Chapter 7 – Kinetic energy, potential energy, work

I. Kinetic energy.

II. Work.

III. Work - Kinetic energy theorem.

IV. Work done by a constant force: Gravitational force

V. Work done by a variable force.

- Spring force.
- General: 1D, 3D, Work-Kinetic Energy Theorem

VI. Power

VII. Potential energy → Energy of configuration

VIII. Work and potential energy

IX. Conservative / Non-conservative forces

X. Determining potential energy values: gravitational potential energy, elastic potential energy

Energy: scalar quantity associated with a state (or condition) of one or more objects.

I. Kinetic energy

Energy associated with the state of motion of an object.

$$K = \frac{1}{2}mv^2 \qquad (7.1)$$

Units: 1 Joule = 1J = 1 kgm^2/s^2 = N m

II. Work

Energy transferred "to" or "from" an object by means of a force acting on the object.

To → +W

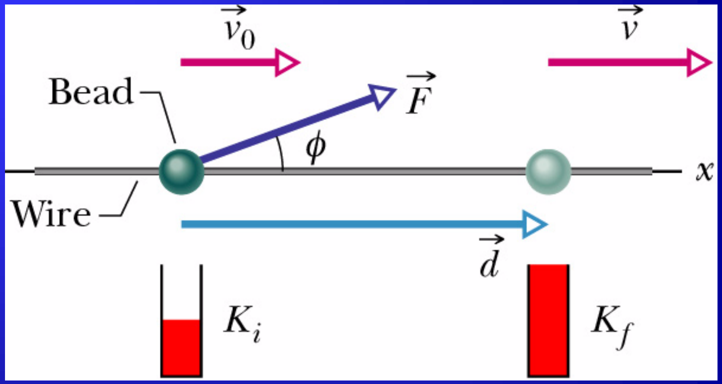
From → -W

- Constant force: $F_x = ma_x$

$$v^2 = v_0^2 + 2a_x d \rightarrow a_x = \frac{v^2 - v_0^2}{2d}$$

$$F_x = ma_x = \frac{1}{2}m(v^2 - v_0^2)\frac{1}{d} \rightarrow ma_x d = \frac{1}{2}m(v^2 - v_0^2)$$

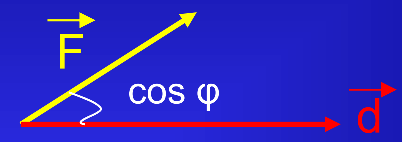
$$\rightarrow \frac{1}{2}m(v^2 - v_0^2) = K_f - K_i = F_x d \rightarrow \boxed{W = F_x d}$$



Work done by the force = Energy transfer due to the force.

- *To calculate the work done on an object by a force during a displacement, we use only the force component along the object's displacement. The force component perpendicular to the displacement does zero work.*

$$W = F_x d = F \cos\varphi \cdot d = \vec{F} \cdot \vec{d} \qquad (7.3)$$



- Assumptions: 1) F=cte, 2) Object particle-like.

Units: 1 Joule = 1J = 1 kgm^2/s^2

$$\varphi < 90^\circ \rightarrow +W$$
$$180^\circ > \varphi > 90^\circ \rightarrow -W$$
$$\varphi = 90^\circ \rightarrow 0$$

A force does +W when it has a vector component in the same direction as the displacement, and –W when it has a vector component in the opposite direction. W=0 when it has no such vector component.

Net work done by several forces = Sum of works done by individual forces.

Calculation: 1) $W_{net} = W_1 + W_2 + W_3 + \ldots$

2) $\vec{F}_{net} \rightarrow W_{net} = \vec{F}_{net} \cdot \vec{d}$

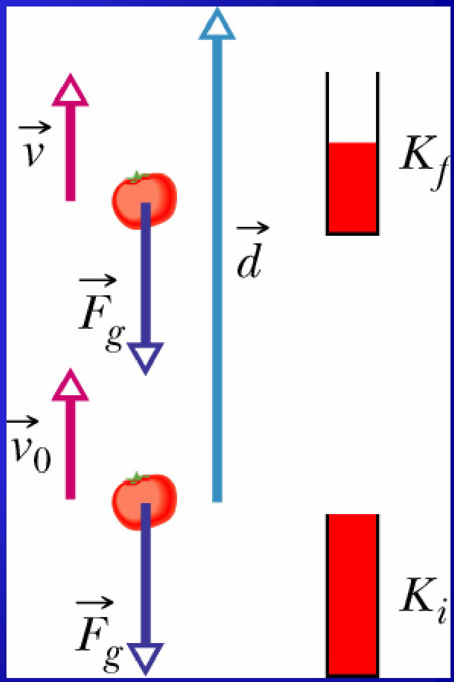
II. Work-Kinetic Energy Theorem

$$\Delta K = K_f - K_i = W \qquad (7.4)$$

Change in the kinetic energy of the particle = Net work done on the particle

III. Work done by a constant force

- Gravitational force: $W = \vec{F} \cdot \vec{d} = mgd \cos\varphi \qquad (7.5)$

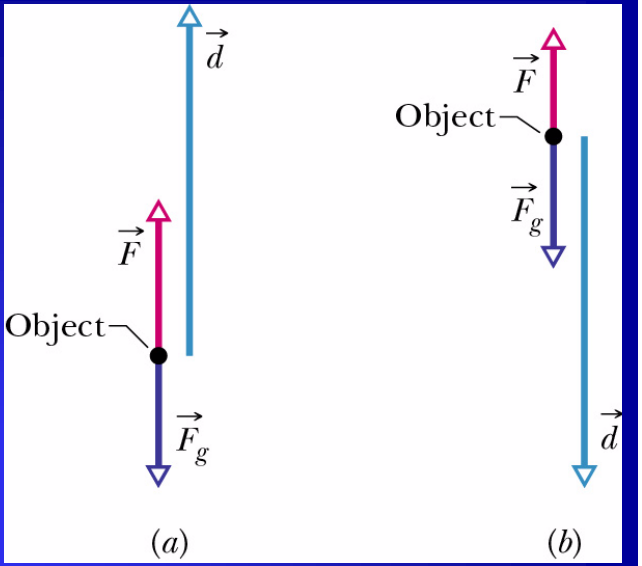


Rising object: W= mgd cos180º = -mgd → F_g transfers mgd energy from the object's kinetic energy.

Falling object: W= mgd cos 0º = +mgd → F_g transfers mgd energy to the object's kinetic energy.

