CHAPTER 7
Interest Rate Models and Bond Pricing

The riskless interest rate has been assumed to be constant in most of the pricing models discussed in previous chapters. Such an assumption is acceptable when the interest rate is not the dominant state variable that determines the option payoff, and the life of the option is relatively short. In recent decades, we have witnessed a proliferation of new interest rate dependent securities, like bond futures, options on bonds, swaps, bonds with option features, etc., whose payoffs are strongly dependent on the interest rates. Note that interest rates are used for discounting as well as for defining the payoff of the derivative. The values of these interest rate derivative products depend sensibly on the level of interest rates. In the construction of valuation models for these securities, it is crucial to incorporate the stochastic movement of interest rates into consideration. Several approaches for pricing interest rate derivatives have been proposed in the literature. Unfortunately, up to this time, no definite consensus has been reached with regard to the best approach for these pricing problems.

The correct modelling of the stochastic behaviors of interest rates, or more specifically, the term structure of the interest rate through time is important for the construction of realistic and reliable valuation models for interest rate derivatives. The extension of the Black-Scholes valuation framework to bond options and other bond derivatives is doomed to be difficult because of the pull-to-par phenomenon, where the bond price converges to par at maturity, thus causing the instantaneous rate of return on the bond to be distributed with a diminishing variance through time. The earlier approaches attempt to model the prices of the interest rate securities as functions of one or a few state variables, say, spot interest rate, long-term interest rate, spot forward rate, etc. In the so called *no arbitrage* or *term structure interest rate models*, the consistencies with the observed initial term structures of interest rates and/or volatilities of interest rates are enforced.

In Sec. 7.1, we introduce the terminologies commonly used in bond pricing models and discuss several one-factor models that are widely used in the literature. However, the empirical tests on the applicability of some of these interest rate models in pricing derivatives are not quite promising. We run into the dilemma that the simple models cannot capture the essence of the term structure movement while the more sophisticated models are too cumbersome to be applied in actual pricing procedures. We examine and analyze the term structure of interest rates obtained from a few of these prototype models. It is commonly observed that the interest rate term structure and the volatility term structure derived from the interest rate models in general do not fit with the observed initial term structures. Such discrepancies are definitely undesirable. In Sec. 7.2, we consider yield curve fitting procedures where the initial term structures are taken as inputs to the models and so values of contingent claims obtained from these models are automatically consistent with these inputs. These no arbitrage models contain parameters which are functions of time, and these parameter functions are to be determined from the current market data. Fortunately, some of these no-arbitrage models have good analytic tractability, (like the Hull-White model). In Sec. 7.3, we consider the Heath-Jarrow-Morton appraoch of modeling the stochastic movement of interest rate. Most earlier interest rate models can be visualized as special cases within the Heath-Jarrow-Morton framework. However, the Heath-Jarrow-Morton type models are in general non-Markovian. This would lead to much tedious numerical implementation, thus limit their practical use. In Sec. 7.4, we consider other common types of interest rate models, like the multi-factor models and market rate models.

7.1 Short rate models

A bond is a long-term contract under which the issuer (or borrower) promises to pay the bondholder coupon interest payments (usually periodic) and principal on specific dates as stated in the bond indenture. If there is no coupon payment, the bond is said to be a *zero-coupon bond*. A bond issue is generally advertised publicly and sold to different investors. A bond is a common financial instrument used by firms or governments to raise capital. The upfront premium paid by the bondholders can be considered as a loan to the issuer. The face value of the bond is usually called the *par value* and the *maturity date* of the bond is the specified date on which the par value of a bond must be repaid. A natural question: how much premium should be paid by the bondholder at the initiation of the contract so that it is fair to both the issuer and bondholder? The amount of premium is the value of the bond. From another perspective, the value of a bond is simply the present value of the cash flows that the bondholder expects to realize throughout the life of the bond. In addition, the possible default of the bond issuer should also be taken into account in the pricing consideration.

Since the life span of a bond is usually 10 years or even longer, it is unrealistic to assume the interest rate to remain constant throughout the whole life of the bond. After the bond is being launched, the value of the bond changes over time until maturity due to the change in its life span, fluctuations in interest rates, and other factors, like coupon payments outstanding and change in credit quality of the bond issuer. First, we assume the interest rate to be a known function of time, and derive the corresponding bond price formula. Next, we discuss various terminologies that describe the term structures of interest rates. In the later parts of this section, we present various stochastic models for the interest rates and discuss the associated bond pricing models.

7.1.1 Basic bond price mathematics

Let $r(t)$ be the *deterministic* riskless interest rate function defined for $t \in [0, T]$, where t is the time variable and T is the maturity date of the bond. Normally, the bond price is a function of the interest rate and time. At this point, we assume that the interest rate is not an independent state variable but itself is a known function of time. Hence, the bond price can be assumed to be a function of time only. Let $B(t)$ and $k(t)$ denote the bond price and the known coupon rate, respectively. The final condition is given by $B(T) = F$, where F is the par value. The derivation of the governing equation for $B(t), t < T$, leads to a simple first order linear ordinary differential equation. Over time increment dt from the current time t, the change in value of the bond is $\frac{dB}{dt}dt$ and the coupon received is $k(t)\ dt$. By no-arbitrage principle, the above sum must equal the riskless interest return $r(t)B(t)\ dt$ in time interval dt; hence

$$\frac{dB}{dt} + k(t) = r(t)B, \quad t < T. \tag{7.1.1a}$$

By multiplying both sides by the integrating factor $e^{\int_t^T r(s)\ ds}$, we obtain

$$\frac{d}{dt}\left[B(t)e^{\int_t^T r(s)\ ds}\right] = -k(t)e^{\int_t^T r(s)\ ds}. \tag{7.1.1b}$$

Together with the final condition: $B(T) = F$, the bond price function is found to be

$$B(t) = e^{-\int_t^T r(s)\ ds}\left[F + \int_t^T k(u)e^{\int_u^T r(s)\ ds} du\right]. \tag{7.1.2}$$

The above bond price formula has nice financial interpretation. The coupon amount $k(u)\ du$ received over the period $[u, u + du]$ will grow to the amount $k(u)e^{\int_u^T r(s)\ ds}\ du$ at maturity time T. The future value at T of all coupons received is given by $\int_t^T k(u)e^{\int_u^T r(s)\ ds}\ du$. The present value of the par value and coupons is obtained by discounting the sum by the discount factor $e^{-\int_t^T r(s)\ ds}$, and this gives the current bond value at time t. Depending on the relative magnitude of $r(t)B$ and $k(t)$, the bond price function can be an increasing or decreasing function of time. A bond is called a *discount bond* if the bond price falls below its par value, and called a *premium bond* if otherwise. Also, the market value of a bond will always approach its par value as maturity is approached. This is known as the *pull-to-par phenomenon*.

Term structure of interest rates
The interest rate market is where the price of rising capital is set. Bonds are traded securities and their prices are observed in the market. The bond price over a term depends crucially on the random fluctuations of the interest rate market. Readers are reminded that interest rate, unlike bonds, cannot be traded. We only trade bonds and other instruments that depend on interest rates.

The bond price $B(t,T)$ is a function of both the current time t and the time of maturity T. Therefore, the plot of $B(t,T)$ is indeed a two-dimensional surface over varying values of t and T. For a given fixed $t = t_0$, the plot of $B(t,T)$ against T represents the whole spectrum of bond prices of different maturities at time t_0 (see Fig. 7.1). The prices of bonds with different maturity dates are different, but they are correlated.

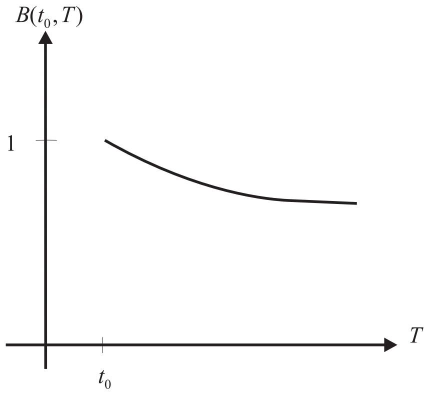

Fig. 7.1 Plot of the whole spectrum of bond prices of maturities beyond t_0. Generally, the bond prices $B(t_0,T)$ decrease monotonically with maturity T.

On the other hand, we can plot $B(t,T_0)$ for a bond of given fixed maturity date T_0 and observe the evolution of the price of a bond with a known maturity T_0 (see Fig. 7.2). However, unlike stock, each bond with a given fixed maturity cannot be treated in isolation. The evolution of the bond price as a function of time t can be considered as a stochastic process with infinite degrees of freedom corresponding to the infinite number of possible maturity dates.

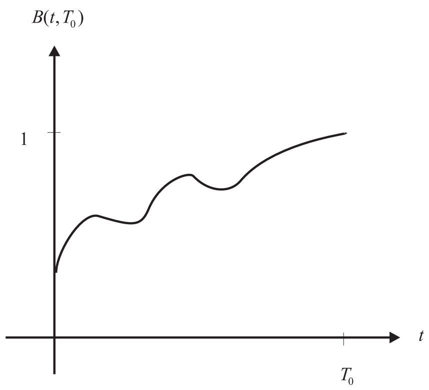

Fig. 7.2 Evolution of the price of a bond with known maturity T_0. Observe that $B(t,T_0)\big|_{t=T_0} = 1$ due to the pull-to-par phenomenon.

To prepare ourselves for the discussion of interest rate models, it is necessary to give precise definitions of the following terms: yield to maturity, yield curve, term structure of interest rates, forward rate and spot rate. All these quantities can be expressed explicitly in terms of traded bond prices, $B(t,T)$, which is the price at time t of a zero-coupon bond maturing at time T. For simplicity, we assume unit value, where $B(T,T) = 1$. The market bond prices indicate the market expectation of the interest rate at future dates.

