9

Modeling of Mechanical Systems for Mechatronics Applications

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9.1 Introduction

Mechatronics applications are distinguished by controlled motion of mechanical systems coupled to actuators and sensors. Modeling plays a role in understanding how the properties and performance of mechanical components and systems affect the overall mechatronic system design. This chapter reviews methods for modeling systems of interconnected mechanical components, initially restricting the

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application to basic translational and rotational elements, which characterize a wide class of mechatronic applications. The underlying basis of mechanical motion (kinematics) is presumed known and not reviewed here, with more discussion and emphasis placed on a system dynamics perspective. More advanced applications requiring two- or three-dimensional motion is presented in section 9.6.

Mechanical systems can be conceptualized as rigid and/or elastic bodies that may move relative to one another, depending on how they are interconnected by components such as joints, dampers, and other passive devices. This chapter focuses on those systems that can be represented using lumped-parameter descriptions, wherein bodies are treated as rigid and no dependence on spatial extent need be considered in the elastic effects. The modeling of mechanical systems in general has reached a fairly high level of maturity, being based on classical methods rooted in the Newtonian laws of motion. One benefits from the extensive and overwhelming knowledge base developed to deal with problems ranging from basic mass-spring systems to complex multibody systems. While the underlying physics are well understood, there exist many different means and ways to arrive at an end result. This can be especially true when the need arises to model a multibody system, which requires a considerable investment in methods for formulating and solving equations of motion. Those applications are not within the scope of this chapter, and the immediate focus is on modeling basic and moderately complex systems that may be of primary interest to a mechatronic system designer/analyst.

9.2 Mechanical System Modeling in Mechatronic Systems

Initial steps in modeling any physical system include defining a system boundary, and identifying how basic components can be partitioned and then put back together. In mechanical systems, these analyses can often be facilitated by identifying points in a system that have a distinct velocity. For purposes of analysis, active forces and moments are "applied" at these points, which could represent energetic interactions at a system boundary. These forces and moments are typically applied by actuators but might represent other loads applied by the environment.

A mechanical component modeled as a point mass or rigid body is readily identified by its velocity, and depending on the number of bodies and complexity of motion there is a need to introduce a coordinate system to formally describe the kinematics (e.g., see [12] or [15]). Through a kinematic analysis, additional (relative) velocities can be identified that indicate the connection with and motion of additional mechanical components such as springs, dampers, and/or actuators. The interconnection of mechanical components can generally have a dependence on geometry. Indeed, it is dependence of mechanical systems on geometry that complicates analysis in many cases and requires special consideration, especially when handling complex systems.

A preliminary description of a mechanical system should also account for any constraints on the motional states, which may be functions of time or of the states themselves. The dynamics of mechanical systems depends, in many practical cases, on the effect of constraints. Quantifying and accounting for constraints is of paramount importance, especially in multibody dynamics, and there are different schools of thought on how to develop models. Ultimately, the decision on a particular approach depends on the application needs as well as on personal preference.

It turns out that a fairly large class of systems can be understood and modeled by first understanding basic one-dimensional translation and fixed-axis rotation. These systems can be modeled using methods consistent with those used to study other systems, such as those of an electric or hydraulic type. Furthermore, building interconnected mechatronic system models is facilitated, and it is usually easier for a system analyst to conceptualize and analyze these models.

In summary, once an understanding of (a) the system components and their interconnections (including dependence on geometry), (b) applied forces/torques, and (c) the role of constraints, is developed, dynamic equations fundamentally due to Newton can be formulated. The rest of this section introduces the selection of physical variables consistent with a power flow and energy-based approach to modeling basic mechanical translational and rotational systems. In doing so, a bond graph approach [28,3,17] is introduced for developing models of mechanical systems. This provides a basis for introducing the

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concept of causality, which captures the input–output relationship between power-conveying variables in a system. The bond graph approach provides a way to understand and mathematically model basic as well as complex mechanical systems that is consistent with other energetic domains (electric, electromechanical, thermal, fluid, chemical, etc.).

Physical Variables and Power Bonds

Power and Energy Basis

One way to consistently partition and connect subsystem models is by using power and energy variables to quantify the system interaction, as illustrated for a mechanical system in Fig. 9.1(a). In this figure, one **port** is shown at which power flow is given by the product of force and velocity, $F \cdot V$, and another for which power is the product of torque and angular velocity, $T \cdot \omega$. These power-conjugate variables (i.e., those whose product yields power) along with those that would be used for electrical and hydraulic energy domains are summarized in Table 9.1. Similar effort (e) and flow (f) variables can be identified for other energy domains of interest (e.g., thermal, magnetic, chemical). This basis assures energetically correct models, and provides a consistent way to connect system elements together.

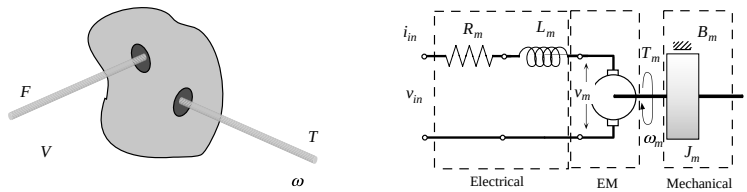
In modeling energetic systems, energy continuity serves as a basis to classify and to quantify systems. Paynter [28] shows how the energy continuity equation, together with a carefully defined port concept, provides a basis for a generalized modeling framework that eventually leads to a bond graph approach. Paynter's reticulated equation of energy continuity,

$$-\sum_{i=1}^{l} P_i = \sum_{j=1}^{m} \frac{dE_j}{dt} + \sum_{k=1}^{n} (P_d)_k \tag{9.1}$$

concisely identifies the l distinct flows of power, P_i, m distinct stores of energy, E_j, and the n distinct dissipators of energy, P_d. Modeling seeks to refine the descriptions from this point. For example, in a simple mass–spring–damper system, the mass and spring store energy, a damper dissipates energy, and

TABLE 9.1 Power and Energy Variables for Mechanical Systems

Energy Domain	Effort, e	Flow, f	Power, P
General	e	f	$e \cdot f$ [W]
Translational	Force, F [N]	Velocity, V [m/sec]	$F \cdot V$ [N m/sec, W]
Rotational	Torque, T or τ [N m]	Angular velocity, ω [rad/sec]	$T \cdot \omega$ [N m/sec, W]
Electrical	Voltage, v [V]	Current, i [A]	$v \cdot i$ [W]
Hydraulic	Pressure, P [Pa]	Volumetric flowrate, Q [m^3/sec]	$P \cdot Q$ [W]



(a) (b)

FIGURE 9.1 Basic interconnection of systems using power variables.

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the interconnection of these elements would describe how power flows between them. Some of the details for accomplishing these modeling steps are presented in later sections.

One way to proceed is to define and categorize types of system elements based on the reticulated energy continuity Eq. (9.1). For example, consider a system made up only of rigid bodies as energy stores (in particular of kinetic energy) for which $P_d = 0$ (we can add these later), and in general there can be l ports that could bring energy into this purely (kinetic)energy-storing system which has m distinct ways to put energy into the rigid bodies. This is a very general concept, consistent with many other ways to model physical systems. Howevever, it is this foundation that provides for a generalized way to model and integrate different types of energetic systems.

The schematic of a permanent-magnet dc (PMDC) motor shown in Fig. 9.1(b) illustrates how power variables would be used to identify inteconnection points. This example also serves to identify the need for modeling mechanisms, such as the electromechanical (EM) interaction, that can represent the exchange of energy between two parts of a system. This model represents a simplified relationship between electrical power flow, $v \cdot i$, and mechanical power flow, $T \cdot \omega$, which forms the basis for a motor model. Further, this is an ideal power-conserving relationship that would only contain the power flows in the energy continuity equation; there are no stores or dissipators. Additional physical effects would be included later.

Power and Signal Flow

In a bond graph formulation of the PMDC motor, a **power bond** is used to identify flow of power. Power bonds quantify power flow via an effort-flow pair, which can label the bonds as shown in Fig. 9.2(a) (convention calls for the effort to take the position above for any orientation of bond). This is a **word bond graph** model, a form used to identify the essential components in a complex system model. At this stage in a model, only the interactions of multiport systems are captured in a general fashion. Adding half-arrows on power bonds defines a power flow direction between two systems (positive in the direction of the arrow). **Signal bonds**, used in control system diagrams, have full-arrows and can be used in bond graph models to indicate interactions that convey only information (or negligible power) between multiports. For example, the word bond graph in Fig. 9.2(b) shows a signal from the mechanical block to indicate an ideal measurement transferred to a controller as a pure signal. The controller has both signal and power flow signals, closing the loop with the electrical side of the model. These conceptual diagrams are useful for understanding and communicating the system interconnections but are not complete or adequate for quantifying system performance.

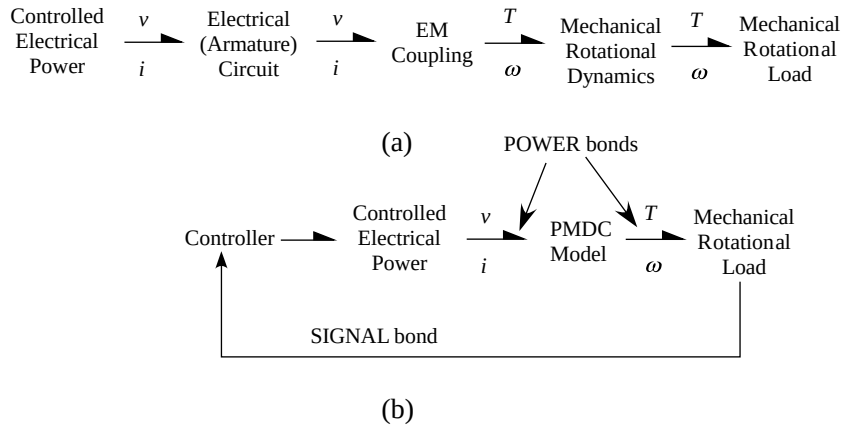

FIGURE 9.2 Power-based bond graph models: (a) PMDC motor word bond graph, (b) PMDC motor word bond graph with controller.

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While it is convenient to use power and energy in formulating system models for mechanical systems, a **motional** basis is critical for identifying interconnections and when formulating quantifiable mathematical models. For many mechanical, translational, and rotational systems, it is sufficient to rely on basic one-dimensional motion and relative motion concepts to identify the interrelation between many types of practical components. Identifying network-like structure in these systems has been the basis for building electrical analogies for some time. These methods, as well as signal-flow analysis techniques, are not presented here but are the method of choice in some approaches to system dynamics [33]. Bond graph models are presented, and it will be shown in later sections how these are consistent even with more complex mechanical system formulations of three-dimensional dynamics as well as with the use of Lagrangian models.

Need for Motional Basis

In modeling mechanical translational or rotational systems, it is important to identify how the configuration changes, and a coordinate system should be defined and the effect of geometric changes identified. It is assumed that the reader is familiar with these basic concepts [12]. Usually a reference configuration is defined from which coordinates can be based. This is essential even for simple one-dimensional translation or fixed-axis rotation. The minumum number of geometrically independent coordinates required to describe the configuration of a system is traditionally defined as the **degrees of freedom**. Constraints should be identified and can be used to choose the most convenient set of coordinates for description of the system. We distinguish between degrees of freedom and the minimum number of **dynamic state variables** that might be required to describe a system. These may be related, but they are not necessarily the same variables or the same in number (e.g., a second-order system has two states but is also referred to as a single degree of freedom system).

An excellent illustration of the relevance of degrees of freedom, constraints, and the role these concepts play in modeling and realizing a practical system is shown in Fig. 9.3. This illustration (adapted from Matschinsky [22]) shows four different ways to configure a wheel suspension. Case (a), which also forms the basis for a 1/4-car model clearly has only one degree of freedom. The same is true for cases (b) and (c), although there are constraints that reduce the number of coordinates to just one in each of these designs. Finally, the rigid beam axle shows how this must have two degrees of freedom in vertical and rotational motion of the beam to achieve at least one degree of freedom at each wheel.

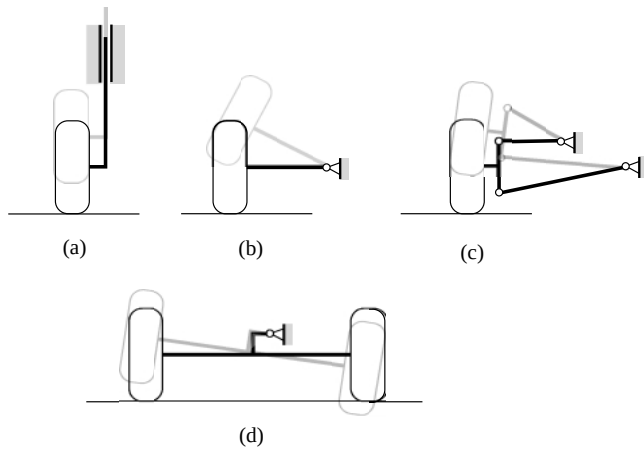

FIGURE 9.3 Wheel suspensions: (a) vertical travel only, 1 DOF; (b) swing-axle with vertical and lateral travel, 1 DOF; (c) four-bar linkage design, constrained motion, 1 DOF; (d) rigid beam axle, two wheels, vertical, and rotation travel, 2 DOF.

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Interconnection of Components

In this chapter, we will use bond graphs to model mechanical systems. Like other graph representations used in system dynamics [33] and multibody system analysis [30,39], bond graphs require an understanding of basic model elements used to represent a system. However, once understood, graph methods provide a systematic method for representing the interconnection of multi-energetic system elements. In addition, bond graphs are unique in that they are not linear graph formulations: power bonds replace branches, multiports replace nodes [28]. In addition, they include a systematic approach for computational causality.

Recall that a single line represents power flow, and a half-arrow is used to designate positive power flow direction. Nodes in a linear graph represent across variables (e.g., velocity, voltage, flowrate); however, the multiport in a bond graph represents a system element that has a physical function defined by an energetic basis. System model elements that represent masses, springs, and other components are discussed in the next section. Two model elements that play a crucial role in describing how model elements are interconnected are the 1-junction and 0-junction. These are ideal (power-conserving) multiport elements that can represent specific physical relations in a system that are useful in interconnecting other model elements.

A point in a mechanical system that has a distinct velocity is represented by a 1-junction. When one or more model elements (e.g., a mass) have the same velocity as a given 1-junction, this is indicated by connecting them to the 1-junction with a power bond. Because the 1-junction is constrained to conserve power, it can be shown that efforts (forces, torques) on all the connected bonds must sum to zero; i.e., $\hat{\mathrm{A}} e_i = 0$. This is illustrated in Fig. 9.4(a). The 1-junction enforces kinematic compatibility and introduces a way to graphically express force summation! The example in Fig. 9.4(b) shows three systems (the blocks labeled 1, 2, and 3) connected to a point of common velocity. In the bond graph, the three systems would be connected by a 1-junction. Note that sign convention is incorporated into the sense of the power arrow.

For the purpose of analogy with electrical systems, the 1-junction can be thought of as a series electrical connection. In this way, elements connected to the 1-junction all have the same current (a flow variable) and the effort summation implied in the 1-junction conveys the Kirchhoff voltage law. In mechanical systems, 1-junctions may represent points in a system that represent the velocity of a mass, and the effort summation is a statement of Newton's law (in D'Alembert form), $\hat{\mathrm{A}} F - \dot{p} = 0$.

Figure 9.4 illustrates how components with common velocity are interconnected. Many physical components may be interconnected by virtue of a common effort (i.e., force or torque) or 0-junction. For example, two springs connected serially deflect and their ends have distinct rates of compression/extension; however, they have the same force across their ends (ideal, massless springs). System components that have this type of relationship are graphically represented using a 0-junction. The basic 0-junction definition is shown in Fig. 9.5(a). Zero junctions are especially helpful in mechanical system modeling because they can also be used to model the connection of components having relative motion. For example, the device in Fig. 9.5(b), like a spring, has ends that move relative to one another, but the force

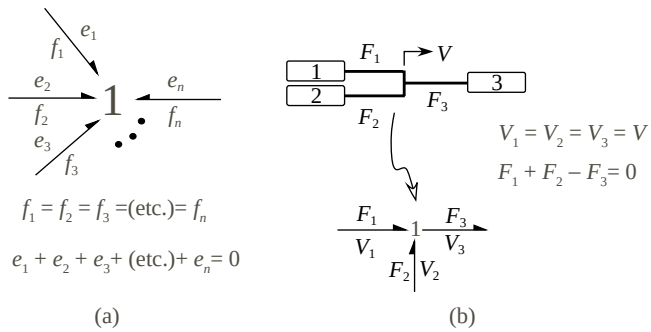

FIGURE 9.4 Mechanical 1-junction: (a) basic definition, (b) example use at a massless junction.

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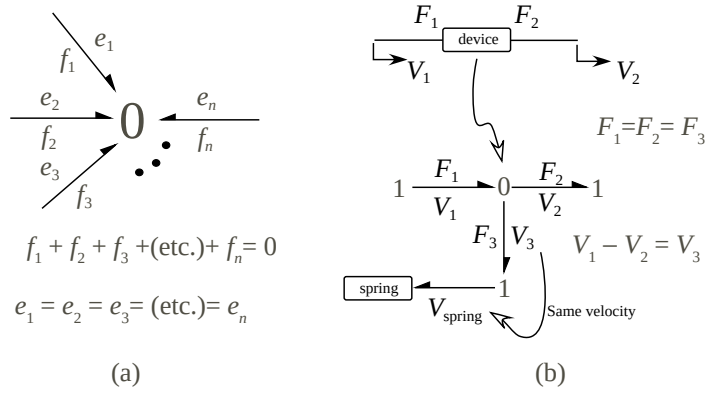

FIGURE 9.5 Mechanical 0-junction: (a) basic definition, (b) example use at a massless junction.

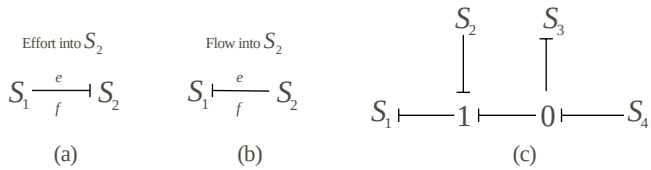

FIGURE 9.6 (a) Specifying effort from S_1 into S_2. (b) Specifying flow from S_1 into S_2. (c) A contrived example showing the constraint on causality assignment imposed by the physical definitions of 0- and 1-junctions.

on each end is the same (note this assumes there is negligible mass). The definition of the 0-junction implies that all the bonds have different velocities, so a flow difference can be formed to construct a relative velocity, V_3. All the bonds have the same force, however, and this force would be applied at the 1-junctions that identify the three distinct velocities in this example. A spring, for example, would be connected on a bond connected to the V_3 junction, as shown in Fig. 9.5(b), and $V_{spring} = V_3$.

The 1- and 0-junction elements graphically represent algebraic structure in a model, with distinct physical attributes from compatibility of kinematics (1-junction) and force or torque (0-junction). The graph should reflect what can be understood about the interconnection of physical devices with a bond graph. There is an advantage in forming a bond graph, since causality can then be used to form mathematical models. See the text by Karnopp, Margolis, and Rosenberg [17] for examples. There is a relation to through and across variables, which are used in linear graph methods [33].

Causality

Bond graph modeling was conceived with a consistent and algorithmic methodology for assignment of causality (see Paynter [28], p. 126). In the context of bond graph modeling, causality refers to the input–output relationship between variables on a power bond, and it depends on the systems connected to each end of a bond. Paynter identified the need for this concept having been extensively involved in analog computing, where solutions rely on well-defined relationships between signals. For example, if system S_1 in Fig. 9.6(a) is a known source of effort, then when connected to a system S_2, it must specify effort into S_2, and S_2 in turn must return the flow variable, f, on the bond that connects the two systems. In a bond graph, this causal relationship is indicated by a vertical stroke drawn on the bond, as shown in Fig. 9.6(a). The vertical stroke at one end of a bond indicates that effort is specified into the multiport element connected at that end. In Fig. 9.6(b), the causality is reversed from that shown in (a).

