# Chapter 6

# **Eigenvalues and Eigenvectors**

# 6.1 Introduction to Eigenvalues

An eigenvector x lies along the same line as Ax : Ax = λx. The eigenvalue is λ.
 If Ax = λx then A<sup>2</sup>x = λ<sup>2</sup>x and A<sup>-1</sup>x = λ<sup>-1</sup>x and (A + cI)x = (λ + c)x: the same x.
 If Ax = λx then (A-λI)x = 0 and A-λI is singular and det(A-λI) = 0. n eigenvalues.
 Check λ's by det A = (λ<sub>1</sub>)(λ<sub>2</sub>) ··· (λ<sub>n</sub>) and diagonal sum a<sub>11</sub> + a<sub>22</sub> + ··· + a<sub>nn</sub> = sum of λ's.
 Projections have λ=1 and 0. Reflections have 1 and -1. Rotations have e<sup>iθ</sup> and e<sup>-iθ</sup>: complex!

This chapter enters a new part of linear algebra. The first part was about Ax = b: balance and equilibrium and steady state. Now the second part is about **change**. Time enters the picture—continuous time in a differential equation du/dt = Au or time steps in a difference equation  $u_{k+1} = Au_k$ . Those equations are NOT solved by elimination.

The key idea is to avoid all the complications presented by the matrix A. Suppose the solution vector u(t) stays in the direction of a fixed vector x. Then we only need to find the number (changing with time) that multiplies x. A number is easier than a vector. We want "eigenvectors" x that don't change direction when you multiply by A.

A good model comes from the powers  $A, A^2, A^3, \ldots$  of a matrix. Suppose you need the hundredth power  $A^{100}$ . Its columns are very close to the *eigenvector* (.6, .4):

$$A, A^{2}, A^{3} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \begin{bmatrix} .650 & .525 \\ .350 & .475 \end{bmatrix} \qquad A^{100} \approx \begin{bmatrix} .6000 & .6000 \\ .4000 & .4000 \end{bmatrix}$$

 $A^{100}$  was found by using the *eigenvalues* of A, not by multiplying 100 matrices. Those eigenvalues (here they are  $\lambda = 1$  and 1/2) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by A. Certain exceptional vectors x are in the same direction as Ax. Those are the "eigenvectors". Multiply an eigenvector by A, and the vector Ax is a number  $\lambda$  times the original x.

#### The basic equation is $Ax = \lambda x$ . The number $\lambda$ is an eigenvalue of A.

The eigenvalue  $\lambda$  tells whether the special vector x is stretched or shrunk or reversed or left unchanged—when it is multiplied by A. We may find  $\lambda = 2$  or  $\frac{1}{2}$  or -1 or 1. The eigenvalue  $\lambda$  could be zero! Then Ax = 0x means that this eigenvector x is in the nullspace.

If A is the identity matrix, every vector has Ax = x. All vectors are eigenvectors of I. All eigenvalues "lambda" are  $\lambda = 1$ . This is unusual to say the least. Most 2 by 2 matrices have *two* eigenvector directions and *two* eigenvalues. We will show that det $(A - \lambda I) = 0$ .

This section will explain how to compute the x's and  $\lambda$ 's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use  $det(A - \lambda I) = 0$  to find the eigenvalues for this first example, and then derive it properly in equation (3).

**Example 1** The matrix A has two eigenvalues  $\lambda = 1$  and  $\lambda = 1/2$ . Look at det $(A - \lambda I)$ :

$$A = \begin{bmatrix} .8 & .3\\ .2 & .7 \end{bmatrix} \quad \det \begin{bmatrix} .8-\lambda & .3\\ .2 & .7-\lambda \end{bmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)\left(\lambda - \frac{1}{2}\right).$$

I factored the quadratic into  $\lambda - 1$  times  $\lambda - \frac{1}{2}$ , to see the two eigenvalues  $\lambda = 1$  and  $\lambda = \frac{1}{2}$ . For those numbers, the matrix  $A - \lambda I$  becomes *singular* (zero determinant). The eigenvectors  $x_1$  and  $x_2$  are in the nullspaces of A - I and  $A - \frac{1}{2}I$ .

- $(A I)\mathbf{x}_1 = 0$  is  $A\mathbf{x}_1 = \mathbf{x}_1$  and the first eigenvector is (.6, .4).
- $(A \frac{1}{2}I)\mathbf{x}_2 = 0$  is  $A\mathbf{x}_2 = \frac{1}{2}\mathbf{x}_2$  and the second eigenvector is (1, -1):

$$\boldsymbol{x}_{1} = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_{1} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \boldsymbol{x}_{1} \quad (A\boldsymbol{x} = \boldsymbol{x} \text{ means that } \lambda_{1} = 1)$$
$$\boldsymbol{x}_{2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad A\boldsymbol{x}_{2} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \quad (\text{this is } \frac{1}{2}\boldsymbol{x}_{2} \text{ so } \lambda_{2} = \frac{1}{2}).$$

If  $x_1$  is multiplied again by A, we still get  $x_1$ . Every power of A will give  $A^n x_1 = x_1$ . Multiplying  $x_2$  by A gave  $\frac{1}{2}x_2$ , and if we multiply again we get  $(\frac{1}{2})^2$  times  $x_2$ .

When A is squared, the eigenvectors stay the same. The eigenvalues are squared.