The *yield to maturity* $R(t,T)$ is defined by

$$R(t,T) = -\frac{1}{T-t}\ln B(t,T), \tag{7.1.3}$$

which gives the *internal rate of return* at time t on the bond. The *yield curve* is the plot of $R(t,T)$ against T and the dependence of the yield curve on the time to maturity $T-t$ is called the *term structure of interest rates*. The term structure reveals market beliefs about future interest rates at different maturities. Normally, the yield increases with maturity due to higher uncertainties with longer time horizon. However, if the current rates are high, the longer-term bond yield may be lower than the shorter-term bond yield.

Next, we consider the price of a forward contract at time t where the holder agrees to purchase at later time T_1 one zero-coupon bond with maturity date $T_2 (> T_1)$. The bond forward price is given by $B(t,T_2)/B(t,T_1)$, since the underlying asset is the T_2-maturity bond and the growth factor (reciprocal of the discount factor) over the time period $[t,T_1]$ is $1/B(t,T_1)$. We define the *forward rate* $f(t,T_1,T_2)$ as seen at time t for the period between T_1 and $T_2 (> T_1)$ in terms of bond forward price by

$$f(t, T_1, T_2) = -\frac{1}{T_2 - T_1} \ln \frac{B(t, T_2)}{B(t, T_1)}. \tag{7.1.4}$$

The forward rate is the rate of interest over a time period in the future implied by today's zero-coupon bonds. By taking $T_1 = T$ and $T_2 = T + \Delta T$, the *instantaneous forward rate* as seen at time t for a bond maturing at time T is given by

$$F(t, T) = -\lim_{\Delta T \to 0} \frac{\ln B(t, T + \Delta T) - \ln B(t, T)}{\Delta T} = -\frac{1}{B(t, T)} \frac{\partial B}{\partial T}(t, T). \tag{7.1.5a}$$

Here, $F(t, T)$ can be interpreted as the marginal rate of return from committing a bond investment for an additional instant. Conversely, by integrating Eq. (7.1.5a) with respect to T, the bond price $B(t, T)$ can be expressed in terms of the forward rate as follows:

$$B(t, T) = \exp\left(-\int_t^T F(t, u)\, du\right). \tag{7.1.5b}$$

Furthermore, by combining Eqs. (7.1.3) and (7.1.5a), $F(t, T)$ can be expressed as

$$F(t, T) = \frac{\partial}{\partial T}[R(t, T)(T - t)] = R(t, T) + (T - t)\frac{\partial R}{\partial T}(t, T), \tag{7.1.6a}$$

or equivalently,

$$R(t, T) = \frac{1}{T - t} \int_t^T F(t, u)\, du. \tag{7.1.6b}$$

Equations (7.1.5b, 7.1.6b) indicate, respectively, that the bond price and bond yield can be recovered from the knowledge of the term structure of the forward rate. On the other hand, the forward rate provides the sense of instantaneity as dictated by the nature of its definition. In Eq. (7.1.6b), $F(t, u)$ gives the internal rate of return as seen at time t over the future period $(u, u + du)$, and its average over (t, T) gives the yield to maturity. The instantaneous *spot rate or short rate* $r(t)$ is simply

$$r(t) = \lim_{T \to t} R(t, T) = R(t, t) = F(t, t). \tag{7.1.7}$$

The plot of $B(t, T)$ against T is inevitably a downward sloping curve since bonds with longer maturity always have lower prices (see Fig. 7.1). However, the yield curve [plot of $R(t, T)$ against T] can be an increasing or decreasing curve, which reveals the average return of the bonds. Therefore, yield curves provide more visual information compared to bond price curves. As deduced from Eq. (7.1.6a), the forward rate curve [plot of $F(t, T)$ against T] will be above the yield curve if the yield curve is increasing or below the yield curve otherwise.

Theories of term structures

Several theories of term structures have been proposed to explain the shape of a yield curve. One of them is the *expectation theory*, which states that long-term interest rates reflect expected future short term interest rates. Let $E_t[r(s)]$ denote the expected value at time t of the spot rate at time s. The yield to maturity for the expectation theory can be expressed as [comparing Eq. (7.1.6b)]

$$R(t,T) = \frac{1}{T-t}\int_t^T E_t[r(s)]\ ds. \qquad (7.1.8a)$$

The other theory is the *market segmentation theory*, which states that each borrower or lender has a preferred maturity so that the slope of the yield curve will depend on the supply and demand conditions for funds in the long-term market relative to the short-term market. The third theory is the *liquidity preference theory*. It conjectures that lenders prefer to make short-term loans rather than long-term loans since liquidity of capital is in general preferred. Hence, long-term bonds normally have a better yield than short-term bonds. The representation equations of the term structures for the market segmentation theory and the liquidity preference theory have similar form, namely,

$$R(t,T) = \frac{1}{T-t}\left[\int_t^T E_t[r(s)]\ ds + \int_t^T L(s,T)\ ds\right], \qquad (7.1.8b)$$

where $L(s,T)$ is interpreted as the instantaneous term premium at time s of a bond maturing at time T. The premium represents the deviation from the expectation theory, which could be irregular as implied by the market segmentation theory or monotonically increasing as implied by the liquidity preference theory.

7.1.2 One-factor short rate models

We would like to derive the governing equation for the bond price using the arbitrage pricing approach. The method of applying the riskless hedging principle is similar but slightly different from that used in equity option pricing model. Suppose the short rate $r(t)$ follows the Ito stochastic process, which is described by the following stochastic differential equation

$$dr = u(r,t)\ dt + w(r,t)\ dZ, \qquad (7.1.9)$$

where dZ is the standard Wiener process, $u(r,t)$ and $w(r,t)^2$ are the instantaneous drift and variance of the process for $r(t)$. The price of a zero-coupon bond is expected to be dependent on $r(t)$. Also, there are other factors which affect the price of the bond, like tax effects, default risk, marketability, seniority and other features associated with the bond indenture. For the present analysis framework, we assume that the bond price depends only on the spot interest rate r, current time t and maturity time T. Note that the present framework corresponds to *one-factor short rate models* since the interest rate movement as assumed by Eq. (7.1.9) depends on a single stochastic variable $r(t)$ only.

If we write the bond price as $B(r,t)$ (suppressing T when there is no ambiguity), then Ito's lemma gives the dynamics of the bond price as

$$dB = \left(\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial r} + \frac{1}{2}w^2\frac{\partial^2 B}{\partial r^2}\right) dt + w\frac{\partial B}{\partial r}\ dZ. \tag{7.1.10}$$

If we write

$$\frac{dB}{B} = \mu_B(r,t)\ dt + \sigma_B(r,t)\ dZ, \tag{7.1.11a}$$

then

$$\mu_B(r,t) = \frac{1}{B}\left(\frac{\partial B}{\partial t} + u\frac{\partial B}{\partial r} + \frac{1}{2}w^2\frac{\partial^2 B}{\partial r^2}\right) \tag{7.1.11b}$$

and

$$\sigma_B(r,t) = \frac{1}{B}w\frac{\partial B}{\partial r}. \tag{7.1.11c}$$

Here, $\mu_B(r,t)$ and $\sigma_B(r,t)^2$ are the respective drift rate and variance rate of the stochastic process of $B(r,t)$. Since interest rate is not a traded security, it cannot be used to hedge with the bond, like the role of the underlying asset in an equity option. Instead we try to hedge bonds of different maturities. The following portfolio is constructed: we buy a bond of dollar value V_1 with maturity T_1 and sell another bond of dollar value V_2 with maturity T_2. The portfolio value Π is given by

$$\Pi = V_1 - V_2. \tag{7.1.12a}$$

According to the bond price dynamics defined by Eq. (7.1.11a), the change in portfolio value in time dt is

$$\begin{aligned} d\Pi =& [V_1\mu_B(r,t;T_1) - V_2\mu_B(r,t;T_2)]\, dt \\ &+ [V_1\sigma_B(r,t;T_1) - V_2\sigma_B(r,t;T_2)]\, dZ. \end{aligned} \tag{7.1.12b}$$

Suppose V_1 and V_2 are chosen such that

$$V_1 = \frac{\sigma_B(r,t;T_2)}{\sigma_B(r,t;T_2) - \sigma_B(r,t;T_1)}\Pi \quad \text{and} \quad V_2 = \frac{\sigma_B(r,t;T_1)}{\sigma_B(r,t;T_2) - \sigma_B(r,t;T_1)}\Pi, \tag{7.1.13}$$

then the stochastic term in Eq. (7.1.12b) vanishes and the equation becomes

$$d\Pi = \Pi\frac{\mu_B(r,t;T_1)\sigma_B(r,t;T_2) - \mu_B(r,t;T_2)\sigma_B(r,t;T_1)}{\sigma_B(r,t;T_2) - \sigma_B(r,t;T_1)}dt. \tag{7.1.14}$$

Since the portfolio is instantaneously riskless, it must earn the riskless short interest rate, that is, $d\Pi = r(t)\Pi\ dt$. Combining with the result in Eq. (7.1.14), we obtain

$$\frac{\mu_B(r,t,T_1)-r(t)}{\sigma_B(r,t;T_1)} = \frac{\mu_B(r,t;T_2)-r(t)}{\sigma_B(r,t;T_2)}. \tag{7.1.15}$$

The above relation is valid for arbitrary maturity dates T_1 and T_2, so the ratio $\dfrac{\mu_B(r,t)-r(t)}{\sigma_B(r,t)}$ should be independent of maturity T. Let the common ratio be defined by $\lambda(r,t)$, that is,

$$\frac{\mu_B(r,t)-r(t)}{\sigma_B(r,t)} = \lambda(r,t). \tag{7.1.16}$$

The quantity $\lambda(r,t)$ is called the *market price of risk* (see Problem 7.1), since it gives the extra increase in expected instantaneous rate of return on a bond per an additional unit of risk. Since the interest rate is not a traded asset, we are unable to eliminate the dependence of $B(r,t)$ on preferences, as what has been done in stock/option hedge. If we substitute $\mu_B(r,t)$ and $\sigma_B(r,t)$ defined in Eqs. (7.1.11b,c) into Eq. (7.1.16), we obtain the following governing equation for the price of a zero-coupon bond

$$\frac{\partial B}{\partial t} + \frac{1}{2}w^2\frac{\partial^2 B}{\partial r^2} + (u-\lambda w)\frac{\partial B}{\partial r} - rB = 0, \quad t < T, \tag{7.1.17}$$

with final condition: $B(T,T)=1$. Once the diffusion process for $r(t)$ is described and the market price of risk $\lambda(r,t)$ is specified, the value of a bond can be obtained by solving Eq. (7.1.17).

