Chapter 7: Logarithmic Functions

Section 7.1: Exploring Characteristics of Logarithmic Functions

<u>Terminology</u>:

- **Logarithmic Functions:**
  A function of the form:
  $y = a\ log_b x$
  Where $a \neq 0, b > 0, b \neq 1,$ *and a and b are real numbers*.
  Note: a is the coefficient, b is the base, and x is the argument.
  Note: $\log_{10} x = \log x$ *as logarithms are base 10 by default.*

Logarithmic Functions and Their Characteristics

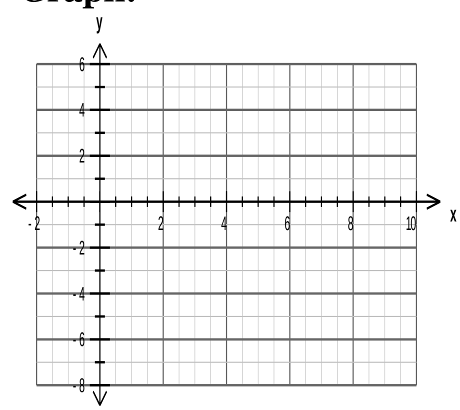
Graph the following exponential function and interpret its characteristics.

(a) $f(x) = \log_{10} x$

Table:

x	$f(x)$
-1	
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Graph:



Characteristics:

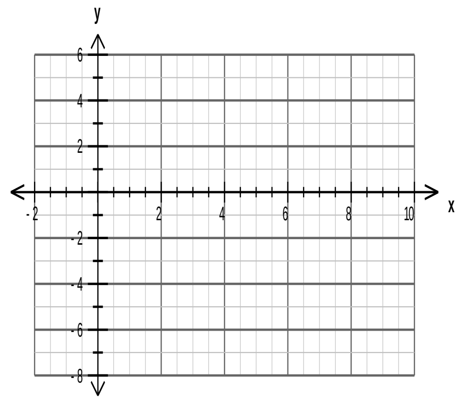
1. *Number of x-intercept:*____________
2. *Coordinates of y-intercept:*___________
3. *End Behaviour:*_________________
4. *Domain:* ____________________
5. *Range:* ______________________
6. *Number of Turning Points:* _______________________

(b) $f(x) = 2\log_{10} x$

Table:

x	$f(x)$
-1	
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Graph:



Characteristics:

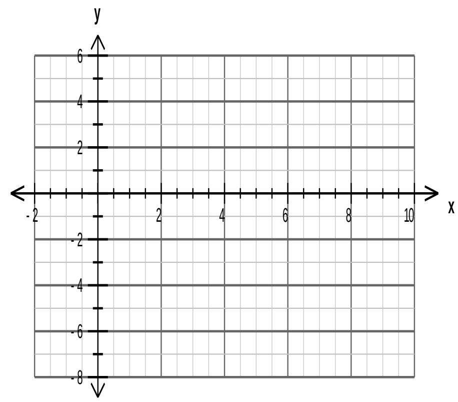
7. *Number of x-intercept:*____________
8. *Coordinates of y-intercept:*___________
9. *End Behaviour:*_________________
10. *Domain:* ____________________
11. *Range:* ______________________
12. *Number of Turning Points:* _______________________

(c) $j(x) = -2\log_{10} x$

Table:

x	$f(x)$
-1	
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

Graph:



Characteristics:

13. Number of x-intercept:____________
14. Coordinates of y-intercept:___________
15. End Behaviour:________________
16. Domain: ____________________
17. Range: _____________________
18. *Number of Turning Points: ______________________*

NOTE: A logarithm with no base shown is assumed to be base ten.
Hence, $\log_{10} x = \log x$. The same is true of what is known as a natural log, $\log_e x = \ln x$.