- External applied force + Gravitational force:

$$\Delta K = K_f - K_i = W_a + W_g \qquad (7.6)$$



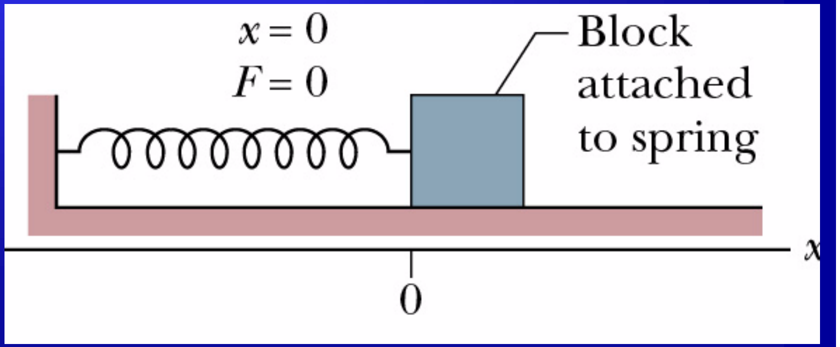
Object stationary before and after the lift: $W_a+W_g=0$

The applied force transfers the same amount of energy to the object as the gravitational force transfers from the object.

IV. Work done by a variable force

- Spring force: $$\vec{F} = -k\vec{d} \qquad (7.7)$$

Hooke's law

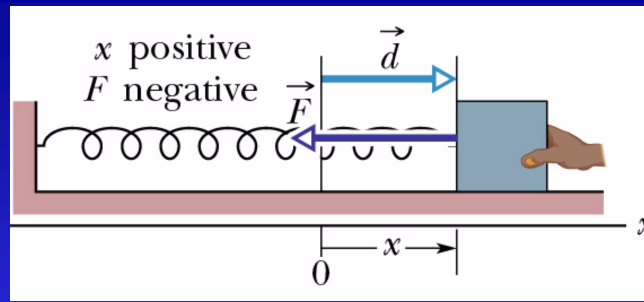


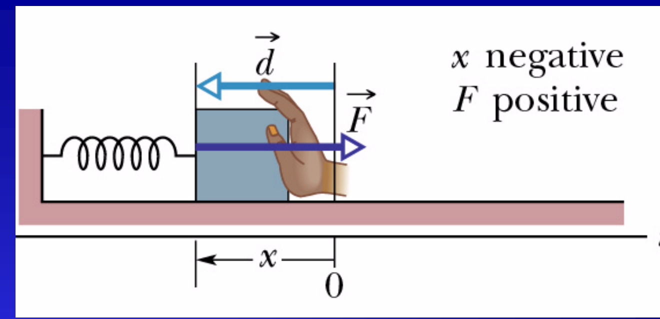
k = spring constant → measures spring's stiffness.

Units: N/m

Hooke's law

$$1D \rightarrow F_x = -kx$$

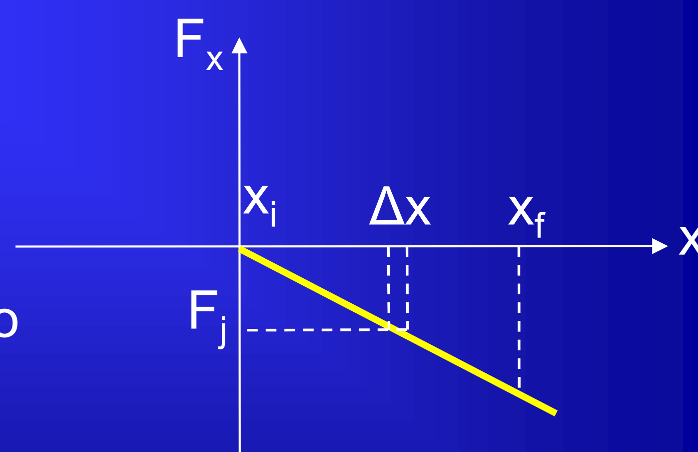




Work done by a spring force:

- Assumptions:

- Spring is massless → $m_{spring} << m_{block}$
- Ideal spring → obeys Hooke's law exactly.
- Contact between the block and floor is frictionless.
- Block is particle-like.



- Calculation:

1) The block displacement must be divided into many segments of infinitesimal width, Δx.

2) $F(x) \approx$ cte within each short Δx segment.

$$W_s = \sum F_j \Delta x \Rightarrow \; \Delta x \to 0 \;\; \Rightarrow W_s = \int_{x_i}^{x_f} F \, dx = \int_{x_i}^{x_f} (-kx) \;\, dx$$

$$W_S = -k \int_{x_i}^{x_f} x \; dx = \left(-\frac{1}{2} k \right) \left[x^2 \right]_{x_i}^{x_f} = \left(-\frac{1}{2} k \right) (x_f^2 - x_i^2)$$

$$W_s = \frac{1}{2} k \; x_i^2 - \frac{1}{2} k \; x_f^2$$

W_s=0 → If Block ends up at x_f=x_i.

$$W_s = -\frac{1}{2} k \; x_f^2 \qquad\qquad if \;\; x_i = 0$$

Work done by an applied force + spring force:

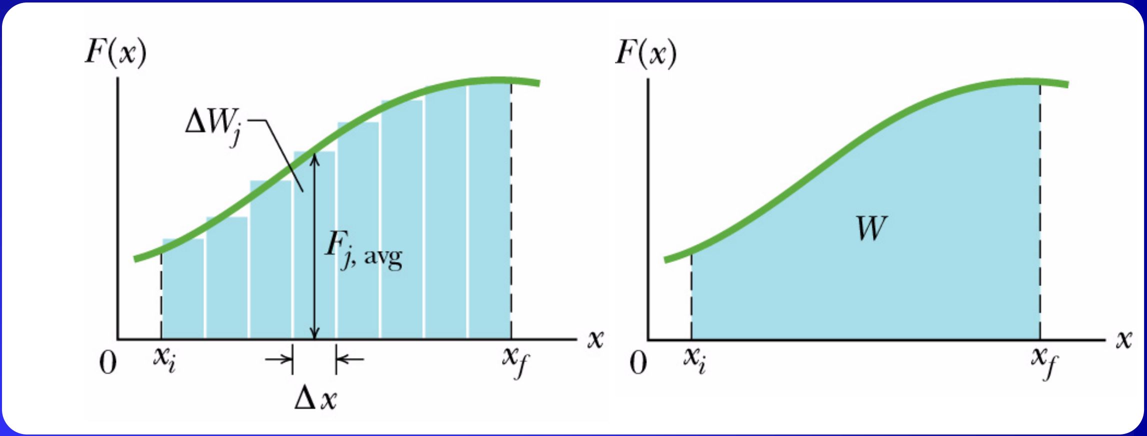
$$\Delta K = K_f - K_i = W_a + W_s$$

Block stationary before and after the displacement: ΔK=0 →W_a= -W_s

→The work done by the applied force displacing the block is the negative of the work done by the spring force.