Representation of the bond price solution in stochastic integrals

The solution of the bond price Eq. (7.1.17) can be formally represented in an integral form in terms of the underlying stochastic process, namely,

$$B(r,t;T) = E_t\left[\exp\left(-\int_t^T r(u) - \frac{\lambda^2(r(u),u)}{2}\,du - \int_t^T \lambda(r(u),u)\,dZ(u)\right)\right], \quad t \le T. \tag{7.1.18}$$

To show the claim, we define the auxiliary function

$$V(r,t;\xi) = \exp\left(-\int_t^\xi r(u) - \frac{\lambda^2(r(u),u)}{2}\,du - \int_t^\xi \lambda(r(u),u)\,dZ(u)\right), \quad t \le \xi, \tag{7.1.19}$$

and apply Ito's differential rule to compute $B(r,\xi;T)V(r,t;\xi)$. This gives

$$
\begin{aligned}
d(BV) &= V\ dB + B\ dV + dB\ dV \\
&= V\left(\frac{\partial B}{\partial \xi} + u\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2}\right)\ d\xi + Vw\frac{\partial B}{\partial r}dZ \\
&\quad + BV\left(-r - \frac{\lambda^2}{2}\right)\ d\xi - BV\lambda\ dZ + \frac{\lambda^2}{2}BV d\xi - V\lambda w\frac{\partial B}{\partial r}d\xi \\
&= V\left[\frac{\partial B}{\partial \xi} + (u - \lambda w)\frac{\partial B}{\partial r} + \frac{w^2}{2}\frac{\partial^2 B}{\partial r^2} - rB\right] d\xi \\
&\quad - BV\lambda\ dZ + Vw\frac{\partial B}{\partial r}dZ \\
&= -\ BV\lambda\ dZ + Vw\frac{\partial B}{\partial r}dZ.
\end{aligned}
\tag{7.1.20}
$$

Next, we integrate the above equation from t to T and take the expectation. Since $E_t\left[\int_t^T\ dZ\right] = 0$, we obtain

$$
E_t[B(r,T;T)V(r,t;T) - B(r,t;T)V(r,t;t)] = 0. \tag{7.1.21}
$$

Applying the terminal conditions: $B(r,T;T) = 1$ and $V(r,t;t) = 1$, we then obtain

$$
B(r,t;T) = E_t[V(r,t;T)]. \tag{7.1.22}
$$

Market price of risk

The drift parameter $u(r,t)$ and variance parameter $w(r,t)$ in the bond price equation may be obtained by statistical analysis of the observable process of $r(t)$. Once u and w are available, the market price of risk $\lambda(r,t)$ can be estimated using the following relation (see Problem 7.3)

$$
\left.\frac{\partial R}{\partial T}\right|_{T=t} = \frac{1}{2}\left[u(r,t) - w(r,t)\lambda(r,t)\right], \tag{7.1.23}
$$

where $\left.\frac{\partial R}{\partial T}\right|_{T=t}$ is the slope of the yield curve $R(t,T)$ at immediate maturity.

We now explore the solution of the above one-factor short rate model with different assumptions of the stochastic process for $r(t)$. The two most popular models are the Vasicek mean reversion model and the Cox-Ingersoll-Ross square root process model.

7.1.3 Vasicek mean reversion model

Vasicek (1977) proposed the stochastic process for the short rate $r(t)$ to be governed by the Ornstein-Uhlenbeck process

$$
dr = \alpha(\gamma - r)\ dt + \rho\ dZ, \quad \alpha > 0. \tag{7.1.24}
$$

The above process is sometimes called the elastic random walk or *mean reversion process*. The instantaneous drift $\alpha(\gamma - r)$ represents the effect of pulling the process towards its long-term mean γ with magnitude proportional to the deviation of the process from the mean. Such mean reversion assumption agrees with the economic phenomenon that interest rates appear over time to be pulled back to some long-run average value. To explain the mean reversion phenomenon, we argue that when interest rates increase, the economy slows down and that there is less demand for loans and a natural tendency for rates to fall. The reverse situation of rates dropping can be argued similarly in the reverse sense. Also, the reversion phenomena in interest rates do not violate the principle of market efficiency. One may devise trading rules on a stock that give above average returns if mean reversion occurs in the stock price movement. However, interest rate is not the price of a traded security. We only trade on interest rate instruments whose prices depend on interest rates, where the mean reversion features in interest rates have been incorporated in the derivative prices.

Given the current level of short rate $r(t)$, the mean of short rate at T, $\bar{r}(T)$, can be obtained by integrating the differential equation: $d\overline{r}(t) = \alpha(r - \overline{r})\, dt$ (see Problem 7.4). This gives

$$\bar{r}(T) = \gamma + [\overline{r}(t) - \gamma]e^{-\alpha(T-t)}. \tag{7.1.25}$$

The variance of the mean reversion process at time T can be obtained by solving

$$\frac{d}{dt}\,\mathrm{var}[r(t)] = -2\alpha\;\mathrm{var}[r(t)] + \rho^2 \tag{7.1.26a}$$

and taking the variance at the current time to be zero (see Problem 7.4). We then obtain

$$\mathrm{var}[r(T)] = \frac{\rho^2}{2\alpha}\left[1 - e^{-2\alpha(T-t)}\right], \quad T > t. \tag{7.1.26b}$$

Analytic bond price formula

Suppose we assume the market price of risk λ to be constant, independent of r and t, then it is possible to derive analytic bond price formula. The Vasicek mean reversion model corresponds to $u = \alpha(\gamma - r)$ and $w = \rho$ in Eq. (7.1.9), so the equation for the bond price becomes

$$\frac{\partial B}{\partial t} + \frac{\rho^2}{2}\frac{\partial^2 B}{\partial r^2} + [\alpha(\gamma - r) - \lambda\rho]\,\frac{\partial B}{\partial r} - rB = 0. \tag{7.1.27}$$

Suppose we assume the solution to be of the form

$$B(r,t;T) = a(t,T)e^{-b(t,T)r}, \tag{7.1.28}$$

and substitute into Eq. (7.1.27), the following pair of differential equations are obtained:

$$\frac{da}{dt} + (\lambda\rho - \alpha\gamma)ab + \frac{1}{2}\rho^2 ab^2 = 0, \quad t < T \qquad (7.1.29a)$$

$$\frac{db}{dt} - \alpha b + 1 = 0, \qquad t < T, \qquad (7.1.29b)$$

with final conditions: $a(T,T) = 1$ and $b(T,T) = 0$. First, we solve for $b(t,T)$ from Eq. (7.1.29b). Substituting the known solution of $b(t,T)$ into Eq. (7.1.29b), we then subsequently solve for $a(t,T)$ [also see Eqs. (7.2.9a,b)]. Combining the solution, we obtain

$$B(r,t;T) = \exp\left(\frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right](R_\infty - r) - R_\infty(T-t) - \frac{\rho^2}{4\alpha^3}\left[1 - e^{-\alpha(T-t)}\right]^2\right), \quad t < T, \quad (7.1.30)$$

where $R_\infty = \gamma - \frac{\rho\lambda}{\alpha} - \frac{\rho^2}{2\alpha^2}$ [R_∞ is actually equal to $\lim_{T\to t} R(t,T)$, see Eq. (7.1.32) below]. Using Eqs. (7.1.11b,c), the mean and standard deviation of the instantaneous rate of return of a bond maturing at time T are found to be

$$\mu_B(r,t;T) = r(t) - \frac{\rho\lambda}{\alpha}\left[1 - e^{-\alpha(T-t)}\right] \qquad (7.1.31a)$$

$$\sigma_B(r,t;T) = \frac{\rho}{\alpha}\left[1 - e^{-\alpha(T-t)}\right]. \qquad (7.1.31b)$$

Also, according to the definition in Eq. (7.1.3), the yield to maturity or the term structure of interest rates can be easily found to be

$$R(t,T) = R_\infty + \frac{[r(t) - R_\infty][1 - e^{-\alpha(T-t)}]}{\alpha(T-t)} + \frac{\rho^2}{4\alpha^3(T-t)}[1 - e^{-\alpha(T-t)}]^2. \qquad (7.1.32)$$

The long-term internal rate of return deduced from the present model is seen to be constant. Note that $R(t,T)$ and $\ln B(r,t;T)$ are linear functions of $r(t)$; and since $r(t)$ is normally distributed, it then follows that $R(t,T)$ is also normally distributed and $B(r,t;T)$ is lognormally distributed. Suppose we set $T = T_1$ and $T = T_2$ in Eq. (7.1.32), and subsequently eliminate $r(t)$, we obtain a relation between $R(t,T_1)$ and $R(t,T_2)$ that is dependent only on the parameter values. This implies that under the present Vasicek model, the instantaneous returns on bonds of different maturities are perfectly correlated. However, in reality, bond returns over a finite period are not correlated perfectly.

Readers are invited to explore more properties on the term structures of rates associated with the Vasicek model in Problem 7.5. Also, a discussion on a discrete version of the Vasicek model is presented in Problem 7.6.

