Chapter 7 Linear Momentum

Definition of linear momentum
An object with mass m and moving with velocity $\vec{\mathbf{v}}$ has linear momentum $\vec{\mathbf{p}}$

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}$$

Many Particles
If many particles m_1, m_2, etc. are moving with velocities $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, etc., the total linear momentum of the system is the vector sum of the individual momenta,

$$\vec{\mathbf{p}} = \sum_i \vec{\mathbf{p}}_i = \sum_i m_i \vec{\mathbf{v}}_i$$

Linear momentum is a vector quantity. We have to use our familiar rules for vector addition when dealing with momentum.

Impulse-Momentum Theorem
Starting with Newton's second law

$$\sum \vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

and using the definition of acceleration

$$\vec{\mathbf{a}} = \frac{\Delta \vec{\mathbf{v}}}{\Delta t}$$

we have

$$\sum \vec{\mathbf{F}} = m\frac{\Delta \vec{\mathbf{v}}}{\Delta t}$$

If the mass is constant, we can pull it inside the Δ operator

$$\sum \vec{\mathbf{F}} = \frac{\Delta(m\vec{\mathbf{v}})}{\Delta t}$$

$$\sum \vec{\mathbf{F}} = \frac{\Delta \vec{\mathbf{p}}}{\Delta t}$$

$$\boxed{\Delta \vec{\mathbf{p}} = \sum \vec{\mathbf{F}} \Delta t}$$

This states that the change in linear momentum is caused by the impulse. The quantity

$$\sum \vec{\mathbf{F}} \Delta t$$

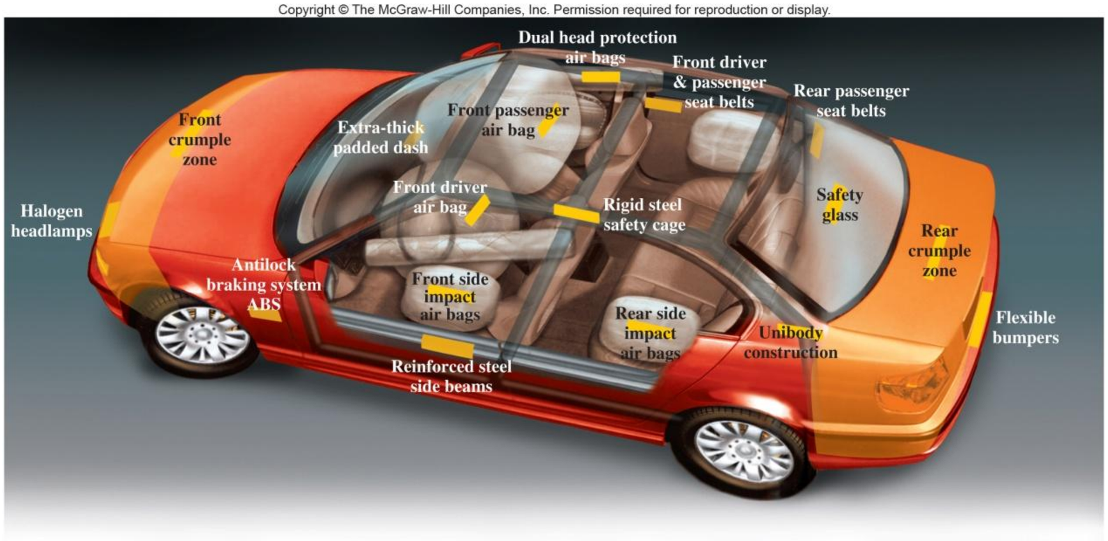
is called the impulse. For situations where the force is not constant, we use the average force,

$$\text{impulse} = \vec{\mathbf{F}}_{av} \Delta t$$

Restatement of Newton's second law
There is a more general form of Newton's second law

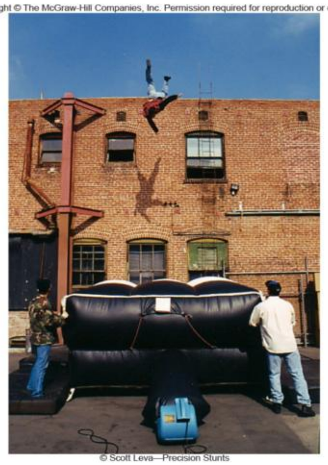
$$\sum \vec{\mathbf{F}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{p}}}{\Delta t}$$

The net force is the rate of change of momentum.