This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of  $A^{100}$  are the same  $x_1$  and  $x_2$ . The eigenvalues of  $A^{100}$  are  $1^{100} = 1$  and  $(\frac{1}{2})^{100} =$  very small number.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of A is the combination  $x_1 + (.2)x_2$ :

Separate into eigenvectors 
$$\begin{bmatrix} .8 \\ .2 \end{bmatrix} = x_1 + (.2)x_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .2 \\ -.2 \end{bmatrix}.$$
 (1)

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$$\lambda = 1 \qquad Ax_1 = x_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \qquad \lambda^2 = 1 \qquad A^2 x_1 = (1)^2 x_1$$

$$\lambda^2 = .25 \qquad A^2 x_2 = (.5)^2 x_2 = \begin{bmatrix} .25 \\ -.25 \end{bmatrix}$$

$$\lambda = .5 \qquad Ax_2 = \lambda_2 x_2 = \begin{bmatrix} .5 \\ -.5 \end{bmatrix} \qquad Ax = \lambda x$$

$$x_2 = \begin{bmatrix} .1 \\ -1 \end{bmatrix} \qquad Ax = \lambda x$$

$$A^2 x = \lambda^2 x$$

Figure 6.1: The eigenvectors keep their directions.  $A^2 x = \lambda^2 x$  with  $\lambda^2 = 1^2$  and  $(.5)^2$ .

When we multiply separately for  $x_1$  and  $(.2)x_2$ , A multiplies  $x_2$  by its eigenvalue  $\frac{1}{2}$ :

**Multiply each** 
$$\boldsymbol{x}_i$$
 by  $\boldsymbol{\lambda}_i \quad A \begin{bmatrix} .8 \\ .2 \end{bmatrix}$  is  $\boldsymbol{x}_1 + \frac{1}{2}(.2)\boldsymbol{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} .1 \\ -.1 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$ .

**Each eigenvector is multiplied by its eigenvalue**, when we multiply by A. At every step  $x_1$  is unchanged and  $x_2$  is multiplied by  $(\frac{1}{2})$ , so 99 steps give the small number  $(\frac{1}{2})^{99}$ :

$$A^{99}\begin{bmatrix} .8\\.2\end{bmatrix} \text{ is really } \boldsymbol{x}_1 + (.2)(\frac{1}{2})^{99}\boldsymbol{x}_2 = \begin{bmatrix} .6\\.4\end{bmatrix} + \begin{bmatrix} \text{very}\\ \text{small}\\ \text{vector} \end{bmatrix}.$$

This is the first column of  $A^{100}$ . The number we originally wrote as .6000 was not exact. We left out  $(.2)(\frac{1}{2})^{99}$  which wouldn't show up for 30 decimal places.

The eigenvector  $x_1$  is a "steady state" that doesn't change (because  $\lambda_1 = 1$ ). The eigenvector  $x_2$  is a "decaying mode" that virtually disappears (because  $\lambda_2 = .5$ ). The higher the power of A, the more closely its columns approach the steady state.

This particular A is a *Markov matrix*. Its largest eigenvalue is  $\lambda = 1$ . Its eigenvector  $x_1 = (.6, .4)$  is the *steady state*—which all columns of  $A^k$  will approach. Section 10.3 shows how Markov matrices appear when you search with Google.

For projection matrices P, we can see when Px is parallel to x. The eigenvectors for  $\lambda = 1$  and  $\lambda = 0$  fill the column space and nullspace. The column space doesn't move (Px = x). The nullspace goes to zero (Px = 0x).

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**Example 2** The projection matrix  $P = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$  has eigenvalues  $\lambda = 1$  and  $\lambda = 0$ .

Its eigenvectors are  $x_1 = (1, 1)$  and  $x_2 = (1, -1)$ . For those vectors,  $Px_1 = x_1$  (steady state) and  $Px_2 = 0$  (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special  $\lambda$ 's and x's:

- **1.** Markov matrix : Each column of P adds to 1, so  $\lambda = 1$  is an eigenvalue.
- **2.** *P* is singular, so  $\lambda = 0$  is an eigenvalue.
- **3.** *P* is symmetric, so its eigenvectors (1, 1) and (1, -1) are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1. The eigenvectors for  $\lambda = 0$  (which means  $P\mathbf{x} = 0\mathbf{x}$ ) fill up the nullspace. The eigenvectors for  $\lambda = 1$  (which means  $P\mathbf{x} = \mathbf{x}$ ) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

**Project each part** 
$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$
 **projects onto**  $Pv = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ 

Projections have  $\lambda = 0$  and 1. Permutations have all  $|\lambda| = 1$ . The next matrix R is a reflection and at the same time a permutation. R also has special eigenvalues.

# **Example 3** The reflection matrix $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ has eigenvalues 1 and -1.

The eigenvector (1,1) is unchanged by R. The second eigenvector is (1,-1)—its signs are reversed by R. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for R are the same as for P, because reflection = 2(projection) - I:

$$\boldsymbol{R} = 2\boldsymbol{P} - \boldsymbol{I} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 2 \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$
(2)

When a matrix is shifted by I, each  $\lambda$  is shifted by 1. No change in eigenvectors.



Figure 6.2: Projections P have eigenvalues 1 and 0. Reflections R have  $\lambda = 1$  and -1. A typical x changes direction, but an eigenvector stays along the same line.

Chapter 6. Eigenvalues and Eigenvectors

## The Equation for the Eigenvalues

For projection matrices we found  $\lambda$ 's and x's by geometry: Px = x and Px = 0. For other matrices we use determinants and linear algebra. *This is the key calculation in the chapter*—almost every application starts by solving  $Ax = \lambda x$ .

