Chapter 8. Process and Measurement System Capability Analysis

Process Capability

1. The natural or inherent variability in a critical-to-quality characteristic at a specified time; that is, "instantaneous" variability
2. The variability in a critical-to-quality characteristic over time

Natural tolerance limits are defined as follows:

$$\text{UNTL} = \mu + 3\sigma$$
$$\text{LNTL} = \mu - 3\sigma$$

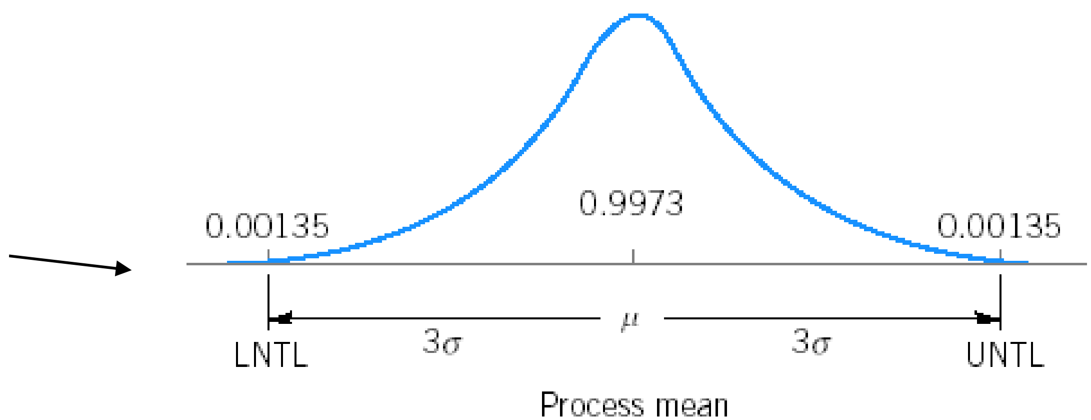

Figure 7-1 Upper and lower natural tolerance limits in the normal distribution.

Uses of process capability

1. Predicting how well the process will hold the tolerances
2. Assisting product developers/designers in selecting or modifying a process
3. Assisting in establishing an interval between sampling for process monitoring
4. Specifying performance requirements for new equipment
5. Selecting between competing suppliers and other aspects of supply chain management
6. Planning the sequence of production processes when there is an interactive effect of processes on tolerances
7. Reducing the variability in a manufacturing process

Reasons for Poor Process Capability

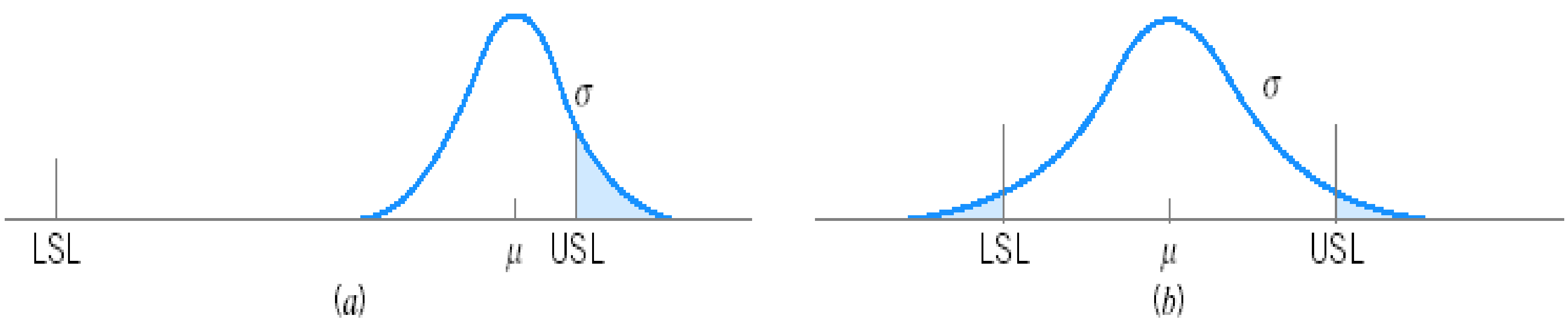

Figure 7-3 Some reasons for poor process capability. (*a*) Poor process centering. (*b*) Excess process variability.

Process may have good potential capability

Data collection steps:

1. Choose the machine or machines to be used. If the results based on one (or a few) machines are to be extended to a larger population of machines, the machine selected should be representative of those in the population. Furthermore, if the machine has multiple workstations or heads, it may be important to collect the data so that head-to-head variability can be isolated. This may imply that designed experiments should be used.
2. Select the process operating conditions. Carefully define conditions, such as cutting speeds, feed rates, and temperatures, for future reference. It may be important to study the effects of varying these factors on process capability.
3. Select a representative operator. In some studies, it may be important to estimate *operator* variability. In these cases, the operators should be selected at random from the population of operators.
4. Carefully monitor the data-collection process, and record the time order in which each unit is produced.

Example 7-1, Glass Container Data

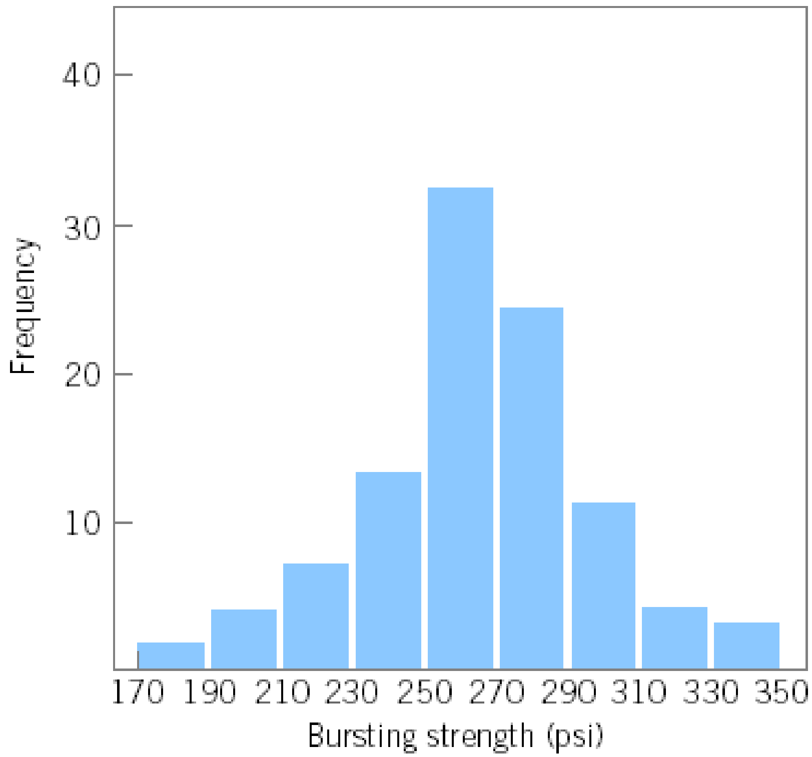

Figure 7-2 Histogram for the bursting-strength data.