Comparing Exponential and Logarithmic Functions

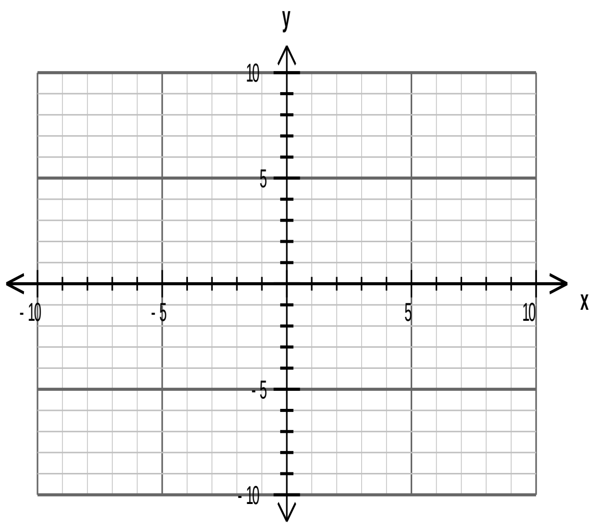
On the graph below, graph the following:

$y = 10^x$

x	$f(x)$
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	
6	

$y = \log_{10} x$

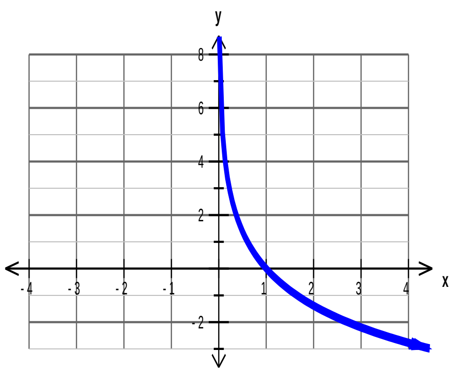
x	$f(x)$
-1	
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	

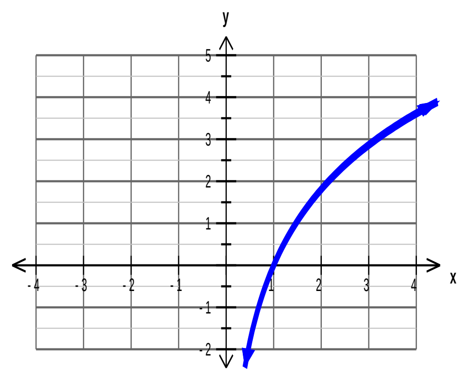


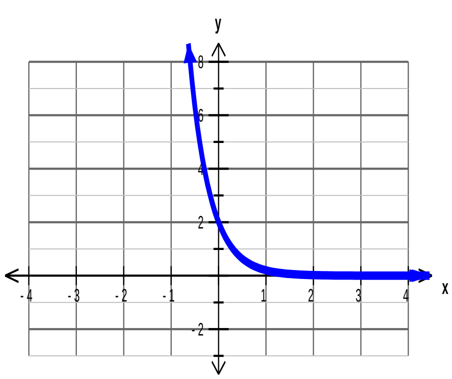
Matching Graphs and Equations

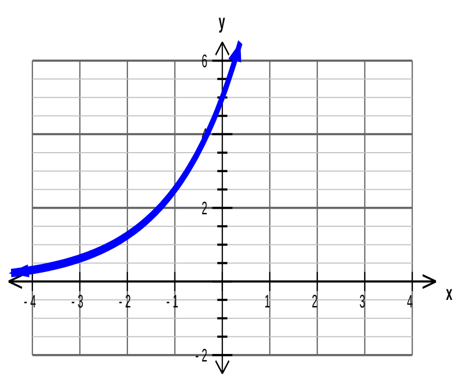
Which function matches each graph below. Provide your reasoning.

i) $y = 5(2)^x$
ii) $y = 2(0.1)^x$
iii) $y = 6\log x$
iv) $y = -2\ln x$









Section 7.2: Evaluating Logarithmic Expressions

Connecting Logarithms and Exponents

Use the function $y = \log x$ to complete the table of values below. Then express the function in an exponential form

Amplitude of Vibrations Measured by Seismograph in exponential form base 10	10^0	10^1	10^2	10^3	10^4	10^5	10^6
Amplitude of Vibrations Measured by Seismograph, x (units)	1	10	100	1000	10 000	100000	1000000
Richter Scale Magnitude, y $y = \log x$							

