CHAPTER 6
NAVIER-STOKES SOLUTION FOR BLASIUS

We are discussion about 2-D laminar boundary layer.
From previous lesson, we could write the x-component Navier-Stokes equation as:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = g_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Flow is laminar (steady flow)	$\frac{\partial u}{\partial t} = 0$
2-D flow, z axis is not existing	$w = 0$
Gravity is not acting in x-axis	$g_x = 0$
2-D flow, z axis is not existing	$\frac{\partial^2 u}{\partial z^2} = 0$

Then, Navier-Stokes equation becomes:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

It can write as:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial x^2}+\nu\frac{\partial^2 u}{\partial y^2}$$

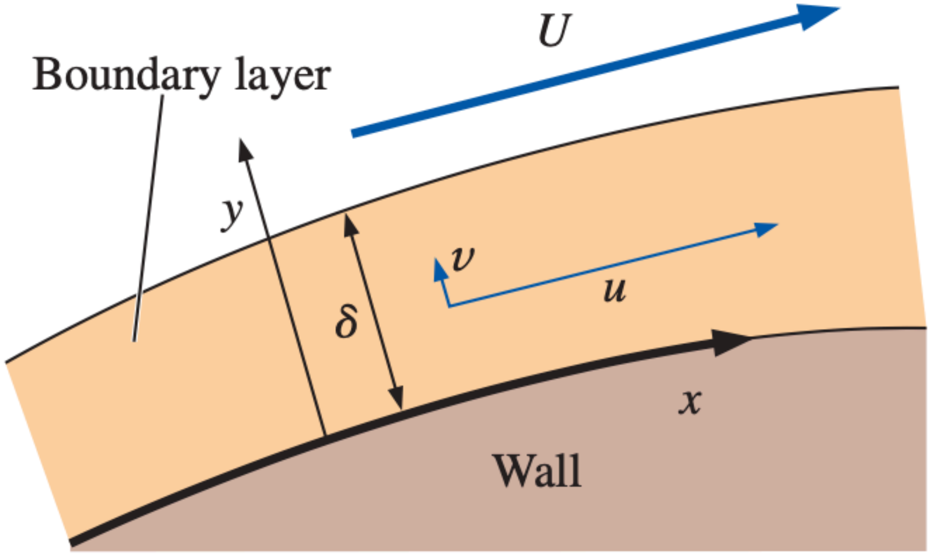
Here ;

$$\nu = \text{kinematic viscosity}$$

We substitute these nondimensional variables:

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{\delta} \qquad u^* = \frac{u}{U} \qquad v^* = \frac{vL}{U\delta} \qquad P^* = \frac{P-P_\infty}{\rho U^2}$$

< Note >



From the continuity equation, we know that ;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

We can make an equivalence relation as ;

$$\frac{U}{L} \sim \frac{v}{\delta}$$

Read as, equivalence relation between $\frac{U}{L}$ and $\frac{v}{\delta}$.

Nondimensional means unity (or one).

Then, we could have the nondimensional variable for v as :

$$v^* = \frac{vL}{U\delta}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2} + \nu \frac{\partial^2 u}{\partial y^2}$$

$$u^* U \frac{\partial}{\partial x^*}\frac{u^* U}{L} + u^* \frac{U\delta}{L} \frac{\partial}{\partial y^*}\frac{u^* U}{\delta} = -\frac{1}{\rho}\frac{\partial}{\partial x^*}\frac{P^* \rho U^2}{L} + \nu \frac{\partial^2}{\partial x^{*2}}\frac{u^* U}{L^2} + \nu \frac{\partial^2}{\partial y^{*2}}\frac{u^* U}{\delta^2}$$

Rearrange the terms, and multiplying each term by $\frac{L}{U^2}$, we can get ;

$$u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\frac{\partial P^*}{\partial x^*} + \left(\frac{\nu}{UL}\right)\frac{\partial^2 u^*}{\partial x^{*2}} + \left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \frac{\partial^2 u^*}{\partial y^{*2}}$$

We focus on the last term. Because above parameters are nondimensional (or unity), the extra term must also be of order unity. We can write as ;

$$\left(\frac{\nu}{UL}\right)\left(\frac{L}{\delta}\right)^2 \sim 1$$

Reynolds number can be written as ;

$$Re = \frac{UL}{\nu}$$

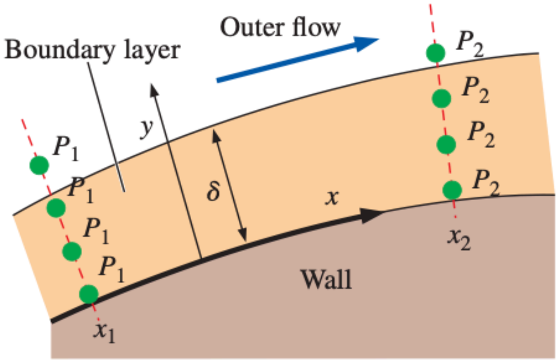
We could get ;

