Chapter 7

Impulse and Momentum

1) Linear momentum

$\mathbf{p} = m\mathbf{v}$ (units: kg m / s)

2) Impulse *(produces a finite change in momentum)*

(a) Constant force:

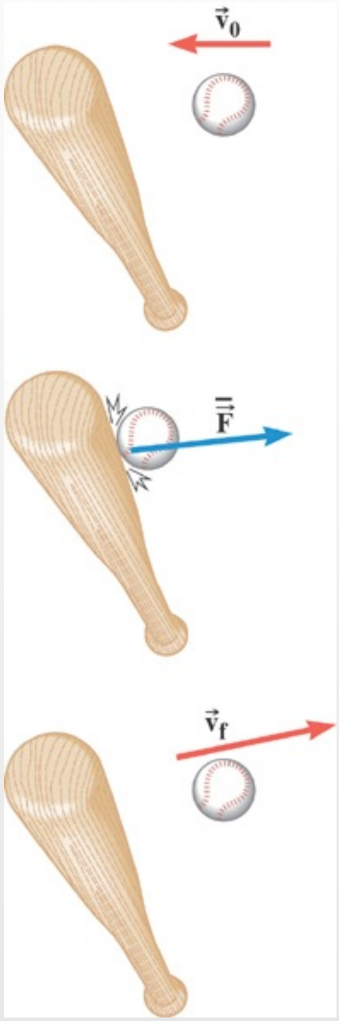
$$\vec{J} = \vec{F}\Delta t$$

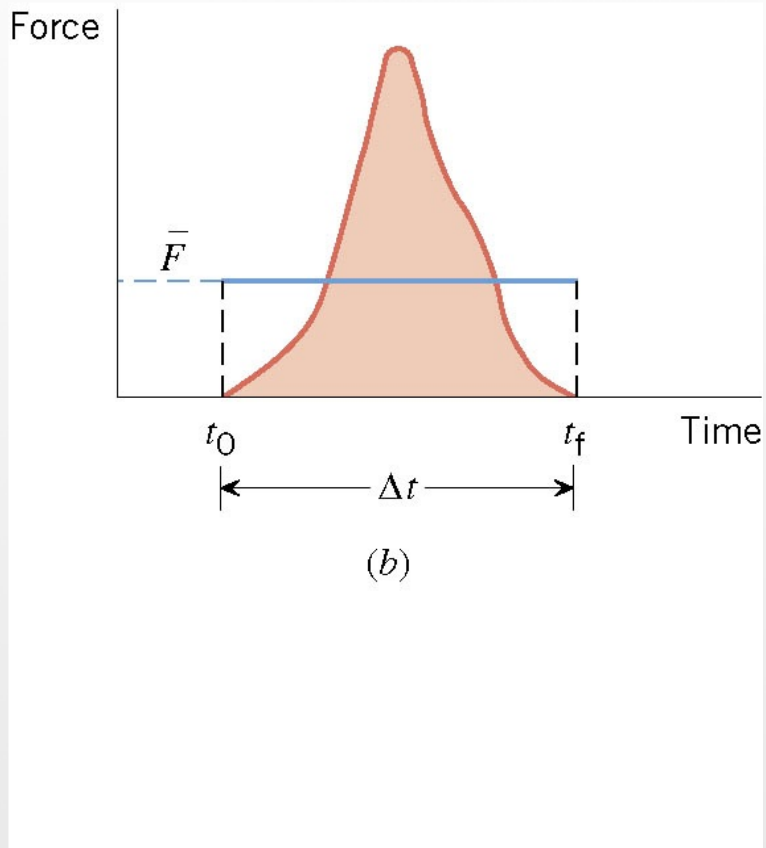
From the 2nd law, $\vec{F} = \dfrac{\Delta(m\vec{v})}{\Delta t} = \dfrac{\Delta\vec{p}}{\Delta t}$, so

$$\boxed{\vec{J} = \vec{F}\Delta t = \Delta\vec{p}}$$

(Units: Ns)

(b) Variable force:





If the force is not constant, use the average force

$$\boxed{\vec{J} = \overline{\vec{F}}\Delta t = \Delta\vec{p}}$$

Impulse-momentum theorem

$$\boxed{\vec{J} = \overline{\vec{F}}\Delta t = \Delta\vec{p}}$$

Impulse

Change in momentum

Example

C&J 7.9 A space probe is traveling in outer space with a momentum that has a magnitude of 7.5×10^7 kg·m/s. A retrorocket is fired to slow down the probe. It applies a force to the probe that has a magnitude of 2.0×10^6 N and a direction opposite to the probe's motion. It fires for a period of 12 s. Determine the momentum of the probe after the retrorocket ceases to fire.

3) Conservation of Momentum (and Newton's laws)

Second law: $$\vec{F} = \frac{\Delta \vec{p}}{\Delta t}$$

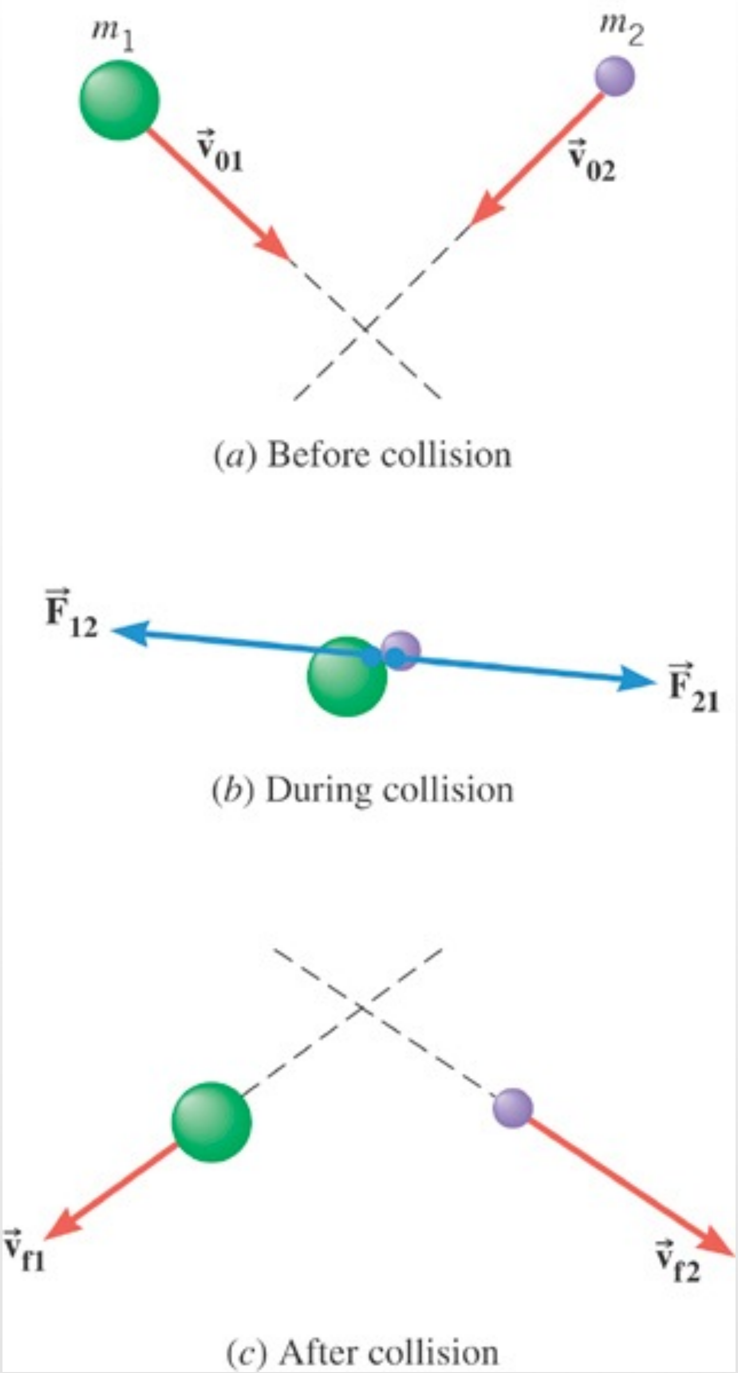
Superposition: $$\vec{F}_{sys} = \frac{\Delta \vec{P}_{sys}}{\Delta t}$$

If the net external force on a system is zero:

$$\frac{\Delta \vec{P}_{sys}}{\Delta t} = 0 \qquad \Rightarrow \boxed{\vec{P}_{sys} = \text{const}}$$

$$\vec{P}_{sys} = \text{const}$$

- Momentum of an isolated system is constant
- Always conserved; *cannot be randomized (internalized) like energy*



$$\vec{P}_0 = m_1\vec{v}_{01} + m_2\vec{v}_{02}$$

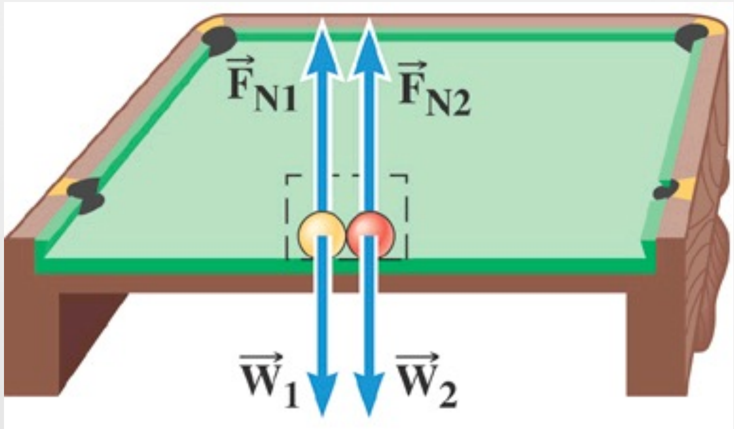
Internal forces are equal and opposite, and do not change momentum of the system.

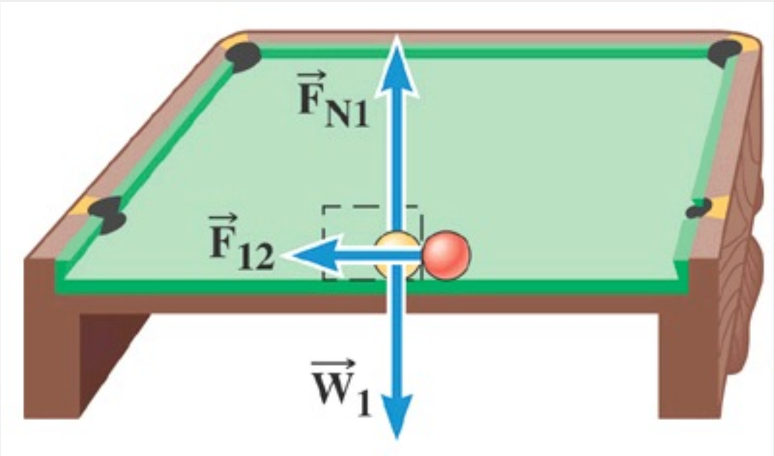
$$\vec{P}_f = m_1\vec{v}_{f1} + m_2\vec{v}_{f2}$$

$$\boxed{\vec{P}_f = \vec{P}_0}$$

Example

Imagine two balls colliding on a billiard table that is friction-free. Use the momentum conservation principle in answering the following questions. (a) Is the total momentum of the two-ball system the same before and after the collision? (b) Answer part (a) for a system that contains only one of the two colliding balls.