The applications of the impulse-momentum theorem are unlimited. In an automobile we have crumple zones, air bags, and bumpers. What do seat belts do?

Here is another example:



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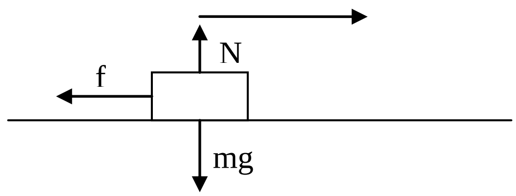
The idea is to lengthen the time Δt during which the force acts, so that the force is diminished while changing the momentum a defined amount.

Problem 10. What average force is necessary to bring a 50.0-kg sled from rest to a speed of 3.0 m/s in a period of 20.0 s. Assume frictionless ice.

Solution: A sketch of the situation would be



We would use the impulse-momentum theorem

$$\Delta\vec{\mathbf{p}} = \sum \vec{\mathbf{F}}\Delta t$$

To use a vector equation we need to take components

$$\Delta p_x = \sum F_x \Delta t$$

The change in momentum is

$$\Delta p_x = p_{fx} - p_{ix} = mv_{fx} - mv_{ix} = 0 - (50\,\text{kg})(3.0\,\text{m/s}) = -150\,\text{kg}\cdot\text{m/s}$$

$$\begin{aligned}\Delta p_x &= \sum F_x \Delta t \\ &= F_{av}\Delta t \\ F_{av} &= \frac{\Delta p_x}{\Delta t} = \frac{-150\,\text{kg}\cdot\text{m/s}}{20\text{s}} = -7.5\,\text{kg}\cdot\text{m/s}^2 = -7.5\,\text{N}\end{aligned}$$

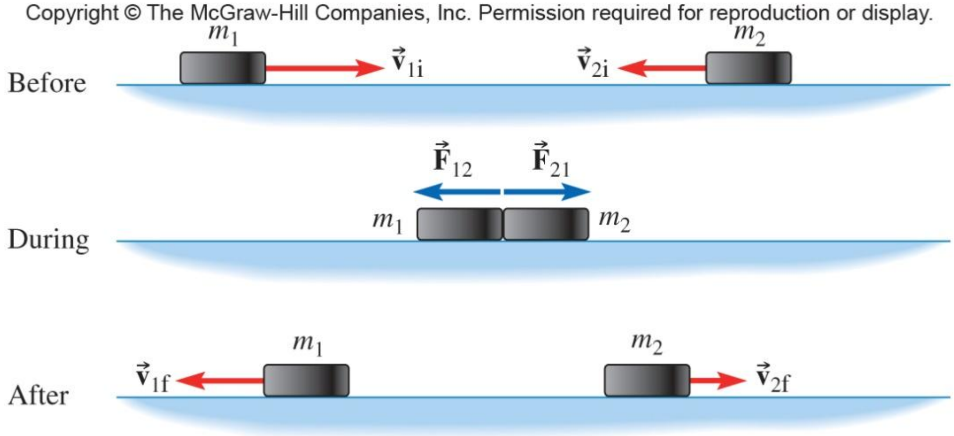
What is the meaning of the negative sign?

Conservation of Linear Momentum
Consider the collision between two pucks. When they collide, they exert forces on each other,

$$\vec{\mathbf{F}}_{12} \text{ and } \vec{\mathbf{F}}_{21}$$

By Newton's third law,

$$\vec{\mathbf{F}}_{12} = -\vec{\mathbf{F}}_{21}$$



The total force acting on both pucks is then

$$\sum \vec{\mathbf{F}} = \vec{\mathbf{F}}_{12} + \vec{\mathbf{F}}_{21} = 0$$

By the impulse-momentum theorem

$$\begin{aligned}\Delta \vec{\mathbf{p}} &= \sum \vec{\mathbf{F}} \Delta t \\ &= 0\end{aligned}$$

Which leads to

$$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$$

In a system composed of more than two objects, interactions between objects inside the system do not change the total momentum of the system – they just transfer some momentum from one part of the system to another. Only external interactions can change the total momentum of the system.

