Chapter 6 Partial Differential Equations (PDE)

6-1 Classification of Partial Differential Equations

The first-order linear PDE: $a(x,y)\frac{\partial u}{\partial x}+b(x,y)\frac{\partial u}{\partial y}+f(x,y)u+g(x,y)=0$

The second-order linear PDE: $a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}$

$$+d(x,y)\frac{\partial u}{\partial x}+e(x,y)\frac{\partial u}{\partial y}+f(x,y)u+g(x,y)=0$$

$$\begin{cases} hyperbolic\ at\ (x_0,y_0):\ \ \Delta(x_0,y_0)=b(x_0,y_0)^2-4a(x_0,y_0)c(x_0,y_0)>0 \\ elliptic\ at\ (x_0,y_0):\ \ \Delta(x_0,y_0)<0 \\ parabolic\ at\ (x_0,y_0):\ \ \Delta(x_0,y_0)=0 \end{cases}$$

Eg. Which is the type of partial differential equation for $4\frac{\partial^2 u}{\partial x^2}=3\frac{\partial^2 u}{\partial y^2}$? [2018 台大電研]

(Sol.) $4\frac{\partial^2 u}{\partial x^2}-3\frac{\partial^2 u}{\partial y^2}=0$, $\because$ Δ=0-4·4·(-3)=48>0, $\therefore$ it is hyperbolic.

Wave equation: $\frac{\partial^2 u}{\partial t^2}=a^2(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})+b\frac{\partial u}{\partial t}+F$

Heat equation or Diffusion equation: $\frac{\partial u}{\partial t}=a^2(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})=a^2\nabla^2 u$

Laplace's and Poisson's equations: $\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\begin{cases}0\\ \rho\end{cases}$

Schrodinger's equation: $i\hbar\frac{\partial \psi}{\partial t}=\frac{-\hbar^2}{2m}\nabla^2\psi+V\psi$ in quantum mechanics.

6-2 Separation-of-Variable Method

Eg. Solve $\frac{\partial^2\theta}{\partial x^2}=\frac{\partial^2\theta}{\partial t^2}$**, $\theta(x,0)=x$, $\theta(0,t)=0$,** $\frac{\partial\theta}{\partial t}_{|t=0}=0$**, and** $\frac{\partial\theta}{\partial x}_{|x=1}=0$**.**

(Sol.) Let $\theta(x,t)=X(x)T(t)$, $X''(x)T(t)=X(x)T''(t)$, $\frac{X''(x)}{X(x)}=\frac{T''(t)}{T(t)}=\lambda$

$$X(0)=0,\ X'(1)=0 \Rightarrow -\lambda=\left[\frac{(2n-1)\pi}{2}\right]^2,\quad X_n=C_n\sin\left[\frac{(2n-1)\pi}{2}x\right]$$

$$\frac{T''(t)}{T(t)}=-\left[\frac{(2n-1)\pi}{2}\right]^2,\ T(0)=\text{constant},\ T'(0)=0 \Rightarrow T_n=d_n\cos\left[\frac{(2n-1)\pi}{2}t\right],$$

$$\therefore\quad \theta(x,t)=\sum_{n=1}^{\infty}A_n\cos\left[\frac{(2n-1)\pi}{2}t\right]\cdot\sin\left[\frac{(2n-1)\pi}{2}x\right]$$

$$\theta(x,0)=x=\sum_{n=1}^{\infty}A_n\sin\left[\frac{(2n-1)\pi}{2}x\right]\Rightarrow A_n=\frac{\int_{-1}^{1}x\sin\left[\frac{(2n-1)\pi}{2}x\right]dx}{\int_{-1}^{1}\sin^2\left[\frac{(2n-1)\pi}{2}x\right]dx}=\frac{8(-1)^{n+1}}{(2n-1)^2\pi^2}$$

Eg. Solve $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$**, $u(0,t)=u(1,t)=u(x,0)=0$, and** $\frac{\partial u}{\partial t}_{|t=0}=\sin(\pi x)$**.**