7.1.4 Cox-Ingersoll-Ross model

In the Vasicek model, it may happen that interest rates become negative. To rectify the problem, Cox, Ingersoll and Ross (1985) proposed the following square root diffusion process for the short rate:

$$dr = \alpha(\gamma - r)\ dt + \rho\sqrt{r}\ dZ, \quad \alpha, \gamma > 0. \tag{7.1.33}$$

With an initially non-negative interest rate, $r(t)$ will never be negative. This is attributed to the mean-reverting drift rate that tends to pull $r(t)$ towards the long-run average γ and the diminishing volatility as $r(t)$ declines to zero (recall that volatility is constant in the Vasicek model). It can be shown that $r(t)$ can reach zero only if $\rho^2 > 2\alpha\gamma$; while the upward drift is sufficiently strong to make $r(t) = 0$ impossible when $2\alpha\gamma \geq \rho^2$. Whenever $r(t)$ becomes zero, it bounces up into the positive region instantaneously. The probability density of the interest rate at time T, conditional on its value at the current time t, is given by

$$g(r(T); r(t)) = ce^{-u-v}\left(\frac{v}{u}\right)^{q/2} I_q\left(2(uv)^{1/2}\right) \tag{7.1.34a}$$

where

$$c = \frac{2\alpha}{\rho^2\left[1 - e^{-\alpha(T-t)}\right]}, \quad u = cr(t)e^{-\alpha(T-t)}, \quad v = cr(T), \quad q = \frac{2\alpha\gamma}{\rho^2} - 1, \tag{7.1.34b}$$

and I_q is the modified Bessel function of the first kind of order q [see Feller (1951) for details]. The mean and variance of $r(T)$ are given by

$$E[r(T)|r(t)] = r(t)e^{-\alpha(T-t)} + \gamma\left[1 - e^{-\alpha(T-t)}\right] \tag{7.1.35a}$$

$$\begin{aligned}\text{var}[r(T)|r(t)] = {} & r(t)\left(\frac{\rho^2}{\alpha}\right)\left[e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right] \\ & + \frac{\gamma\rho^2}{2\alpha}\left[1 - e^{-\alpha(T-t)}\right]^2.\end{aligned} \tag{7.1.35b}$$

The distribution of the future interest rates has the following properties:
(i) as $\alpha \to \infty$, the mean tends to γ and the variance to zero,
(ii) as $\alpha \to 0^+$, the mean tends to $r(t)$ and the variance to $\rho^2(T-t)r(t)$.

To find the price of a zero-coupon bond based on the present squqre root diffusion process for the short rate, we assume the same form of solution as given in Eq. (7.1.28). The corresponding new pair of differential equations for $a(t,T)$ and $b(t,T)$ become

$$\frac{da}{dt} - \alpha\gamma ab = 0, \quad t < T, \tag{7.1.36a}$$

$$\frac{db}{dt} - (\alpha + \lambda\rho)b - \frac{1}{2}\rho^2 b^2 + 1 = 0, \quad t < T, \tag{7.1.36b}$$

where the market price of risk is taken to be $\lambda\sqrt{r}$, and λ is assumed to be constant. The final conditions are: $a(T,T) = 1$ and $b(T,T) = 0$. The solutions to the above equations are found to be (Cox *et al.*, 1985)

$$a(t,T) = \left\{ \frac{2\theta e^{(\theta+\psi)(T-t)/2}}{(\theta+\psi)\left[e^{\theta(T-t)}-1\right]+2\theta} \right\}^{2\alpha\gamma/\rho^2} \quad (7.1.37a)$$

$$b(t,T) = \frac{2\left[e^{\theta(T-t)}-1\right]}{(\theta+\psi)\left[e^{\theta(T-t)}-1\right]+2\theta}, \quad (7.1.37b)$$

where

$$\psi = \alpha + \lambda\rho, \quad \theta = \sqrt{\psi^2 + 2\rho^2}. \quad (7.1.37c)$$

Note that the market price of risk λ appears only through the sum ψ in the above solution. The properties of the comparative statics for the bond price and the yield to maturity of the Cox-Ingersoll-Ross model are addressed in Problems 7.8 and 7.9.

7.1.5 Generalized one-factor short rate models

Besides the Vasicek and Cox-Ingersoll-Ross models, an array of one-factor short rate models have also been proposed in the literature. Many of these models can be nested within the stochastic process represented by

$$dr = (\alpha + \beta r)\ dt + \rho r^\gamma\ dZ, \quad (7.1.38)$$

where the parameters α, β, γ and ρ are constants. For example, the Vasicek and Cox-Ingersoll-Ross models correspond to $\gamma = 0$ and $\gamma = 1/2$, respectively, and the Geometric Brownian model corresponds to $\alpha = 0$ and $\gamma = 1$. The stochastic interest rate model used by Merton (1973, Chap. 1) can be nested within the Vasicek model with $\beta = 0$ and $\gamma = 0$. Other examples of one-factor interest rate models nested within the stochastic process represented by Eq. (7.1.38) are

Dothan model (1978)	$dr = \rho r\ dZ$
Brennan-Schwartz model (1979)	$dr = (\alpha + \beta r)\ dt + \rho r\ dZ$
Cox-Ingersoll-Ross variable rate model (1980)	$dr = \rho r^{3/2}\ dZ$
Constant elasticity of variance model	$dr = \beta r\ dt + \rho r^\gamma\ dZ$

Note that when $\gamma > 0$, the volatility increases with the level of interest rate. This is called the *level effect*.

Chan *et al.* (1992) did a comprehensive empirical analysis on the above list of one-factor short rate models. They found that the most successful models which capture the dynamics of the short-term interest rate are those that allow the volatility of interest rate changes to be highly sensitive to the level of the interest rate. The findings confirm the financial intuition that the volatility of the term structure is an important factor governing the value of contingent claims. Using the data of one-month Treasury bill yields, they discovered that those models with $\gamma \geq 1$ can capture the dynamics of the short-term interest rate better than those models with $\gamma < 1$. The relation between interest rate volatility and the level of r is more important, as compared to the mean reversion feature, in the characterization of dynamic interest rate models. The incorporation of mean reversion feature usually causes much complexity in the analysis of the term structure; and since mean reversion plays the lesser role, the additional generality of adding mean reversion in a model may not be well justified. The Vasicek and Merton models have always been criticized for allowing negative interest rate values. However, their far more serious deficiency is the assumption of $\gamma = 0$ in the models. This assumption implies the conditional volatility of changes in the interest rate to be constant, independent on the level of r. Another disquieting conclusion deduced from their empirical studies is that the range of possible call option values varies significantly across various models. This indicates that the present framework of the one-factor diffusion process for the short rate may not be adequate to describe the true term structure of the interest rates over time.

One serious concern for these one-factor interest rate models is that the term structures derived from these models only provide a limited family which cannot correctly price many traded bonds. This stems from the inherent shortcomings that these models price interest rate derivatives with reference to a theoretical yield curve rather than the actually observed curve. Once the process for r is fully defined, everything about the initial term structure and how it can evolve at future times are then fully defined. The initial term structure is an output from the model rather than an input to the model. In the next section, we discuss the construction of no arbitrage interest rate models where the current market information about the term structure of interest rates and the term structure of interest rate volatilities are incorporated into the models.

7.2 Yield curve fitting and no-arbitrage models

In the short rate models discussed in Sec. 7.1, there are several unobservable parameters but there is no mechanism for which the parameters in the models are chosen such that the term structure obtained from the model fits today's observable term structure. Since the information of the current term structure is available, an interest rate model should take the data on initial term structure as an input rather than an output. Arbitrage exists if the theoretical bond prices solved from the model do not agree with the observed bond prices. In this section, we discuss no-arbitrage short rate models that contain time dependent parameter functions and the functions are determined in such a way that the current bond prices obtained from the model coincide with the observed market prices. The initial term structure may be prescribed as term structure of bond prices or forward rates. The most popular models in this class include:

Ho-Lee (HL) model
This is the first no arbitrage model proposed in the literature (Ho and Lee, 1986), where the initial formulation is in the form of a binomial tree. The continuous time limit of the model takes the form

$$dr = \theta(t)\ dt + \sigma_r\ dZ, \tag{7.2.1}$$

where r is the short rate and σ_r is the constant instantaneous standard deviation of the short rate. The time dependent drift function $\theta(t)$ is chosen to ensure that the model fits the initial term structure (see Problem 7.11).

Hull-White (HW) model
The Ho-Lee model assumes constant volatility structure and incorporates no mean reversion. Hull and White (1990) proposed the following model for the short rate

$$dr = [\theta(t) - \alpha(t)r]\ dt + \sigma(t)r^\beta\ dZ. \tag{7.2.2}$$

The mean reversion level is given by $\dfrac{\theta(t)}{\alpha(t)}$. The model can be considered as the extended Vasicek model when $\beta = 0$ and the extended Cox-Ingersoll-Ross model when $\beta = 1/2$.

Black-Derman-Toy (BDT) model
Similar to the Ho-Lee model, the original formulation of the Black-Derman-Toy model (Black *et al.*, 1990) is in the form of a binomial tree. The continuous time equivalent of the model can be shown to be

$$d\ln r = \left[\theta(t) - \frac{\sigma_r'(t)}{\sigma_r(t)}\ln r\right]\ dt + \sigma_r(t)\ dZ. \tag{7.2.3}$$

In this model, the changes in the short rate in the model are lognormally distributed, and that the short rates are always non-negative. The first short $\theta(t)$ is chosen so that the model fits the term structure of short rates and the second function $\sigma_r(t)$ is chosen to fit the term structure of short rate volatilities. When the volatility function $\sigma_r(t)$ is taken to be constant, the BDT model reduces to a lognormal version of the HL model.

Suppose the reversion rate and volatility in the BDT model are decoupled, we then have

$$d\ln r = [\theta(t) - \alpha(t)\ln r]\ dt + \sigma_r(t)\ dZ. \tag{7.2.4}$$

The new version is called the Black-Karasinski (BK) model (1991).

7.2.1 Hull-White model

A special class of the generalized Hull-White model is now considered, where the drift term containing $\theta(t)$ is the only time dependent function in the model. First, we examine the analytic procedure to determine $\theta(t)$ using the information of the initial term structure of bond prices or forward rates.

This special class of models take the following form

$$dr = [\theta(t) - \alpha r]\; dt + \sigma r^\beta \; dZ, \tag{7.2.5}$$

where α, σ and $\beta \geq 0$ are constant, $\theta(t)$ is an unknown function of time. The particular case $\beta = 0$ deserves the most special attention here. With $\beta = 0$, the model can be considered an extension of the Vasicek model, and it becomes the continuous version of the HL model when $\alpha = 0$ (see Problem 7.11). Likewise, it possesses the mean reversion property and good analytic tractability.

