CHAPTER 7   *The Time Value of Money*

After studying this chapter, you should be able to:

1. *Explain the concept of the time value of money.*
2. *Calculate the present value and future value of a stream of cash flows using Excel.*
3. *Explain the types of cash flows encountered in financial analysis and how to adjust for each type in making time value calculations in Excel.*
4. *Differentiate between the alternative compounding periods, and use Excel to compare present and future values under different compounding schemes.*

"A bird in the hand is worth more than two in the bush." That old aphorism, when translated into the language of finance, becomes "A dollar today is worth more than a dollar tomorrow." Intuitively, it probably makes sense, but why? Stated very simply, you can take that dollar today and invest it with the expectation of having more than a dollar tomorrow.

Because money can be invested to grow to a larger amount in the future, we say that money has a "time value." This concept of a time value of money underlies much of the theory of financial decision making, and you will be required to understand this material in order to complete the remaining chapters.

Future Value

Imagine that you have $1,000 available to invest. If you earn interest at the rate of 10% per year, then you will have $1,100 at the end of one year. The mathematics behind this example is quite simple:

$$1{,}000 + 1{,}000(0.10) \ = \ 1{,}100$$

In other words, after one year you will have your original $1,000 (the *principal* amount) plus the 10% interest earned on the principal. Because you won't have the $1,100 until one year in the future, we refer to this amount as the *future value*. The amount that you have today, $1,000, is referred to as the *present value*. If, at the end of the year, you choose to make the same investment again, then at the end of the second year you will have:

$$1{,}000 + 1{,}000(0.10) + 100(0.10) + 1{,}000(0.10) \ = \ 1{,}210$$

The $1,210 at the end of the second year can be broken down into its components: the original principal, the first year's interest, the interest earned in the second year on the first year's interest, and the second year's interest on the original principal. Note that we could restate the second year calculation to be:

$$1{,}100 + 1{,}100(0.10) \ = \ 1{,}210$$

Or, by factoring out the 1,100 we get:

$$1{,}100(1 + 0.10) \ = \ 1{,}210$$

Notice that in the second year the interest is earned on both the original principal and the interest earned during the first year. The idea of earning interest on previously earned interest is known as *compounding*. This is why the total interest earned in the second year is $110 versus only $100 the first year.

Returning to our original one-year example, we can generalize the formula for any one-year investment as follows:

$$FV_1 \ = \ PV + PV(i)$$

where FV_1 is the future value at the end of year 1, PV is the present value, and i is the one-year interest rate (compounding rate). The above equation is not in its simplest form. We can factor PV from both terms on the right-hand side, simplifying the future value equation to:

$$FV_1 \ = \ PV(1 + i) \tag{7-1}$$

Recall that in our two-year example, we earned interest on both the principal and interest from the first year. In other words, the first year *FV* became the second year *PV*. Symbolically, the second year *FV* is:

$$FV_2 = FV_1(1 + i)$$

Substituting $PV(1 + i)$ for FV_1 and simplifying, we have:

$$FV_2 = PV(1 + i)(1 + i) = PV(1 + i)^2$$

We can actually further generalize our future value equation. Realize that the exponent (on the right-hand side) is the same as the subscript (on the left-hand side) in the future value equation. When we were solving for the future value at the end of the first year, the exponent was 1. When we were solving for the future value at the end of the second year, the exponent was 2. In general, the exponent will be equal to the number of the period for which we wish to find the future value.

$$FV_N = PV(1 + i)^N \tag{7-2}$$

Equation (7-2) is the basis for all of the time value equations that we will look at in the sections ahead. Using this version of the equation, you can see that investing $1,000 for two years at 10% per year will leave you with $1,210 at the end of two years. In other words:

$$FV_2 = 1,000(1.10)^2 = 1,210$$

Using Excel to Find Future Values

It is easy enough to calculate future values with a hand calculator, especially a financial calculator. But, as we will see in the sections and chapters ahead, it is often necessary to use future values in worksheets. Excel makes these calculations easy with the use of the built-in **Fv** function:

Fv(*RATE*, *NPER*, *PMT*, *PV*, *TYPE*)

There are five arguments to the **Fv** function. *RATE* is the interest rate per period (year, month, day, etc.), *NPER* is the total number of periods, and *PV* is the present value. *PMT* and *TYPE* are included to handle annuities (a series of equal payments, equally spaced over time), which we will deal with later. For the problem types we are currently solving, we will set both *PMT* and *TYPE* to 0.[1]

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1. The *TYPE* argument tells Excel whether the cash flows occur at the end (0) or beginning (1) of the period.

Let's set up a simple worksheet to calculate the future value of a single sum. Starting with a blank worksheet, enter the labels and numbers as shown in Exhibit 7-1.

EXHIBIT 7-1
FUTURE VALUE OF A SINGLE CASH FLOW

We want to use the **Fv** function to calculate the future value of $1,000 for one year at 10% per year. In B5 enter the formula: `=FV(B4,B3,0,-B2,0)`. The result, $1,100, is exactly the same as we found earlier. Note that we have entered –B2 for the *Pv* argument. The reason for the negative sign is because Excel realizes that either the *Pv* or *Fv* must be a cash outflow. If we had not used the negative sign, the result (*Fv*) would have been negative. Users of financial calculators will recognize this as the *cash flow sign convention*.

You can now experiment with different values for the arguments. Try replacing the 1 in B3 with a 2. Excel immediately updates the result in B5 with $1,210, just as we found in the second part of our example. To see just how powerful compounding can be, insert 30 into B3. The result, $17,449.40, indicates that each $1,000 invested at 10% per year will grow to $17,449.40 after just 30 years. If we double the investment to $2,000, then we should double the future value. Try it; you should get a result of $34,898.80, exactly twice what we got with a $1,000 investment. In general, any money invested for 30 years at 10% per year will grow to 17.449 times its initial value. To see even more powerful examples of compounding, try increasing the interest rate.

Present Value

Our future value equation can be solved for any of its variables. We may wish to turn our example problem around to solve for the present value. Suppose that the problem is restated as, "What initial investment is required so that you will accumulate $1,210 after two years if you expect to earn an interest rate of 10% per year?" In this case, we want to solve for the present value—we already know the future value.

Mathematically, all that we need to do is to solve the future value equation (7-2) for the present value:

$$PV = \frac{FV_N}{(1+i)^N}$$

Of course, we already know that the answer must be $1,000:

$$PV = \frac{1{,}210}{(1.10)^2} = 1{,}000$$

In Excel, we can solve problems of this type by using the built-in **Pv** function:

$$\textbf{PV(}\textsc{Rate, Nper, Pmt, Fv, Type}\textbf{)}$$

The arguments to the **Pv** function are exactly the same as those for the **Fv** function, except that *PV* is replaced by *FV*. For this example, modify your worksheet by entering data into cells D1:E5 as shown in Exhibit 7-2.

