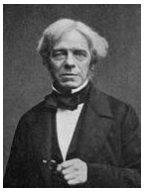
CHAPTER 7
ELECTRODYNAMICS



Michael Faraday

James C. Maxwell

Outlines

1. Electromotive Force
2. Electromagnetic Induction
3. Maxwell's Equations

Summary of Electrostatics and Magnetostatics

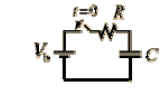
$$\nabla \cdot \vec{E} = \rho/\varepsilon_o \qquad \nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = 0 \qquad \nabla \times \vec{B} = \mu_o \vec{J}$$

This semester, we will study E&M phenomena that are time-dependent. We start with Faraday's law of induction and time dependent Maxwell's equations to derive wave equations that govern the electromagnetic wave. We then investigate the potential generated by a time-dependent charge distribution and radiation from accelerating charge distribution. Last we study the relationship between electrodynamics and special relativity.

Time dependent E&M phenomena

Charging a capacitor



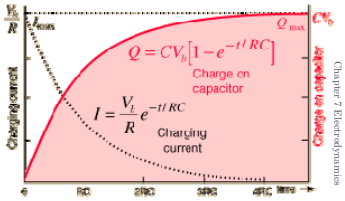
$$V_b = V_R + V_C$$

$$V_b = IR + \frac{Q}{C}$$

As charging progresses,

$$V_b = IR + \frac{Q}{C}$$

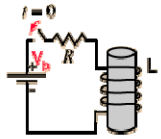
current decreases and charge increases.



At $t = 0$: $Q = 0$, $V_C = 0$, $I = \frac{V_b}{R}$

As $t \to \infty$: $Q \to CV_b$, $V_C \to V_b$, $I \to 0$

Applying a step voltage to an inductor



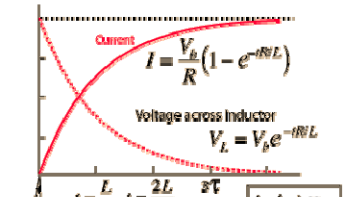
$$V_b = V_R + V_L$$

$$V_b = IR + L\frac{\Delta I}{\Delta t}$$

As the current increases

$$V_b = IR + L\frac{\Delta I}{\Delta t}$$

the rate of change of current decreases.



Time constant: $\tau = \frac{L}{R}$

Electromotive force

Ohm's Law

In electrostatics, we said that $\vec{E} = 0$ inside a conductor because if $\vec{E} \neq 0$, the charges will re-arrange themselves until $\vec{E} = 0$.

In magnetostatics, we assume that there is a "steady current" flows in a conducting wire and treated this "steady current" as a source in magnetostatics. Under such condition, the $\vec{E}$ is not required to be zero inside a conductor.

For most materials,

$$\vec{J} \propto \vec{f} \quad \text{(force per unit charge)}$$

$$\vec{J} = \sigma \vec{f}$$

σ -------- conductivity

$\rho = 1/\sigma$ ------- resistivity

The force experienced by the charge is giving by the Coulomb force and the Lorentz force,

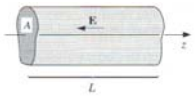
$$\vec{J} = \sigma[\vec{E} + \vec{v} \times \vec{B}]$$

Since in general $\vec{B} \ll \vec{E}$, so we can ignore the magnetic field

$$\vec{J} = \sigma\vec{E}$$

Example 1 A wire with a conductivity σ

$$I = JA = \sigma EA$$
$$= \sigma\left(\frac{V}{L}\right)A = \frac{\sigma A}{L}V = \frac{A}{\rho L}V$$



Define $\frac{A}{\rho L} = \frac{1}{R}$

$$V = IR$$

Example 2 Two long coaxial metal cylindrical tubes with radii of a and b, are separated by material of conductivity σ. If they are maintained at a potential difference V, find the current flows between the cylinders.

