Chapter 8 HW Solution

Review Questions.

1. What is a root locus? A plot of the possible closed-loop pole locations as some parameter varies from 0 to ∞.

4. Do the zeros of a system change with a change in gain? No.

5. Where are the zeros of the closed-loop transfer function? They are the roots of the **numerator** of the closed-loop transfer function.

7. How can you tell from the root locus if a system in unstable? If any of the poles are in the right-half s-plane.

Problem 1. The potential root loci for the stated parts are shown below:

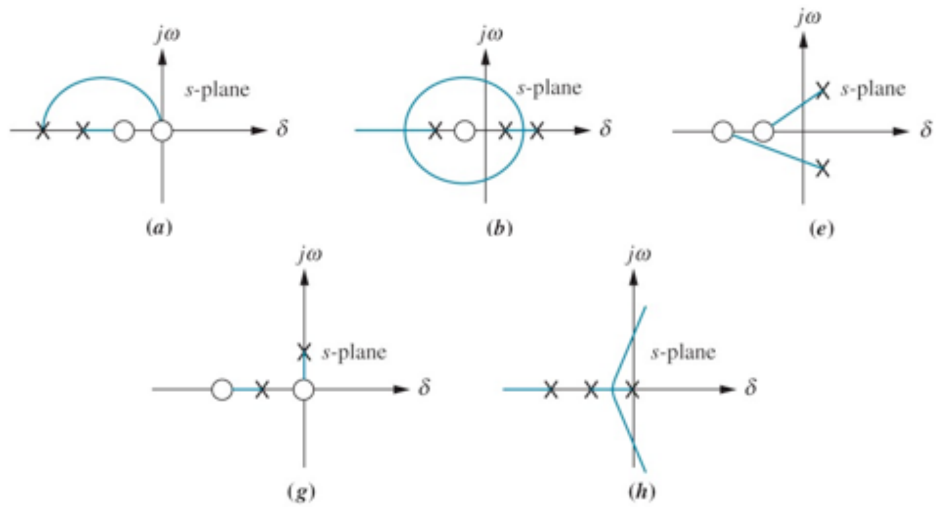
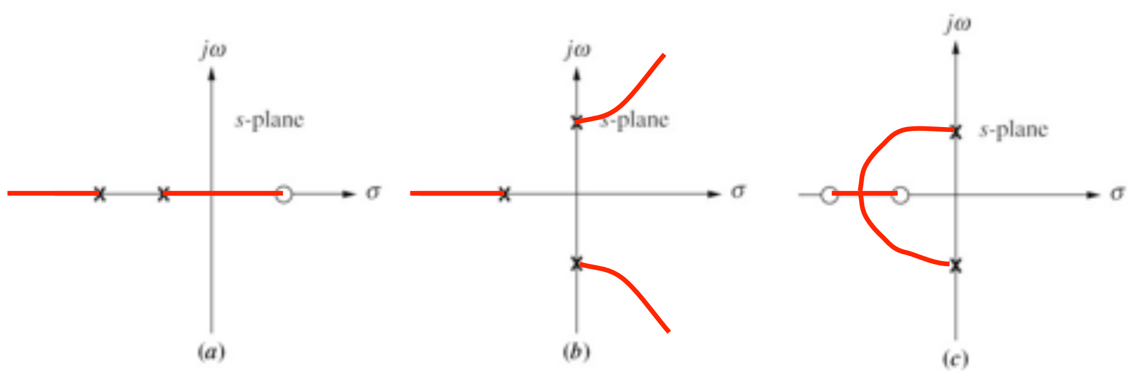


Figure 1: Potential root loci for Problem 1.

- **(a) CANNOT** be root locus: not symmetric about Real axis, and real axis branches not shown.
- **(b) CANNOT** be root locus: Real axis branch is to the left of **FOUR** poles/zeros (even number).
- **(e) CANNOT** be root locus: not symmetric about Real axis, and no real axis branch between zeros.
- **(g) CANNOT** be root locus: not symmetric about Real axis (poles must occur in conjugate pairs), and real axis branch is to the left of **TWO** poles/zeros (even number).
- **(h) CAN** be root locus: everything looks correct to me.

Problem 2. Do parts (a), (b), and (c). In accordance with my comments, my drawing attempts are shown below:



Problem 8. The characteristic equation (denominator of the closed-loop transfer function set equal to zero) is:

$$s^3 + 2s^2 + (20K + 7)s + 100K = 0 \tag{1}$$

When this CE is put into "root locus" for versus parameter K, it is

$$1 + K\frac{20(s+5)}{s\underbrace{(s^2+2s+7)}_{s+1\pm j2.45}} = 0 \tag{2}$$

In the process of sketching the root locus, I asked for the following:

(a) Real-axis branches. As shown on the sketch, there is a real axis branch between the pole at $s = 0$ and the zero at $s = -5$.