Table 7-1 Bursting Strengths for 100 Glass Containers

265	197	346	280	265	200	221	265	261	278
205	286	317	242	254	235	176	262	248	250
263	274	242	260	281	246	248	271	260	265
307	243	258	321	294	328	263	245	274	270
220	231	276	228	223	296	231	301	337	298
268	267	300	250	260	276	334	280	250	257
260	281	208	299	308	264	280	274	278	210
234	265	187	258	235	269	265	253	254	280
299	214	264	267	283	235	272	287	274	269
215	318	271	293	277	290	283	258	275	251

EXAMPLE 7-1

To illustrate the use of a histogram to estimate process capability, consider Fig. 7-2, which presents a histogram of the bursting strength of 100 glass containers. The data are shown in Table 7-1. Analysis of the 100 observations gives

$$\bar{x} = 264.06 \qquad s = 32.02$$

Consequently, the process capability would be estimated as

$$\bar{x} \pm 3s$$

or

$$264.06 \pm 3(32.02) \simeq 264 \pm 96 \text{ psi}$$

Furthermore, the shape of the histogram implies that the distribution of bursting strength is approximately normal. Thus, we can estimate that approximately 99.73% of the bottles manufactured by this process will burst between 168 and 360 psi. Note that we can estimate process capability *independent of the specifications on bursting strength*.

Probability Plotting

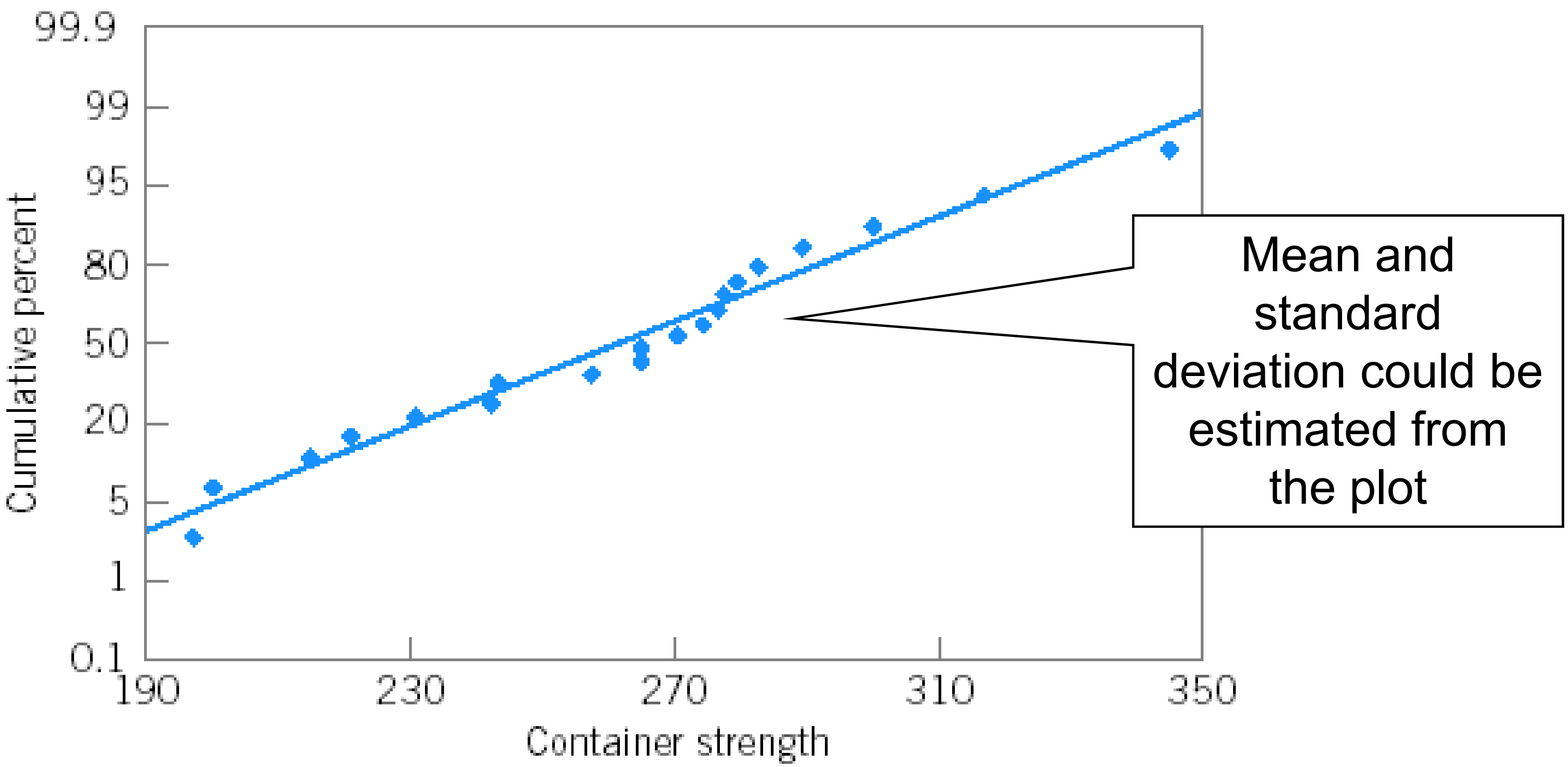

Figure 7-4 Normal probability plot of the container-strength data.

- The distribution may not be normal; other types of probability plots can be useful in determining the appropriate distribution.