First we use Gauss's law to find out the E field between the two cylinders to be

$$\vec{E} = \frac{1}{2\pi\epsilon}\frac{\lambda}{r}\hat{r}$$

The current and voltage between the inner cylinder and the outer cylinder are giving by:

$$I = \int \vec{J}\cdot\overrightarrow{da} = \sigma\int \vec{E}\cdot\overrightarrow{da} = \sigma\left(\frac{\lambda}{2\pi\epsilon r}\right)(2\pi r\cdot L) = \sigma\frac{\lambda L}{\epsilon}$$

$$V = -\int_a^b \vec{E}\cdot\overrightarrow{dl} = \frac{\lambda}{2\pi\epsilon}ln\left(b/a\right) \quad\Rightarrow\quad I = \frac{2\pi\sigma L}{ln\left(b/a\right)}\cdot V$$

For steady current inside a conductor that has a uniform conductivity, we can see that

$$\nabla\cdot\vec{E} = \nabla\cdot\left(\frac{\vec{J}}{\sigma}\right) = \frac{1}{\sigma}(\nabla\cdot\vec{J}) = 0$$

Since

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon} \quad (\rho \text{ is the charge density})$$

So the above equation implies that $\rho = 0$ inside a conductor. All the charges reside on the surface of a conductor.

Electromotive Force

Electromotive force (emf) is not a force, it is the ability to drive current through a circuit. So it is kind of like a voltage. In the circuit below, the chemical energy in the battery generates the emf to drive a current through the circuit.

The current in the circuit is the same everywhere around the loop. There is no charge accumulation anywhere as Kirchhoff's law indicated.

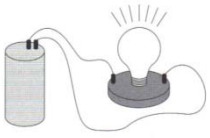


Figure 7.7

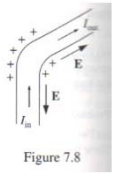


Figure 7.8

$\vec{f}$ — force per unit charge
$\vec{f}_s$ — force due to source

The Ohm's Law stated that

$$\vec{J} = \sigma\vec{f} = \sigma[\vec{E} + \vec{f}_s]$$

$$\oint \vec{f}\cdot\overrightarrow{dl} = \oint \vec{f}_s\cdot\overrightarrow{dl} = \varepsilon$$

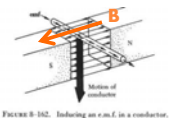
ε (emf), the electromotive force is defined as the line integral of a force per unit charge. (Energy per unit charge)

$$\varepsilon = \oint \vec{f}\cdot\overrightarrow{dl} = \oint \frac{\vec{J}}{\sigma}\cdot\overrightarrow{dl} = \oint \frac{I}{\sigma A}\cdot dl = I\oint\frac{dl}{\sigma A} = I\left(\frac{L}{\sigma A}\right)$$

Let $R = \frac{L}{\sigma A}$ $\quad\Rightarrow\quad \varepsilon = IR$

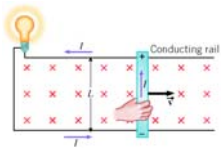
Motional emf

The motional emf is the electromotive force due to the motion of a conducting wire through a magnetic field.



This emf is caused by the Lorentz force acting on the charge carriers.

Charges will pile up at the two ends of the falling rod until the potential difference between the two ends is equal to the electromotive force generated by the movement of the rod inside the magnetic field.



$$\vec{f}_{mag} = \frac{q\vec{v}\times\vec{B}}{q}$$

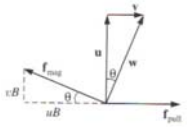
$$\varepsilon = \oint \vec{f}_{mag}\cdot\vec{dl} = vBh$$

Now if we make the rod as part of a circuit, we can see that the motional emf generated will drive a current through the circuit.

Note: The Lorentz force does not do work because $\vec{f}_{mag} \perp \vec{V}$.

The person pulling the rod is doing the work, because as emf generate the current I, moving upward, the force per unit charge is

$$\vec{f} = \vec{u}\times\vec{B} = -\vec{f}_{pull}$$



Now if we look from the charge point of view, its velocity is $\vec{w}$ as shown on the left. The work done by $\vec{f}_{pull}$ is

See Fig 7.12

$$\int \vec{f}_{pull}\cdot\vec{dl} = (uB)\left(\frac{h}{cos\theta}\right)(sin\theta) = vBh = \varepsilon$$

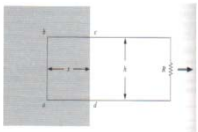
So the work done per unit charge is exactly equal to the emf.