EXHIBIT 7-2
**PRESENT VALUE OF A SINGLE CASH FLOW**

In cell E5 place the formula: `=PV(E4,E3,0,-E2,0)`. Again, we enter the future value reference as negative so that the present value result will be positive. The result will be $1,000, exactly as expected.

We have purposely constructed our future value and present value examples side-by-side in the worksheet to demonstrate that present value and future value are inverse functions. Let's change our worksheet to make this concept clear. We want to link the references in the present value function to the cells used in the future value function. This will allow changes in the future value arguments to change the present value arguments. First, select E2 and enter: `=B5`, in E3 type: `=B3`, and in E4 enter: `=B4`. Now, regardless of the changes made to the future value side of the worksheet, the present value should be equal to the value in B2. Try making some changes to the inputs in B2, B3, and B4. No matter what changes you make, the calculated present value (in E5) is always the same as the present value input in B2. This is because the present value and future value are inverse functions.

Annuities

Thus far, we have examined the present and future values of single cash flows (also referred to as lump sums). These are powerful concepts that will allow us to deal with more complex cash flows. *Annuities* are a series of nominally equal cash flows, equally spaced in time. Examples of annuities abound. Your car payment is an annuity, so is your mortgage (or rent) payment. If you don't already, you may someday own annuities as part of a retirement program. The cash flow pictured in Figure 7-1 is another example.

FIGURE 7-1
A TIMELINE FOR AN ANNUITY CASH FLOW
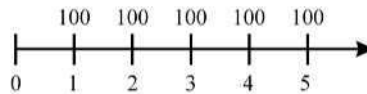
How do we find the value of a stream of cash flows such as that pictured in Figure 7-1? The answer involves the *principle of value additivity*. This principle says that "the value of a stream of cash flows is equal to the sum of the values of the components." As long as the cash flows occur at the same time, they can be added together. Therefore, if we can move each of the cash flows to the same time period (any time period), we can add them to find the value as of that time period. Cash flows can be moved around in time by compounding or discounting.

Present Value of an Annuity

One way to find the present value of an annuity is to find the present value of each of the cash flows separately and then add them together. Equation (7-4) summarizes this method:

$$PV_A = \sum_{t=1}^{N} \frac{Pmt_t}{(1+i)^t} \tag{7-4}$$

where PV_A is the present value of the annuity, t is the time period, N is the total number of payments, Pmt_t is the payment in period t, and i is the discount rate.

Of course, this equation works fine for any annuity (or any stream of cash flows), but it can be very tedious for annuities with more than just a few payments. Imagine finding the current balance (i.e., present value) of a mortgage with more than 300 payments to go before it is paid off! We can find a closed-form solution (the above equation is an "open-form" solution because the number of additions is indefinite) by taking the summation:

$$PV_A = Pmt \left[\frac{1 - \dfrac{1}{(1 + i)^N}}{i} \right]$$

(7-5)

where all terms are as previously defined. Notice that we have dropped the subscript t because this solution does not depend on our taking the present values separately. Instead, because each payment is the same, we can value the entire stream of cash flows in one step.

Let's find the present value of the cash flow pictured in Figure 7-1. Assuming that the discount rate for this cash flow is 8%, the equation is:

$$PV_A = 100 \left[\frac{1 - \dfrac{1}{(1.08)^5}}{0.08} \right] = 399.27$$

This means that if you were to deposit $399.27 into an account today that pays 8% interest per year, you could withdraw $100 at the end of each year and be left with a balance of $0.00 at the end of the five years.

Recall from our earlier discussion of single cash flows that we can use Excel's built-in **Pv** function to find present values. To recap, the **Pv** function is defined as:

Pv(*Rate, Nper, Pmt, Fv, Type*)

When dealing with single cash flows we set ***Pmt*** and *Type* to 0. Those arguments are used only in the case of annuities. ***Pmt*** will be set to the dollar amount of the periodic payment. *Type* is an optional binary (0 or 1) variable that controls whether Excel assumes the payment occurs at the end (0) or the beginning (1) of the period. For the time being, we will assume that all payments occur at the end of the period (i.e., they are *regular* annuities).

EXHIBIT 7-3
PRESENT VALUE OF AN ANNUITY

Set up a worksheet with the data pictured in Exhibit 7-3 in cells A1:B5. In B5 we wish to find the present value of the annuity presented in Figure 7-1, so enter: =PV(B3,B4,B2,0,0). Note that we have entered the payment as a positive number and the result is –$399.27. The interpretation is that if you were to make a deposit (a cash outflow) of $399.27 today, you could make a withdrawal of (a cash inflow) $100 each year for the next five years. Had we made the payment (B2) negative instead, the present value would have been a positive $399.27. The answer is the same, except for the sign, but the interpretation is different. In this case, the interpretation is that if you were to take out a loan of $399.27 (a cash inflow) today, you would need to repay $100 (a cash outflow) per year for each of the next five years to retire the loan.

We can, of course, experiment with various arguments. For example, suppose that instead of five withdrawals of $100 each you wanted to make ten withdrawals of $50 each. How much would you need to deposit into this account in order to deplete the account after 10 withdrawals? Change the number of payments in B4 to: 10 and the payment in B2 to: 50. After these changes, you will see that an initial deposit of only $335.50 will allow you to achieve your goal.

Returning now to our original example, reset the payment amount to: 100 and the number of payments to: 5. How much would you have to deposit if you want to make your first withdrawal today, rather than one year from today? (An annuity that begins paying immediately is known as *annuity due*.) To answer this question, realize that the only thing we have changed is the timing of the first withdrawal. We will still make a total of five withdrawals of $100 each, but they occur at the beginning of each period. In B5, change the *TYPE* argument to 1, from 0 originally, so that the formula is now: =PV(B3,B4,B2,0,1). The result is –$431.21, indicating that, because the first withdrawal occurs immediately, you will have to make a larger initial deposit. Note that the amount of the deposit must be larger because you will not earn the first year's interest before making the first withdrawal.

Another way to look at this is that we are effectively depositing $331.21 (= $431.21 deposit –$100 withdrawal) in order to be able to make four future withdrawals of $100 each. To see that this is the case, change the **Pv** formula back to its original form (*TYPE* = 0) and change the number of payments to 4. The present value is then shown to be $331.21, exactly as claimed.

Future Value of an Annuity

Imagine that you have recently begun planning for retirement. One of the attractive options available is to set up a traditional Individual Retirement Account (IRA). What makes the IRA so attractive is that you can deposit up to $5,000 per year, and the investment gains will accrue tax-free until you begin to make withdrawals after age 59½. Furthermore, depending on your situation, the IRA deposits may reduce your taxable income.