Work done by a general variable force:

1D-Analysis



$$\Delta W_j = F_{j,avg}\Delta x$$

$$W = \sum \Delta W_j = \sum F_{j,avg}\Delta x$$

$$better\ approximation \Rightarrow more\ \Delta x,\ \ \Delta x \to 0$$

$$\Rightarrow W = \lim_{\Delta x \to 0} F_{j,avg}\Delta x \Rightarrow \boxed{W = \int_{x_i}^{x_f} F(x)dx} \qquad (7.10)$$

Geometrically: Work is the area between the curve F(x) and the x-axis.

3D-Analysis

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \ ; \qquad F_x = F(x), \ F_y = F(y), \ F_z = F(z)$$

$$d\vec{r} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k}$$

$$dW = \vec{F}\cdot d\vec{r} = F_x dx + F_y dy + F_z dz \Rightarrow \boxed{W = \int_{r_i}^{r_f} dW = \int_{x_i}^{x_f} F_x dx + \int_{y_i}^{y_f} F_y dy + \int_{z_i}^{z_f} F_z dz}$$

Work-Kinetic Energy Theorem - Variable force

$$W = \int_{x_i}^{x_f} F(x)dx = \int_{x_i}^{x_f} ma\ dx$$

$$ma\ dx = m\boxed{\frac{dv}{dt}}dx \quad \rightarrow \ m\frac{dv}{dx}v\ dx = mvdv$$

$$\left(\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v\right)$$

$$W = \int_{v_i}^{v_f} mv\ dv = m\int_{v_i}^{v_f} v\ dv = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = K_f - K_i = \Delta K$$

V. Power

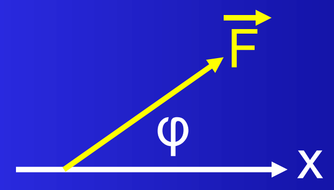
Time rate at which the applied force does work.

- Average power: amount of work done in an amount of time Δt by a force.

$$P_{avg} = \frac{W}{\Delta t} \qquad (7.12)$$

- Instantaneous power: instantaneous time rate of doing work.

$$P = \frac{dW}{dt} \qquad (7.13)$$

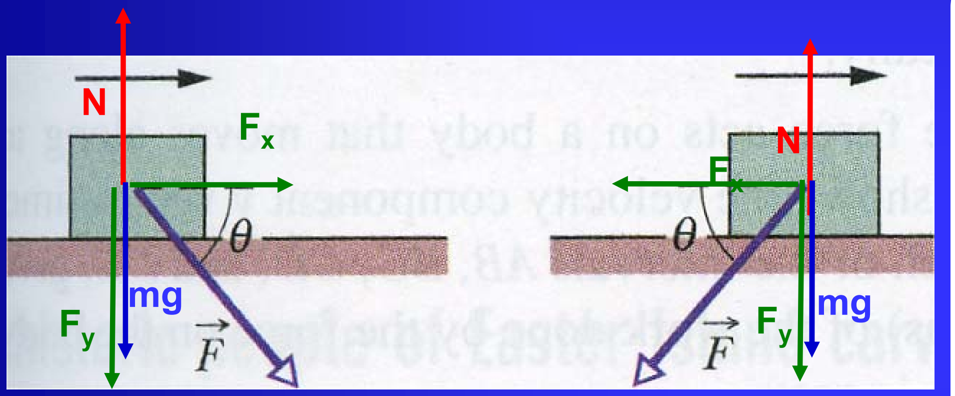


$$P = \frac{dW}{dt} = \frac{F\cos\varphi\, dx}{dt} = F\cos\varphi\left(\frac{dx}{dt}\right) = Fv\cos\varphi = \vec{F}\cdot\vec{v} \qquad (7.14)$$

Units: 1 Watt= 1 W = 1J/s

1 kilowatt-hour = 1 kW·h = 3.60 x 10^6 J = 3.6 MJ

54. In the figure (a) below a 2N force is applied to a 4kg block at a downward angle θ as the block moves rightward through 1m across a frictionless floor. Find an expression for the speed v_f at the end of that distance if the block's initial velocity is: (a) 0 and (b) 1m/s to the right. (c) The situation in (b) is similar in that the block is initially moving at 1m/s to the right, but now the 2N force is directed downward to the left. Find an expression for the speed of the block at the end of the 1m distance.



$$W = \vec{F}\cdot\vec{d} = (F\cos\theta)d$$

$$W = \Delta K = 0.5m(v_f^2 - v_0^2)$$

$$(a)\ \ v_0 = 0 \rightarrow \Delta K = 0.5mv_f^2$$

$$(2N)\cos\theta = 0.5(4kg)v_f^2$$

$$\rightarrow v_f = \sqrt{\cos\theta}\ \ m/s$$

$$(b)\ \ v_0 = 1m/s \rightarrow \Delta K = 0.5mv_f^2 - 0.5\cdot(4kg)\cdot(1m/s)^2$$

$$(2N)\cos\theta = 0.5(4kg)v_f^2 - 2J$$

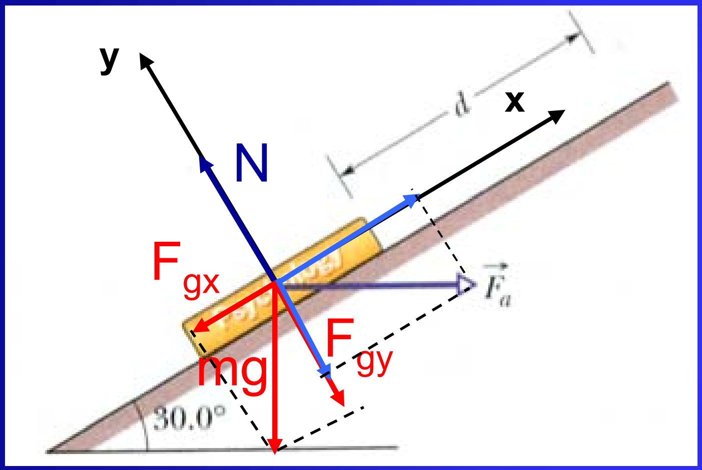
$$\rightarrow v_f = \sqrt{1+\cos\theta}\ \ m/s$$

$$(c)\ \ v_0 = 1m/s \rightarrow \Delta K = 0.5mv_f^2 - 2J$$

$$-(2N)\cos\theta = 0.5(4kg)v_f^2 - 2J$$

$$\rightarrow v_f = \sqrt{1-\cos\theta}\ \ m/s$$

18. In the figure below a horizontal force F_a of magnitude 20N is applied to a 3kg psychology book, as the book slides a distance of d=0.5m up a frictionless ramp. (a) During the displacement, what is the net work done on the book by F_a, the gravitational force on the book and the normal force on the book? (b) If the book has zero kinetic energy at the start of the displacement, what is the speed at the end of the displacement?