The force of pull, $\vec{f}_{pull}$ contributes indirectly to the emf, even though it is perpendicular to the wire, whereas $\vec{f}_{mag}$ contributes indirectly to the work done, even it is perpendicular to the motion of the charge. (See AJP 42 295 1974)

Another way to look at motional emf

Define flux ϕ of a magnetic field $\vec{B}$ through an area as

$$\phi = \int \vec{B}\cdot\vec{da}$$



So the flux in the previous case is

$$\phi = BhS$$

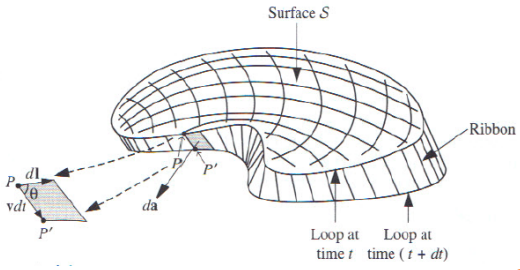
And

$$\frac{d\phi}{dt} = B\cdot h\cdot\frac{ds}{dt} = -vBh = -\varepsilon$$

$$\varepsilon = -\frac{d\phi}{dt} \quad \textbf{Flux rule}$$

The minus sign is called Lenz's law, i.e. the induced emf will go against the flux change.

Proof of the flux rule for an arbitrary shaped loop



The flux change is

$$d\phi = \phi(t+dt) - \phi(t) = \int_{ribbon} \vec{B}\cdot\vec{da}$$

$\vec{v}$ --- velocity of the point P on the loop
$\vec{u}$ --- velocity of the charge moving alone $\vec{dl}$

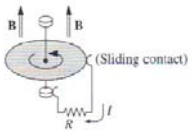
$$\vec{da} = (\vec{v}\times\vec{dl})dt \qquad \frac{d\phi}{dt} = \oint \vec{B}\cdot(\vec{v}\times\vec{dl})$$

Since $\vec{u} \parallel \vec{dl}$, so $\vec{u}\times\vec{dl} = 0$, let $\vec{w} = \vec{v} + \vec{u}$, $\quad \vec{w}\times\vec{dl} = \vec{v}\times\vec{dl}$

$$\frac{d\phi}{dt} = \oint \vec{B}\cdot(\vec{w}\times\vec{dl}) = -\oint(\vec{w}\times\vec{B})\cdot\vec{dl}$$

$$\frac{d\phi}{dt} = -\oint \vec{f}_{mag}\cdot\vec{dl} = -\varepsilon$$

Flux rule is very useful for finding motional emf. However, occasionally we have to use Lorentz force law to find the motional emf.



Example 7-4

A metal disk rotates with $\vec{\omega}$ in a uniform magnetic field $\vec{B}$, find the emf between the edge and the center of the disk.

The magnetic force per unit charge is given by

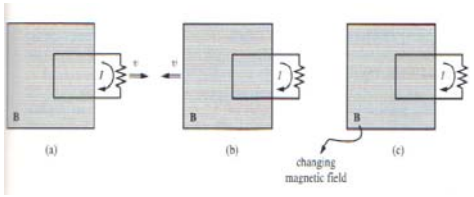
$$\vec{f}_{mag} = \vec{v}\times\vec{B} = \omega rB\hat{r}$$

The emf can be considered as the work done per unit charge

$$\varepsilon = \int_0^a f_m\cdot dr = \frac{\omega Ba^2}{2} \quad \Longrightarrow \quad I = \frac{\varepsilon}{R} = \frac{\omega Ba^2}{2R}$$

Electromagnetic Induction

In 1831, Faraday did a series of experiments showing the effects of motional emf on a closed loop of circuit.



A change of magnetic field induces an "electric field" that produces emf or current in a closed loop circuit.

The above experiments can be expressed by the flux rule:

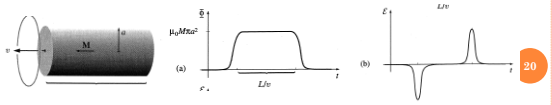
$$\varepsilon = \oint \vec{E}\cdot\vec{dl} = -\frac{d\Phi}{dt}$$

From Stoke's theorem

$$\oint \vec{E}\cdot\vec{dl} = \int (\nabla\times\vec{E})\cdot\vec{da} = -\frac{d}{dt}\int \vec{B}\cdot\vec{da} = \int\left(-\frac{\partial\vec{B}}{\partial t}\right)\cdot\vec{da}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$ This is Faraday's law

Example 7-5



The induced electric filed

In Ampere's law we have

$$\nabla\times\vec{B} = \mu_o\vec{j}, \qquad \oint \vec{B}\cdot\vec{dl} = \mu_o I_{enc}$$

In Faraday's law of induction, we have

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \oint \vec{E}\cdot\vec{dl} = -\frac{\partial\phi}{\partial t}$$

We can see that there are certain similarities and differences between the two cases. For example, in the 1st case, $\nabla\cdot\vec{B} = 0$ in all situations, because there is no magnetic monopole; while in the 2nd case, $\nabla\cdot\vec{E} = 0$ is only true when no source is involved.