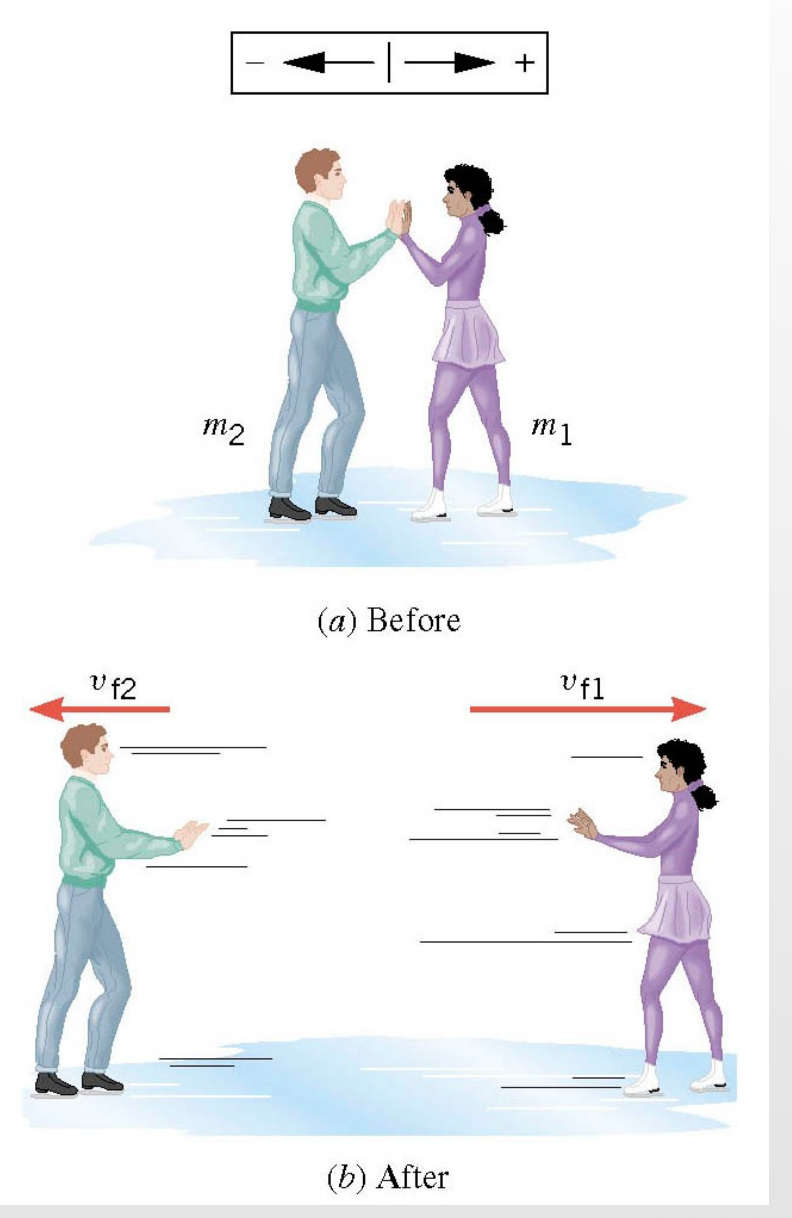




Example Ice Skaters

Starting from rest, two skaters push off against each other on ice where friction is negligible.

One is a 54-kg woman and one is a 88-kg man. The woman moves away with a speed of +2.5 m/s. Find the recoil velocity of the man.



(*a*) Before

(*b*) After

Applying the Principle of Conservation of Linear Momentum

1. Decide which objects are included in the system.

2. Relative to the system, identify the internal and external forces.

3. Verify that the system is isolated.

4. Set the final momentum of the system equal to its initial momentum.
Remember that momentum is a vector.

Example

For tests using a ballistocardiograph, a patient lies on a horizontal platform that is supported on jets of air. Each time the heart beats, blood is pushed out from the heart in a direction that is nearly parallel to the platform. The body and platform recoil, and this recoil can be detected. Suppose that 0.050 kg of blood is pushed out of the heart with a velocity of 0.25 m/s and that the mass of the patient + platform is 85 kg. Assuming that the patient does not slip wrt the platform, and that the patient and platform start from rest, determine the recoil velocity of the platform.

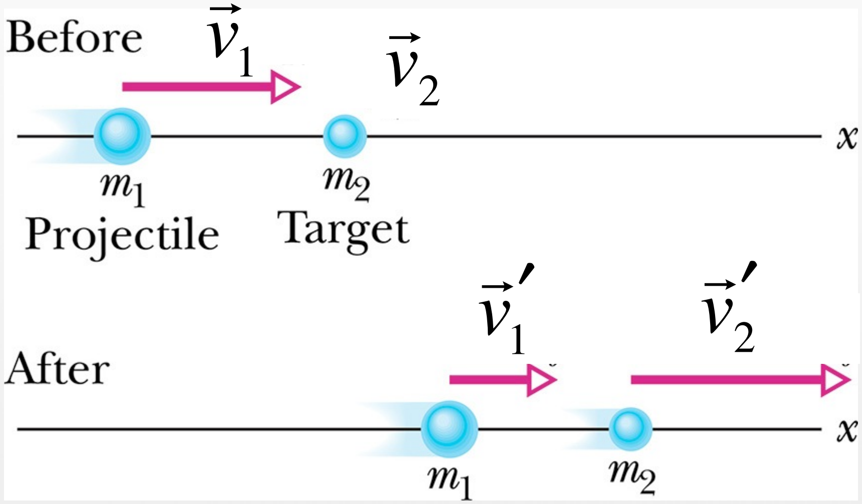
Example (Homework)

Two friends, Al and Jo, have a combined mass of 168 kg.

At an ice skating rink they stand close together on skates, at rest and facing each other, with a compressed spring between them. The spring is kept from pushing them apart because they are holding each other.

When they release their arms, Al moves off in one direction at a speed of 0.90 m/s, while Jo moves off in the opposite direction at a speed of 1.2 m/s. Assuming that friction is negligible, find Al's mass.

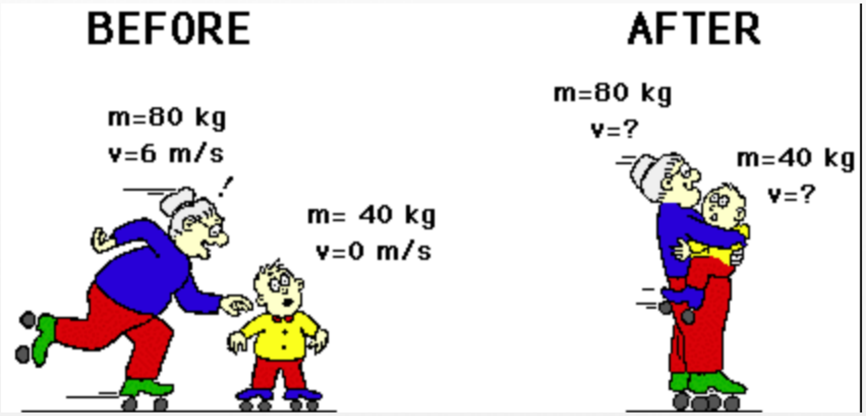
4) 1-d collisions



$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$

One equation, two unknowns;
initial conditions are not enough (even in 1d)

a) Completely inelastic collisions



$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$

$$\vec{v}_1' = \vec{v}_2' = \vec{v}'$$

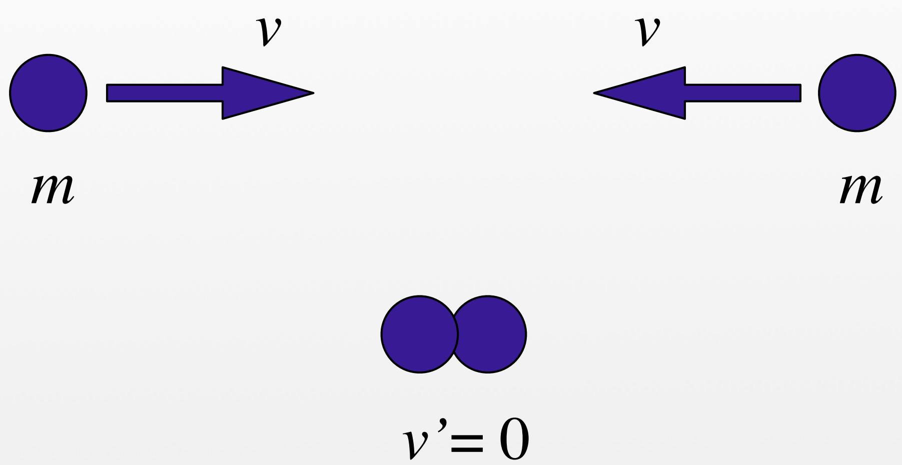
Example

$m_1 = 1.0$ kg, $v_{1x} = 10$ m/s

$m_2 = 2.0$ kg, $v_{2x} = -8$ m/s

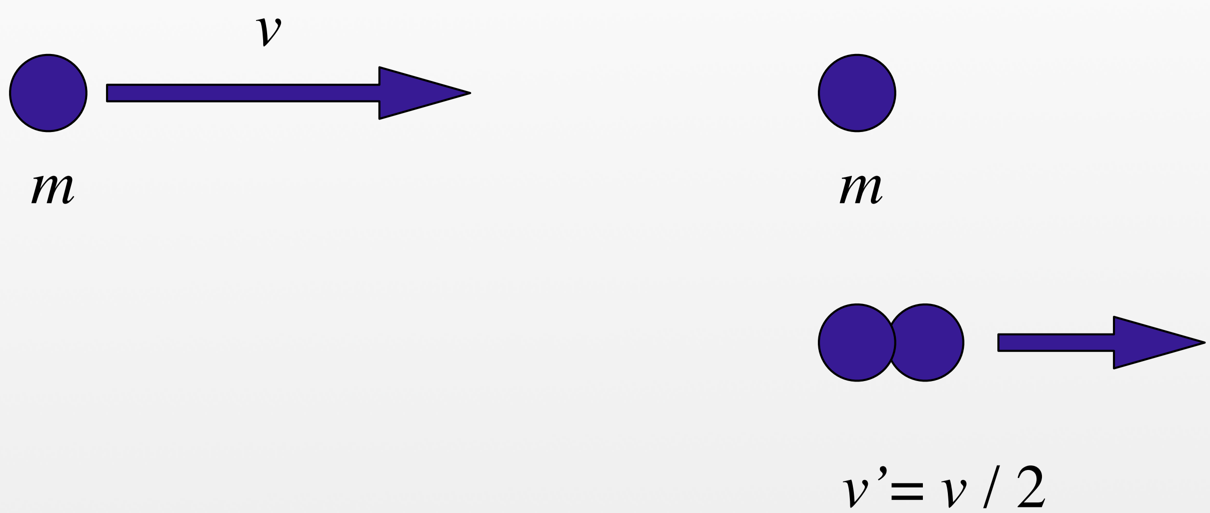
If they stick together, find final velocity:

- Equal masses, opposite momenta



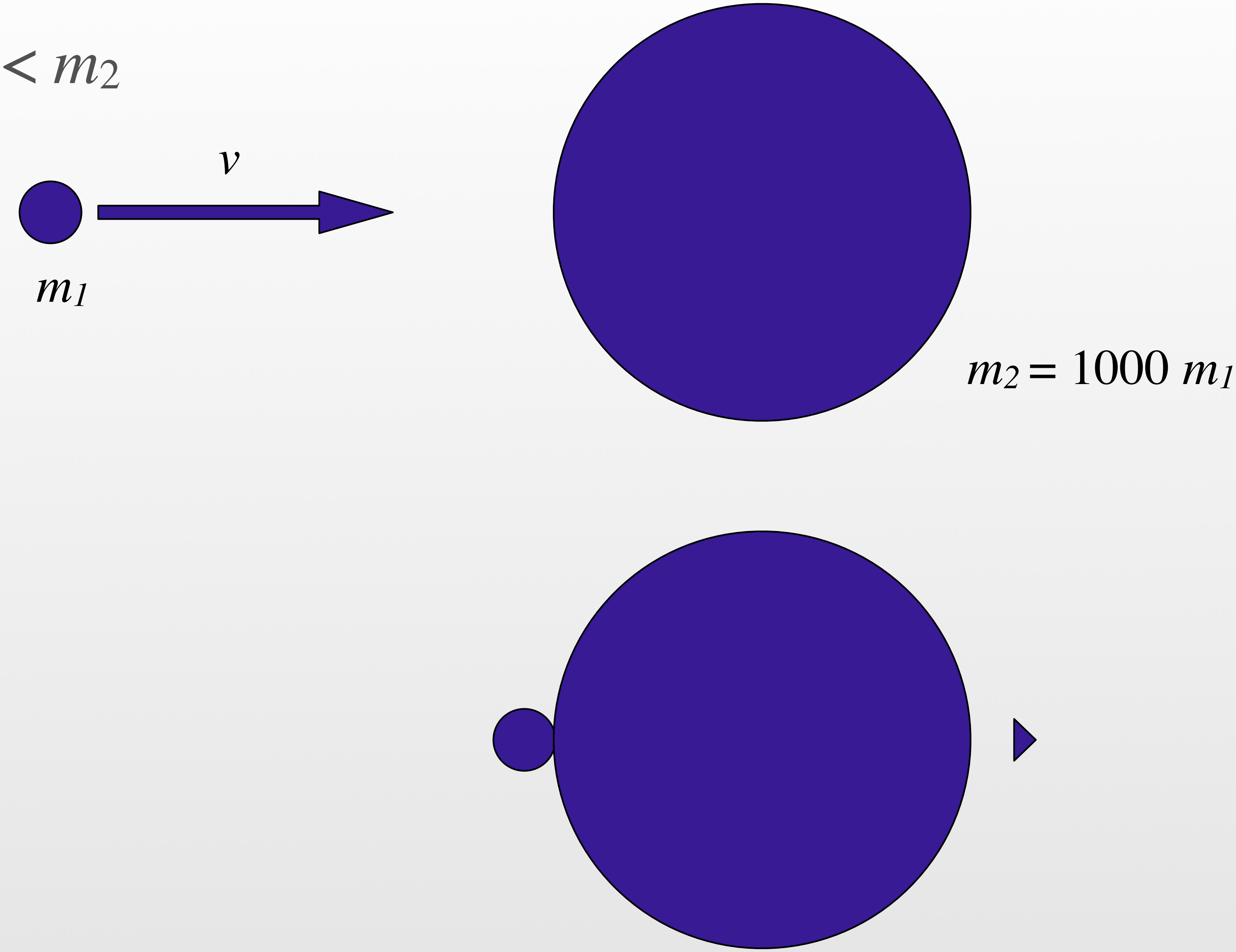
All kinetic energy lost
Opposite of explosion

- Equal masses, one at rest



1/2 kinetic energy is lost

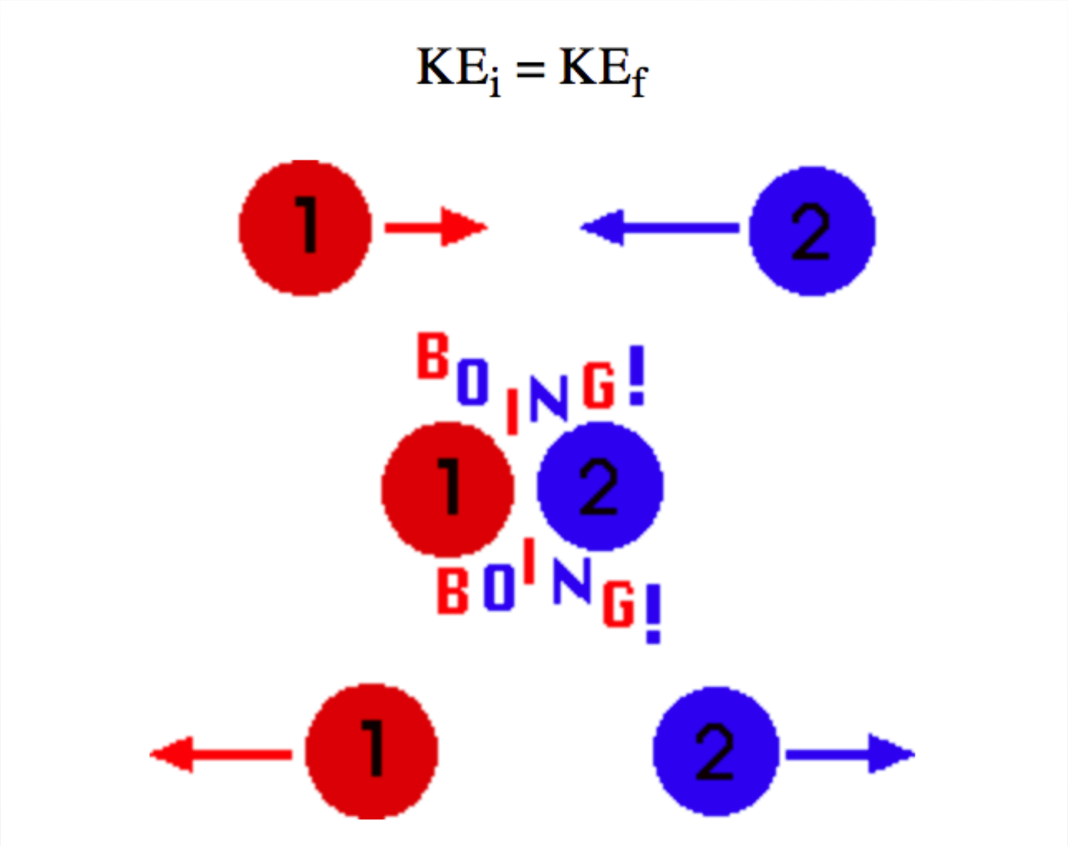
- $m_1 << m_2$



$$v' = \frac{m_1}{m_1 + m_2} v$$

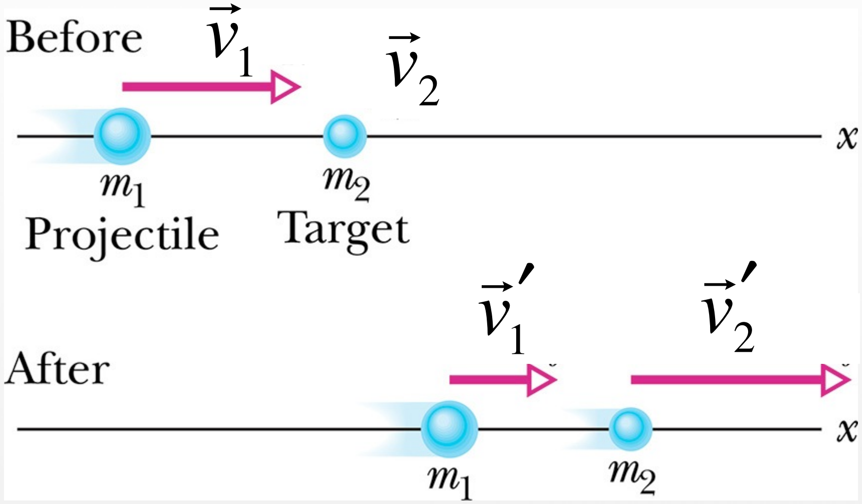
1 thousandth of KE survives

b) Completely elastic collisions



KE conserved

Elastic collision soluble:



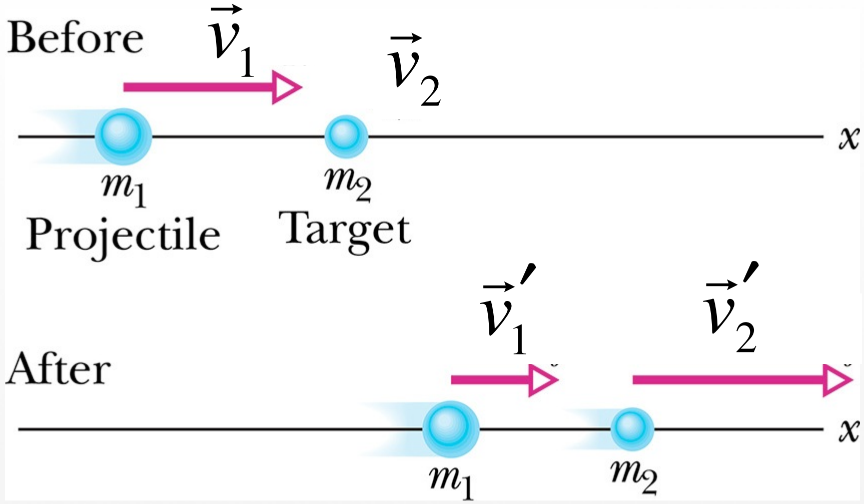
Conservation of momentum:

$$m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1' + m_2\vec{v}_2'$$

Conservation of energy:

$$\tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_1v_1'^2 + \tfrac{1}{2}m_2v_2'^2$$

Two equations, two unknowns (v_1', v_2')



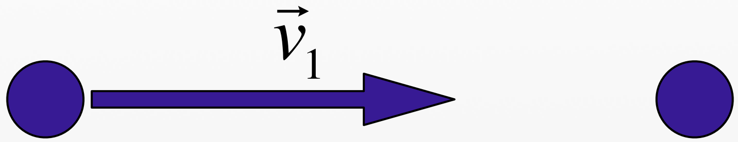
Result for elastic collision in one dimension with $v_2 = 0$.