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The example in Fig. 9.6(c) illustrates how causality "propagates" through a bond graph of interconnected bonds and systems. Note that a 1-junction with multiple ports can only have one bond specifying flow at that junction, so the other bonds specify effort into the 1-junction. A 0-junction requires one bond to specify effort, while all others specify flow. Also note that a direction for positive power flow has not been assigned on these bonds. This is intentional to emphasize the fact that power sense and causality assignment on a bond are **independent** of each other.

Causality assignment in system models will be applied in examples that follow. An extensive discussion of the successive cauality assignment procedure (sometimes referred to as SCAP) can be found in Rosenberg and Karnopp [32] or Karnopp, Margolis, and Rosenberg [17]. By using the defined bond graph elements, causality assignment is made systematically. The procedure has been programmed into several commercially available software packages that use bond graphs as formal descriptions of physical system models.

Because it reveals the input–output relationship of variables on all the bonds in a system model, causality can infer computational solvability of a bond graph model. The results are used to indicate the number of dynamic states required in a system, and the causal graph is helpful in actually deriving the mathematical model. Even if equations are not to be derived, causality can be used to derive physical insight into how a system works.

9.3 Descriptions of Basic Mechanical Model Components

Mechanical components in mechatronic systems make their presence known through motional response and by force and torque (or moment) reactions notably on support structures, actuators, and sensors. Understanding and predicting these response attributes, which arise due to combinations of frictional, elastic, and inertial effects, can be gained by identifying their inherent dissipative and energy storing nature. This emphasis on dissipation and energy storage leads to a systematic definition of constitutive relations for basic mechanical system modeling elements. These model elements form the basis for building complex nonlinear system models and for defining impedance relations useful in transfer function formulation. In the following, it is assumed that the system components can be well represented by lumped-parameter formulations.

It is presumed that a modeling decision is made so that dissipative and energy storing (kinetic and potential) elements can be identified to faithfully represent a system of interest. The reticulation is an essential part of the modeling process, but sometimes the definition and interconnection of the elements is not easy or intuitive. This section first reviews mechanical system input and output model elements, and then reviews passive dissipative elements and energy-storing elements. The section also discusses coupling elements used for modeling gears, levers, and other types of power-transforming elements. The chapter concludes by introducing impedance relationships for all of these elements.

Defining Mechanical Input and Output Model Elements

In dynamic system modeling, initial focus requires defining a **system boundary**, a concept borrowed from basic thermodynamics. In isolating mechanical systems, a system boundary identifies ports through which power and signal can pass. Each port is described either by a force–velocity or torque–angular velocity power conjugate pair. It is helpful, when focusing on the mechanical system modeling, to make a judgement on the causality at each port. For example, if a motor is to be attached to one port, it may be possible to define torque as the input variable and angular velocity as the output (back to the motor).

It is important to identify that these are model assumptions. We define specific elements as **sources** of effort or flow that can be attached at the boundary of a system of interest. These inputs might be known and or idealized, or they could simply be "placeholders" where we will later attach a model for an actuator or sensor. In this case, the causality specified at the port is fixed so that the (internal) system model will not change. If the causality changes, it will be necessary to reformulate a new model.

In bond graph terminology, the term **effort source** is used to define an element that specifies an effort, such as this force or torque. The symbol S_e or E can be used to represent the effort source on a bond graph.

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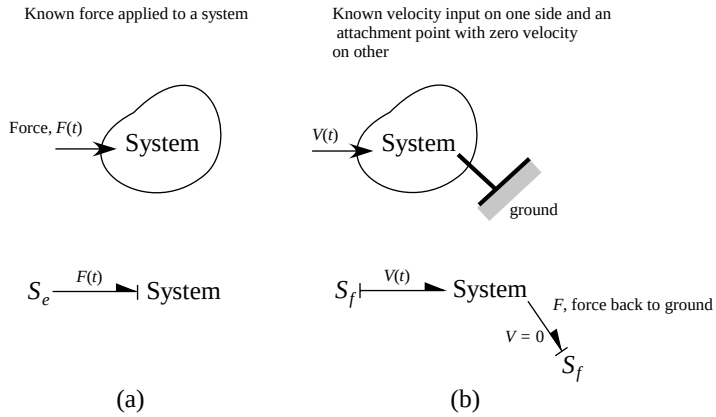

FIGURE 9.7 Two cases showing effort and flow sources on word bond graphs.

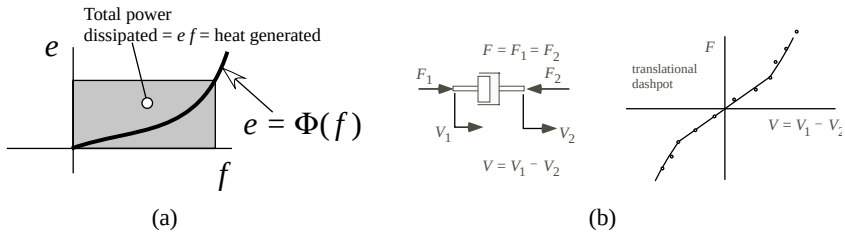

FIGURE 9.8 (a) Resistive constitutive relation. (b) Example dashpot resistive model.

A **flow source** is an element that specifies a flow on a bond, such as a translational velocity or angular or rotational velocity. The bond graph symbol is S_f or F. Two basic examples of sources are shown in Fig. 9.7. Note that each bond has a defined effort or flow, depending on the source type. The causality on these model elements is always known, as shown. Further, each bond carries both pieces of information: (1) the effort or flow variable specified by the source, and (2) the *back reaction* indicated by the causality. So, for example, at the ground connection in Fig. 9.7(b), the source specifies the zero velocity constraint into the system, and the system, in turn, specifies an effort *back* to the ground. The symbolic representation emphasizes the causal nature of bond graph models and emphasizes which variables are available for examination. In this case, the force back into the ground might be a critical output variable.

Dissipative Effects in Mechanical Systems

Mechanical systems will dissipate energy due to friction in sliding contacts, dampers (passive or active), and through interaction with different energy domains (e.g., fluid loading, eddy current damping). These irreversible effects are modeled by constitutive functions between force and velocity or torque and angular velocity. In each case, the product of the effort-flow variables represents power dissipated, $P_d = e \cdot f$, and the total energy dissipated is $E_d = \int P_d \, dt = \int (e \cdot f) \, dt$. This energy can be determined given knowledge of the constitutive function, $e = \Phi(f)$, shown graphically in Fig. 9.8(a). We identify this as a basic *resistive* constitutive relationship that must obey the restriction imposed by the second law of thermodynamics; namely that, $e \cdot f \geq 0$. A typical mechanical dashpot that follows a resistive-type model description is summarized in Fig. 9.8(b).

In a bond graph model, resistive elements are symbolized by an **R** element, and a generalized, multiport **R**-element model is shown in Fig. 9.9(a). Note that the **R** element is distinguished by its ability to represent entropy production in a system. On the **R** element, a *thermal port* and bond are shown, and the power direction is always positive *away* from the **R**. In thermal systems, temperature, T, is the effort variable

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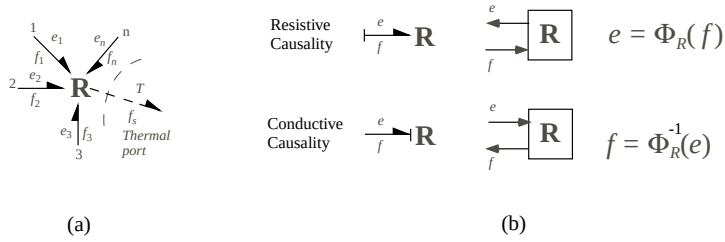

FIGURE 9.9 (a) Resistive bond graph element. (b) Resistive and conductive causality.

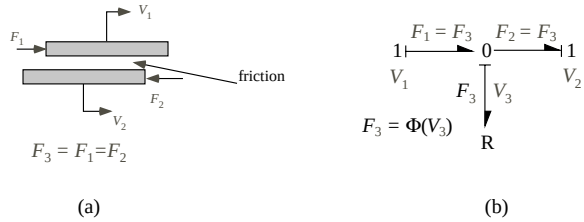

FIGURE 9.10 (a) Two sliding surfaces. (b) Bond graph model with causality implying velocities as known inputs.

and entropy flow rate, f_s is the flow variable. To compute heat generated by the **R** element, compose the calculation as Q (heat in watts) $= T \cdot f_s = \sum_i e_i \cdot f_i$ over the n ports.

The system attached to a resistive element through a power bond will generally determine the causality on that bond, since resistive elements generally have no preferred causal form.[1] Two possible cases on a given **R**-element port are shown in Fig. 9.9(b). A block diagram emphasizes the computational aspect of causality. For example, in a resistive case the flow (e.g., velocity) is a known input, so power dissipated is $P_d = e \cdot f = \Phi(f) \cdot f$. For the linear damper, $F = b \cdot V$, so $P_d = F \cdot V = bV^2$ (W).

In mechanical systems, many frictional effects are driven by relative motion. Hence, identifying how a dissipative effect is configured in a mechanical system requires identifying critical motion variables. Consider the example of two sliding surfaces with distinct velocities identified by 1-junctions, as shown in Fig. 9.10(a). Identifying one surface with velocity V_1, and the other with V_2, the simple construction shown in Fig. 9.10(b) shows how an **R** element can be connected at a *relative* velocity, V_3. Note the relevance of the causality as well. Two velocities join at the 0-junction to form a relative velocity, which is a causal input to the **R**. The causal output is a force, F_3, computed using the constitutive relation, $F = \Phi(V_3)$. The 1-junction formed to represent V_3 can be eliminated when there is only a single element attached as shown. In this case, the **R** would replace the 1-junction.

When the effort-flow relationship is linear, the proportionality constant is a **resistance**, and in mechanical systems these quantities are typically referred to as **damping constants**. Linear damping may arise in cases where two surfaces separated by a fluid slide relative to one another and induce a viscous and strictly laminar flow. In this case, it can be shown that the force and relative velocity are linearly related, and the material and geometric properties of the problem quantify the linear damping constant. Table 9.2 summarizes both translational and rotational damping elements, including the linear cases. These components are referred to as dampers, and the type of damping described here leads to the term viscous friction in mechanical applications, which is useful in many applications involving lubricated surfaces. If the relative speed is relatively high, the flow may become turbulent and this leads to nonlinear damper behavior. The constitutive relation is then a nonlinear function, but the **structure** or interconnection of

[1]This is true in most cases. Energy-storing elements, as will be shown later, have a causal form that facilitates equation formulation.

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TABLE 9.2 Mechanical Dissipative Elements

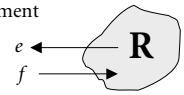
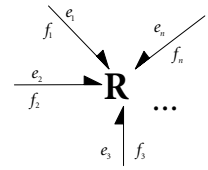
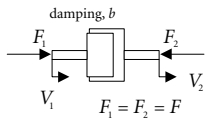
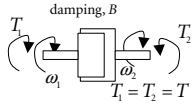
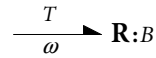
Physical System	Fundamental Relations	Bond Graph
Generalized Dissipative Element  • Resistive element • **Resistance,** R	Dissipation: $\mathbf{e}\cdot\mathbf{f}=\sum_i e_i f_i = T\cdot f_s$ Resistive law: $e=\Phi_R(f)$ Conductive law: $f=\Phi_R^{-1}(e)$ Content: $P_f=\int e\cdot df$ Co-content: $P_e=\int f\cdot de$	 **Generalized multiport R-element**
Mechanical Translation  • Damper • **damping,** b	Constitutive: $F=\Phi(V)$ Content: $P_V=\int F\cdot dV$ Co-energy: $P_F=\int V\cdot dF$ Dissipation: $P_d=P_V+P_F$	 Linear: $F=b\cdot V$ Dissipation: $P_d=bV^2$
Mechanical Rotation  • Torsional damper • **damping,** B	Constitutive: $T=\Phi(\omega)$ Content: $P_\omega=\int T\cdot d\omega$ Co-energy: $P_T=\int \omega\cdot dT$ Dissipation: $P_d=P_\omega+P_T$	 Linear: $T=B\cdot\omega$ Dissipation: $P_d=B\omega^2$

TABLE 9.3 Typical Coefficient of Friction Values. Note, Actual Values Will Vary Significantly Depending on Conditions

Contacting Surfaces	Static, μ_s	Sliding or Kinetic, μ_k
Steel on steel (dry)	0.6	0.4
Steel on steel (greasy)	0.1	0.05
Teflon on steel	0.04	0.04
Teflon on teflon	0.04	—
Brass on steel (dry)	0.5	0.4
Brake lining on cast iron	0.4	0.3
Rubber on asphalt	—	0.5
Rubber on concrete	—	0.6
Rubber tires on smooth pavement (dry)	0.9	0.8
Wire rope on iron pulley (dry)	0.2	0.15
Hemp rope on metal	0.3	0.2
Metal on ice	—	0.02

the model in the system does not change. Dampers are also constructed using a piston/fluid design and are common in shock absorbers, for example. In those cases, the force–velocity characteristics are often tailored to be nonlinear.

The viscous model will not effectively model friction between dry solid bodies, which is a much more complex process and leads to performance bounds especially at lower relative velocities. One way to capture this type of friction is with the classic Coulomb model, which depends on the normal load between surfaces and on a coefficient of friction, typically denoted μ (see Table 9.3). The Coulomb model quantifies the friction force as $F = \mu N$, where N is the normal force. This function is plotted in Fig. 9.11(a) to illustrate how it models the way the friction force always opposes motion. This model still qualifies as a resistive constitutive function relating the friction force and a relative velocity of the surfaces. In this case,

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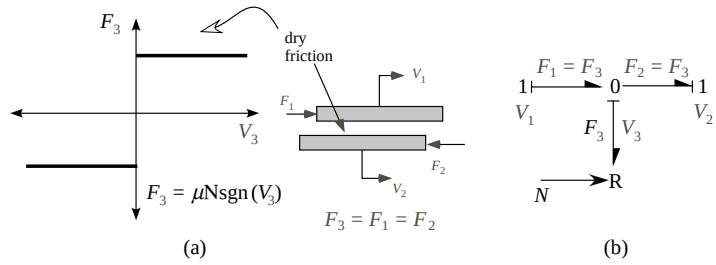

FIGURE 9.11 (a) Classic coulomb friction for sliding surfaces. (b) Bond graph showing effect of normal force as a modulation of the **R**-element law.

however, the velocity comes into effect only to determine the sign of the force; i.e., $F = \mu N\ \mathrm{sgn}(V)$, where sgn is the signum function (value of 1 if $V > 0$ and -1 if $V < 0$).

This model requires a special condition when $V \rightarrow 0$. Dry friction can lead to a phenomenon referred to as stick-slip, particularly common when relative velocities between contacting surfaces approach low values. Stick-slip, or stiction, friction forces are distinguished by the way they vary as a result of other (modulating) variables, such as the normal force or other applied loads. Stick-slip is a type of system response that arises due to frictional effects. On a bond graph, a signal bond can be used to show that the normal force is determined by an external factor (e.g., weight, applied load, etc.). This is illustrated in Fig. 9.11(b). When the basic properties of a physical element are changed by signal bonds in this way, they are said to be **modulated**. This is a modeling technique that is very useful, but care should be taken so it is not applied in a way that violates basic energy principles.

Another difficulty with the standard dry friction model is that it has a preferred causality. In other words, if the causal input is velocity, then the constitutive relation computes a force. However, if the causal input is force then there is no unique velocity output. The function is not bi-unique. Difficulties of this sort usually indicate that additional underlying physical effects are not modeled. While the effort-flow constitutive relation is used, the form of the constitutive relation may need to be parameterized by other critical variables (temperature, humidity, etc.). More detailed models are beyond the scope of this chapter, but the reader is referred to Rabinowicz (1995) and Armstrong-Helouvry (1991) who present thorough discussions on modeling friction and its effects. Friction is usually a dominant source of uncertainty in many predictive modeling efforts (as is true in most energy domains).

Potential Energy Storage Elements

Part of the energy that goes into deforming any mechanical component can be associated with pure (lossless) storage of potential energy. Often the decision to model a mechanical component this way is identified through a basic constitutive relationship between an effort variable, e (force, torque), and a displacement variable, q (translational displacement, angular displacement). Such a relationship may be derived either from basic mechanics [29] or through direct measurement. An example is a translational spring in which a displacement of the ends, x, is related to an applied force, F, as $F = F(x)$.

In an energy-based lumped-parameter model, the generalized displacement variable, q, is used to define a state-determined potential energy function,

$$E = E(q) = U_q$$

This energy is related to the constitutive relationship, $e = \mathrm{F}(q)$, by

$$U(q) = {}_qU = \int e\,dq = \int \Phi(q)\,d$$

It is helpful to generalize in this way, and to identify that practical devices of interest will have at least one connection (or port) in which power can flow to store potential energy. At this port the displacement

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TABLE 9.4 Mechanical Potential Energy Storage Elements (Integral Form)

Physical System	Fundamental Relations	Bond Graph
Generalized Potential Energy Storage Element • Capacitive element • **Capacitance,** C	State: $\mathbf{q}$ = displacement Rate: $\dot{\mathbf{q}} = \mathbf{f}$ Constitutive: $\mathbf{e} = \Phi(\mathbf{q})$ Energy: $U_{\mathbf{q}} = \int \mathbf{e} \cdot d\mathbf{q}$ Co-energy: $U_{\mathbf{e}} = \int \mathbf{q} \cdot d\mathbf{e}$	Generalized multiport C-element
Mechanical Translation • spring $V_1 - V_2 = V$ • **stiffness,** k**, compliance,** C	State: x = displacement Rate: $\dot{x} = V$ Constitutive: $F = F(x)$ Energy: $U_x = \int F \cdot dx$ Co-energy: $U_F = \int x \cdot dF$	F, $\dot{x} = V$ **C:**$1/C=k$ Linear: $F = k \cdot x$ Energy: $U_x = \frac{1}{2} k x^2$ Co-energy: $U_F = F^2/2k$
Mechanical Rotation • Torsional spring • **stiffness,** K**, compliance,** C	State: θ = angle Rate: $\dot{\theta} = \omega$ Constitutive: $T = T(\theta)$ Energy: $U_\theta = \int T \cdot d\theta$ Co-energy: $U_T = \int \theta \cdot dT$	T, $\dot{\theta} = \omega$ **C:**$1/C=K$ Linear: $T = K \cdot \theta$ Energy: $U_\theta = \frac{1}{2} k \theta^2$ Co-energy: $U_T = T^2/2K$

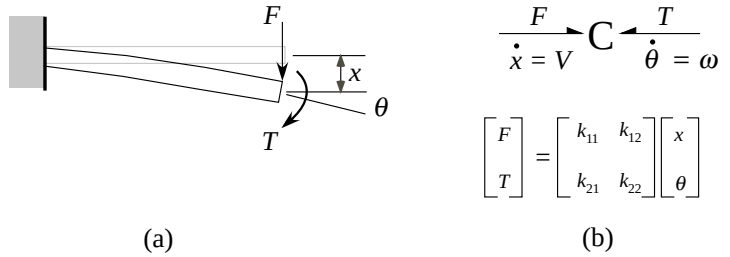

FIGURE 9.12 Example of two-port potential energy storing element: (a) cantilevered beam with translational and rotational end connections, (b) **C**-element, 2-port model.

variable of interest is either translational, x, or angular, θ, and the associated velocities are $V = \dot{x}$ and $\omega = \dot{\theta}$, respectively. A generalized potential energy storage element is summarized in Table 9.4, where examples are given for the translational and rotational one-port.