$$\vec{N} \perp \vec{d} \rightarrow W = 0$$

$$Only\ F_{gx},\ F_{ax}\ do\ work$$

$$(a)\ W = W_{Fa_x} - W_{Fg_x} \qquad or \qquad W_{net} = \vec{F}_{net} \cdot \vec{d}$$

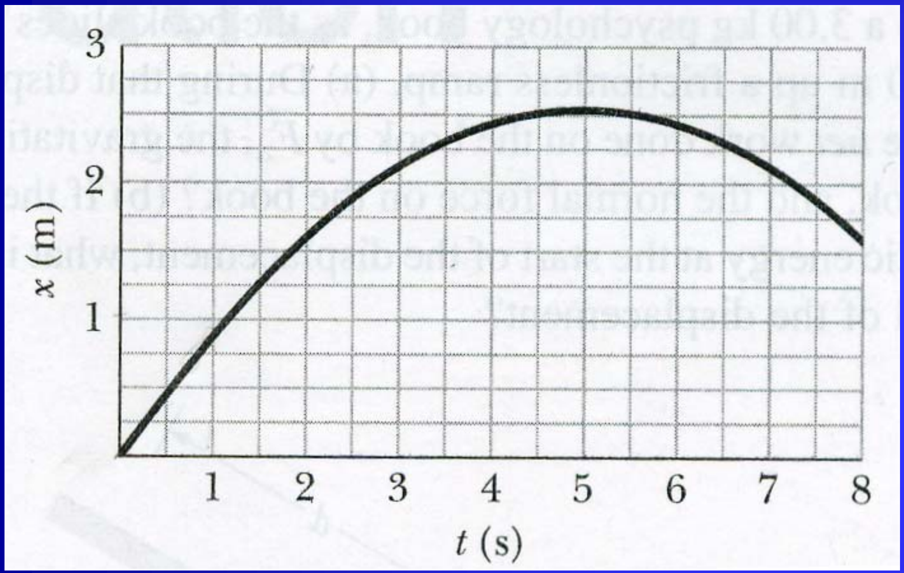
$$F_{net} = Fa_x - Fg_x = 20\cos 30^\circ - mg\sin 30^\circ$$

$$W_{net} = (17.32N - 14.7N)0.5m = 1.31J$$

$$(b)\ K_0 = 0 \rightarrow W = \Delta K = K_f$$

$$W = 1.31J = 0.5mv_f^2 \rightarrow v_f = 0.93m/s$$

55. A 2kg lunchbox is sent sliding over a frictionless surface, in the positive direction of an x axis along the surface. Beginning at t=0, a steady wind pushes on the lunchbox in the negative direction of x, Fig. below. Estimate the kinetic energy of the lunchbox at (a) t=1s, (b) t=5s. (c) How much work does the force from the wind do on the lunch box from t=1s to t=5s?



$$Motion \rightarrow concave\ downward\ \ parabola$$

$$x = \frac{1}{10}t(10-t)$$

$$v = \frac{dx}{dt} = 1 - \frac{2}{10}t$$

$$a = \frac{dv}{dt} = -\frac{2}{10} = -0.2m/s^2$$

$$F = cte = ma = (2kg)(-0.2m/s^2) = -0.4N$$

$$W = F \cdot x = (-0.4N)(t - 0.1t^2)$$

$$(a)\ \ t = 1s \rightarrow v_f = 0.8m/s$$

$$K_f = 0.5(2kg)(0.8m/s)^2 = 0.64J$$

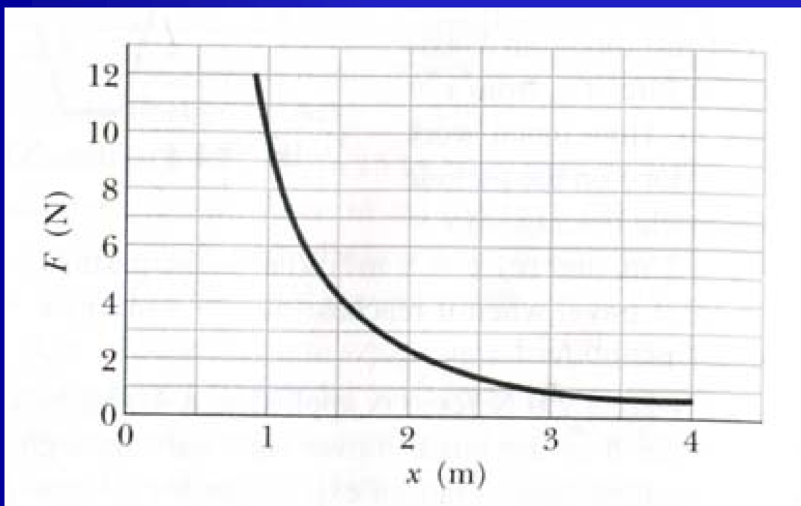
$$(b)\ \ t = 5s \rightarrow v_f = 0$$

$$K_f = 0J$$

$$(c)\ \ W = \Delta K = K_f(5s) - K_f(1s)$$

$$W = 0 - 0.64 = -0.64J$$

74. (a) Find the work done on the particle by the force represented in the graph below as the particle moves from x=1 to x=3m. (b) The curve is given by $F=a/x^2$, with $a=9Nm^2$. Calculate the work using integration

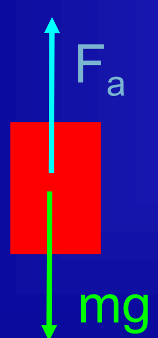


$$(a)\ W = Area\ under\ curve$$

$$W = (11.5 squares)(0.5m)(1N) = 5.75J$$

$$(b)\ W = \int_1^3 \frac{9}{x^2}dx = -9\left[\frac{1}{x}\right]_1^3 = -9(\frac{1}{3}-1) = 6J$$

73. An elevator has a mass of 4500kg and can carry a maximum load of 1800kg. If the cab is moving upward at full load at 3.8m/s, what power is required of the force moving the cab to maintain that speed?



