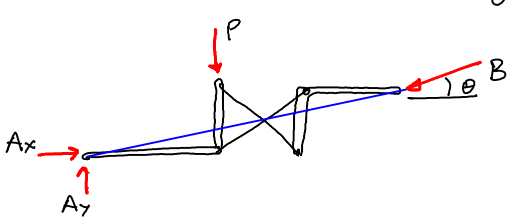


3-force member

$\Rightarrow$



$\Rightarrow$ Now consider the whole thing:-



$\sum M_A = 0 \Rightarrow P \times 2a = 0 \Rightarrow \underline{P = 0}$

Thus the structure cannot support any load

$\Rightarrow$ partially constrained.

OR (long way)

FBD 1:

$\sum F_x = 0 \Rightarrow A_x + T_1 \cos 45° + T_2 \cos 45° = 0$

$\sum F_y = 0 \Rightarrow A_y - P - T_1 \sin 45° + T_2 \sin 45° = 0$

$\sum M_O = 0 \Rightarrow A_x\, 0.5a - A_y\, 2.5a + P \times 0.5a = 0$

FBD 2:

$\sum F_x = 0 \Rightarrow -T_1 \cos 45° - T_2 \cos 45° - B \cos\theta = 0$

$\sum F_y = 0 \Rightarrow T_1 \sin 45° - T \sin 45° - B \sin\theta = 0$

5 equations

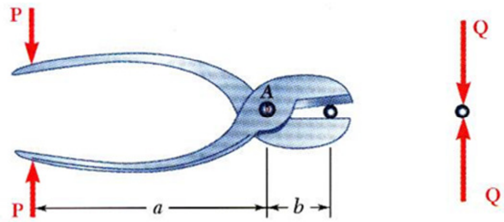
5 unknowns

(A_x, A_y, T_1, T_2, B)

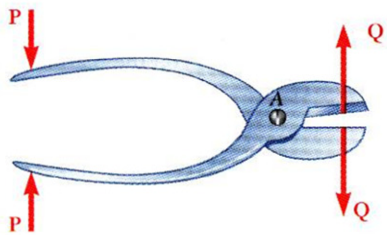
Try to solve for all unknowns.

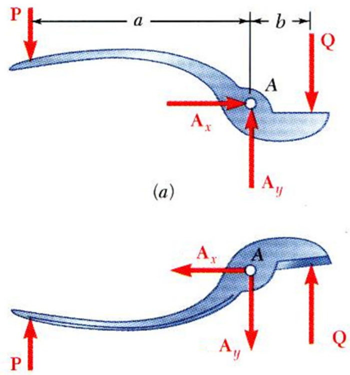
You will see that you cant (unless $P = 0$). $\Rightarrow$ Partially constrained.

6.12 Machines



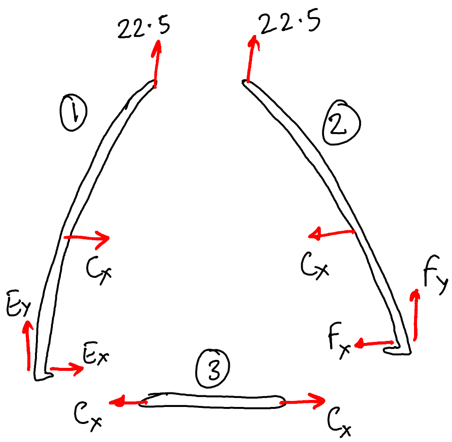
- Machines are structures designed to transmit and modify forces. Their main purpose is to transform *input forces* into *output forces*.
- Machines are usually non-rigid internally. So we use the components of the machine as a free-body.
- Given the magnitude of ***P***, determine the magnitude of ***Q***.





Exercise 6.143

6.143 The tongs shown are used to apply a total upward force of 45 kN on a pipe cap. Determine the forces exerted at *D* and *F* on tong *ADF*.



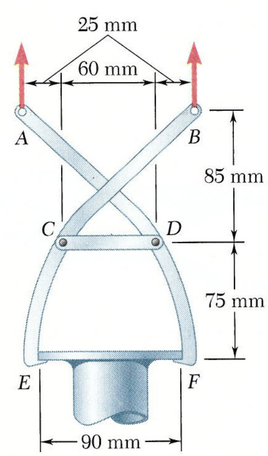


Fig. P6.143

2

$\Sigma M_F = 0$

$\Rightarrow -22.5 \times 100$

$+ C_x \times 75 = 0$

$\Rightarrow C_x = \cancel{22.5}^{7.5} \times \frac{4}{\cancel{3}} = \boxed{30\ kN}$ $\Rightarrow$ $\Sigma F_x = 0 \Rightarrow -C_x - F_x \Rightarrow \boxed{F_x = -30\,kN}$

$\Sigma F_y \Rightarrow \boxed{F_y = -22.5\,kN}$

Determinate vs. Indeterminate Structures

Structures such as Trusses and Frames can be broadly classified as:

- Determinate:
  When all the unknowns (external reactions and internal forces) can be found using "Statics" i.e. Drawing FBDs and writing equilibrium equations.

