## Chapter 6

## Eigenvalues and Eigenvectors

### 6.1 Introduction to Eigenvalues

An eigenvector $\boldsymbol{x}$ lies along the same line as $A \boldsymbol{x}: \quad A \boldsymbol{x}=\lambda \boldsymbol{x}$. The eigenvalue is $\lambda$.
2 If $A \boldsymbol{x}=\lambda \boldsymbol{x}$ then $A^{2} \boldsymbol{x}=\lambda^{2} \boldsymbol{x}$ and $A^{-1} \boldsymbol{x}=\lambda^{-1} \boldsymbol{x}$ and $(A+c I) \boldsymbol{x}=(\lambda+c) \boldsymbol{x}$ : the same $\boldsymbol{x}$.
3 If $A \boldsymbol{x}=\lambda \boldsymbol{x}$ then $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$ and $A-\lambda I$ is singular and $\operatorname{det}(\boldsymbol{A}-\boldsymbol{\lambda I})=\mathbf{0}$. $n$ eigenvalues.
4 Check $\lambda$ 's by $\operatorname{det} A=\left(\lambda_{1}\right)\left(\lambda_{2}\right) \cdots\left(\lambda_{n}\right)$ and diagonal sum $a_{11}+a_{22}+\cdots+a_{n n}=$ sum of $\lambda$ 's.
5 Projections have $\lambda=1$ and 0 . Reflections have 1 and -1 . Rotations have $e^{i \theta}$ and $e^{-i \theta}$ : complex!.
This chapter enters a new part of linear algebra. The first part was about $A \boldsymbol{x}=\boldsymbol{b}$ : balance and equilibrium and steady state. Now the second part is about change. Time enters the picture-continuous time in a differential equation $\boldsymbol{d u} / d t=A \boldsymbol{u}$ or time steps in a difference equation $\boldsymbol{u}_{k+1}=A \boldsymbol{u}_{k}$. Those equations are NOT solved by elimination.

The key idea is to avoid all the complications presented by the matrix $A$. Suppose the solution vector $\boldsymbol{u}(t)$ stays in the direction of a fixed vector $\boldsymbol{x}$. Then we only need to find the number (changing with time) that multiplies $\boldsymbol{x}$. A number is easier than a vector. We want "eigenvectors" $\boldsymbol{x}$ that don't change direction when you multiply by $\boldsymbol{A}$.

A good model comes from the powers $A, A^{2}, A^{3}, \ldots$ of a matrix. Suppose you need the hundredth power $A^{100}$. Its columns are very close to the eigenvector (.6, .4) :

$$
A, A^{2}, A^{3}=\left[\begin{array}{ll}
.8 & .3 \\
.2 & .7
\end{array}\right] \quad\left[\begin{array}{ll}
.70 & .45 \\
.30 & .55
\end{array}\right] \quad\left[\begin{array}{ll}
.650 & .525 \\
.350 & .475
\end{array}\right] \quad A^{\mathbf{1 0 0}} \approx\left[\begin{array}{ll}
. \mathbf{6 0 0 0} & \mathbf{. 6 0 0 0} \\
.4000 & \mathbf{. 4 0 0 0}
\end{array}\right]
$$

$A^{100}$ was found by using the eigenvalues of $A$, not by multiplying 100 matrices. Those eigenvalues (here they are $\lambda=1$ and $1 / 2$ ) are a new way to see into the heart of a matrix.

To explain eigenvalues, we first explain eigenvectors. Almost all vectors change direction, when they are multiplied by $A$. Certain exceptional vectors $x$ are in the same direction as $A x$. Those are the "eigenvectors". Multiply an eigenvector by $A$, and the vector $A \boldsymbol{x}$ is a number $\lambda$ times the original $\boldsymbol{x}$.

The basic equation is $A x=\lambda x$. The number $\lambda$ is an eigenvalue of $A$.
The eigenvalue $\lambda$ tells whether the special vector $x$ is stretched or shrunk or reversed or left unchanged-when it is multiplied by $A$. We may find $\lambda=2$ or $\frac{1}{2}$ or -1 or 1 . The eigenvalue $\lambda$ could be zero! Then $A \boldsymbol{x}=0 \boldsymbol{x}$ means that this eigenvector $\boldsymbol{x}$ is in the nullspace.

If $A$ is the identity matrix, every vector has $A \boldsymbol{x}=\boldsymbol{x}$. All vectors are eigenvectors of $I$. All eigenvalues "lambda" are $\lambda=1$. This is unusual to say the least. Most 2 by 2 matrices have two eigenvector directions and two eigenvalues. We will show that $\operatorname{det}(A-\lambda I)=0$.

This section will explain how to compute the $\boldsymbol{x}$ 's and $\lambda$ 's. It can come early in the course because we only need the determinant of a 2 by 2 matrix. Let me use $\operatorname{det}(A-\lambda I)=0$ to find the eigenvalues for this first example, and then derive it properly in equation (3).
Example 1 The matrix $A$ has two eigenvalues $\lambda=1$ and $\lambda=1 / 2$. Look at $\operatorname{det}(A-\lambda I)$ :

$$
A=\left[\begin{array}{ll}
.8 & .3 \\
.2 & .7
\end{array}\right] \quad \operatorname{det}\left[\begin{array}{ll}
.8-\lambda & .3 \\
.2 & .7-\lambda
\end{array}\right]=\lambda^{2}-\frac{3}{2} \lambda+\frac{1}{2}=(\lambda-1)\left(\lambda-\frac{1}{2}\right)
$$

I factored the quadratic into $\lambda-1$ times $\lambda-\frac{1}{2}$, to see the two eigenvalues $\boldsymbol{\lambda}=\mathbf{1}$ and $\boldsymbol{\lambda}=\frac{1}{2}$. For those numbers, the matrix $A-\lambda I$ becomes singular (zero determinant). The eigenvectors $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ are in the nullspaces of $A-I$ and $A-\frac{1}{2} I$.
$(A-I) \boldsymbol{x}_{1}=0$ is $A \boldsymbol{x}_{1}=\boldsymbol{x}_{1}$ and the first eigenvector is $(. \boldsymbol{6}, . \mathbf{4})$.
$\left(A-\frac{1}{2} I\right) \boldsymbol{x}_{2}=0$ is $A \boldsymbol{x}_{2}=\frac{\mathbf{1}}{2} \boldsymbol{x}_{2}$ and the second eigenvector is $(\mathbf{1}, \mathbf{1})$ :

$$
\begin{aligned}
& \boldsymbol{x}_{1}=\left[\begin{array}{l}
.6 \\
.4
\end{array}\right] \quad \text { and } \quad A \boldsymbol{x}_{1}=\left[\begin{array}{ll}
.8 & .3 \\
.2 & .7
\end{array}\right]\left[\begin{array}{l}
.6 \\
.4
\end{array}\right]=\boldsymbol{x}_{1} \quad\left(A \boldsymbol{x}=\boldsymbol{x} \text { means that } \lambda_{1}=1\right) \\
& \left.\boldsymbol{x}_{2}=\left[\begin{array}{r}
1 \\
-1
\end{array}\right] \quad \text { and } \quad A \boldsymbol{x}_{2}=\left[\begin{array}{ll}
.8 & .3 \\
.2 & .7
\end{array}\right]\left[\begin{array}{r}
1 \\
-1
\end{array}\right]=\left[\begin{array}{r}
.5 \\
-.5
\end{array}\right] \quad \text { (this is } \frac{1}{2} \boldsymbol{x}_{2} \text { so } \lambda_{2}=\frac{1}{2}\right)
\end{aligned}
$$

If $\boldsymbol{x}_{1}$ is multiplied again by $A$, we still get $\boldsymbol{x}_{1}$. Every power of $A$ will give $A^{n} \boldsymbol{x}_{1}=\boldsymbol{x}_{1}$. Multiplying $\boldsymbol{x}_{2}$ by $A$ gave $\frac{1}{2} \boldsymbol{x}_{2}$, and if we multiply again we get $\left(\frac{1}{2}\right)^{2}$ times $\boldsymbol{x}_{2}$.