The linear translational spring is one in which $F = F(x) = kx = (1/C)x$, where k is the stiffness and $C \equiv 1/k$ is the *compliance* of the spring (compliance is a measure of "softness"). As shown in Table 9.4, the potential energy stored in a linear spring is $U_x = \int F\,dx = \int kx\,dx = \frac{1}{2} kx^2$, and the co-energy is $U_F = \int F\,dx = \int (F/k)\,dF = F^2/2k$. Since the spring is linear, you can show that $U_x = U_F$. If the spring is nonlinear due to, say, plastic deformation or work hardening, then this would not be true.

Elastic potential energy can be stored in a device through multiple ports and through different energy domains. A good example of this is the simple cantilevered beam having both tip force and moment (torque) inputs. The beam can store energy either by translational or rotational displacement of the tip. A constitutive relation for this 2-port **C**-element relates the force and torque to the linear and rotational displacments, as shown in Fig. 9.12. A stiffness (or compliance) matrix for small deflections is derived by linear superposition.

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Kinetic Energy Storage

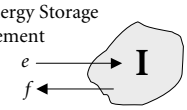
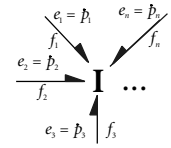
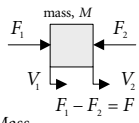
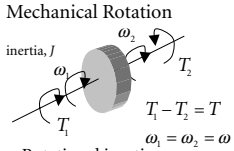
All components that constitute mechanical systems have mass, but in a system analysis, where the concern is dynamic performance, it is often sufficient to focus only on those components that may store relevant amounts of kinetic energy through their motion. This presumes that an energetic basis is used for modeling, and that the tracking of kinetic energy will provide insight into the system dynamics. This is the focus of this discussion, which is concerned for the moment with one-dimensional translation and fixed-axis rotation. Later it will be shown how the formulation presented here is helpful for understanding more complex systems.

The concept of mass and its use as a model element is faciliated by Newton's relationship between the rate of change of momentum of the mass to the net forces exerted on it, $F = \dot{p}$, where p is the momentum. The energy stored in a system due to translational motion with velocity V is the kinetic energy. Using the relation from Newton's law, $dp = Fdt$, this energy is $E(p) = T(p) = T_p = \int P dt = \int FV\ dt = \int V\ dp$. If the velocity is expressed solely as a function of the momentum, p, this system is a pure translational mass, $V = \Phi(p)$. If the velocity is linearly proportional to the momentum, then $V = p/m$, where m is the mass. Similar basic definitions are made for a body in rotation about a fixed axis, and these elements are summarized in Table 9.5.

For many applications of practical interest to engineering, the velocity–momentum relation, $V = V(p)$ (the constitutive relation), is linear. Only in relativistic cases might there be a nonlinear relationship in the constitutive law for a mass. Nevertheless, this points out that for the general case of kinetic energy storage a constitutive relation is formed between the flow variable and the momentum variable, $f = f(p)$. This should help build appreciation for analogies with other energy domains, particularly in electrical systems where inductors (the mass analog) can have nonlinear relationships between current (a flow) and flux linkage (momentum).

The rotational motion of a rigid body considered here is constrained thus far to the simple case of planar and fixed-axis rotation. The mass moment of intertia of a body about an axis is defined as the sum of the products of the mass-elements and the squares of their distance from the axis. For the discrete case, $I = \sum r^2 \Delta m$, which for continuous cases becomes, $I = \int r^2 dm$ (units of kg m^2). Some common shapes

TABLE 9.5 Mechanical Kinetic Energy Storage Elements (Integral Form)

Physical System	Fundamental Relations	Bond Graph
Generalized Kinetic Energy Storage Element  • Inertive element • **Inertance,** I	State: $\mathbf{p}$ = momentum Rate: $\dot{\mathbf{p}} = \mathbf{e}$ Constitutive: $\mathbf{f} = \Phi(\mathbf{p})$ Energy: $T_{\mathbf{p}} = \int \mathbf{f} \cdot d\mathbf{p}$ Co-energy: $T_{\mathbf{f}} = \int \mathbf{p} \cdot d\mathbf{f}$	 **Generalized multiport 1-element**
Mechanical Translation  • Mass • **mass,** m	State: p = momentum Rate: $\dot{p} = F$ Constitutive: $V = V(p)$ Energy: $T_p = \int f \cdot dp$ Co-energy: $T_V = \int p \cdot dV$	$\frac{\dot{p} = F}{V}$ **I:**M Linear: $V = p/M$ Energy: $T_p = p^2/2M$ Co-energy: $T_V = \frac{1}{2} MV^2$
Mechanical Rotation  • Rotational inertia • **mass moment of inertia,** J	State: h = angular momentum Rate: $\dot{h} = T$ Constitutive: $\omega = \omega(h)$ Energy: $T_h = \int \omega \cdot dh$ Co-energy: $T_\omega = \int h \cdot d\omega$	$\frac{\dot{h} = T}{\omega}$ **I:**J Linear: $\omega = h/J$ Energy: $T_h = h^2/2J$ Co-energy: $T_\omega = \frac{1}{2} J\omega^2$

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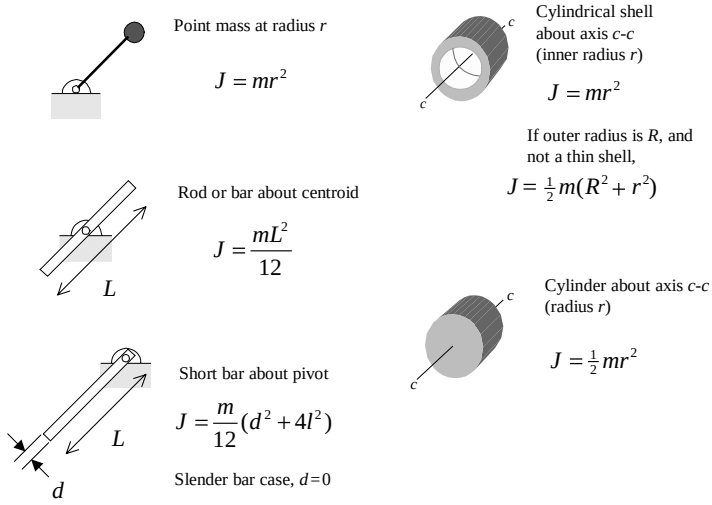

FIGURE 9.13 Mass moments of inertia for some common bodies.

and associated mass moments of inertia are given in Fig. 9.13. General rigid bodies are discussed in section "Inertia Properties."

There are several useful concepts and theorems related to the properties of rigid bodies that can be helpful at this point. First, if the mass moment of inertia is known about an axis through its center of mass (I_G), then Steiner's theorem (parallel axis theorem) relates this moment of inertia to that about another axis a distance d away by $I = I_G + md^2$, where m is the mass of the body. It is also possible to build a moment of inertia for composite bodies, in those situations where the individual motion of each body is negligible. A useful concept is the radius of gyration, k, which is the radius of an imaginary cylinder of infinitely small wall thickness having the same mass, m, and the same mass moment of inertia, I, as a body in question, and given by, $k = \sqrt{I/m}$. The radius of gyration can be used to find an equivalent mass for a rolling body, say, using $m_{eq} = I/k^2$.

Coupling Mechanisms

Numerous types of devices serve as couplers or power transforming mechanisms, with the most common being levers, gear trains, scotch yokes, block and tackle, and chain hoists. Ideally, these devices and their analogs in other energy domains are power conserving, and it is useful to represent them using a 2-port model. In such a model element, the power in is equal to the power out, or in terms of effort-flow pairs, $e_1 f_1 = e_2 f_2$. It turns out that there are two types of basic devices that can be represented this way, based on the relationship between the power variables on the two ports. For either type, a relationship between two of the variables can usually be identified from geometry or from basic physics of the device. By imposing the restriction that there is an ideal power-conserving transformation inherent in the device, a second relationship is derived. Once one relation is established the device can usually be classified as a **transformer** or **gyrator**. It is emphasized that these model elements are used to represent the ideal power-conserving aspects of a device. Losses or dynamic effects are added to model real devices.

A device can be modeled as a **transformer** when $e_1 = me_2$ and $mf_1 = f_2$. In this relation, m is a transformer **modulus** defined by the device physics to be constant or in some cases a function of states of the system. For example, in a simple gear train the angular velocities can be ideally related by the ratio of pitch radii, and in a slider crank there can be formed a relation between the slider motion and the crank angle. Consequently, the two torques can be related, so the gear train is a transformer. A device can be modeled as a **gyrator** if $e_1 = rf_2$ and $rf_1 = e_2$, where r is the gyrator modulus. Note that this model can represent

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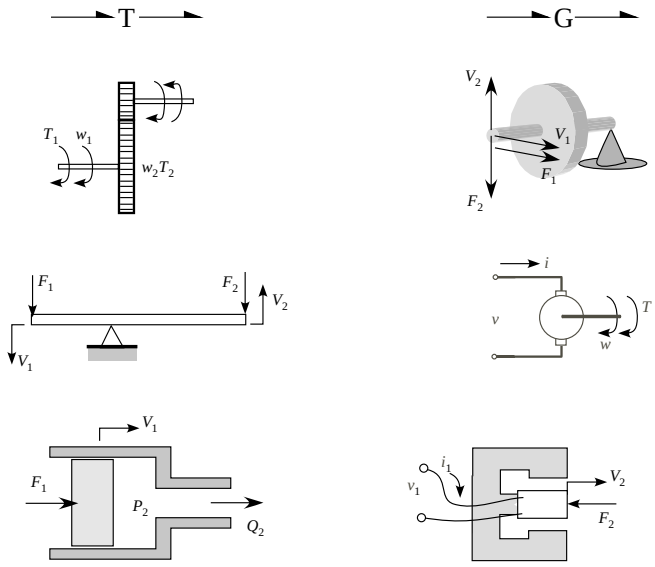

FIGURE 9.14 Common devices that can be modeled as transformers and gyrators in mechatronic systems.

the power-conserving transformation in devices for which a cross-relationship between power variables (i.e., effort related to flow) has been identified.[2]

Some examples of transformers and gyrators are shown in Fig. 9.14. In a bond graph model, the transformer can be represented by a **TF** or **T**, while a gyrator is represented by a **GY** or **G** (note, the two letter symbol is common). The devices shown in Fig. 9.14 indicate a modulus *m* or *r*, which may or may not be a constant value. Many devices may have power-conserving attributes; however, the relationship between the effort-flow variables may not be constant, so the relationship is said to be *modulated* when the modulus is a function of a dynamic variable (preferably a state of the system). On a bond graph, this can be indicated using a signal bond directed into the **T** or **G** modulus.

Examples of a modulated transformer and gyrator are given in Fig. 9.15. These examples highlight useful techniques in modeling of practical devices. In the slider crank, note that the modulation is due to a change in the angular position of the crank. We can get this information from a bond that is adjacent to the transformer in question; that is, if we integrate the angular velocity found on a neighboring bond, as shown in Fig. 9.15(a). For the field excited dc motor shown in Fig. 9.15(b), the torque–current relation in the motor depends on a flux generated by the field; however, this field is excited by a circuit that is powered *independent* of the armature circuit. The signal information for modulation does not come from a neighboring bond, as in the case for the slider crank. These two examples illustrate two ways that constraints are imposed in coupling mechanisms. The modulation in the slider crank might be said to represent a holonomic constraint, and along these same lines the field excitation in the motor imposes a non-holonomic constraint. We cannot relate torque and current in the latter case without solving for the dynamics of an independent system—the field circuit. In the slider crank, the angular position required for the modulation is obtained simply by integrating the velocity, since $\mathsf{q} = \omega$. Additional discussion on constraints can be found in section 9.7.

The system shown in Fig. 9.16(a) is part of an all-mechanical constant-speed drive. A mechanical feedback force, F_2, will adjust the position of the middle rotor, x_2. The effect is seen in the bond graph

[2]It turns out that the gyrator model element is essential in all types of systems. The need for such an element to represent gyroscopic effects in mechanical systems was first recognized by Thomson and Tait in the late 1900s. However, it was G. D. Birkhoff (1927) and B. D. H. Tellegen (1948) who independently identified the need for this element in analysis and synthesis of systems.

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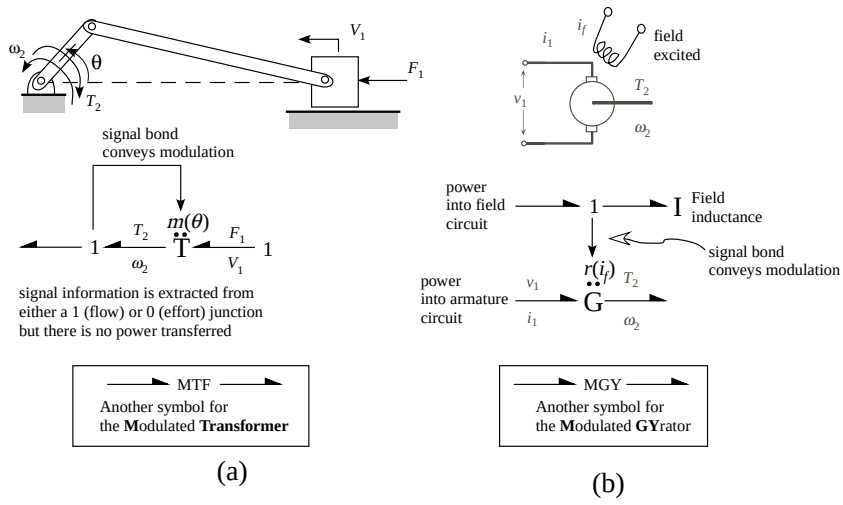

FIGURE 9.15 Concept of modulation in transformers and gyrators.

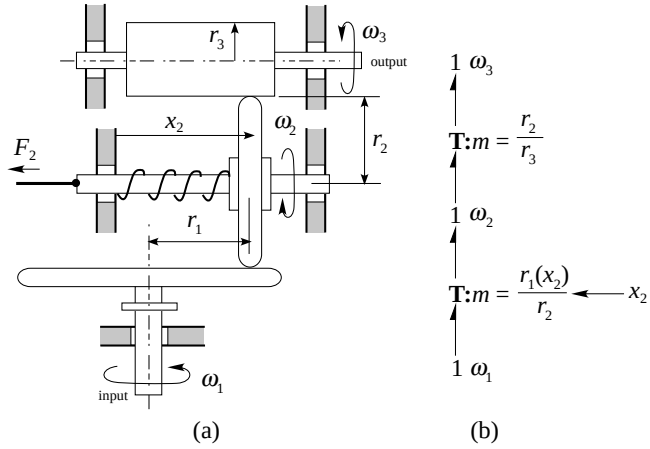

FIGURE 9.16 A nonholonomic constraint in a transformer model.

model of Fig. 9.16(b), which has two transformers to represent the speed ratio between the input (turntable) 1 and the mid-rotor 2, and the speed ratio between the mid-rotor and the output roller 3. The first transformer is a mechanical version of a nonholonomic transformation. Specifically, we would have to solve for the dynamics of the rotor position (x_2) in order to transform power between the input and output components of this device.

Impedance Relationships

The basic component descriptions presented so far are the basis for building basic models, and a very useful approach relies on impedance formulations. An impedance function, Z, is a ratio of effort to flow variables at a given system port of a physical device, and the most common application is for linear systems where $Z = Z(s)$, where s is the complex frequency variable (sometimes called the Laplace operator). An admittance is the inverse of the impedance, or $Y = 1/Z$. For each basic element defined, a linear impedance relation can be derived for use in model development. First, recall that the derivative operator can be represented by the s operator, so that dx/dt in s-domain is simply sx and $\int x\,dt$ is x/s, and so on.

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TABLE 9.6 Basic Mechanical Impedance Elements

System	Resistive, Z_R	Capacitive, Z_C	Inertive, Z_I
Translation	b	k/s	$m \cdot s$
Rotation	B	K/s	$J \cdot s$

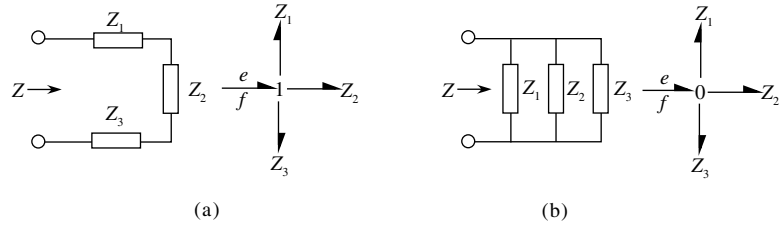

FIGURE 9.17 (a) Impedance of a series connection. (b) Admittance for a parallel combination.

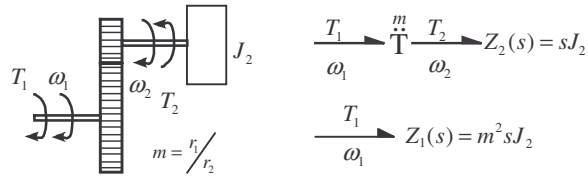

FIGURE 9.18 Rotational inertia attached to gear train, and corresponding model in impedance form. This example illustrates how a transformer can scale the gain of an impedance.

For the basic inertia element in rotation, for example, the basic rate law (see Table 9.5) is $\dot{h} = T$. In s-domain, $sh = T$. Using the linear constitutive relation, $h = J\omega$, so $sJ\omega = T$. We can observe that a rotation inertial impedance is defined by taking the ratio of effort to flow, or $T/\omega \equiv Z_I = sJ$. A similar exercise can be conducted for every basic element to construct Table 9.6.

Using the basic concept of a 0 junction and a 1 junction, which are the analogs of parallel and series circuit connections, respectively, basic impedance formulations can be derived for bond graphs in a way analogous to that done for circuits. Specifically, when impedances are connected in series, the total impedance is the sum, while admittances connected in parallel sum to give a total admittance. These basic relations are illustrated in Fig. 9.17, for which

$$Z = \underbrace{Z_1 + Z_2 + \cdots + Z_n}_{\substack{n \text{ impedances in series sum} \\ \text{to form a total impedance}}}, \qquad Y = \underbrace{Y_1 + Y_2 + \cdots + Y_n}_{\substack{n \text{ admittances in parallel sum} \\ \text{to form a total admittance}}} \tag{9.2}$$

Impedance relations are useful when constructing transfer functions of a system, as these can be developed directly from a circuit analog or bond graph. The transformer and gyrator elements can also be introduced in these models. A device that can be modeled with a transformer and gyrator will exhibit impedance-scaling capabilities, with the moduli serving a principal role in adjusting how an impedance attached to one "side" of the device appears when "viewed" from the other side. For example, for a device having an impedance Z_2 attached on port 2, the impedance as viewed from port 1 is derived as

$$Z_1 = \frac{e_1}{f_1} = \left[\frac{e_1}{e_2}\right]\left[\frac{e_2}{f_2}\right]\left[\frac{f_2}{f_1}\right] = [m][Z_2(s)][m] = m^2 Z_2(s) \tag{9.3}$$

This concept is illustrated by the gear-train system in Fig. 9.18. A rotational inertia is attached to the output shaft of the gear pair, which can be modeled as a transformer (losses, and other factors ignored here).

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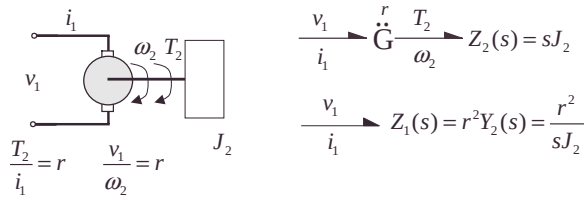

FIGURE 9.19 Rotational inertial attached to a basic rotational machine modeled as a simple gyrator. This example illustrates how a gyrator can scale the gain but also convert the impedance to an admittance form.

