8-1
Introduction

Area Moment of Inertia
The second moment of area is also known as the moment of inertia of a shape. The second moment of area is a measure of the "efficiency" of a cross-sectional shape to resist bending caused by loading.

The Symbol for Moment of Inertia is I
The units of moment of inertia are length raised to the fourth power, such as, in^4 or mm^4

Moments of inertia of areas are used in calculating the stresses and deflections of beams, the torsion of shafts, and the buckling of columns.

The location of the centroid of an area involves the quantity $\Sigma x \Delta A$, which represents the first moment of the area. The area moment of inertia involves the quantity $\Sigma x^2 \Delta A$, which represent the second moment of an area (because x is squared).

The moment of inertia is always computed with respect to an axis; its value is greatly affected by the distribution of the area relative to the axis.
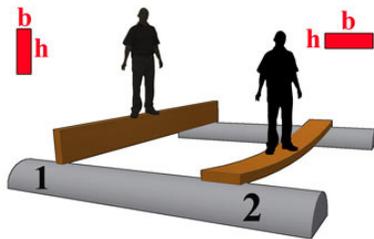
Both beams have the same area and even the same shape.

Beam 1 is stronger than Beam 2 because it has a larger second moment of inertia (I).

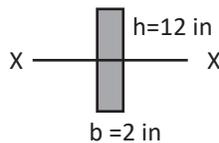Orientation can change the second moment of area (I).

For a rectangle,

$$\overline{I}_x = \frac{1}{12} bh^3$$

where b is the breadth (horizontal) and h is the height (vertical) if the load is vertical i.e. gravity load

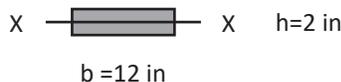If Beam 1 and Beam 2 are 2 in x 12 in,

Beam 1
I_x = 1/12 (2 in)(12in)³ = 288 in⁴

Beam 2
I_x = 1/12 (12 in)(2in)³ = 8 in⁴

Under the same loading conditions, Beam 2 will bend before Beam 1.

8-2
Moments of Inertia and Radii of Gyration

To study the strength of beams and columns the moment of inertia or second moment of a plane area is required. Numerical value of this quantity is used to indicate how the area is distributed about a specified axis.

If the axis lies within the plane of an area, that area's second moment about the axis is called the Rectangular Moment of Inertia, I

For any axis that is perpendicular to the plane of an area, the area's second moment in known as the Polar Moment of Inertia, J

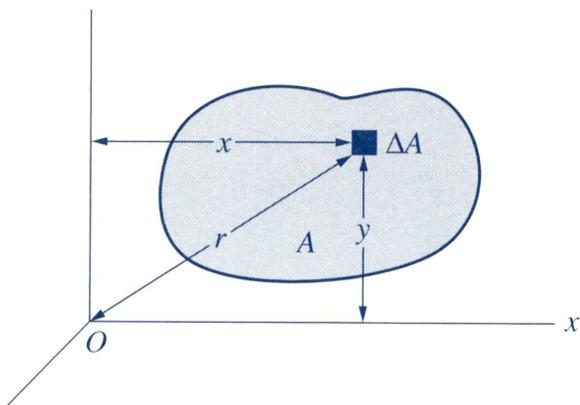Low values for I or J – describes an area whose elements are closely grouped about an axis
High values for I or J – indicates that much of an area is located at some distance from the selected axis

Moments of Inertia
The moments of inertia for the entire area A with respect to the x and y axis are:

$$I_x = \Sigma\ y^2\ \Delta A \qquad\qquad I_y = \Sigma\ x^2\ \Delta A$$

Moment of Inertia is always positive.
Units of Moment of Inertia are length raised to the fourth power, such as in^4 or m^4.