$$m_{total} = 4500kg + 1800kg = 6300kg$$

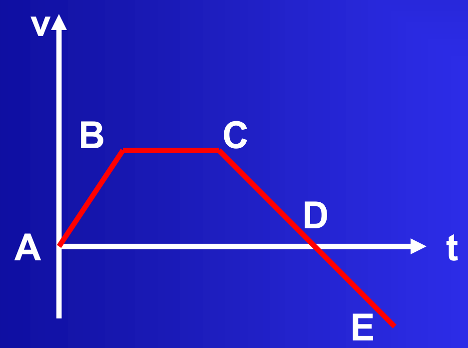
$$\vec{F}_a + m\vec{g} = \vec{F}_{net} = \vec{0} \rightarrow F_a - F_g = 0 \rightarrow$$

$$F_a = mg = (6300kg)(9.8m/s^2) = 61.74kN$$

$$P = \vec{F}\cdot\vec{v} = (61.74kN)(3.8m/s)$$

$$P = 234.61kW$$

A single force acts on a body that moves along an x-axis. The figure below shows the velocity component versus time for the body. For each of the intervals AB, BC, CD, and DE, give the sign (plus or minus) of the work done by the force, or state that the work is zero.



$$W = \Delta K = K_f - K_0 = \frac{1}{2}m\left(v_f^2 - v_0^2\right)$$

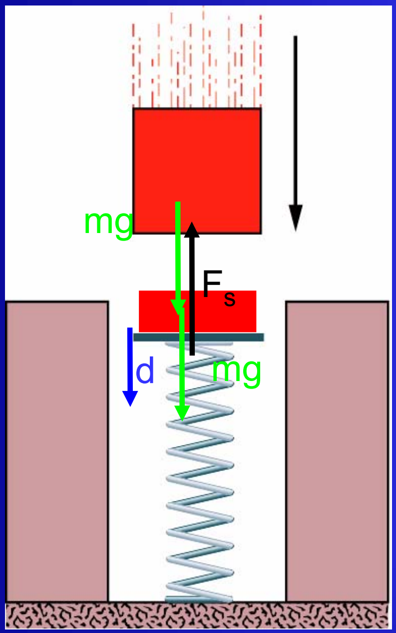
$$AB \rightarrow v_B > v_A \rightarrow W > 0$$

$$BC \rightarrow v_C = v_B \rightarrow W = 0$$

$$CD \rightarrow v_D < v_C \rightarrow W < 0$$

$$DE \rightarrow v_E < 0,\ \ v_D = 0 \rightarrow W > 0$$

50. A 250g block is dropped onto a relaxed vertical spring that has a spring constant of k=2.5N/cm. The block becomes attached to the spring and compresses the spring 12 cm before momentarily stopping. While the spring is being compressed, what work is done on the block by (a) the gravitational force on it and (b) the spring force? (c) What is the speed of the block just before it hits the spring? (Friction negligible) (d) If the speed at impact is doubled, what is the maximum compression of the spring?



$$(a)\ W_{Fg} = \vec{F}_g \vec{d} = mgd = (0.25kg)(9.8m/s^2)(0.12m) = 0.29J$$

$$(b)\ W_s = -\frac{1}{2}kd^2 = -0.5\cdot(250N/m)(0.12m)^2 = -1.8J$$

$$(c)\ W_{net} = \Delta K = 0.5mv_f^2 - 0.5mv_i^2$$

$$v_f = 0 \rightarrow K_f = 0 \rightarrow \Delta K = -K_i = -0.5mv_i^2 = W_{Fg} + W_s$$

$$0.29J - 1.8J = -0.5\cdot(0.25kg)v_i^2$$

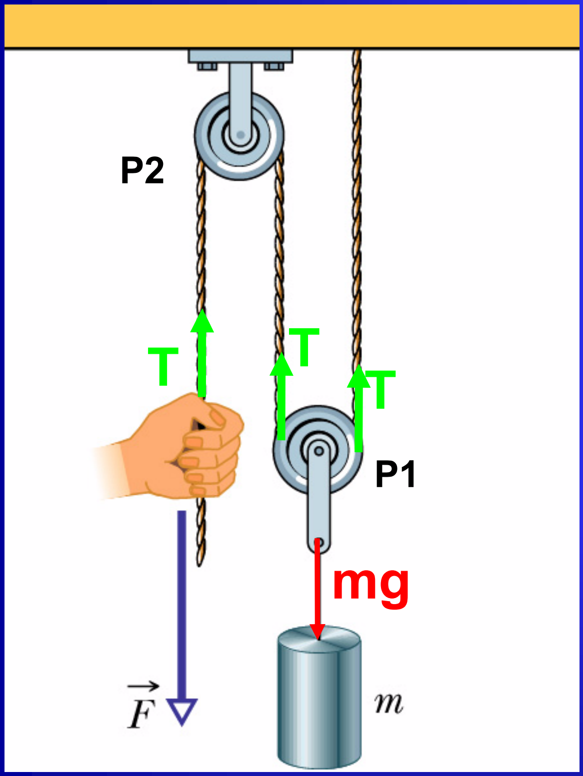
$$\rightarrow v_i = 3.47m/s$$

$$(d)\ If\ v_i' = 6.95m/s \rightarrow Maximum\ \ spring\ \ compression? v_f = 0$$

$$W_{net} = mgd' - 0.5kd'^2 = \Delta K = -0.5mv_i'^2$$

$$d' = 0.23m$$

62. In the figure below, a cord runs around two massless, frictionless pulleys; a canister with mass m=20kg hangs from one pulley; and you exert a force F on the free end of the cord. (a) What must be the magnitude of F if you are to lift the canister at a constant speed? (b) To lift the canister by 2cm, how far must you pull the free end of the cord? During that lift, what is the work done on the canister by (c) your force (via the cord) and (d) the gravitational force on the canister?



$$(a)\ \ Pulley\ 1:\ \ v = cte \rightarrow F_{net} = 0 \rightarrow 2T - mg = 0 \rightarrow T = 98N$$

$$Hand - cord:\ \ T - F = 0 \rightarrow F = \frac{mg}{2} = 98N$$

(b) To rise "m" 0.02m, two segments of the cord must be shorten by that amount. Thus, the amount of the string pulled down at the left end is: 0.04m

$$(c)\ \ W_F = F \cdot d = (98N)(0.04m) = 3.92J$$

$$(d)\ \ W_{Fg} = -mgd = (-0.02m)(20kg)(9.8m/s^2) = -3.92J$$

$W_F + W_{Fg} = 0$ There is no change in kinetic energy.