With $\beta = 0$ in the short rate process defined in Eq. (7.2.5), the governing equation for the bond price $B(r, t; T)$ can be found to be [see Eq. (7.1.27)]

$$\frac{\partial B}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 B}{\partial r^2} + [\phi(t) - \alpha r]\frac{\partial B}{\partial r} - rB = 0, \tag{7.2.6}$$

where $\phi(t) = \theta(t) - \sigma\lambda(t), \lambda(t)$ is the market price of risk. Here, we make the simplifying assumption that the market price of risk is a function of time only. The nice analytic tractability of the Hull-White model derives from the property that the bond price admits solution of the form

$$B(r, t; T) = a(t, T)e^{-b(t,T)r}. \tag{7.2.7}$$

By substituting the assumed form into Eq. (7.2.6), we obtain a pair of ordinary differential equations

$$\frac{db}{dt} - \alpha b + 1 = 0 \tag{7.2.8a}$$

$$\frac{da}{dt} - \phi(t)ab + \frac{\sigma^2}{2}ab^2 = 0, \tag{7.2.8b}$$

with terminal conditions at $t = T$:

$$a(T, T) = 1 \quad \text{and} \quad b(T, T) = 0. \tag{7.2.8c}$$

Solving the above system of ordinary differential equations, we obtain

$$b(t, T) = \frac{1}{\alpha}\left[1 - e^{-\alpha(T-t)}\right] \tag{7.2.9a}$$

$$\ln a(t, T) = \frac{\sigma^2}{2}\int_t^T b^2(u, T)\; du - \int_t^T \phi(u)b(u, T)\; du. \tag{7.2.9b}$$

In Problem 7.20, we outline an alternative approach using stochastic calculus method to solve for $B(r, t; T)$.

Our goal is to determine $\phi(T)$ in terms of the current bond prices $B(r,t;T)$. From Eq. (7.2.8b) and the relation: $\ln B(r,t;T) + rb(t,T) = \ln a(t,T)$, we have

$$\int_t^T \phi(u)b(u,T)\,du = \frac{\sigma^2}{2}\int_t^T b^2(u,T)\,du - \ln B(r,t;T) - rb(t,T). \quad (7.2.10)$$

To solve for $\phi(u)$, the first goal is to obtain an explicit expression for $\int_t^T \phi(u)\,du$. This may be achieved by differentiating $\int_t^T \phi(u)b(u,T)\,du$ with respect to T and subtracting those terms involving $\int_t^T \phi(u)e^{-\alpha(T-t)}\,du$. The derivative of the left hand side of Eq. (7.2.10) with respect to T gives

$$\begin{aligned}\frac{\partial}{\partial T}\int_t^T \phi(u)b(u,T)\,du &= \phi(u)b(u,T)\Big|_{u=T} + \int_t^T \phi(u)\frac{\partial}{\partial T}b(u,T)\,du \\ &= \int_t^T \phi(u)e^{-\alpha(T-u)}\,du. \end{aligned} \quad (7.2.11a)$$

In a similar manner, we differentiate the right hand side with respect to T. By equating the derivatives on both sides,

$$\begin{aligned}\int_t^T \phi(u)e^{-\alpha(T-u)}\,du &= \frac{\sigma^2}{\alpha}\int_t^T [1-e^{-\alpha(T-u)}]e^{-\alpha(T-u)}\,du \\ &\quad - \frac{\partial}{\partial T}\ln B(r,t;T) - re^{-\alpha(T-t)}. \end{aligned} \quad (7.2.11b)$$

We multiply Eq. (7.2.10) by α and add it to Eq. (7.2.11b) to obtain

$$\begin{aligned}\int_t^T \phi(u)\,du &= \frac{\sigma^2}{2\alpha}\int_t^T [1-e^{-2\alpha(T-u)}\,du] - r \\ &\quad - \frac{\partial}{\partial T}\ln B(r,t;T) - \alpha\ln B(r,t;T). \end{aligned} \quad (7.2.12)$$

Lastly, by differentiating the above equation with respect to T again, we obtain

$$\begin{aligned}\phi(T) &= \frac{\sigma^2}{2\alpha}[1-e^{-2\alpha(T-t)}] - \frac{\partial^2}{\partial T^2}\ln B(r,t;T) \\ &\quad - \alpha\frac{\partial}{\partial T}\ln B(r,t;T). \end{aligned} \quad (7.2.13)$$

One may express $\phi(T)$ in terms of the current forward rate $F(t,T)$. Recall that $-\frac{\partial}{\partial T}\ln B(r,t;T) = F(t,T)$ [see Eq. (7.1.5b)] so that we may rewrite $\phi(T)$ in the form

$$\phi(T) = \frac{\sigma^2}{2\alpha}[1-e^{-2\alpha(T-t)}] + \frac{\partial}{\partial T}F(t,T) + \alpha F(t,T). \quad (7.2.14)$$

An alternative representation of the drift function $\phi(T)$ in terms of current yield curve $R(t, T)$ can also be derived [see Problem 7.13].

From Eq. (7.1.11c), the bond price volatility $\sigma_B(t, T)$ is given by

$$\sigma_B(t, T) = \frac{\sigma}{\beta} \frac{\partial B}{\partial r} = -\sigma b(t, T) = -\frac{\sigma}{\alpha}[1 - e^{-\alpha(T-t)}]. \qquad (7.2.14)$$

By virtue of Ito's lemma and applying the relation (7.2.7), the stochastic differential equation for the bond price $B(r, t; T)$ is found to be

$$\frac{dB(r, t; T)}{B(r, t; T)} = r\ dt - \sigma b(t, T)\ dZ. \qquad (7.2.15a)$$

Since the bond is a traded security, the drift rate of bond price under the risk neutral measure is simply given by r. To be consistent with the usual convention, we choose to express the bond price volatility as a positive quantity. We then have

$$\frac{dB(r, t; T)}{B(r, t; T)} = r\ dt + \frac{\sigma}{\alpha}[1 - e^{-\alpha(T-t)}]\ dZ. \qquad (7.2.15b)$$

As the bond volatility is independent of r, so the distribution of the bond price at any given time conditional on its price at an earlier time is lognormal.

The above analytic tractability property of determining the drift term by matching data of the initial bond price and/or initial forward rate can be extended to the following generalized Vasicek mean reversion short rate model of the form (Hull and White, 1990)

$$dr = [\theta(t) + \alpha(t)(d - r)]\ dt + \sigma_r(t)\ dZ. \qquad (7.2.16)$$

The details are found in Problem 7.14.

7.3 Heath-Jarrow-Morton framework

Recall that the bond prices $B(t, T)$, the yields $R(t, T)$ and the forward rates $F(t, T)$ all provide the same information on the term structure of interest rates. The Heath-Jarrow-Morton (HJM) framework (Heath *et al.*, 1990a, 1990b, 1992) attempts to construct a family of continuous time stochastic processes for the term structure, consistent with the observed initial term structure data. The driving state variable of the model is chosen to be $F(t, T)$, the forward rate at time t for instantaneous borrowing at a later time T. In the most general form of the model, the stochastic process for $F(t, T)$ is assumed to be

$$F(t, T) = F(0, T) + \int_0^t \alpha_F(u, T)\ du + \sum_{i=1}^n \int_0^t \sigma_F^i(u, T)\ dZ_i(u), \quad (7.3.1a)$$

or in differential form,

$$dF(t,T) = \alpha_F(t,T)\ dt + \sum_{i=1}^{n} \sigma_F^i(t,T)\ dZ_i(t), \quad 0 \le t \le T. \quad (7.3.1b)$$

Here, $F(0,T)$ is the known initial forward rate curve, $\alpha_F(t,T)$ is the instantaneous forward rate's drift, $\sigma_F^i(t,T)$ are the volatilities of the forward rates, and dZ_i is the ith Wiener process. Note that there are n independent Wiener processes determining the stochastic fluctuation of the forward rate curve, $\alpha_F(T,T)$ and $\sigma_F^i(t,T)$ are adapted processes. The forward rate process starts with initial value $F(0,T)$, then evolves under a drift and several Wiener processes.

For arbitrary set of drift and volatility structures, the interest rate dynamics as posed in Eq. (7.3.1a) is not necessarily arbitrage free. For the existence of a unique equivalent martingale measure, it is necessary that the drift $\alpha_F(t,T)$ must be related to the volatility functions $\sigma_F^i(t,T)$. In Sec. 7.3.1, we illustrate how to relate the drift with the volatility in order that the derived system of bond prices admits no arbitrage opportunities. In Sec. 7.3.2, we demonstrate how short rate models can be formulated into the HJM framework.

7.3.1 No-arbitrage restrictions on drifts

For simplicity of discussion, we assume that a single stochastic state variable drives the whole term structure. Under the risk netural measure, the drift rate of the discount bond price $B(r,t)$ must be $r(t)$. We express the price dynamics for $B(t,T)$ under the risk neutral measure as follows:

$$\frac{dB(t,T)}{B(t,T)} = r(t)\ dt + \sigma_B(t,T)\ dZ, \quad (7.3.2)$$

where $\sigma_B(t,T)$ is an adapted process so that bond price volatility at time t are dependent on the past and present values of bond prices. Also, the bond price volatility must decline to zero at maturity due to the pull-to-par phenomenon so that

$$\sigma_B(T,T) = 0. \quad (7.3.3)$$

According to the definition of $f(t,T_1,T_2)$ defined in Eq. (7.1.4), its differential is given by

$$df(t,T_1,T_2) = \frac{d\ln B(t,T_1) - d\ln B(t,T_2)}{T_2 - T_1}. \quad (7.3.4)$$

By Ito's lemma, the log derivative of the bond price is given by

$$d\ln B(t,T_i) = \left[r(t) - \frac{\sigma_B(t,T_i)^2}{2}\right]\ dt + \sigma_B(t,T_i)\ dZ, \quad i = 1,2, (7.3.5)$$

so that Eq. (7.3.4) can be rewritten as

$$df(t,T_1,T_2) = \frac{\sigma_B(t,T_2)^2 - \sigma_B(t,T_1)^2}{2(T_2-T_1)}dt + \frac{\sigma_B(t,T_1)-\sigma_B(t,T_2)}{T_2-T_1}dZ. \tag{7.3.6}$$

Equation (7.3.6) indicates that the risk neutral process for f depends only on the volatilities σ_B's. It depends on r and B only through the dependence of σ_B's on these variables.