- Indeterminate:
  When, not all the unknowns can be found using Statics.
  Note: Some/most unknowns can still be found.

Structures can also be classified as:
- Completely restrained
- Partially restrained
- Improperly restrained

For trusses, we have been using "formulas" such as ($2n = m+r$) for planar trusses, and ($3n = m+r$) for space trusses to judge the type of structure. For frames, this can be much more complicated. We need to write and solve the equilibrium equations and only if a solution exists, we can conclude that the structure is determinate. Otherwise the structure may be partially constrained or indeterminate or both.

IMPORTANT:

One of the best ways (and mathematically correct way) to conclude determinacy of any structure is by using Eigen-values. Eigen-values tell us how many independent equations we have and whether can or can't solve a system of equations written in the form of Matrices.

$$[\mathbf{A}]\ \underline{\mathbf{x}} = \underline{\mathbf{b}}$$

$$\overbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & \ddots & \ddots & \\ a_{31} & \ddots & \ddots & \ddots \\ a_{41} & & \ddots & \ddots \end{bmatrix}}^{A} \overbrace{\begin{bmatrix} A_x \\ A_y \\ B_x \\ B_y \end{bmatrix}}^{\underline{x}} = \overbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}^{\underline{b}}$$

To do this,

Draw the FBDs of all rigid components of the structure
Write out the all the possible equilibrium equations.

Case 1: Number of Equations (E) < Number of Unknowns (U) <=> INDETERMINATE

Case 2: Number of Equations (E) > Number of Unknowns (U) <=> PARTIALLY RESTRAINED

Case 3: Number of Equations (E) = Number of Unknowns (U)

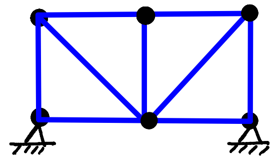
Find the number of **non-zero Eigen-values** (V_1) of the square matrix **[A]**.
Find the number of **non-zero Eigen-values** (V_2) of the rectangular matrix **[A|b]**.

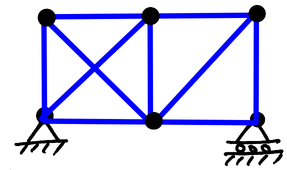
Case 3(a): $V_1 = E = U$	=> Unique Solution DETERMINATE
Case 3(b): $V_1 < E$ Number of INDEPENDENT equations = $V_1 < U$	=> Improperly constrained Indeterminate & Partially constrained
(i) $V_1 = V_2 \quad < U$	=> Infinitely many solutions possible
(ii) $V_1 < V_2$	=> No solution exists

Note: In this procedure, it is better not to reduce the number of unknowns or number of equations by using properties of 2-force or 3-force members.

Examples:

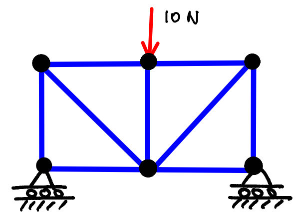
Completely constrained and indeterminate



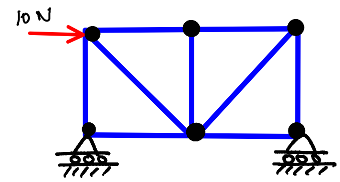


(Infinitely many solutions possible)

Partially constrained

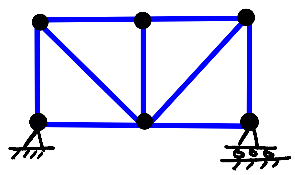


Solution exists

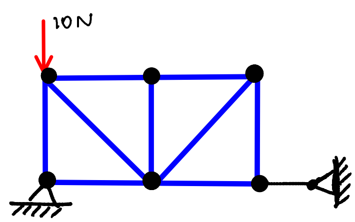


NO solution
(10 = 0?)

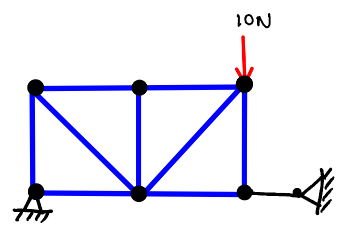
Determinate



Improperly restrained

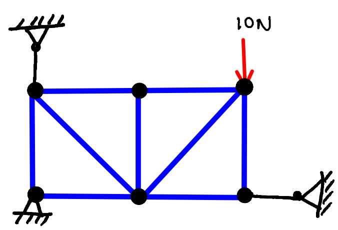


(Infinitely many solutions possible)



NO solution

Improperly restrained



NO solution