I. Potential energy

Energy associated with the <u>arrangement</u> of a system of objects that exe
forces on one another.

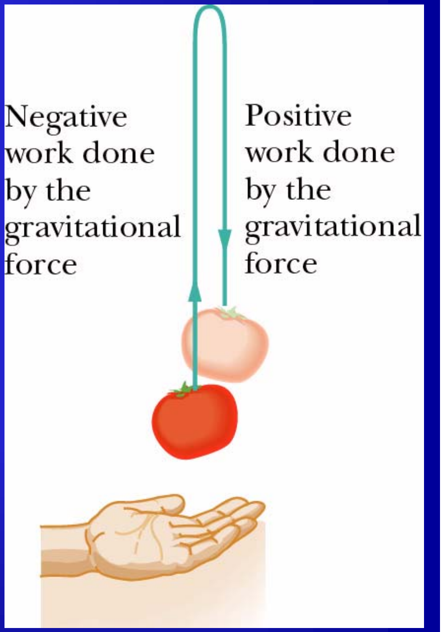
Units: J

Examples:

- Gravitational potential energy: associated with the state of separation
  between objects which can attract one another via the gravitational for

- Elastic potential energy: associated with the state
of
  compression/extension of an elastic object.

II. Work and potential energy

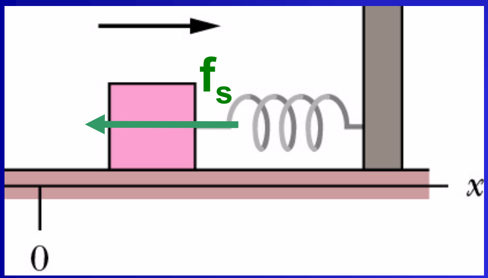


If tomato <u>rises</u> → gravitational force transfers energy "from" tomato's kinetic energy "to" the gravitational potential energy of the tomato-Earth system.

If tomato <u>falls down</u> → gravitational force transfers energy "from" the gravitational potential energy "to" the tomato's kinetic energy.

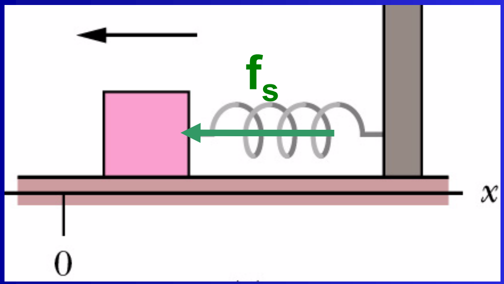
$\Delta U = -W$ Also valid for elastic potential energy

Spring compression



Spring force does –W on block → energy transfer from kinetic energy of the block to potential elastic energy of the spring.

Spring extension



Spring force does +W on block → energy transfer from potential energy of the spring to kinetic energy of the block.

General:

- System of two or more objects.

- A force acts between a particle in the system and the rest of the system.

- When system configuration changes → force does work on the object (W_1) transferring energy between KE of the object and some other form of energy of the system.

- When the configuration change is reversed → force reverses the energy transfer, doing W_2.

III. Conservative / Nonconservative forces

- If $W_1=W_2$ always → conservative force.

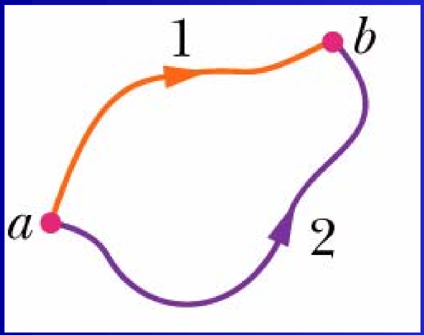
Examples: Gravitational force and spring force → associated potential energies.

- If $W_1 \neq W_2$ → nonconservative force.

Examples: Drag force, frictional force → KE transferred into thermal energy. Non-reversible process.

- **Thermal energy:** Energy associated with the random movement of atoms and molecules. This is not a potential energy.

- **Conservative force:** The net work it does on a particle moving around every closed path, from an initial point and then back to that point is zero.
- The net work it does on a particle moving between two points does not depend on the particle's path.

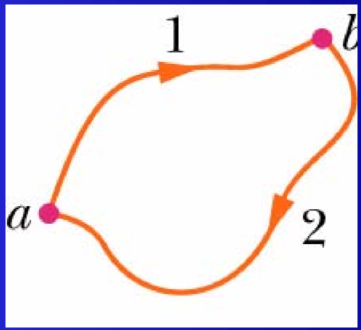


Conservative force → $W_{ab,1} = W_{ab,2}$

Proof:

$W_{ab,1} + W_{ba,2} = 0 \rightarrow W_{ab,1} = -W_{ba,2}$

$W_{ab,2} = -W_{ba,2} \rightarrow W_{ab,2} = W_{ab,1}$



IV. Determining potential energy values

$$W = \int_{x_i}^{x_f} F(x)dx = -\Delta U$$

Force F is conservative

Gravitational potential energy:

$$\Delta U = -\int_{y_i}^{y_f} (-mg)dy = mg[y]_{y_i}^{y_f} = mg(y_f - y_i) = mg\Delta y$$

Change in the gravitational potential energy of the particle-Earth system.

$$U_i = 0, \quad y_i = 0 \rightarrow U(y) = mgy$$ Reference configuration

The gravitational potential energy associated with particle-Earth system depends <u>only</u> on particle's vertical position "y" relative to the reference position y=0, not on the horizontal position.

Elastic potential energy: $$\Delta U = -\int_{x_i}^{x_f} (-kx)dx = \frac{k}{2}\left[x^2\right]_{x_i}^{x_f} = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2$$

Change in the elastic potential energy of the spring-block system.

Reference configuration → when the spring is at its relaxed length and th block is at x_i=0.

$$U_i = 0, \quad x_i = 0 \rightarrow U(x) = \frac{1}{2}kx^2$$

Remember! Potential energy is always associated with a system.

V. Conservation of mechanical energy

Mechanical energy of a system: Sum of its potential (U) and kinetic (K) energies.

$E_{mec} = U + K$

Assumptions: - Only conservative forces cause energy transfer within the system.

- The system is isolated from its environment → No external force from an object outside the system causes energy changes inside the system.