(Sol.) $u(x,t)=X(x)T(t)$, $X''(x)T(t)=X(x)T''(t)$, $\frac{X''(x)}{X(x)}=\frac{T''(t)}{T(t)}=\lambda$,

$X(0)=0=X(1)\Rightarrow\lambda=-(n\pi)^2$ and $X(x)=C_n\sin(n\pi x)$,

$\frac{T''(t)}{T(t)}=-(n\pi)^2$ and $T(0)=0$, $T'(0)=\text{constant}\Rightarrow T(t)=d_n\sin(n\pi t)$,

$$\therefore\quad u(x,t)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\cdot\sin(n\pi t),\quad \frac{\partial u(x,t)}{\partial t}=\sum_{n=1}^{\infty}n\pi A_n\sin(n\pi x)\cdot\cos(n\pi t)$$

$\frac{\partial u}{\partial t}_{|t=0}=\sin(\pi x)\Rightarrow \pi A_1=1\Rightarrow A_1=1/\pi$ but $A_n=0$ for $n\neq1\Rightarrow u(x,t)=\frac{1}{\pi}\sin(\pi x)\cdot\sin(\pi t)$

Eg. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$**, $u(x,0)=3\sin(2\pi x)$, $u(0,t)=u(1,t)=0$, $0<x<1$, $t\geq 0$.**

(Sol.) $u(x,t)=X(x)T(t)$, $X(x)T'(t)=X''(x)T(t)$, $\frac{X''(x)}{X(x)}=\frac{T'(t)}{T(t)}=\lambda$,

$X(0)=0=X(1)\Rightarrow\lambda=-(n\pi)^2$ and $X(x)=C_n\sin(n\pi x)$, $\frac{T'(t)}{T(t)}=-(n\pi)^2$ and $T(0)=\text{constant}$

$\Rightarrow T(t)=d_n e^{-n^2\pi^2 t}$, $\therefore\quad u(x,t)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)\cdot e^{-n^2\pi^2 t}$ and $u(x,0)=\sum_{n=1}^{\infty}A_n\sin(n\pi x)$

$u(x,0)=3\sin(2\pi x)\Rightarrow A_2=3$ but $A_n=0$ for $n\neq2\Rightarrow u(x,t)=3e^{-4\pi^2 t}\cdot\sin(2\pi x)$

Eg. Solve $\dfrac{\partial^2 y}{\partial t^2}=\dfrac{\partial^2 y}{\partial x^2}$ **,** ***y*(*x*,0)=*f*(*x*)=** $0.1\left(1-\left|\dfrac{2x}{\pi}-1\right|\right)=\begin{cases}\dfrac{0.2x}{\pi} & ,0<x<\dfrac{\pi}{2}\\ \left(1-\dfrac{x}{\pi}\right)0.2 & ,\dfrac{\pi}{2}<x<\pi\end{cases}$ **,**

and *y*(*x*,*t*)= $\sum_{n=1}^{\infty}b_n(t)\sin(nx)$**. (a) Find the Fourier sine series for *f*(*x*) on [0,*π*]. (b) Find the ordinary differential equation and the initial condition for $b_n(t)$. (c) Find *y*(*x*,*t*). [1990 中央電研]**

(Sol.) (a) $f(x)=\sum_{n=1}^{\infty}a_n\sin\dfrac{n\pi x}{L}=\sum_{n=1}^{\infty}a_n\sin(nx)$, $2L=2\pi$, $L=\pi$, $\dfrac{n\pi x}{L}=nx$

$$a_n=\frac{1}{L}\int_{-\pi}^{\pi}f(x)\sin(nx)dx=\frac{2}{L}\int_{0}^{\pi}f(x)\sin(nx)dx$$

$$=\frac{2}{\pi}\left[\int_0^{\pi/2}\frac{0.2x}{\pi}\cdot\sin(nx)dx+\int_{\pi/2}^{\pi}0.2\left(1-\frac{x}{\pi}\right)\sin(nx)dx\right]=\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\therefore\quad f(x)=\sum_{n=1}^{\infty}\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cdot\sin(nx)$$