Conversions Between Exponential and Logarithmic Forms

An exponential function of the form $y = b^x$ is the equivalent to the logarithmic function $x = \log_b y$ and vice versa.

$$y = b^x \leftrightarrow x = \log_b x$$

Convert each of the following:

Exponential Form	Logarithmic Form
$81 = 10^y$	
$25 = e^y$	
$32 = 10^y$	
	$y = \log_2 16$
	$y = \log_4 64$
	$y = \log_3 27$

Estimating and Evaluating Logarithmic Expressions

Given a logarithmic expression, we can estimate it's value on a calculator using the property:

$$y = \log_b x = \frac{\log(x)}{\log(b)}$$

Determine the value of each of the following:

(a) $\log_2 16$

(b) $\log_4 64$

(c) $\log_3 27$

(d) $\log_3 \left(\frac{1}{27}\right)$

(e) $\log_{\frac{1}{4}} 64$

(f) $\log_2 -4$

(g) $\log_5 \left(\frac{1}{25}\right)$

(h) $\log_2 16 + \log_2 2$

(i) $\log_2 1 - \log_2 \left(\frac{1}{8}\right)$

(j) $\log_2 64 \div \log_2 8$

(k) $81 = 10^y$

(l) $25 = e^y$

(m) $32 = 10^y$

PH Scale Questions

The pH scale in chemistry is used to measure the acidity of a solution. The pH scale is logarithmic, with base 10. A logarithmic scale is useful for comparing numbers that vary greatly in size. The pH, $p(x)$, is defined by the equation:

$$p(x) = -\log x$$

Where the concentration of hydrogen ions, x, in a solution is measured in moles per litre (mol/L).

(a) The hydrogen ion concentration, x, of a solution is 0.0001 mol/L. Calculate the pH of the solution.

(b) The hydrogen ion concentration, x, of a solution is 0.0000001 mol/L. Calculate the pH of the solution.

(c) The pH of lemon juice is 2. Determine the concentration of hydrogen ion for lemon juice.

(d) The pH of baking soda is 9. Determine the concentration of hydrogen ion for baking soda.

(e) In terms of hydrogen ion concentration, how much more acidic is solution A with a pH of 1.6, than solution B with a pH of 2.5? Round your answer to the nearest tenth.

Section 7.3: Laws of Logarithms

Laws of Logarithms

1. **Addition of Logarithms**
   When two logarithms with the same base are added, their arguments are multiplied.

$$\log_{\mathrm{b}}(X) + \log_{\mathrm{b}}(Y) = \log_{\mathrm{b}}(X \cdot Y)$$

2. **Subtraction of Logarithms**
   When two logarithms with the same base are subtracted, their arguments are divided.

$$\log_{\mathrm{b}}(X) - \log_{\mathrm{b}}(Y) = \log_{\mathrm{b}}\left(\frac{X}{Y}\right)$$

3. **Power Law of Logarithms**
   When an argument has an exponent, that exponent can be written as a coefficient of a logarithm and vice versa.

$$\log_b(X^c) = c \cdot \log_b X$$

Using Laws of Logarithms to Simplify and Evaluate an Expression

Ex. Simplify and then evaluate each logarithmic expression.

(a) $\log_2 5 + \log_2 6.4$

(b) $\log_5 100 - \log_5 4$

(c) $\log_3 27^5$

(d) $\log_2 48 - \log_2 3$

Ex. Write each expression as a single logarithm, and then evaluate.

(a) $\log_3 18 + \log_3 \left(\frac{3}{2}\right)$

(b) $\log_5 40 - 3\log_5 2$

(c) $2\log_3 6 + \log_3 \left(\frac{3}{4}\right)$

(d) $\log_4 144 - \left(\frac{1}{2}\right)\log_4 9$

(e) $2\log_5 3 + \log_5 96 - \frac{1}{3}\log_5 64$

(f) $\frac{1}{4}\log_3 64 - (\log_3 8 - 2\log_3 16)$

Section 7.4: Solving Equations using Logarithms

How to Solve an Exponential Equation Using Logarithms

1. Add or subtract any constants in the equation if any exist
2. Divide by any coefficient on the same side as the exponent and base
3. Ensure that there is only one base on each side of the equation
4. Convert to logarithmic form
5. Use the estimation method to evaluate the logarithm: $\log_b A = \frac{\log A}{\log b}$
6. Multiply through by any denominator that exists from the original exponent.