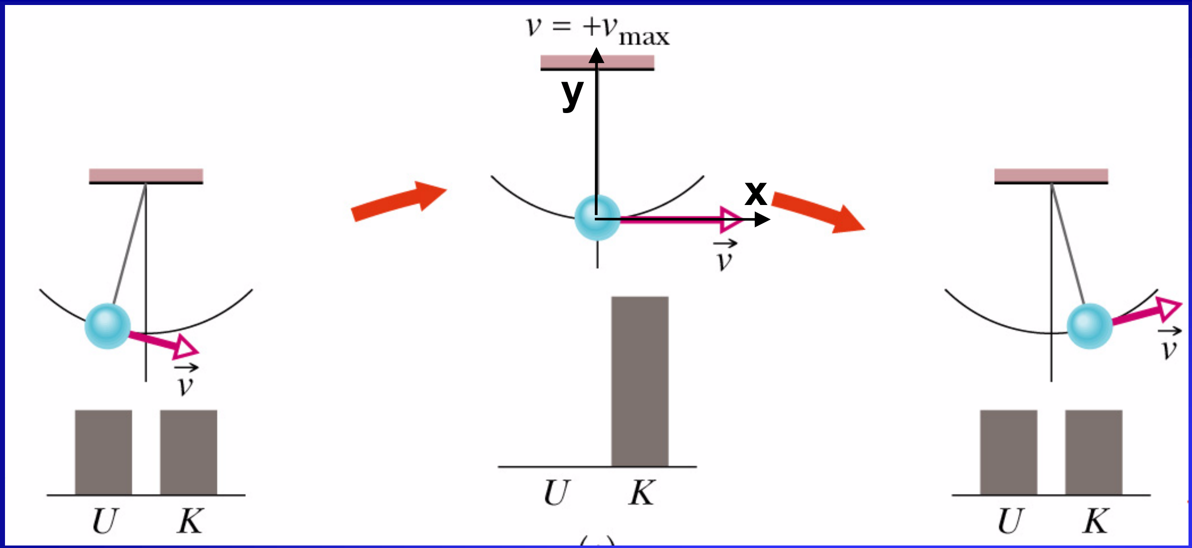
$$W = \Delta K$$
$$W = -\Delta U$$

$$\Delta K + \Delta U = 0 \rightarrow (K_2 - K_1) + (U_2 - U_1) = 0 \rightarrow K_2 + U_2 = K_1 + U_1$$

$\Delta E_{mec} = \Delta K + \Delta U = 0$

- *In an isolated system where only conservative forces cause energy changes, the kinetic energy and potential energy can change, but their sum, the mechanical energy of the system cannot change.*

- *When the mechanical energy of a system is conserved, we can relate the sum of kinetic energy and potential energy at one instant to that at another instant without considering the intermediate motion and without finding the work done by the forces involved.*



E_{mec}= constant

$$\Delta E_{mec} = \Delta K + \Delta U = 0$$

$$K_2 + U_2 = K_1 + U_1$$

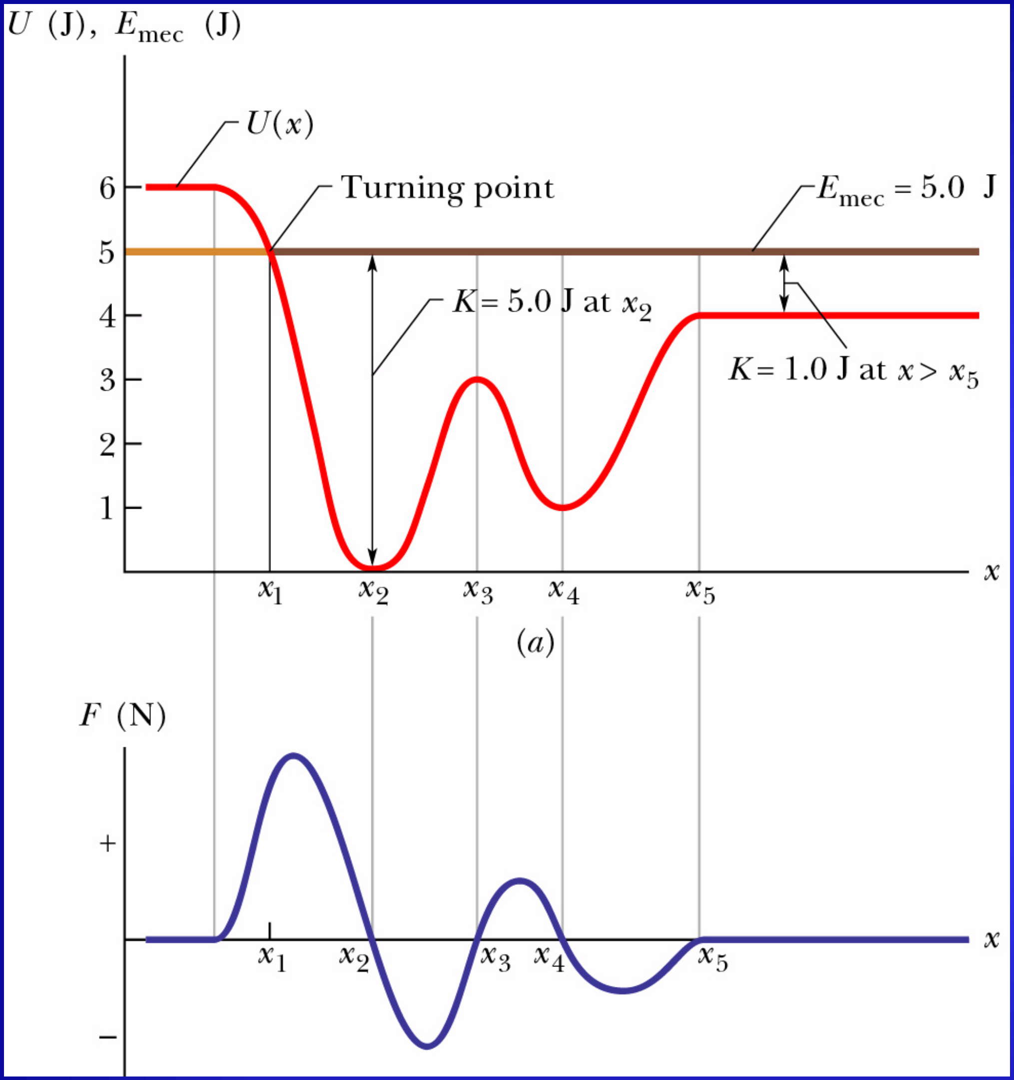
Potential energy curves

Finding the force analytically:

$$\Delta U(x) = -W = -F(x)\Delta x \rightarrow F(x) = -\frac{dU(x)}{dx} \quad (1D\ motion)$$

- The force is the negative of the slope of the curve U(x) versus x.

- The particle's kinetic energy is: K(x) = E_{mec} – U(x)



Turning point: a point x at which the particle reverses its motion (K=0).