(b) $y(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin(nx)$, $y(x,0)=\sum_{n=1}^{\infty}b_n(0)\sin(nx)=f(x)$

$$\Rightarrow\sum_{n=1}^{\infty}b_n''(t)\sin(nx)=\sum_{n=1}^{\infty}b_n(t)\cdot n^2\cdot(-\sin(nx))$$

$\Rightarrow b_n''(t)+n^2b_n(t)=0$ is the ordinary differential equation.

$$\Rightarrow b_n(t)=\alpha_n\cos(nt)+\beta_n\sin(nt)$$

$$y(x,0)=f(x)=\sum_{n=1}^{\infty}\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cdot\sin(nx)=\sum_{n=1}^{\infty}b_n(0)\sin(nx)$$

$\Rightarrow b_n(0)=\dfrac{0.8}{n^2\pi^2}\cdot\sin\left(\dfrac{n\pi}{2}\right)=\alpha_n$ (the initial condition), $\beta_n=0$

(c) $y(x,t)=\sum_{n=1}^{\infty}b_n(t)\sin(nx)=\sum_{n=1}^{\infty}\alpha_n\cos(nt)\cdot\sin(nx)$

$$=\sum_{n=1}^{\infty}\frac{0.8}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\cos(nt)\cdot\sin(nx).$$

6-3 Laplace Transform Solutions of Partial Differential Equations

Eg. Solve $\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial y}=y$**,** $\theta(x,0)=0$ **for** $x\geq 0$**, and** $\theta(0,y)=y$ **for** $y\geq 0$**. [1990 台大化工研]**

(Sol.) **Method 1: By Laplace transform**

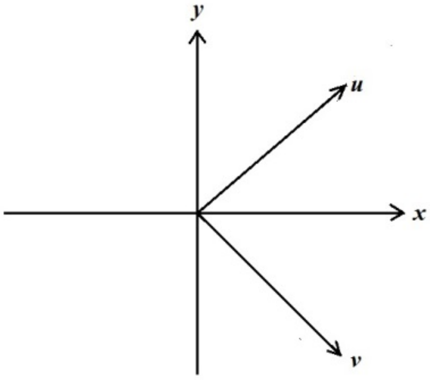
$$L[\theta(x,y)]=\int_0^{\infty}\theta(x,y)e^{-sy}dy=\Theta(x,s),\quad L[y]=\frac{1}{s^2}=L[\theta(0,y)]=\Theta(0,s)$$

$$\Rightarrow \frac{d\Theta(x,s)}{dx}+s\Theta(x,s)-\theta(x,0)=\frac{1}{s^2}\Rightarrow \Theta(x,s)=A(s)\cdot e^{-sx}+\frac{1}{s^3}$$

$$\Theta(0,s)=A(s)+\frac{1}{s^3}=\frac{1}{s^2}\Rightarrow A(s)=\frac{1}{s^2}-\frac{1}{s^3}\Rightarrow \Theta(x,s)=\left(\frac{1}{s^2}-\frac{1}{s^3}\right)e^{-sx}+\frac{1}{s^3}$$

$$\Rightarrow \theta(x,y)=\left[y-x-\frac{1}{2}(y-x)^2\right]\cdot u(y-x)+\frac{y^2}{2}$$

$$=\begin{cases}\frac{y^2}{2},\ y\leq x\\ y-x-\frac{1}{2}(y-x)^2+\frac{y^2}{2},\quad y\geq x\end{cases}$$