Ex1. Solve

(a) $3(2)^x = 15$

Solution:

$\frac{3(2)^x}{3} = \frac{15}{3}$ → Divide both sides by the coefficient of 3

$(2)^x = 5$ → Now that there is one base on each side convert to log form

$x = \log_2 5$ → By converting to log form we now have an equation that is rearranged to solve for x. We will now apply the estimation rule to evaluate.

$x = \frac{\log(5)}{\log(2)}$ → Put this into your calculator to determine the estimated value

$x \doteq 2.3219$ → Always give your answer to 4 decimal places when dealing with logarithms

(b) $180 = 5(1.5)^{\frac{x}{2}}$

Solution:

$\frac{180}{5} = \frac{5(1.5)^{\frac{x}{2}}}{5}$ → Divide both sides by the coefficient of 5

$36 = (1.5)^{\frac{x}{2}}$ → Now that there is one base on each side convert to log form

$\frac{x}{2} = \log_{1.5} 36$ → By converting to log form we now have an equation that is rearranged to solve for x. We will now apply the estimation rule to evaluate. Note the /2 is still attached to the x.

$\frac{x}{2} = \frac{\log(36)}{\log(1.5)}$ → Put this into your calculator to determine the estimated value

$\frac{x}{2} \doteq 8.8380$ → Always give your answer to 4 decimal places when dealing with logarithms

$2\left(\frac{x}{2}\right) \doteq 2(8.8380)$ → Now we multiply both sides by the denominator of 2, this will provide us with the value of x.

$x \doteq 17.6760$ → Final Answer

Word Problems:

Sometimes you will have to take the same approach to word problems. In such situations you may have to create the equation such as we did in chapter 6 or the equation will be given to you.

When dealing with a word problem, we approach it in exactly the same way with exactly the same steps as those highlighted above.

Ex1. The half life of substance is 15 hours. Given that there is originally 150 mg of the substance, how long will it take for there to be 25 mg of the substance remaining. Keep in mind that the equation for the half life of a substance is given by $A = A_o\left(\frac{1}{2}\right)^{\frac{t}{h}}$.

Step 1: Set up the problem

We know the original amount is 150 this represents A_o, the half life is 15 hours, this represents h, and the amount remaining is 25 mg, this will represent A. So our equation becomes:

$$A = A_o\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$25 = 150\left(\frac{1}{2}\right)^{\frac{t}{15}}$$

Step 2: Work it out using the same steps we applied in the previous examples

$25 = 150\left(\frac{1}{2}\right)^{\frac{t}{15}}$

$\frac{25}{150} = \frac{\cancel{150}\left(\frac{1}{2}\right)^{\frac{t}{15}}}{\cancel{150}}$ → Divide by 150

$\frac{1}{6} = \left(\frac{1}{2}\right)^{\frac{t}{15}}$ → Ensure there is only one base on each side. Note I kept the answers as fractions to reduce the amount of estimation needed.

$\frac{x}{15} = \log_{\frac{1}{2}}\left(\frac{1}{6}\right)$ → Converted to log. The /15 stays with the x until the end.

$\frac{x}{15} = \frac{\log\left(\frac{1}{6}\right)}{\log\left(\frac{1}{2}\right)}$ → Use the rule for estimating logs.

$\frac{x}{15} \doteq 2.5850$ → State the answer to 4 decimal places when estimating logs

$x \doteq 15(2.5850)$ → Multiply through by 15 to determine the value of x.

$x \doteq 38.7750$

Step 3: Write a statement.

It would take 39 hours for the substance to decay to 25 mg.