When $A$ is squared, the eigenvectors stay the same. The eigenvalues are squared.
This pattern keeps going, because the eigenvectors stay in their own directions (Figure 6.1) and never get mixed. The eigenvectors of $A^{100}$ are the same $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$. The eigenvalues of $A^{100}$ are $1^{100}=1$ and $\left(\frac{1}{2}\right)^{100}=$ very small number.

Other vectors do change direction. But all other vectors are combinations of the two eigenvectors. The first column of $A$ is the combination $\boldsymbol{x}_{1}+(.2) \boldsymbol{x}_{2}$ :
$\begin{aligned} & \text { Separate into eigenvectors } \\ & \text { Then multiply by } \boldsymbol{A}\end{aligned} \quad\left[\begin{array}{l}.8 \\ .2\end{array}\right]=\boldsymbol{x}_{1}+(.2) \boldsymbol{x}_{2}=\left[\begin{array}{l}.6 \\ .4\end{array}\right]+\left[\begin{array}{r}.2 \\ -.2\end{array}\right]$.


Figure 6.1: The eigenvectors keep their directions. $A^{2} \boldsymbol{x}=\lambda^{2} \boldsymbol{x}$ with $\lambda^{2}=1^{2}$ and $(.5)^{2}$.

When we multiply separately for $\boldsymbol{x}_{1}$ and (.2) $\boldsymbol{x}_{2}, A$ multiplies $\boldsymbol{x}_{2}$ by its eigenvalue $\frac{1}{2}$ :

Multiply each $\boldsymbol{x}_{i}$ by $\boldsymbol{\lambda}_{i} \quad A\left[\begin{array}{l}.8 \\ .2\end{array}\right] \quad$ is $\quad \boldsymbol{x}_{1}+\frac{1}{2}(.2) \boldsymbol{x}_{2}=\left[\begin{array}{l}.6 \\ .4\end{array}\right]+\left[\begin{array}{c}.1 \\ -.1\end{array}\right]=\left[\begin{array}{l}.7 \\ .3\end{array}\right]$.
Each eigenvector is multiplied by its eigenvalue, when we multiply by $A$. At every step $\boldsymbol{x}_{1}$ is unchanged and $\boldsymbol{x}_{2}$ is multiplied by $\left(\frac{1}{2}\right)$, so 99 steps give the small number $\left(\frac{1}{2}\right)^{99}$ :

$$
A^{99}\left[\begin{array}{l}
.8 \\
.2
\end{array}\right] \quad \text { is really } \quad \boldsymbol{x}_{1}+(.2)\left(\frac{1}{2}\right)^{99} \boldsymbol{x}_{2}=\left[\begin{array}{l}
.6 \\
.4
\end{array}\right]+\left[\begin{array}{c}
\text { very } \\
\text { small } \\
\text { vector }
\end{array}\right]
$$

This is the first column of $A^{100}$. The number we originally wrote as .6000 was not exact. We left out $(.2)\left(\frac{1}{2}\right)^{99}$ which wouldn't show up for 30 decimal places.

The eigenvector $\boldsymbol{x}_{1}$ is a "steady state" that doesn't change (because $\lambda_{1}=1$ ). The eigenvector $\boldsymbol{x}_{2}$ is a "decaying mode" that virtually disappears (because $\lambda_{2}=.5$ ). The higher the power of $A$, the more closely its columns approach the steady state.

This particular $A$ is a Markov matrix. Its largest eigenvalue is $\lambda=1$. Its eigenvector $\boldsymbol{x}_{1}=(.6, .4)$ is the steady state-which all columns of $A^{k}$ will approach. Section 10.3 shows how Markov matrices appear when you search with Google.

For projection matrices $P$, we can see when $P x$ is parallel to $\boldsymbol{x}$. The eigenvectors for $\lambda=1$ and $\lambda=0$ fill the column space and nullspace. The column space doesn't move $(P \boldsymbol{x}=\boldsymbol{x})$. The nullspace goes to zero $(P \boldsymbol{x}=0 \boldsymbol{x})$.

Example 2 The projection matrix $P=\left[\begin{array}{cc}.5 & .5 \\ .5 & .5\end{array}\right]$ has eigenvalues $\lambda=1$ and $\lambda=0$.

Its eigenvectors are $\boldsymbol{x}_{1}=(1,1)$ and $\boldsymbol{x}_{2}=(1,-1)$. For those vectors, $P \boldsymbol{x}_{1}=\boldsymbol{x}_{1}$ (steady state) and $P \boldsymbol{x}_{2}=\mathbf{0}$ (nullspace). This example illustrates Markov matrices and singular matrices and (most important) symmetric matrices. All have special $\lambda$ 's and $\boldsymbol{x}$ 's:

1. Markov matrix : Each column of $P$ adds to 1 , so $\lambda=1$ is an eigenvalue.
2. $P$ is singular, so $\lambda=0$ is an eigenvalue.
3. $P$ is symmetric, so its eigenvectors $(1,1)$ and $(1,-1)$ are perpendicular.

The only eigenvalues of a projection matrix are 0 and 1 . The eigenvectors for $\lambda=0$ (which means $P \boldsymbol{x}=0 \boldsymbol{x}$ ) fill up the nullspace. The eigenvectors for $\lambda=1$ (which means $P \boldsymbol{x}=\boldsymbol{x}$ ) fill up the column space. The nullspace is projected to zero. The column space projects onto itself. The projection keeps the column space and destroys the nullspace:

Project each part $\quad \boldsymbol{v}=\left[\begin{array}{r}1 \\ -1\end{array}\right]+\left[\begin{array}{l}2 \\ 2\end{array}\right] \quad$ projects onto $P \boldsymbol{v}=\left[\begin{array}{l}\mathbf{0} \\ \mathbf{0}\end{array}\right]+\left[\begin{array}{l}\mathbf{2} \\ \mathbf{2}\end{array}\right]$.
Projections have $\lambda=0$ and 1. Permutations have all $|\lambda|=1$. The next matrix $R$ is a reflection and at the same time a permutation. $R$ also has special eigenvalues.

## Example 3 The reflection matrix $R=\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ has eigenvalues 1 and -1 .

The eigenvector $(1,1)$ is unchanged by $R$. The second eigenvector is $(1,-1)$-its signs are reversed by $R$. A matrix with no negative entries can still have a negative eigenvalue! The eigenvectors for $R$ are the same as for $P$, because reflection $=2$ (projection) $-I$ :

$$
\boldsymbol{R}=2 \boldsymbol{P}-\boldsymbol{I} \quad\left[\begin{array}{cc}
0 & 1  \tag{2}\\
1 & 0
\end{array}\right]=2\left[\begin{array}{cc}
.5 & .5 \\
.5 & .5
\end{array}\right]-\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] .
$$

When a matrix is shifted by $I$, each $\lambda$ is shifted by 1 . No change in eigenvectors.