Process Capability Ratios

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{7-4}$$

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}} \tag{7-5}$$

Example 5-1: USL = 1.00 microns, LSL = 2.00 microns

$$\sigma = \frac{\overline{R}}{d_2} = 0.1398$$

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6\hat{\sigma}} = \frac{2.00 - 1.00}{6(0.1398)} = 1.192$$

Practical interpretation

PCR = proportion of tolerance interval used by process

$$P = \left(\frac{1}{C_p}\right)100 \qquad (7\text{-}6)$$

For the hard bake process:

$$P = \left(\frac{1}{1.192}\right)100 = 83.89$$

One-Sided PCR

$$C_{pu} = \frac{\text{USL} - \mu}{3\sigma} \quad \text{(upper specification only)} \qquad (7\text{-}7)$$

$$C_{pl} = \frac{\mu - \text{LSL}}{3\sigma} \quad \text{(lower specification only)} \qquad (7\text{-}8)$$

Table 7-2 Values of the Process Capability Ratio (C_p) and Associated Process Fallout for a Normally Distributed Process (in Defective ppm) That Is in Statistical Control

	Process Fallout (in defective ppm)	
PCR	One-Sided Specifications	Two-Sided Specifications
0.25	226,628	453,255
0.50	66,807	133,614
0.60	35,931	71,861
0.70	17,865	35,729
0.80	8,198	16,395
0.90	3,467	6,934
1.00	1,350	2,700
1.10	484	967
1.20	159	318
1.30	48	96
1.40	14	27
1.50	4	7
1.60	1	2
1.70	0.17	0.34
1.80	0.03	0.06
2.00	0.0009	0.0018

Table 7-3 Recommended Minimum Values of the Process Capability Ratio

	Two-Sided Specifications	One-Sided Specifications
Existing processes	1.33	1.25
New processes	1.50	1.45
Safety, strength, or critical parameter, existing process	1.50	1.45
Safety, strength, or critical parameter, new process	1.67	1.60

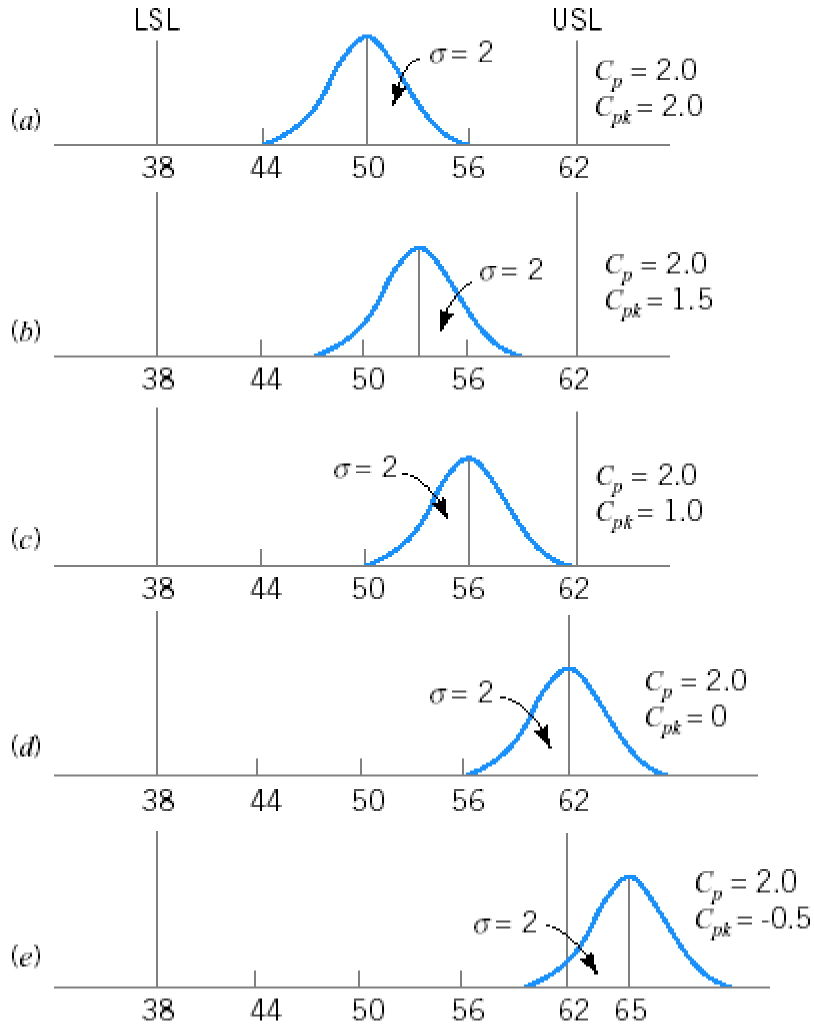

Figure 7-8 Relationship of C_p and C_{pk}.

- C_p does not take process centering into account
- It is a measure of *potential* capability, not *actual* capability

Measure of Actual Capability

$$C_{pk} = \min\left(C_{pu}, C_{pl}\right) \tag{7-9}$$

C_{pk}: one-sided PCR for specification limit nearest to process average

$$\begin{aligned}
C_{pk} &= \min\left(C_{pu}, C_{pl}\right) \\
&= \min\left(C_{pu} = \frac{\text{USL} - \mu}{3\sigma}, C_{pl} = \frac{\mu - \text{LSL}}{3\sigma}\right) \\
&= \min\left(C_{pu} = \frac{62 - 53}{3(2)} = 1.5, C_{pl} = \frac{53 - 38}{3(2)} = 2.5\right) \\
&= 1.5
\end{aligned}$$

Normality and Process Capability Ratios

- The assumption of normality is critical to the interpretation of these ratios.
- For non-normal data, options are:
  1. Transform non-normal data to normal.
  2. Extend the usual definitions of PCRs to handle non-normal data.
  3. Modify the definitions of PCRs for general families of distributions.

- Confidence intervals are an important way to express the information in a PCR

$$\hat{C}_p\sqrt{\frac{\chi^2_{1-\alpha/2,n-1}}{n-1}} \le C_p \le \hat{C}_p\sqrt{\frac{\chi^2_{\alpha/2,n-1}}{n-1}} \qquad (7\text{-}20)$$

EXAMPLE 7-4

Suppose that a stable process has upper and lower specifications at USL = 62 and LSL = 38. A sample of size $n = 20$ from this process reveals that the process mean is centered approximately at the midpoint of the specification interval and that the sample standard deviation $s = 1.75$. Therefore, a point estimate of C_p is

$$\hat{C}_p = \frac{\text{USL} - \text{LSL}}{6s} = \frac{62 - 68}{6(1.75)} = 2.29$$

The 95% confidence interval on C_p is found from equation 7–20 as follows:

$$\hat{C}_p \sqrt{\frac{\chi^2_{1-0.025,n-1}}{n-1}} \le C_p \le \hat{C}_p \sqrt{\frac{\chi^2_{0.025,n-1}}{n-1}}$$

$$2.29\sqrt{\frac{8.91}{19}} \le C_p \le 2.29\sqrt{\frac{32.85}{19}}$$

$$1.57 \le C_p \le 3.01$$

Confidence interval is wide because the sample size is small

where $\chi^2_{0.975,19} = 8.91$ and $\chi^2_{0.025,19} = 32.85$ were taken from Appendix Table III.