To determine the amount that you will have accumulated in your IRA at retirement, you need to understand the future value of an annuity. Recalling the principle of value additivity, we could simply find the future value of each year's investment and add them together at retirement. Mathematically this is:

$$FV_A = \sum_{t=1}^{N} [Pmt_t(1+i)^{N-t}] \tag{7-6}$$

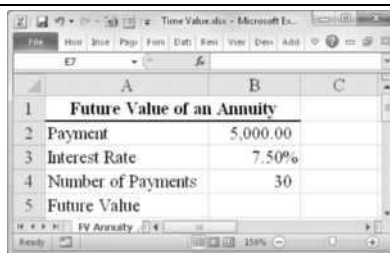Alternatively, we could use the closed-form solution of equation (7-6):

$$FV_A = Pmt\left[\frac{(1+i)^N - 1}{i}\right] \tag{7-7}$$

Assume that you are planning on retirement in 30 years. If you deposit $5,000 each year into an IRA account that will earn an average of 7.5% per year, how much will you have after 30 years? Because of the large number of deposits, equation (7-7) will be easier to use than equation (7-6), though we could use either one. The solution is:

$$FV_A = 5,000\left[\frac{(1.075)^{30} - 1}{0.075}\right] = 516,997.01$$

As usual, Excel provides a built-in function to handle problems such as this one. The **Fv** function, which we used to find the future value of a single sum earlier, will also find the future value of an annuity. Its use is nearly identical to the **Pv** function; the only difference is the substitution of **Pv** for **Fv**. Set up a new worksheet like the one in Exhibit 7-4.

EXHIBIT 7-4
FUTURE VALUE OF AN ANNUITY


In B5 place the formula: `=FV(B3,B4,-B2,0,0)`. The result of $516,997.01 agrees exactly with the result from the formula. What if that amount is less than what you had hoped for? One solution is to start making the investments this year, rather than next (i.e., the beginning of this period rather than the end of this period). To see the effect of this change

all that needs to be done is to change the *TYPE* argument to `1` so that the formula is now: `=FV(B3,B4,-B2,0,1)`. That minor change in your investment strategy will net you nearly $39,000 extra at retirement. Perhaps a better alternative is to accept a little extra risk (we assume that you are young enough that this makes sense) by investing in stock mutual funds that will return an average of about 10% per year over the 30-year horizon. In this case, still assuming that you start investing right now, you will have $904,717.12 at retirement. Significantly better!

Solving for the Annuity Payment

Suppose that we want to know the amount that we have to deposit in order to accumulate a given sum after a number of years. For example, assume that you are planning to purchase a house five years from now. Because you are currently a student, you will begin saving for the $10,000 down payment one year from today. How much will you need to save each year, if your savings will earn a rate of 4% per year? Figure 7-2 diagrams the problem.

<center>

FIGURE 7-2
A TIMELINE FOR ANNUAL SAVINGS TO OBTAIN $10,000 IN FIVE YEARS

</center>
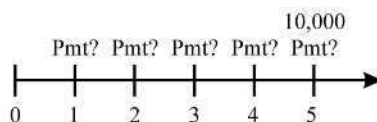
In this case, we wish to solve for the payment that you would have to make each year. The future value of the annuity is already known, so the **FV** function would be inappropriate. What we need is Excel's **PMT** function:

<center>

PMT(*RATE*, *NPER*, *PV*, *FV*, *TYPE*)

</center>

The arguments for the **PMT** function are similar to those for the **PV** and **FV** functions, except that it has *PV* and *FV* arguments in place of the *PMT* argument.

Enter the information from Exhibit 7-5 into cells A1:B6 of a new worksheet. In cell B6 enter the **PMT** function: `=PMT(B5,B4,-B2,B3,0)`. The result indicates that you will have to save $1,846.27 per year (a cash outflow) in order to accumulate $10,000 for the down payment in five years.

The **PMT** function allows *both PV* and *FV* to be inputs. In the previous example, it was assumed that *PV* was 0. However, let's suppose that you have recently inherited $3,000 from your uncle and that you want to use this money to begin saving now for that down payment. Because the $3,000 will grow to only $3,649.96 after five years at 4% per year (this can be verified by using the worksheet created for Exhibit 7-1), you will still need to save some amount every year. How much will you need to save each year? To find out, simply set the present value, in B2, to `3000` leaving the other values unchanged. Because the initial investment *reduces* the total amount that you need to save to $6,350.04 (why?), your annual saving requirement is reduced to $1,172.39.

Solving for the Number of Periods in an Annuity

Solving for the present value, future value, and payment for annuities are fairly simple problems. That is, the formulas are straightforward and easy to apply. Solving for the number of periods, *N*, is not as obvious mathematically. To do so requires knowledge of logarithms. If you know the present value of the annuity, then solving equation (7-5) for *N* we get:

$$N = \frac{\ln\left(\frac{-iPV_A}{Pmt} + 1\right)}{-\ln(1 + i)} \tag{7-8}$$

where $\ln(\cdot)$ is the natural logarithm operator. If you know the future value, then solving equation (7-7) for *N* results in:

$$N = \frac{\ln\left(\frac{iFV_A}{Pmt} + 1\right)}{\ln(1 + i)} \tag{7-9}$$

Return now to our example of saving for the down payment for a house. Recall that it was determined that by saving $1,846.27 per year you could afford the down payment after five years, assuming no initial investment. Set up the worksheet in Exhibit 7-6.

EXHIBIT 7-6
NUMBER OF ANNUITY PAYMENTS WHEN PV OR FV IS KNOWN

We can solve this problem using equation (7-9) and the built-in **LN** function:[2]

LN(*NUMBER*)

In B6 enter the formula: `=LN(B5*B3/B4+1)/LN(1+B5)`. The result is five years as we would expect.

Excel also offers the built-in **NPER** function, which also works with lump sums, to solve problems of this type directly. This function is defined as:

NPER(*RATE*, *PMT*, *PV*, *FV*, *TYPE*)

where all of the arguments are as previously defined. To use this function, you must know the payment, per period interest rate, and either the present value or future value or both.

Because we want to solve for the number of periods, insert the **NPER** function into B6: `=NPER(B5,-B4,-B2,B3,0)`. Notice that both the *PV* and *PMT* arguments are made negative in this function. Again, this is because of the cash flow sign convention. In this case, we wish to be able to withdraw the future value (a cash inflow and, therefore, positive) and deposit the *PV* and *PMT*s (cash outflows, therefore negative). The result is five years, exactly as we would expect. If you enter the $3,000 inheritance into B2, then you will have the down payment in only 3.39 years.

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2. Logarithms are often useful tools, and Excel offers functions to handle them. In addition to **LN**, the other logarithm functions are **LOG10** and **LOG**. The former calculates the base 10 logarithm, and the latter can calculate a logarithm with any base.