- The total momentum of a system is the vector sum of the momenta of each object in the system
- External interactions can change the total momentum of a system.
- Internal interactions do not change the total momentum of a system.

The Law of Conservation of Linear Momentum
If the net external force acting on a system is zero, then the momentum of the system is conserved.

$$\text{If } \sum \vec{\mathbf{F}}_{ext} = 0, \qquad \vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$$

Linear momentum is always conserved for an isolated system. Of course, we deal with components

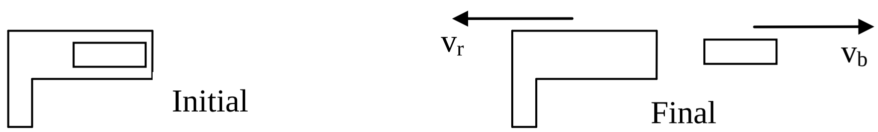
$$p_{ix} = p_{fx} \qquad \text{and} \qquad p_{iy} = p_{fy}$$

I like to say that momentum is conserved in collisions and explosions.

Note: The total momentum of the system is conserved. The momentum of an individual particle can change.

Problem 18. A rifle has a mass of 4.5 kg and it fires a bullet of mass 10.0 g at a muzzle speed of 820 m/s. What is the recoil speed of the rifle as the bullet leaves the gun barrel?

Solution: Draw a sketch of the initial and final situations.



Since only internal forces act, linear momentum is conserved.

$$\vec{\mathbf{p}}_i = \vec{\mathbf{p}}_f$$

The motion is in the *x*-direction,

$$p_{ix} = p_{fx}$$

Initially, everything is at rest and $p_{ix} = 0$. The final momentum is the vector sum of the momenta of the bullet and the rifle. From the diagram,

$$p_{fx} = m_b v_b - m_r v_r$$

Using conservation of momentum

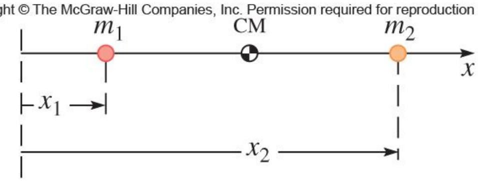
$$\begin{aligned} p_{ix} &= p_{fx} \\ 0 &= m_b v_b - m_r v_r \\ v_r &= \frac{m_b v_b}{m_r} = \frac{(0.010\,\text{kg})(820\,\text{m/s})}{4.5\,\text{kg}} = 1.82\,\text{m/s} \end{aligned}$$

The heavier the rifle, the smaller the recoil speed.

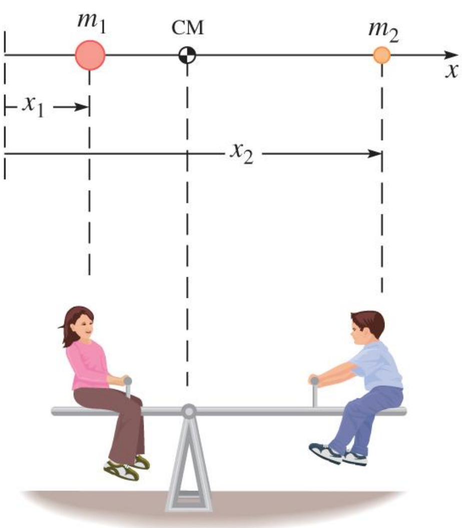
Center of Mass

We have seen that the momentum of an isolated system is conserved even though parts of the system may interact with other parts; internal interactions transfer momentum between parts of the system but do not change the total momentum of the system. We can define point called the center of mass (CM) that serves as an average location of the system.

What if a system is not isolated, but has external interactions? Again imagine all of the mass of the system concentrated into a single point particle located at the CM. The motion of this fictitious point particle is determined by Newton's second law, where the net force is the sum of all the external forces acting on any part of the system. In the case of a complex system composed of many parts interacting with each other, the motion of the CM is considerably simpler than the motion of an arbitrary particle of the system.