Suppose we let $T_1 = T$ and $T_2 = T + \Delta T$ and take the limit $\Delta T \to 0$, $f(t,T_1,T_2)$ becomes the instantaneous forward rate $F(t,T)$. In addition, we observe

$$\lim_{\Delta T\to 0} \frac{\sigma_B(t,T)-\sigma_B(t,T+\Delta T)}{\Delta T} = -\frac{\partial \sigma_B}{\partial T}(t,T) \tag{7.3.7a}$$

and

$$\lim_{\Delta T\to 0} \frac{\sigma_B(t,T+\Delta T)^2-\sigma_B(t,T)^2}{2\Delta T} = \sigma_B(t,T)\frac{\partial \sigma_B}{\partial T}(t,T). \tag{7.3.7b}$$

Since dZ is a Wiener process, there is no loss of generality to change the sign of dZ. Therefore, we can express $dF(t,T)$ in the form

$$dF(t,T) = \sigma_B(t,T)\frac{\partial \sigma_B}{\partial T}(t,T)\ dt + \frac{\partial \sigma_B}{\partial T}(t,T)\ dZ. \tag{7.3.8}$$

Note that the risk neutral processes for all instantaneous forward rates are known once the volatilities $\sigma_B(t,T)$ are specified for all t and T. Hence, $\sigma_B(t,T)$'s are sufficient to fully define a one-factor interest rate model.

The one-factor version of Eq. (7.3.16) can be expressed as

$$dF(t,T) = \alpha_F(t,T)\ dt + \sigma_F(t,T)\ dZ. \tag{7.3.9}$$

Comparing Eqs. (7.3.8) and (7.3.9), we obtain the following relation between the volatilities of the forward rate and bond price

$$\sigma_F(t,T) = \frac{\partial \sigma_B}{\partial T}(t,T), \tag{7.3.10}$$

and the drift $\alpha_F(t,T)$ is related to the volatility $\sigma_F(t,T)$ by

$$\alpha_F(t,T) = \sigma_B(t,T)\frac{\partial \sigma_B}{\partial T}(t,T) = \sigma_F(t,T)\int_t^T \sigma_F(t,u)\ du. \tag{7.3.11}$$

The class of HJM interest rate models specify the volatilities of all instantaneous forward rates $\sigma_F(t,T)$ at all future times. Once the *volatility structure* $\sigma_F(t,T)$ is specified, the drift $\alpha_F(t,T)$ can be found using Eq. (7.1.11). Somewhat similar in spirit as the Black-Scholes equity option model, the only inputs to HJM are an underlying and a measure of its volatility. The underlying is the entire term structure and the volatility structure describes how this term structure evolves over time. The initial term structure plays the same role as the asset price in the Black-Scholes model.

7.3.2 Formulation of short rate models in HJM framework

Recall that the short rate is given by

$$r(t) = F(t,t) = F(0,t) + \int_0^t dF(u,t)\ du. \tag{7.3.12}$$

From Eq. (7.3.8), we have

$$r(t) = F(0,t) + \int_0^t \sigma_B(u,t)\frac{\partial \sigma_B}{\partial t}(u,t)\ du + \int_0^t \frac{\partial \sigma_B}{\partial t}(u,t)\ dZ(u). \tag{7.3.13}$$

Differentiating the above result and observing that $\sigma_B(t,t) = 0$, we obtain

$$\begin{aligned} dr(t) = {} & \frac{\partial F}{\partial t}(0,t)\ dt + \left\{\int_0^t \left[\sigma_B(u,t)\frac{\partial^2 \sigma_B}{\partial t^2}(u,t) + \left(\frac{\partial \sigma_B}{\partial t}(u,t)\right)^2\right]\ du\right\} dt \\ & + \left\{\int_0^t \frac{\partial^2 \sigma_B}{\partial t^2}(u,t)\ dZ(u)\right\} dt + \left[\frac{\partial \sigma_B}{\partial t}(u,t)\bigg|_{u=t}\right] dZ(t). \end{aligned} \tag{7.3.14}$$

Some special cases
Consider the following simple choice for $\sigma_B(t,T)$

$$\sigma_B(t,T) = \sigma(T-t), \quad \sigma \text{ is a constant}, \tag{7.3.15}$$

Eq. (7.3.14) is then reduced to

$$dr(t) = \left[\frac{\partial F}{\partial t}(0,t) + \sigma^2 t\right] dt + \sigma\ dZ(t). \tag{7.3.16}$$

This happens to have similar functional representation as that of the continuous time version of the HL model (see Problem 7.11). The short rate process is seen to be Markovian.

Consider the more general form of $\sigma_B(t,T)$ as follows:

$$\sigma_B(t,T) = x(t)[y(T) - y(t)]. \tag{7.3.17}$$

Substituting into Eq. (7.3.11) and performing the differentiation accordingly, it can be shown that Eq. (7.3.14) can be expressed in the form

$$dr = [\theta(t) - \alpha(t)r]\ dt + \sigma_r(t)\ dZ(t), \tag{7.3.18}$$

for some appropriate functions $\theta(t), \alpha(t)$ and $\sigma_r(t)$ (see Problem 7.12). This is precisely the extended Vasicek model within the class of HW models. The process for r as indicated by Eq. (7.3.18) is obviously Markovian.

Indeed, Hull and White (1993b) showed a stronger result. For non-stochastic $\sigma_B(t,T)$, the process for r is Markovian if and only if $\sigma_B(t,T)$ assumes the form given in Eq. (7.3.17).

Non-Markovian nature

By examining Eq. (7.3.14), the second term shows the dependence on the history of $\sigma_B(u,t)$, while the third term shows the dependence on the history of both $\sigma_B(u,t)$ and $dZ(u)$. Note that $\sigma_B(u,t)$, $u < t$, may depend on the value of stochastic variables observed at times earlier than t and this may cause the nature of the second term to be non-Markovian. When $\sigma_B(u,t)$ is dependent only on u and t, the third term becomes the only source which may cause the process of r to become non-Markovian since it depends also on the history of $dZ(u), u < t$.

Hence, the evolution of the term structure under the HJM framework is in general path dependent. In this way, an upward movement followed by a downward movement of rates does not necessarily lead to the same term structure as a downward movement followed by an upward movement. Consequently, the discrete HJM tree is non-recombining and the number of nodes grows exponentially causing the amount of computation insurmountable even at a moderate number of time steps. This "bushy tree" phenomenon poses problems of computational infeasibility which makes the HJM approach less popular compared to other short rate models. A review on the development process of the HJM term structure models can be found in Jarrow's paper (1997).

7.5 Problems

7.1 (Market price of risk) Consider two securities, both of them are dependent on the short rate. Suppose security A has an expected return of 4% per annum and a volatility of 10% per annum, while security B has a volatility of 20% per annum. Suppose the riskless interest rate is 7% per annum. Find the market price of interest rate risk and the expected returns from security B per annum. Give a financial argument why the market price of the interest rate risk is usually negative.

Hint: The returns on the stocks and bonds are negatively correlated to changes in interest rates.

7.2 Suppose the price of a bond is dependent on the price of a commodity, denoted by Q. Let the stochastic process followed by Q be governed by

$$\frac{dQ}{Q} = \mu \; dt + \sigma \; dZ.$$

Following the arbitrage pricing approach, show that the governing equation for the bond price $B(Q, t)$ is given by [see Eq. (7.1.17)]

$$\frac{\partial B}{\partial t} + \frac{\sigma^2}{2} Q^2 \frac{\partial^2 B}{\partial Q^2} + (\mu - \lambda\sigma) Q \frac{\partial B}{\partial Q} - rB = 0,$$

where λ is the market price of risk and r is the riskless interest rate. Since the commodity is a *traded security* (unlike the interest rate), so the price of the commodity should also satisfy the same governing differential equation as that of the bond price. Substituting Q for B into the differential equation, show that

$$\mu - \lambda\sigma = r.$$

Argue why the governing equation for the bond price now resembles the Black-Scholes equation, which exhibits independence of the risk preference.

7.3 From the bond price representation formula (7.1.18), use Ito's differentiation to show

$$\left.\frac{\partial^2 B}{\partial T^2}\right|_{T=t} = r^2(t) - u(r, t) + w(r, t)\lambda(r, t).$$

Next, deduce from Eq. (7.1.3) that

$$\left.\frac{\partial^2 B}{\partial T^2}\right|_{T=t} = r^2(t) - 2\left.\frac{\partial R}{\partial T}\right|_{T=t}.$$

Lastly, try to relate the market price of risk $\lambda(r, t)$ to $\left.\frac{\partial R}{\partial T}\right|_{T=t}$ [see Eq. (7.1.23)].