The impedance of the inertial is $Z_2 = sJ_2$, where J_2 is the mass moment of inertia. The gear train has an impedance-scaling capability, which can be designed through selection of the gear ratio, m.

The impedance change possible with a transformer is only in gain. The gyrator can affect gain and in addition can change the impedance into an admittance. Recall the basic gyrator relation, $e_1 = rf_2$ and $e_2 = rf_1$, then for a similar case as before,

$$\begin{matrix}1\\ \cdot\\ 1\end{matrix} = \left[\frac{e_1}{f_2}\right]\left[\frac{f_2}{e_2}\right]\left[\frac{e_2}{f_1}\right] = [r][Y_2(s)][r] = r^2 Y_2(s \tag{9.4}$$

This functional capability of gyrators helps identify basic motor-generator designs as integral parts of a flywheel battery system. A very simplified demonstration is shown in Fig. 9.19, where a flywheel (rotational inertia) is attached to the mechanical port of a basic electromechanical gyrator. When viewed from the electrical port, you can see that the gyrator makes the inertia "look" like a potential energy storing device, since the impedance goes as $1/(sC)$, like a capacitive element, although here C is a mechanical inertia.

9.4 Physical Laws for Model Formulation

This section will illustrate basic equation formulation for systems ranging in complexity from mass-spring-damper models to slightly more complex models, showing how to interface with nonmechanical models.

Previous sections of this chapter provide descriptions of basic elements useful in modeling mechanical systems, with an emphasis on a dynamic system approach. The power and energy basis of a bond graph approach makes these formulations consistent with models of systems from other energy domains. An additional benefit of using a bond graph approach is that a systematic method for causality assignment is available. Together with the physical laws, causal assignment provides insight into how to develop computational models. Even without formulating equations, causality turns out to be a useful tool.

Kinematic and Dynamic Laws

The use of basic kinematic and dynamic equations imposes a structure on the models we build to represent mechanical translation and rotation. Dynamic equations are derived from Newton's laws, and we build free-body diagrams to understand how forces are imposed on mechanical systems. In addition, we must use geometric aspects of a system to develop kinematic equations, relying on properly defined coordinate systems. If the goal is to analyze a mechanical system alone, typically the classical application of conservation of momentum or energy methods and/or the use of kinematic analysis is required to arrive at solutions to a given problem. In a mechatronic system, it is implied that a mechanical system is coupled to other types of systems (hydraulics, electromechanical devices, etc.). Hence, we focus here on how to build models that will be easily integrated into overall system models. A detailed classical discussion of kinematics and dynamics from a fundamental perspective can be found in many introductory texts such as Meriam and Kraige [23] and Bedford and Fowler [5], or in more advanced treatments by Goldstein [11] and Greenwood [12].

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When modeling simple translational systems or fixed-axis rotational systems, the basic set of laws summarized below are sufficient to build the necessary mathematical models.

Basic Dynamic and Kinematic Laws

System	Dynamics	Kinematics
Translational	$\sum_i^N F_i = 0$	$\sum_i^N V_i = 0$
Rotational	$\sum_i^N T_i = 0$	$\sum_i^N \omega_i = 0$
Junction type	1-junction	0-junction

There is a large class of mechanical systems that can be represented using these basic equations, and in this form it is possible to see how: (a) bond graph junction elements can be used to structure these models and (b) how these equations support circuit analog equations, since they are very similar to the Kirchhoff circuit laws for voltage and current. We present here the bond graph approach, which graphically communicates these physical laws through the 0- and 1-junction elements.

Identifying and Representing Motion in a Bond Graph

It is helpful when studying a mechanical system to focus on identifying points in the system that have distinct velocities (V or ω). One simply can associate a 1-junction with these points. Once this is done, it becomes easier to identify connection points for other mechanical components (masses, springs, dampers, etc.) as well as points for attaching actuators or sensors. Further, it is critical to identify and to define additional velocities associated with relative motion. These may not have clear, physically identifiable points in a system, but it is necessary to localize these in order to attach components that rely on relative motion to describe their operation (e.g., suspensions).

Figure 9.20 shows how identifying velocities of interest can help identify 1-junctions at which mechanical components can be attached. For the basic mass element in part (a), the underlying premise is that a component of a system under study is idealized as a pure translational mass for which momentum and velocity are related through a constitutive relation. What this implies is that the velocity of the mass is the same throughout this element, so a 1-junction is used to identify this distinct motion. A bond attached to this 1-junction represents how any power flowing into this junction can flow into a kinetic energy storing element, **I**, which represents the mass, m. Note that the force on the bond is equal to the rate of change of momentum, $\dot{p}$, where $p = mV$.

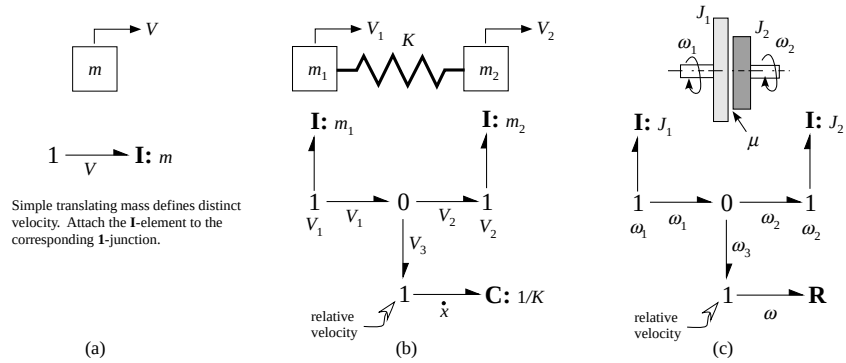

FIGURE 9.20 Identifying velocities in a mechanical system can help identify correct interconnection of components and devices: (a) basic translating mass, (b) basic two-degree of freedom system, (c) rotational frictional coupling between two rotational inertias.

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The two examples in Figs. 9.20(b) and 9.20(c) demonstrate how a relative velocity can be formed. Two masses each identify the two distinct velocity points in these systems. Using a 0-junction allows construction of a *velocity difference*, and in each case this forms a relative velocity. In each case the relative velocity is represented by a 1-junction, and it is critical to identify that this 1-junction is essentially an attachment point for a basic mechanical modeling element.

Assigning and Using Causality

Bond graphs describe how modeling decisions have been made, and how model elements (**R**, **C**, etc.) are interconnected. A power bond represents power flow, and assigning power convention using a half-arrow is an essential part of making the graph useful for modeling. A sign convention is essential for expressing the algebraic summation of effort and flow variables at 0- and 1-junctions. Power is generally assigned positive sense flowing into passive elements (resistive, capacitive, inertive), and it is usually safe to always adopt this convention. Sign convention requires consistent and careful consideration of the reference conditions, and sometimes there may be some arbitrariness, not unlike the definition of reference directions in a free-body diagram.

Causality involves an augmentation of the bond graph, but is strictly independent of power flow convention. As discussed earlier, an assignment is made on each bond that indicates the input–output relationship of the effort-flow variables. The assignment of causality follows a very consistent set of rules. A system model that has been successfully assigned causality on all bonds essentially communicates solvability of the underlying mathematical equations. To understand where this comes from, we can begin by examining the contents of Tables 9.4 and 9.5. These tables refer to the *integral* form of the energy storage elements. An energy storage element is in integral form if it has been assigned integral causality. Integral causality implies that the causal input variable (effort or flow) leads to a condition in which the state of the energy stored in that element can be determined only by *integrating* the fundamental rate law. As shown in Table 9.7, integral causality for an **I** element implies effort is the input, whereas integral causality for the **C** element implies flow is the input.

TABLE 9.7 Table Summarizing Causality for Energy Storage Elements

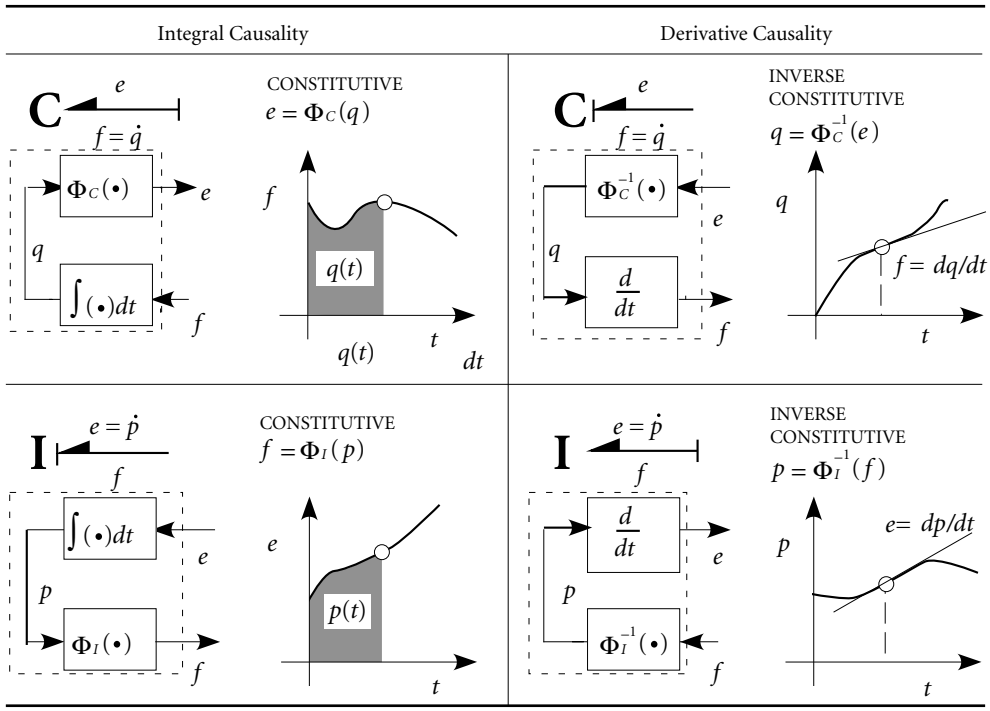

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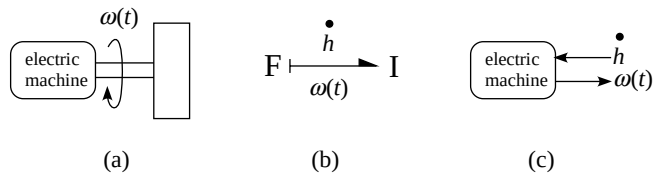
TABLE 9.8 Table of Causality Assignment Guidelines

Sources	Junctions	Ideal Coupling Elements
E $e(t)$	**0** Only one bond specifies effort.	e_1 **T** e_2 (f_1, f_2; $e_1 = me_2$, $mf_1 = f_2$) e_1 **G** e_2 (f_1, f_2; $e_1 = rf_2$, $e_2 = rf_1$)
F $f(t)$	**1** Only one bond specifies flow.	e_1 **T** e_2 (f_1, f_2) e_1 **G** e_2 (f_1, f_2)



(a) (b) (c)

FIGURE 9.21 Driving a rotational inertia with a velocity source: (b) simple bond graph with causality, (c) explanation of back effect.

As shown in this table, the alternative causality for each element leads to *derivative causality*, a condition in which the state of the energy storage element is known instantaneously and as such is said to be *dependent* on the input variable, and is in a state of dependent causality. The implication is that energy storage elements in integral causality require one differential equation (the rate law) to be solved in order to determine the value of the *state variable* (p or q). Energy storage elements in derivative causality don't require a differential equation; however, they still make their presence known through the back reaction implied. For example, if an electric machine shown in Fig. 9.21(a) is assumed to drive a rotational inertial with a known velocity, ω, then the inertia is in derivative causality. There will also be losses, but the problem is simplified to demonstrate the causal implications. The energy is always known since, $h = J\omega$, so $T_h = h^2/2J$. However, the machine will feel an inertial back torque, $\dot{h}$, whenever a change is made to ω. This effect cannot be neglected.

Causality assignment on some of the other modeling elements is very specific, as shown in Table 9.8. For example, for sources of effort or flow, the causality is implied. On the two-port transformer and gyrator, there are two possible causality arrangements for each. Finally, for 0- and 1-junctions, the causality is also very specific since in each case only one bond can specify the effort or flow at each.

With all the guidelines established, a basic causality assignment procedure can be followed that will make sure all bonds are assigned causality (see also Rosenberg and Karnopp [32] and Karnopp, Margolis, and Rosenberg [17]).

1. For a given system, assign causality to any effort or flow sources, and for each one assign the causality as required through 0- and 1-junctions and transformer and gyrator elements. The causality should be spread through the model until a point is reached where no assignment is implied. Repeat this procedure until all sources have been assigned causality.
2. Assign causality to any **C** or **I** element, trying to assign integral causality if possible. For each assignment, propagate the causality through the system as required. Repeat this procedure until all storage elements are assigned causality.

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3. Make any final assignments on **R** elements that have not had their causality assigned through steps 1 and 2, and again propagate causality as required. Any arbitrary assignment on an **R** element will indicate need for solving an algebraic equation.
4. Assign any remaining bonds arbitrarily, propagating each case as necessary.

Causality can provide information about system operation. In this sense, the bond graph provides a picture of how inputs to a system lead to certain outputs. The use of causality with a bond graph replaces ad hoc assignment of causal notions in a system. This type of information is also useful for understanding how a system can be split up into modules for simulation and/or it can confirm the actual physical boundaries of components.

Completing the assignment of causality on a bond graph will also reveal information about the solvability of the system model. The following are key results from causality assignment.

- Causality assignment will reveal the order of the system, which is equal to the number of independent energy storage elements (i.e., those with integral causality). The state variable (p or q) for any such element will be a state of the system, and one first-order differential equation will be required to describe how this state propagates through time.
- Any arbitrary assignment of causality on an **R** element indicates there is an algebraic loop. The number of arbitrary assignments can be related to the number of algebraic equations required in the model.

Developing a Mathematical Model

Mathematical models for lumped-parameter mechanical systems will take the form of coupled ordinary differential equations or, for a linear or linearized system, transfer functions between variables of interest and system inputs. The form of the mathematical model should match the application, and one can readily convert between the different forms. A classical approach to developing the mathematical model will involve applying Newton's second law directly to each body, taking account of the forces and torques. Commonly, the result is a second-order ordinary differential equation for each body in a system. An alternative is to use Lagrange's equations, and for multidimensional dynamics, where bodies may have combined translation and rotation, additional considerations are required as will be discussed in Section 9.6. At this point, consider those systems where a given body is either under translation or rotation.

Mass-Spring-Damper: Classical Approach

A basic mechanical system that consists of a rigid body that can translate in the z-direction is shown in Fig. 9.22(a). The system is modeled using a mass, a spring, and a damper, and a force, $F(t)$, is applied

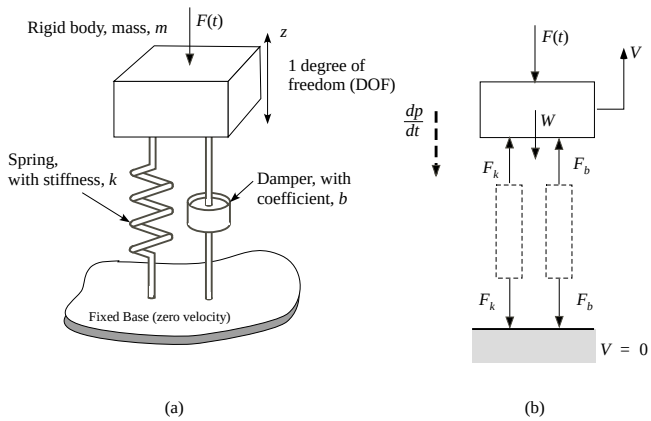

FIGURE 9.22 Basic mass-spring-damper system: (a) schematic, (b) free-body diagram.

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directly to the mass. A free-body diagram in part (b) shows the forces exerted on the system. The spring and damper exert forces F_k and F_b on the mass, and these same forces are also exerted on the fixed base since the spring and damper are assumed to be massless. A component of the weight, W, resolved along the axis of motion is included. The sum of applied forces is then, $\sum F = F(t) + W - F_k - F_b$. The dashed arrow indicates the "inertial force" which is equal to the rate of change of the momentum in the z-direction, p_z, or, $dp_z/dt = \dot{p}_z = m\dot{V}_z$. This term is commonly used in a D'Alembert formulation, one can think of this force as opposing or resisting the effect of applied forces to accelerate the body. It is common to use the inertial force as an "applied force," especially when performing basic analysis (e.g., see Chapter 3 or 6 of [23]).

Newton's second law relates rate of change of momentum to applied forces, $\dot{p} = \sum F$, so, $\dot{p}_z = F(t) + W - F_k - F_b$. To derive a mathematical model, form a basic coordinate system with the z-axis positive upward. Recall the constitutive relations for each of the modeling elements, assumed here to be linear, $p_z = mV_z$, $F_k = kz_k$, and $F_b = bV_b$. In each of these elements, the associated velocity, V, or displacement, z, must be identified. The mass has a velocity, $V_z = \dot{z}$, relative to the inertial reference frame. The spring and damper have the same *relative* velocity since one end of each component is attached to the mass and the other to the base. The change in the spring length is z and the velocity is $\dot{z} - V_{\text{base}}$. However, $V_{\text{base}} = 0$ since the base is fixed, so putting this all together with Newton's second law, $m\ddot{z} = F(t) + W - kz - b\dot{z}$. A second order ordinary differential equation (ODE) is derived for this single degree of freedom (DOF) system as

$$m\ddot{z} + b\dot{z} + kz = F(t) + W$$

In this particular example, if W is left off, z is the "oscillation" about a position established by static equilibrium, $z_{\text{static}} = W/k$.

If a transfer function is desired, a simple Laplace transform leads to (assuming zero initial conditions for motion about z_{static})

$$\frac{Z(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

The simple mass-spring-damper example illustrates that models can be readily derived for mechanical systems with direct application of kinematics and Newton's laws. As systems become more complex either due to number of bodies and geometry, or due to interaction between many types of systems (hydraulic, electromechanical, etc.), it is helpful to employ tools that have been developed to facilitate model development. In a subsequent section, multibody problems and methods of analysis are briefly discussed. It has often been argued that the utility of bond graphs can only be seen when a very complex, multi-energetic system is analyzed. This need not be true, since a system (or mechatronics) analyst can see that a consistent formulation and efficacy of causality are very helpful in analyzing many different types of physical systems. This should be kept in mind, as these basic bond graph methods are used to re-examine the simple mass-spring-damper system.

Mass-Spring-Damper: Bond Graph Approach

Figure 9.23 illustrates the development of a bond graph model for a mass-spring-damper system. In part (a), the distinct velocity points are identified and 1-junctions are used to represent them on a bond graph. Even though the base has zero velocity, and there will be no power flow into or out of that point, it is useful to identify it at this point. A relative velocity is formed using a 0-junction, and note that all bonds have sign convention applied, so at the 0-junction, $V_{\text{mass}} - V_{\text{relative}} - V_{\text{base}} = 0$, which gives, $V_{\text{relative}} = V_{\text{mass}} - V_{\text{base}}$ as required.