Solving for the Interest Rate in an Annuity

Unlike the present value, future value, payment, and number of periods, there is no closed-form solution for the rate of interest of an annuity. The only way to solve this problem is to use a trial-and-error approach, perhaps an intelligent one such as the Newton–Raphson technique or the bisection method.[3]

Excel, however, offers the **RATE** function that will solve for the interest rate. It is defined as:

$$\text{RATE}(\textit{NPER, PMT, PV, FV, TYPE, GUESS})$$

where the arguments are as defined earlier, and *GUESS* is your optional first guess at the correct answer. Ordinarily, the *GUESS* can be safely omitted.

> Suppose that you are approached with an offer to purchase an investment that will provide cash flows of $1,500 per year for 10 years. The cost of purchasing this investment is $10,500. If you have an alternative investment opportunity, of equal risk, that will yield 8% per year, which one should you accept?

There are actually several ways that a problem such as this could be solved. One method is to realize that 8% is your opportunity cost of funds and should therefore be used as your discount rate. Using the worksheet created in Exhibit 7-3 we find that the present value (i.e., current worth to you) of the investment is only $10,065.12. Because the price ($10,500) is greater than the value, you should reject the investment and accept your alternative.[4]

Another method of solving this dilemma is to compare the yields (i.e., compound annual return) offered by the investments. All other things being equal, the investment with the highest yield should be accepted. We already know that your alternative investment offers an 8% yield, but what is the yield of your new opportunity? We will use the worksheet in Exhibit 7-7 to find out.

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3. These are powerful techniques for solving these types of problems. The bisection method, briefly, involves choosing two initial guesses at the answer that are sure to bracket the true answer. Each successive guess is halfway between the two previous guesses that bracket the solution. The Newton–Raphson technique requires calculus and is beyond the scope of this book. For more information, consult any numerical methods textbook.

4. Note that we are simply comparing the cost of the investment to its perceived benefit (present value). If the cost is greater than the benefit, the investment should be rejected. We will expand on this method in future chapters.

EXHIBIT 7-7
YIELD ON AN ANNUITY

Into B6 place the function: `=RATE(B5,B4,B2,B3,0,0.1)`. The result is 7.07% per year, so you should reject the new investment in favor of the alternative that offers 8% per year. This is the same result we obtained with the present value methodology, as expected. Later, we will see that this will always be the case when comparing mutually exclusive investment opportunities.[5]

Deferred Annuities

Not all annuities begin their payments during the year following the analysis period. For example, if you are planning for your retirement, you will probably start by determining the amount of income that you will need each year during retirement. However, if you are a student, you will probably not retire for many years. Your retirement income, then, is an annuity that won't begin until you retire. In other words, it is a *deferred annuity*. How do we determine the value of a deferred annuity?

Assume that you own a time machine (made of a super-strong futuristic metal that can withstand the gravitational forces of a black hole in space). This machine can transport you to any time period that you choose. If we use this time machine to transport you to the year just prior to retirement, then valuing the stream of retirement income becomes a simple matter. Just use Excel's **Pv** function. The year before the first withdrawal is now considered to be year 0, the first year of retirement is year 1, and so on. Figure 7-3 demonstrates this time-shifting technique.

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5. Mutually exclusive investment opportunities are those in which you may choose one investment or the other, but not both. That is, the choice of one precludes you also choosing the other.
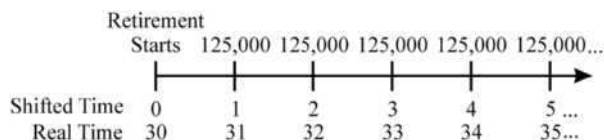
In constructing Figure 7-3, we have assumed that you will retire 30 years from now, and will require an income of $125,000 per year during retirement. If we further assume that you will need your retirement income for 35 years and expect to earn 6% per year, you will need to have accumulated $1,812,280.80 by year 30 to provide this income. In other words, $1,812,280.80 is the present value, at year 30, of $125,000 per year for 35 years at 6%. You can use the worksheet created for Exhibit 7-3 to verify these numbers.[6]

The problem in Figure 7-3 is that knowing the amount that we will need 30 years from now doesn't directly tell us how much we need to save today. The present value function in Excel, or the PV_A equation (7-5), must be thought of as a transformation function. That is, it transforms a series of payments into a lump sum. That lump sum ($1,812,280.80 in our example) is then placed *one period before the first payment occurs*. In our earlier examples, the annuities began payment at the end of period 1, so the present value was at time period 0 (one period earlier than period 1). In the current example, the present value is at time period 30, also one period before the first payment.

In order to determine the amount that we need to invest today, we must treat the required savings at retirement as a future value. This sum must then be discounted back to period 0. For example, if we assume that we can earn 8% per year before retirement, we would need to invest $180,099.63 today in order to meet our retirement goals.

Exhibit 7-8 presents a simple worksheet to determine the investment required today in order to provide a particular income during retirement. Open a new worksheet and enter the data and labels from Exhibit 7-8.

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6. Although $125,000 per year may seem like a lot of money, we arrived at this figure by assuming that you would need $50,000 per year in today's dollars. We then adjusted that amount for an average inflation rate of 3% per year to arrive at $125,000. In fact, that won't be enough because inflation will continue during retirement. So, your retirement income will need to rise each year to keep pace with inflation. We will deal with this in the next section.

EXHIBIT 7-8
PLANNING FOR RETIREMENT

To complete our retirement worksheet, we need to enter functions into cells B7:B9. Recall that the first step in our retirement income problem was to determine the present value of your retirement income at period 30. To do this in our worksheet, enter the **PV** function into B7: `=PV(B6,B4,-B2,0,0)`. The result, $1,812,280.80, tells us that you will need to have saved this amount in order to provide the income indicated in B2 for the number of years indicated in B4. To determine the amount that you would need to invest today (a lump sum), you need to determine the present value, at time period 0, of the amount in B7. To do this, in B8 enter the formula: `=PV(B5,B3,0,-B7,0)`. As before, the amount required today is $180,099.63.

Another feature of the retirement planning worksheet is that it will calculate the annual savings required to reach your goal. To make Excel do this calculation, we need to use the **PMT** function. In B9 enter: `=PMT(B5,B3,0,-B7,0)`. The result is $15,997.79, which means that if you can save this amount each year for the next 30 years, and earn an average of 8% interest each year, you will reach your goal. As difficult as that may be, it is much easier than investing the lump sum today.

We have ignored the effects of inflation and taxes on your retirement planning for this worksheet. But if we assume that you save the amount in B9 in a tax-deferred account, the results are a bit more realistic. Experiment with this worksheet. You may be surprised at the difficulty of saving for a comfortable retirement.

Graduated Annuities

Previously, we defined an annuity as a series of nominally equal cash flows, equally spaced in time. However, not everything that is called an annuity has equal cash flows each period.