7.4 Consider the linear stochastic differential equation

$$dr = [a(t)r + b(t)] \; dt + \rho(t) \; dZ.$$

Show that the mean, $E[r(t)]$, is governed by the following deterministic linear differential equation

$$\frac{d}{dt} E[r(t)] = a(t)E[r(t)] + b(t),$$

while the variance, $\text{var}[r(t)]$, is governed by (Arnold, 1974)

$$\frac{d}{dt} \text{var}[r(t)] = 2a(t)\text{var}[r(t)] + \rho(t)^2.$$

7.5 For the yield curve associated with the Vasicek model [see Eq. (7.1.32)], show that the yield curve is monotonically increasing when $r(t) \leq R_\infty - \frac{\rho^2}{4\alpha^2}$, monotonically decreasing when $r(t) \geq R_\infty + \frac{\rho^2}{2\alpha^2}$, and it is a bumped curve when $R_\infty - \frac{\rho^2}{4\alpha^2} < r(t) < R_\infty + \frac{\rho^2}{2\alpha^2}$. Further, suppose we define the *liquidity premium* of the term structure as

$$\pi(\tau) = F(t, t+\tau) - E_t[r(t+\tau)], \quad \tau \geq 0,$$

where $F(t, t+\tau)$ is the forward rate and E_t is the expectation operator. Show that the liquidity premium for the Vasicek model is given by (Vasicek, 1977)

$$\pi(\tau) = \left(R_\infty - \gamma + \frac{\rho^2}{2\alpha^2}e^{-\alpha\tau}\right)\left(1 - e^{-\alpha\tau}\right), \quad \tau \geq 0.$$

7.6 Consider the following discrete version of the Vasicek model when the spot interest rate $r(t)$ follows the discrete mean reversion binary random walk

$$r(t+1) = \rho r(t) + \alpha \pm \sigma.$$

Let $V(t)$ denote the value of an interest rate contingent claim at current time $t, V_u(t+1)$ and $V_d(t+1)$ be the corresponding values of the contingent claim at time $t+1$ when the interest rate moves up and down, respectively. Let $B(t,T)$ be the price of a zero-coupon bond that pays one unit at time T; and observe that $B(t, t+1) = e^{-r(t)}$. By adopting a similar approach as the Cox-Ross-Rubinstein binomial pricing model, show that the binomial formula that relates $V(t), V_u(t+1)$ and $V_d(t+1)$ is given by (Heston, 1995)

$$V(t) = \frac{pV_u(t+1) + (1-p)V_d(t+1)}{e^{r(t)}},$$

where

$$p = \frac{e^{r(t)} - d}{u - d}, \quad u = \frac{e^{-[\alpha + \rho r(t) + \sigma]}}{B(t, t+2)}, \quad d = \frac{e^{-[\alpha + \rho r(t) - \sigma]}}{B(t, t+2)}.$$

7.7 Show that the steady state density function of the short rate at time T in the Cox-Ingersoll-Ross model [see Eq. (7.1.34a)] is given by (Cox *et al.*, 1985)

$$\lim_{T\to\infty} g(r(T); r(t)) = \frac{\omega^\nu}{\Gamma(\nu)} r^{\nu-1} e^{-\omega r},$$

where $\omega = 2\alpha/\rho^2$ and $\nu = 2\alpha\gamma/\rho^2$. Show that the corresponding steady state mean and variance of $r(T)$ are γ and $\frac{\gamma\rho^2}{2\alpha}$, respectively.

7.8 Show that the bond price for the Cox-Ingersoll-Ross model [see Eqs. (7.1.37a,b,c)] is a decreasing convex function of the interest rate and a decreasing function of time to maturity. Further, show that the bond price is a decreasing convex function of the mean interest rate level γ and an increasing concave function of the speed of adjustment α if $r(t) > \gamma$. What would be the effects on the bond price when the interest rate variance ρ^2 and the market price of risk λ increase?

7.9 Consider the yield to maturity $R(t,T)$ corresponding to the Cox-Ingersoll-Ross model. Show that [see Eqs. (7.1.37a,b,c)]

$$\lim_{T\to\infty} R(t,T) = \frac{2\alpha\gamma}{\theta+\psi}.$$

Explain why an increase in the current interest rate increases yields for all maturities, but the effect is more significant for shorter maturities, while an increase in the steady state mean γ increases all yields but the effect is more significant for longer maturities. What would be the effect on the yields when the interest rate variance ρ^2 and the market price of risk λ increase?

7.10 Suppose the *duration* D of a coupon-bearing coupon bond B at the current time t is defined by

$$D(B,t) = \left[\sum_{i=1}^{n} c_i(t_i - t)e^{-R(t_i-t)} + (t_n - t)Fe^{-R(t_n-t)}\right] \Big/ B,$$

where $c_i, i = 1, 2, \cdots, n$, is the ith coupon on the bond paid at time t_i, F is the face value. Here, R is the yield to maturity on the bond, which is given by the solution of the following equation

$$B = \sum_{i=1}^{n} c_i e^{-R(t_i-t)} + Fe^{-R(t_n-t)}.$$

Show that

$$D(B,t) = -\frac{1}{B}\frac{\partial B}{\partial R}.$$

Give a financial interpretation of duration.

7.11 Consider the continuous time equivalent of the Ho-Lee model can be considered as a degenerate case of the Hull-White model, the diffusion process for the short rate $r(t)$ is

$$dr = \theta(t)\ dt + \sigma_r\ dZ.$$

Let $F(t,T)$ be the instantaneous forward rate for a contract maturing at T and let $B(t,T)$ be the price at time t of a discount bond maturing at time T. Show that the parameter $\theta(t)$ is related to the slope of the initial forward rate curve through the following formula

$$\theta(t) = \frac{\partial F}{\partial T}(0,T)\Big|_{T=t} + \sigma_r^2 t.$$

Suppose we assume the bond price to be of the form

$$B(t,T) = a(t,T)e^{-b(t,T)r},$$

show that

$$b(t,T) = T - t$$
$$\ln a(t,T) = \ln \frac{B(0,T)}{B(0,t)} + (T-t)F(0,t) - \frac{\sigma_r^2}{2}t(T-t)^2.$$

Since the short rates are normally distributed, the discount bond prices are lognormally distributed; so the value of option on discount bonds can be expressed as a variant of the Black-Scholes formula. Show that the value of a European call option on a discount bond is given by (Hull and White, 1994)

$$c(B,t;T_B,T) = B(t,T_B)N(d) - XB(t,T)N(d-\sigma_B),$$

where T_B is the maturity date of the bond underlying the option, X is the strike price, T is the expiration date of the option,

$$d = \frac{1}{\sigma_B}\ln\frac{B(t,T_B)}{XB(t,T)} + \frac{\sigma_B}{2}, \quad \sigma_B^2 = \sigma_r^2(T_B - T)^2(T-t).$$

7.12 The expression for $\ln a(t,T)$ derived from in Eq. (7.2.9b) involves $\phi(t)$. It may be desirable to express $\ln a(t,T)$ solely in terms of $B(0,T)$, the initial bond prices for all maturities. Show that (Hull and White, 1994)

$$\ln a(t,T) = \ln\frac{B(0,T)}{B(0,t)} - b(t,T)\frac{\partial}{\partial t}\ln B(0,t)$$
$$- \frac{\sigma^2}{4\alpha^3}(e^{-\alpha T} - e^{-\alpha t})^2(e^{2\alpha t} - 1).$$

Hence, the bond prices $B(t,T)$ for all maturities at time t can be determined from known initial bond prices for all maturities.

7.13 Consider the extended Vasicek model where the short rate is defined by

$$dr = [\theta(t) - \alpha r]\ dt + \sigma\ dZ.$$

Suppose we define a new variable $x(t)$ where

$$dx(t) = -\alpha x(t)\ dt + \sigma\ dZ,$$

and let $\psi(t) = r(t) - x(t)$, show that $\theta(t)$ and $\psi(t)$ are related by

$$\psi'(t) + \alpha\psi(t) = \theta(t), \quad \psi(0) = r(0).$$

Let $Y(t) = R(0,t)$ where $R(t,T)$ is the yield to maturity. Show that

$$\psi(t) = \frac{d}{dt}[tY(t)] + \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2.$$

Also, show that the bond price $B(t,T)$ can be expressed as (Kijima and Nagayama, 1994)

$$\begin{aligned}\ln B(t,T) = &\ln \frac{B(0,T)}{B(0,t)} + \frac{1}{\alpha}\left[e^{-\alpha(T-t)} - 1\right][r(t) - \psi(t)] \\ &+ \frac{\sigma^2}{4\alpha^3}\left\{1 - [2 - e^{-\alpha(T-t)}]^2 + (2 - e^{-\alpha T})^2 - (2 - e^{-\alpha t})^2\right\}.\end{aligned}$$

7.14 An extended version of the Vasicek model takes the form (Hull and White, 1990)

$$dr = [\theta(t) + \alpha(t)(d - r)]\, dt + \sigma_r(t)\, dZ.$$

Let $\lambda(t)$ denote the time dependent market price of risk. Show that the bond price equation is given by

$$\frac{\partial B}{\partial t} + [\phi(t) - \alpha(t)r]\frac{\partial B}{\partial r} + \frac{\sigma_r(t)^2}{2}\frac{\partial^2 B}{\partial r^2} - rB = 0,$$

where

$$\phi(t) = \alpha(t)d + \theta(t) - \lambda(t)\sigma(t).$$

Suppose we write the bond price $B(r,t;T)$ in the form

$$B(r,t;T) = a(t,T)e^{-b(t,T)r},$$

show that $a(t,T)$ and $b(t,T)$ are governed by

$$\begin{aligned}&\frac{\partial a}{\partial t} - \phi(t)ab + \frac{\sigma_r(t)^2}{2}ab^2 = 0 \\ &\frac{\partial b}{\partial t} - \alpha(t)b + 1 = 0,\end{aligned}$$

with auxiliary conditions: $a(T,T) = 1$ and $b(T,T) = 0$. Solve for $a(t,T)$ and $b(t,T)$ in terms of $\alpha(t), \phi(t)$ and $\sigma_r(t)$. It is desirable to express $a(t,T)$ and $b(t,T)$ in terms of $a(0,t)$ and $b(0,t)$ instead of $\alpha(t)$ and $\phi(t)$. Show that the new set of governing equations for $a(t,T)$ and $b(t,T)$, independent of $\alpha(t)$ and $\phi(t)$, are given by

$$\frac{\partial b}{\partial t}\frac{\partial b}{\partial T} - b\frac{\partial^2 b}{\partial t \partial T} + \frac{\partial b}{\partial T} = 0$$
$$ab\frac{\partial^2 a}{\partial t \partial T} - b\frac{\partial a}{\partial t}\frac{\partial a}{\partial t} - a\frac{\partial a}{\partial t}\frac{\partial b}{\partial T} + \frac{\sigma(t)^2}{2}a^2 b^2 \frac{\partial b}{\partial T} = 0.$$