EXAMPLE 7-5

A sample of size $n = 20$ from a stable process is used to estimate C_{pk}, with the result that $\hat{C}_{pk} = 1.33$. Using equation 7–21, an approximate 95% confidence interval on C_{pk} is

$$\hat{C}_{pk}\left[1 - Z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right] \le C_{pk} \le \hat{C}_{pk}\left[1 + Z_{\alpha/2}\sqrt{\frac{1}{9n\hat{C}_{pk}^2} + \frac{1}{2(n-1)}}\right]$$

$$1.33\left[1 - 1.96\sqrt{\frac{1}{9(20)(1.33)^2} + \frac{1}{2(19)}}\right] \le C_{pk} \le 1.33\left[1 + 1.96\sqrt{\frac{1}{9(20)(1.33)^2} + \frac{1}{2(19)}}\right]$$

or

$$0.88 \le C_{pk} \le 1.78$$

Confidence interval is wide because the sample size is small

Process Performance Index: $\hat{P}_p = \frac{\text{USL} - \text{LSL}}{6s}$

Use only when the process is not in control.

If the process is normally distributed and in control:

$$\hat{P}_p = \hat{C}_p \text{ and } \hat{P}_{pk} = \hat{C}_{pk} \text{ because } \hat{\sigma} \approx \frac{\bar{R}}{d_2}$$

Process Capability Analysis Using Control Chart

Table 7-5 Glass Container Strength Data (psi)

Sample	Data					$\bar{x}$	R
1	265	205	263	307	220	252.0	102
2	268	260	234	299	215	255.2	84
3	197	286	274	243	231	246.2	89
4	267	281	265	214	318	269.0	104
5	346	317	242	258	276	287.8	104
6	300	208	187	264	271	246.0	113
7	280	242	260	321	228	266.2	93
8	250	299	258	267	293	273.4	49
9	265	254	281	294	223	263.4	71
10	260	308	235	283	277	272.6	73
11	200	235	246	328	296	261.0	128
12	276	264	269	235	290	266.8	55
13	221	176	248	263	231	227.8	87
14	334	280	265	272	283	286.8	69
15	265	262	271	245	301	268.8	56
16	280	274	253	287	258	270.4	34
17	261	248	260	274	337	276.0	89
18	250	278	254	274	275	266.2	28
19	278	250	265	270	298	272.2	48
20	257	210	280	269	251	253.4	70
						$\bar{\bar{x}} = 264.06$	$\bar{R} = 77.3$

- Specifications are not needed to estimate parameters.

***R* Chart**

Center line $= \bar{R} = 77.3$

$\text{UCL} = D_4\bar{R} = (2.115)(77.3) = 163.49$

$\text{LCL} = D_3\bar{R} = (0)(77.3) = 0$

$\bar{x}$ Chart

Center line $= \bar{\bar{x}} = 264.06$

$\text{UCL} = \bar{\bar{x}} + A_2\bar{R} = 264.06 + (0.577)(77.3) = 308.66$

$\text{LCL} = \bar{\bar{x}} - A_2\bar{R} = 264.06 - (0.577)(77.3) = 219.46$

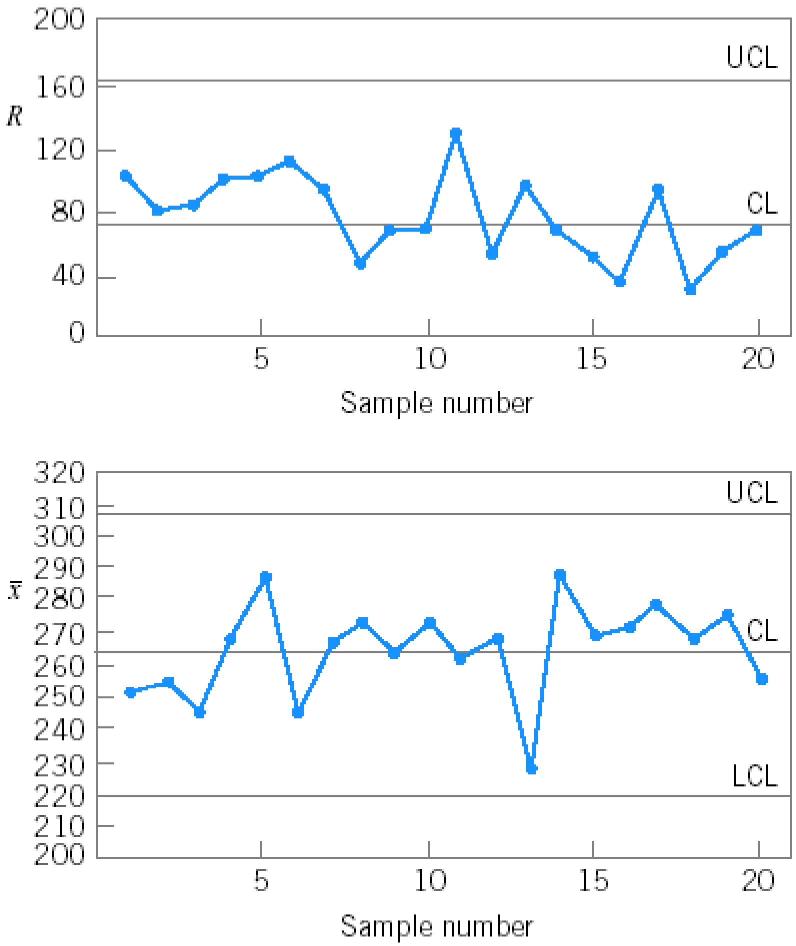

Figure 7-12 $\bar{x}$ and R charts for the bottle-strength data.

$$\hat{\mu} = \bar{\bar{x}} = 264.06$$

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{77.3}{2.326} = 33.23$$

Since LSL = 200

$$\hat{C}_{pl} = \frac{\mu - \text{LSL}}{3\hat{\sigma}} = \frac{264.06 - 200}{3(33.23)} = 0.64$$