We can even assume that the $-\frac{\partial\vec{B}}{\partial t}$ term as a "current density" term, and use an expression similar to the "Biot-Savart law as follow:

$$\vec{B} = \frac{\mu_o}{4\pi}\int\frac{\vec{j}\times\hat{r}}{r^2}d\tau \qquad \vec{E} = -\frac{1}{4\pi}\int\frac{\frac{\partial\vec{B}}{\partial t}\times\hat{r}}{r^2}d\tau$$

Re-arrange the equation on the right

$$\vec{E} = -\frac{\partial}{\partial t}\left[\frac{1}{4\pi}\int\frac{\vec{B}\times\hat{r}}{r^2}d\tau\right] = -\frac{\partial\vec{A}}{\partial t}$$

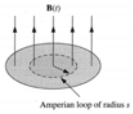
And

$$\nabla\times\vec{E} = -\frac{\partial}{\partial t}\nabla\times\vec{A} = -\frac{\partial\vec{B}}{\partial t}$$

Note: This is only true for Coulomb gauge, where $\nabla\cdot\vec{A} = 0$.

Example 7-7 A uniform magnetic field fills the shaded circular region as shown. If $\vec{B}$ field is changing with time, what is the induced $\vec{E}$ field at $\vec{r}$?

Use equation (7.18)



$$\oint \vec{E}\cdot\vec{dl} = E(2\pi s) = -\frac{d\phi}{dt} = -\pi s^2\frac{dB}{dt}$$

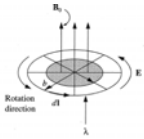
Therefore

$$\vec{E} = -\frac{s}{2}\frac{dB}{dt}\hat{\phi}$$

Example 7-8 A line charge λ is glued on the rim of a wheel of radius b, in the center region out to radius a, there is a uniform magnetic field pointing up a shown. What happen if the field is turn off?

From Faraday's law

$$\oint \vec{E}\cdot\vec{dl} = -\pi a^2\frac{dB_o}{dt} \Rightarrow \vec{E} = -\frac{\pi a^2}{2\pi b}\frac{dB_o}{dt}\hat{\phi}$$



The torque on $\vec{dl}$

$$\vec{dN} = \vec{b}\times\vec{df} = b\lambda E\cdot dl\hat{z}$$

Total torque on the wheel

$$\vec{N} = \int\vec{dN} = -b\lambda\cdot\pi a^2\frac{dB_o}{dt}\hat{z}$$

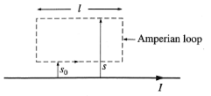
Total angular momentum gained

$$\Delta L = \int N\cdot dt = b\lambda\cdot\pi a^2\cdot B_o = b\lambda\phi$$

Example 7.9

An infinitely long straight wire carries a slowly varying current I(t). Determine the induced electric field as a function of the distance s from the wire.

Apply Ampere law to the Ampere loop in the right figure.



$$\oint \vec{E}\cdot d\vec{l} = E(s_o)l - E(s)l = -\frac{d}{dt}\int \vec{B}\cdot d\vec{a}$$

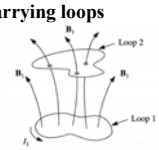
$$= -\frac{\mu_o l}{2\pi}\frac{dI}{dt}\int_{s_o}^{s}\frac{1}{s'}ds' = -\frac{\mu_o l}{2\pi}\frac{dI}{dt}(lns - lns_o)$$

$$\vec{E}(s) = \left[\frac{\mu_o}{2\pi}\frac{dI}{dt}lns + K\right]\hat{z}$$

Induction

Here we will look into how two current-carrying loops interact with each other.

We ask the question, what is the magnetic flux through loop 2, when there is a current I_1 on the loop 1?



$$\vec{B}_1 = \frac{\mu_o}{4\pi}I_1\oint\frac{\vec{dl}_1\times\hat{r}}{r^2} \quad (1); \qquad \phi_2 = \int \vec{B}_1\cdot\vec{da}_2 \quad (2)$$

From definition of vector potential, $\vec{B} = \nabla\times\vec{A}$, and apply Stoke's theorem, we obtain

$$\phi_2 = \int(\nabla\times\vec{A}_1)\cdot\vec{da}_2 = \oint\vec{A}_1\cdot\vec{dl}_2 \quad (3)$$

From Eq. 5-64

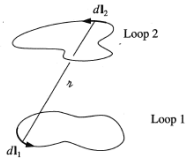
$$\vec{A}_1 = \frac{\mu_o I_1}{4\pi}\oint\frac{\vec{dl}_1}{r}$$