Figure 6.2: Projections $P$ have eigenvalues 1 and 0 . Reflections $R$ have $\lambda=1$ and -1 . A typical $\boldsymbol{x}$ changes direction, but an eigenvector stays along the same line.

## The Equation for the Eigenvalues

For projection matrices we found $\lambda$ 's and $\boldsymbol{x}$ 's by geometry: $P \boldsymbol{x}=\boldsymbol{x}$ and $P \boldsymbol{x}=\mathbf{0}$. For other matrices we use determinants and linear algebra. This is the key calculation in the chapter-almost every application starts by solving $A \boldsymbol{x}=\lambda \boldsymbol{x}$.

First move $\boldsymbol{\lambda} \boldsymbol{x}$ to the left side. Write the equation $A \boldsymbol{x}=\lambda \boldsymbol{x}$ as $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$. The matrix $A-\lambda I$ times the eigenvector $\boldsymbol{x}$ is the zero vector. The eigenvectors make up the nullspace of $\boldsymbol{A}-\boldsymbol{\lambda I}$. When we know an eigenvalue $\lambda$, we find an eigenvector by solving $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$.

Eigenvalues first. If $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$ has a nonzero solution, $A-\lambda I$ is not invertible. The determinant of $A-\lambda I$ must be zero. This is how to recognize an eigenvalue $\lambda$ :

Eigenvalues The number $\lambda$ is an eigenvalue of $A$ if and only if $A-\lambda I$ is singular.

Equation for the eigenvalues

$$
\begin{equation*}
\operatorname{det}(A-\lambda I)=0 \tag{3}
\end{equation*}
$$

This "characteristic polynomial" $\operatorname{det}(A-\lambda I)$ involves only $\lambda$, not $\boldsymbol{x}$. When $A$ is $n$ by $n$, equation (3) has degree $n$. Then $A$ has $n$ eigenvalues (repeats possible!) Each $\lambda$ leads to $\boldsymbol{x}$ :

$$
\text { For each eigenvalue } \lambda \text { solve }(A-\lambda I) \boldsymbol{x}=\mathbf{0} \text { or } A \boldsymbol{x}=\lambda \boldsymbol{x} \text { to find an eigenvector } \boldsymbol{x} .
$$

Example $4 \quad A=\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]$ is already singular (zero determinant). Find its $\lambda$ 's and $\boldsymbol{x}$ 's.
When $A$ is singular, $\lambda=0$ is one of the eigenvalues. The equation $A \boldsymbol{x}=0 \boldsymbol{x}$ has solutions. They are the eigenvectors for $\lambda=0$. But $\operatorname{det}(A-\lambda I)=0$ is the way to find all $\lambda$ 's and $\boldsymbol{x}$ 's. Always subtract $\lambda I$ from $A$ :

$$
\text { Subtract } \lambda \text { from the diagonal to find } \quad A-\lambda I=\left[\begin{array}{cc}
1-\lambda & 2  \tag{4}\\
2 & 4-\lambda
\end{array}\right]
$$

Take the determinant " $a d-b c$ " of this 2 by 2 matrix. From $1-\lambda$ times $4-\lambda$, the " $a d$ " part is $\lambda^{2}-5 \lambda+4$. The " $b c$ " part, not containing $\lambda$, is 2 times 2 .

$$
\operatorname{det}\left[\begin{array}{cc}
1-\lambda & 2  \tag{5}\\
2 & 4-\lambda
\end{array}\right]=(1-\lambda)(4-\lambda)-(2)(2)=\lambda^{2}-5 \lambda
$$

Set this determinant $\boldsymbol{\lambda}^{\mathbf{2}}-\mathbf{5} \boldsymbol{\lambda}$ to zero. One solution is $\lambda=0$ (as expected, since $A$ is singular). Factoring into $\lambda$ times $\lambda-5$, the other root is $\lambda=5$ :

$$
\operatorname{det}(A-\lambda I)=\lambda^{2}-5 \lambda=0 \quad \text { yields the eigenvalues } \quad \lambda_{1}=0 \quad \text { and } \quad \lambda_{2}=5
$$

Now find the eigenvectors. Solve $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$ separately for $\lambda_{1}=0$ and $\lambda_{2}=5$ :

$$
\begin{aligned}
& (A-0 I) \boldsymbol{x}=\left[\begin{array}{ll}
1 & 2 \\
2 & 4
\end{array}\right]\left[\begin{array}{l}
y \\
z
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \text { yields an eigenvector }\left[\begin{array}{l}
y \\
z
\end{array}\right]=\left[\begin{array}{r}
2 \\
-1
\end{array}\right] \quad \text { for } \lambda_{1}=0 \\
& (A-5 I) \boldsymbol{x}=\left[\begin{array}{rr}
-4 & 2 \\
2 & -1
\end{array}\right]\left[\begin{array}{l}
y \\
z
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \text { yields an eigenvector }\left[\begin{array}{l}
y \\
z
\end{array}\right]=\left[\begin{array}{l}
1 \\
2
\end{array}\right] \quad \text { for } \lambda_{2}=5
\end{aligned}
$$

The matrices $A-0 I$ and $A-5 I$ are singular (because 0 and 5 are eigenvalues). The eigenvectors $(2,-1)$ and $(1,2)$ are in the nullspaces: $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$ is $A \boldsymbol{x}=\lambda \boldsymbol{x}$.

We need to emphasize: There is nothing exceptional about $\lambda=0$. Like every other number, zero might be an eigenvalue and it might not. If $A$ is singular, the eigenvectors for $\lambda=0$ fill the nullspace: $A \boldsymbol{x}=0 \boldsymbol{x}=\mathbf{0}$. If $A$ is invertible, zero is not an eigenvalue. We shift $A$ by a multiple of $I$ to make it singular.

In the example, the shifted matrix $A-5 I$ is singular and 5 is the other eigenvalue.
Summary To solve the eigenvalue problem for an $n$ by $n$ matrix, follow these steps:

1. Compute the determinant of $A-\lambda I$. With $\lambda$ subtracted along the diagonal, this determinant starts with $\lambda^{n}$ or $-\lambda^{n}$. It is a polynomial in $\lambda$ of degree $n$.
2. Find the roots of this polynomial, by solving $\operatorname{det}(A-\lambda I)=0$. The $n$ roots are the $n$ eigenvalues of $A$. They make $A-\lambda I$ singular.
3. For each eigenvalue $\lambda$, solve $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$ to find an eigenvector $\boldsymbol{x}$.

A note on the eigenvectors of 2 by 2 matrices. When $A-\lambda I$ is singular, both rows are multiples of a vector $(a, b)$. The eigenvector is any multiple of $(b,-a)$. The example had
$\lambda=0$ : rows of $A-0 I$ in the direction $(1,2)$; eigenvector in the direction $(2,-1)$
$\lambda=5$ : rows of $A-5 I$ in the direction $(-4,2)$; eigenvector in the direction $(2,4)$.
Previously we wrote that last eigenvector as $(1,2)$. Both $(1,2)$ and $(2,4)$ are correct. There is a whole line of eigenvectors-any nonzero multiple of $\boldsymbol{x}$ is as good as $\boldsymbol{x}$. MATLAB's eig $(A)$ divides by the length, to make the eigenvector into a unit vector.