The auxiliary conditions are the known values of $a(0,T)$ and $b(0,T)$, $a(T,T) = 1$ and $b(T,T) = 0$. Finally, show that the solutions for $b(t,T)$ and $a(t,T)$, expressed in terms of $b(0,T)$ and $a(0,T)$, are given by

$$b(t,T) = \frac{b(0,T) - b(0,t)}{\frac{\partial b}{\partial T}(0,T)\big|_{T=t}}$$
$$\ln a(t,T) = \ln \frac{a(0,T)}{a(0,t)} - b(t,T)\frac{\partial}{\partial T}[\ln a(0,T)]\big|_{T=t}$$
$$- \frac{1}{2}\left[b(t,T)\frac{\partial b}{\partial T}(0,T)\big|_{T=t}\right]^2 \int_0^t \left[\frac{\sigma(u)}{\frac{\partial b}{\partial T}(0,T)\big|_{T=u}}\right]^2 du.$$

7.15 Consider the Vasicek short rate model of the form

$$dr = (\theta - \alpha r)\ dt + \sigma\ dZ,$$

where α, θ and σ are constant. Suppose we translate the above Vasicek model into the Heath-Jarrow-Morton framework, show that the volatility structure and the initial term structure of the forward rate are given by

$$\sigma_F(t,T) = \sigma e^{-\alpha(T-t)}$$
$$f(0,T) = \frac{\theta}{\alpha} + e^{-\alpha T}\left[r(0) - \frac{\theta}{\alpha}\right] - \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha T})^2.$$

7.16 Suppose the bond price volatility $\sigma_B(t,T)$ takes the form

$$\sigma_B(t,T) = x(t)[y(T) - y(t)],$$

show that the differential of the short rate can be represented in the following form (Hull and White, 1993a)

$$dr = [\theta(t) - \alpha(t)r]\ dt + \sigma_r(t)\ dZ.$$

Express $\theta(t), \alpha(t)$ and $\sigma_r(t)$ in terms of $x(t)$ and $y(t)$.

Hint: Use Eqs. (7.3.14-15). In particular, explore the relation between the terms involving the integration of $dZ(\tau)$, namely,

$$\int_0^t \frac{\partial \sigma_B}{\partial t}(\tau,t)\ dZ(\tau) \text{ and } \int_0^t \frac{\partial^2 \sigma_B}{\partial t^2}(\tau,t)\ dZ(\tau).$$

These two terms appear in $r(t)$ and $dr(t)$, respectively.

7.17 Empirical evidence reveals that the long interest rate and the spread (short interest rate minus long interest rate) are almost uncorrelated. Suppose we choose the stochastic state variables in the two-factor interest rate model to be the spread s and the long rate ℓ, where

$$\begin{aligned} ds &= \beta_s(s,\ell,t)\ dt + \eta_s(s,\ell,t)\ dZ_s, \quad s = r - \ell, \\ d\ell &= \beta_\ell(s,\ell,t)\ dt + \eta_\ell(s,\ell,t)\ dZ_\ell. \end{aligned}$$

Assuming zero correlation between the above processes, show that the price of a default free bond $B(s,\ell,\tau)$ is governed by

$$\begin{aligned} \frac{\partial B}{\partial \tau} &= \frac{1}{2}\eta_s^2 \frac{\partial^2 B}{\partial s^2} + \frac{1}{2}\eta_\ell^2 \frac{\partial^2 B}{\partial \ell^2} + (\beta_s - \lambda \eta_s)\frac{\partial B}{\partial s} \\ &\quad + \left(\frac{\eta_\ell^2}{\ell} - s\ell\right)\frac{\partial B}{\partial \ell} - (s+\ell)B, \end{aligned}$$

where λ is the market price of spread risk. Schaefer and Schwartz (1984) proposed the following stochastic processes for s and ℓ

$$\begin{aligned} ds &= m(\mu - s)\ dt + \gamma\ dZ_s \\ d\ell &= \beta_\ell(s,\ell,t)\ dt + \sigma\sqrt{\ell}\ dZ_\ell. \end{aligned}$$

Show that the above bond price equation becomes

$$\begin{aligned} \frac{\partial B}{\partial \tau} &= \frac{1}{2}\gamma^2 \frac{\partial^2 B}{\partial s^2} + \frac{1}{2}\sigma^2 \ell \frac{\partial^2 B}{\partial \ell^2} + (m\mu - \lambda\gamma - ms)\frac{\partial B}{\partial s} \\ &\quad + (\sigma^2 - \ell s)\frac{\partial B}{\partial \ell} - (s+\ell)B. \end{aligned}$$

The payoff function at maturity is $B(s,\ell,0) = 1$. Schaefer and Schwartz (1984) proposed the following analytic approximation procedure to solve the above equation. They take the coefficient of $\dfrac{\partial B}{\partial \ell}$ to be constant by treating s as a frozen constant $\widehat{s}$. Now, we write the bond price as the product of two functions, namely,

$$B(s,\ell,\tau) = X(s,\tau)Y(\ell,\tau).$$

Show that the bond price equation can be split into the following pair of equations:

$$\frac{\partial X}{\partial \tau} = \frac{1}{2}\gamma^2 \frac{\partial^2 X}{\partial s^2} + (m\mu - \lambda\gamma - ms)\frac{\partial X}{\partial s} - sX, \quad X(s,0) = 1,$$

and

$$\frac{\partial Y}{\partial \tau} = \frac{1}{2}\sigma^2 \ell \frac{\partial^2 Y}{\partial \ell^2} + (\sigma^2 - \ell\widehat{s})\frac{\partial Y}{\partial \ell} - \ell Y, \quad Y(\ell,0) = 1.$$

Assuming that all parameters are constant, solve the above two equations for $X(s,\tau)$ and $Y(\ell,\tau)$.

7.18 Consider the continuous version of the Ho-Lee short rate model

$$dr = \theta(t)\ dt + \sigma\ dZ,$$

where $\theta(t)$ is deterministic and σ is constant. Show that the HJM formulation of the above model is given by

$$dF(t,T) = \sigma^2(T-t)\ dt + \sigma\ dZ$$

with

$$F(0,T) = r(0) - \frac{\sigma^2 T^2}{2} + \int_0^T \theta(u)\ du.$$

7.19 Consider the pricing of a futures contract on a zero-coupon bond, where the interest rate r is assumed to follow the Vasicek process defined by Eq. (24) in Sec. 7.1. On the expiration date T_F of the futures, a bond of unit par value with maturity date T_B is delivered. Let $B(r,t;T_B)$ and $V(r,t;T_F,T_B)$ denote, respectively, the bond price and futures price at the current time t. Show that the governing equation for the futures price is given by

$$\frac{\partial V}{\partial t} + \frac{\rho^2}{2}\frac{\partial^2 V}{\partial r^2} + [\alpha(\gamma - r) - \lambda\rho]\frac{\partial V}{\partial r} = 0, \quad t < T_F,$$

with terminal payoff condition: $V(r,T_F;T_F,T_B) = B(r,T_F;T_B)$. By assuming the solution of the futures price to be the form:

$$V(r,t;T_F,T_B) = e^{-rX(t)-Y(t)},$$

show that (Chen, 1992)

$$\begin{aligned} X(t) &= E(t,T_B) - E(t,T_F) \\ Y(t) &= D[T_B - T_F - X(t)] - \frac{\rho^2}{2\alpha^2}\left[X(t) - \frac{\alpha}{2}X(t)^2 - E(T_F,T_B)\right] \end{aligned}$$

where

$$D = \gamma - \frac{\rho\lambda}{\alpha} - \frac{\rho^2}{2\alpha^2} \quad \text{and} \quad E(t,T) = \frac{1 - e^{-\alpha(T-t)}}{\alpha}.$$

7.20 Consider the Hull-White model where the interest rate process follows

$$dr(t) = [\phi(t) - \alpha r]\ dt + \sigma\ dZ,$$

where $Z(t)$ is a Brownian process under the probability measure P. Using the relation

$$d[r(t)e^{\alpha t}] = \phi(t)e^{\alpha t}\ dt + \sigma e^{\alpha t}\ dZ,$$

show that

$$\begin{aligned}\int_t^T r(u)\,du &= r(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \int_t^u \phi(s)e^{-\alpha(u-s)}ds\,du \\ &\quad + \int_t^T \int_t^u \sigma e^{-\alpha(u-s)}dZ(s)\,du \\ &= r(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \phi(s)\frac{1-e^{-\alpha(T-s)}}{\alpha}\,ds \\ &\quad + \int_t^T \frac{\sigma}{\alpha}[1-e^{-\alpha(T-s)}]\,dZ(s).\end{aligned}$$

Accordingly, the mean and variance of $\displaystyle\int_t^T r(u)\,du$ are found to be

$$\begin{aligned}E_P\left[\int_t^T r(u)\,du\right] &= r(t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \int_t^T \phi(s)\frac{1-e^{-\alpha(T-s)}}{\alpha}\,ds \\ \text{var}\left(\int_t^T r(u)\,du\right) &= \int_t^T \frac{\sigma^2}{\alpha^2}[1-e^{-\alpha(T-s)}]^2\,ds.\end{aligned}$$

The bond price $B(r,t;T)$ is given by [see Eq. (2.3.18)]

$$\begin{aligned}B(r,t;T) =&\ E_P[e^{-\int_t^T r(u)\,du}] \\ =&\ \exp\left(-E_P\left[\int_t^T r(u)\,du\right] + \frac{1}{2}\text{var}\left(\int_t^T r(u)\,du\right)\right).\end{aligned}$$

Show that the above result agrees with that given by Eq. (7.2.9a,b).

7.21 Suppose the forward rate as a function of time t evolves as

$$df(t,T) = \mu(t,T)\,dt + \sigma\,dZ_t$$

where $\mu(t,T)$ is a deterministic function of t and T. Explain why the forward rates at different maturities are perfectly correlated.
Hint: Show that the forward rate is normally distributed where

$$f(t,T) = f(0,T) + \int_0^t \mu(u,T)\,du + \sigma Z_t.$$

Hence, $f(t,T) - f(t,S)$ is purely deterministic.