Substitute above eq. into eq. (3) on the previous page

$$\phi_2 = \frac{\mu_o I_1}{4\pi}\oint\oint\frac{\vec{dl}_1\cdot\vec{dl}_2}{r} = M_{21}I_1$$

And

$$M_{21} = \frac{\mu_o}{4\pi}\oint\oint\frac{\vec{dl}_1\cdot\vec{dl}_2}{r}$$



M_{21} is the ***mutual inductance*** between loop 1 and loop 2. It is a purely geometric factor and from the symmetry of the equation, we can see that $M_{21} = M_{12}$

This means that a current I in loop 1 produce a flux in loop 2, and the same current in loop 2 will produce the same flux in loop 1, even if loop 1 and loop 2 are very different.

$$\phi_2 = MI_1 \quad \text{and} \quad \phi_1 = MI_2$$

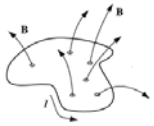
Now if the current is time-dependent, the magnetic flux is also depended on time.

$$\varepsilon_2 = -\frac{d\phi_2}{dt} = -M\frac{dI_1}{dt}$$

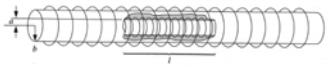
A varying current in loop 1 will induce a varying current in a near-by loop 2. Not only that, it can induct an ε on itself !!!

$$\phi = LI \quad \text{and}$$

$$\varepsilon = -\frac{d\phi}{dt} = -L\frac{dI}{dt}$$



Example 7.10. A short solenoid (n_1 turns per unit length) lies on the axis of a long solenoid (n_2 turns per unit length) as shown.



Current I flows in the short solenoid, find the flux through the long solenoid.

First, the magnetic field inside the short solenoid due to current I on the solenoid is $B_1 = \mu_o n_1 I,$

The flux in one loop of solenoid 2 will be

$$\phi_2 = B_1\cdot\pi a^2 = \mu_o n_1\cdot I\pi a^2$$

This flux passing through $n_2\cdot l$ loops on the long solenoid, so the total flux of long solenoid is

$$\phi_2 = \mu_o\pi a^2 n_1 n_2 lI \quad \Rightarrow \quad M = \mu_o\pi a^2 n_1 n_2 l$$

Now we want to show that the mutual inductance of the long solenoid on the smaller solenoid is also the same.

First we will find the magnetic field due to current I on the outer long solenoid.

$$B_2 = \mu_o n_2 I$$

The flux passing through one loop of the smaller solenoid due to this magnetic field is

$$\phi_1 = \mu_o n_2 I\cdot\pi a^2$$

The flux passing through the "whole" solenoid is

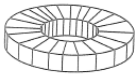
$$\Phi_1 = n_1 l\cdot\phi_1 = \mu_o\pi a^2 n_1 n_2 lI = MI$$

where M is $M = \mu_o\pi a^2 n_1 n_2 l$

Example 7.11. Find the self-inductance of a toroid coil.

From Eq. 5-60, page 239

$$B = \frac{\mu_o NI}{2\pi s}\hat{\phi}$$

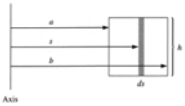


The flux for one loop is

$$\phi = \int \vec{B}\cdot\vec{da} = \int \frac{\mu_o NI}{2\pi s}\cdot h\cdot ds = \frac{\mu_o NIh}{2\pi} ln\left({}^{b}/_{a}\right)$$

Total flux

$$\phi_{total} = N\phi = \frac{\mu_o N^2 hI}{2\pi} ln\left({}^{b}/_{a}\right) = LI$$



$$L = \frac{\mu_o N^2 h}{2\pi} ln\left(\frac{b}{a}\right)$$

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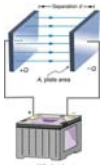
Energy stored in the magnetic field

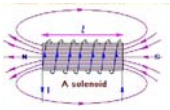
In Chap. 2, we discussed the energy storage in the $\vec{E}$ field of a capacitor is given by:

$$W = \frac{\varepsilon_o}{2}\int E^2\cdot d\tau$$



Here we want to derive the following expression for the energy stored in a solenoid.

$$W = \frac{1}{2\mu_o}\int B^2\cdot d\tau$$



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Because of the self-induction, as the switch is turn on, the solenoid will induce a back-emf to oppose the current change, the work done to overcome this back-emf is

$$W = q\varepsilon \Rightarrow \frac{dW}{dt} = \frac{dq}{dt}\varepsilon = I\cdot\left[L\frac{dI}{dt}\right]$$

$$\Rightarrow W = L\int IdI = \frac{1}{2}LI^2$$

This is the work required to run a current through an inductor with a self-inductance L. This is also the energy stored in the magnetic field $\vec{B}$ inside the inductor.

$$\frac{1}{2}LI^2 = \frac{1}{2\mu_o}\int B^2\cdot d\tau$$

Next, we are going to derive the above equation.