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1$$

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1$$

(x-components, not magnitudes)

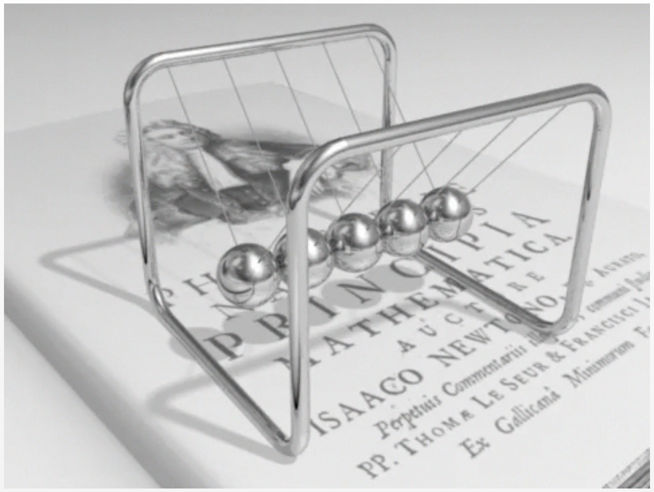
- *elastic collision with* $m_1 = m_2$, $v_2 = 0$



$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 = 0$$

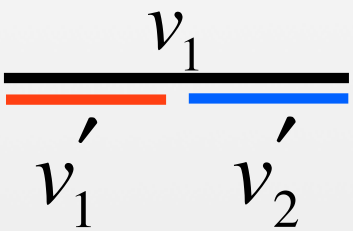
$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 = v_1$$

Newton's cradle



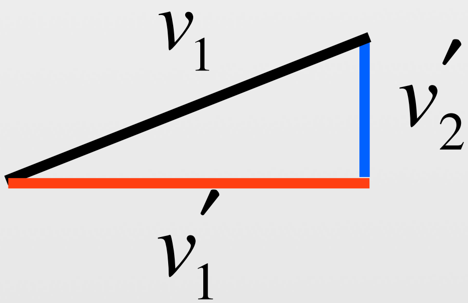
Conservation of momentum:

$$v_1 = v_1' + v_2'$$



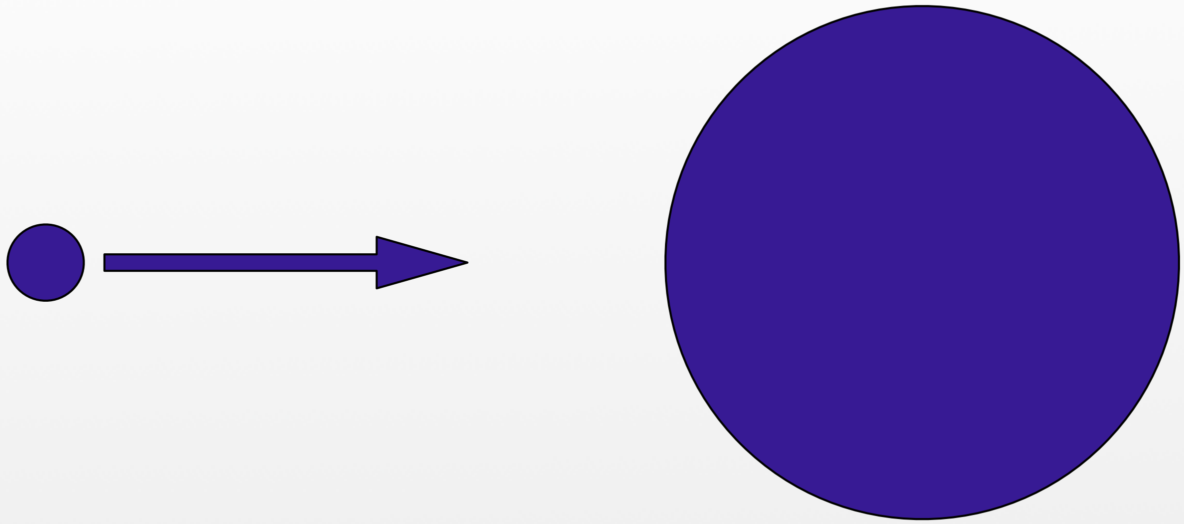
Conservation of mech energy:

$$v_1^2 = v_1'^2 + v_2'^2$$



For both to be true, one of the final velocities must be zero.

- *elastic collision with* $m_1 << m_2$, $v_2 = 0$

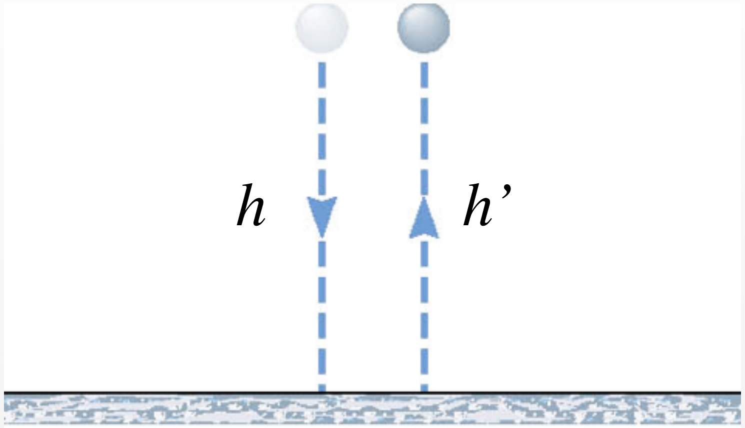


$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \cong -v_1$$

$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 \cong 0$$

Larger mass acquires negligible KE

- Bouncing ball (floor has infinite mass)



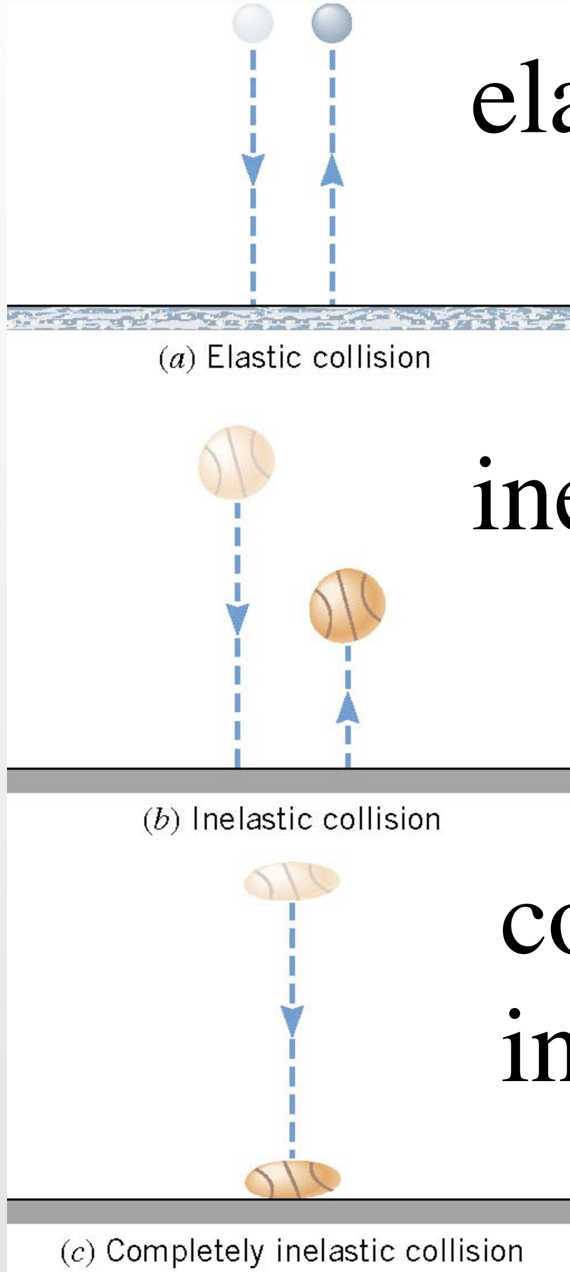
(a) Elastic collision

Energy conserved ==> $mgh = mgh'$

If $h' < h$, some mechanical energy lost in the collision

Velocities: $v = \sqrt{2gh}$ $v' = \sqrt{2gh'}$

$$\frac{h'}{h} = \left(\frac{v'}{v}\right)^2$$



(a) Elastic collision

(b) Inelastic collision

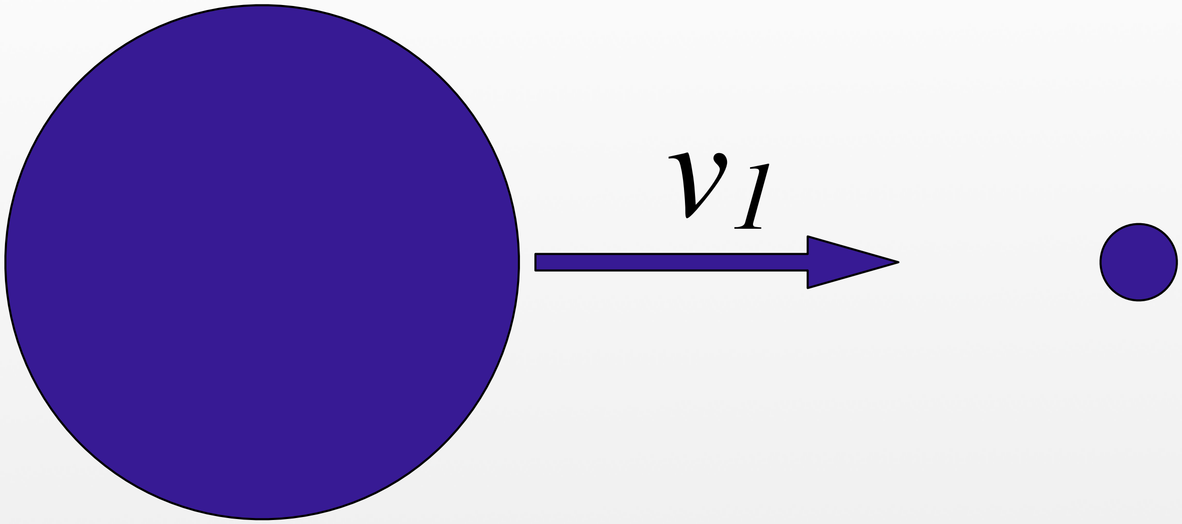
(c) Completely inelastic collision

elastic

inelastic

completely inelastic

- *elastic collision with* $m_1 >> m_2$, $v_2 = 0$



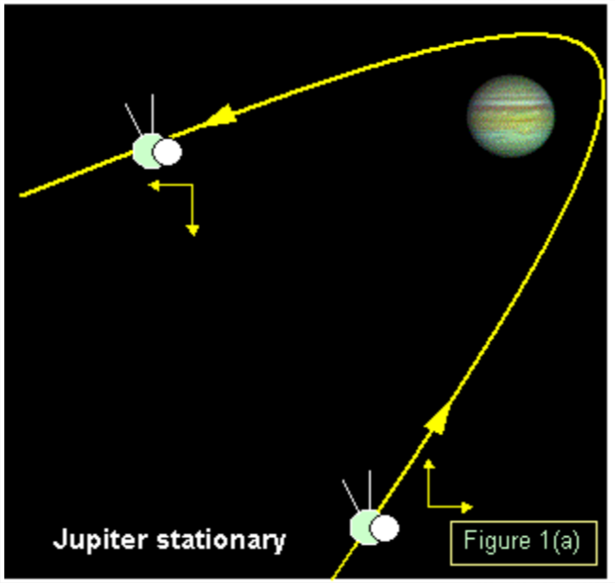
$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \cong v_1$$