Process Capability Analysis Using Designed Experiments

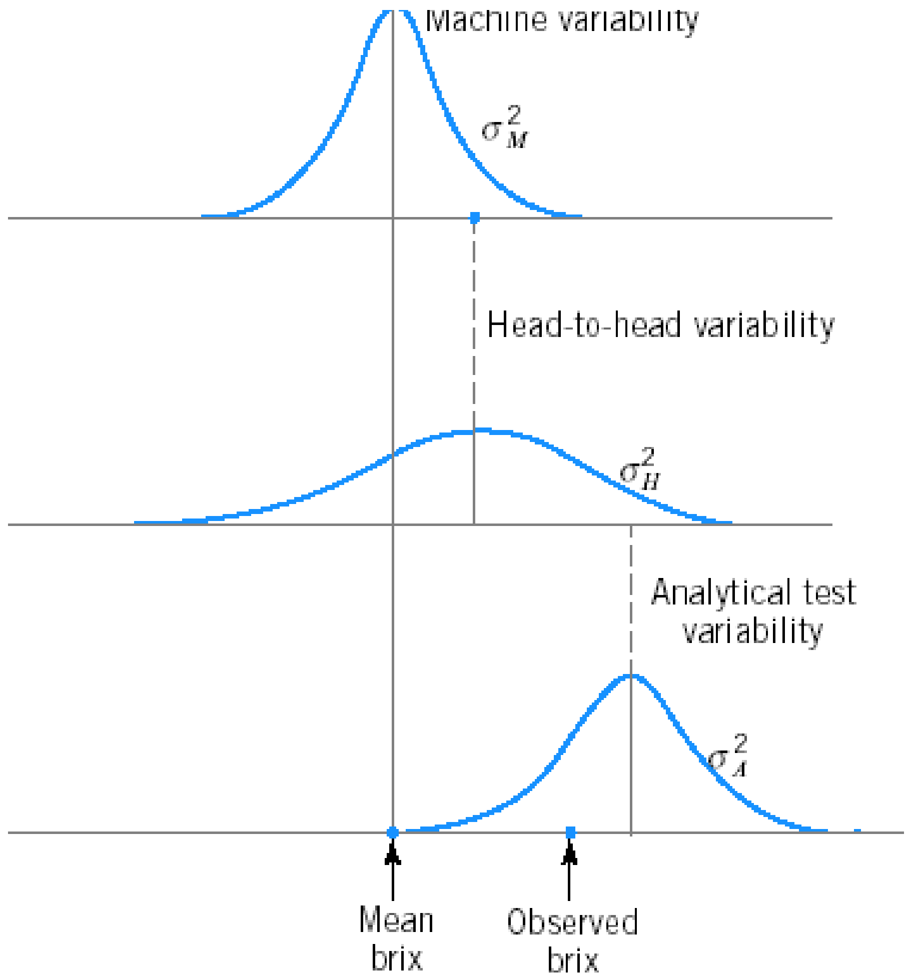

Figure 7-13 Sources of variability in the bottling line example.

Gauge and Measurement System Capability

1. Determine how much of the total observed variability is due to the gauge or instrument
2. Isolate the components of variability in the measurement system
3. Assess whether the instrument or gauge is capable (that is, is it suitable for the intended application)

Simple model:

$$y = x + \varepsilon \qquad (7\text{-}23)$$

y: total observed measurement
x: true value of measurement
ε: measurement error

x and ε are independent. $x \sim N\left(\mu,\ \sigma_P^2\right)$ and $\varepsilon \sim N\left(0,\ \sigma_{\text{Gauge}}^2\right)$

$$\sigma_{\text{Total}}^2 = \sigma_P^2 + \sigma_{\text{Gauge}}^2 \qquad (7\text{-}24)$$

EXAMPLE 7-7

Measuring Gauge Capability

An instrument is to be used as part of a proposed SPC implementation. The quality-improvement team involved in designing the SPC system would like to get an assessment of gauge capability. Twenty units of the product are obtained, and the process operator who will actually take the measurements for the control chart uses the instrument to measure each unit of product twice. The data are shown in Table 7-6.

Figure 7-14 shows the $\bar{x}$ and R charts for these data. Note that the $\bar{x}$ chart exhibits many out-of-control points. This is to be expected, because in this situation the $\bar{x}$ chart has an interpretation that is somewhat different from the usual interpretation. The $\bar{x}$ chart in this example shows the **discriminating power** of the instrument—literally, the ability of the gauge to distinguish between units of product. The R chart directly shows the magnitude of measurement error, or the gauge capability. The R values represent the difference between measurements made on the same unit using the same instrument. In this example, the R chart is in control. This indicates that the operator is having no difficulty in making consistent measurements. Out-of-control points on the R chart could indicate that the operator is having difficulty using the instrument.

Table 7-6 Parts Measurement Data

Part Number	Measurements 1	Measurements 2	$\bar{x}$	R
1	21	20	20.5	1
2	24	23	23.5	1
3	20	21	20.5	1
4	27	27	27.0	0
5	19	18	18.5	1
6	23	21	22.0	2
7	22	21	21.5	1
8	19	17	18.0	2
9	24	23	23.5	1
10	25	23	24.0	2
11	21	20	20.5	1
12	18	19	18.5	1
13	23	25	24.0	2
14	24	24	24.0	0
15	29	30	29.5	1
16	26	26	26.0	0
17	20	20	20.0	0
18	19	21	20.0	2
19	25	26	25.5	1
20	19	19	19.0	0
			$\bar{\bar{x}} = 22.3$	$\bar{R} = 1.0$

$$\hat{\sigma}_{\text{Gauge}} = \frac{\bar{R}}{d_2} = \frac{1.0}{1.128} = 0.887$$