For example, it is common today for people to invest a lump sum today in exchange for a series of payments that will escalate over time to maintain constant purchasing power. An insurance company may offer such an investment opportunity to retirees who are concerned about inflation in the future, and some lotteries offer an "annuitized" payout that increases each year.

Return to our example in the previous section. Your stated goal was to receive $125,000 each year for the 35 years that you expect to spend in retirement. However, if your income doesn't rise each year then your purchasing power will decline dramatically by the end of the 35 years. In fact, if inflation averages 3% per year, your income in the last year will have the same purchasing power as only about $44,423 would have had at the beginning of your retirement. Clearly, then, it would be beneficial if your retirement income would grow to keep up with the rate of inflation.

Present Value of a Graduated Annuity

Suppose that you expect inflation to be 3% per year during retirement, so you have revised your retirement income needs so that your income grows by 3% each year. Figure 7-4 shows the revised timeline.

FIGURE 7-4
RETIREMENT INCOME GROWING AT 3% PER YEAR
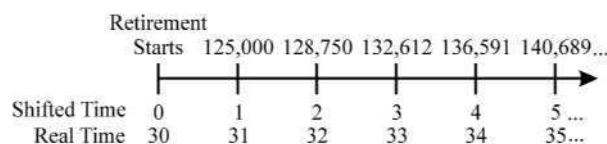
How do we find the present value of such a cash flow stream? It would appear that we cannot use the **Pv** function because it assumes that all of the cash flows are the same in each period. That leaves us with the fallback position of using the principle of value additivity. In other words, calculate the present value of each cash flow separately, and then add them together to get the total present value. This method requires a table that lists all of the cash flows and the use of equation (7-4) on page 194.

Fortunately, if we can assume that the growth rate of the cash flows will be constant, as it is in this case, then there is a closed-form solution for the present value of a graduated annuity when the first payment occurs one period from now:

$$PV_{GA} = \frac{Pmt_1}{i - g}\left[1 - \left(\frac{1 + g}{1 + i}\right)^{N}\right] \tag{7-10}$$

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where Pmt_1 is the first cash flow, i is the interest rate, g is the growth rate, and N is the number of cash flows.

$$PV_{GA} = \frac{125,000}{0.06 - 0.03}\left[1 - \left(\frac{1.03}{1.06}\right)^{35}\right] = 2,641,257.55$$

As before, if we assume that you will earn 6% per year during retirement and your withdrawals grow 3% per year to keep up with inflation, then you will need to accumulate $2,641,257.55 in order to meet your income goals.

We can use the **Pv** function to calculate the same result, but only after modifying the interest rate. In the case of a graduated annuity we have two rates: a discount rate and a growth rate. These rates work against each other, so you might expect that we could use a "net" interest rate. In this case that would be 6% - 3% = 3%, but that won't work properly because interest rates compound. In other words, they are multiplicative instead of additive. Therefore, we need a slightly more complicated formula:

$$\text{Net Rate} = \frac{(1 + i)}{(1 + g)} - 1 \qquad\qquad \textbf{(7-11)}$$

Instead of subtracting, we need to divide. In this case, the net interest rate is:

$$\text{Net Rate} = \frac{1.06}{1.03} - 1 \cong 0.02913$$

One final adjustment is necessary; we must also divide the result of the **Pv** function by $(1 + g)$. This adjustment effectively changes the payment at period 1 into its inflation-adjusted value at period 0.

We can now easily modify the retirement worksheet presented in Exhibit 7-8 to handle our new assumption regarding inflation. Make a copy of that worksheet, and then select row 5 and insert a new row. In A5 enter the label: `Expected Inflation Rate`, and in B5 enter: `3%`. Now, we need to change the formula in B8 to: `=-PV((1+B7)/(1+B5)-1,B4,B2)/(1+B5)`. You will see that the result is $2,641,257.55, exactly the same as before. Furthermore, the results in B9:B10 have automatically updated to include our new assumption. As you can see, you will now need to save $23,315.53 per year before retirement. That is about $7,318 more per year than the original projection, but this will allow you to maintain constant purchasing power during retirement. Exhibit 7-9 shows the new retirement planning worksheet.

PLANNING FOR RETIREMENT ADJUSTED FOR INFLATION

In this example, the annuity payments begin at the end of the period. However, in many cases (such as lotteries) the first cash flow occurs immediately. This type of cash flow is called a *graduated annuity due*. We can easily modify equation (7-10) using the principal of value additivity to handle cash flows that start immediately.

Realize that we can treat a graduated annuity due as a regular graduated annuity with one less payment plus an additional payment today:

$$PV_{GAD} = Pmt_0 + \frac{Pmt_0(1+g)}{i-g}\left[1 - \left(\frac{1+g}{1+i}\right)^{N-1}\right] \tag{7-12}$$

where Pmt_0 is the cash flow that occurs immediately. Again, we can use the interest rate adjustment from equation (7-11) in the **Pv** function. In this case, we need to use only the interest rate adjustment and set the *TYPE* argument to 1. So, if you plan to start withdrawals on the day of your retirement, you can change the formula in B8 to: `=-PV((1+B7)/(1+B5)-1,B4,B2,0,1)`. Because you will be making the first withdrawal one year earlier, you will need to have accumulated $2,799,733 at retirement, and you will need to save more each year.

Because equations (7-10) and (7-12) are complex, we have written an add-in function to do the calculations. An add-in function is similar to a built-in function, but you must have the add-in installed and open to use the function.[7] The FAME_PVGA function is defined as:

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7. You can download the Famefncs.xlam add-in from the official Web site. Installation of the add-in is described in Chapter 1 on page 25.

$$\textbf{FAME\_PVGA}(\textbf{\textit{PMT}}, \textbf{\textit{NPER}}, \textbf{\textit{GROWTHRATE}}, \textbf{\textit{DISCRATE}}, \textit{BEGEND})$$

Where **_PMT_** is the first cash flow, **_NPER_** is the number of cash flows, **_GROWTHRATE_** is the rate at which the cash flows grow over time, and **_DISCRATE_** is the required return. _BEGEND_ is an optional argument that specifies if the cash flow occurs at the end (0) or beginning (1) of the period. (Note that _BEGEND_ works just like the _TYPE_ argument in the built-in **PV** function.)

Using this add-in function, you can replace the formula in B8 with: `=FAME_PVGA(B2,B4,B5,B7,0)`. You will get the same answer, but it is a bit easier to use the add-in function.