We must add a warning. Some 2 by 2 matrices have only one line of eigenvectors. This can only happen when two eigenvalues are equal. (On the other hand $A=I$ has equal eigenvalues and plenty of eigenvectors.) Without a full set of eigenvectors, we don't have a basis. We can't write every $\boldsymbol{v}$ as a combination of eigenvectors. In the language of the next section, we can't diagonalize a matrix without $n$ independent eigenvectors.

## Determinant and Trace

Bad news first: If you add a row of $A$ to another row, or exchange rows, the eigenvalues usually change. Elimination does not preserve the $\lambda$ 's. The triangular $U$ has its eigenvalues sitting along the diagonal-they are the pivots. But they are not the eigenvalues of $A$ ! Eigenvalues are changed when row 1 is added to row 2 :

$$
U=\left[\begin{array}{ll}
1 & 3 \\
0 & 0
\end{array}\right] \quad \text { has } \lambda=0 \text { and } \lambda=1 ; \quad A=\left[\begin{array}{cc}
1 & 3 \\
2 & 6
\end{array}\right] \quad \text { has } \lambda=0 \text { and } \lambda=7
$$

Good news second: The product $\lambda_{1}$ times $\lambda_{2}$ and the sum $\lambda_{1}+\lambda_{2}$ can be found quickly from the matrix. For this $A$, the product is 0 times 7. That agrees with the determinant (which is 0 ). The sum of eigenvalues is $0+7$. That agrees with the sum down the main diagonal (the trace is $1+6$ ). These quick checks always work:

## The product of the $n$ eigenvalues equals the determinant. <br> The sum of the neigenvalues equals the sum of the $n$ diagonal entries.

The sum of the entries along the main diagonal is called the trace of $A$ :

$$
\begin{equation*}
\lambda_{1}+\lambda_{2}+\cdots+\lambda_{n}=\boldsymbol{t r a c e}=a_{11}+a_{22}+\cdots+a_{n n} \tag{6}
\end{equation*}
$$

Those checks are very useful. They are proved in Problems 16-17 and again in the next section. They don't remove the pain of computing $\lambda$ 's. But when the computation is wrong, they generally tell us so. To compute the correct $\lambda$ 's, go back to $\operatorname{det}(A-\lambda I)=0$.

The trace and determinant $d o$ tell everything when the matrix is 2 by 2 . We never want to get those wrong! Here trace $=\mathbf{3}$ and $\operatorname{det}=\mathbf{2}$, so the eigenvalues are $\lambda=\mathbf{1}$ and $\mathbf{2}$ :

$$
A=\left[\begin{array}{ll}
1 & 9  \tag{7}\\
0 & 2
\end{array}\right] \quad \text { or }\left[\begin{array}{rr}
3 & 1 \\
-2 & 0
\end{array}\right] \text { or }\left[\begin{array}{rr}
7 & -3 \\
10 & -4
\end{array}\right]
$$

And here is a question about the best matrices for finding eigenvalues : triangular.
Why do the eigenvalues of a triangular matrix lie along its diagonal?

## Imaginary Eigenvalues

One more bit of news (not too terrible). The eigenvalues might not be real numbers.

$$
\begin{aligned}
& \text { Example } 5 \text { The } 90^{\circ} \text { rotation } Q=\left[\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right] \quad \text { has no real eigenvectors. Its eigenvalues } \\
& \text { are } \lambda_{1}=i \text { and } \lambda_{2}=-i \text {. Then } \lambda_{1}+\lambda_{2}=\text { trace }=0 \text { and } \lambda_{1} \lambda_{2}=\text { determinant }=1
\end{aligned}
$$

After a rotation, no real vector $Q \boldsymbol{x}$ stays in the same direction as $\boldsymbol{x}$ ( $\boldsymbol{x}=\mathbf{0}$ is useless). There cannot be an eigenvector, unless we go to imaginary numbers. Which we do.

To see how $i=\sqrt{-1}$ can help, look at $Q^{2}$ which is $-I$. If $Q$ is rotation through $90^{\circ}$, then $Q^{2}$ is rotation through $180^{\circ}$. Its eigenvalues are -1 and -1 . (Certainly $-I \boldsymbol{x}=-1 \boldsymbol{x}$.) Squaring $Q$ will square each $\lambda$, so we must have $\lambda^{2}=-1$. The eigenvalues of the $90^{\circ}$ rotation matrix $Q$ are $+i$ and $-i$, because $i^{2}=-1$.

Those $\lambda$ 's come as usual from $\operatorname{det}(Q-\lambda I)=0$. This equation gives $\lambda^{2}+1=0$. Its roots are $i$ and $-i$. We meet the imaginary number $i$ also in the eigenvectors:

## Complex

 eigenvectors$$
\left[\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
i
\end{array}\right]=-i\left[\begin{array}{l}
1 \\
i
\end{array}\right] \quad \text { and } \quad\left[\begin{array}{rr}
0 & -1 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
i \\
1
\end{array}\right]=i\left[\begin{array}{l}
i \\
1
\end{array}\right]
$$

Somehow these complex vectors $\boldsymbol{x}_{1}=(1, i)$ and $\boldsymbol{x}_{2}=(i, 1)$ keep their direction as they are rotated. Don't ask me how. This example makes the all-important point that real matrices can easily have complex eigenvalues and eigenvectors. The particular eigenvalues $i$ and $-i$ also illustrate two special properties of $Q$ :

1. $Q$ is an orthogonal matrix so the absolute value of each $\lambda$ is $|\lambda|=1$.
2. $Q$ is a skew-symmetric matrix so each $\lambda$ is pure imaginary.

A symmetric matrix $\left(S^{\mathrm{T}}=S\right)$ can be compared to a real number. A skew-symmetric matrix $\left(A^{\mathrm{T}}=-A\right)$ can be compared to an imaginary number. An orthogonal matrix $\left(Q^{\mathrm{T}} Q=I\right)$ corresponds to a complex number with $|\lambda|=1$. For the eigenvalues of $S$ and $A$ and $Q$, those are more than analogies-they are facts to be proved in Section 6.4.

The eigenvectors for all these special matrices are perpendicular. Somehow $(i, 1)$ and $(1, i)$ are perpendicular (Chapter 9 explains the dot product of complex vectors).

## Eigenvalues of $A B$ and $A+B$

The first guess about the eigenvalues of $A B$ is not true. An eigenvalue $\lambda$ of $A$ times an eigenvalue $\beta$ of $B$ usually does not give an eigenvalue of $A B$ :

False proof

$$
\begin{equation*}
A B \boldsymbol{x}=A \beta \boldsymbol{x}=\beta A \boldsymbol{x}=\beta \lambda \boldsymbol{x} . \tag{8}
\end{equation*}
$$

It seems that $\beta$ times $\lambda$ is an eigenvalue. When $\boldsymbol{x}$ is an eigenvector for $A$ and $B$, this proof is correct. The mistake is to expect that $A$ and $B$ automatically share the same eigenvector $x$. Usually they don't. Eigenvectors of $A$ are not generally eigenvectors of $B$. $A$ and $B$ could have all zero eigenvalues while 1 is an eigenvalue of $A B$ :

$$
A=\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{ll}
0 & 0 \\
1 & 0
\end{array}\right] ; \quad \text { then } \quad A B=\left[\begin{array}{cc}
1 & 0 \\
0 & 0
\end{array}\right] \quad \text { and } \quad A+B=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] .
$$

For the same reason, the eigenvalues of $A+B$ are generally not $\lambda+\beta$. Here $\lambda+\beta=0$ while $A+B$ has eigenvalues 1 and -1 . (At least they add to zero.)