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re}}$$

Or

$$\delta \sim \frac{1 \times L}{\sqrt{Re}}$$

Pressure in the boundary layer

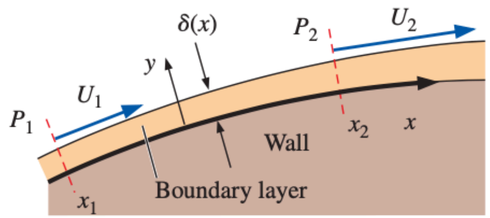


Pressure may change *along* a boundary layer (x-direction), but the change in pressure *across* a boundary layer (y-direction) is negligible.

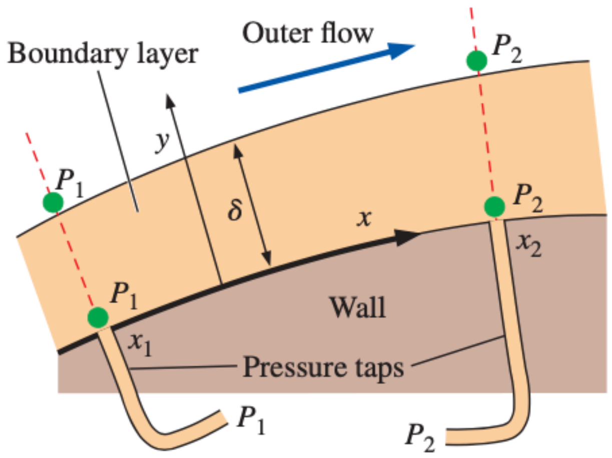
The pressure across boundary layer (y-direction) is nearly constant.

$$\frac{\partial P}{\partial y} \approx 0$$

However, pressure across the x-direction of the boundary layer cannot be assumed as zero. The value of the pressure can be calculated but it is make the equation more complicated.



Outer flow speed parallel to the wall is $U(x)$ and is obtained from the outer flow pressure, $P(x)$. This speed appears in the x-component of the boundary layer momentum equation



The pressure in the irrotational region of flow outside of a boundary layer can be measured by static pressure taps in the surface of the wall. Two such pressure taps are sketched.

The pressure at the outer edge of a boundary layer can be measured experimentally by a static pressure tap at the wall directly beneath the boundary layer

Potential flow analysis and Bernoulli equation may also be used to calculate the pressure of the boundary layer.

From Bernoulli equation:

$$\frac{P}{\rho} + \frac{1}{2}U^2 = \text{constant}$$

Then,

$$\frac{1}{\rho}\frac{dP}{dx} = -U\frac{dU}{dx}$$

Boundary layer equation from Navier-Stokes equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$$