(b) Asymptote intersection and angle. Since there are 3 poles and 1 zero, there are 2 asymptotes. The asymptote angle and intersection are:

$$\boxed{\theta_a = \frac{\pm 180^\circ}{3-1} = \pm 90^\circ, \quad \sigma_a = \frac{-1-1-(-5)}{3-1} = \frac{3}{2} = 1.5} \tag{3}$$

(c) Departure angle from complex poles. Let $s \approx -1 + j2.45$ (near the upper complex pole), and evaluate the angle condition (angle of vectors from poles to s minus angle of vectors from zeros to s):

$$\boxed{112.2^\circ + \theta + 90^\circ - 31.5^\circ = \pm 180^\circ \quad \Longrightarrow \quad \theta = 9.3^\circ} \tag{4}$$

(d) Imaginary axis crossing point and corresponding K. Substituting $s = j\omega$ into the characteristic equation of (1), you can find

$$j\omega(-\omega^2 + j2\omega + 7) + K[20(j\omega + 5)] = 0. \tag{5}$$

so

$$j\omega^3 - 2\omega^2 + j7\omega + j20K\omega + 100K = 0. \tag{6}$$

Equation (6) yields a Real equation and an Imag equation:

$$\text{Real:} \; -2\omega^2 + 100K = 0 \tag{7}$$
$$\text{Imag:} \; -\omega^3 + 7\omega + 20K\omega = 0 \tag{8}$$

Solving (7) and (8) simultaneously (not hard), you find that

$$\boxed{K = \frac{7}{30} = 0.2333, \quad \omega = \sqrt{50K} = 3.42} \tag{9}$$

My root locus sketch is shown in Figure 2 on the next page.

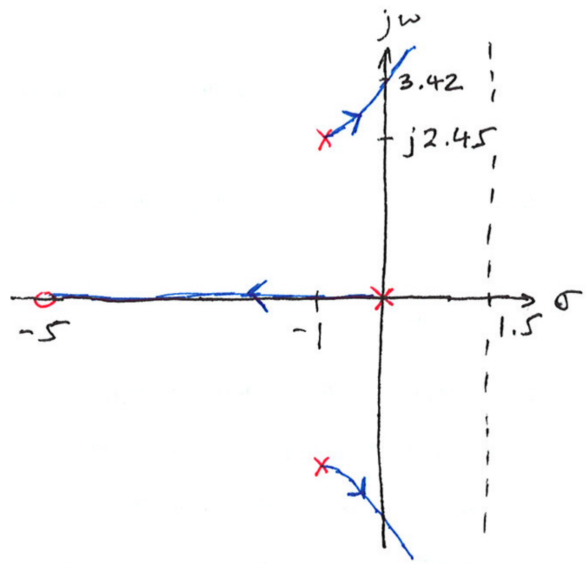
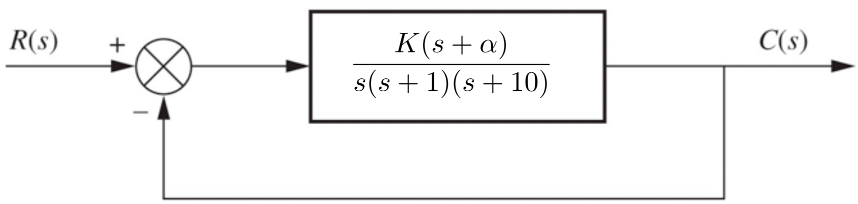


Figure 2: Root locus sketch for Problem 8, using asymptotes, $j\omega$ axis crossing, and departure angles.

Problem 30. A unity-feedback system is shown in the block diagram below:



a. Find α so the system will have a settling time of 4 seconds for large values of K. The expression for 2% settling time T_s is

$$T_s = \frac{4}{\zeta\omega_n} = 4 \text{ sec} \quad \Longrightarrow \quad \zeta\omega_n = 1, \tag{10}$$

where $\zeta\omega_n$ is the REAL part of a complex pole location. The closed-loop TF is

$$\frac{C(s)}{R(s)} = \frac{K(s+\alpha)}{s(s+1)(s+10) + K(s+\alpha)}. \tag{11}$$

The corresponding characteristic equation and root locus form are

$$s(s+1)(s+10) + K(s+\alpha) = 0 \quad \Longrightarrow \quad 1 + K\frac{s+\alpha}{s(s+1)(s+10)} = 0 \tag{12}$$

The root locus diagram will have 3 poles and 1 zeros, hence **TWO** asymptotes at angles of $\pm 90°$. The complex RL branches will converge to these asymptotes, so if these asymptotes intersect the REAL axis at $s = -1$, the settling time condition of (10) will be satisfied.