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We start with the definition of self-inductance

$$\phi = LI$$

$$\phi = \int \vec{B}\cdot\vec{da} = \int(\nabla\times\vec{A})\cdot\vec{da} = \oint \vec{A}\cdot\vec{dl}$$

$$\Rightarrow L = \frac{1}{I}\oint \vec{A}\cdot\vec{dl}$$

$$W = \frac{1}{2}LI^2 = \frac{I}{2}\oint \vec{A}\cdot\vec{dl} = \frac{1}{2}\oint(\vec{A}\cdot\vec{I})dl = \frac{1}{2}\int(\vec{A}\cdot\vec{J})\,d\tau$$

From Ampere's law, the above equation becomes

$$W = \frac{1}{2\mu_o}\int \vec{A}\cdot(\nabla\times\vec{B})d\tau$$

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From the vector product rules on page 21 of Griffiths

$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$$

$$= B^2 - \vec{A}\cdot(\nabla\times\vec{B})$$

$$W = \frac{1}{2\mu_o}\left[\int B^2 d\tau - \int \underbrace{\nabla\cdot(\vec{A}\times\vec{B})}_{0} d\tau\right]$$

$$W = \frac{1}{2\mu_o}\int B^2\cdot d\tau$$

Keep in mind that in electrostatics, we have (eq. 2-45)

$$W = \frac{\varepsilon_o}{2}\int E^2\cdot d\tau$$

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Maxwell equations

Electrodynamics before Maxwell

$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_o}$ (Gauss Law)

$\nabla\cdot\vec{B} = 0$ (No magnetic point charge)

$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ (Faraday's Law)

$\nabla\times\vec{B} = \mu_o\vec{J}$ (Ampere's Law)

There are several things that would be desirable if we can add a few new terms in these equations.

First, it would be desirable, if the above equations are more symmetrical. However, we know that ρ_m has not been found yet, so conventional wisdom assumes that

$$\rho_m = 0, \text{ and } \vec{J}_m = 0$$

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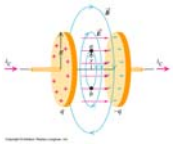
But a term $\mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$ that can be added to the Ampere's law seems to be in order.

For example during the charging of a capacitor, the magnetic field inside the capacitor, $\vec{B}_{in} \neq 0$. Why?



The $\vec{B}_{in}$ comes from the time-varying electric field inside the capacitor.

This is one indication that Ampere's law should be modified as follow:

$$\nabla \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Another way to look at this is to explore the inconsistency in the Ampere's law. By taking the divergence of the Ampere's law we end up with

$$\nabla \cdot (\nabla \times \vec{B}) = \mu_o (\nabla \cdot \vec{J})$$

The left hand side always equals to zero from vector identity, eq. (1.46). However on the right hand side, $\nabla \cdot \vec{J}$ is not always equal to zero. From the continuity equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

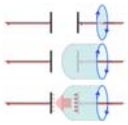
We can see that only in the electrostatics case, where the charge density does not change with time, we have $\nabla \cdot \vec{J} = 0$.

→ So there is a need to add an extra term in Ampere's Law to deal with dynamics.

There is another way to see that Ampere's law is bound to fail for the non-stead current such as charging a capacitor.

Ampere's law in integral form

$$\int \vec{B} \cdot \vec{dl} = \mu_o \int \vec{J} \cdot \vec{da} = \mu_o I_{enc}$$



As we can see from the figure above, the figure on the top make sense. However, when we re-draw the surface as a balloon-shape, there is no current going through the capacitor. So Maxwell solve this difficulty by assuming that "the rate of change of $\vec{E}$ *field*" is equivalent to a displacement current, $\vec{J}_d$.

$$\vec{J}_d = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

How Maxwell Fixed Ampere's Law

He started with the continuity equation

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Substitute Gauss law into the above equation

$$\nabla \cdot \vec{J} + \varepsilon_o \frac{\partial}{\partial t} (\nabla \cdot \vec{E}) = 0$$

$$\rightarrow \quad \nabla \cdot \left[\vec{J} + \varepsilon_o \frac{\partial \vec{E}}{\partial t} \right] = 0$$