The false proof suggests what is true. Suppose $\boldsymbol{x}$ really is an eigenvector for both $A$ and $B$. Then we do have $A B \boldsymbol{x}=\lambda \beta \boldsymbol{x}$ and $B A \boldsymbol{x}=\lambda \beta \boldsymbol{x}$. When all $n$ eigenvectors are shared, we can multiply eigenvalues. The test $A B=B A$ for shared eigenvectors is important in quantum mechanics-time out to mention this application of linear algebra:
$A$ and $B$ share the same $\boldsymbol{n}$ independent eigenvectors if and only if $\boldsymbol{A B}=\boldsymbol{B} \boldsymbol{A}$.
Heisenberg's uncertainty principle In quantum mechanics, the position matrix $P$ and the momentum matrix $Q$ do not commute. In fact $Q P-P Q=I$ (these are infinite matrices). To have $P \boldsymbol{x}=\mathbf{0}$ at the same time as $Q \boldsymbol{x}=\mathbf{0}$ would require $\boldsymbol{x}=I \boldsymbol{x}=\mathbf{0}$. If we knew the position exactly, we could not also know the momentum exactly. Problem 36 derives Heisenberg's uncertainty principle $\|P \boldsymbol{x}\|\|Q \boldsymbol{x}\| \geq \frac{1}{2}\|\boldsymbol{x}\|^{2}$.

## - REVIEW OF THE KEY IDEAS

1. $A \boldsymbol{x}=\lambda \boldsymbol{x}$ says that eigenvectors $\boldsymbol{x}$ keep the same direction when multiplied by $A$.
2. $A \boldsymbol{x}=\lambda \boldsymbol{x}$ also says that $\operatorname{det}(A-\lambda I)=0$. This determines $n$ eigenvalues.
3. The eigenvalues of $A^{2}$ and $A^{-1}$ are $\lambda^{2}$ and $\lambda^{-1}$, with the same eigenvectors.
4. The sum of the $\lambda$ 's equals the sum down the main diagonal of $A$ (the trace). The product of the $\lambda$ 's equals the determinant of $A$.
5. Projections $P$, reflections $R, 90^{\circ}$ rotations $Q$ have special eigenvalues $1,0,-1, i,-i$. Singular matrices have $\lambda=0$. Triangular matrices have $\lambda$ 's on their diagonal.
6. Special properties of a matrix lead to special eigenvalues and eigenvectors. That is a major theme of this chapter (it is captured in a table at the very end).

## - WORKED EXAMPLES

6.1 A Find the eigenvalues and eigenvectors of $A$ and $A^{2}$ and $A^{-1}$ and $A+4 I$ :

$$
A=\left[\begin{array}{rr}
2 & -1 \\
-1 & 2
\end{array}\right] \quad \text { and } \quad A^{2}=\left[\begin{array}{rr}
5 & -4 \\
-4 & 5
\end{array}\right]
$$

Check the trace $\lambda_{1}+\lambda_{2}=4$ and the determinant $\lambda_{1} \lambda_{2}=3$.
Solution $\quad$ The eigenvalues of $A$ come from $\operatorname{det}(A-\lambda I)=0$ :

$$
A=\left[\begin{array}{rr}
2 & -1 \\
-1 & 2
\end{array}\right] \quad \operatorname{det}(A-\lambda I)=\left|\begin{array}{cc}
\mathbf{2}-\boldsymbol{\lambda} & \mathbf{- 1} \\
\mathbf{- 1} & \mathbf{2 - \boldsymbol { \lambda }}
\end{array}\right|=\lambda^{2}-4 \lambda+3=0
$$

This factors into $(\lambda-1)(\lambda-3)=0$ so the eigenvalues of $A$ are $\lambda_{1}=\mathbf{1}$ and $\lambda_{2}=\mathbf{3}$. For the trace, the sum $2+2$ agrees with $1+3$. The determinant 3 agrees with the product $\lambda_{1} \lambda_{2}$.

The eigenvectors come separately by solving $(A-\lambda I) \boldsymbol{x}=\mathbf{0}$ which is $A \boldsymbol{x}=\lambda \boldsymbol{x}$ :

$$
\begin{aligned}
& \boldsymbol{\lambda}=\mathbf{1}: \quad(A-I) \boldsymbol{x}=\left[\begin{array}{rr}
1 & -1 \\
-1 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \text { gives the eigenvector } \boldsymbol{x}_{1}=\left[\begin{array}{l}
1 \\
1
\end{array}\right] \\
& \boldsymbol{\lambda}=\mathbf{3}: \quad(A-3 I) \boldsymbol{x}=\left[\begin{array}{ll}
-1 & -1 \\
-1 & -1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
0 \\
0
\end{array}\right] \text { gives the eigenvector } \boldsymbol{x}_{2}=\left[\begin{array}{r}
1 \\
-1
\end{array}\right]
\end{aligned}
$$

$A^{2}$ and $A^{-1}$ and $A+4 I$ keep the same eigenvectors as $A$. Their eigenvalues are $\lambda^{2}$ and $\lambda^{-1}$ and $\lambda+4$ :

$$
A^{2} \text { has eigenvalues } 1^{2}=1 \text { and } 3^{2}=9 \quad A^{-1} \text { has } \frac{1}{1} \text { and } \frac{1}{3} \quad A+4 I \text { has } \begin{aligned}
& 1+4=5 \\
& 3+4=7
\end{aligned}
$$

Notes for later sections: $A$ has orthogonal eigenvectors (Section 6.4 on symmetric matrices). $A$ can be diagonalized since $\lambda_{1} \neq \lambda_{2}$ (Section 6.2). $A$ is similar to any 2 by 2 matrix with eigenvalues 1 and 3 (Section 6.2). $A$ is a positive definite matrix (Section 6.5) since $A=A^{\mathrm{T}}$ and the $\lambda$ 's are positive.

### 6.1 B How can you estimate the eigenvalues of any $\boldsymbol{A}$ ? Gershgorin gave this answer.

Every eigenvalue of $A$ must be "near" at least one of the entries $a_{i i}$ on the main diagonal. For $\lambda$ to be "near $a_{i i}$ " means that $\left|a_{i i}-\lambda\right|$ is no more than the $\operatorname{sum} \boldsymbol{R}_{\boldsymbol{i}}$ of all other $\left|\boldsymbol{a}_{\boldsymbol{i} \boldsymbol{j}}\right|$ in that row $i$ of the matrix. Then $R_{i}=\Sigma_{j \neq i}\left|a_{i j}\right|$ is the radius of a circle centered at $a_{i i}$.

## Every $\boldsymbol{\lambda}$ is in the circle around one or more diagonal entries $a_{i i}:\left|a_{i i}-\lambda\right| \leq \boldsymbol{R}_{\boldsymbol{i}}$.