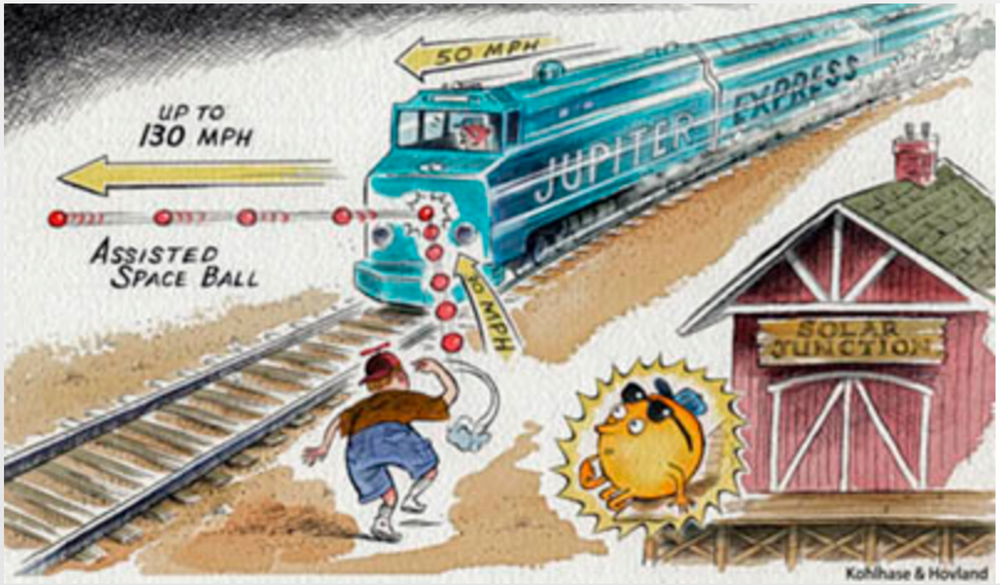
$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 \cong 2v_1$$

Example

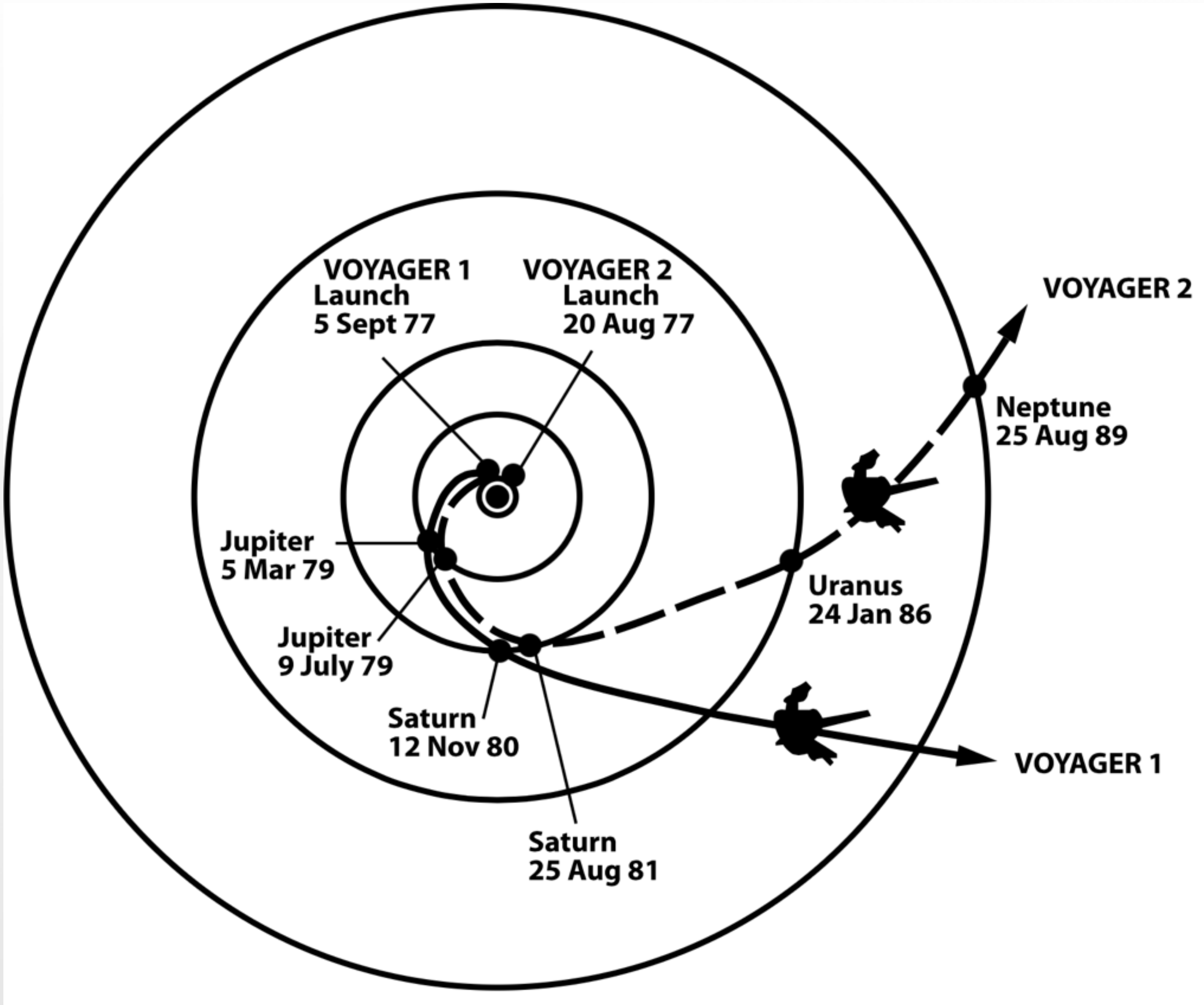
- Sling-shot effect

  If a small mass with speed v collides elastically with a large mass at speed V, find the final speed of the small mass.





Answer: $v + 2V$



The trajectories that enabled NASA's twin Voyager spacecraft to tour the four gas giant planets and achieve velocity to escape our solar system (http://en.wikipedia.org/wiki/Slingshot_effect)

Example

If a small ball is dropped with and above a larger ball from a height h, and all collisions are elastic, how high does the smaller ball rebound?

A) h
B) $2h$
C) $3h$
D) $6h$
E) $9h$

Example

A 1055 kg van, stopped at a traffic light, is hit directly in the rear by a 715 kg car traveling with a velocity of +2.25 m/s. Assume that the transmission of the van is in neutral, the brakes are not being applied and that the collision is elastic. What is the final velocity of the car and the van?

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 \qquad = -0.43 \text{ m/s}$$

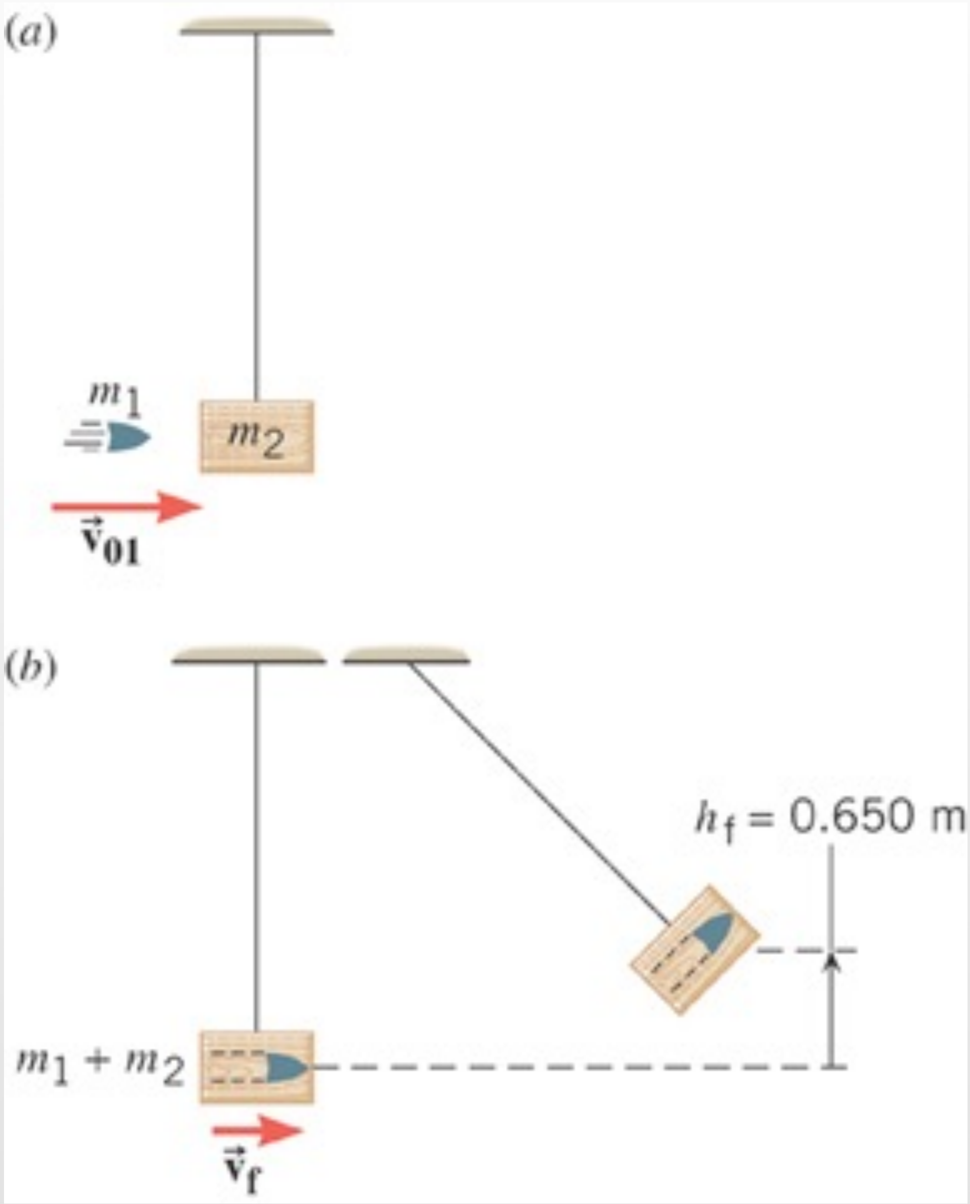
$$v_2' = \frac{2m_1}{m_1 + m_2} v_1 \qquad = 1.82 \text{ m/s}$$