Method 2: Define $u=x+y$ and $v=x-y$, $\therefore x=\frac{u+v}{2}$, $y=\frac{u-v}{2}$,

we have $\frac{\partial\theta}{\partial x}=\frac{\partial\theta}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\theta}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial\theta}{\partial u}+\frac{\partial\theta}{\partial v}$

and $\frac{\partial\theta}{\partial y}=\frac{\partial\theta}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\theta}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial\theta}{\partial u}-\frac{\partial\theta}{\partial v}$

$$\frac{\partial\theta}{\partial x}+\frac{\partial\theta}{\partial y}=2\frac{\partial\theta}{\partial u}=y=\frac{u-v}{2},\quad \frac{\partial\theta}{\partial u}=\frac{u-v}{4},\ \therefore\ \theta(u,v)=\frac{1}{4}\left(\frac{u^2}{2}-uv\right)+C(v)$$

1° $\theta(x,0)=0$ for $x\geq 0$ $\Rightarrow$ $\theta\left(\frac{u+v}{2},\frac{u-v}{2}=0\right)=0$, $u=v$ and we have

$\theta=\frac{1}{4}\left(\frac{v^2}{2}-v^2\right)+C(v)=0$,

$\therefore C(v)=\frac{v^2}{8}$ for $v\geq 0$ or $x\geq y$ $\Rightarrow \theta=\frac{1}{4}\left(\frac{u^2}{2}-uv\right)+\frac{v^2}{8}=\frac{1}{8}(u-v)^2=\frac{y^2}{2}$ for $x\geq y$

2° $\theta(0,y)=y$ for $y\geq 0\Rightarrow$ $\theta\left(\frac{u+v}{2}=0,\frac{u-v}{2}\right)=\frac{u-v}{2}$, $u=-v$ and we have

$\theta=\frac{1}{4}\left(\frac{v^2}{2}+v^2\right)+C(v)=-v$, $\therefore C(v)=-v-\frac{3v^2}{8}$ for $v\leq 0$ or $x\leq y$

$\Rightarrow \theta=\frac{1}{4}\left(\frac{u^2}{2}-uv\right)-v-\frac{3v^2}{8}=\frac{1}{8}(u-v)^2-v-\frac{v^2}{2}=\frac{y^2}{2}+(y-x)-\frac{(x-y)^2}{2}$ for $x\leq y$

Note: This partial differential equation cannot be solved by separation of variables.

Eg. Solve $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial^2 u}{\partial t^2}$**, $u(0,t)=u(1,t)=u(x,0)=0$, and** $\dfrac{\partial u}{\partial t}_{|t=0}=\sin(\pi x)$**.**

(Sol.) $L[u(x,t)]=\int_0^\infty u(x,t)e^{-st}dt=U(x,s)$

$$\frac{d^2U(x,s)}{dx^2}=s^2U(x,s)-su(x,0)-\frac{\partial u}{\partial t}_{|t=0}=s^2U(x,s)-\sin(\pi x)$$

$$\Rightarrow \frac{d^2U(x,s)}{dx^2}-s^2U(x,s)=-\sin(\pi x)\Rightarrow U(x,s)=c_1e^{sx}+c_2e^{-sx}+\frac{\sin\pi x}{s^2+\pi^2}$$

$$\begin{cases}u(0,t)=0\\u(1,t)=0\end{cases}\Rightarrow\begin{cases}U(0,s)=0\\U(1,s)=0\end{cases}\Rightarrow\begin{cases}c_1=0\\c_2=0\end{cases}\Rightarrow u(x,t)=\frac{1}{\pi}\sin(\pi x)\cdot\sin(\pi t)$$

Eg. Solve $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2}$**, $u(x,0)=3\sin(2\pi x)$, $u(0,t)=u(1,t)=0$, $0<x<1$, $t\geq 0$.**