K always ≥0 $(K=0.5mv^2 \geq 0)$

Examples:

$x = x_1 \rightarrow E_{mec} = 5J = 5J + K \rightarrow K = 0$

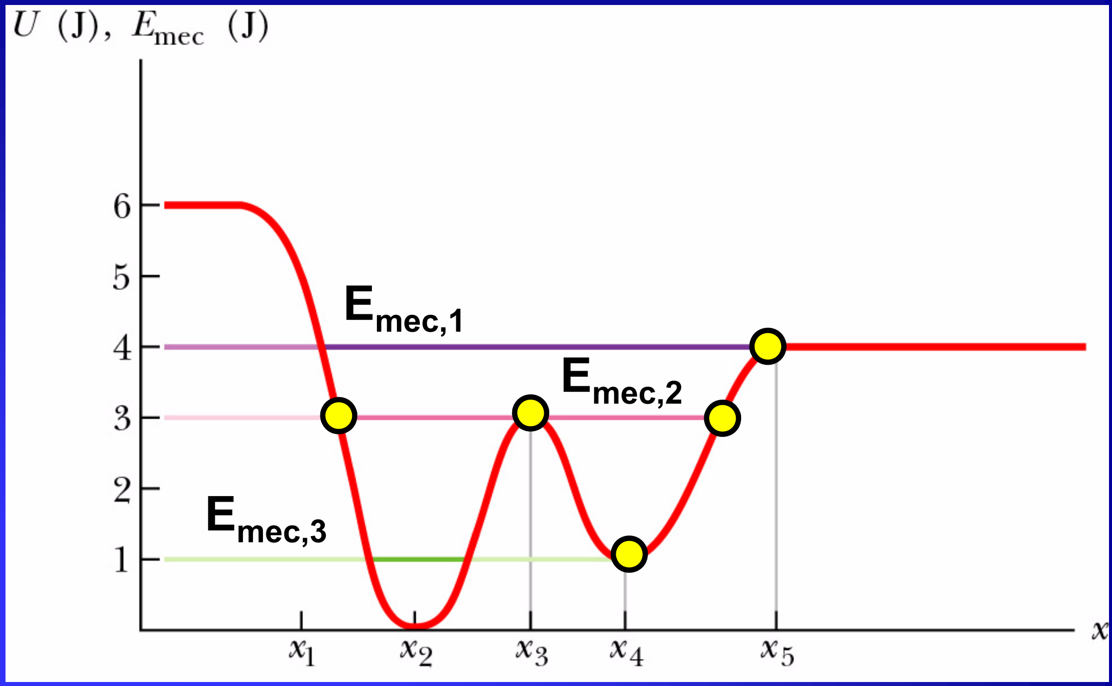
$x < x_1 \rightarrow E_{mec} = 5J = >5J + K \rightarrow K < 0 \rightarrow$ impossible

Equilibrium points: where the slope of the U(x) curve is zero → F(x)=0

$\Delta U = -F(x)\,dx \rightarrow \Delta U/dx = -F(x)$

$\Delta U(x)/dx = -F(x) \rightarrow$ **Slope**

Equilibrium points



Example: $x \geq x_5 \rightarrow E_{mec,1} = 4J = 4J + K \rightarrow K = 0$ and also $F = 0 \rightarrow x_5$ **neutral equilibrium**

$x_2 > x > x_1$, $x_5 > x > x_4 \rightarrow E_{mec,2} = 3J = 3J + K \rightarrow K = 0 \rightarrow$ **Turning points**

$x_3 \rightarrow K = 0, F = 0 \rightarrow$ particle stationary $\rightarrow$ **Unstable equilibrium**

$x_4 \rightarrow E_{mec,3} = 1J = 1J + K \rightarrow K = 0, F = 0$, it cannot move to $x > x_4$ or $x < x_4$, since then $K < 0$ $\rightarrow$

Stable equilibrium

Review: Potential energy

$W = -\Delta U$

- The zero is arbitrary → Only potential energy differences have physical meaning.

- The potential energy is a scalar function of the position.

- The force (1D) is given by: $F = -dU/dx$

P1. The force between two atoms in a diatomic molecule can be represented by the following potential energy function:

$$U(x) = U_0\left[\left(\frac{a}{x}\right)^{12} - 2\left(\frac{a}{x}\right)^{6}\right]$$

where U_0 and *a* are constants.

i) Calculate the force F_x

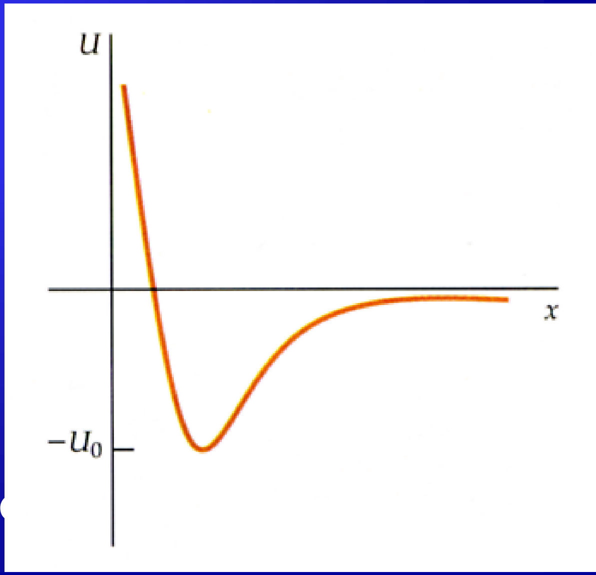
$$F(x) = -\frac{dU(x)}{dx} = -U_0\left[12\left(\frac{-a}{x^2}\right)\left(\frac{a}{x}\right)^{11} - 2\left(\frac{-a}{x^2}\right)6\left(\frac{a}{x}\right)^{5}\right] =$$

$$-U_0\left[-12a^{12}x^{-13} + 12a^{6}x^{-7}\right] = \frac{12U_0}{a}\left[\left(\frac{a}{x}\right)^{13} - \left(\frac{a}{x}\right)^{7}\right]$$

ii) Minimum value of U(x).

$$U(x)_{\min}\ \ if\ \ \frac{dU(x)}{dx} = -F(x) = 0 \rightarrow \frac{-12U_0}{a}\left[\left(\frac{a}{x}\right)^{13} - \left(\frac{a}{x}\right)^{7}\right] = 0$$

$$\rightarrow x = a \qquad U(a) = U_0[1-2] = -U_0$$



U_0 is approx. the energy necessary to dissociate the two atoms.