Future Value of a Graduated Annuity

Recall that the principle of value additivity states that we can find the value of a series of cash flows by moving the cash flows to a given period and then summing them together. The resulting lump sum is economically equivalent to the original series of cash flows. We can make use of this fact (further illustrated in the next section) to derive a formula for the future value of a graduated annuity. Specifically, we will first find the present value using equation (7-10) and then multiply it by $(1 + i)^N$. After simplifying, the formula for the future value of a graduated annuity is:

$$FV_{GA} = \frac{Pmt_1(1 + i)^N}{i - g}\left[1 - \left(\frac{1+g}{1+i}\right)^N\right] \tag{7-13}$$

Let's look at an example:

> Imagine that you are going to make an investment of $1,000 next year and then increase your investment each year by 6% per year for four additional years. If you can earn 8% per year on your investment, how much will you have accumulated at the end of year 5?

Using equation (7-13) we can see that the answer is $6,555.13:

$$FV_{GA} = \frac{1,000(1.08)^5}{0.08 - 0.06}\left[1 - \left(\frac{1.06}{1.08}\right)^5\right] = 6,555.13$$

As before, we have created a user-defined function to do this calculation:

$$\textbf{FAME\_FVGA}(\textbf{\textit{PMT}}, \textbf{\textit{NPER}}, \textbf{\textit{GROWTHRATE}}, \textbf{\textit{DISCRATE}}, \textit{BEGEND})$$

All of the arguments are defined identically to those for the **FAME_PVGA** function.

Uneven Cash Flow Streams

Annuities are very neat from a cash flow point of view, but most investments don't have cash flows that are the same in each period. When the cash flows are different in each period, we refer to them as *uneven cash flow streams*. Investments of this type are not as easy to deal with, though conceptually they are the same.

Recall our discussion of the principle of value additivity. This principle says that as long as cash flows occur in the same period, we can add them together to determine their combined value. The principle applies to any time period, not just to time period 0. So, to determine the present value of an uneven stream of cash flows, one option is to determine the present value of each cash flow separately, and then add them together. The same technique applies to the future value of an uneven stream. Simply find the future value of each cash flow separately, and then add them together.

Excel's **Pv** and **Fv** functions cannot be used for uneven cash flow streams because they assume equal (annuity) payments or a lump sum. Set up the worksheet in Exhibit 7-10 and we'll see what needs to be done.

EXHIBIT 7-10
Pv and Fv for Uneven Cash Flows

First, we want to solve for the present value of the cash flows in B3:B7. To do this, we need to use the net present value (NPV) function. This function will be especially valuable for capital budgeting in Chapter 11. The **NPV** function is defined as:

$$NPV(RATE, VALUE1, VALUE2, \ldots)$$

where **RATE** is the per period rate of return (i.e., the discount rate), **VALUE1** is the first cash flow (or range of cash flows), **VALUE2** is the second cash flow, and so on. Excel will accept

up to 255 cash flows in the list. To find the present value of the cash flows, enter: =NPV(B8,B3:B7) into B9.[8] Note that we have entered the cash flows as a range, rather than as individual values. Excel will accept the arguments either way, though a range is generally easier to enter. The result is $10,319.90. To verify this result, you can find the present value of each cash flow at 11% per year and then add them together.

Finding the future value of an uneven stream is a bit more difficult because Excel has no built-in function to perform this calculation. Recall, however, the principle of value additivity. If we can get all of the cash flows into the same period, we can add them together and then move the result to the desired period. Figure 7-5 shows this solution.

FIGURE 7-5
FINDING THE FUTURE VALUE OF AN UNEVEN CASH FLOW STREAM
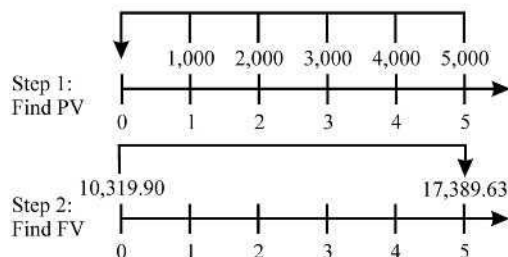
First, we find the present value of the uneven stream of cash flows, perhaps using the **NPV** function, and then we find the future value of the present value of the cash flows. The easiest way to implement this method in Excel is to use the **NPV** function *nested* within the **FV** function. A nested function is one that is used as an input to another function. In B10 enter: =FV(B8,A7,0,-NPV(B8,B3:B7),0). The future value is found to be $17,389.63. Notice that we have used the **NPV** function inside the **FV** function. As an alternative, we could have entered –B9 for the present value argument because we have already calculated it, but the result would be the same. Using nested functions can often simplify a worksheet by making use of fewer cells, though the formulas tend to be more complex.

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8. If you are familiar with the definition of Net Present Value (NPV), you should know that Excel's NPV function does not calculate the NPV as it is normally defined. Instead, it merely calculates the present value of uneven cash flows. This is covered in more depth in Chapter 11.

Solving for the Yield in an Uneven Cash Flow Stream

Often in financial analysis, it is necessary to determine the yield of an investment given its price and cash flows. For example, we have already seen that one way to choose between alternative investments is to compare their yields, and we will see more examples in Chapter 11. This was easy when dealing with annuities and lump-sum investments, where we could use the **RATE** function. But, what about the case of uneven cash flow investments? We will use the worksheet in Exhibit 7-11 to find out.

<div align="center">

EXHIBIT 7-11
YIELD ON AN UNEVEN CASH FLOW STREAM

</div>

To solve for the yield in problems such as this, we need to make use of the **IRR** (internal rate of return) function. The IRR is defined as the rate of return that equates the present value of future cash flows with the cost of the investment ($10,319.90 in this problem). In Excel, the **IRR** function is defined as:

<div align="center">

IRR(*VALUES*, *GUESS*)

</div>

where *VALUES* is a range of cash flows (including the cost), and *GUESS* is the optional first guess at the correct interest rate. We will study this function in depth in Chapter 11, but for now we will just make use of it.

Before we find the solution, notice a couple of things about the worksheet. The cash flows are listed separately, so we cannot use the **IRR** function like we used the **FV**, **PV**, and **PMT** functions. Also, we must include the cost of the investment as one of the cash flows. To find the yield on this investment, insert into B9: =IRR(B3:B8,0.10). The result is 11%, which means that if you purchase this investment you will earn a compound annual rate of 11%.

We have used one form of the **IRR** function in B9. Another option is to omit the *GUESS* (0.10 in our example). In this case, either form will work. Sometimes, however, Excel will not be able to converge on a solution without a *GUESS* being specified. Remember that this is essentially a trial-and-error process, and sometimes Excel needs a little help to go in the right direction.

A few situations may cause an error when using the **IRR** function. One that we've already discussed is that Excel may not converge to a solution. In this case, you can usually find the answer by supplying Excel with a different *GUESS*. Another occurs if you have no negative cash flows. As an example, change the purchase price to a positive `10,319.90`. Excel will return the #NUM! error message indicating that there is a problem. In this case the problem is that your return is infinite (why?). A third problem can result from more than one negative cash flow in the stream. In general, there will be one solution to the problem for each sign change in the cash flow stream. In our original example, there is only one sign change (from negative to positive after the initial purchase.)