Example

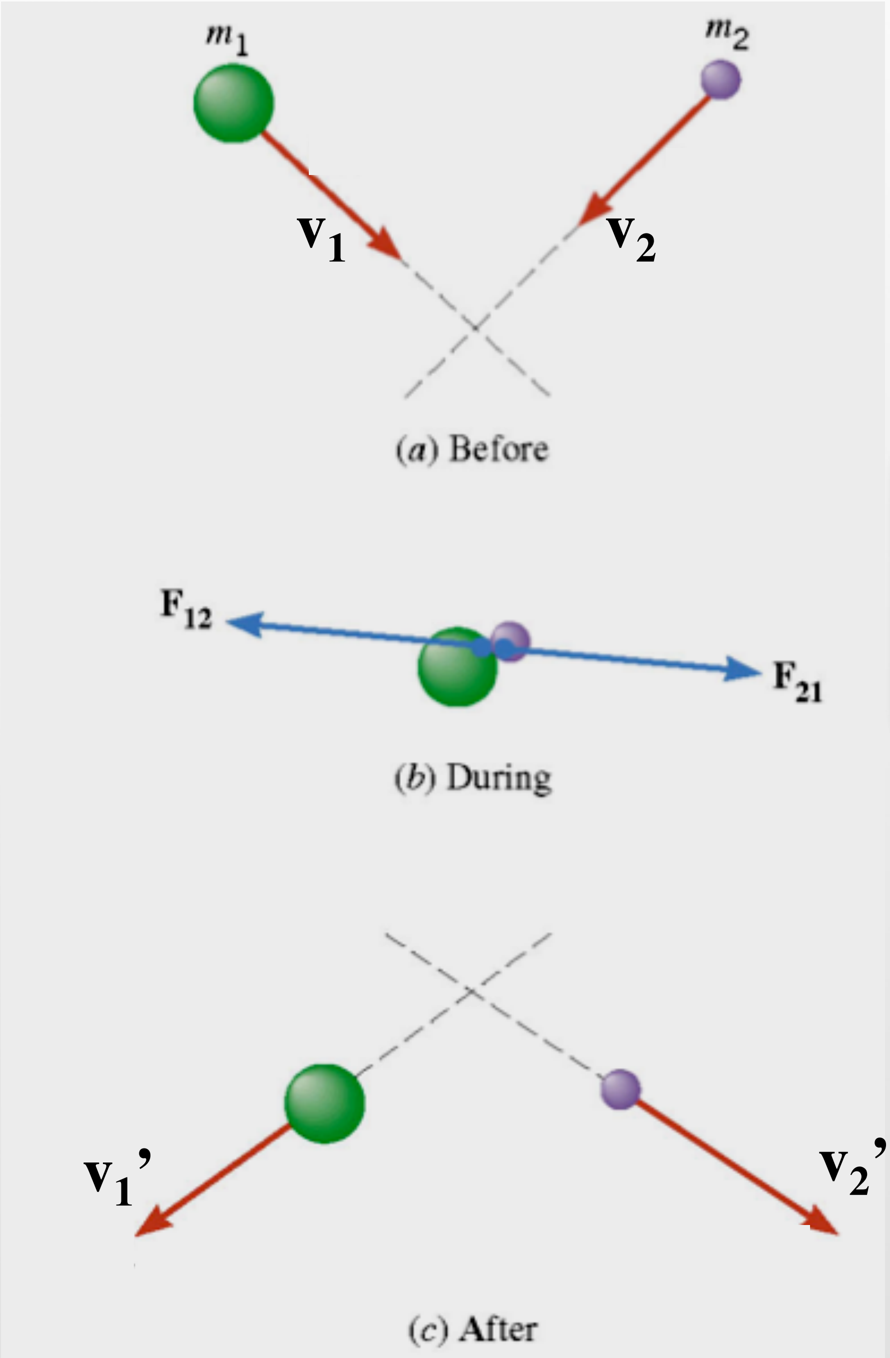
Ballistic Pendulum

The mass of the block of wood is 2.50-kg and the mass of the bullet is 0.0100-kg. The block swings to a maximum height of 0.650 m above the initial position.

Find the initial speed of the bullet.



6) Conservation of momentum in 2d



$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'$$

$$\boxed{m_1\vec{v}_1 + m_2\vec{v}_2 = m_1\vec{v}_1{}' + m_2\vec{v}_2{}'}$$

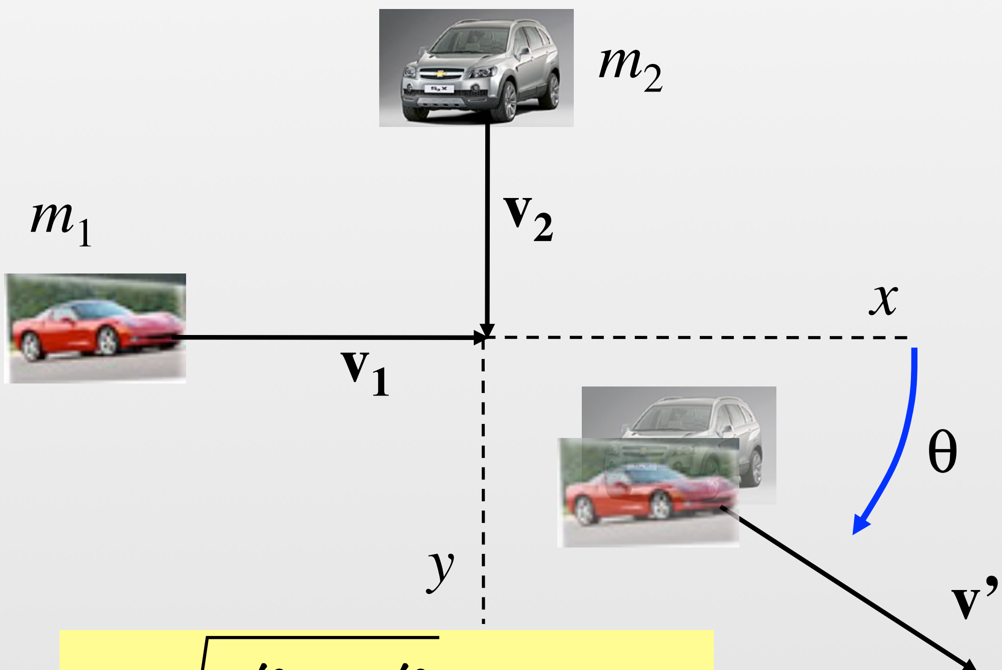
$$m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}{}' + m_2 v_{2x}{}'$$

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v_{1y}{}' + m_2 v_{2y}{}'$$

If initial conditions are known, this gives 2 equations, with 4 unknowns, so more information is needed.

*a) **Inelastic collision**:* $\mathbf{p}_1' = \mathbf{p}_2' = \mathbf{p}'$ reduces unknowns to p_x' and p_y'

Example: Find $\mathbf{v}'$ if $m_1 = 1450$ kg, $m_2 = 1750$ kg, $v_1 = 11.5$ m/s, $v_2 = 15.5$ m/ss



- conservation of x momentum

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v_{1x}' + m_2 v_{2x}'$$

$$m_1 v_{1x} + 0 = (m_1 + m_2) v_x'$$

$$\boxed{v_x' = \frac{m_1}{m_1 + m_2} v_{1x} = 5.21 \text{ m/s}}$$

- conservation of y momentum

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v_{1y}' + m_2 v_{2y}'$$

$$0 + m_2 v_{2y} = (m_1 + m_2) v_y'$$

$$\boxed{v_y' = \frac{m_2}{m_1 + m_2} v_{2y} = 8.48 \text{ m/s}}$$

$$v' = \sqrt{v_x'^2 + v_y'^2} = 9.95 \text{ m/s}$$

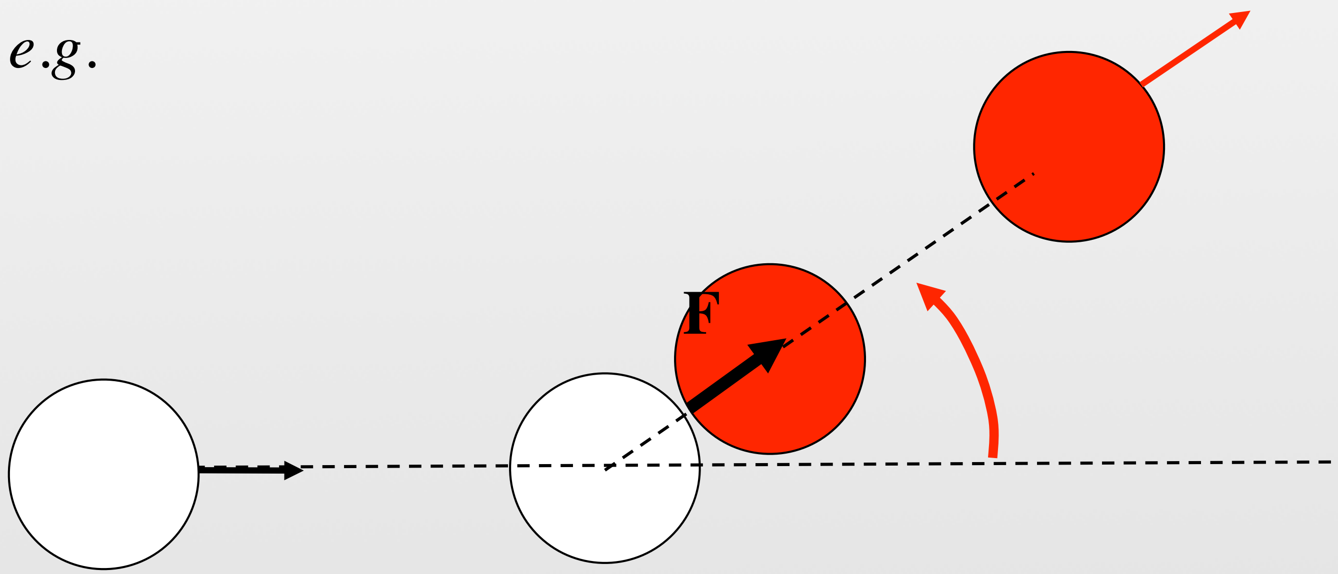
$$\tan\theta = \frac{v_y'}{v_x'} \rightarrow \theta = 58.4^\circ$$

b) ***Elastic collision****:* Energy conservation adds 3rd equation:

$$\tfrac{1}{2}mv_1^{\,2} + \tfrac{1}{2}mv_2^{\,2} = \tfrac{1}{2}mv_1'^{\,2} + \tfrac{1}{2}mv_2'^{\,2}$$