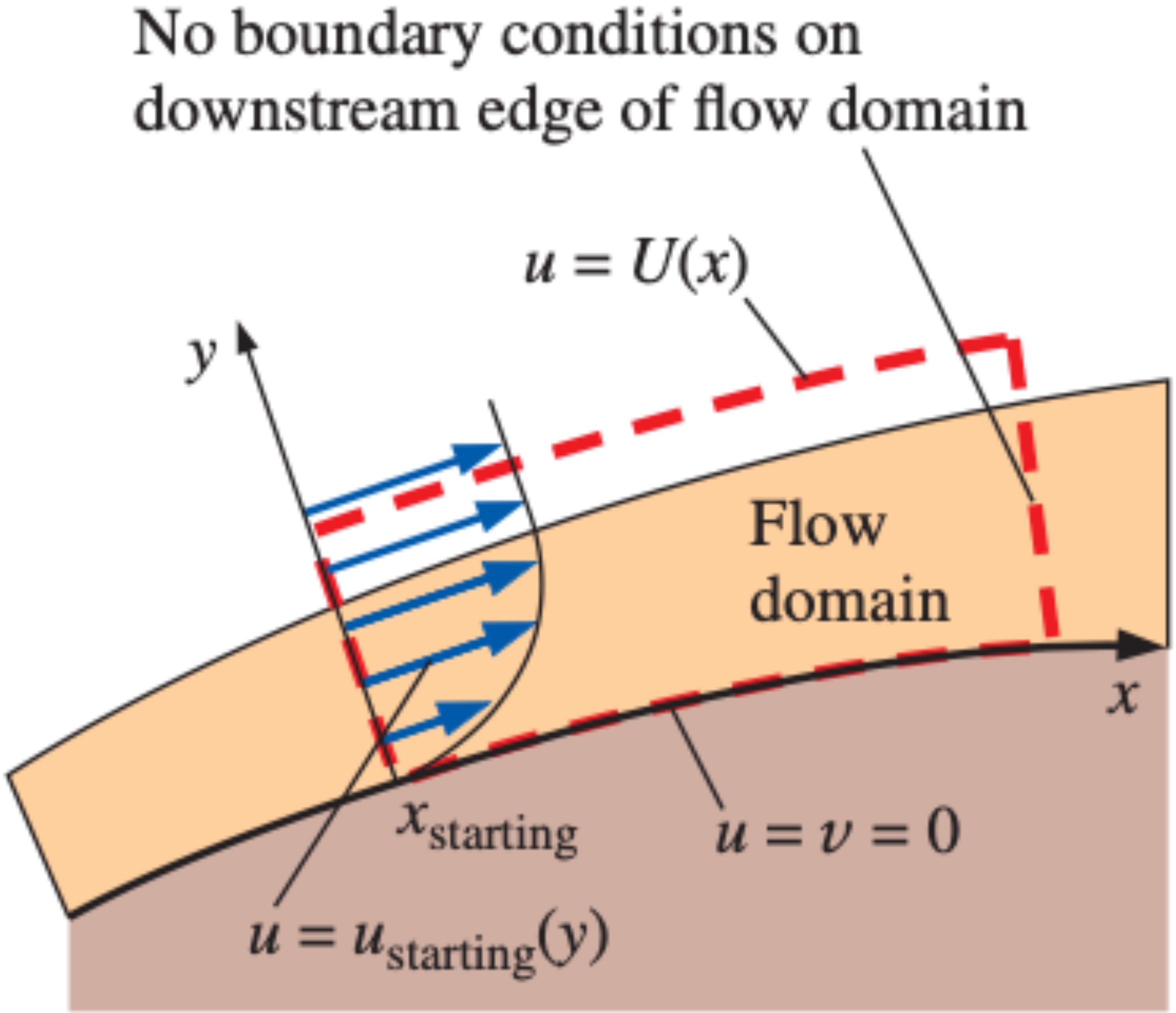
$\frac{1}{\rho}\frac{\partial P}{\partial x}$	This term can be replaced by: $$-U\frac{dU}{dx}$$
$\nu\frac{\partial^2 u}{\partial x^2}$	This term can be ignored. Since, kinematic viscosity, ν , is too small. The changes of velocity, u , along the x-direction also too small. The sum will be too small.

The boundary layer equations become:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

Boundary condition



There are three boundary conditions that can be specified from the boundary layer phenomenon:

$$u = v = 0 \quad at \quad y = 0$$

$$u = U(x) \quad as \quad y \to \infty$$

$$u = u_{starting}(y) \quad at \quad x = x_{starting}$$

where $x_{starting}$ may or may not be zero

Blasius solution

Assumption for Blasius solution:

1. The flow is steady, incompressible and two-dimensional in the xy-plane.
2. The Reynolds number is high enough that the boundary layer approximation is reasonable.
3. The boundary layer remains laminar over the range of interest.
4. No pressure gradient remain in the x-direction boundary layer.

So that, boundary layer equations become:

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=0$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

Boundary conditions:

$$u=0 \quad at \quad y=0 \qquad\qquad u=U \quad as \quad y\to\infty$$

$$v=0 \quad at \quad y=0 \qquad\qquad u=U \quad for\ all \quad y \quad at \quad x=0$$

Blasius used the idea of similarity to solve these equations. In similarity terms, there is no characteristic length scale in the geometry of the problem. This means that we will see the same flow pattern no matter how much we zoom in or zoom out.

Blasius introduced a similarity variable called “eta”, written as η, that combines independent variables x and y into one nondimensional independent variable.

$$\eta = y\sqrt{\frac{U}{\nu x}} \tag{2}$$

And Blasius solved for a nondimensionalized form of the x-component of velocity,

$$f' = \frac{u}{U} = \text{function of } \eta \tag{3}$$

Blasius substitutes Eq.(2) and Eq.(3) into Eq.(1), subjected to the boundary conditions.

He gets an ordinary differential equation for nondimensional speed

$$f'(\eta) = \frac{u}{U}$$

as a function of similarity variable of η .