(Sol.) $L[u(x,t)]=\int_0^\infty u(x,t)e^{-st}dt=U(x,s)\Rightarrow sU(x,s)-u(x,0)=\dfrac{d^2}{dx^2}U(x,s)$

$$\frac{d^2}{dx^2}U(x,s)-sU(x,s)=-3\sin(2\pi x)\Rightarrow U(x,s)=c_1e^{-\sqrt{s}x}+c_2e^{\sqrt{s}x}+\frac{3}{s+4\pi^2}\cdot\sin(2\pi x)$$

$$L[u(0,t)]=U(0,s)=0,\quad L[u(1,t)]=U(1,s)=0\Rightarrow c_1=0,\quad c_2=0$$

$$U(x,s)=\frac{3}{s+4\pi^2}\cdot\sin(2\pi x)\Rightarrow u(x,t)=L^{-1}\left[\frac{3}{s+4\pi^2}\cdot\sin(2\pi x)\right]=3e^{-4\pi^2t}\cdot\sin(2\pi x)$$

6-4 Fourier Transform Solutions of Partial Differential Equations

Eg. Solve $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$**,** $u(x,0)=e^{-x^2}$**,** $-\infty<x<\infty$, $t>0$. **[2011 成大電研、2001 台大電研類似題]**

(Sol.) $\Im[u(x,t)]=U(\omega,t)$, $\Im\left[\frac{\partial^2 u(x,t)}{\partial x^2}\right]=-\omega^2 U(\omega,t)$

$$\frac{d}{dt}U(\omega,t)=-\omega^2 U(\omega,t)\Rightarrow U(\omega,t)=Ae^{-\omega^2 t}$$

According to $\Im[e^{-a^2x^2}]=\frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}}$,

$$U(\omega,0)=A=\Im[u(x,0)]=\Im[e^{-x^2}]=\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}\Rightarrow A=\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}$$

Let $b^2=t+1/4$ and according to $\Im^{-1}[e^{-b^2\omega^2}]=\frac{e^{-\frac{x^2}{4b^2}}}{2b\sqrt{\pi}}$,

$$\Rightarrow u(x,t)=\Im^{-1}[U(\omega,t)]=\Im^{-1}[\sqrt{\pi}\cdot e^{-\frac{\omega^2}{4}}e^{-\omega^2 t}]=\sqrt{\pi}\cdot\Im^{-1}[e^{-(t+\frac{1}{4})\omega^2}]=\frac{e^{-\frac{x^2}{1+4t}}}{\sqrt{1+4t}}$$

Eg. Solve $\frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}$**,** $u(x,0)=4e^{-5|x|}$**,** $\frac{\partial u(x,0)}{\partial t}=0$**,** $-\infty<x<\infty$, $t>0$.

(Sol.) $\Im[u(x,t)]=U(\omega,t)$, $\Im\left[9\frac{\partial^2 u(x,t)}{\partial x^2}\right]=-9\omega^2 U(\omega,t)$

$$\frac{d^2}{dt^2}U(\omega,t)=-9\omega^2 U(\omega,t)\Rightarrow U(\omega,t)=A\cos(3\omega t)+B\sin(3\omega t)$$

$$\frac{d}{dt}U(\omega,t)=-3\omega A\sin(3\omega t)+3\omega B\cos(3\omega t)$$

According to $\Im\left[e^{-a|x|}\right]=\frac{2a}{a^2+\omega^2}$, $U(\omega,0)=A=\Im[u(x,0)]=\Im[4e^{-5|x|}]=\frac{40}{25+\omega^2}$

$\Im\left[\frac{\partial u(x,0)}{\partial t}\right]=\frac{d}{dt}U(\omega,0)=0=3\omega B\Rightarrow B=0$, $\therefore$ $U(\omega,t)=\frac{40}{\omega^2+25}\cdot\cos(3\omega t)$

By $\Im^{-1}[e^{ja\omega}F(\omega)]=f(x+a)$ and $\Im^{-1}[e^{-ja\omega}F(\omega)]=f(x-a)$,

$$u(x,t)=\Im^{-1}[U(\omega,t)]=\Im^{-1}[\frac{40}{\omega^2+25}\cdot\cos(3\omega t)]=\Im^{-1}[\frac{40}{\omega^2+25}\cdot\frac{e^{i3\omega t}+e^{-i3\omega t}}{2}]$$

$$=\Im^{-1}[\frac{20}{\omega^2+25}\cdot(e^{i3\omega t}+e^{-i3\omega t})]=2\Im^{-1}[\frac{2\cdot 5}{\omega^2+25}\cdot(e^{i3\omega t}+e^{-i3\omega t})]=2e^{-5|x-3t|}+2e^{-5|x+3t|}$$