Polar Moment of Inertia
The second moment of Area A with respect to the pole O or the z-axis.
$\Delta J_O = \Sigma\ r^2\ \Delta A$ \qquad\qquad r = distance from the pole (or the z-axis) to ΔA

$J_O = \Sigma\ r^2\ \Delta A$

Since $r^2 = x^2 + y^2$ \qquad\qquad (Pythagorean's Theorem, right triangle)

$$J_O = \Sigma\ r^2\ \Delta A \quad = \Sigma\ (x^2 + y^2)\ \Delta A$$
$$= \Sigma\ x^2\ \Delta A + \Sigma\ y^2\ \Delta A$$
$$= I_x + I_y \qquad \text{Polar Moment of Inertia}$$

Radius of Gyration

The analysis and design of columns requires the radius of gyration of the cross-sectional area of the column. [See Table A-1(a) through Table A-6(a) in the textbook (pgs. 762-776)]

Radius of Gyration r of an area with respect to a given axis is defined by the relationship:

$$I_x = A r_x^2$$

Similarly we have, $\quad I_y = A r_y^2$

Where,
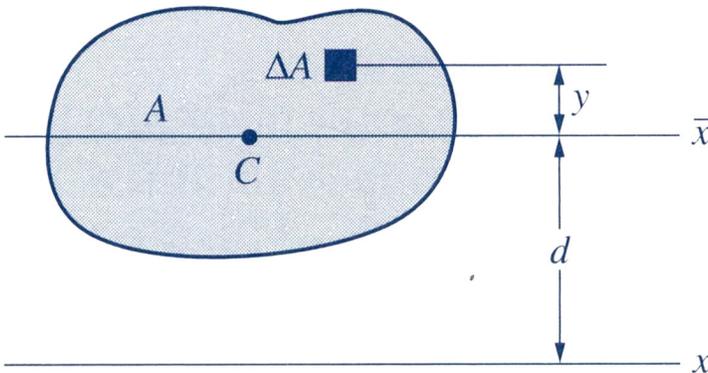I = Moment of Inertia wrt the given axis
A = cross-sectional area

Parallel-Axis Theorem

The moment of inertia of an area with respect to a noncentroidal axis may be expressed in terms of the moment of inertia with respect to the parallel centroidal axis.

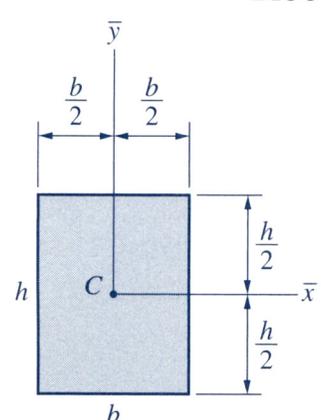$$I_x = \Sigma(y + d)^2 \Delta A = \Sigma(y^2 + 2dy + d^2)\Delta A$$

or

$$I_x = \Sigma y^2 \Delta A + 2d \Sigma y \Delta A + d^2 \Sigma \Delta A$$
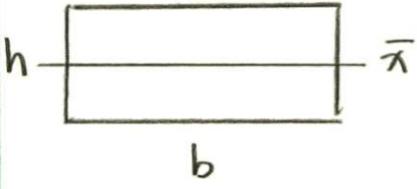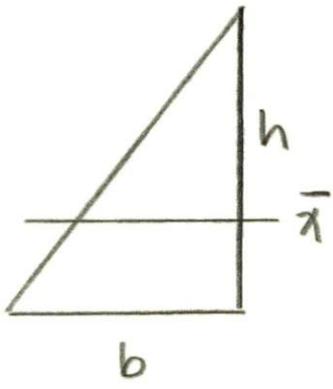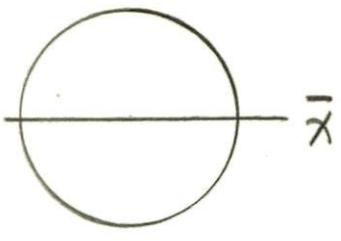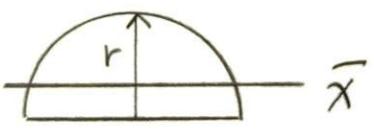

In the above equation, the first term represents moment of inertia $\bar{I}_x$ of the area about the centroidal $\bar{x}$ axis. The second term is zero since $\Sigma y \Delta A = A\bar{y}$, and $\bar{y}$ is zero with respect to the centroidal $\bar{x}$ axis. The third term is simply Ad^2. Thus, the expression for I_x becomes