The model elements needed to represent the system are connected to the 1-junctions, as shown in Fig. 9.23(b). Two sources are required, one to represent the applied force (effort, S_e) due to weight, and a second to represent the fixed based velocity (a flow source, S_f). The flow source is directly attached to

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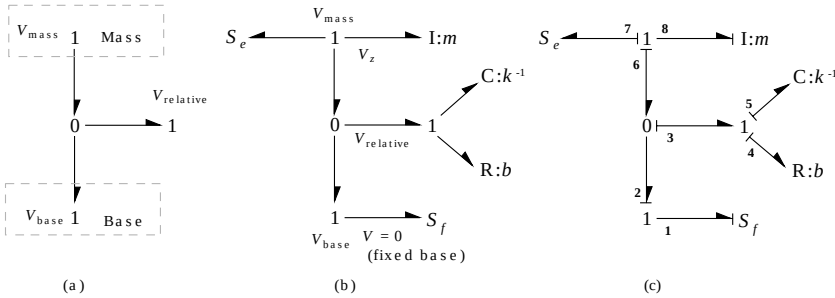
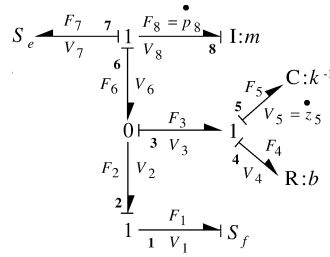


FIGURE 9.23 Basic mass-spring-damper system: (a) identifying velocity 1-junctions, (b) attaching model elements, (c) assignment of causality.



$\mathbf{C}_5$ rate law:	$\dot{z}_5$	=	V_5
			$V_5 = V_3$
			$V_3 = V_6 - V_2$
			$V_6 = V_8 = p_8/m_8$*
			$V_2 = V_1 = 0$*
$\mathbf{I}_8$ rate law:	$\dot{p}_8$	=	$-F_7 - F_6$
			$F_7 = W$*; $F_6 = F_3$
			$F_3 = F_4 + F_5$
			$F_4 = bV_4 = bV_3$**
			$F_5 = k_5 z_5$*

FIGURE 9.24 Equation derivation for mass-spring-damper. The '*' indicates these relations are reduced to functions of state or input. A '**' shows an intermediate variable has been reached that has elsewhere been reduced to '*'.

the 1-junction (the extra bond could be eliminated). An **I** element represents mass, a **C** represents the spring, and an **R** represents the losses in the damper. Note how the mass and the source of effort are attached to the 1-junction representing the mass velocity (the weight is always applied at that velocity). The spring and damper are attached via a power bond to the relative velocity between the mass and base.

Finally, in Fig. 9.23(c) the eight bonds are labeled and causality is assigned. First, the fixed base source fixes the causality on bond 1, specifying the velocity at the 1-junction, and thus constraining the causality of bond 2 to have effort into the 1-junction. Since bond 2 did not specify effort into the 0-junction, causality assignment should proceed to other sources, and the effort source fixes causality on bond 7. This bond does not specify the flow at the adjoining 1-junction, so at this point we could look for other specified sources. Since there are none, we assign causality to any energy-storing elements which have a preferred integral causality. The bond 8 is assigned to give the **I** element integral causality (see Table 9.7), which then specifies the velocity at the 1-junction and thus constrains bond 6. At this point, bonds 6 and 2 both specify flow into the 0-junction, so the remaining bond 3 must specify the effort. This works out well because now bond 3 specifies flow into the remaining 1-junction (the relative velocity), which specifies velocity into the **C** and **R** elements. For the **C** element, this gives integral causality.

In summary, the causality is assigned and there are no causal conflicts (e.g., two bonds trying to specify velocity into a 1-junction). Both energy-storing elements have integral causality. This indicates that the states for the **I** (mass) and **C** (spring) will contribute to the state variables of the system. This procedure assures a minimum-size state vector, which in this case is of order 2 (a 2nd-order system). Figure 9.24 shows a fully annotated bond graph, with force-velocity variables labeling each bond. The state for an **I** element is a momentum, in this case the translational momentum of the mass, p_8. For a **C** element, a

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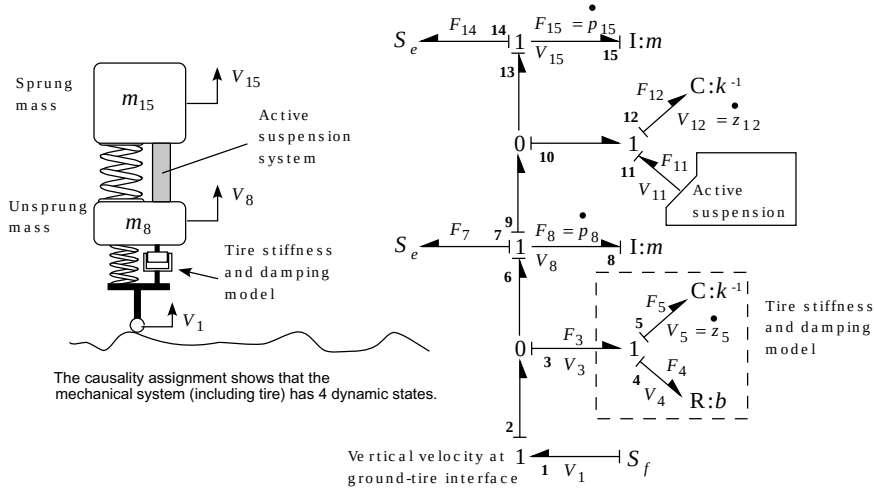

FIGURE 9.25 Example of model for vertical vibration in a quarter-car suspension model with an active suspension element. This example builds on the simple mass-spring-damper model, and shows how to integrate an actuator into a bond graph model structure.

displacement variable is the state z_5, which here represents the change in length of the spring. The state vector is $\mathbf{x}^T = [p_8, z_5]$.

A mathematical model can be derived by referring to this bond graph, focusing on the independent energy storage elements. The **rate law** (see Tables 9.4 and 9.5) for each energy storage element in integral causality constitutes one first-order ordinary differential **state equation** for this system. In order to formulate these equations, the right-hand side of each rate law must be a function only of states or inputs to the system. The process is summarized in the table of Fig. 9.24. Note that the example assumes linear constitutive relations for the elements, but it is clear in this process that this is not necessary. Of course, in some cases nonlinearity complicates the analysis as well as the modeling process in other ways.

Quarter-car Active Suspension: Bond Graph Approach

The simple mass-spring-damper system forms a basis for building more complex models. A model for the vertical vibration of a quarter-car suspension is shown in Fig. 9.25. The bond graph model illustrates the use of the mass-spring-damper model, although there are some changes required. In this case, the base is now moving with a velocity equal to the vertical velocity of the ground-tire interface (this requires knowledge of the terrain height over distance traveled as well as the longitudinal velocity of the vehicle). The power direction has changed on many of the bonds, with many now showing positive power flowing from the ground up into the suspension system.

The active suspension system is isolated to further illustrate how bond graph modeling promotes a modular approach to the study of complex systems. Most relevant is that the model identifies the required causal relation at the interface with the active suspension, specifying that the relative velocity is a causal input, and force is a causal output of the active suspension system. The active force is exerted in an equal and opposite fashion onto the sprung and unsprung mass elements.

The causality assignment identifies four states (two momentum states and two spring displacement states). Four first-order state equations can be derived using the rate laws of each of the independent energy-storing elements ($\mathbf{C}_5$, $\mathbf{I}_8$, $\mathbf{C}_{12}$, $\mathbf{I}_{15}$). At this point, depending on the goals of the analysis, either the nonlinear equations could be derived (which might include an active suspension force that depends on the velocity input), or a linearized model could be developed and impedance methods applied to derive a transfer function directly.

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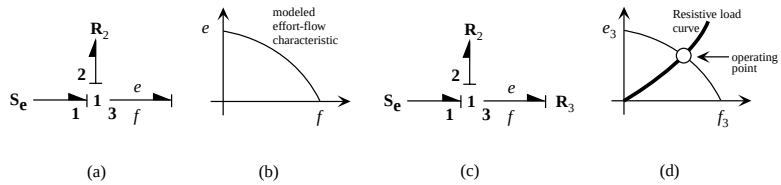

FIGURE 9.26 Algebraic loop in a simple source-load model.

Note on Some Difficulties in Deriving Equations

There are two common situations that can lead to difficulties in the mathematical model development. These issues will arise with any method, and is not specific to bond graphs. Both lead to a situation that may require additional algebraic manipulation in the equation derivation, and it may not be possible to accomplish this in closed form. There are also some ways to change the model in order to eliminate these problems, but this could introduce additional problems. The two issues are (1) derivative causality, and (2) algebraic loops. Both of these can be detected during causality assignment, so that a problem can be detected before too much time has been spent.

The occurence of derivative causality can be described in bond graph terms using Table 9.7. The issue is one in which the state of an energy-storing element (**I** or **C**) is dependent on the system to which it is attached. This might not seem like a problem, particularly since this implies that no differential equation need be solved to find the state. It is necessary to see that there is still a need to compute the back-effect that the system will feel in forcing the element into a given state. For example, if a mass is to be driven by a velocity, V, then it is clear that we know the energy state, $p = mV$, so all is known. However, there is an inertial force computed as $\dot{p} = m\dot{V} = ma$. Many times, it is possible to resolve this problem by performing the algebraic manipulations required to include the effect of this element (difficulty depends on complexity of the system). Sometimes, these dependent states arise because the system is not modeled in sufficient detail, and by inserting a compliance between two gears, for example, the dependence is removed. This might solve the problem, costing only the introduction of an additional state. A more serious drawback to this approach would occur if the compliance was actually very small, so that numerical stiffness problems are introduced (with modern numerical solver routines, even this problem can be tolerated). Yet another way to resolve the problem of derivative causality in mechanical systems is to employ a Lagrangian approach for mechanical system modeling. This will be discussed in section 9.7.

Another difficulty that can arise in developing solvable systems of equations is the presence of an algebraic loop. Algebraic loops are relatively easy to generate, especially in a block diagram modeling environment. Indeed, it is often the case that algebraic loops arise because of modeling decisions, and in this way a bond graph's causality provides quick feedback regarding the system solvability. Algebraic loops imply that there is an arbitrary way to make computations in the model, and in this way they reveal themselves when an arbitrary decision must be made in assigning causality to an **R** element.[3]

As an example, consider the basic model of a Thevenin source in Fig. 9.26(a). This model uses an effort source and a resistive element to model an effort-flow (steady-state) characteristic curve, such as a motor or engine torque-speed curve or a force-velocity curve for a linear actuator. A typical characteristic is shown in Fig. 9.26(b). When a resistive load is attached to this source as shown in Fig. 9.26(c), the model is purely algebraic. When the causality is assigned, note that after applying the effort causality on bond 1, there are two resistive elements remaining. The assignment of causality is arbitrary. The solution

[3]The arbitrary assignment on an **R** element is not unlike the arbitrariness in assigning integral or derivative causality to energy-storing elements. An "arbitrary" decision to assign integral causality on an energy-storing element leads to a requirement that we solve a *differential* equation to find a state of interest. In the algebraic loop, a similar arbitary decision to assign a given causality on an **R** element implies that at least one *algebraic* equation must be solved along with any other system equations. In other words, the system is described by differential algebraic equations (DAEs).

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requires analytically solving algebraic relations for the operating point, or by using a graphical approach as shown in Fig. 9.26(d).

This is a simple example indicating how algebraic loops are detected with a bond graph, and how the solution requires solving algebraic relations. In complex systems, this might be difficult to achieve. Sometimes it is possible to introduce or eliminate elements that are "parasitic," meaning they normally would be neglected due to their relatively small effect. However, such elements can relieve the causal bind. While this might resolve the problem, as in the case of derivative causality there are cases where such a course could introduce numerical stiffness problems. Sometimes a solution is reached by using energy methods to resolve some of these problems, as shown in the next section.

9.5 Energy Methods for Mechanical System Model Formulation

This section describes methods for using energy functions to describe basic energy-storing elements in mechanical systems, as well as a way to describe collections of energy-storing elements in multiport fields. Energy methods can be used to simplify model development, providing the means for deriving constitutive relations, and also as a basis for eliminating dependent energy storage (see last section). The introduction of these methods provides a basis for introducing the Lagrange equations in section 9.7 as a primary approach for system equation derivation or in combination with the bond graph formulation.

Multiport Models

The energy-storing and resistive models introduced in section 9.3 were summarized in Tables 9.2, 9.4, and 9.5 as multiport elements. In this section, we review how multiport elements can be used in modeling mechanical systems, and outline methods for deriving the constitutive relations. Naturally, these methods apply to the single-port elements as well.

An example of a **C** element with two-ports was shown in Fig. 9.12 as a model for a cantilevered beam that can have both translational and rotational deflections at its tip. A 2-port is required in this model because there are two independent ways to store potential energy in the beam. A distinguishing feature in this example is that the model is based on relationships between efforts and displacement variables (for this case of a capacitive element). Multiport model elements developed in this way are categorized as explicit fields to distinguish them from implicit fields [17]. Implicit fields are formed by assembling energy-storing 1-port elements with junction structure (i.e., 1, 0, and TF elements) to form multiport models.

Explicit fields are often derived using physical laws directly, relying on an understanding of how the geometric and material properties affect the basic constitutive relation between physical variables. Geometry and material properties always govern the parametric basis of all constitutive relations, and for some cases these properties may themselves be functions of state. Indeed, these cases require the multiport description, which finds extensive use in modeling of many practical devices, especially sensors and actuators. Multiport models should follow a strict energetic basis, as described in the following.

Restrictions on Constitutive Relations

Energy-storing multiports must follow two basic restrictions, which are also useful in guiding the derivation of energetically-correct constitutive relations. The definition of the energy-storing descriptions summarized in Tables 9.4 and 9.5 specifies that there exists an energy state function, $E = E(\mathbf{x})$, where $\mathbf{x}$ is either a generalized displacement, $\mathbf{q}$, for capacitive (**C**) elements or a generalized momentum, $\mathbf{p}$, for inertive (**I**) elements. For the multiport energy-storing element, the specification requires the following specifications [2,3].

1. There exists a rate law, $\dot{x}_i = u_i$, where u_i as input specifies integral causality on port i.
2. The energy stored in a multiport is determined by

$$E(\mathbf{x}) = \int \sum_{i=1}^{n} y_i \, d\mathbf{x}_i \tag{9.5}$$

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3. A first restriction on a multiport constitutive relation requires that the causal output at any port is given by

$$y_i = \Phi_{si}(\mathbf{x}) = \frac{\partial E(\mathbf{x})}{\partial x_i} \tag{9.6}$$

where $\Phi_{si}()$ is a single-valued function.

4. A second restriction on a multiport constitutive relation requires that the constitutive relations obey Maxwell reciprocity, or

$$\frac{\partial y_i}{\partial x_j} = \frac{\partial^2 E(\mathbf{x})}{\partial x_j \partial x_i} = \frac{\partial y_j}{\partial x_i} \tag{9.7}$$

Deriving Constitutive Relations

The first restriction on the constitutive relations, Eq. (9.6), establishes how constitutive relations can be derived for a multiport if an energy function can be formulated. This restriction forms the basis for a method used in many practical applications to find constitutive relationships from energy functions (e.g., strain-energy, electromechanics, etc.). In these methods, it is assumed that at least one of the constitutive relations for an energy-storing multiport is given. Then, the energy function is formed using Eq. (9.5) where, after interchanging the integral and sum,

$$\mathbf{x}) = \sum_{i=1}^{n} \int y_i \, dx_i = \int y_1 dx_1 + \circ + \int y_n d$$ (9.8)

Presume that y_1 is a known function of the states, $y_1 = \Phi_{si}(\mathbf{x})$. Since the element is conservative, any energetic state can be reached via a convenient path where $dx_i = 0$ for all i except $i = 1$. This allows the determination of $E(\mathbf{x})$.

To illustrate, consider the simple case of a rack and pinion system, shown in Fig. 9.27. The pinion has rotational inertia, J, about its axis of rotation, and the rack has mass, m. The kinetic co-energy is easily formulated here, considering that the pinion angular velocity, ω, and the rack velocity, V, are constrained by the relationship $V = R\omega$, where R is the pinion base radius. If this basic subsystem is modeled directly, it will be found that one of the inertia elements (pinion, rack) will be in derivative causality. Say, it is desired to connect to this system through the rotational port, T - ω. To form a single-port **I** element that includes the rack, form the kinetic co-energy as $T = T(\omega, V) = J\omega^2/2 + mV^2/2$. Use the constraint relation to write, $T = T(\omega) = (J + mR^2)\omega^2/2$. To find the constitutive relation for this 1-port rotational **I** element, let $h = \partial T(\omega)/\partial\omega = (J + mR^2)\omega$, where we can now define an equivalent rotational inertia as $J_{eq} = J + mR^2$.

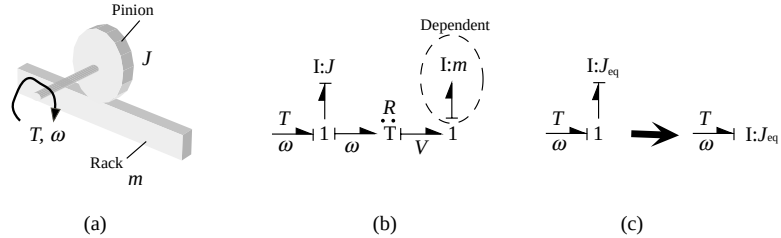

FIGURE 9.27 (a) Rack and pinion subsystem with torque input. (b) Direct model, showing dependent mass. (c) Equivalent model, derived using energy principles.

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The rack and pinion example illustrates a basic method for relieving derivative causality, which can be used to build basic energy-storing element models. Some problems might arise when the kinetic co-energy depends on system configuration. In such a case, a more systematic method employing Lagrange's equations may be more suitable (see Section 9.7).

The approach described here for deriving constitutive relations is similar to Castigliano's theorom [6,9]. Castigliano's theorem relies on formulation of a strain-energy function in terms of the forces or moments, and as such employs a potential co-energy function. Specifically, the results lead to displacements (translational, rotational) as functions of efforts (forces, torques). As in the case above, these functions are found by taking partial derivatives of the co-energy with respect to force or moment. Castigliano's theorem is especially well-suited for finding force-displacment functions for curved and angled beam structures (see [6]).

Formulations using energy functions to derive constitutive relations are found in other application areas, and some references include Lyshevski [21] for electromechanics, and Karnopp, Margolis, and Rosenberg [17] for examples and applications in the context of bond graph modeling.

Checking the Constitutive Relations

The second restriction on the constitutive relations, Eq. (9.7), provides a basis for testing or checking if the relationships are correct. This is a reciprocity condition that provides a check for energy conservation in the energy-storing element model, and a quick check for linear mechanical systems shows that either the inertia or stiffness matrix must be symmetrical.

Recall the example of the 2-port cantilevered beam, shown again in Fig. 9.12. For small deflections, the total tip translational and angular deflections due to a tip force and torque can be added (using flexibility influence coefficients), which can be expressed in matrix form,

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} \frac{1}{3}l^3 & \frac{1}{2}l^2 \\ \frac{1}{2}l^2 & l \end{bmatrix} \begin{bmatrix} F \\ T \end{bmatrix} = \mathbf{C} \begin{bmatrix} F \\ T \end{bmatrix} = \mathbf{K}^{-1} \begin{bmatrix} F \\ T \end{bmatrix}$$

where **C** and **K** are the compliance and stiffness matrices, respectively. This constitutive relation satisfies the Maxwell reciprocity since, $\partial x/\partial T = \partial\theta/\partial F$. This 2-port **C** element is used to model the system shown in Fig. 9.28(a), which consists of a bar-bell rigidly attached to the tip of the beam. Under small deflection, a bond graph shown in Fig. 9.28(b) is assembled. Causality applied to this system reveals that each port of the 2-port **C** element has integral causality. On a multiport energy storing element, each port is independently assigned causality following the same rules as for 1-ports. It is possible that a multiport could have a mixed causality, where some of the ports are in derivative causality. If a multiport has mixed causality, part of the state equations will have to be inverted. This algebraic difficulty is best avoided by trying to assign integral causality to all multiport elements in a system model if possible.