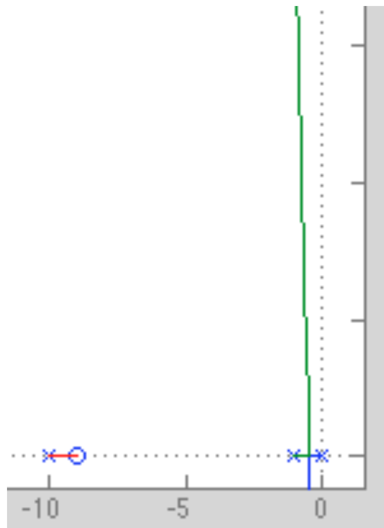
The equation for asymptote intersection is

$$\sigma_a = \frac{\Sigma\text{pole locations} - \Sigma\text{zero locations}}{\text{\#poles - \#zeros}} = \frac{0 - 1 - 10 - (-\alpha)}{3 - 1} = \frac{-11 + \alpha}{2} = -1. \tag{13}$$

Solving (13) for α yields

$$\boxed{\alpha = 9} \tag{14}$$

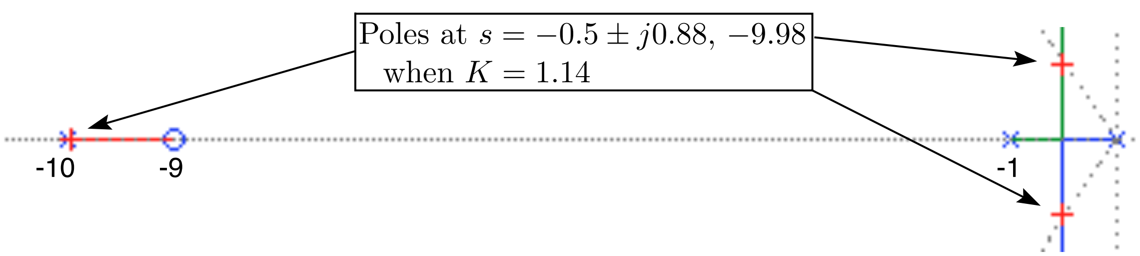
The MATLAB RL diagram of the resulting system is shown below. At "large" values of K the complex poles will have a REAL part near -1, as desired.



b. Find gain K such that the complex closed-loop poles have damping ratio $\zeta \approx 0.5$. With $\alpha = 9$, the RL form of the characteristic equation is

$$1 + K\frac{s+9}{s(s+1)(s+10)} = 0 \tag{15}$$

A portion of the MATLAB root locus is shown below, with the pole locations indicated when $\boxed{K \approx 1.14}$ (this is approximately where $\zeta = 0.5$).



c. Using the K from part (b), find the factored form of the resulting closed-loop transfer function and—using either MATLAB or Simulink —plot the unit step response. With the gain K and the pole locations shown in the previous root locus diagram, the **factored form** of the CLTF is

$$\boxed{\frac{C(s)}{R(s)} = \frac{1.14(s+9)}{(s+0.5 \pm j0.88)(s+9.98)} = \frac{1.14(s+9)}{(s^2+s+1.02s)(s+9.98)}} \tag{16}$$

where the second TF in (16) has the complex terms multiplied together to form a quadratic.

The unit step response is shown in Figure 3 on the next page.

d. Is the overshoot what you expect? Is the closed-loop system dominated by a pair of complex poles? Yes, a damping ratio of $\zeta = 0.5$ should produce an overshoot of about 15%. And yes, the system response **is** dominated by a pair of complex poles. The complex poles of the root locus diagram are **MUCH** closer to the origin than either the zero at -9 or the third pole at -9.98.

e. Is the settling time what the text problem statement requested? Why not? No, the settling time is longer than 4 seconds. That's because we **didn't** use a "large value of K."

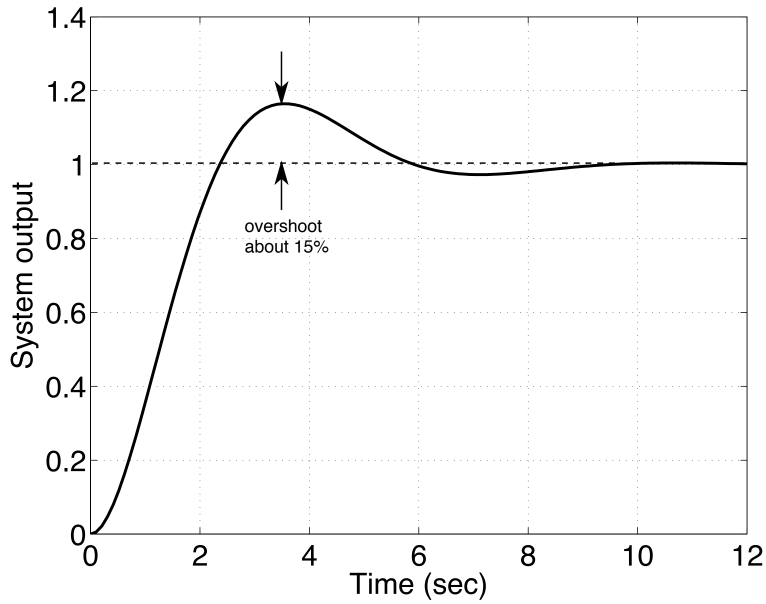


Figure 3: Unit step response for Problem 30 with K set for $\zeta = 0.5$.

f. Use the root-locus diagram of (a) to select a K that should satisfy the settling time requirement. By "clicking" on the complex branch high enough so the real part is nearly -1, I found the following value for K and the corresponding pole locations:

$$\boxed{\begin{aligned} K &\approx 199 \\ \text{poles at } s &= -0.86 \pm j13.9, -9.2787 \end{aligned}} \quad \begin{matrix}(17)\\(18)\end{matrix}$$

g. Use either MATLAB or Simulink to plot the step response with the K from (f). Does this step response look desirable? The unit step response is shown below. With this large K there is a **LOT** of oscillation—doesn't look desirable to me.