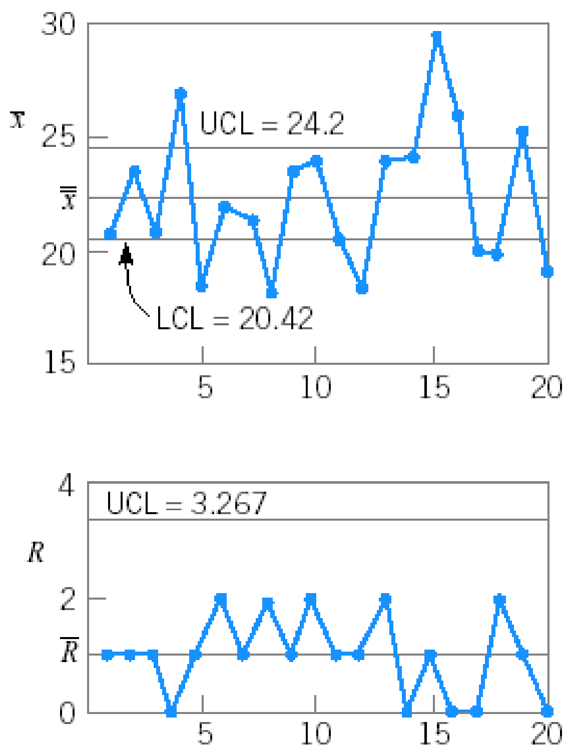

Figure 7-14 Control charts for the gauge capability analysis in Example 7-7.

$$P/T = \frac{k\hat{\sigma}_{\text{Gauge}}}{\text{USL} - \text{LSL}} \qquad (7\text{-}25)$$

k = 5.15 → number of standard deviation between 95% tolerance interval Containing 99% of normal population.
k = 6 → number of standard deviations between natural tolerance limit of normal population.

For Example 7-7, USL = 60 and LSL = 5. With *k =6*

$$P/T = \frac{6(0.887)}{60 - 5} = \frac{5.32}{55} = 0.097$$

P/T ≤ 0.1 is taken as appropriate.

Estimating Variances

$$\hat{\sigma}^2_{\text{Total}} = s^2 = (3.17)^2 = 10.05$$

$$\sigma^2_{\text{Total}} = \sigma^2_P + \sigma^2_{\text{Gauge}}$$

Because we estimate $\hat{\sigma}^2_{\text{Gauge}} = (0.887)^2 = 0.79$:

$$\hat{\sigma}^2_P = \hat{\sigma}^2_{\text{Total}} - \hat{\sigma}^2_{\text{Gauge}} = 10.05 - 0.79 = 9.26$$

$$\longrightarrow \qquad \hat{\sigma}_P = \sqrt{9.26} = 3.04$$

Signal to Noise Ration (*SNR*)

$$SNR = \sqrt{\frac{2\rho_P}{1-\rho_P}} \tag{7-28}$$

SNR: number of distinct levels that can be reliably obtained from measurements. *SNR* should be ≥ 5

For Example 7-7: $\rho_M = 0.0786$ and $\hat{\rho}_P = 1 - \hat{\rho}_M = 0.9214.$

$$\widehat{SNR} = \sqrt{\frac{2\hat{\rho}_P}{1-\hat{\rho}_P}} = \sqrt{\frac{2(0.9214)}{1-0.9214}} = 4.84$$

The gauge (with estimated *SNR* < 5) is not capable according to *SNR* criterion.

Discrimination Ratio

$$DR = \frac{1+\rho_P}{1-\rho_P} \qquad (7\text{-}29)$$

DR > 4 for capable gauges.
For example, 7-7:

$$\widehat{DR} = \frac{1+\rho_P}{1-\rho_P} = \frac{1+0.9214}{1-0.9214} = 24.45$$

The gauge is capable according *to DR* criterion.

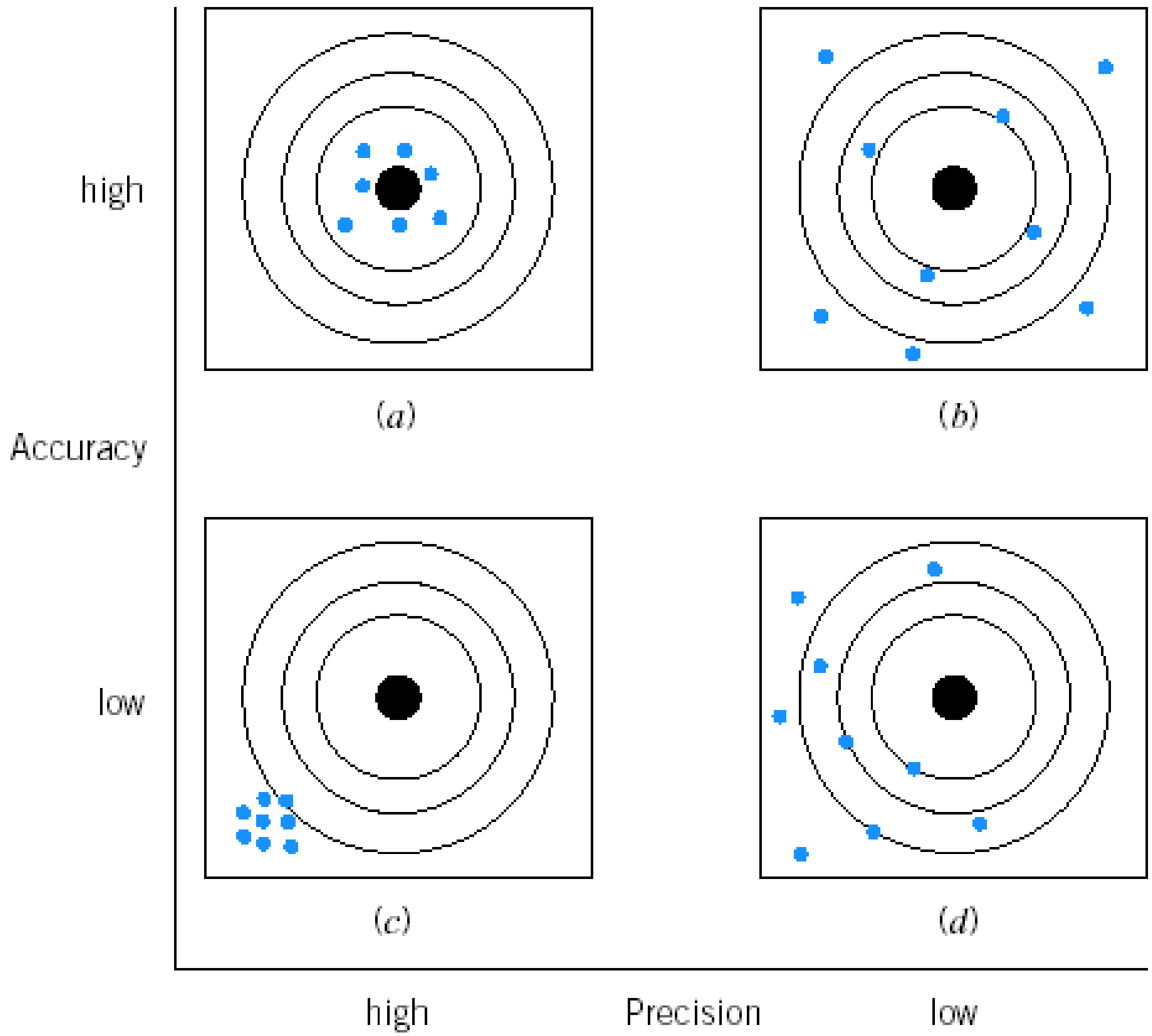

Figure 7-15 The concepts of accuracy and precision. (*a*) The gauge is accurate and precise. (*b*) The gauge is accurate but not precise. (*c*) The gauge is not accurate but it is precise. (*d*) The gauge is neither accurate nor precise.