Here is the reasoning. If $\lambda$ is an eigenvalue, then $A-\lambda I$ is not invertible. Then $A-\lambda I$ cannot be diagonally dominant (see Section 2.5). So at least one diagonal entry $a_{i i}-\lambda$ is not larger than the sum $R_{i}$ of all other entries $\left|a_{i j}\right|$ (we take absolute values!) in row $i$.

Example 1. Every eigenvalue $\lambda$ of this $A$ falls into one or both of the Gershgorin circles: The centers are $a$ and $d$, the radii are $R_{1}=|b|$ and $R_{2}=|c|$.

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \quad \begin{array}{ll}
\text { First circle: } & |\lambda-a| \leq|b| \\
\text { Second circle: } & |\lambda-d| \leq|c|
\end{array}
$$

Those are circles in the complex plane, since $\lambda$ could certainly be complex.
Example 2. All eigenvalues of this $A$ lie in a circle of radius $R=3$ around one or more of the diagonal entries $d_{1}, d_{2}, d_{3}$ :

$$
A=\left[\begin{array}{rcc}
d_{1} & 1 & 2 \\
2 & d_{2} & 1 \\
-1 & 2 & d_{3}
\end{array}\right] \quad \begin{aligned}
& \left|\lambda-d_{1}\right| \leq 1+2=R_{1} \\
& \left|\lambda-d_{2}\right| \leq 2+1=R_{2} \\
& \left|\lambda-d_{3}\right| \leq 1+2=R_{3}
\end{aligned}
$$

You see that "near" means not more than 3 away from $d_{1}$ or $d_{2}$ or $d_{3}$, for this example.
6.1 C Find the eigenvalues and eigenvectors of this symmetric 3 by 3 matrix $S$ :

Symmetric matrix
Singular matrix
Trace $\mathbf{1}+\mathbf{2}+\mathbf{1}=4$

$$
S=\left[\begin{array}{rrr}
1 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 1
\end{array}\right]
$$

Solution Since all rows of $S$ add to zero, the vector $\boldsymbol{x}=(1,1,1)$ gives $S \boldsymbol{x}=\mathbf{0}$. This is an eigenvector for $\lambda=0$. To find $\lambda_{2}$ and $\lambda_{3}$ I will compute the 3 by 3 determinant:

$$
\operatorname{det}(S-\lambda I)=\left|\begin{array}{ccc}
1-\lambda & -1 & 0 \\
-1 & 2-\lambda & -1 \\
0 & -1 & 1-\lambda
\end{array}\right|=\begin{aligned}
& =(1-\lambda)(2-\lambda)(1-\lambda)-2(1-\lambda) \\
& =(1-\lambda)[(2-\lambda)(1-\lambda)-2] \\
& \\
& =(\mathbf{1})(\mathbf{\lambda})(\mathbf{3}-\boldsymbol{\lambda})
\end{aligned}
$$

Those three factors give $\lambda=0,1,3$. Each eigenvalue corresponds to an eigenvector (or a line of eigenvectors):

$$
\boldsymbol{x}_{1}=\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right] \quad S \boldsymbol{x}_{1}=\mathbf{0} \boldsymbol{x}_{1} \quad \boldsymbol{x}_{2}=\left[\begin{array}{r}
1 \\
0 \\
-1
\end{array}\right] \quad \boldsymbol{\boldsymbol { x } _ { 2 }}=\mathbf{1} \boldsymbol{x}_{2} \quad \boldsymbol{x}_{3}=\left[\begin{array}{r}
1 \\
-2 \\
1
\end{array}\right] \quad S \boldsymbol{x}_{3}=\boldsymbol{3}_{3}
$$

I notice again that eigenvectors are perpendicular when $S$ is symmetric. We were lucky to find $\lambda=0,1,3$. For a larger matrix I would use $\mathbf{e i g}(A)$, and never touch determinants.

The full command $[\boldsymbol{X}, \boldsymbol{E}]=\mathbf{e i g}(A)$ will produce unit eigenvectors in the columns of $X$.

## Problem Set 6.1

1 The example at the start of the chapter has powers of this matrix $A$ :

$$
A=\left[\begin{array}{ll}
.8 & .3 \\
.2 & .7
\end{array}\right] \quad \text { and } \quad A^{2}=\left[\begin{array}{ll}
.70 & .45 \\
.30 & .55
\end{array}\right] \quad \text { and } \quad A^{\infty}=\left[\begin{array}{ll}
.6 & .6 \\
.4 & .4
\end{array}\right]
$$

Find the eigenvalues of these matrices. All powers have the same eigenvectors.
(a) Show from $A$ how a row exchange can produce different eigenvalues.
(b) Why is a zero eigenvalue not changed by the steps of elimination?

2 Find the eigenvalues and the eigenvectors of these two matrices:

$$
A=\left[\begin{array}{ll}
1 & 4 \\
2 & 3
\end{array}\right] \quad \text { and } \quad A+I=\left[\begin{array}{ll}
2 & 4 \\
2 & 4
\end{array}\right]
$$

$A+I$ has the $\qquad$ eigenvectors as $A$. Its eigenvalues are $\qquad$ by 1.
3 Compute the eigenvalues and eigenvectors of $A$ and $A^{-1}$. Check the trace!

$$
A=\left[\begin{array}{ll}
0 & 2 \\
1 & 1
\end{array}\right] \quad \text { and } \quad A^{-1}=\left[\begin{array}{rr}
-1 / 2 & 1 \\
1 / 2 & 0
\end{array}\right]
$$

$A^{-1}$ has the $\qquad$ eigenvectors as $A$. When $A$ has eigenvalues $\lambda_{1}$ and $\lambda_{2}$, its inverse has eigenvalues $\qquad$

4 Compute the eigenvalues and eigenvectors of $A$ and $A^{2}$ :

$$
A=\left[\begin{array}{rr}
-1 & 3 \\
2 & 0
\end{array}\right] \quad \text { and } \quad A^{2}=\left[\begin{array}{rr}
7 & -3 \\
-2 & 6
\end{array}\right]
$$

$A^{2}$ has the same $\qquad$ as $A$. When $A$ has eigenvalues $\lambda_{1}$ and $\lambda_{2}, A^{2}$ has eigenvalues
$\qquad$ . In this example, why is $\lambda_{1}^{2}+\lambda_{2}^{2}=13$ ?

5 Find the eigenvalues of $A$ and $B$ (easy for triangular matrices) and $A+B$ :

$$
A=\left[\begin{array}{ll}
3 & 0 \\
1 & 1
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{ll}
1 & 1 \\
0 & 3
\end{array}\right] \quad \text { and } \quad A+B=\left[\begin{array}{ll}
4 & 1 \\
1 & 4
\end{array}\right]
$$

Eigenvalues of $A+B$ (are equal to)(are not equal to) eigenvalues of $A$ plus eigenvalues of $B$.

6 Find the eigenvalues of $A$ and $B$ and $A B$ and $B A$ :
$A=\left[\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right]$ and $B=\left[\begin{array}{ll}1 & 2 \\ 0 & 1\end{array}\right]$ and $A B=\left[\begin{array}{ll}1 & 2 \\ 1 & 3\end{array}\right]$ and $B A=\left[\begin{array}{ll}3 & 2 \\ 1 & 1\end{array}\right]$.
(a) Are the eigenvalues of $A B$ equal to eigenvalues of $A$ times eigenvalues of $B$ ?
(b) Are the eigenvalues of $A B$ equal to the eigenvalues of $B A$ ?