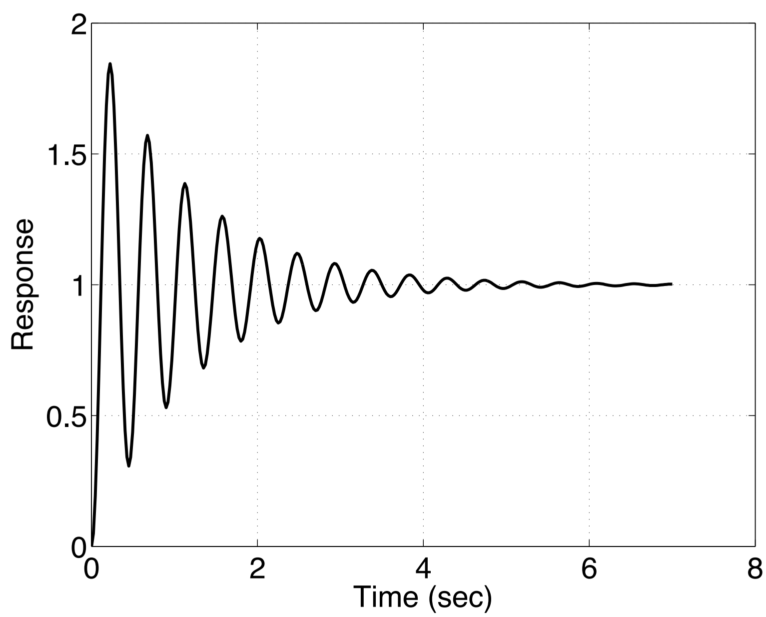


Figure 4: Unit step response for Problem 30 with "large" K set for $T_s = 4$ Not too desirable.

Problem 60. I modified the block diagram of the system to be more descriptive, as shown below:

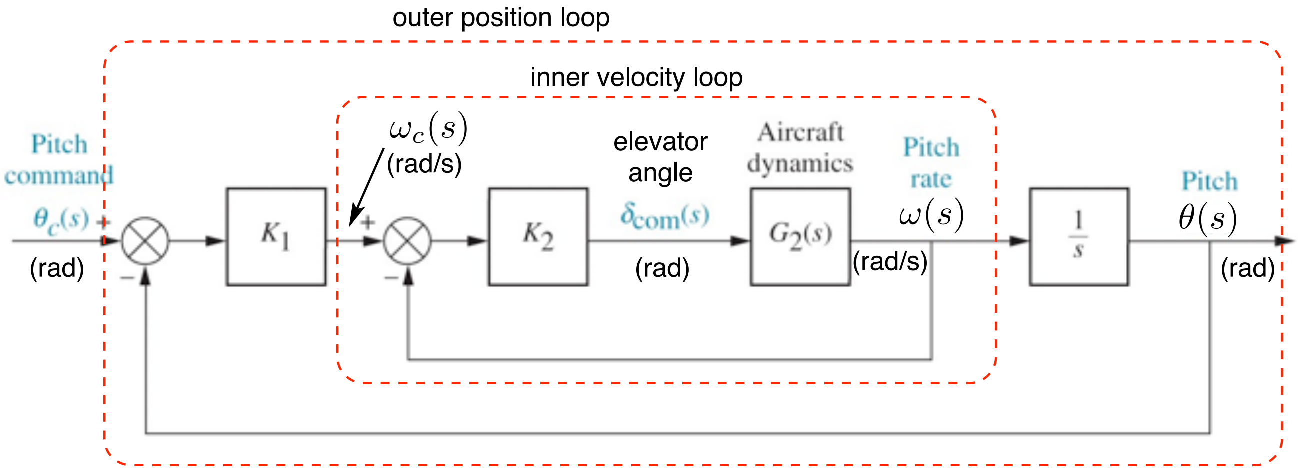
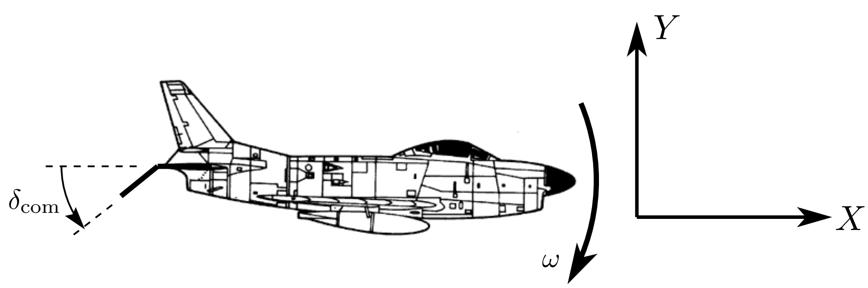


Figure 5: F4E Pitch Stabilization Loop.