(a)



(b)

For the two particles pictured above, the CM is

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

Notice that the CM is closer to the more massive object.

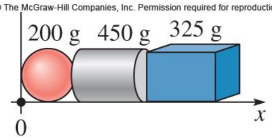
For many particles the definition is generalized to

$$\vec{\mathbf{r}}_{CM} = \frac{\sum m_i \vec{\mathbf{r}}_i}{M}$$

The more useful component form is

$$x_{CM} = \frac{\sum m_i x_i}{M} \qquad y_{CM} = \frac{\sum m_i y_i}{M} \qquad z_{CM} = \frac{\sum m_i z_i}{M}$$

Problem 33. Find the *x*-coordinate of the CM of the composite object shown in the figure. The sphere, cylinder, and rectangular solid all have uniform composition. Their masses and dimensions are: sphere: 200 g, diameter = 10 cm; cylinder: 450 g, length = 17 cm, radius = 5.0 cm; rectangular solid: 325 g, length in *x*-direction = 16 cm, height = 10 cm, depth = 12 cm.



Solution: Because the different pieces have uniform composition, the center of mass of each piece is located at the geometrical center of that piece. (See the note at the top of page 240.) For the sphere, m_s = 200 g, x_s = 5 cm. For the cylinder, m_c = 450 g, x_c = 10 cm + 8.5 cm = 18.5 cm. For the rectangular solid, m_r = 325 g, x_r = 10 cm + 17 cm + 8 cm = 35 cm. The center of mass is

$$\begin{aligned} x_{CM} &= \frac{\sum m_i x_i}{M} \\ &= \frac{m_s x_s + m_c x_c + m_r x_r}{m_s + m_c + m_r} \\ &= \frac{(200\text{ g})(5\text{ cm}) + (450\text{ g})(18.5\text{ cm}) + (325\text{ g})(35\text{ cm})}{(200\text{ g}) + (450\text{ g}) + (325\text{ g})} \\ &= 21.2\text{ cm} \end{aligned}$$

Motion of the Center of Mass

How is the velocity of the CM related to the velocities of the various parts of the system of particles?

During a short time interval Δt, each particle changes position by $\Delta\vec{\mathbf{r}}_i$. The CM changes

$$\Delta\vec{\mathbf{r}}_{CM} = \frac{\sum m_i \Delta\vec{\mathbf{r}}_i}{M}$$

Divide each side by the time interval Δt

$$\frac{\Delta\vec{\mathbf{r}}_{CM}}{\Delta t} = \frac{\sum m_i \dfrac{\Delta\vec{\mathbf{r}}_i}{\Delta t}}{M}$$

In general, the velocity is defined as

$$\vec{\mathbf{v}} = \frac{\Delta \vec{\mathbf{r}}}{\Delta t}$$

Using the definition of velocity

$$\vec{\mathbf{v}}_{CM} = \frac{\sum m_i \vec{\mathbf{v}}_i}{M}$$
$$M\vec{\mathbf{v}}_{CM} = \sum m_i \vec{\mathbf{v}}_i$$

The right hand side is the total linear momentum of the system of particles. We have the very useful relation

$$\vec{\mathbf{p}}_{total} = M\vec{\mathbf{v}}_{CM}$$

For a complicated system consisting of many particles moving in different directions, the total linear momentum of the system can be found from the total mass of the system and the velocity of the center of mass.