Gauge R&R Studies

$$\sigma^2_{\text{Measurement Error}} = \sigma^2_{\text{Gauge}} = \sigma^2_{\text{Repeatability}} + \sigma^2_{\text{Reproducibility}} \qquad (7\text{-}30)$$

- Usually conducted with a factorial experiment (*p*: number of randomly selected parts, *o*: number of randomly selected operators, *n*: number of times each operator measures each part)

$$y_{ijk} = \mu + P_i + O_j + (PO)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, o \\ k = 1, 2, \ldots, n \end{cases} \qquad (7\text{-}30)$$

$P_i \sim N\left(0, \sigma_P^2\right)$: effects of parts

$O_j \sim N\left(0, \sigma_O^2\right)$: effects of operators

$(PO)_{ij} \sim N\left(0, \sigma_{PO}^2\right)$: joint effects of parts and operators

$\varepsilon_{ijk} \sim N\left(0, \sigma^2\right)$: effects of operators

$$V\left(y_{ijk}\right) = \sigma_P^2 + \sigma_O^2 + \sigma_{PO}^2 + \sigma^2 \qquad (7\text{-}31)$$

Table 7-7 Thermal Impedance Data (°C/W × 100) for the Gauge R & R Experiment

Part Number	Inspector 1			Inspector 2			Inspector 3		
	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3	Test 1	Test 2	Test 3
1	37	38	37	41	41	40	41	42	41
2	42	41	43	42	42	42	43	42	43
3	30	31	31	31	31	31	29	30	28
4	42	43	42	43	43	43	42	42	42
5	28	30	29	29	30	29	31	29	29
6	42	42	43	45	45	45	44	46	45
7	25	26	27	28	28	30	29	27	27
8	40	40	40	43	42	42	43	43	41
9	25	25	25	27	29	28	26	26	26
10	35	34	34	35	35	34	35	34	35

- This is a two-factor factorial experiment.
- ANOVA methods are used to conduct the R&R analysis.

$$SS_{\text{Total}} = SS_{\text{Parts}} + SS_{\text{Operators}} + SS_{P\times O} + SS_{\text{Error}} \tag{7-32}$$

$$MS_P = \frac{SS_{\text{Parts}}}{p-1}$$

$$MS_O = \frac{SS_{\text{Operators}}}{o-1}$$

$$MS_{PO} = \frac{SS_{P\times O}}{(p-1)(o-1)}$$

$$MS_E = \frac{SS_{\text{Error}}}{po(n-1)}$$

$$E(MS_P) = \sigma^2 + n\sigma_{PO}^2 + bn\sigma_P^2$$

$$E(MS_O) = \sigma^2 + n\sigma_{PO}^2 + an\sigma_O^2$$

$$E(MS_{PO}) = \sigma^2 + n\sigma_{PO}^2$$

$$\hat{\sigma}^2 = MS_E$$

$$\hat{\sigma}_{PO}^2 = \frac{MS_{PO} - MS_E}{n}$$

$$\hat{\sigma}_O^2 = \frac{MS_O - MS_{PO}}{pn}$$

$$\hat{\sigma}_P^2 = \frac{MS_P - MS_{PO}}{on}$$

Table 7-8 ANOVA: Thermal Impedance versus Part Number, Operator

Factor	Type	Levels	Values						
Part Num	random	10	1	2	3	4	5	6	7
			8	9	10				
Operator	random	3	1	2	3				

Analysis of Variance for Thermal

Source	DF	SS	MS	F	P
Part Num	9	3935.96	437.33	162.27	0.000
Operator	2	39.27	19.63	7.28	0.005
Part Num*Operator	18	48.51	2.70	5.27	0.000
Error	60	30.67	0.51		
Total	89	4054.40			

Source	Variance component	Error term	Expected Mean Square for Each Term (using unrestricted model)
1 Part Num	48.2926	3	(4) + 3(3) + 9(1)
2 Operator	0.5646	3	(4) + 3(3) + 30(2)
3 Part Num*Operator	0.7280	4	(4) + 3(3)
4 Error	0.5111		(4)

$$\hat{\sigma}_P^2 = \frac{437.33 - 2.70}{(3)(3)} = 48.29 \qquad \hat{\sigma}_{PO}^2 = \frac{2.70 - 0.51}{3} = 0.73$$

$$\hat{\sigma}_O^2 = \frac{19.63 - 2.70}{(10)(3)} = 0.56 \qquad \sigma^2 = 0.51$$

- Negative estimates of a variance component would lead to filling a reduced model, such as, for example:

$$y_{ijk} = \mu + P_i + O_j + \varepsilon_{ijk}$$

σ2: repeatability variance component

$$\longrightarrow \quad \sigma^2_{\text{Reproducibility}} = \sigma^2_O + \sigma^2_{PO}$$

$$\longrightarrow \quad \sigma^2_{\text{Gauge}} = \sigma^2_{\text{Reproducibility}} + \sigma^2_{\text{Repeatability}}$$

For the example:

$$\begin{aligned}\hat{\sigma}^2_{\text{Gauge}} &= \hat{\sigma}^2 + \hat{\sigma}^2_O + \hat{\sigma}^2_{PO} \\ &= 0.51 + 0.56 + 0.73 \\ &= 1.80\end{aligned}$$

With LSL = 18 and USL = 58:

$$\widehat{P/T} = \frac{6\hat{\sigma}_{\text{Gauge}}}{\text{USL} - \text{LSL}} = \frac{6(1.34)}{58 - 18} = 0.27$$

This gauge is not capable since *P/T* > 0.1.