Here is how this pitch angle control system is supposed to work:

- Block "Aircraft Dynamics" $G_2(s)$ has input of **elevator angle** δ_{com}, and output **pitch rate** which I have renamed ω. This **pitch rate** is simply the angular velocity of the aircraft as it pitches up/down, and h aas units of rad/s.
- The "inner loop" with gain K_2 is intended to control this pitch rate ω. Therefore its input should be commanded pitch rate (rad/s), which I have labeled as ω_c.
- The "outer loop" with gain K_1 is intended to control the pitch angle, which I have relabeled as θ.
- This structure of an "inner loop" controlling velocity, and an "outer loop" controlling position is very common in motion control. At least *this* part of the problem was reasonable.

Behavior of "aircraft dynamics." Consider the sketch shown below, where the Z axis defines positive angle.



The aircraft elevator is shown with a positive displacement δ_{com} (positive around Z axis by RH rule). By intuitive reasoning you can see that this elevator displacement will cause the aircraft to pitch **down**, as shown by the angular direction ω.

By the sense of positive angle, this resulting ω is in the **negative** direction. This is the reason for the "minus" sign in the numerator of "aircraft dynamics" transfer function $G_s(s)$.

a. Simulate the behavior of the aircraft dynamics. Apply a 1° positive elevator angle (*i.e.* downward elevator angular displacement as shown in the sketch). Use MATLAB to plot the pitch rate ω (rad/s) *vs* time for 1 second. What is the pitch rate (rad/s) after 1 second? Does this seem reasonable?

The transfer function of the aircraft dynamics from commanded elevator angle input δ_{com} (rad) to pitch rate output ω (rad/s) is

$$G_2(s) = \frac{\omega(s)}{\delta_{com}(s)} = \frac{-508(s+1.6)}{(s+14)(s-1.8)(s+4.9)} \ \frac{\text{rad/s}}{\text{rad}} \tag{19}$$

An input of 1° is 1/57.3 radians, and the resulting step response is shown below

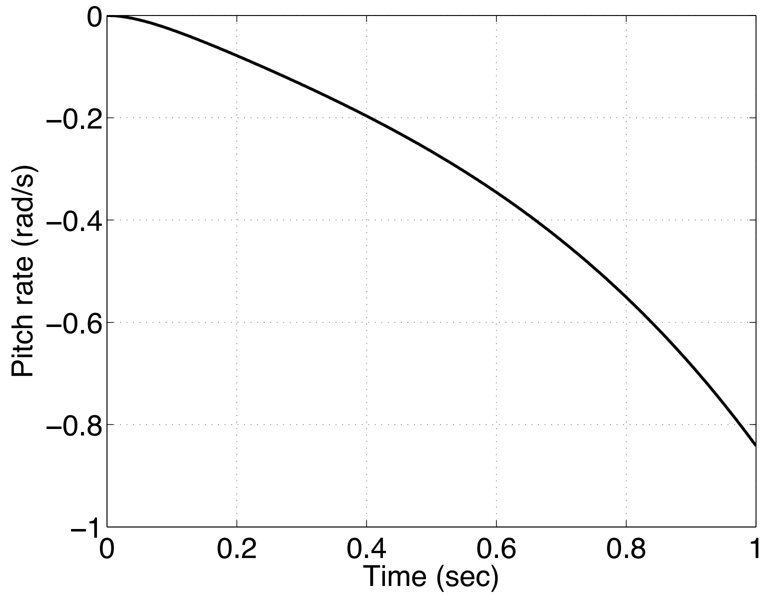


Figure 6: Aircraft pitch rate in response to 1° elevator angle.

At time of 1.0 second, the pitch rate $\omega = -0.8415$ rad/s ($-48°$/sec). For a **ONE DEGREE** elevator angle, that seems awfully large to me, so I would say that it **doesn't look reasonable, but that's a judgment call...**

b. Inner loop CL TF. Using standard block diagram reduction, the closed-loop transfer function of the inner loop is

$$\boxed{\frac{\omega(s)}{\omega_c(s)} = \frac{-508K_2(s+1.6)}{(s+4.9)(s+14)(s-1.8) - 508K_2(s+1.6)} \ \frac{\text{rad/s}}{\text{rad/s}}} \tag{20}$$

c. Inner loop RL form. The **root-locus** form of the inner-loop characteristic equation of (1) based on parameter K_2 is

$$\boxed{1 - 508K_2 \frac{s+1.6}{(s+4.9)(s+14)(s-1.8)} = 0} \tag{21}$$

d. Selection of inner loop gain. Define $\boxed{K \triangleq -508K_2}$, the inner loop RL form is now

$$1 + K\frac{s+1.6}{(s+4.9)(s+14)(s-1.8)} = 0 \tag{22}$$

Use MATLAB to draw the root locus diagram of (3), and select K_2 (actually K) to place the complex poles to yield a damping ratio of $\zeta = 0.5$. I found that $-0.4 > K_2 > -0.5$. The MATLAB root locus is shown on the next page in Figure 8.