In the present example, causality assignment on the **I** elements is also integral. In all, there are four independent energy-storing elements, so there are four state variables, $\mathbf{x} = [x, \theta, p, h]^\top$. Four state equations can be derived using the rate laws indicated in Fig. 9.28.

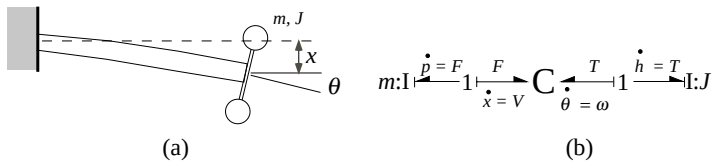

FIGURE 9.28 Model of beam rigidly supporting a bar- or dumb-bell: (a) schematic, (b) bond graph model using a 2-port **C** to represent beam. Dumb-bell is represented by translational mass, *m*, and rotational inertia, *J*.

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9.6 Rigid Body Multidimensional Dynamics

The modeling of bodies in mechanical systems presumes adoption of a "rigid body" that can involve rotation as well as translation, and in this case the dynamic properties are more complex than those for a point mass. In earlier sections of this chapter, a simple rigid body has already been introduced, and it is especially useful for a large class of problems with rotation about a single fixed axis.

In the rigid body, the distance between any two elements of mass within a body is a constant. In some cases, it is convenient to consider a continuous distribution of mass while in others a system of discrete mass particles rigidly fixed together helps conceptualize the problem. In the latter, the rigid body properties can be found by summing over all the discrete particles, while in the continuous mass concept an integral formulation is used. Either way, basic concepts can be formulated and relations derived for use in rigid body dynamic analysis. Finally, the modeling in most engineering systems is restricted to classical Newtonian mechanics, where the linear velocity–momentum relation holds (so energy and coenergy are equal).

Kinematics of a Rigid Body

In this section, a brief overview is given of three-dimensional motion calculations for a rigid body. The focus here is to present methods for analyzing rotation of a rigid body about a fixed axis and methods for analyzing relative motion of a rigid body using translating and rotating axes. These concepts introduce the basis for understanding more complex formulations. While vector descriptions (denoted using an arrow over the symbol, $\vec{a}$) are useful for understanding basic problems, more complex multibody systems usually adopt a matrix formulation. The presentation here is brief and included for reference. A more extensive discussion and examples can be found in introductory dynamics textbooks (e.g., [23]), where a separate discussion is usually given on the special case of plane motion.

Rotation of a Body About a Fixed Point

Basic concepts are introduced here in relation to rotation of a rigid body about a fixed point. This basic motion specifies that any point on the body lies on the surface of a sphere with a radius centered at the fixed point. The body can be said to have spherical motion.

Euler's Theorem. Euler's theorem states that any displacement of a body in spherical motion can be expressed as a rotation about a line that passes through the center of the spherical motion. This axis can be referred to as the orientational axis of rotation [26]. For example, two rotations about different axes passing through a fixed point of rotation are equivalent to a single resultant rotation about an axis passing through that point.

Finite Rotations. If the rotations used in Euler's theorem are finite, the order of application is important because finite rotations do not obey the law of vector addition.

Infinitesimal Rotations. Infinitesimally small rotations can be added vectorially in any manner, and these are generally considered when defining rigid body motions.

Angular Velocity. A body subjected to rotation $d\vec{\theta}$ about a fixed point will have an angular velocity $\vec{w}$ defined by the time derivative $d\vec{\theta}/dt$, in a direction collinear with $d\vec{\theta}$. If the body is subjected to two component angular motions that define w_1 and w_2, then the body has a resultant angular velocity, $w = \vec{w}_1 + \vec{w}_2$.

Angular Acceleration. A body's angular acceleration is found from the time derivative of the angular velocity, $\vec{\alpha} = \vec{\omega}$, and in general the acceleration is not collinear with velocity.

Motion of Points in the Body. Given ω, the velocity of a point on the body is $\vec{v} = \vec{\omega} \times \vec{r}$, where $\vec{r}$ is a position vector to the point as measured relative to the fixed point of rotation. The acceleration of a point on the body is then, $\vec{\alpha} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$.

Relating Vector Time Derivatives in Coordinate Systems

It is often the case that we need to determine the time rate of change of a vector such as $\vec{A}$ in Fig. 9.29 relative to different coordinate systems. Specifically, it may be easier to determine $\vec{A}$ in x_a, y_a, z_a, but we

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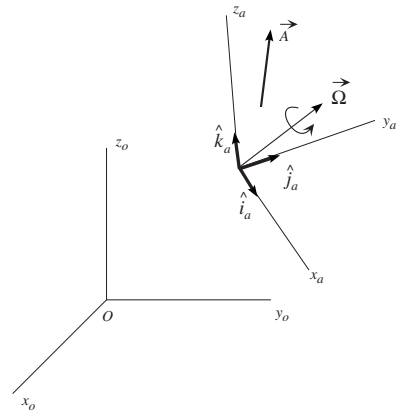

FIGURE 9.29 Often it is necessary to find the time derivative of vector $\vec{A}$ relative to a axes, x_o, y_o, z_o, given its value in the translating-rotating system x_a, y_a, z_a.

need to find its value in x_o, y_o, z_o. The vector $\vec{A}$ is expressed in the axes x_a, y_a, z_a using the unit vectors shown as

$$\vec{A} = A_x\hat{i}_a + A_y\hat{j}_a + A_z\hat{k}_a$$

To find the time rate of change, we identify that in the moving reference the time derivative of $\vec{A}$ is

$$\left(\frac{d\vec{A}}{dt}\right)_a = \frac{dA_x}{dt}\hat{i}_a + \frac{dA_y}{dt}\hat{j}_a + \frac{dA_z}{dt}\hat{k}_a$$

Relative to the x_o, y_o, z_o axes, the direction of the unit vectors $\hat{i}_a$, $\hat{j}_a$, and $\hat{k}_a$ change only due to rotation Ω, so,

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{A}}{dt}\right) + A_x\frac{d\hat{i}_a}{dt} + A_y\frac{d\hat{j}_a}{dt} + A_z\frac{d\hat{k}_a}{dt}$$

$$\frac{d\hat{i}_a}{dt} = \vec{\Omega}\times\hat{i}_a, \quad \frac{d\hat{j}_a}{dt} = \vec{\Omega}\times\hat{j}_a, \quad \frac{d\hat{k}_a}{dt} = \vec{\Omega}\times\hat{k}_a$$

then,

$$\frac{d\vec{A}}{dt} = \left(\frac{d\vec{A}}{dt}\right)_a + \vec{\Omega}\times\vec{A} \tag{9.9}$$

This relationship is very useful not only for calculating derivatives, as derived here, but also for formulating basic bond graph models. This is shown in the section titled "Rigid Body Dynamics."

Motion of a Body Relative to a Coordinate System

Translating Coordinate Axes

The origin of a set of axes x_a, y_a, z_a is fixed in a rigid body at A as shown in Fig. 9.30(a), and translates without rotation relative to the axes x_o, y_o, z_o with known velocity and acceleration. The rigid body is subjected to angular velocity $\vec{\omega}$ and angular acceleration $\vec{\alpha}$ in three dimensions.

Motion of Point *B* Relative to *A*. The motion of point B relative to A is the same as motion about a fixed point, so $\vec{v}_{B/A} = \vec{\omega}\times\vec{r}_{B/A}$, and $\vec{a}_{B/A} = \vec{\alpha}\times\vec{r}_{B/A} + \vec{\omega}\times(\vec{\omega}\times\vec{r}_{B/A})$.

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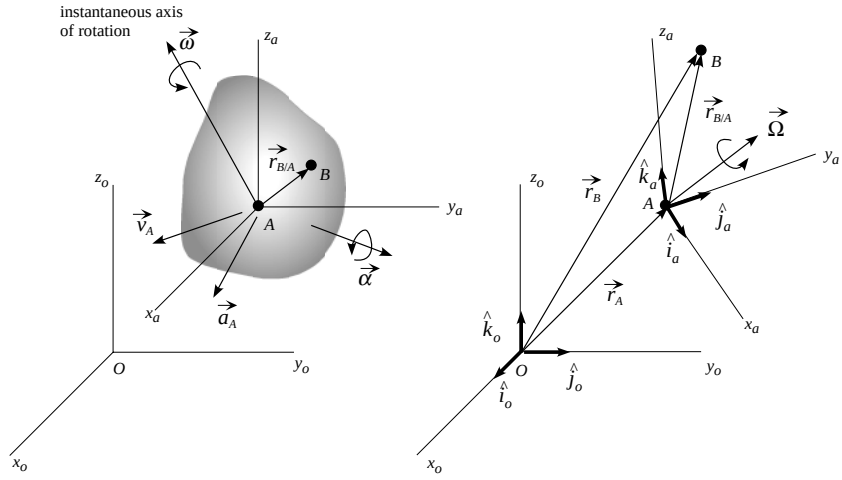

FIGURE 9.30 General rigid body motion: (a) rigid body with translating coordinate system, (b) translating and rotating coordinate system.

Motion of Point *B* Relative to *O*. For translating axes with no rotation, the velocity and acceleration of point B relative to system 0 is simply, $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ and $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ respectively, or,

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \tag{9.10}$$

$$\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}) \tag{9.11}$$

Translating and Rotating Coordinate Axes

A general way of describing the three-dimensional motion of a rigid body uses a set of axes that can translate and rotate relative to a second set of axes, as illustrated in Fig. 9.30(b). Position vectors specify the locations of points A and B on the body relative to x_o, y_o, z_o, and the axes x_a, y_a, z_a have angular velocity $\vec{\Omega}$ and angular acceleration $\dot{\vec{\Omega}}$. With the position of point B given by

$$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A} \tag{9.12}$$

the velocity and acceleration are found by direct differentiation as

$$\vec{v}_B = \vec{v}_A + \vec{\Omega} \times \vec{r}_{B/A} + (v_{B/A})_a \tag{9.13}$$

and

$$_B = \vec{a}_A + \dot{\vec{\Omega}} \times \vec{r}_{B/A} + \Omega \times (\Omega \times \vec{r}_{B/A}) + 2\Omega \times (v_{B/A})_a + (\vec{a}_{B/A}) \tag{9.14}$$

where $(v_{B/A})_a$ and $(a_{B/A})_a$ are the velocity and acceleration, respectively, of B relative to A in the x_a, y_a, z_a coordinate frame.

These equations are applicable to plane motion of the rigid body for which the analysis is simplified since $\vec{\Omega}$ and $\dot{\vec{\Omega}}$ have a constant direction. Note that for the three-dimensional case, $\dot{\vec{\Omega}}$ must be computed by using Eq. (9.9).

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Matrix Formulation and Coordinate Transformations

A vector in three-dimensional space characterized by the right-handed reference frame x_a, y_a, z_a, $\vec{A} = A_x\hat{i}_a + A_y\hat{j}_a + A_z\hat{k}_a$, can be represented as an ordered triplet,

$$\vec{A} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_a = \begin{bmatrix} A_x & A_y & A_z \end{bmatrix}_a^{\mathsf{T}}$$

where the elements of the column vector represent the vector projections on the unit axes. Let $\underline{A}_a$ denote the column vector relative to the axes x_a, y_a, z_a. It can be shown that the vector $\vec{A}$ can be expressed in another right-handed reference frame x_b, y_b, z_b, by the transformation relation

$$\underline{A}_b = \overline{\underline{C}}_{ab}\,\underline{A}_a \tag{9.15}$$

where $\overline{\underline{C}}_{ab}$ is a 3×3 matrix,

$$\overline{\underline{C}}_{ab} = \begin{bmatrix} cx_ax_b & cx_ay_b & cx_az_b \\ cy_ax_b & cy_ay_b & cy_az_b \\ cz_ax_b & cz_ay_b & cz_az_b \end{bmatrix} \tag{9.16}$$

The elements of this matrix are the cosines of the angles between the respective axes. For example, cz_ay_b is the cosine of the angle between z_a and y_b. This is the rotational transformation matrix and it must be orthogonal, or

$$C_{ab}^{-1} = C_{ab}^{\mathsf{T}} = C_{ba}$$

and for right-handed systems, let $C_{ab} = +1$.

Angle Representations of Rotation

The six degrees of freedom needed to describe general motion of a rigid body are characterized by three degrees of freedom each for translation and for rotation. The focus here is on methods for describing rotation.

Euler's theorem (11) confirms that only three parameters are needed to characterize rotation. Two parameters define an axis of rotation and another defines an angle about that axis. These parameters define three positional degrees of freedom for a rigid body. The three rotational parameters help construct a rotation matrix, $\overline{\underline{C}}$. The following discussion describes how the rotation matrix, or direction cosine matrix, can be formulated.

General Rotation. Unit vectors for a system a, $\hat{u}_a$, are said to be carried into b, as $\hat{u}_b = \overline{\underline{C}}_{ba}\hat{u}_a$. It can be shown that a direction cosine matrix can be formulated by [30]

$$\overline{\underline{C}} = \underline{\lambda}\underline{\lambda}^{\mathsf{T}} + (\overline{\underline{E}} - \underline{\lambda}\underline{\lambda}^{\mathsf{T}})\cos\psi - \overline{\underline{S}}(\underline{\lambda})\sin\psi \tag{19.17}$$

where $\overline{\underline{E}}$ is the identity matrix, and $\underline{\lambda}$ represents a unit vector, $\underline{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^{\mathsf{T}}$, which is parallel to the axis of rotation, and ψ is the angle of rotation about that axis [30]. In this relation, $\overline{\underline{S}}(\underline{\lambda})$ is a **skew-symmetric matrix**, which is defined by the form

$$\overline{\underline{S}}(\underline{\lambda}) = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}$$

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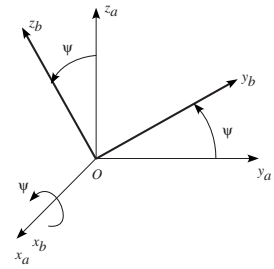

FIGURE 9.31 An elementary rotation by angle ϕ about axis x.

The matrix elements of $\underline{\bar{C}}$ can be found by expanding the relation given above, using $\underline{\bar{S}}(\lambda)$, to give

$$\underline{\bar{C}} = \begin{bmatrix} (1-\cos\psi)\lambda_1^2 + \cos\psi & (1-\cos\psi)\lambda_1\lambda_2 + \lambda_3\sin\psi & (1-\cos\psi)\lambda_1\lambda_3 + \lambda_2\sin\psi \\ (1-\cos\psi)\lambda_2\lambda_1 + \lambda_3\sin\psi & (1-\cos\psi)\lambda_2^2 + \cos\psi & (1-\cos\psi)\lambda_2\lambda_3 + \lambda_1\sin\psi \\ (1-\cos\psi)\lambda_3\lambda_1 + \lambda_2\sin\psi & (1-\cos\psi)\lambda_3\lambda_2 + \lambda_1\sin\psi & (1-\cos\psi)\lambda_3^2 + \cos\psi \end{bmatrix} \tag{9.18}$$

The value of this formulation is in identifying that there are formally defined principle axes, characterized by the $\underline{\lambda}$, and angles of rotation, ψ, that taken together define the body orientation. These rotations describe classical angular variables formed by elementary (or principle) rotations, and it can be shown that there are two cases of particular and practical interest, formed by two different axis rotation sequences.

Elementary Rotations. Three elementary rotations are formed when the rotation axis (defined by the eigenvector) coincides with one of the base vectors of a defined coordinate system. For example, letting $\underline{\lambda} = [1,\ 0,\ 0]^T$ define an axis of rotation x, as in Fig. 9.31, with an elementary rotation of ϕ gives the rotation matrix,

$$\underline{\bar{C}}_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

The two elementary rotations about the other two axes, y and z, are

$$\underline{\bar{C}}_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \quad \text{and} \quad \underline{\bar{C}}_{z,\psi} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These three elementary rotation matrices can be used in sequence to define a direction cosine matrix, for example,

$$\underline{\bar{C}} = \underline{\bar{C}}_{z,\psi}\underline{\bar{C}}_{y,\theta}\underline{\bar{C}}_{x,\phi}$$

and the elementary rotations and the direction cosine matrix are all orthogonal; i.e.,

$$\underline{\bar{C}}\ \underline{\bar{C}}^T = \underline{\bar{C}}^T\underline{\bar{C}} = \underline{\bar{E}}$$

where $\underline{\bar{E}}$ is the identity matrix. Consequently, the inverse of the rotation or coordinate transformation matrix can be found by $\underline{\bar{C}}^{-1} = \underline{\bar{C}}^T$.

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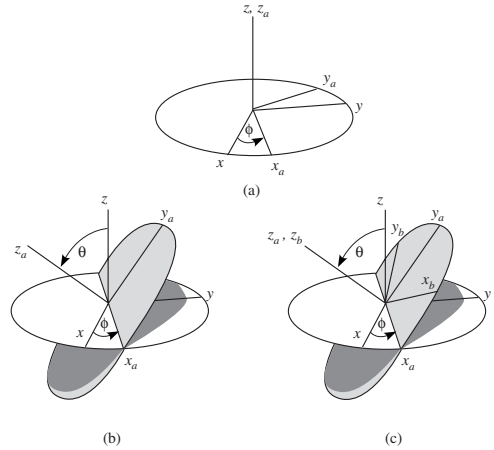

FIGURE 9.32 The rotations defining the Euler angles (adapted from Goldstein [11]).

It can be shown that there exist two sequences that have independent rotation sequences, and these lead to the well known Euler angle and Tait-Bryan or Cardan angle rotation descriptions [30].

Euler Angles. Euler angles are defined by a specific rotation sequence. Consider a right-handed axes system defined by the base vectors, x, y, z, as shown in Fig. 9.32(a). The rotation sequence of interest involves rotations about the axes in the following sequence: (1) ϕ about z, (2) θ about x_a, then (3) ψ about z_b. This set of rotation sequences is defined by the elementary rotation matrices,

$$\overline{C}_{z,\phi} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{C}_{x_a,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}, \quad \overline{C}_{z_b,\psi} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where the subscript on each $\overline{C}$ denotes the axis and angle of rotation. Using these transformations relates the quantity $\underline{A}$ in x, y, z to $\underline{A}_b$ in x_b, y_b, z_b, or

$$\underline{A}_b = \overline{C}_{\text{Euler}}\,\underline{A} = \overline{C}_{z_b,\psi}\overline{C}_{x_a,\theta}\overline{C}_{z,\phi}\,\underline{A}$$

where $\overline{C}_{\text{Euler}}$ is given by

$$\overline{C}_{\text{Euler}} = \begin{bmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix} \tag{9.19}$$

Since $\overline{C}_{\text{Euler}}$ is orthogonal, transforming between the two coordinate systems is relatively easy since the inverse can be found simply by the transpose of Eq. (9.19).

In some applications, it is desirable to derive the angles given the direction cosine matrix. So, if the (3,3) element of $\overline{C}_{\text{Euler}}$ is given, then θ is easily found, but there can be difficulties in discerning small angles. Also, if θ goes to zero, there is a singularity in solving for ϕ and ψ, so determining body orientation becomes difficult. The problem also makes itself known when transforming angular velocities between the coordinate systems. If the problem at hand avoids this case (i.e., θ never approaches zero), then Euler angles are a viable solution. Many applications that cannot tolerate this problem adopt other representations, such as the Euler parameters to be discussed later.