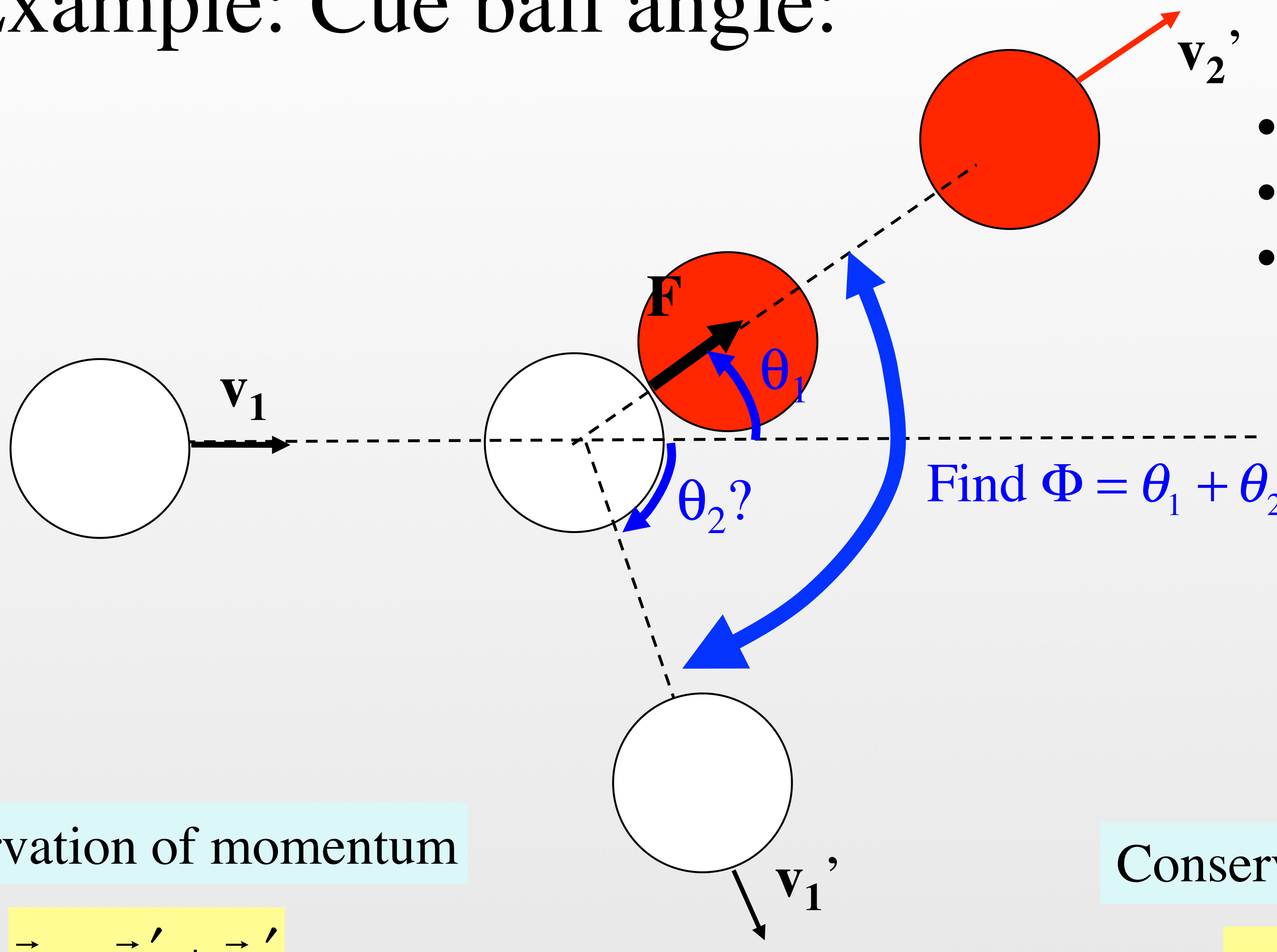
The last condition is determined by the shape & location of impact:

e.g.



For a billiard ball collision, the angle of the object ball is determined by the line through the centres at the point of contact.

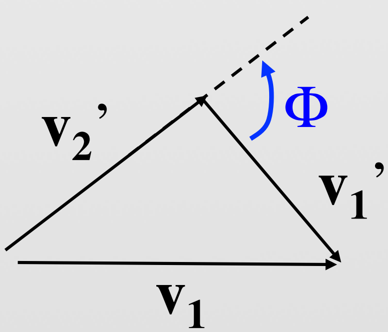
Example: Cue ball angle:



- identical masses
- elastic collision
- m_2 initially at rest

Conservation of momentum

$$\vec{v}_1 = \vec{v}_1{}' + \vec{v}_2{}'$$



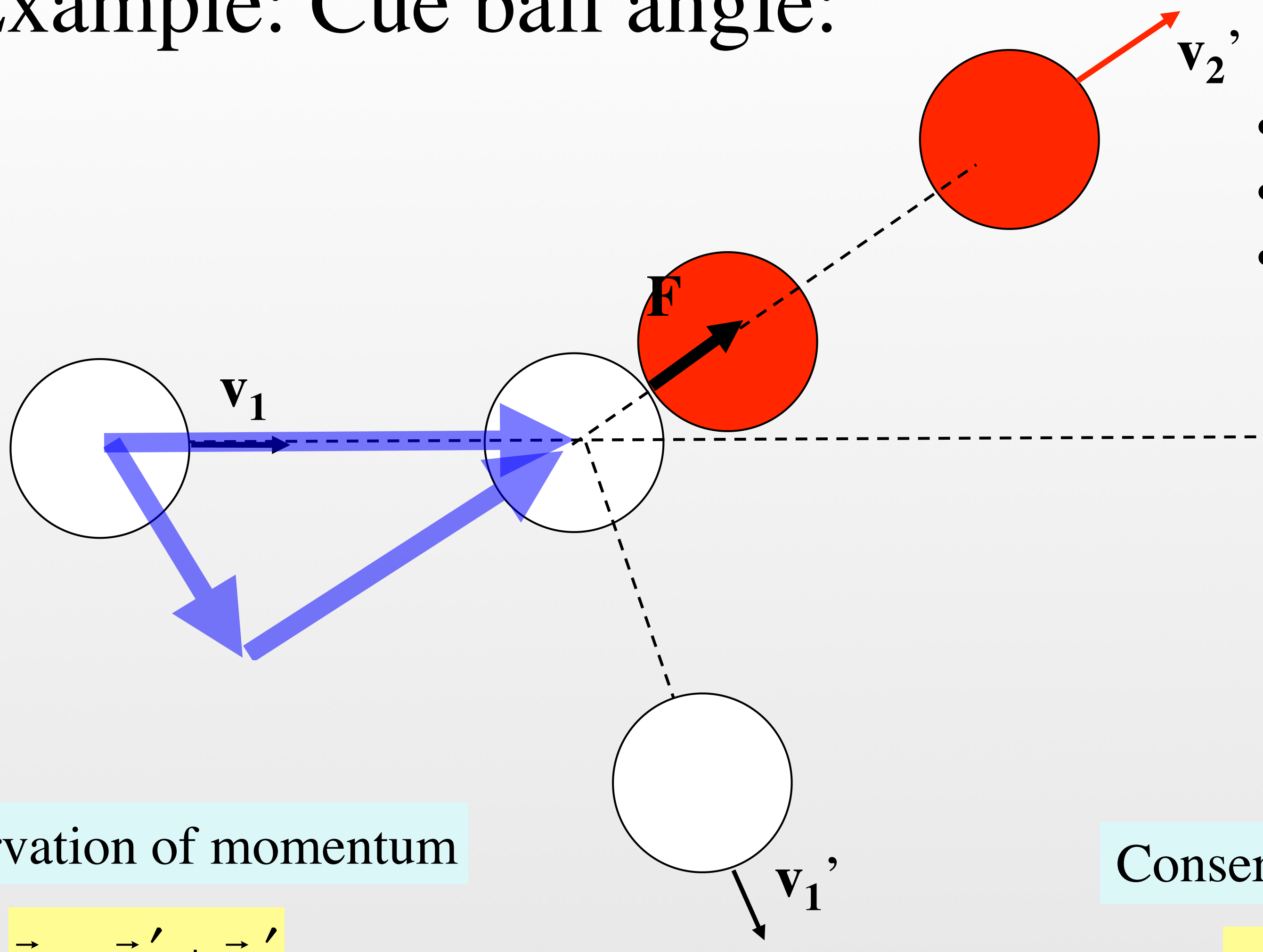
Conservation of energy

$$v_1{}^2 = v_1'^2 + v_2'^2$$

Therefore, by Pythagoras

$$\Phi = 90^\circ$$

Example: Cue ball angle:



- identical masses
- elastic collision
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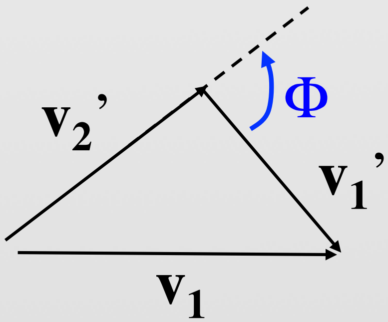

Conservation of energy

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Therefore, by Pythagoras

$$\Phi = 90^{\circ}$$

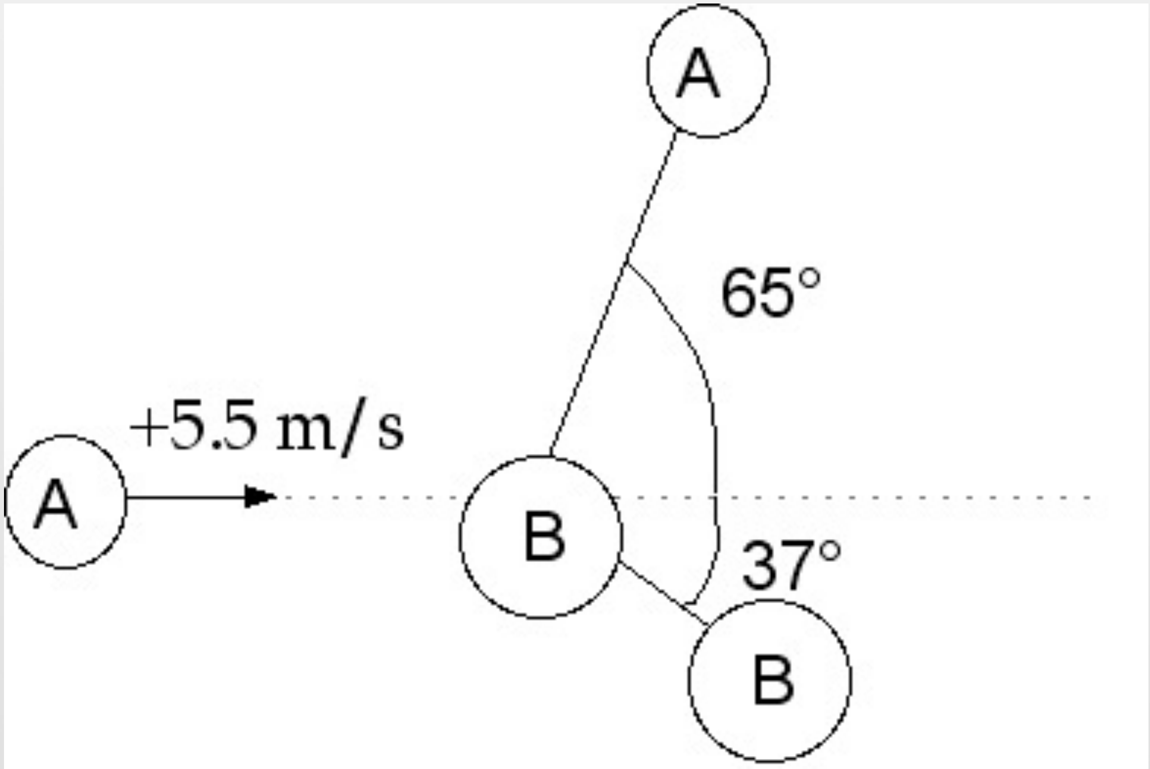
Example

2 pucks collide on an air hockey table.

$m_A = 0.025$ kg and A is initially moving with a velocity of +5.5 m/s.

It collides with B (mass 0.050 kg) which is initially at rest.

After the collision the 2 pucks fly apart with the angles shown. Find the final speeds of A and B.