First move  $\lambda x$  to the left side. Write the equation  $Ax = \lambda x$  as  $(A - \lambda I)x = 0$ . The matrix  $A - \lambda I$  times the eigenvector x is the zero vector. The eigenvectors make up the nullspace of  $A - \lambda I$ . When we know an eigenvalue  $\lambda$ , we find an eigenvector by solving  $(A - \lambda I)x = 0$ .

Eigenvalues first. If  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  has a nonzero solution,  $A - \lambda I$  is not invertible. The determinant of  $A - \lambda I$  must be zero. This is how to recognize an eigenvalue  $\lambda$ :

**Eigenvalues** The number  $\lambda$  is an eigenvalue of A if and only if  $A - \lambda I$  is singular.

$$\det(A - \lambda I) = 0. \tag{3}$$

This "characteristic polynomial" det $(A - \lambda I)$  involves only  $\lambda$ , not x. When A is n by n, equation (3) has degree n. Then A has n eigenvalues (repeats possible!) Each  $\lambda$  leads to x:

For each eigenvalue  $\lambda$  solve  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  or  $A\mathbf{x} = \lambda \mathbf{x}$  to find an eigenvector  $\mathbf{x}$ .

**Example 4**  $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$  is already singular (zero determinant). Find its  $\lambda$ 's and x's.

When A is singular,  $\lambda = 0$  is one of the eigenvalues. The equation Ax = 0x has solutions. They are the eigenvectors for  $\lambda = 0$ . But  $det(A - \lambda I) = 0$  is the way to find all  $\lambda$ 's and x's. Always subtract  $\lambda I$  from A:

Subtract 
$$\lambda$$
 from the diagonal to find  $A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 2 & 4 - \lambda \end{bmatrix}$ . (4)

*Take the determinant* "ad - bc" of this 2 by 2 matrix. From  $1 - \lambda$  times  $4 - \lambda$ , the "ad" part is  $\lambda^2 - 5\lambda + 4$ . The "bc" part, not containing  $\lambda$ , is 2 times 2.

$$\det \begin{bmatrix} 1-\lambda & 2\\ 2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) - (2)(2) = \lambda^2 - 5\lambda.$$
(5)

Set this determinant  $\lambda^2 - 5\lambda$  to zero. One solution is  $\lambda = 0$  (as expected, since A is singular). Factoring into  $\lambda$  times  $\lambda - 5$ , the other root is  $\lambda = 5$ :

 $\det(A - \lambda I) = \lambda^2 - 5\lambda = 0$  yields the eigenvalues  $\lambda_1 = 0$  and  $\lambda_2 = 5$ .

**Equation for the eigenvalues** 

Now find the eigenvectors. Solve  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  separately for  $\lambda_1 = 0$  and  $\lambda_2 = 5$ :

$$(A - 0I)\boldsymbol{x} = \begin{bmatrix} 1 & 2\\ 2 & 4 \end{bmatrix} \begin{bmatrix} y\\ z \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y\\ z \end{bmatrix} = \begin{bmatrix} 2\\ -1 \end{bmatrix} \text{ for } \lambda_1 = 0$$
$$(A - 5I)\boldsymbol{x} = \begin{bmatrix} -4 & 2\\ 2 & -1 \end{bmatrix} \begin{bmatrix} y\\ z \end{bmatrix} = \begin{bmatrix} 0\\ 0 \end{bmatrix} \text{ yields an eigenvector } \begin{bmatrix} y\\ z \end{bmatrix} = \begin{bmatrix} 1\\ 2 \end{bmatrix} \text{ for } \lambda_2 = 5.$$

The matrices A - 0I and A - 5I are singular (because 0 and 5 are eigenvalues). The eigenvectors (2, -1) and (1, 2) are in the nullspaces:  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  is  $A\mathbf{x} = \lambda \mathbf{x}$ .

We need to emphasize: There is nothing exceptional about  $\lambda = 0$ . Like every other number, zero might be an eigenvalue and it might not. If A is singular, the eigenvectors for  $\lambda = 0$  fill the nullspace:  $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ . If A is invertible, zero is not an eigenvalue. We shift A by a multiple of I to make it singular.

In the example, the shifted matrix A - 5I is singular and 5 is the other eigenvalue.

**Summary** To solve the eigenvalue problem for an *n* by *n* matrix, follow these steps:

- 1. Compute the determinant of  $A \lambda I$ . With  $\lambda$  subtracted along the diagonal, this determinant starts with  $\lambda^n$  or  $-\lambda^n$ . It is a polynomial in  $\lambda$  of degree *n*.
- 2. Find the roots of this polynomial, by solving  $det(A \lambda I) = 0$ . The *n* roots are the *n* eigenvalues of *A*. They make  $A \lambda I$  singular.
- **3.** For each eigenvalue  $\lambda$ , solve  $(A \lambda I)x = 0$  to find an eigenvector x.

A note on the eigenvectors of 2 by 2 matrices. When  $A - \lambda I$  is singular, both rows are multiples of a vector (a, b). The eigenvector is any multiple of (b, -a). The example had

 $\lambda = 0$  : rows of A - 0I in the direction (1, 2); eigenvector in the direction (2, -1)

 $\lambda = 5$  : rows of A - 5I in the direction (-4, 2); eigenvector in the direction (2, 4).

Previously we wrote that last eigenvector as (1,2). Both (1,2) and (2,4) are correct. There is a whole *line of eigenvectors*—any nonzero multiple of x is as good as x. MATLAB's **eig**(A) divides by the length, to make the eigenvector into a unit vector.