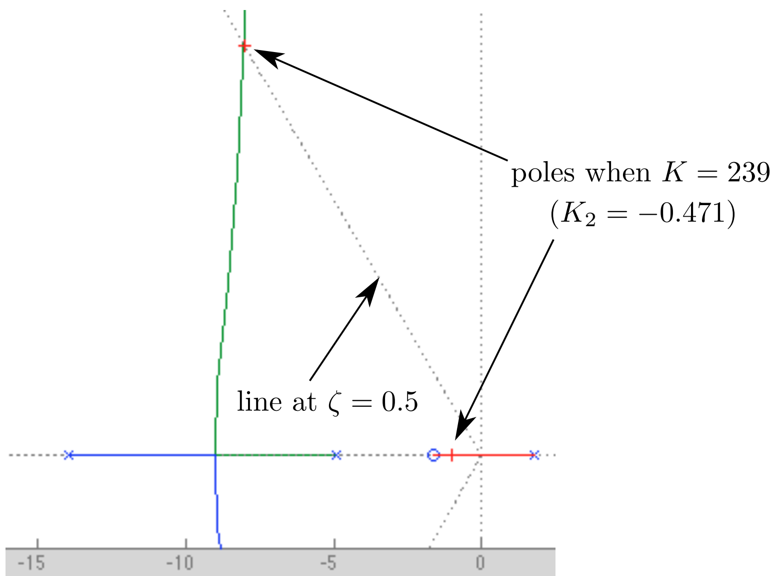


Figure 7: Root locus of inner loop for part (d).

In Figure 8, to place the complex poles with $\zeta = 0.5$ requires (recall $K = -508K_2$)

$$\boxed{K = 239 \implies K_2 = -0.471} \tag{23}$$

The reason for the negative K_2 is because of the "minus" sign in the numerator of $G_2(s)$ (downward pitching for positive elevator angle).

e. Factored form of inner loop TF. With the value of K_2 from part (**d**), the **factored form** of inner loop transfer function $\dfrac{\omega(s)}{\omega_c(s)}$ (using the pole location from the root locus) is:

$$\boxed{\frac{\omega(s)}{\omega_c(s)} = \frac{239(s+1.6)}{(s+1.006)\underbrace{(s+8.05 \pm j13.89)}_{s^2+16s+257}} \quad \frac{\text{rad/s}}{\text{rad/s}}} \tag{24}$$

Note that the inner loop is a **Type 0** system (no free integration in forward path), and so the DC gain of (24) is **NOT** one, but is in fact about 1.47.

f. Response of controlled inner loop. Use either MATLAB or Simulink to find the step response of the controlled inner loop to a commanded pitch rate of $\omega_c = 1°$/sec. Plot the resulting pitch rate ω (degrees/sec) versus time. Does it look reasonable? What is the final value of ω in degrees/sec? Does this agree with the DC gain of the inner loop?

Remember that a step input of $1°$/sec is $1/57.3$ rad/sec. The response of the controlled inner loop is shown in Figure 9.