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In classical rigid body dynamics, ϕ is called the *precession angle,* θ is the *nutation angle*, and ψ is the *spin angle*. The relationship between the time derivative of the Euler angles, $\dot{\varphi} = [\dot{\phi}, \dot{\theta}, \dot{\psi}]^T$, and the body angular velocity, $\underline{\omega} = [\omega_x, \omega_y, \omega_z]_b^T$, is given by [11]

$$\underline{\omega}_b = \overline{T}(\underline{\varphi})\dot{\underline{\varphi}} \tag{9.20}$$

where the transformation matrix, $\overline{T}(\underline{\varphi})$, is given by

$$\overline{T}(\underline{\varphi}) = \begin{bmatrix} \sin\theta\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & -\sin\psi & 0 \\ \cos\theta & 0 & 1 \end{bmatrix}$$

Note here again that $\overline{T}(\varphi)$ will become singular at $\theta = \pm\pi/2$.

Tait-Bryan or Cardan Angles. The Tait-Bryan or Cardan angles are formed when the three rotation sequences each occur about a different axis. This is the sequence preferred in flight and vehicle dynamics. Specifically, these angles are formed by the sequence: (1) ϕ about z (yaw), (2) θ about y_a (pitch), and (3) ϕ about the final x_b axis (roll), where a and b denote the second and third stage in a three-stage sequence or axes (as used in the Euler angle description). These rotations define a transformation,

$$\underline{A}_b = \overline{C}\,\underline{A} = \overline{C}_{x_b,\psi}\overline{C}_{y_a,\theta}\overline{C}_{z,\phi}\,\underline{A}$$

where

$$\overline{C}_{z,\phi} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \overline{C}_{y_a,\theta} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}, \quad \overline{C}_{x_b,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{bmatrix}$$

and the final coordinate transformation matrix for Tait-Bryan angles is

$$\overline{C}_{\text{Tait-Bryan}} = \begin{bmatrix} \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \cos\theta\sin\psi \\ \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi & \cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\theta\cos\psi \end{bmatrix} \tag{9.21}$$

A linearized form of $\overline{C}_{\text{Trait-Bryan}}$ gives a form preferred to that derived for Euler angles, making it useful in some forms of analysis and control. There remains the problem of a singularity, in this case when θ approaches $\pm\pi/2$.

For the Tait-Bryan angles, the transformation matrix relating $\dot{\underline{\varphi}}$ to $\underline{\omega}_b$ is given by

$$\overline{T}(\underline{\varphi}) = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\psi & \cos\psi & 0 \\ \cos\theta\cos\psi & -\sin\psi & 0 \end{bmatrix}$$

which becomes singular at $\theta = 0, \pi$.

Euler Parameters and Quaternions

The degenerate conditions in coordinate transformations for Euler and Tait-Bryan angles can be avoided by using more than a minimal set of parameterizing variables (beyond the three angles). The most notable

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set are referred to as Euler parameters, which are unit quaternions. There are many other possibilities, but this four-parameter method is used in many areas, including spacecraft/flight dynamics, robotics, and computational kinematics and dynamics. The term "quaternion" was coined by Hamilton in about 1840, but Euler himself had devised the use of Euler parameters 70 years before. Quaternions are discussed by Goldstein [11], and their use in rigid body dynamics and attitude control dates back to the late 1950s and early 1960s [13,24]. Application of quaternions is common in control applications in aerospace applications [38] as well as in ocean vehicles [10]. More recently (past 20 years or so), these methods have found their way into motion and control descriptions for robotics [34] and computational kinematics and dynamics [14,25,26]. An overview of quaternions and Euler parameters is given by Wehage [37]. Quaternions and rotational sequences and their role in a wide variety of applications areas, including sensing and graphics, are the subject of the book by Kuipers [19]. These are representative references that may guide the reader to an application area of interest where related studies can be found. In the following only a brief overview is given.

Quaternion. A quaternion is defined as the sum of a scalar, q_0, and a vector, $\vec{q}$, or,

$$q = q_0 + \vec{q} = q_0 + q_1\hat{i} + q_2\hat{j} + q_3\hat{k}$$

A specific algebra and calculus exists to handle these types of mathematical objects [7,19,37]. The conjugate is defined as $q = q_0 - \vec{q}$.

Euler Parameters. Euler parameters are normalized (unit) quaternions, and thus share the same properties, algebra and calculus. A principal eigenvector of rotation has an eigenvalue of 1 and defines the Euler axis of rotation (see Euler's theorem discussion and [11]), with angle of rotation α. Let this eigenvector be $\underline{e} = [e_1,\ e_2,\ e_3]^\mathsf{T}$. Recall from Eq. (9.17), the direction cosine matrix is now

$$\underline{\underline{C}} = \underline{e}\underline{e}^\mathsf{T} + (I - \underline{e}\underline{e}^\mathsf{T})\cos\alpha - \underline{\underline{S}}(\underline{e})\sin\alpha$$

where $\underline{\underline{S}}(\underline{e})$ is a skew-symmetric matrix. The Euler parameters are defined as

$$\underline{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\alpha/2) \\ e_1\sin(\alpha/2) \\ e_2\sin(\alpha/2) \\ e_3\sin(\alpha/2) \end{bmatrix}$$

where

$$q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$$

Relating Quaternions and the Coordinate Transformation Matrix. The direction cosine matrix in terms of Euler parameters is now

$$\underline{\underline{C}}_q = (q_0^2 - \underline{q}^\mathsf{T}\underline{q})\,\underline{\underline{E}} + 2\underline{q}\underline{q}^\mathsf{T} - 2q_0\underline{\underline{S}}(\underline{q})$$

where $\underline{q} = [q_1,\ q_2,\ q_3]^\mathsf{T}$, and $\underline{\underline{E}}$ is the identity matrix. The direction cosine matrix is now written in terms of quaternions

$$\underline{\underline{C}}_q = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 + q_3q_0) & 2(q_1q_3 - q_2q_4) \\ 2(q_1q_2 - q_3q_0) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 + q_1q_4) \\ 2(q_1q_3 + q_2q_0) & 2(q_1q_2 + q_3q_0) & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

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It is possible to find the quaternions and the elements of the direction cosine matrix independently by integrating the angular rates about the principal axes of a body. Given the direction cosine matrix elements, we can find the quaternions, and vice versa. For a more extended discussion and application, the reader is referred to the listed references.

Dynamic Properties of a Rigid Body

Inertia Properties

The moments and products of inertia describe the distribution of mass for a body relative to a given coordinate system. This description relies on the specific orientation and reference frame. It is presumed that the reader is familiar with basic properties such as mass center, and the focus here is on those properties essential in understanding the general motion of rigid bodies, and particularly the rotational dynamics.

Moment of Inertia. For the rigid body shown in Fig. 9.33(a), the moment of inertia for a differential element, dm, about any of the three coordinate axes is defined as the product of the mass of the differential element and the square of the shortest distance from the axis to the element. As shown, $r_x = \sqrt{y^2 + z^2}$, so the contribution to the moment of inertia about the x-axis, I_{xx}, from dm is

$$dI_{xx} = r_x^2 = (y^2 + z^2)\,dm$$

The total I_{xx}, I_{yy}, and I_{zz} are found by integrating these expressions over the entire mass, m, of the body. In summary, the three moments of inertia about the x, y, and z axes are

$$\begin{aligned} I_{xx} &= \int_m r_x^2\,dm = \int_m (y^2 + z^2)\,dm \\ I_{yy} &= \int_m r_y^2\,dm = \int_m (x^2 + z^2)\,dm \\ I_{zz} &= \int_m r_z^2\,dm = \int_m (x^2 + y^2)\,dm \end{aligned} \tag{9.22}$$

Note that the moments of inertia, by virtue of their definition using squared distances and finite mass elements, are always positive quantities.

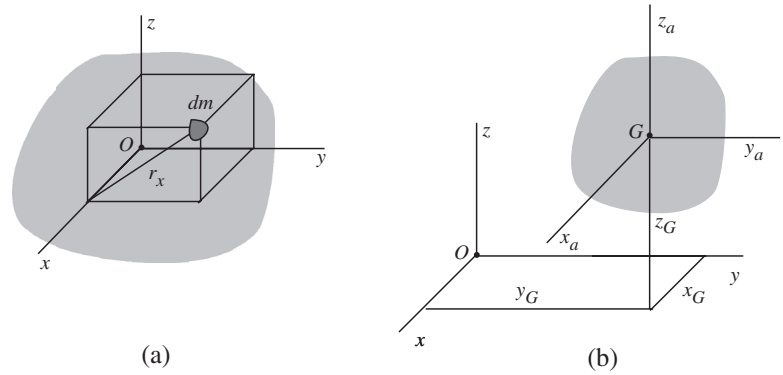

FIGURE 9.33 Rigid body properties are defined by how mass is distributed throughout the body relative to a specified coordinate system. (a) Rigid body used to describe moments and products of inertia. (b) Rigid body and axes used to describe parallel-axis and parallel-plane theorem.

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Product of Inertia. The product of inertia for a differential element dm is defined with respect to a set of two orthogonal planes as the product of the mass of the element and the perpendicular (or shortest) distances from the planes to the element. So, with respect to the $y-z$ and $x-z$ planes (z common axis to these planes), the contribution from the differential element to I_{xy} is dI_{xy} and is given by $dI_{xy} = xy\,dm$.

As for the moments of inertia, by integrating over the entire mass of the body for each combination of planes, the products of inertia are

$$\begin{aligned} I_{xy} &= I_{yx} = \int_m xy\,dm \\ I_{yz} &= I_{zy} = \int_m yz\,dm \\ I_{xz} &= I_{zx} = \int_m xz\,dm \end{aligned} \tag{9.23}$$

The product of inertia can be positive, negative, or zero, depending on the sign of the coordinates used to define the quantity. If either one or both of the orthogonal planes are planes of symmetry for the body, the product of inertia with respect to those planes will be zero. Basically, the mass elements would appear as pairs on each side of these planes.

Parallel-Axis and Parallel-Plane Theorems. The parallel-axis theorem can be used to transfer the moment of inertia of a body from an axis passing through its mass center to a parallel axis passing through some other point (see also the section "Kinetic Energy Storage"). Often the moments of inertia are known for axes fixed in the body, as shown in Fig. 9.33(b). If the center of gravity is defined by the coordinates (x_G, y_G, z_G) in the x, y, z axes, the parallel-axis theorem can be used to find moments of inertia relative to the x, y, z axes, given values based on the body-fixed axes. The relations are

$$\begin{aligned} I_{xx} &= (I_{xx})_a + m(y_G^2 + z_G^2) \\ I_{yy} &= (I_{yy})_a + m(x_G^2 + z_G^2) \\ I_{zz} &= (I_{zz})_a + m(x_G^2 + y_G^2) \end{aligned}$$

where, for example, $(I_{xx})_a$ is the moment of inertia relative to the x_a axis, which passes through the center of gravity. Transferring the products of inertia requires use of the parallel-plane theorem, which provides the relations

$$\begin{aligned} I_{xy} &= (I_{xy})_a + mx_G y_G \\ I_{yz} &= (I_{yz})_a + my_G z_G \\ I_{zx} &= (I_{zx})_a + mz_G x_G \end{aligned}$$

Inertia Tensor. The rotational dynamics of a rigid body rely on knowledge of the inertial properties, which are completely characterized by nine terms of an inertia tensor, six of which are independent. The inertia tensor is

$$\underline{\bar{I}} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

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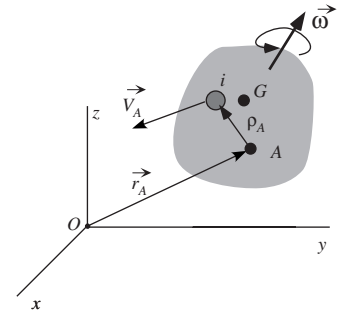

FIGURE 9.34 Rigid body in general motion relative to an inertial coordinate system, *x*, *y*, *z*.

and it relies on the specific location and orientation of coordinate axes in which it is defined. For a rigid body, an origin and axes orientation can be found for which the inertia tensor becomes diagonalized, or

$$\underline{\bar{I}} = \begin{bmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{bmatrix}$$

The orientation for which this is true defines the principal axes of inertia, and the principal moments of inertia are now $I_x = I_{xx}$, $I_y = I_{yy}$, and $I_z = I_{zz}$ (one should be a maximum and another a minimum of the three). Sometimes this orientation can be determined by inspection. For example, if two of the three orthogonal planes are planes of symmetry, then all of the products of inertia are zero, so this would define principal axes of inertia.

The principal axes directions can be interpreted as an eigenvalue problem, and this allows you to find the orientation that will lead to principal directions, as well as define (transform) the inertia tensor into any orientation. For details on this method, see Crandall et al. [8].

Angular Momentum

For the rigid body shown in Fig. 9.34, conceptualized to be composed of particles, *i*, of mass, m_i, the angular momentum about the point *A* is defined as

$$(\vec{h}_A)_i = \vec{\rho}_A \times m_i \vec{V}_i$$

where $\vec{V}_i$ is the velocity measured relative to the inertial frame. Since $\vec{V}_i = \vec{V}_A + \vec{\omega} \times \vec{\rho}_A$, then

$$(\vec{h}_A)_i = \vec{\rho}_A \times m_i \vec{V}_i = m_i \vec{\rho}_A \times \vec{V}_A + m_i \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A)$$

Integrating over the mass of the body, the total angular momentum of the body is

$$\vec{h}_A = \left(\int_m \vec{\rho}_A dm\right) \times \vec{V}_A + \int_m \vec{\rho}_A \times (\vec{\omega} \times \vec{\rho}_A)\, dm \tag{9.24}$$

This equation can be used to find the angular momentum about a point of interest by setting the point *A*: (1) fixed, (2) at the center of mass, and (3) an arbitrary point on the mass. A general form arises in cases 1 and 2 that take the form

$$\vec{h} = \int_m \vec{\rho} \times (\vec{\omega} \times \vec{\rho})\, dm$$

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When this form is expanded for either case into x, y, z components, then

$$\vec{h} = h_x\hat{i} + h_y\hat{j} + h_z\hat{k} = \int_m (x\hat{i} + y\hat{j} + z\hat{k}) \times [(\omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}) \times (x\hat{i} + y\hat{j} + z\hat{k})]\, dm$$

which can be expanded to

$$\begin{aligned} h_x\hat{i} + h_y\hat{j} + h_z\hat{k} &= \left[\omega_x \int_m (y^2 + z^2)\, dm - \omega_y \int_m xy\, dm - \omega_z \int_m xz\, dm\right]\hat{i} \\ &= \left[-\omega_x \int_m xy\, dm + \omega_y \int_m (x^2 + z^2)\, dm - \omega_z \int_m yz\, dm\right]\hat{j} \\ &= \left[-\omega_x \int_m xy\, dm - \omega_y \int_m zy\, dm - \omega_z \int_m (x^2 + y^2)\, dm\right]\hat{k} \end{aligned}$$

The expression for moments and products of inertia can be identified here, and then this expression leads to the three angular momentum components, written in matrix form

$$\underline{h} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \underline{\underline{I}}\,\underline{\omega} \tag{9.25}$$

Note that the case where principal axes are defined leads to the much simplified expression

$$\vec{h} = I_{xx}\omega_x\hat{i} + I_{yy}\omega_y\hat{j} + I_{zz}\omega_z\hat{k}$$

This shows that when the body rotates so that its axis of rotation is parallel to a principal axis, the angular momentum vector, $\vec{h}$, is parallel to the angular velocity vector. In general, this is not true (this is related to the discussion at the end of the section "Inertia Properties").

The angular momentum about an arbitrary point, Case 3, is the resultant of the angular momentum about the mass center (a free vector) and the moment of the *translational momentum* through the mass center,

$$\vec{p} = mV_x\hat{i} + mV_y\hat{j} + mV_z\hat{k} = m\vec{V}$$

or

$$\vec{h} = \vec{h}_G + \vec{r} \times \vec{p}$$

where $\vec{r}$ is the position vector from the arbitary point of interest to the mass center, G. This form can also be expanded into its component forms, as in Eq. (9.25).

Kinetic Energy of a Rigid Body

Several forms of the kinetic energy of a rigid body are presented in this section. From the standpoint of a bond graph formulation, where kinetic energy storage is represented by an **I** element, Eq. (9.25) demonstrates that the rigid body has at least three ports for rotational energy storage. Adding the three translational degrees of freedom, a rigid body can have up to six independent energy storage "ports."

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A 3-port **I** element can be used to represent the rotational kinetic energy for the case of rotation about a fixed point (no translation). The constitutive relation is simply Eq. (9.25). The kinetic energy is then

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{h}$$

where $\vec{h}$ is the angular momentum with an inertia tensor defined about the fixed point. If the axes are aligned with principal axes, then

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

The total kinetic energy for a rigid body that can translate and rotate, with angular momentum defined with reference to the center of gravity, is given by

$$T = \frac{1}{2}mV_G^2 + \frac{1}{2}\vec{\omega} \cdot \vec{h}_G$$

where $V_G^2 = V_x^2 + V_y^2 + V_z^2$.

Rigid Body Dynamics

Given descriptions of inertial properties, translational and angular momentum, and kinetic energy of a rigid body, it is possible to describe the dynamics of a rigid body using the equations of motion using Newton's laws. The classical Euler equations are presented in this section, and these are used to show how a bond graph formulation can be used to integrate rigid body elements into a bond graph model.

Basic Equations of Motion

The translational momentum of the body in Fig. 9.30 is $\underline{p} = m\underline{V}$, where m is the mass, and $\underline{V}$ is the velocity of the mass center with three components of velocity relative to the inertial reference frame x_o, y_o, z_o. In three-dimensional motion, the net force on the body is related to the rate of change of momentum by Newton's law, namely,

$$\underline{F} = \frac{d}{dt}\underline{p}$$

which can be expressed as (using Eq. (9.9)),

$$\underline{F} = \left.\frac{\partial \underline{p}}{\partial t}\right|_{\text{rel}} + \underline{\Omega} \times \underline{p}$$

with $\underline{p}$ now relative to the moving frame x_a, y_a, z_a, and $\underline{\Omega}$ is the absolute angular velocity of the rotating axes.

A similar expression can be written for rate of change of the angular momentum, which is related to applied torques $\underline{T}$ by

$$\underline{T} = \left.\frac{\partial \underline{h}}{\partial t}\right|_{\text{rel}} + \underline{\Omega} \times \underline{h}$$

where $\underline{h}$ is relative to the moving frame x_a, y_a, z_a.

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In order to use these relations effectively, the motion of the axes x_a, y_a, z_a, must be chosen to fit the problem at hand. This choice usually comes down to three cases described by how $\underline{\Omega}$ relates to the body angular velocity $\underline{\omega}$.

1. $\underline{\Omega} = 0$. If the body has general motion and the axes are chosen to translate with the center of mass, then this case will lead to a simple set of equations with $\Omega = 0$, although it will be necessary to describe the inertia properties of the body as functions of time.
2. $\underline{\Omega} \neq 0 \neq \underline{\omega}$. In this case, axes have an angular velocity different from that of the body, a form convenient for bodies that are symmetrical about their spinning axes. The moments and products of inertia will be constant relative to the rotating axes. The equations become

$$\begin{aligned} F_x &= m\dot{V}_x - mV_y\Omega_z + mV_z\Omega_y \\ F_y &= m\dot{V}_y - mV_z\Omega_x + mV_x\Omega_z \\ F_z &= m\dot{V}_z - mV_x\Omega_y + mV_y\Omega_x \\ T_x &= I_x\dot{\omega}_x - I_y\omega_y\Omega_z + I_z\Omega_y\omega_z \\ T_y &= I_y\dot{\omega}_y - I_z\omega_z\Omega_x + I_x\Omega_z\omega_x \\ T_z &= I_z\dot{\omega}_z - I_x\omega_x\Omega_y + I_y\Omega_x\omega_y \end{aligned} \tag{9.26}$$

3. $\underline{\Omega} = \underline{\omega}$. Here the axes are fixed and moving with the body. The moments and products of intertia relative to the moving axes will be constant. A particularly convenient case arises if the axes are chosen to be the principal axes of inertia (see the section titled "Inertia Properties"), which leads to the *Euler equations*,[4]

$$\begin{aligned} F_x &= m\dot{V}_x - mV_y\omega_z + mV_z\omega_y \\ F_y &= m\dot{V}_y - mV_z\omega_x + mV_x\omega_z \\ F_z &= m\dot{V}_z - mV_x\omega_y + mV_y\omega_x \\ T_x &= I_x\dot{\omega}_x - (I_y - I_z)\omega_y\omega_z \\ T_y &= I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x \\ T_z &= I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y \end{aligned} \tag{9.27}$$

These equations of motion can be used to determine the forces and torques, given motion of the body. Textbooks on dynamics [12,23] provide extensive examples on this type of analysis. Alternatively, these can be seen as six nonlinear, coupled ordinary differential equations (ODEs). Case 3 (the Euler equations) could be solved in such a case, since these can be rewritten as six first-order ODEs. A numerical solution may need to be implemented. Modern computational software packages will readily handle these equations, and some will feature a form of these equations in a form suitable for immediate use. Case 2 requires knowledge of the axes' angular velocity, $\underline{\Omega}$.