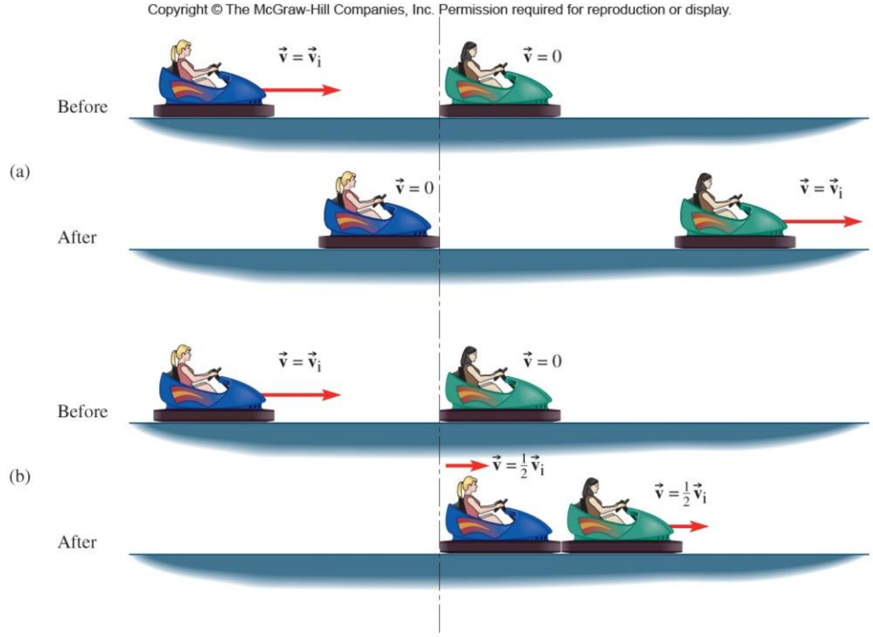
We have shown that for an isolated system, the total linear momentum of the system is conserved. In such system, the equation above implies that the CM must move with constant velocity regardless of the motions of the individual particles. On the other hand, what if the system is not isolated? If a net force acts on a system, the CM does not move with constant velocity. Instead, it moves as if all the mass were concentrated there into a fictitious point particle with all the external forces acting on that point. The motion of the CM obeys the following statement of Newton's second law:

$$\sum \vec{\mathbf{F}}_{ext} = M\vec{\mathbf{a}}_{CM}$$

A complicated system is reduced to treating the system as a point particle located at its center of mass reacting only to external forces!

Collisions in One Dimension

In general, conservation of momentum is not enough to predict what will happen after a collision. Here is a collision between to objects with the same mass. Many other outcomes are possible besides the two given,



Some vocabulary

- A collision in which the total kinetic energy is the same before and after is called **elastic**.
- When the final kinetic energy is less than the initial kinetic energy, the collision is said to be **inelastic**. Collisions between two macroscopic objects (for example, billiard balls) are generally inelastic to some degree, but sometimes the change in kinetic energy is so small that we treat them as elastic.
- When a collision results in two objects sticking together, the collision is **perfectly inelastic**. The decrease of kinetic energy is a perfectly inelastic collision is as large as possible (consistent with the conservation of momentum).

Problem-Solving Strategy for Collisions Involving Two Objects (page 245)

1. Draw before and after diagrams of the collision
2. Collect and organize information on the masses and velocities of the two objects before and after the collision. Express the velocities in component form (with correct algebraic signs).
3. Set the sum of the momenta of the two before and after the collision equal to the sum of the momenta after the collision. Write one equation for each direction:

$$m_1 v_{1ix} + m_2 v_{2ix} = m_1 v_{1fx} + m_2 v_{2fx}$$

$$m_1 v_{1iy} + m_2 v_{2iy} = m_1 v_{1fy} + m_2 v_{2fy}$$

4. If the collision is known to be perfectly inelastic, set the final velocities equal:

$$v_{1fx} = v_{2fx} \text{ and } v_{1fy} = v_{2fy}$$

5. If the collision is known to be perfectly elastic, then set the final kinetic energy equal to the initial kinetic energy:

$$\tfrac{1}{2} m_1 v_{1i}{}^2 + \tfrac{1}{2} m_2 v_{2i}{}^2 = \tfrac{1}{2} m_1 v_{1f}{}^2 + \tfrac{1}{2} m_2 v_{2f}{}^2$$

6. Solve for the unknown quantities.

Example: A one-dimensional elastic collision.

initial final

Conservation of momentum gives

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

Elastic collision means that kinetic energy is conserved

$$\tfrac{1}{2} m_1 v_{1i}{}^2 + \tfrac{1}{2} m_2 v_{2i}{}^2 = \tfrac{1}{2} m_1 v_{1f}{}^2 + \tfrac{1}{2} m_2 v_{2f}{}^2$$