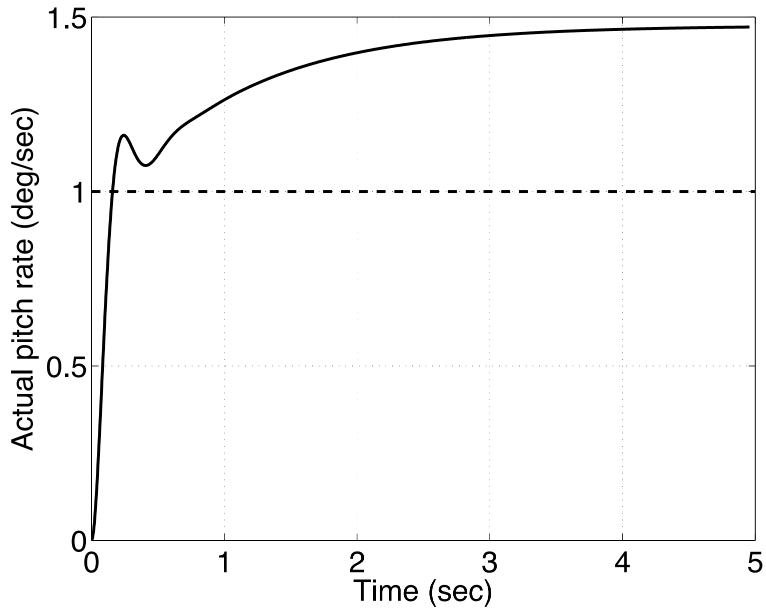


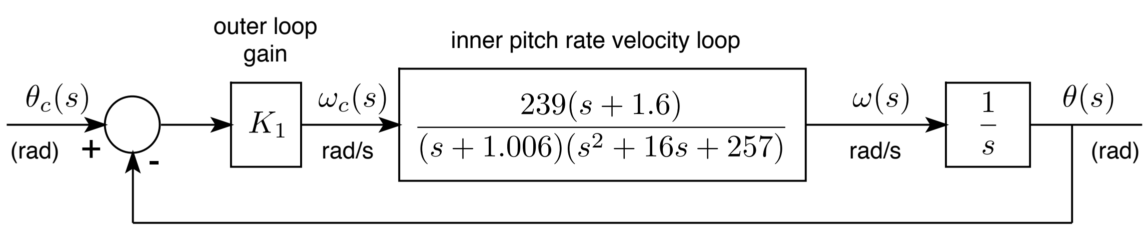
Figure 8: Response of controlled inner loop to desired pitch rate of $1°$/sec.

I would say that the response of Figure 9 $\boxed{\textbf{does}\text{ look reasonable}}$. The initial response quite fast. There is a slow "exponential" final convergence—this is the mode associated with the closed-loop pole at $s = -1$ in the inner-loop transfer function of (24).

In response to a desired pitch rate of $1°$/sec, the actual aircraft pitch rate of Figure 9 goes to a final value of $\boxed{1.47°/\text{sec.}}$ This $\boxed{\textbf{does agree}}$ with the inner loop DC gain of 1.47.

g. Block diagram of outer loop. Using the factored form of the inner loop from part (**e**), draw a **block diagram** of the outer loop (similar to Figure 1 on the previous page). The outer loop forward path will have three blocks in series: (1) gain K_1, (2) factored form of inner loop TF from part (**e**), and (3) the integrator $1/s$. What is the **TYPE** of the outer loop system?

The block diagram is shown below:



Since there is an integration in the forward path (the integration from ω to θ), this system is $\boxed{\textbf{TYPE 1}}$.

h. Closed-loop TF of outer loop. From the block diagram of part (**g**), find the closed-loop transfer function $\dfrac{\theta(s)}{\theta_c(s)}$.

Using the standard unity-feedback closed-loop reduction, it is

$$\boxed{\frac{\theta(s)}{\theta_c(s)} = \frac{239K_1(s+1.6)}{s(s+1)(s^2+16s+257)) + 239K_1(s+1.6)}} \tag{25}$$

Note that I approximated the denominator term $(s + 1.006)$ by $(s + 1)$. As I've said before, we usually don't know these parameters that accurately.

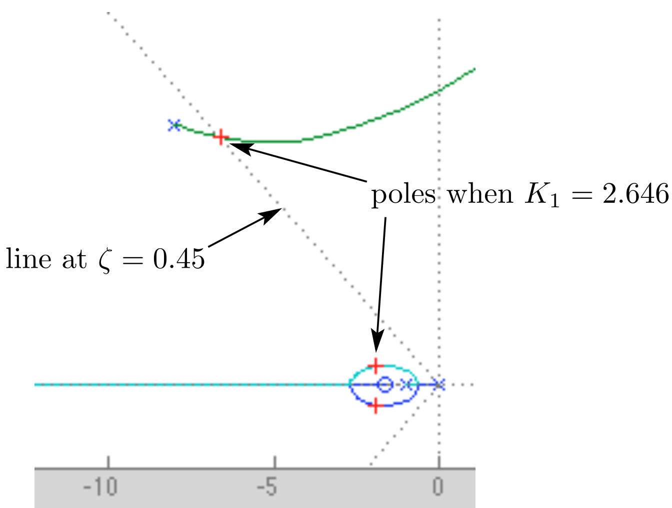
i. Outer loop CE and RL form. Find the characteristic equation of the outer loop, and put it in root-locus form using parameter K_1.

The CE of the outer loop, with corresponding root locus form are:

$$\boxed{\begin{aligned} \text{CE: } & s(s+1)(s^2+16s+257)) + 239K_1(s+1.6) = 0 && (26)\\ \text{RL form: } & 1 + K_1\frac{239(s+1.6)}{s(s+1)(s^2+16s+257)} = 0 && (27)\end{aligned}}$$

j. Selection of outer loop gain. Use MATLAB to draw the root-locus diagram of part (**i**) versus gain K_1. Select K_1 so the complex poles have a damping ratio of $\zeta = 0.45$.

The MATLAB root locus diagram of (27) is shown below (lower complex branch not shown):



The desired CL poles are where the line at $\zeta = 0.45$ intersects the complex branches. The gain at this point is $\boxed{K_1 = 2.646}$.

The closed-loop pole locations (from the RL diagram above) corresponding to that K_1 are:

$$s = -1.9 \pm j1.02 \;\; \text{(2 poles)} \tag{28}$$
$$= -6.6 \pm j13.2 \;\; \text{(2 poles)} \tag{29}$$

k. Factored form of outer loop. With the value of K_1 you selected in part (**j**), what is the **factored form** of the outer-loop transfer function (the entire system)? This will be $\frac{\theta(s)}{\theta_c(s)}$.