We must add a warning. Some 2 by 2 matrices have only *one* line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand A = I has equal eigenvalues and plenty of eigenvectors.) Without a full set of eigenvectors, we don't have a basis. We can't write every v as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without n independent eigenvectors.

Chapter 6. Eigenvalues and Eigenvectors

# **Determinant and Trace**

Bad news first: If you add a row of A to another row, or exchange rows, the eigenvalues usually change. *Elimination does not preserve the*  $\lambda$ 's. The triangular U has *its* eigenvalues sitting along the diagonal—they are the pivots. But they are not the eigenvalues of A! Eigenvalues are changed when row 1 is added to row 2:

$$U = \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 1; \quad A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \text{ has } \lambda = 0 \text{ and } \lambda = 7.$$

Good news second: The product  $\lambda_1$  times  $\lambda_2$  and the sum  $\lambda_1 + \lambda_2$  can be found quickly from the matrix. For this A, the product is 0 times 7. That agrees with the determinant (which is 0). The sum of eigenvalues is 0 + 7. That agrees with the sum down the main diagonal (the **trace** is 1 + 6). These quick checks always work:

The product of the *n* eigenvalues equals the determinant. The sum of the *n* eigenvalues equals the sum of the *n* diagonal entries.

The sum of the entries along the main diagonal is called the *trace* of A:

$$\lambda_1 + \lambda_2 + \dots + \lambda_n = trace = a_{11} + a_{22} + \dots + a_{nn}.$$
(6)

Those checks are very useful. They are proved in Problems 16–17 and again in the next section. They don't remove the pain of computing  $\lambda$ 's. But when the computation is wrong, they generally tell us so. To compute the correct  $\lambda$ 's, go back to det $(A - \lambda I) = 0$ .

The trace and determinant do tell everything when the matrix is 2 by 2. We never want to get those wrong! Here trace = 3 and det = 2, so the eigenvalues are  $\lambda = 1$  and 2:

$$A = \begin{bmatrix} 1 & 9 \\ 0 & 2 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 7 & -3 \\ 10 & -4 \end{bmatrix}.$$
(7)

And here is a question about the best matrices for finding eigenvalues : *triangular*.

Why do the eigenvalues of a triangular matrix lie along its diagonal?

#### **Imaginary Eigenvalues**

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

<b>Example 5</b> The 90° rotation	$Q = \left[ egin{smallmatrix} 0 & -1 \ 1 & 0 \end{smallmatrix}  ight]$	has no real eigenvectors. Its eigenvalues								
are $\lambda_1 = i$ and $\lambda_2 = -i$ . Then $\lambda_1 + \lambda_2 = \text{trace} = 0$ and $\lambda_1 \lambda_2 = \text{determinant} = 1$										

After a rotation, no real vector Qx stays in the same direction as x (x = 0 is useless). There cannot be an eigenvector, unless we go to *imaginary numbers*. Which we do.

To see how  $i = \sqrt{-1}$  can help, look at  $Q^2$  which is -I. If Q is rotation through 90°, then  $Q^2$  is rotation through 180°. Its eigenvalues are -1 and -1. (Certainly -Ix = -1x.) Squaring Q will square each  $\lambda$ , so we must have  $\lambda^2 = -1$ . The eigenvalues of the 90° rotation matrix Q are +i and -i, because  $i^2 = -1$ .

Those  $\lambda$ 's come as usual from det $(Q - \lambda I) = 0$ . This equation gives  $\lambda^2 + 1 = 0$ . Its roots are *i* and -i. We meet the imaginary number *i* also in the eigenvectors:

Complex	0	-1][	1]	_ [1]	and	0	$-1$ $\begin{bmatrix} i \end{bmatrix}$	[	i	
eigenvectors	1	0][	i	$= -i \lfloor i \rfloor$	anu	1	$0 \rfloor \lfloor 1 \rfloor$	$= \iota$	1	•

Somehow these complex vectors  $x_1 = (1, i)$  and  $x_2 = (i, 1)$  keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues i and -i also illustrate two special properties of Q:

- **1.** Q is an orthogonal matrix so the absolute value of each  $\lambda$  is  $|\lambda| = 1$ .
- **2.** Q is a skew-symmetric matrix so each  $\lambda$  is pure imaginary.

A symmetric matrix  $(S^T = S)$  can be compared to a real number. A skew-symmetric matrix  $(A^T = -A)$  can be compared to an imaginary number. An orthogonal matrix  $(Q^TQ = I)$  corresponds to a complex number with  $|\lambda| = 1$ . For the eigenvalues of S and A and Q, those are more than analogies—they are facts to be proved in Section 6.4.

The eigenvectors for all these special matrices are perpendicular. Somehow (i, 1) and (1, i) are perpendicular (Chapter 9 explains the dot product of complex vectors).

### Eigenvalues of AB and A+B

The first guess about the eigenvalues of AB is not true. An eigenvalue  $\lambda$  of A times an eigenvalue  $\beta$  of B usually does *not* give an eigenvalue of AB:

**False proof**  $ABx = A\beta x = \beta Ax = \beta \lambda x.$  (8)

It seems that  $\beta$  times  $\lambda$  is an eigenvalue. When x is an eigenvector for A and B, this proof is correct. *The mistake is to expect that* A and B automatically share the same eigenvector x. Usually they don't. Eigenvectors of A are not generally eigenvectors of B. A and B could have all zero eigenvalues while 1 is an eigenvalue of AB:

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \text{ then } AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

For the same reason, the eigenvalues of A + B are generally not  $\lambda + \beta$ . Here  $\lambda + \beta = 0$  while A + B has eigenvalues 1 and -1. (At least they add to zero.)