So if we broaden the concept of current density to include the $\varepsilon_o \frac{\partial \vec{E}}{\partial t}$ term, and re-write the Ampere's law as

$$\nabla \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Maxwell called this extra term the displacement current:

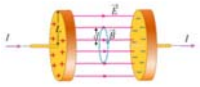
$$\vec{J}_d = \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \frac{\partial \vec{D}}{\partial t} \quad \text{(In free space)}$$

With this extra term, several inconsistencies in electrodynamics are resolved

1. Ampere's law is fixed,
2. Electrodynamics equations are more symmetric,
3. Charging capacitor puzzle is resolved.

When charging a capacitor,

$$\vec{E} = \frac{\sigma}{\varepsilon_o} \hat{\imath} = \frac{Q}{\varepsilon_o A} \hat{\imath}$$



$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_o A} \frac{\partial Q}{\partial t} \hat{\imath} = \frac{I_d}{\varepsilon_o A} \hat{\imath}$$

$$\rightarrow \quad \varepsilon_o \frac{\partial \vec{E}}{\partial t} = \frac{I_d}{A} \hat{\imath} = \vec{J}_d$$

$$\int \vec{B} \cdot \vec{dl} = \mu_o I_{enc} + \mu_o \varepsilon_o \int \frac{\partial \vec{E}}{\partial t} \cdot \vec{da}$$

Maxwell Equations

After re-arrangement, the Maxwell equation can be written as follow:

$$(1)\ \nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_o},\quad (3)\ \nabla\times\vec{E}+\frac{\partial\vec{B}}{\partial t}=0,$$
$$(2)\ \nabla\cdot\vec{B}=0,\quad (4)\ \nabla\times\vec{B}-\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}=\mu_o\vec{J}.$$

Together with the force equation,

$$\vec{F}=q\left[\vec{E}+\vec{v}\times\vec{B}\right]$$

They can describe the entire theory of classical electrodynamics. Even the continuity equation is contained in the modified Ampere's law

$$\nabla\cdot\left[\vec{J}+\varepsilon_o\frac{\partial\vec{E}}{\partial t}\right]=0$$

Magnetic charges

In free space, without any sources (ρ=0, $\vec{J}=0$), the Maxwell equations are even simpler

$$\nabla\cdot\vec{E}=0,\quad \nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=0,\quad \nabla\times\vec{B}=\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

There is a striking symmetry between the $\vec{E}$ and $\vec{B}$, if we replace $\vec{E}$ by $\vec{B}$, and replace $\vec{B}$ by $-\mu_o\varepsilon_o\vec{E}$, the above Maxwell equations in free space remain the same.

This gives us a hint that $\vec{E}$ and $\vec{B}$ are closely related or maybe "equivalent". But of course, if we put the sources into the equations, the symmetry breaks down.

We know that so far no magnetic monopole has been observed. However, it has not stopped people imaging what will happen if we do have magnetic monopole. For one thing the Maxwell equation will be "perfectly" symmetrical.

$$\nabla\cdot\vec{E}=\frac{\rho_e}{\varepsilon_o}\qquad \nabla\times\vec{E}=-\mu_o\vec{J}_m-\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B}=\mu_o\rho_m\qquad \nabla\times\vec{B}=\mu_o\vec{J}_e+\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

and

$$\nabla\cdot\vec{J}_e+\frac{\partial\rho_e}{\partial t}=0,\qquad \nabla\cdot\vec{J}_m+\frac{\partial\rho_m}{\partial t}=0$$

No experimental data support the above view so far.

Paul Dirac showed in 1931 that the existence of magnetic charge could explain why electric charge is quantized. (See chapter 8 for details)

Maxwell Equations in Matter

When we deal with field inside matter, we want to deal with $\vec{D}$ and $\vec{H}$, because polarization $\vec{P}$ and magnetization $\vec{M}$ are harder to deal with. So we link them with the bound charge ρ_b and bound current $\vec{J}_b$ as follow:

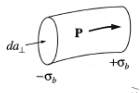
$$\rho_b=-\nabla\cdot\vec{P},\qquad \vec{J}_b=\nabla\times\vec{M}$$
$$\sigma_b=\vec{P}\cdot\hat{n},\qquad \vec{K}_b=\vec{M}\times\hat{n}$$

Note: The time derivative of ρ_b has nothing to do with $\vec{J}_b$. ρ_b comes from polarization, while $\vec{J}_b$ comes from magnetization.