Cancelling the ½ and grouping like masses on the same side of the equation,

$$m_1 (v_{1i}{}^2 - v_{1f}{}^2) = m_2 (v_{2f}{}^2 - v_{2i}{}^2)$$

Doing the same with the linear momentum equation

$$m_1 (v_{1i} - v_{1f}) = m_2 (v_{2f} - v_{2i})$$

Divide the two equations

$$\frac{m_1 (v_{1i}{}^2 - v_{1f}{}^2)}{m_1 (v_{1i} - v_{1f})} = \frac{m_2 (v_{2f}{}^2 - v_{2i}{}^2)}{m_2 (v_{2f} - v_{2i})}$$

$$v_{1i} + v_{1f} = v_{2f} + v_{2i}$$

This can be rewritten as

$$v_{1i} - v_{2i} = v_{2f} - v_{1f}$$

The equation states that m_1 approaches m_2 with the same speed as m_2 moves away from m_1 after the collision. Important result. Solve the above for v_{2f}

$$v_{2f} = v_{1f} + v_{1i} - v_{2i}$$

and substitute into the momentum equation

$$\begin{aligned}
m_1 v_{1i} + m_2 v_{2i} &= m_1 v_{1f} + m_2 v_{2f} \\
&= m_1 v_{1f} + m_2 (v_{1f} + v_{1i} - v_{2i}) \\
&= m_1 v_{1f} + m_2 v_{1f} + m_2 v_{1i} - m_2 v_{2i} \\
(m_1 - m_2) v_{1i} + 2 m_2 v_{2i} &= (m_1 + m_2) v_{1f} \\
v_{1f} &= \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} + \left(\frac{2 m_2}{m_1 + m_2} \right) v_{2i}
\end{aligned}$$

Using very similar reasoning, we can find an expression for the final speed of m_2

$$v_{2f} = \left(\frac{2 m_1}{m_1 + m_2} \right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2} \right) v_{2i}$$

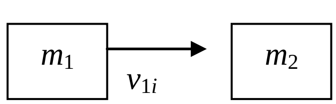
Be careful when you use these equations. A mass moving to the left would have a negative speed.

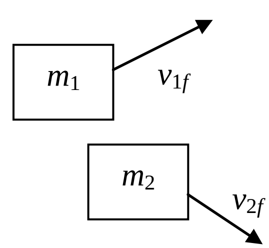
If m_2 is initially at rest, $v_{2i} = 0$ and we have

$$\begin{aligned}
v_{1f} &= \left(\frac{m_1 - m_2}{m_1 + m_2} \right) v_{1i} \\
v_{2f} &= \left(\frac{2 m_1}{m_1 + m_2} \right) v_{1i}
\end{aligned}$$

- After the collision, m_2 moves to the right.
- If $m_1 > m_2$, m_1 continues to move to the right.
- If $m_1 = m_2$, m_1 stops and m_2 to move to the right with speed v_{1i}.
- If $m_1 < m_2$, m_1 bounces back to the left.

Example: Two dimensional elastic collision between two identical masses. One mass is initially at rest.





Elastic collisions conserve kinetic energy

$$\tfrac{1}{2}m_1 v_{1i}^{\,2} = \tfrac{1}{2}m_1 v_{1f}^{\,2} + \tfrac{1}{2}m_2 v_{2f}^{\,2}$$

Cancelling the ½ and the masses, since $m_1 = m_2$,

$$\boxed{v_{1i}^{\,2} = v_{1f}^{\,2} + v_{2f}^{\,2}}$$

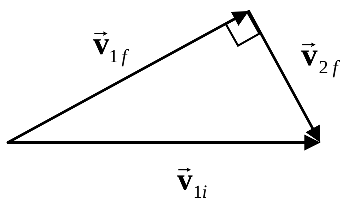
The conservation of linear momentum condition is

$$m_1\vec{\mathbf{v}}_{1i} = m_1\vec{\mathbf{v}}_{1f} + m_2\vec{\mathbf{v}}_{2f}$$

Again, we can cancel the masses

$$\boxed{\vec{\mathbf{v}}_{1i} = \vec{\mathbf{v}}_{1f} + \vec{\mathbf{v}}_{2f}}$$

The two boxed equations can be interpreted by the picture



The velocities of the two masses after collision are perpendicular to each other.