$$I_x = \bar{I}_x + Ad^2 \qquad\qquad (8\text{–}7)$$

155

TABLE 8–1 Properties of Areas of Common Shapes

Rectangle

$$A = bh$$

$$\overline{I}_x = \tfrac{1}{12} bh^3$$

$$\overline{I}_y = \tfrac{1}{12} hb^3$$

$$\overline{J} = \tfrac{1}{12} bh(h^2 + b^2)$$

$$\overline{r}_x = \frac{h}{\sqrt{12}}$$

$$\overline{r}_y = \frac{b}{\sqrt{12}}$$

Triangle
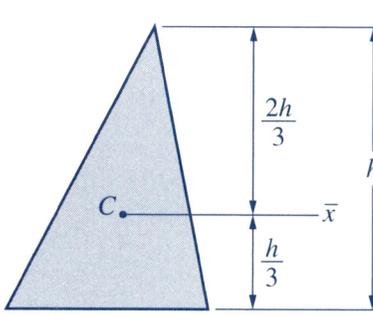
$$A = \tfrac{1}{2} bh$$

$$\overline{I}_x = \tfrac{1}{36} bh^3$$

$$\overline{r}_x = \frac{h}{\sqrt{18}}$$

Circle
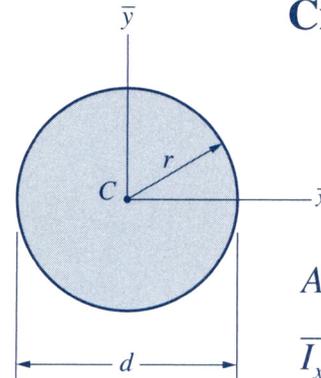
$$A = \tfrac{1}{4}\pi d^2 = \pi r^2$$

$$\overline{I}_x = \overline{I}_y = \tfrac{1}{64}\pi d^4 = \tfrac{1}{4}\pi r^4$$

$$\overline{J} = \tfrac{1}{32}\pi d^4 = \tfrac{1}{2}\pi r^4$$

$$\overline{r}_x = \overline{r}_y = \tfrac{1}{4} d$$

Circular Ring
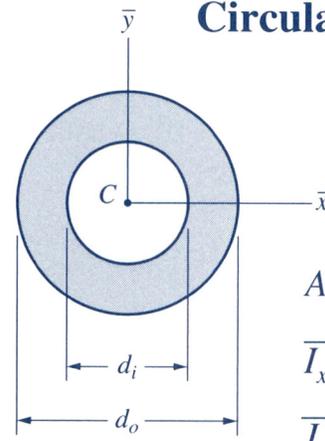
$$A = \tfrac{1}{4}\pi(d_o^2 - d_i^2)$$

$$\overline{I}_x = \overline{I}_y = \tfrac{1}{64}\pi(d_o^4 - d_i^4)$$

$$\overline{J} = \tfrac{1}{32}\pi(d_o^4 - d_i^4)$$

$$\overline{r}_x = \overline{r}_y = \tfrac{1}{4}\sqrt{d_o^2 + d_i^2}$$

Semicircle
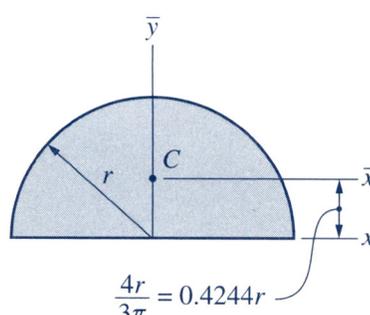
$$A = \tfrac{1}{2}\pi r^2$$

$$\overline{I}_x = 0.1098 r^4$$

$$\overline{I}_y = \overline{I}_x = \tfrac{1}{8}\pi r^4$$

$$\overline{J} = 0.5025 r^4$$

$$\overline{r}_x = 0.2644 r$$

$$\overline{r}_y = r_x = \tfrac{1}{2} r$$

$$\frac{4r}{3\pi} = 0.4244r$$

Quarter-Circle
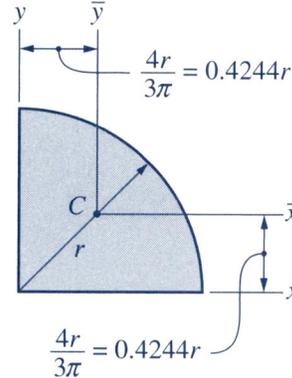
$$A = \tfrac{1}{4}\pi r^2$$