6-5 Miscellaneous Methods of Solving Partial Differential Equations

Error function: $erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$

Complementary error function: $erfc(x) = 1 - erf(x)$

Eg. Solve $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$, $u(0,y)=0$ and $u(x,1)=x^2$.

(Sol.) Set $u(x,y)=f(y+mx)$ and $\varepsilon \equiv y+mx$ $\Rightarrow$ $m^2 f''(\varepsilon)+2mf''(\varepsilon)+f''(\varepsilon)=0$, $m^2+2m+1=0$, $m=-1, -1$

$\therefore$ $u(x,y)=f(y-x)+xg(y-x)$. $u(0,y)=0= f(y-0)+0\cdot g(y-0)=f(y)$ $\Rightarrow$ $f(1-x)=0$ and $f(y-x)=0$

$u(x,1)=x^2=f(1-x)+xg(1-x)=xg(1-x)$ $\Rightarrow$ $g(1-x)=x$, $g(x)=1-x$ $\Rightarrow$ $g(y-x)=1-y+x$

$$\therefore\ u(x,y)=x\cdot(1-y+x)$$

Eg. Solve $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} + u$. [2015 台大電子所工數 K]

(Sol.) $u(x,y) = X(x)Y(y)$, $[X''(x)+2X'(x)]\cdot Y(y) = X(x)\cdot[Y'(y)+Y(y)]$,

$\frac{X''(x)+2X'(x)}{X(x)} = \frac{Y'(y)+Y(y)}{Y(y)} = \lambda \Rightarrow X''(x)+2X'(x)-\lambda X(x)=0$ and $Y'(y)+(1-\lambda)Y(y)=0$

$X''(x)+2X'(x)-\lambda X(x)=0$, $r^2+2r-\lambda=0$, $r=-1\pm\sqrt{1+\lambda}$

1° If $1+\lambda>0$, $\sqrt{1+\lambda}=k \in R$, $\lambda=k^2-1$ and $r=-1\pm k \Rightarrow X(x)=Ae^{(-1+k)x}+Be^{(-1-k)x}$, $1-\lambda=2-k^2$,

$Y'(y)+(1-\lambda)Y(y)=0 \Rightarrow Y(y)=C'e^{-(1-\lambda)y}=C'e^{(k^2-2)y}$

$\Rightarrow u(x,y)= e^{-x-2y}\cdot[\int_0^\infty C(k)e^{-kx+k^2 y}dk + \int_0^\infty D(k)e^{kx+k^2 y}dk]$

2° If $1+\lambda=0$, $\lambda=-1$ and $r=-1, -1 \Rightarrow X(x)=Ae^{-x}+Bxe^{-x}$, $Y'(y)+2Y(y)=0 \Rightarrow Y(y)=C'e^{-2y}$

$\Rightarrow u(x,y)= e^{-x-2y}\cdot(C+Dx)$

3° If $1+\lambda<0$, $\sqrt{1+\lambda}=\pm i\omega$, $r=-1\pm i\omega \Rightarrow X(x)=Ae^{-x}\cos(\omega x)+Be^{-x}\sin(\omega x)$,

$\lambda=-\omega^2-1$, $1-\lambda=2+\omega^2$, $Y'(y)+(2+\omega^2)Y(y)=0$, $Y(y)=C'e^{(-\omega^2-2)y}$

$\Rightarrow u(x,y)= e^{-x-2y}\cdot[\int_0^\infty C(\omega)e^{-\omega^2 y}\cos(\omega x)d\omega + \int_0^\infty D(\omega)e^{-\omega^2 y}\sin(\omega x)d\omega]$