From the K_1 and the closed-loop poles of the above RL diagram, this will be

$$\boxed{\frac{\theta(s)}{\theta_c(s)} = \frac{632(s+1.6)}{(s+1.9\pm j1.02)(s+6.6\pm j13.2)} \; \frac{\text{rad}}{\text{rad}}} \tag{30}$$

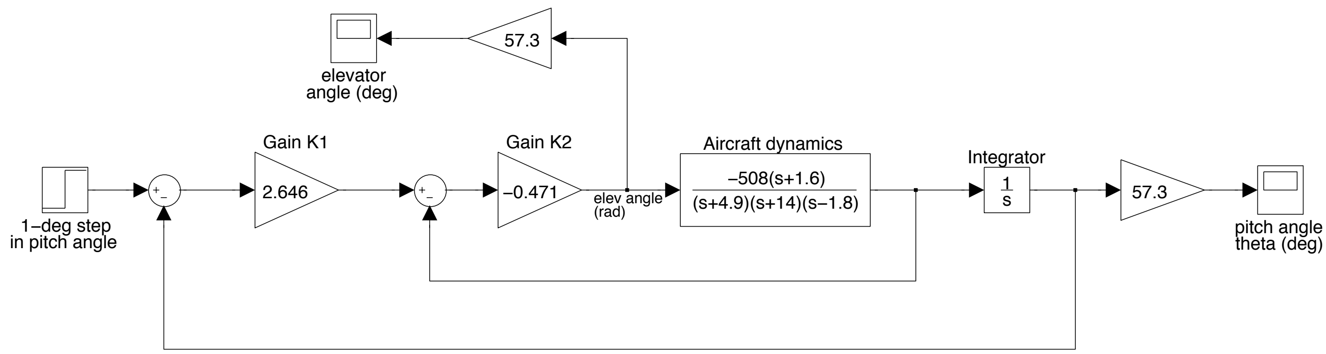
l. Response of controlled outer loop. Using Simulink , construct a model of the entire system. You will have the two nested loops. Use the two numerical values you've found for K_2 and K_1, plus all other numerical values, of course.

Let the commanded pitch angle input be a step of 1°, and produce the following two plots:

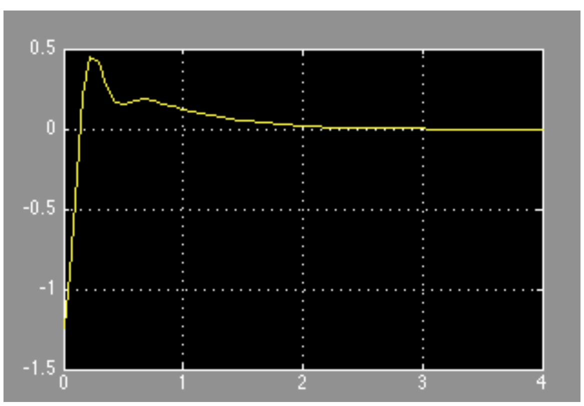
- Plot the actual pitch angle θ (degrees) of the aircraft versus time
- Plot the elevator angle δ_{com} (degrees) versus time

Does the response of the aircraft look reasonable? Does the response of the elevator angle look reasonable (you may have to think about that a little)?

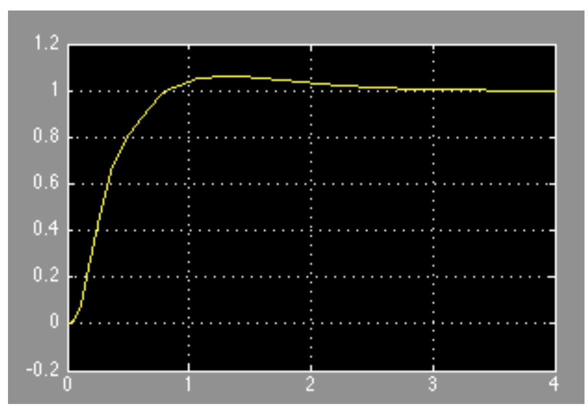
Simulink Model: My Simulink block diagram is shown below:



Response Plots: The plots of elevator angle and aircraft pitch angle in response to a commanded pitch angle of 1° are shown below:



(a) Elevator angle response.



(b) Pitch angle response.

Figure 9: Responses of pitch angle control system to 1° input.

Comments on response:

- **Aircraft response (pitch angle):** The overall pitch angle of the aicraft responds pretty quickly—it is mostly all the way there in less than 1.0 second. And since this is a **Type 1** system, the final value goes to 1° as desired...no steady-state error. I would say the aircraft response $\boxed{\textbf{DOES LOOK REASONABLE}}$.
- **Elevator angle:** The initial response of the elevator angle is strongly **negative** (look closely); this is to get the aircraft pitching **positively**. The elevator angle then quickly goes **positive** to stop the aircraft before it pitches too far. This is the same thing you would do as a human pilot of your intuition and reflexes were good enough. So again, the elevator response $\boxed{\textbf{DOES LOOK REASONABLE}}$.
- Whew! I'm tired...