$$\overline{I}_x = \overline{I}_y = 0.0549 r^4$$

$$I_x = I_y = \tfrac{1}{16}\pi r^4$$

$$\overline{J} = 0.1098 r^4$$

$$\overline{r}_x = \overline{r}_y = 0.2644 r$$

$$r_x = r_y = \tfrac{1}{2} r$$

$$\frac{4r}{3\pi} = 0.4244r$$

Moment of Inertia for Simple shapes about the Centroidal axis

Shape	Area	Moment of Inertia
h [rectangle] b, $\bar{x}$	bh	$\dfrac{bh^3}{12}$
[triangle] h, $\bar{x}$, b	$\dfrac{bh}{2}$	$\dfrac{bh^3}{36}$
[circle] $\bar{x}$	$\dfrac{\pi d^2}{4}$	$\dfrac{\pi r^4}{4}$
[semicircle] r, $\bar{x}$	$\dfrac{\pi d^2}{8}$	$\dfrac{\pi r^4}{8}$

Example 1.

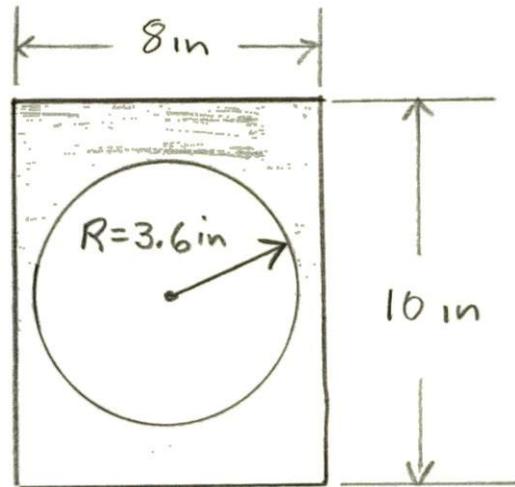For the shape shown, calculate the
Moment of Inertia about the x-axis.

centroidal axis of
the given shape
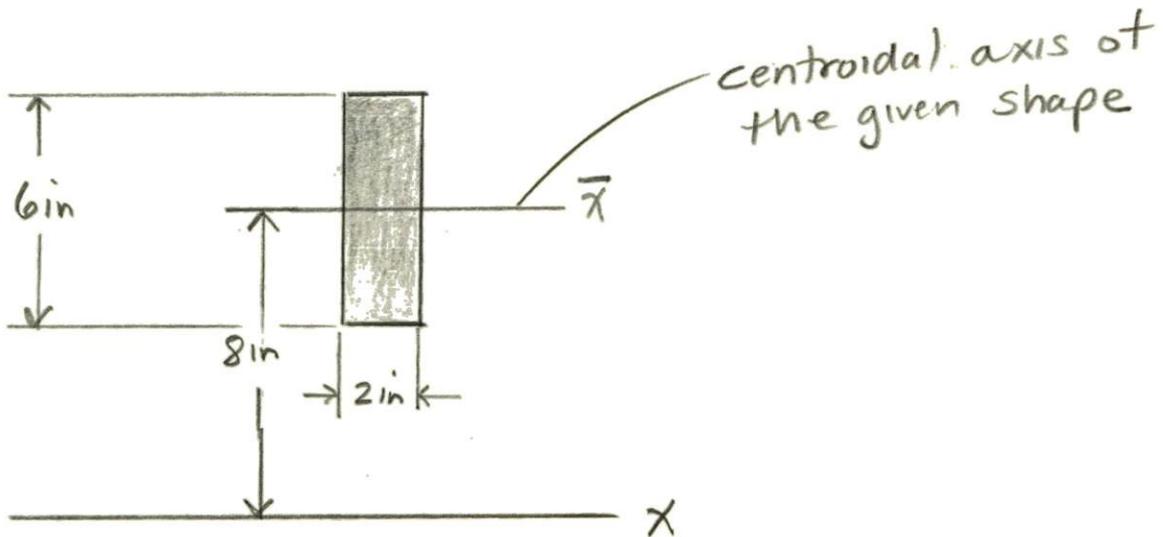
Solution.

$$\bar{I}_x = \frac{bh^3}{12} = \frac{(2 in)(6 in)^3}{12} = 36 \text{ in.}^4$$

From the Parallel-Axis Theorem

$$I_x = \bar{I}_x + A d^2$$

$$= 36 \text{ in.}^4 + (2in)(6in)(8in)^2$$

$$= 36 \text{ in.}^4 + 768 \text{ in.}^4$$

$$= \underline{\underline{804 \text{ in}^4}}$$

Example 2. Calculate the moment of inertia $\bar{I}_x$ about the centroidal x-axis for the shape shown.