7 Elimination produces $A=L U$. The eigenvalues of $U$ are on its diagonal; they are the $\qquad$ . The eigenvalues of $L$ are on its diagonal; they are all $\qquad$ . The eigenvalues of $A$ are not the same as $\qquad$ —.

8 (a) If you know that $\boldsymbol{x}$ is an eigenvector, the way to find $\lambda$ is to $\qquad$ -
(b) If you know that $\lambda$ is an eigenvalue, the way to find $\boldsymbol{x}$ is to $\qquad$ .

9 What do you do to the equation $A \boldsymbol{x}=\lambda \boldsymbol{x}$, in order to prove (a), (b), and (c)?
(a) $\lambda^{2}$ is an eigenvalue of $A^{2}$, as in Problem 4.
(b) $\lambda^{-1}$ is an eigenvalue of $A^{-1}$, as in Problem 3.
(c) $\lambda+1$ is an eigenvalue of $A+I$, as in Problem 2 .

10 Find the eigenvalues and eigenvectors for both of these Markov matrices $A$ and $A^{\infty}$. Explain from those answers why $A^{100}$ is close to $A^{\infty}$ :

$$
A=\left[\begin{array}{ll}
.6 & .2 \\
.4 & .8
\end{array}\right] \quad \text { and } \quad A^{\infty}=\left[\begin{array}{ll}
1 / 3 & 1 / 3 \\
2 / 3 & 2 / 3
\end{array}\right]
$$

11 Here is a strange fact about 2 by 2 matrices with eigenvalues $\lambda_{1} \neq \lambda_{2}$ : The columns of $A-\lambda_{1} I$ are multiples of the eigenvector $\boldsymbol{x}_{2}$. Any idea why this should be?

12 Find three eigenvectors for this matrix $P$ (projection matrices have $\lambda=1$ and 0 ):

$$
\text { Projection matrix } \quad P=\left[\begin{array}{ccc}
.2 & .4 & 0 \\
.4 & .8 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

If two eigenvectors share the same $\lambda$, so do all their linear combinations. Find an eigenvector of $P$ with no zero components.
13 From the unit vector $\boldsymbol{u}=\left(\frac{1}{6}, \frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right)$ construct the rank one projection matrix $P=\boldsymbol{u} \boldsymbol{u}^{\mathrm{T}}$. This matrix has $P^{2}=P$ because $\boldsymbol{u}^{\mathrm{T}} \boldsymbol{u}=1$.
(a) $P \boldsymbol{u}=\boldsymbol{u}$ comes from $\left(\boldsymbol{u} \boldsymbol{u}^{\mathrm{T}}\right) \boldsymbol{u}=\boldsymbol{u}\left(\_\right)$. Then $\boldsymbol{u}$ is an eigenvector with $\lambda=1$.
(b) If $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ show that $P \boldsymbol{v}=\mathbf{0}$. Then $\lambda=0$.
(c) Find three independent eigenvectors of $P$ all with eigenvalue $\lambda=0$.

14 Solve $\operatorname{det}(Q-\lambda I)=0$ by the quadratic formula to reach $\lambda=\cos \theta \pm i \sin \theta$ :

$$
Q=\left[\begin{array}{cc}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{array}\right] \quad \text { rotates the } x y \text { plane by the angle } \theta . \text { No real } \lambda \text { 's. }
$$

Find the eigenvectors of $Q$ by solving $(Q-\lambda I) \boldsymbol{x}=\mathbf{0}$. Use $i^{2}=-1$.
15 Every permutation matrix leaves $\boldsymbol{x}=(1,1, \ldots, 1)$ unchanged. Then $\lambda=1$. Find two more $\lambda$ 's (possibly complex) for these permutations, from $\operatorname{det}(P-\lambda I)=0$ :

$$
P=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right] \quad \text { and } \quad P=\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right]
$$

16 The determinant of $A$ equals the product $\lambda_{1} \lambda_{2} \cdots \lambda_{n}$. Start with the polynomial $\operatorname{det}(A-\lambda I)$ separated into its $n$ factors (always possible). Then set $\lambda=0$ :

$$
\operatorname{det}(A-\lambda I)=\left(\lambda_{1}-\lambda\right)\left(\lambda_{2}-\lambda\right) \cdots\left(\lambda_{n}-\lambda\right) \quad \text { so } \quad \operatorname{det} A=
$$

$\qquad$ -.
Check this rule in Example 1 where the Markov matrix has $\lambda=1$ and $\frac{1}{2}$.
17 The sum of the diagonal entries (the trace) equals the sum of the eigenvalues:

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \quad \text { has } \quad \operatorname{det}(A-\lambda I)=\lambda^{2}-(a+d) \lambda+a d-b c=0
$$

The quadratic formula gives the eigenvalues $\lambda=(a+d+\sqrt{ }) / 2$ and $\lambda=$ $\qquad$ -. Their sum is $\qquad$ . If $A$ has $\lambda_{1}=3$ and $\lambda_{2}=4$ then $\operatorname{det}(A-\lambda I)=$ $\qquad$ .
18 If $A$ has $\lambda_{1}=4$ and $\lambda_{2}=5$ then $\operatorname{det}(A-\lambda I)=(\lambda-4)(\lambda-5)=\lambda^{2}-9 \lambda+20$. Find three matrices that have trace $a+d=9$ and determinant 20 and $\lambda=4,5$.

19 A 3 by 3 matrix $B$ is known to have eigenvalues $0,1,2$. This information is enough to find three of these (give the answers where possible):
(a) the rank of $B$
(b) the determinant of $B^{\mathrm{T}} B$
(c) the eigenvalues of $B^{\mathrm{T}} B$
(d) the eigenvalues of $\left(B^{2}+I\right)^{-1}$.

20 Choose the last rows of $A$ and $C$ to give eigenvalues 4,7 and $1,2,3$ :

## Companion matrices

$$
A=\left[\begin{array}{ll}
0 & 1 \\
* & *
\end{array}\right] \quad C=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
* & * & *
\end{array}\right]
$$

21 The eigenvalues of $\boldsymbol{A}$ equal the eigenvalues of $\boldsymbol{A}^{\mathbf{T}}$. This is because $\operatorname{det}(A-\lambda I)$ equals $\operatorname{det}\left(A^{\mathrm{T}}-\lambda I\right)$. That is true because $\qquad$ . Show by an example that the eigenvectors of $A$ and $A^{\mathrm{T}}$ are not the same.

22 Construct any 3 by 3 Markov matrix $M$ : positive entries down each column add to 1 . Show that $M^{\mathrm{T}}(1,1,1)=(1,1,1)$. By Problem $21, \lambda=1$ is also an eigenvalue of $M$. Challenge: A 3 by 3 singular Markov matrix with trace $\frac{1}{2}$ has what $\lambda$ 's ?

23 Find three 2 by 2 matrices that have $\lambda_{1}=\lambda_{2}=0$. The trace is zero and the determinant is zero. $A$ might not be the zero matrix but check that $A^{2}=0$.