Linear Combinations

For $x_1, x_2, \ldots, x_n,$ assume $x_i \sim N\left(\mu_i, \sigma_i^2\right)$ and independent from each other.

Let $y = a_1 x_1 + a_2 x_2 + \ldots + a_n x_n$.

Then $y \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^{\ 2}\right)$

EXAMPLE 7-8

A linkage consists of four components as shown in Fig. 7-17. The lengths of x_1, x_2, x_3, and x_4 are known to be $x_1 \sim N(2.0, 0.0004)$, $x_2 \sim N(4.5, 0.0009)$, $x_3 \sim N(3.0, 0.0004)$, and $x_4 \sim N(2.5, 0.0001)$. The lengths of the components can be assumed independent, because they are produced on different machines. All lengths are in inches.

The design specifications on the length of the assembled linkage are 12.00 ± 0.10. To find the fraction of linkages that fall within these specification limits, note that y is normally distributed with mean

$$\mu_y = 2.0 + 4.5 + 3.0 + 2.5 = 12.0$$

and variance

$$\sigma_y^2 = 0.0004 + 0.0009 + 0.0004 + 0.0001 = 0.0018$$

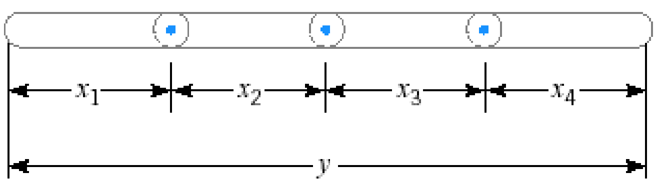

Figure 7-17 A linkage assembly with four components.

$$\begin{aligned}
P\{11.90 \le y \le 12.10\} &= P\{y \le 12.10\} - P\{y \le 11.90\} \\
&= \Phi\left(\frac{12.10 - 12.00}{\sqrt{0.0018}}\right) - \Phi\left(\frac{11.90 - 12.00}{\sqrt{0.0018}}\right) \\
&= \Phi(2.36) - \Phi(-2.36) \\
&= 0.99086 - 0.00914 \\
&= 0.98172
\end{aligned}$$

Nonlinear Combinations

$$y = g(x_1, x_2, \ldots, x_n)$$

$g(\bullet)$: nonlinear function of $x_1, x_2, \ldots, x_n$

μ_i : nominal (i.e., average) dimension for x_i (for $i = 1, 2, \ldots n$)

Taylor series expansion of $g(\bullet)$:

$$\longrightarrow \quad y = g(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} (x_i - \mu_i) \left. \frac{\partial g}{\partial x_i} \right|_{\mu_1, \mu_2, \ldots, \mu_n} + R$$

R is higher order (2 or higher) remainder of the expansion.
$R \rightarrow 0$

$$\longrightarrow \quad \mu_y \simeq g(\mu_1, \mu_2, \ldots, \mu_n)$$

$$\sigma_y^2 \simeq \sum_{i=1}^{n} \left(\left. \frac{\partial g}{\partial x_i} \right|_{\mu_1, \mu_2, \ldots, \mu_n} \right)^2 \sigma_i^2$$

EXAMPLE 7-11

Consider the simple DC circuit components shown in Fig. 7-20. Suppose that the voltage across the points (a, b) is required to be 100 ± 2 V. The specifications on the current and the resistance in the circuit are shown in Fig. 7-20. We assume that the component random variables I and R are normally and independently distributed with means equal to their nominal values.

From Ohm's law, we know that the voltage is

$$V = IR$$

Since this involves a nonlinear combination, we expand V in a Taylor series about mean current μ_I and mean resistance μ_R, yielding

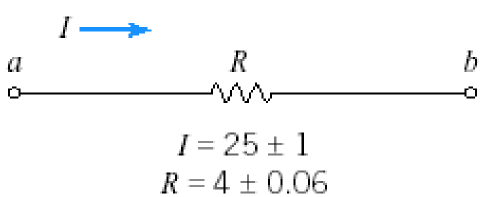

Figure 7-20 Electrical circuit for Example 7-11.

$$V \simeq \mu_I \mu_R + (I - \mu_I)\mu_R + (R - \mu_R)\mu_I$$

$$\mu_V \simeq \mu_I \mu_R$$

$$\sigma_V^2 \simeq \mu_R^2 \sigma_I^2 + \mu_I^2 \sigma_R^2$$

Assume *I* and *R* are centered at the nominal values.
α = 0.0027: fraction of values falling outside the natural tolerance limits.
Specification limits are equal to natural tolerance limits.

Assume $I = 25 \pm 1$ Amp or $24 \le I \le 26$ and $\mathrm{I} \sim N\left(25, \sigma_I^2\right)$.

$$Z_{\alpha/2} = Z_{0.00135} = 3.0$$

$$\rightarrow \frac{26-25}{\sigma_I} = 3.0 \qquad \rightarrow \sigma_I = 0.33$$

Assume $R = 4 \pm 0.06$ ohms or $3.94 \le I \le 4.06$ and $\mathrm{I} \sim N\left(4, \sigma_R^2\right)$.

$$\rightarrow \frac{4.06-4.00}{\sigma_R} = 3.0 \qquad \rightarrow \sigma_R = 0.02$$

Assuming *V* is approximately normal:

$$\mu_V \simeq \mu_I \mu_R = (25)(4) = 100 \text{ V}$$

$$\sigma_V^2 \simeq \mu_R^2 \sigma_I^2 + \mu_I^2 \sigma_R^2 = (4)^2 (0.33)^2 + (25)^2 (0.02)^2 = 1.99 \qquad \longrightarrow \quad \sigma_V = \sqrt{1.99} = 1.41$$

$$P\{98 \le V \le 102\} = P\{V \le 102\} - P\{V \le 98\} = \Phi\left(\frac{102-100}{1.41}\right) - \Phi\left(\frac{98-100}{1.41}\right)$$

$$= \Phi(1.42) - \Phi(-1.42) = \Phi(1.42) - \Phi(-1.42)$$

$$= \Phi(1.42) - \Phi(-1.42)$$

Estimating Natural Tolerance Limits

For normal distribution with unknown mean and variance:

$$\bar{x} \pm Z_{\alpha/2} s \qquad (7\text{-}45)$$

- Difference between tolerance limits and confidence limits
- Nonparametric tolerance limits can also be calculated.