Solution.
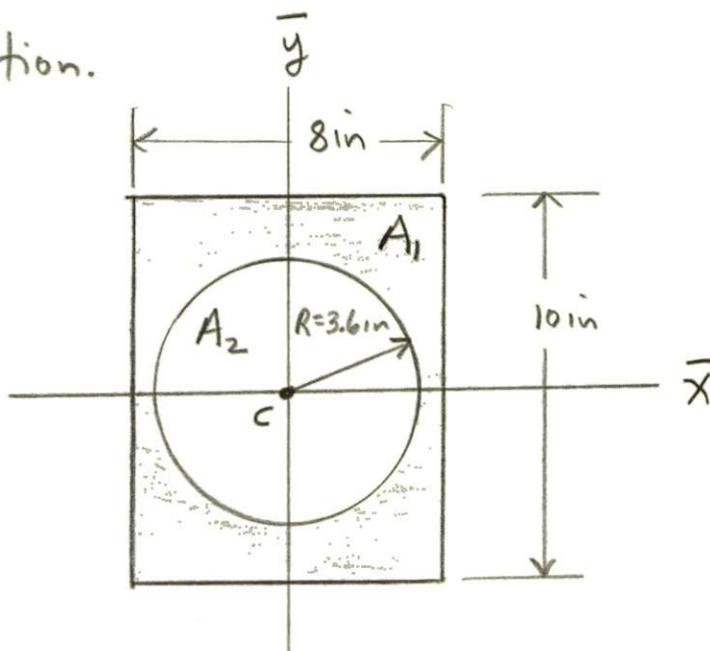
$$\bar{I}_x = \frac{(8\text{in})(10\text{in})^3}{12} - \frac{\pi(3.6\text{in})^4}{4}$$

$$= 666.67 \text{ in}^4 - 131.9 \text{ in}^4$$

$$= \underline{534.7 \text{ in}^4}$$

8-4
Moments of Inertia of Composite Areas

Composite areas are those areas made up of more than one of the common areas shown in Table 8-1.

The moment of inertia of a composite area about an axis is the sum of the moments of inertia of the component parts about the same axis.

The moment of inertia of a composite area about the x-axis may be computed from

$$I_x = \Sigma \left[I + Ay^2 \right]$$

where y = distance from the centroid of a component area to the x-axis

The moment of inertia of a composite area about the centroidal X axis may be computed from
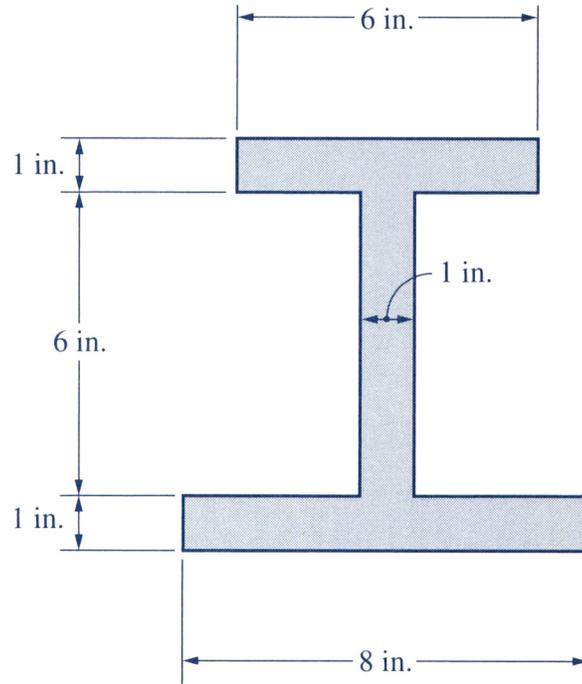
$$\bar{I}_x = \Sigma \left[I + A(\bar{y} - y)^2 \right]$$

where y = distance from the centroid of a component area to the reference x-axis

$\bar{y}$ = distance from the centroid of the entire area to the reference x-axis.

Example 3
Determine the moment of inertia of the area with respect to the horizontal centroidal axis.

FIGURE P8–14

Solution.

(1)	(2)	(3)	(4)	(5)	(6)	(7)
Part	A (in²)	y (in)	Ay (in³)	ȳ-y (in)	A(ȳ-y)² (in⁴)	I (in⁴)
Σ						