24 This matrix is singular with rank one. Find three $\lambda$ 's and three eigenvectors:

$$
A=\left[\begin{array}{l}
1 \\
2 \\
1
\end{array}\right]\left[\begin{array}{lll}
2 & 1 & 2
\end{array}\right]=\left[\begin{array}{lll}
2 & 1 & 2 \\
4 & 2 & 4 \\
2 & 1 & 2
\end{array}\right]
$$

25 Suppose $A$ and $B$ have the same eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$ with the same independent eigenvectors $\boldsymbol{x}_{1}, \ldots, \boldsymbol{x}_{n}$. Then $A=B$. Reason: Any vector $\boldsymbol{x}$ is a combination $c_{1} \boldsymbol{x}_{1}+\cdots+c_{n} \boldsymbol{x}_{n}$. What is $A \boldsymbol{x}$ ? What is $B \boldsymbol{x}$ ?
26 The block $B$ has eigenvalues 1,2 and $C$ has eigenvalues 3,4 and $D$ has eigenvalues 5,7 . Find the eigenvalues of the 4 by 4 matrix $A$ :

$$
A=\left[\begin{array}{ll}
B & C \\
0 & D
\end{array}\right]=\left[\begin{array}{rrrr}
0 & 1 & 3 & 0 \\
-2 & 3 & 0 & 4 \\
0 & 0 & 6 & 1 \\
0 & 0 & 1 & 6
\end{array}\right]
$$

27 Find the rank and the four eigenvalues of $A$ and $C$ :

$$
A=\left[\begin{array}{llll}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{array}\right] \quad \text { and } \quad C=\left[\begin{array}{cccc}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{array}\right]
$$

28 Subtract $I$ from the previous $A$. Find the $\lambda$ 's and then the determinants of

$$
B=A-I=\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{array}\right] \quad \text { and } \quad C=I-A=\left[\begin{array}{rrrr}
0 & -1 & -1 & -1 \\
-1 & 0 & -1 & -1 \\
-1 & -1 & 0 & -1 \\
-1 & -1 & -1 & 0
\end{array}\right]
$$

29 (Review) Find the eigenvalues of $A, B$, and $C$ :

$$
A=\left[\begin{array}{lll}
1 & 2 & 3 \\
0 & 4 & 5 \\
0 & 0 & 6
\end{array}\right] \quad \text { and } \quad B=\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 2 & 0 \\
3 & 0 & 0
\end{array}\right] \quad \text { and } \quad C=\left[\begin{array}{lll}
2 & 2 & 2 \\
2 & 2 & 2 \\
2 & 2 & 2
\end{array}\right]
$$

30 When $a+b=c+d$ show that $(1,1)$ is an eigenvector and find both eigenvalues :

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]
$$

31 If we exchange rows 1 and 2 and columns 1 and 2, the eigenvalues don't change.
Find eigenvectors of $A$ and $B$ for $\lambda=11$. Rank one gives $\lambda_{2}=\lambda_{3}=0$.

$$
A=\left[\begin{array}{lll}
1 & 2 & 1 \\
3 & 6 & 3 \\
4 & 8 & 4
\end{array}\right] \quad \text { and } \quad B=P A P^{\mathrm{T}}=\left[\begin{array}{lll}
6 & 3 & 3 \\
2 & 1 & 1 \\
8 & 4 & 4
\end{array}\right]
$$

32 Suppose $A$ has eigenvalues $0,3,5$ with independent eigenvectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.
(a) Give a basis for the nullspace and a basis for the column space.
(b) Find a particular solution to $A \boldsymbol{x}=\boldsymbol{v}+\boldsymbol{w}$. Find all solutions.
(c) $\boldsymbol{A x}=\boldsymbol{u}$ has no solution. If it did then $\qquad$ would be in the column space.

## Challenge Problems

33 Show that $\boldsymbol{u}$ is an eigenvector of the rank one $2 \times 2$ matrix $A=\boldsymbol{u} \boldsymbol{v}^{\mathrm{T}}$. Find both eigenvalues of $A$. Check that $\lambda_{1}+\lambda_{2}$ agrees with the trace $u_{1} v_{1}+u_{2} v_{2}$.

34 Find the eigenvalues of this permutation matrix $P$ from $\operatorname{det}(P-\lambda I)=0$. Which vectors are not changed by the permutation? They are eigenvectors for $\lambda=1$. Can you find three more eigenvectors?

$$
P=\left[\begin{array}{llll}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right]
$$

35 There are six 3 by 3 permutation matrices $P$. What numbers can be the determinants of $P$ ? What numbers can be pivots? What numbers can be the trace of $P$ ? What four numbers can be eigenvalues of $P$, as in Problem 15?

36 (Heisenberg's Uncertainty Principle) $A B-B A=I$ can happen for infinite matrices with $A=A^{\mathrm{T}}$ and $B=-B^{\mathrm{T}}$. Then

$$
\boldsymbol{x}^{\mathrm{T}} \boldsymbol{x}=\boldsymbol{x}^{\mathrm{T}} A B \boldsymbol{x}-\boldsymbol{x}^{\mathrm{T}} B A \boldsymbol{x} \leq 2\|A \boldsymbol{x}\|\|B \boldsymbol{x}\| .
$$

Explain that last step by using the Schwarz inequality $\left|\boldsymbol{u}^{\mathrm{T}} \boldsymbol{v}\right| \leq\|\boldsymbol{u}\|\|\boldsymbol{v}\|$. Then Heisenberg's inequality says that $\|A x\| /\|x\|$ times $\|B x\| /\|x\|$ is at least $\frac{1}{2}$. It is impossible to get the position error and momentum error both very small.

37 Find a 2 by 2 rotation matrix (other than $I$ ) with $A^{3}=I$. Its eigenvalues must satisfy $\lambda^{3}=1$. They can be $e^{2 \pi i / 3}$ and $e^{-2 \pi i / 3}$. What are the trace and determinant?

38 (a) Find the eigenvalues and eigenvectors of $A$. They depend on $c$ :

$$
A=\left[\begin{array}{cc}
.4 & 1-c \\
.6 & c
\end{array}\right]
$$

(b) Show that $A$ has just one line of eigenvectors when $c=1.6$.
(c) This is a Markov matrix when $c=8$. Then $A^{n}$ will approach what matrix $A^{\infty}$ ?

## Eigshow in MATLAB

There is a MATLAB demo (just type eigshow), displaying the eigenvalue problem for a 2 by 2 matrix. It starts with the unit vector $\boldsymbol{x}=(1,0)$. The mouse makes this vector move around the unit circle. At the same time the screen shows $A \boldsymbol{x}$, in color and also moving. Possibly $A \boldsymbol{x}$ is ahead of $\boldsymbol{x}$. Possibly $A \boldsymbol{x}$ is behind $\boldsymbol{x}$. Sometimes $A \boldsymbol{x}$ is parallel to $\boldsymbol{x}$.

At that parallel moment, $A \boldsymbol{x}=\lambda \boldsymbol{x}$ (at $\boldsymbol{x}_{1}$ and $\boldsymbol{x}_{2}$ in the second figure).


These are not eigenvectors

$A \boldsymbol{x}$ lines up with $\boldsymbol{x}$ at eigenvectors

The eigenvalue $\lambda$ is the length of $A \boldsymbol{x}$, when the unit eigenvector $\boldsymbol{x}$ lines up. The built-in choices for $A$ illustrate three possibilities: 0,1 , or 2 real vectors where $A \boldsymbol{x}$ crosses $\boldsymbol{x}$.

The axes of the ellipse are singular vectors in 7.4 -and eigenvectors if $A^{\mathrm{T}}=A$.