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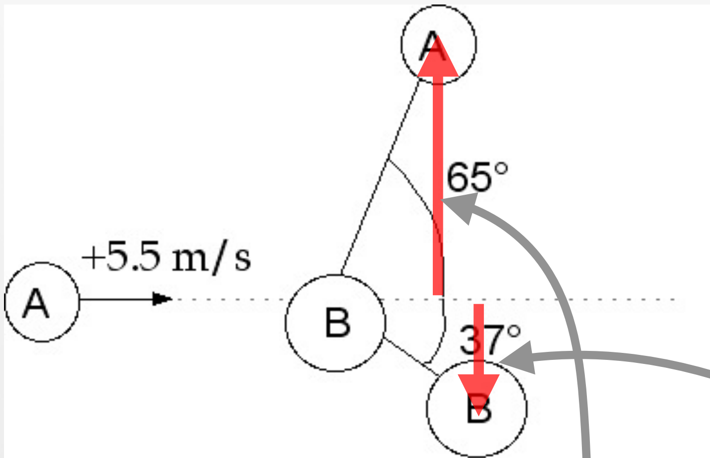


Cons y momentum

$$m_A v_{Ay}' + m_B v_{By}' = 0 \qquad (1)$$

Cons x momentum

$$m_A v_{Ax}' + m_B v_{Bx}' = m_A v_{Ax} \qquad (2)$$

(1) —> $m_A v_A' \sin 65^\circ - m_B v_B' \sin 37^\circ = 0$ —> $v_B' = \dfrac{m_A v_A' \sin 65^\circ}{m_B \sin 37^\circ}$

(2) —> $m_A v_A' \cos 65^\circ + m_B v_B' \cos 37^\circ = m_A v_A$

—> $m_A v_A' \cos 65^\circ + m_B \left(\dfrac{m_A v_A' \sin 65^\circ}{m_B \sin 37^\circ} \right) \cos 37^\circ = m_A v_A$

$$m_A v_A' \cos 65^\circ + m_B \left(\frac{m_A v_A' \sin 65^\circ}{m_B \sin 37^\circ} \right) \cos 37^\circ = m_A v_A$$

Solve for $v_A' = 3.4 \text{ m/s}$

Then

$$v_B' = \frac{m_A v_A' \sin 65^\circ}{m_B \sin 37^\circ} \quad = 2.6 \text{ m/s}$$

8) Centre of Mass

a) *Acceleration and force:*

The *centre-of-mass* of a system of particles (or 3d object) reacts to the total force like a point particle with a mass equal to the total mass.

$$\vec{F}_{total} = m_{total}\vec{a}_{CM}$$

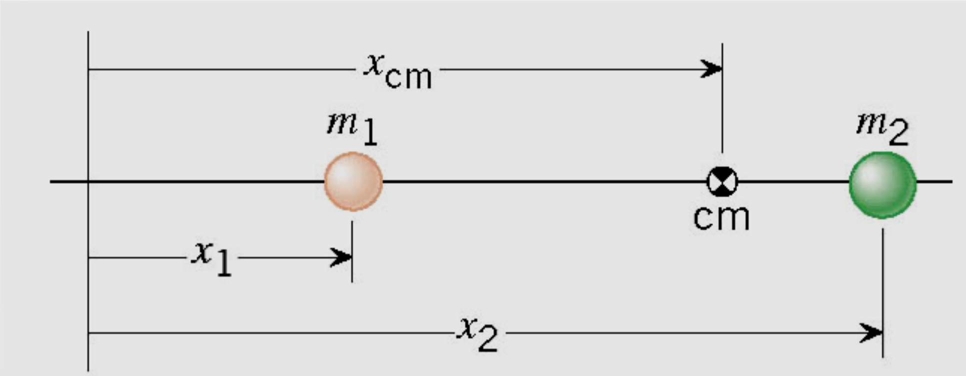
If the total force is zero, the centre-of-mass does not accelerate.

b) *Position of the centre-of-mass*

$$\vec{x}_{CM} = \frac{\sum m_i \vec{x}_i}{\sum m_i}$$

For 2 masses: $$\vec{x}_{CM} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}$$

In one dimension,



For $m_1 = 5.0$ kg, $m_2 = 12$ kg, $x_1 = 2.0$ m, and $x_2 = 6.0$ m,
$x_{CM} = 4.8$ m

$$\vec{x}_{CM} = \frac{m_1\vec{x}_1 + m_2\vec{x}_2}{m_1 + m_2}$$

b) *Velocity of the centre-of-mass*

For 2 masses: $$\vec{v}_{CM} = \frac{\Delta\vec{x}_{CM}}{\Delta t} = \frac{m_1\frac{\Delta\vec{x}_1}{\Delta t} + m_2\frac{\Delta\vec{x}_2}{\Delta t}}{m_1 + m_2}$$

numerator is total momentum
(v_{CM} constant if ext force is zero)

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

CM momentum is simply equal to the total momentum:

$$(m_1 + m_2)\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2$$

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

b) *Acceleration of the centre-of-mass*

For 2 masses: $$\vec{a}_{CM} = \frac{\Delta\vec{v}_{CM}}{\Delta t} = \frac{m_1\frac{\Delta\vec{v}_1}{\Delta t} + m_2\frac{\Delta\vec{v}_2}{\Delta t}}{m_1 + m_2}$$

numerator is total force

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$$

satisfies the definition: $$\vec{F}_{total} = m_{total}\vec{a}_{CM}$$

$$\vec{x}_{CM} = \frac{\sum m_i \vec{x}_i}{\sum m_i}$$

For 2 masses:

$$\vec{x}_{CM} = \frac{m_1 \vec{x}_1 + m_2 \vec{x}_2}{m_1 + m_2}$$

$$\vec{v}_{CM} = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m_1 + m_2}$$

$$\vec{a}_{CM} = \frac{m_1 \vec{a}_1 + m_2 \vec{a}_2}{m_1 + m_2}$$

Example

Two people are standing on a 2.0-m-long platform, one at each end.

The platform floats parallel to the ground on a cushion of air, like a hovercraft.

One person throws a 6.0-kg ball to the other, who catches it.

The ball travels nearly horizontally.

Excluding the ball, the total mass of the platform and people is 118 kg.

Because of the throw, this 118-kg mass recoils.

How far does it move before coming to rest again?

Example

C&J 7.35 A projectile (mass = 0.20 kg) is fired and embeds itself in a target (mass = 2.50 kg). The target, with the projectile in it, flies off after being struck. What percentage of the projectile's incident KE does the target (with the projectile in it) carry off after being struck?

Example

C&J 7.61 Three guns are aimed at the centre of a circle, and each fires a bullet simultaneously. The directions in which they fire are 120° apart. Two of the bullets have the same mass of 4.50×10-3 kg and the same speed of 324 m/s. The other bullet has an unknown mass and a speed of 575 m/s. The bullets collide at the centre and mash into a stationary lump. What is the unknown mass?

Example

Hans Brinker is on skates and there is no friction.

He has two identical snowballs, and wants to get from A to B just by throwing the balls.

Is it better to throw them together, or one after the other?

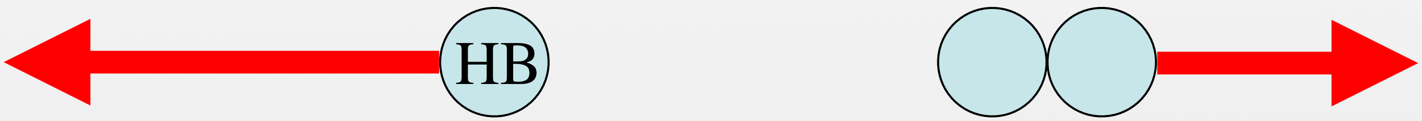
Assume that the relative velocity after release is the same whether he throws one or two snowballs.

For simplicity, consider the mass of the two snowballs and HB to be the same: m

The relative velocity (separation speed) is v

Case 1: throwing them together.

$v_1 = 2v/3$



$v_2 = v/3$

(All v's represent magnitudes.)

$$mv_1 = (2m)v_2$$
$$v = v_1 + v_2$$
$$v = 2v_2 + v_2 = 3v_2$$

$$\vec{v} = \overrightarrow{v_{21}} = \overrightarrow{v_{2g}} + \overrightarrow{v_{g1}}$$
$$\vec{v} = \overrightarrow{v_2} - \overrightarrow{v_1}$$

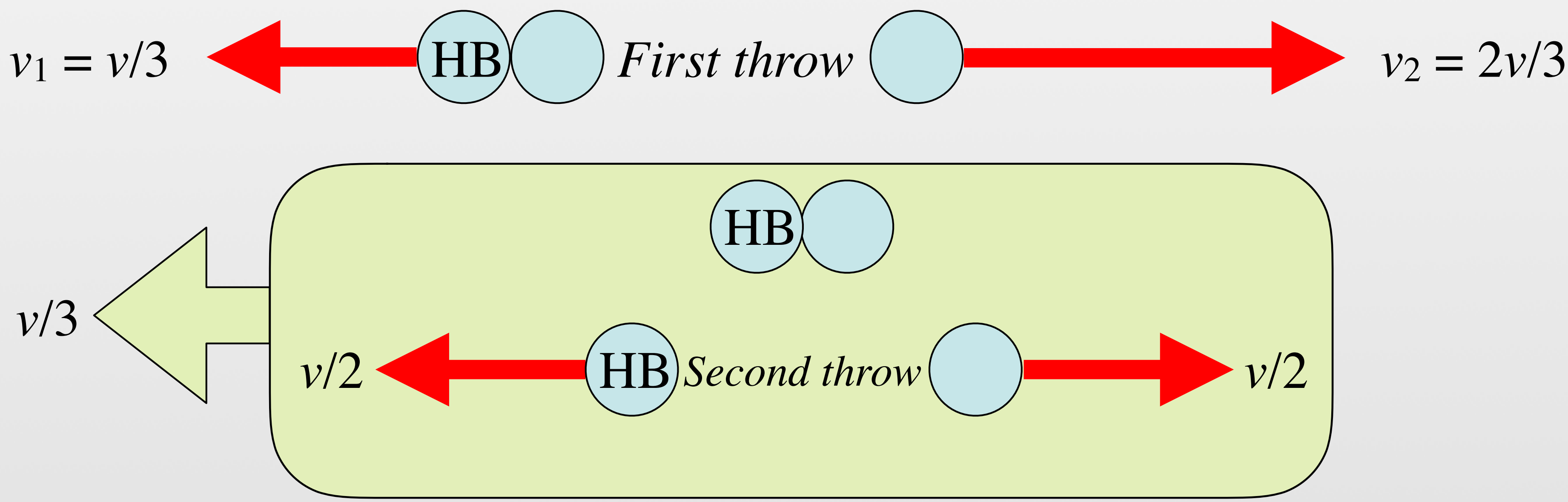
Case 1: $v_{HB} = 2v/3 = 4v/6$

For simplicity, consider the mass of the two snowballs and HB to be the same: m

The relative velocity (separation speed) is v

Case 1: $v_{HB} = 2v/3 = 4v/6$

Case 2: throwing them sequentially.



Case 2: $v_{HB} = v/3 + v/2 = 5v/6$

For simplicity, consider the mass of the two snowballs and HB to be the same: m

The relative velocity (separation speed) is v

Case 1: $v_{HB} = 2v/3 = 4v/6$

Case 2: $v_{HB} = v/3 + v/2 = 5v/6$

sequentially is better