The false proof suggests what is true. Suppose x really is an eigenvector for both A and B. Then we do have  $ABx = \lambda\beta x$  and  $BAx = \lambda\beta x$ . When all n eigenvectors are shared, we *can* multiply eigenvalues. The test AB = BA for shared eigenvectors is important in quantum mechanics—time out to mention this application of linear algebra:

A and B share the same *n* independent eigenvectors if and only if AB = BA.

**Heisenberg's uncertainty principle** In quantum mechanics, the position matrix P and the momentum matrix Q do not commute. In fact QP - PQ = I (these are infinite matrices). To have Px = 0 at the same time as Qx = 0 would require x = Ix = 0. If we knew the position exactly, we could not also know the momentum exactly. Problem 36 derives Heisenberg's uncertainty principle  $||Px|| ||Qx|| \ge \frac{1}{2} ||x||^2$ .

# REVIEW OF THE KEY IDEAS

- **1.**  $Ax = \lambda x$  says that eigenvectors x keep the same direction when multiplied by A.
- **2.**  $A\mathbf{x} = \lambda \mathbf{x}$  also says that  $\det(A \lambda I) = 0$ . This determines *n* eigenvalues.
- **3.** The eigenvalues of  $A^2$  and  $A^{-1}$  are  $\lambda^2$  and  $\lambda^{-1}$ , with the same eigenvectors.
- **4.** The sum of the  $\lambda$ 's equals the sum down the main diagonal of A (*the trace*). The product of the  $\lambda$ 's equals the determinant of A.
- 5. Projections P, reflections R, 90° rotations Q have special eigenvalues 1, 0, -1, i, -i. Singular matrices have  $\lambda = 0$ . Triangular matrices have  $\lambda$ 's on their diagonal.
- **6.** Special properties of a matrix lead to special eigenvalues and eigenvectors. That is a major theme of this chapter (it is captured in a table at the very end).

### WORKED EXAMPLES

**6.1 A** Find the eigenvalues and eigenvectors of A and  $A^2$  and  $A^{-1}$  and A + 4I:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}.$$

Check the trace  $\lambda_1 + \lambda_2 = 4$  and the determinant  $\lambda_1 \lambda_2 = 3$ .

**Solution** The eigenvalues of A come from  $det(A - \lambda I) = 0$ :

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 2 - \lambda & -1 \\ -1 & 2 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 3 = 0.$$

This factors into  $(\lambda - 1)(\lambda - 3) = 0$  so the eigenvalues of A are  $\lambda_1 = 1$  and  $\lambda_2 = 3$ . For the trace, the sum 2+2 agrees with 1+3. The determinant 3 agrees with the product  $\lambda_1 \lambda_2$ .

The eigenvectors come separately by solving  $(A - \lambda I)\mathbf{x} = \mathbf{0}$  which is  $A\mathbf{x} = \lambda \mathbf{x}$ :

$$\boldsymbol{\lambda} = \mathbf{1}: \quad (A - I)\boldsymbol{x} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\boldsymbol{\lambda} = \mathbf{3}: \quad (A - 3I)\boldsymbol{x} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ gives the eigenvector } \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

 $A^2$  and  $A^{-1}$  and A + 4I keep the same eigenvectors as A. Their eigenvalues are  $\lambda^2$  and  $\lambda^{-1}$  and  $\lambda + 4$ :

$$A^{2}$$
 has eigenvalues  $1^{2} = 1$  and  $3^{2} = 9$   $A^{-1}$  has  $\frac{1}{1}$  and  $\frac{1}{3}$   $A + 4I$  has  $\begin{array}{c} 1 + 4 = 5 \\ 3 + 4 = 7 \end{array}$ 

Notes for later sections: A has orthogonal eigenvectors (Section 6.4 on symmetric matrices). A can be diagonalized since  $\lambda_1 \neq \lambda_2$  (Section 6.2). A is similar to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.2). A is a positive definite matrix (Section 6.5) since  $A = A^{T}$  and the  $\lambda$ 's are positive.

#### 6.1 B How can you estimate the eigenvalues of any A? Gershgorin gave this answer.

Every eigenvalue of A must be "near" at least one of the entries  $a_{ii}$  on the main diagonal. For  $\lambda$  to be "near  $a_{ii}$ " means that  $|a_{ii} - \lambda|$  is no more than **the sum**  $R_i$  of all other  $|a_{ij}|$ in that row *i* of the matrix. Then  $R_i = \sum_{j \neq i} |a_{ij}|$  is the radius of a circle centered at  $a_{ii}$ .

#### Every $\lambda$ is in the circle around one or more diagonal entries $a_{ii}$ : $|a_{ii} - \lambda| \leq R_i$ .

Here is the reasoning. If  $\lambda$  is an eigenvalue, then  $A - \lambda I$  is not invertible. Then  $A - \lambda I$  cannot be diagonally dominant (see Section 2.5). So at least one diagonal entry  $a_{ii} - \lambda$  is *not larger* than the sum  $R_i$  of all other entries  $|a_{ij}|$  (we take absolute values!) in row *i*.