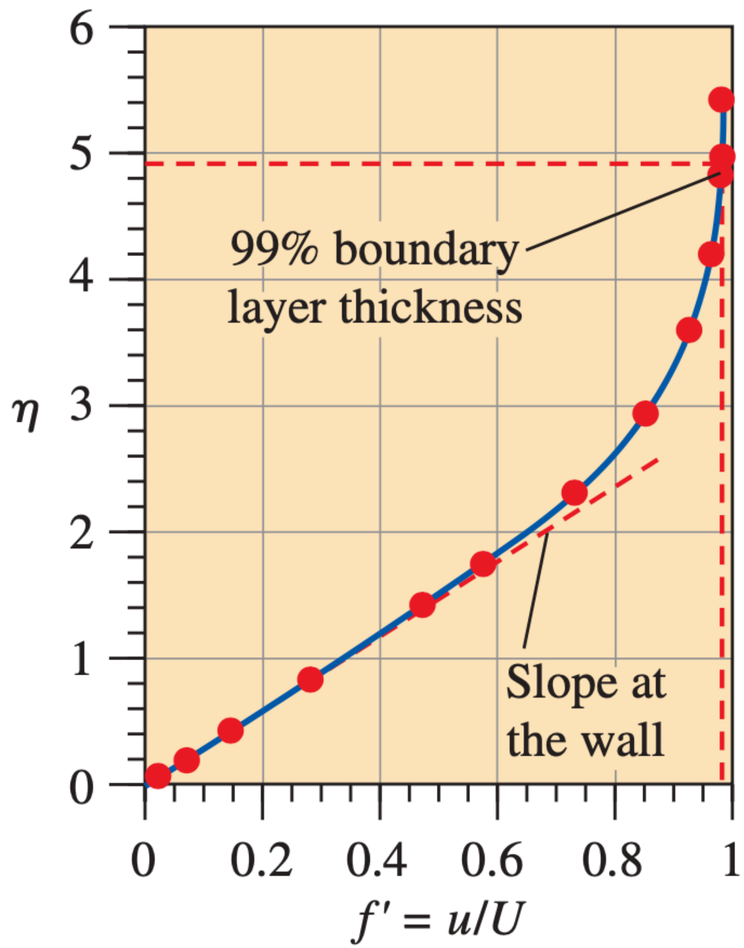
Blasius used the popular Runge–Kutta numerical technique to obtain the results shown in table below.

Solution of the Blasius laminar flat plate boundary layer in similarity variables*

η	f''	f'	f	η	f''	f'	f
0.0	0.33206	0.00000	0.00000	2.4	0.22809	0.72898	0.92229
0.1	0.33205	0.03321	0.00166	2.6	0.20645	0.77245	1.07250
0.2	0.33198	0.06641	0.00664	2.8	0.18401	0.81151	1.23098
0.3	0.33181	0.09960	0.01494	3.0	0.16136	0.84604	1.39681
0.4	0.33147	0.13276	0.02656	3.5	0.10777	0.91304	1.83770
0.5	0.33091	0.16589	0.04149	4.0	0.06423	0.95552	2.30574
0.6	0.33008	0.19894	0.05973	4.5	0.03398	0.97951	2.79013
0.8	0.32739	0.26471	0.10611	5.0	0.01591	0.99154	3.28327
1.0	0.32301	0.32978	0.16557	5.5	0.00658	0.99688	3.78057
1.2	0.31659	0.39378	0.23795	6.0	0.00240	0.99897	4.27962
1.4	0.30787	0.45626	0.32298	6.5	0.00077	0.99970	4.77932
1.6	0.29666	0.51676	0.42032	7.0	0.00022	0.99992	5.27923
1.8	0.28293	0.57476	0.52952	8.0	0.00001	1.00000	6.27921
2.0	0.26675	0.62977	0.65002	9.0	0.00000	1.00000	7.27921
2.2	0.24835	0.68131	0.78119	10.0	0.00000	1.00000	8.27921

η is the similarity variable defined in Eq.(2).
Function $f(\eta)$ is solved using the Runge-Kutta numerical technique.
f is proportional to the stream function.
f' is proportional to the x-component of velocity in the boundary layer $f' = \frac{u}{U}$
f'' is proportional to the shear stress τ .

f' is plotted as a function of η as shown in the figure.



The Blasius profile in similarity variables for the boundary layer growing on a semi-infinite flat plate. Experimental data (show as circles) are at $Re = 3.64 \times 10^5$.

Calculation of the boundary layer thickness, δ

From above mention table, we find that $\frac{u}{U} = f' = 0.99 = 99\%$ occur at $\eta = 4.91$.

$$\eta = 4.91 = \sqrt{\frac{U}{\upsilon x}}\delta \qquad \Rightarrow \qquad \frac{\delta}{x} = \frac{4.91}{\sqrt{Re}}$$

Calculation of the shear stress, τ_w

$$\tau_w = \frac{0.332\rho U^2}{\sqrt{Re}}$$

Calculation of the local drag coefficient, C_d

$$C_d = \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{Re}}$$