When $\vec{P}(t)$ is a functions of time, we can see that there will be a current density associated with the rate of change of $\vec{P}$.

$$\vec{J}_P=\frac{\partial\vec{P}}{\partial t}$$

We look at a rod-shaped volume element that has a polarization $\vec{P}$. On each end there are surface charge density σ_b.



When the polarization increases, the surface charge density also increases, which can be viewed as current

$$dI=\frac{\partial\sigma_b}{\partial t}da_\perp=\frac{\partial P}{\partial t}da_\perp\quad\Rightarrow\quad \vec{J}_P=\frac{\partial\vec{P}}{\partial t}$$

If we take the divergence of the above high-lighted eq.

$$\Rightarrow\quad \nabla\cdot\vec{J}_P=\frac{\partial}{\partial t}(\nabla\cdot\vec{P})=-\frac{\partial\rho_b}{\partial t}$$

This is the continuity equation for the bound charges.

So the total charge density is

$$\rho=\rho_f+\rho_b\qquad \text{Eq. (1)}$$

And the total current density is

$$\vec{J}=\vec{J}_f+\vec{J}_b+\vec{J}_P=\vec{J}_f+\nabla\times\vec{M}+\frac{\partial\vec{P}}{\partial t}\qquad \text{Eq. (2)}$$

Apply the Gauss law to eq. (1)

$$\nabla\cdot\vec{E}=\frac{1}{\varepsilon_o}\left[\rho_f-\nabla\cdot\vec{P}\right]\quad\Rightarrow\quad \nabla\cdot\left[\varepsilon_o\vec{E}+\vec{P}\right]=\rho_f$$

$$\varepsilon_o\vec{E}+\vec{P}=\vec{D}\quad\Rightarrow\quad \nabla\cdot\vec{D}=\rho_f$$

Apply Ampere's law to eq. (2)

$$\nabla \times \vec{B} = \mu_o \left(\vec{J}_f + \nabla \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \left[\vec{B} - \mu_o \vec{M} \right] = \mu_o \vec{J}_f + \mu_o \frac{\partial}{\partial t} \left(\varepsilon_o \vec{E} + \vec{P} \right)$$

$$\Rightarrow \quad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

Define

$$\vec{J}_D = \frac{\partial \vec{D}}{\partial t} \quad \Rightarrow \quad \nabla \times \vec{H} = \vec{J}_f + \vec{J}_D$$

Now combines all four equations

$$\nabla \cdot \vec{D} = \rho_f \; ; \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \; ; \qquad \nabla \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

For linear dielectric materials, we also have

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} \qquad \vec{H} = \frac{1}{\mu_o} \vec{B} - \vec{M}$$

$$\vec{P} = \varepsilon_o \chi_e \vec{E} \qquad \vec{M} = \chi_m \vec{H}$$

$$\vec{D} = \varepsilon_o (1 + \chi_e) \vec{E} \qquad \vec{B} = \mu_o (1 + \chi_m) \vec{H}$$

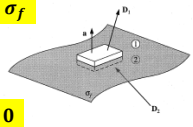
$$\vec{D} = \varepsilon \vec{E} \qquad \vec{H} = \frac{1}{\mu} \vec{B}$$

Boundary conditions

The boundary conditions will remain the same as in the static case. They come out from the integral form of the Maxwell's equations.

Perpendicular components

$$\oint \vec{D} \cdot \vec{da} = Q_f \quad \Rightarrow \quad D_1^{\perp} - D_2^{\perp} = \sigma_f$$



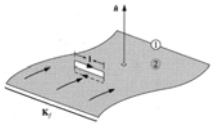
$$\oint \vec{B} \cdot \vec{da} = 0 \quad \Rightarrow \quad B_1^{\perp} - B_2^{\perp} = 0$$

Even though these two Maxwell equations contains time derivatives, the boundary conditions derived still are the same as in the statics case. Namely the area on the right side of the equation can be arbitrarily small.

Parallel components

$$\oint \vec{E} \cdot \vec{dl} = -\frac{d}{dt} \int \vec{B} \cdot \vec{da}$$

$$\Rightarrow \quad E_1^{\parallel} - E_2^{\parallel} = 0$$



$$\oint \vec{H} \cdot \vec{dl} = I_f + \frac{d}{dt} \int \vec{D} \cdot \vec{da}$$

$$\Rightarrow \quad H_1^{\parallel} - H_2^{\parallel} = K_f \times \hat{n}$$