*Example 1.* Every eigenvalue  $\lambda$  of this A falls into one or both of the **Gershgorin circles**: The centers are a and d, the radii are  $R_1 = |b|$  and  $R_2 = |c|$ .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
 First circle:  $|\lambda - a| \le |b|$   
Second circle:  $|\lambda - d| \le |c|$ 

Those are circles in the complex plane, since  $\lambda$  could certainly be complex.

*Example 2.* All eigenvalues of this A lie in a circle of radius R = 3 around *one or more* of the diagonal entries  $d_1, d_2, d_3$ :

$$A = \begin{bmatrix} d_1 & 1 & 2\\ 2 & d_2 & 1\\ -1 & 2 & d_3 \end{bmatrix} \qquad \begin{array}{c} |\lambda - d_1| \le 1 + 2 = R_1\\ |\lambda - d_2| \le 2 + 1 = R_2\\ |\lambda - d_3| \le 1 + 2 = R_3 \end{array}$$

You see that "near" means not more than 3 away from  $d_1$  or  $d_2$  or  $d_3$ , for this example.

**6.1 C** Find the eigenvalues and eigenvectors of this symmetric 3 by 3 matrix S:

Symmetric matrix	<b>[</b> 1	$^{-1}$	0]
Singular matrix	$S = \left  -1 \right $	2	-1
<b>Trace</b> $1 + 2 + 1 = 4$	0	-1	1

**Solution** Since all rows of S add to zero, the vector x = (1,1,1) gives Sx = 0. This is an eigenvector for  $\lambda = 0$ . To find  $\lambda_2$  and  $\lambda_3$  I will compute the 3 by 3 determinant:

$$\det(S - \lambda I) = \begin{vmatrix} 1 - \lambda & -1 & 0\\ -1 & 2 - \lambda & -1\\ 0 & -1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(2 - \lambda)(1 - \lambda) - 2(1 - \lambda)$$
$$= (1 - \lambda)[(2 - \lambda)(1 - \lambda) - 2]$$
$$= (1 - \lambda)(-\lambda)(3 - \lambda).$$

Those three factors give  $\lambda = 0, 1, 3$ . Each eigenvalue corresponds to an eigenvector (or a line of eigenvectors):

$$\boldsymbol{x}_1 = \begin{bmatrix} 1\\1\\1 \end{bmatrix}$$
  $S\boldsymbol{x}_1 = \boldsymbol{0}\boldsymbol{x}_1$   $\boldsymbol{x}_2 = \begin{bmatrix} 1\\0\\-1 \end{bmatrix}$   $S\boldsymbol{x}_2 = \boldsymbol{1}\boldsymbol{x}_2$   $\boldsymbol{x}_3 = \begin{bmatrix} 1\\-2\\1 \end{bmatrix}$   $S\boldsymbol{x}_3 = \boldsymbol{3}\boldsymbol{x}_3$ .

I notice again that eigenvectors are perpendicular when S is symmetric. We were lucky to find  $\lambda = 0, 1, 3$ . For a larger matrix I would use **eig**(A), and never touch determinants.

The full command  $[\mathbf{X}, \mathbf{E}] = \operatorname{eig}(A)$  will produce unit eigenvectors in the columns of X.

# **Problem Set 6.1**

1 The example at the start of the chapter has powers of this matrix A:

$$A = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \text{ and } A^2 = \begin{bmatrix} .70 & .45 \\ .30 & .55 \end{bmatrix} \text{ and } A^{\infty} = \begin{bmatrix} .6 & .6 \\ .4 & .4 \end{bmatrix}.$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.

- (a) Show from A how a row exchange can produce different eigenvalues.
- (b) Why is a zero eigenvalue *not* changed by the steps of elimination?
- 2 Find the eigenvalues and the eigenvectors of these two matrices:

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \quad \text{and} \quad A + I = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}.$$

A + I has the \_\_\_\_\_ eigenvectors as A. Its eigenvalues are \_\_\_\_\_ by 1.

**3** Compute the eigenvalues and eigenvectors of A and  $A^{-1}$ . Check the trace !

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

 $A^{-1}$  has the \_\_\_\_\_ eigenvectors as A. When A has eigenvalues  $\lambda_1$  and  $\lambda_2$ , its inverse has eigenvalues \_\_\_\_\_.

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- 6.1. Introduction to Eigenvalues
- 4 Compute the eigenvalues and eigenvectors of A and  $A^2$ :

$$A = \begin{bmatrix} -1 & 3\\ 2 & 0 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 7 & -3\\ -2 & 6 \end{bmatrix}.$$

 $A^2$  has the same \_\_\_\_\_ as A. When A has eigenvalues  $\lambda_1$  and  $\lambda_2$ ,  $A^2$  has eigenvalues \_\_\_\_\_. In this example, why is  $\lambda_1^2 + \lambda_2^2 = 13$ ?

5 Find the eigenvalues of A and B (easy for triangular matrices) and A + B:

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \quad \text{and} \quad A + B = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$$

Eigenvalues of A + B (are equal to)(are not equal to) eigenvalues of A plus eigenvalues of B.