Nonannual Compounding Periods

There is no reason why we should restrict our analyses to investments that pay cash flows annually. Many investments make payments (e.g., interest) semiannually, monthly, daily, or even more frequently. Everything that we have learned to this point still applies, with only a minor change.

Recall our basic time value of money formula (7-2):

$$FV_N = PV(1+i)^N$$

Originally, we defined i as the annual rate of interest and N as the number of years. Actually, i is the periodic rate of interest and N is the total number of periods. As an example, i might be the weekly interest rate and N the total number of weeks for which we will hold the investment. Because rates are usually quoted in terms of simple (i.e., not compounded) annual rates, we can restate our basic formula as:

$$FV_N = PV\left(1 + \frac{i}{m}\right)^{Nm} \tag{7-14}$$

where i is the annual rate, N is the number of years, and m is the number of periods per year. Because there are 52 weeks in a year ($m = 52$), we would calculate the weekly rate as the annual rate divided by 52. Similarly, the number of weeks would be calculated by multiplying the number of years (perhaps a fractional number of years) by 52.

Excel can handle nonannual compounding just as easily as annual compounding. Just enter the rate and number of periods adjusted for the length of the compounding period. Let's look at an example.

> Assume that you are shopping for a new bank to set up a savings account. As you start shopping, you notice that all of the banks offer the same stated annual interest rate, but different compounding periods.

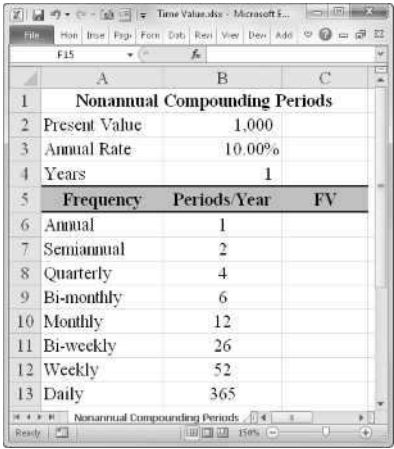To help make your decision, you set up the worksheet in Exhibit 7-12.

EXHIBIT 7-12
NONANNUAL COMPOUNDING PERIODS

Notice that all of the banks are advertising a 10% annual rate. The only difference is how often they credit the interest to your account (i.e., the frequency of compounding). Being economically rational, you will choose the bank that will provide the highest balance at the end of the year. To determine the end of year balances, enter the **FV** formula in B7: =FV(B3/B6,B4*B6,0,-B2). Copy the formula from B7 to both B10 and B13. Note that we have again made use of nested functions. In this case, the rate is defined as the annual rate *divided* by the number of periods in a year, and the number of periods is the number of years *times* the number of periods in a year.

The choice is clear. You should choose the Third National Bank because it offers the highest end of year balance ($1,104.71). All other things being equal, the more frequent the compounding, the higher your future value will be. To see this more clearly, set up the worksheet shown in Exhibit 7-13.

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**EXHIBIT 7-13**
COMPARING VARIOUS NONANNUAL COMPOUNDING PERIODS

To finish the worksheet, use the **Fv** formula in C6: `=FV(B$3/B6,B6*$B$4,0,-B$2)` and copy it down to the other cells. It is important that you insert the dollar signs as indicated so that the references to the present value and interest rate remain fixed when copying.

As before, more frequent compounding leads to higher future values. Furthermore, the future value increases at a decreasing rate as the number of compounding periods increases. This can be seen more easily if we create a chart of the future values. Select the labels in A5:A14 and the numbers in C5:C14 (remember, you can select discontiguous ranges by holding down the Ctrl key while dragging the mouse). Note that you are selecting one extra row because we will use this worksheet again later to add one more data point. Now, click the Insert tab and create a Column chart formatted as shown in Exhibit 7-14.

Continuous Compounding

We have seen that more frequent compounding leads to higher future values. However, our examples extended this idea only as far as daily compounding. There is no reason that we can't also compound every half-day, every hour, or even every minute. In fact, this concept can be extended to the smallest imaginable time period: the instant. This type of compounding is referred to as *continuous compounding*.

EXHIBIT 7-14
NONANNUAL COMPOUNDING RESULTS
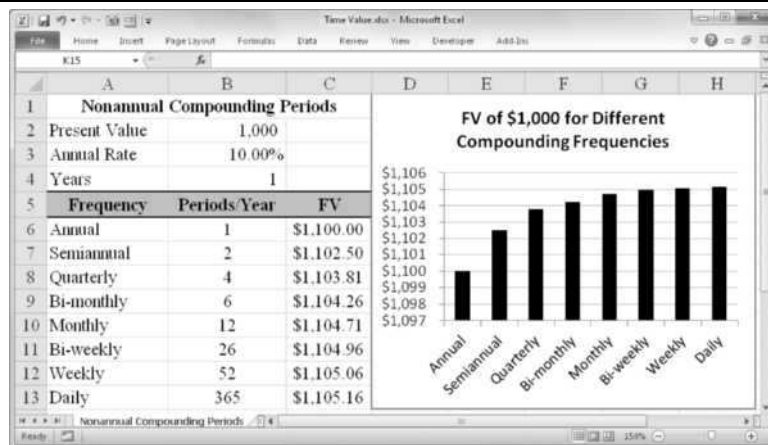
Continuous compounding is an extension of what we have seen already. To recap, recall that we changed the basic future value function:

$$FV_N = PV\left(1 + \frac{i}{m}\right)^{Nm}$$

The more frequently we compound, the larger m is going to be. For example, with semiannual compounding $m = 2$, but with daily compounding $m = 365$. What if we set m equal to infinity? We can't do that because i/∞ is effectively equal to zero. Instead we can take the limit as m approaches infinity. When we do this, we get:

$$\lim_{m \to \infty} FV_N = PVe^{iN} \tag{7-15}$$

where e is the base of the natural logarithm and is approximately equal to 2.718.

Excel does not offer functions to solve for the present or future value when compounding is continuous. However, we can easily create the formulas. To do so requires that you know about the **EXP** function, which raises e to a specified power.[9] This function is defined as:

EXP(*NUMBER*)

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9. e is the base of the natural logarithm, so $\exp(\cdot)$ is the inverse of $\ln(\cdot)$. In other words, $\exp(\ln(x)) = x$.

Using the worksheet in Exhibit 7-14, we can add in cell C14: `=B2*exp(B3*B4)`. Because we have assumed a one year period in this example, the power to which e is raised is simply the interest rate. Add the label: `Continuous` in A14 and the worksheet is complete. Note that continuous compounding doesn't offer much of an increase over daily compounding. The advantage does get larger as the amount invested grows, but it would take huge sums to make a significant difference. To see this, change the present value, B2, to 10,000,000. The difference will also grow as the number of years increases. Experiment by changing B4.