Ex2. An investment of \$200 was placed into a savings account that provided 6%/a interest compounded annually. Determine how long it will take for the investment to reach a value of \$1000.

Step 1: Set up the problem
This is a compounded interest problem, so we must set up the equation using $A = P(1+i)^n$, and we know the principal amount (P) is \$200, we know that it is compounded annually at 6% interest, so i = 0.06, and we know the amount we are looking for, A, is \$1000. So our equation is:

$$A = P(1+i)^n$$
$$1000 = 200(1+0.06)^n$$
$$1000 = 200(1.06)^n$$

Step 2: Work it out using the same steps we applied in the previous examples

$1000 = 200(1.06)^n$

$\frac{1000}{200} = \frac{200(1.06)^n}{200}$ → Divide by 200

$5 = (1.06)^n$ → Ensure there is only one base on each side.

$n = \log_{1.06}(5)$ → Converted to log.

$n = \frac{\log(5)}{\log(1.06)}$ → Use the rule for estimating logs.

$n \doteq 27.6209$ → State the answer to 4 decimal places when estimating logs

Step 3: Write a statement.

It will take 28 years for the investment to reach a value of \$1000

Solving Exponentials Using Common Logarithms

Solve:

(a) $3^{x+1} = 20$

(b) $104 = 5^{x+2}$

(c) $5^{x-2} - 7^{x+1} = 0$

(d) $2^{x-1} = 3^{x+1}$

(e) Allen currently has \$2000 in credit card debt. The interest rate on his credit card is 18.5% yearly. Determine the number of years it would take for hid debt to double if he made no payments against his balance.

(f) \$1000 is invested at 6%/a compounded monthly. How long, in months will it take for the investment to reach a value of \$3000

Section 7.5: Modeling Data Using Logarithmic Functions

Using Logarithmic Regression to Solve a Problem Graphically

Ex1. The flash on most digital cameras requires a charged capacitor in order to operate. The percent charge, Q, remaining on a capacitor was recorded at different times, t, after the flash had gone off.

Percentage Charge, Q (%)	Time, t (s)
100.00	0
90.26	0.01
73.90	0.03
60.51	0.05
49.54	0.07
40.56	0.09

The t5 flash duration represents the time until a capacitor has only 50% of its initial charge. The t5 flash duration also represents the length of time that the flash is effective, to ensure that the object being photographed is properly lit.

(a) Construct a scatter plot for the given data using graphing technology
(b) Determine a logarithmic model for the data

(c) Use your logarithmic model to determine the t5 flash duration to the nearest hundredth of a second.

(d) Use your logarithmic model to determine the duration to the nearest hundredth of a second until the capacitor has just 10% of the initial charge remaining.

Ex2. Caffeine is found in coffee, tea, and soft drinks. Many people find that caffeine makes it difficult for them to sleep. The following data was collected in a study to determine how quickly the human body metabolizes caffeine. Each person started with 200 mg of caffeine in her or his bloodstream, and the caffeine level was measured at various times.

(a) Determine the equation of the logarithmic regression function for the data representing time as a function of caffeine level.

Caffeine Level in Bloodstream, m (mg)	Time after Ingesting, t(h)
168	1.0
167	1.5
113	5.0
145	3.0
90	6.5
125	4.0
138	3.5
77	8.0
83	7.0
50	12.0
150	2.5
55	12.0
112	5.0
84	7.0
136	3.5
180	1.0
110	5.0
75	8.0
76	9.0
49	12.5

Caffeine Level in Bloodstream, m (mg)	Time after Ingesting, t(h)
33	14.0
80	7.5
145	3.0
100	6.0
71	8.5
156	2.0
153	2.5
130	4.0
90	6.5
112	5.0
32	16.0
23	18.0
25	17.5
45	13.0
27	18.5
18	20.0
29	15.0
43	12.0
25	17.5
21	19.0

(b) Determine the time it takes for an average person to metabolize 50% of the caffeine in their bloodstream.

(c) Estimate how much caffeine would be in Paula's bloodstream at 9:00pm if she ingested 200 mg at 10:00 am.