**6** Find the eigenvalues of A and B and AB and BA:

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } AB = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

- (a) Are the eigenvalues of AB equal to eigenvalues of A times eigenvalues of B?
- (b) Are the eigenvalues of AB equal to the eigenvalues of BA?
- 7 Elimination produces A = LU. The eigenvalues of U are on its diagonal; they are the \_\_\_\_\_. The eigenvalues of L are on its diagonal; they are all \_\_\_\_\_. The eigenvalues of A are not the same as \_\_\_\_\_.
- 8 (a) If you know that x is an eigenvector, the way to find  $\lambda$  is to \_\_\_\_\_.
  - (b) If you know that  $\lambda$  is an eigenvalue, the way to find x is to \_\_\_\_\_
- **9** What do you do to the equation  $Ax = \lambda x$ , in order to prove (a), (b), and (c)?
  - (a)  $\lambda^2$  is an eigenvalue of  $A^2$ , as in Problem 4.
  - (b)  $\lambda^{-1}$  is an eigenvalue of  $A^{-1}$ , as in Problem 3.
  - (c)  $\lambda + 1$  is an eigenvalue of A + I, as in Problem 2.
- **10** Find the eigenvalues and eigenvectors for both of these Markov matrices A and  $A^{\infty}$ . Explain from those answers why  $A^{100}$  is close to  $A^{\infty}$ :

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \quad \text{and} \quad A^{\infty} = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues  $\lambda_1 \neq \lambda_2$ : The columns of  $A - \lambda_1 I$  are multiples of the eigenvector  $x_2$ . Any idea why this should be?

12 Find three eigenvectors for this matrix P (projection matrices have  $\lambda = 1$  and 0):

		.2	.4	0	
Projection matrix	P =	.4	.8	0	
		0	0	1	

If two eigenvectors share the same  $\lambda$ , so do all their linear combinations. Find an eigenvector of P with no zero components.

- **13** From the unit vector  $\boldsymbol{u} = (\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6})$  construct the rank one projection matrix  $P = \boldsymbol{u}\boldsymbol{u}^{\mathrm{T}}$ . This matrix has  $P^2 = P$  because  $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = 1$ .
  - (a)  $P \boldsymbol{u} = \boldsymbol{u}$  comes from  $(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}})\boldsymbol{u} = \boldsymbol{u}(\underline{\quad})$ . Then  $\boldsymbol{u}$  is an eigenvector with  $\lambda = 1$ .
  - (b) If v is perpendicular to u show that Pv = 0. Then  $\lambda = 0$ .
  - (c) Find three independent eigenvectors of P all with eigenvalue  $\lambda = 0$ .
- 14 Solve  $det(Q \lambda I) = 0$  by the quadratic formula to reach  $\lambda = \cos \theta \pm i \sin \theta$ :

$$Q = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$
 rotates the *xy* plane by the angle  $\theta$ . No real  $\lambda$ 's.

Find the eigenvectors of Q by solving  $(Q - \lambda I)\mathbf{x} = \mathbf{0}$ . Use  $i^2 = -1$ .

**15** Every permutation matrix leaves x = (1, 1, ..., 1) unchanged. Then  $\lambda = 1$ . Find two more  $\lambda$ 's (possibly complex) for these permutations, from det $(P - \lambda I) = 0$ :

	0	1	0]			[0]	0	1	
P =	0	0	1	and	P =	0	1	0	
	1	0	0			1	0	0	

**16** The determinant of A equals the product  $\lambda_1 \lambda_2 \cdots \lambda_n$ . Start with the polynomial  $det(A - \lambda I)$  separated into its *n* factors (always possible). Then set  $\lambda = 0$ :

$$det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$
 so  $det A =$ \_\_\_\_\_

Check this rule in Example 1 where the Markov matrix has  $\lambda = 1$  and  $\frac{1}{2}$ .

17 The sum of the diagonal entries (the *trace*) equals the sum of the eigenvalues:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ has } \det(A - \lambda I) = \lambda^2 - (a + d)\lambda + ad - bc = 0$$

The quadratic formula gives the eigenvalues  $\lambda = (a+d+\sqrt{-})/2$  and  $\lambda =$ \_\_\_\_\_. Their sum is \_\_\_\_\_\_. If A has  $\lambda_1 = 3$  and  $\lambda_2 = 4$  then det $(A - \lambda I) =$ \_\_\_\_\_\_.

**18** If A has  $\lambda_1 = 4$  and  $\lambda_2 = 5$  then  $\det(A - \lambda I) = (\lambda - 4)(\lambda - 5) = \lambda^2 - 9\lambda + 20$ . Find three matrices that have trace a + d = 9 and determinant 20 and  $\lambda = 4, 5$ .

- 6.1. Introduction to Eigenvalues
- **19** A 3 by 3 matrix B is known to have eigenvalues 0, 1, 2. This information is enough to find three of these (give the answers where possible):
  - (a) the rank of B
  - (b) the determinant of  $B^{\mathrm{T}}B$
  - (c) the eigenvalues of  $B^{\mathrm{T}}B$
  - (d) the eigenvalues of  $(B^2 + I)^{-1}$ .
- **20** Choose the last rows of A and C to give eigenvalues 4, 7 and 1, 2, 3:

Companion matrices	$A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$	C =	$\begin{bmatrix} 0\\ 0\\ * \end{bmatrix}$	1 0 *	0 1 *	•
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- 21 The eigenvalues of A equal the eigenvalues of  $A^{T}$ . This is because  $det(A \lambda I)$  equals  $det(A^{T} \lambda I)$ . That is true because \_\_\_\_\_. Show by an example that the eigenvectors of A and  $A^{T}$  are not the same.
- **22** Construct any 3 by 3 Markov matrix M: positive entries down each column add to 1. Show that  $M^{T}(1,1,1) = (1,1,1)$ . By Problem 21,  $\lambda = 1$  is also an eigenvalue of M. Challenge: A 3 by 3 singular Markov matrix with trace  $\frac{1}{2}$  has what  $\lambda$ 's?
- **23** Find three 2 by 2 matrices that have  $\lambda_1 = \lambda_2 = 0$ . The trace is zero and the determinant is zero. A might not be the zero matrix but check that  $A^2 = 0$ .
- 24 This matrix is singular with rank one. Find three  $\lambda$ 's and three eigenvectors:

	$\begin{bmatrix} 1 \end{bmatrix}$					2	1	2	
A =	2	2	1	2]=	=	4	2	4	
	1	-		-		2	1	2	

- **25** Suppose A and B have the same eigenvalues  $\lambda_1, \ldots, \lambda_n$  with the same independent eigenvectors  $x_1, \ldots, x_n$ . Then A = B. *Reason*: Any vector x is a combination  $c_1x_1 + \cdots + c_nx_n$ . What is Ax? What is Bx?
- **26** The block *B* has eigenvalues 1, 2 and *C* has eigenvalues 3, 4 and *D* has eigenvalues 5, 7. Find the eigenvalues of the 4 by 4 matrix *A*:

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 3 & 0 \\ -2 & 3 & 0 & 4 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 1 & 6 \end{bmatrix}.$$

**27** Find the rank and the four eigenvalues of A and C:

**28** Subtract *I* from the previous *A*. Find the  $\lambda$ 's and then the determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$$

**29** (Review) Find the eigenvalues of A, B, and C:

	[1	2	3			0	0	1			2	2	2	
A =	0	4	5	and	B =	0	2	0	and	C =	2	2	2	
	0	0	6			3	0	0			$\lfloor 2 \rfloor$	2	2	

30 When a + b = c + d show that (1, 1) is an eigenvector and find both eigenvalues :

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

31 If we exchange rows 1 and 2 and columns 1 and 2, the eigenvalues don't change. Find eigenvectors of A and B for  $\lambda = 11$ . Rank one gives  $\lambda_2 = \lambda_3 = 0$ .

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & 4 \end{bmatrix} \quad \text{and} \quad B = PAP^{\mathrm{T}} = \begin{bmatrix} 6 & 3 & 3 \\ 2 & 1 & 1 \\ 8 & 4 & 4 \end{bmatrix}.$$

- **32** Suppose A has eigenvalues 0, 3, 5 with independent eigenvectors u, v, w.
  - (a) Give a basis for the nullspace and a basis for the column space.
  - (b) Find a particular solution to Ax = v + w. Find all solutions.
  - (c) Ax = u has no solution. If it did then \_\_\_\_\_ would be in the column space.

# **Challenge Problems**

- **33** Show that  $\boldsymbol{u}$  is an eigenvector of the rank one  $2 \times 2$  matrix  $A = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ . Find both eigenvalues of A. Check that  $\lambda_1 + \lambda_2$  agrees with the trace  $u_1v_1 + u_2v_2$ .
- **34** Find the eigenvalues of this permutation matrix P from det  $(P \lambda I) = 0$ . Which vectors are not changed by the permutation? They are eigenvectors for  $\lambda = 1$ . Can you find three more eigenvectors?

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

- 6.1. Introduction to Eigenvalues
- **35** There are six 3 by 3 permutation matrices *P*. What numbers can be the *determinants* of *P*? What numbers can be *pivots*? What numbers can be the *trace* of *P*? What *four numbers* can be eigenvalues of *P*, as in Problem 15?
- **36** (Heisenberg's Uncertainty Principle) AB BA = I can happen for infinite matrices with  $A = A^{T}$  and  $B = -B^{T}$ . Then

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}AB\boldsymbol{x} - \boldsymbol{x}^{\mathrm{T}}BA\boldsymbol{x} \leq 2\|A\boldsymbol{x}\| \|B\boldsymbol{x}\|.$$

Explain that last step by using the Schwarz inequality  $|u^{T}v| \leq ||u|| ||v||$ . Then Heisenberg's inequality says that ||Ax||/||x|| times ||Bx||/||x|| is at least  $\frac{1}{2}$ . It is impossible to get the position error and momentum error both very small.

- **37** Find a 2 by 2 rotation matrix (other than I) with  $A^3 = I$ . Its eigenvalues must satisfy  $\lambda^3 = 1$ . They can be  $e^{2\pi i/3}$  and  $e^{-2\pi i/3}$ . What are the trace and determinant?
- **38** (a) Find the eigenvalues and eigenvectors of A. They depend on c:

$$A = \begin{bmatrix} .4 & 1-c \\ .6 & c \end{bmatrix}.$$

- (b) Show that A has just one line of eigenvectors when c = 1.6.
- (c) This is a Markov matrix when c = .8. Then  $A^n$  will approach what matrix  $A^{\infty}$ ?

## **Eigshow in MATLAB**

There is a MATLAB demo (just type **eigshow**), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector x = (1, 0). The mouse makes this vector move around the unit circle. At the same time the screen shows Ax, in color and also moving. Possibly Ax is ahead of x. Possibly Ax is behind x. Sometimes Ax is parallel to x.

At that parallel moment,  $A \boldsymbol{x} = \lambda \boldsymbol{x}$  (at  $\boldsymbol{x}_1$  and  $\boldsymbol{x}_2$  in the second figure).



The eigenvalue  $\lambda$  is the length of Ax, when the unit eigenvector x lines up. The built-in choices for A illustrate three possibilities: 0, 1, or 2 real vectors where Ax crosses x.

The axes of the ellipse are singular vectors in 7.4—and eigenvectors if  $A^{T} = A$ .