We can also calculate the present value of a continuously compounded sum. All that needs to be done is to solve equation (7-15) for PV:[10]

$$\lim_{m \to \infty} PV = FV_N e^{-iN} \tag{7-16}$$

Summary

In this chapter, we have discussed the concept of the time value of money. Present value represents the amount of money that needs to be invested today in order to purchase a future cash flow or stream of cash flows. Future value represents the amount of money that will be accumulated if we invest known cash flows at known interest rates. Further, we discussed various types of cash flows. Annuities are equal cash flows, equally spaced through time. Graduated annuities are similar to normal annuities, but the cash flows grow by a certain amount each period. Uneven cash flows are those in which the periodic cash flows are not equal.

Before continuing with future chapters, you should be comfortable with these concepts. Practice by changing the worksheets presented in this chapter until you develop a sense for the type of results that you will obtain.

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10. Many students find that the continuous compounding equations are easier to recall if we change the notation slightly. Specifically, let P be the present value, F be the future value, r is the annual rate of interest, and T is the number of years (which can be fractional). With this notation, equation (7-15) becomes: $F = Pe^{rT}$, and equation (7-16) becomes: $P = Fe^{-rT}$. This is easier because the formulas can be pronounced. For example, equation (7-15) is pronounced "Pert."

TABLE 7-1
FINANCIAL FUNCTIONS USED IN THIS CHAPTER

Purpose	Function	Page
Find the future value	**FV(RATE, NPER, PMT, PV, TYPE)**	191
Find the present value	**PV(RATE, NPER, PMT, FV, TYPE)**	193
Payment of an annuity	**PMT(RATE, NPER, PV, FV, TYPE)**	198
Number of periods	**NPER(RATE, PMT, PV, FV, TYPE)**	200
Natural logarithm	**LN(NUMBER)**	200
Yield of an annuity	**RATE(NPER, PMT, PV, FV, TYPE, GUESS)**	201
Present value of a graduated annuity	**FAME_PVGA(PMT, NPER, GROWTHRATE, DISCRATE, BEGEND)**	208
Future value of a graduated annuity	**FAME_FVGA(PMT, NPER, GROWTHRATE, DISCRATE, BEGEND)**	208
Present value of unequal cash flows	**NPV(RATE, VALUE1, VALUE2,…)**	209
Find the yield of unequal cash flows	**IRR(VALUES, GUESS)**	211
Raise *e* to a power	**EXP(NUMBER)**	215

Problems

1. Upon starting your new job after college, you've been confronted with selecting the investments for your 401k retirement plan. You have four choices for investing your money:

   - A money market fund that has historically returned about 2% per year.

   - A long-term bond fund that has earned an average annual return of 5%.

   - A conservative common-stock fund that has earned 7% per year.

   - An aggressive common-stock fund that has earned 10% per year.

   a. If you were to contribute $10,000 per year for the next 35 years, how much would you accumulate in each of the above funds?

b. Now, change your worksheet so that it allows for less than annual investments (monthly, biweekly, etc.). Your total annual investment will remain unchanged, but it may be made in smaller, but more frequent, amounts.

c. Set up a scenario analysis that shows your accumulated value in each fund if you were to invest quarterly, monthly, biweekly, and weekly. Create a scenario summary of your results.

d. What relationship do you notice between the frequency of investment and the future value? Create a Column chart of the results that more clearly shows the outcome from more frequently investing.

2. Given the following set of cash flows:

Period	Cash Flow
1	$ 30,000
2	25,000
3	20,000
4	15,000
5	10,000

a. If your required rate of return is 8% per year, what is the present value of the above cash flows? Future value?

b. Now, suppose that you are offered another investment that is identical, except that the cash flows are reversed (i.e., cash flow 1 is 10,000, cash flow 2 is 15,000, etc). Is this worth more, or less, than the original investment? Why?

c. If you paid $75,000 for the original investment, what average annual rate of return would you earn? What return would you earn on the reversed cash flows?

d. Still assuming that your required return is 8%, would you be willing to purchase either of these investments? Explain why, or why not.

3. Your five-year-old daughter has just announced that she would like to attend college. The College Board has reported that the average cost of tuition, room, board, and other expenses at public colleges is $16,140 in the 2010–2011 academic year.[11] The cost has risen 6.1% over the last year. You believe that you can earn a rate of 8% on investments to meet this goal.

    a. If costs continue to rise at 6.1% per year, how much will it cost for the first year of tuition in 13 years?

    b. Assuming that you plan to have enough money saved in 13 years to cover all four years of college costs, how much will you need to have accumulated by that time? Note that the tuition, room, and board is a graduated annuity growing 6.1% per year, and assume that you will pay all costs at the beginning of each year.

    c. If you were to invest a lump sum today in hopes of covering your daughter's college costs, how much would you have to invest?

    d. If you now decided to invest annually instead, how much would you have to invest each year? What if you make investments monthly instead?

    e. You just learned of a $10,000 inheritance and plan to invest it in your daughter's college fund. Given this new source of funds, how much will you now have to invest each year?

4. You have decided to invest in a small commercial office building that has one tenant. The tenant has a lease that calls for annual rent payments of $15,000 per year for the next three years. However, after that lease expires you expect to be able to increase the rent by 5% per year for the next seven years. You plan to sell the building for $250,000 ten years from now.

    a. Create a table showing the projected cash flows for this investment assuming that the next lease payment will be made one year from today.

    b. Assuming that you need to earn 11% per year on this investment, what is the maximum price that you would be willing to pay for the building today? Use the **NPV** function.

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11. See http://trends.collegeboard.org/college_pricing/ for the full results.

c.   Notice that the cash flow stream starts out as a three-year regular annuity, but it then changes into a seven-year graduated annuity plus a lump sum in year 10. Use the principal of value additivity to calculate the present value of the cash flows.

d.   Suppose that the current owner of the building is asking $175,000 for the building. If you paid this price, what annual rate of return would you earn? Should you buy the building at this price?

5.   Congratulations! You have just won the State Lottery. The lottery prize was advertised as an annuitized $85 million paid out in 30 equal annual payments beginning immediately. The annual payment is determined by dividing the advertised prize by the number of payments. You now have up to 60 days to determine whether to take the cash prize or the annuity.

a.   If you were to choose the annuitized prize, how much would you receive each year?

b.   The cash prize is the present value of the annuity payments. If interest rates are 7.5%, how much will you receive if you choose the cash option?

c.   Now suppose that, as many lotteries do, the annuitized cash flows will grow by 3% per year to keep up with inflation, but they still add up to $85 million. In this case, the first payment will be $1,786,637.04 today. If you took the cash prize instead, how much would you receive (before taxes)?