If the rotational motion is coupled to the translational motion such that the forces and torques, say, are related, then a dynamic model is required. In some, it may be desirable to formulate the problem in a bond graph form, especially if there are actuators and sensors and other multienergetic systems to be incorporated.

[4]First developed by the Swiss mathematician L. Euler.

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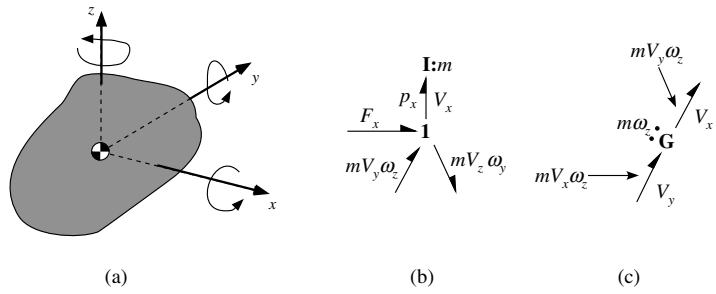

FIGURE 9.35 (a) Rigid body with angular velocity components about x, y, z axes. (b) x-direction translational dynamics in bond graph form. (c) Gyrator realization of coupling forces.

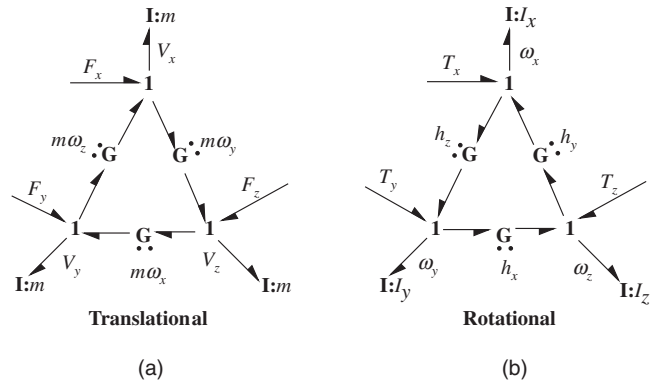

FIGURE 9.36 (a) Bond graph for rigid body translation. (b) Bond graph for rigid body rotation.

Rigid Body Bond Graph Formulation

Due to the body's rotation, there is an inherent coupling of the translational and rotational motion, which can be summarized in a bond graph form. Consider the case of Euler's equations, given in Eqs. (9.27). For the x-direction translational dynamics,

$$F_x = \dot{p}_x - mV_y\omega_z + mV_z\omega_y$$

where $p_x = mV_x$, and F_x is the net "external" applied forces in the x-direction. This equation, a summation of forces (efforts) is represented in bond graph form in Fig. 9.35(b). All of these forces are applied at a common velocity, V_x, represented by the 1-junction. The **I** element represents the storage of kinetic energy in the body associated with motion in the x-direction. The force $mV_y\omega_z$ in Fig. 9.35(b) is induced by the y-direction velocity, V_y, and by the angular velocity component, ω_z. This physical effect is gyrational in nature, and can be captured by the gyrator, as shown in Fig. 9.35(c). Note that this is a modulated gyrator (could also be shown as **MGY**) with a gyrator modulus of $r = m\omega_z$ (verify that the units are force).

The six equations of motion, Eqs. (9.27), can be represented in bond graph form as shown in Fig. 9.36. Note that these two bond graph ring formations, first shown by Karnopp and Rosenberg [18], capture the Euler equations very efficiently and provide a graphical mnemonic for rigid body motion. Indeed, Euler's equations can now be "drawn" simply in the following steps: (1) lay down three 1-junctions representing angular velocity about x, y, z (counter clockwise labeling), with **I** elements attached, (2) between each 1-junction place a gyrator, modulated by the momentum about the axis represented by the

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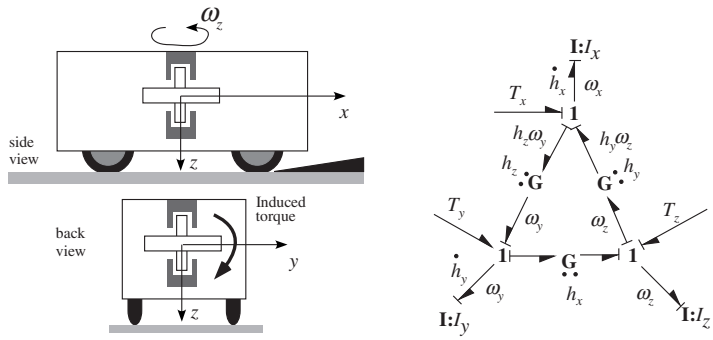

FIGURE 9.37 A cart with a rigid and internally mounted flywheel approaches a ramp.

1-junction directly opposite in the triangle, (3) draw power arrows in a counter clockwise direction. This sketch will provide the conventional Euler equations. The translational equations are also easily sketched.

These bond graph models illustrate the inherent coupling through the gyrator modulation. There are six **I** elements, and each can represent an independent energetic state in the form of the momenta [p_x, p_y, p_z, h_x, h_y, h_z] or alternatively the analyst could focus on the associated velocities [V_x, V_y, V_z, ω_x, ω_y, ω_z].

If forces and torques are considered as inputs, through the indicated bonds representing F_x, F_y, F_z, T_x, T_y, T_z, then you can show that all the **I** elements are in integral causality, and the body will have six independent states described by six first-order nonlinear differential equations.

Example: Cart-Flywheel

A good example of how the rigid body bond graphs represent the basic mechanics inherent to Eqs. (9.27) and of how the graphical modeling can be used for "intuitive" gain is shown in Fig. 9.37. The flywheel is mounted in the cart, and spins in the direction shown. The body-fixed axes are mounted in the vehicle, with the convention that z is positive into the ground (common in vehicle dynamics). The cart approaches a ramp, and the questions which arise are whether any significant loads will be applied, what their sense will be, and on which parameters or variables they are dependent.

The bond graph for rotational motion of the flywheel (assume it dominates the problem for this example) is shown in Fig. 9.37. If the flywheel momentum is assumed very large, then we might just focus on its effect. At the 1-junction for ω_x, let $T_x = 0$, and since ω_z is spinning in a negative direction, you can see that the torque $h_z\omega_y$ is applied in a positive direction about the x-axis. This will tend to "roll" the vehicle to the right, and the wheels would feel an increased normal load. With the model shown, it would not be difficult to develop a full set of differential equations.

Need for Coordinate Transformations

In the cart-flywheel example, it is assumed that as the front wheels of the cart lift onto the ramp, the flywheel will react because of the direct induced motion at the bearings. Indeed, the flywheel-induced torque is also transmitted directly to the cart. The equations and basic bond graphs developed above are convenient if the forces and torques applied to the rigid body are moving with the rotating axes (assumed to be fixed to the body). The orientational changes, however, usually imply that there is a need to relate the body-fixed coordinate frames or axes to inertial coordinates. This is accomplished with a coordinate transformation, which relates the body orientation into a frame that makes it easier to interpret the motion, apply forces, understand and apply measurements, and apply feedback controls.

Example: Torquewhirl Dynamics

Figure 9.38(a) illustrates a cantilevered rotor that can exhibit torquewhirl. This is a good example for illustrating the need for coordinate transformations, and how Euler angles can be used in the modeling process. The whirling mode is conical and described by the angle θ. There is a drive torque, T_s, that is

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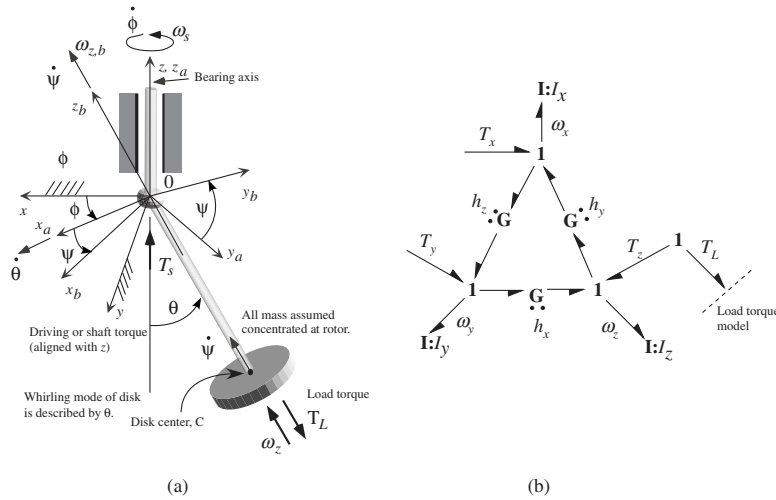

FIGURE 9.38 (a) Cantilevered rotor with flexible joint and rigid shaft (after Vance [36]). (b) Bond graph representing rigid body rotation of rotor.

aligned with the bearing axis, z, where x, y, z is the inertial coordinate frame. The bond graph in Fig. 9.38(b) captures the rigid body motion of the rotor, represented in body-fixed axes x_b, y_b, z_b, which represent principal axes of the rotor.

The first problem seen here is that while the bond graph leads to a very convenient model formulation, the applied torque, T_s, is given relative to the inertial frame x, y, z. Also, it would be nice to know how the rotor moves relative to the inertial frame, since it is that motion that is relevant. Other issues arise, including a stiffness of the rotor that is known relative to the angle θ. These problems motivate the use of Euler angles, which will relate the motion in the body fixed to the inertial frame, and provide three additional state equations for ϕ, θ, and ψ (which are needed to quantify the motion).

In this example, the rotation sequence is (1) x, y, z (inertial) to x_a, y_b, z_c, with ϕ about the z-axis, so note, $\dot{\phi} = \omega_s$, (2) x_a, y_a, z_a to x_b, y_b, z_b, with θ about x_a, (3) ψ rotation about z_b. Our main interest is in the overall transformation from x, y, z (inertia) to x_b, y_b, z_b (body-fixed). In this way, we relate the body angular velocities to inertial velocities using the relation from Eq. (9.20),

$$\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}_b = \begin{bmatrix} \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}$$

where the subscript b on the left-hand side denotes velocities relative to the x_b, y_b, z_b axes. A full and complete bond graph would include a representation of these transformations (e.g., see Karnopp, Margolis, and Rosenberg [17]). Explicit 1-junctions can be used to identify velocity junctions at which torques and forces are applied. For example, at a 1-junction for $\dot{\phi} = \omega_z$, the input torque T_s is properly applied. Once the bond graph is complete, causality is applied. The preferred assignment that will lead to integral causality on all the **I** elements is to have torques and forces applied as causal *inputs*. Note that in transforming the expression above which relates the angular velocities, a problem with Euler angles arises related to the singularity (here at $\theta = \pi/2$, for example).

An alternative way to proceed in the analysis is using a Lagrangian approach as in Section 9.7, as done by Vance [36] (see p. 292). Also, for advanced multibody systems, a multibond formulation can be more efficient and may provide insight into complex problems (see Breedveld [4] or Tiernego and Bos [35]).

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9.7 Lagrange's Equations

The discussion on energy methods focuses on deriving constitutive relations for energy-storing multiports, and this can be very useful in some modeling exercises. For some cases where the constraint relationships between elements are primarily holonomic, and definitely scleronomic (not an explicit function of time), implicit multiport fields can be formulated (see Chapter 7 of [17]). The principal concern arises because of dependent energy storage, and the methods presented can be a solution in some practical cases. However, there are many mechanical systems in which geometric configuration complicates the matter. In this section, Lagrange's equations are introduced to facilitate analysis of those systems.

There are several ways to introduce, derive, and utilize the concepts and methods of Lagrange's equations. The summary presented below is provided in order to introduce fundamental concepts, and a thorough derivation can be found either in Lanczos [20] or Goldstein [11]. A derivation using energy and power flow is presented by Beaman, Paynter, and Longoria [3].

Lagrange's equations are also important because they provide a unified way to model systems from different energy domains, just like a bond graph approach. The use of scalar energy functions and minimal geometric reasoning is preferred by some analysts. It is shown in the following that the particular benefits of a Lagrange approach that make it especially useful for modeling mechanical systems enhance the bond graph approach. A combined approach exploits the benefits of both methods, and provides a methodology for treating complex mechatronic systems in a systematic fashion.

Classical Approach

A classical derivation of Lagrange's equations evolves from the concept of virtual displacement and virtual work developed for analyzing static systems (see Goldstein [11]). To begin with, the Lagrange equations can be derived for dynamic systems by using Hamilton's principle or D'Alembert's principle.

For example, for a system of particles, Newton's second law for the i mass, $\mathbf{F}_i = \mathbf{p}_i$, is rewritten, $\mathbf{F}_i - \mathbf{p}_i = 0$. The forces are classified as either applied or constraint, $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i$. The principle of virtual work is applied over the system, recognizing that constraint forces $\mathbf{f}_i$, do no work and will drop out. This leads to the D'Alembert principle [11],

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta\mathbf{r}_i = 0 \tag{9.28}$$

The main point in presenting this relation is to show that: (a) the constraint forces do not appear in this formulative equation and (b) the need arises for transforming relationships between, in this case, the N coordinates of the particles, $\mathbf{r}_i$, and a set of n **generalized coordinates**, $\mathbf{q}_i$, which are independent of each other (for holonomic constraints), i.e.,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t) \tag{9.29}$$

By transforming to generalized coordinates, D'Alembert's principle becomes [11]

$$\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0 \tag{9.30}$$

where T is the system kinetic energy, and the Q_j are components of the **generalized forces** given by

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}$$

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If the transforming relations are restricted to be holonomic, the constraint conditions are implicit in the transforming relations, and independent coordinates are assured. Consequently, all the terms in Eq. (9.30) must vanish for independent virtual displacements, δq_j, resulting in the n equations:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \tag{9.31}$$

These equations become Lagrange's equations through the following development. Restrict all the applied forces, Q_j, to be derivable from a scalar function, U, where in general, $U = U(q_j, \dot{q}_j)$, and

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

The Lagrangian is defined as $L = T - U$, and substituted into Eq. (9.31) to yield the n Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j \tag{9.32}$$

This formulation yields n second-order ODEs in the q_j.

Dealing with Nonconservative Effects

The derivation of Lagrange's equations assumes, to some extent, that the system is conservative, meaning that the total of kinetic and potential energy remains constant. This is not a limiting assumption because the process of reticulation provides a way to extract nonconservative effects (inputs, dissipation), and then to assemble the system later. It is necessary to recognize that the nonconservative effects can be integrated into a model based on Lagrange's equations using the Q_i's. Associating these forces with the generalized coordinates implies work is done, and this is in accord with energy conservation principles (we account for total work done on system). The generalized force associated with a coordinate, q_i, and due to external forces is then derived from $Q_i = \delta W_i / \delta q_i$, where W_i is the work done on the system by all external forces during the displacement, δq_i.

Extensions for Nonholonomic Systems

In the case of nonholonomic constraints, the coordinates q_j are not independent. Assume you have m nonholonomic constraints ($m \le n$). If the equations of constaint can be put in the form

$$\sum_k \frac{\partial a_l}{\partial q_k} dq_k + \frac{\partial a_l}{\partial t} dt = \sum_k a_{lk} dq_k + a_{lt}\, dt = 0 \tag{9.33}$$

where l indexes up to m such constraints, then the Lagrange equations are formulated with Lagrange undetermined multipliers, λ_l. We maintain n coordinates, q_k, but the n Lagrange equations are now expressed [11] as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \sum_l \lambda_l a_{lk}, \quad k = 1, 2, \ldots, n \tag{9.34}$$

However, since there are now m unknown Lagrange multipliers, λ_l, it is necessary to solve an additional m equations:

$$\sum_k a_{lk}\dot{q}_k + a_{lt} = 0 \tag{9.35}$$

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The terms $\sum_l \lambda_l a_{lk}$ can be interpreted as generalized forces of constraint. These are still workless constraints. The Lagrange equations for nonholonomic constraints can be used to study holonomic systems, and this analysis would provide a solution for the constraint forces through evaluation of the Lagrange multipliers. The use of Lagrange's equations with Lagrange multipliers is one way to model complex, constrained multibody systems, as discussed in Haug [14].

Mechanical Subsystem Models Using Lagrange Methods

The previous sections summarize a classical formulation and application of Lagrange's equations. When formulating models of mechanical systems, these methods are well proven. Lagrange's equations are recognized as an approach useful in handling systems with complex mechanical systems, including systems with constraints. The energy-basis also makes the method attractive from the standpoint of building multi-energetic system models, and Lagrange's equations have been used extensively in electromechanics modeling, for example. For conservative systems, it is possible to arrive at solutions sometimes without worrying about forces, especially since nonconservative effects can be handled "outside" the conservative dynamics. Developing transformation equations between the coordinates, say **x**, used to describe the system and the independent coordinates, **q**, helps assure a minimal formulation. However, it is possible sometimes to lose insight into cause and effect, which is more evident in other approaches. Also, the algebraic burden can become excessive. However, it is the analytical basis of the method that makes it especially attactive. Indeed, with computer-aided symbolic processing techniques, extensive algebra becomes a non-issue.

In this section, the advantages of the Lagrange approach are merged with those of a bond graph approach. The concepts and formulations are classical in nature; however, the graphical interpretation adds to the insight provided. Further, the use of bond graphs assures a consistent formulation with causality so that some insight is provided into how the conservative dynamics described by the energy functions depend on inputs, which typically arrive from the nonconservative dynamics. The latter are very effectively dealt with using bond graph methods, and the combined approach is systematic and yields first-order differential equations, rather than the second-order ODEs in the classical approach. Also, it will be shown that in some cases the combined approach makes it relatively easy to model certain systems that would be very troublesome for a direct approach by either method independently.

A Lagrange bond graph subsystem model will capture the elements summarized with a word bond graph in Fig. 9.39. The key elements are identified as follows: (a) conservative energy storage captured by kinetic and potential energy functions, (b) power-conserving transforming relations, and (c) coupling/interconnections with nonconservative and non-Lagrange system elements. Note that on the nonconservative side of the transforming relations, there are m coordinates that can be identified in the modeling, but these are not independent. The power-conserving transforming relations reduce the coordinates to a set of n independent coordinates, q_i. Associated with each independent coordinate or velocity, $\dot{q}_i$, there is an associated storage of kinetic and potential energy which can be represented by the coupled **IC** in Fig. 9.40(a) [16]. An alternative is the single **C** element used to capture all the coupled energy storage [3], where the gyrator has a modulus of 1 (this is called a symplectic gyrator). In either case, this structure shows that there will be one common flow junction associated with each independent coordinate. Recall the efforts at a 1-junction sum, and at this ith junction,

$$E_{q_i} = \dot{\tilde{p}}_i + e_{q_i} \tag{9.36}$$

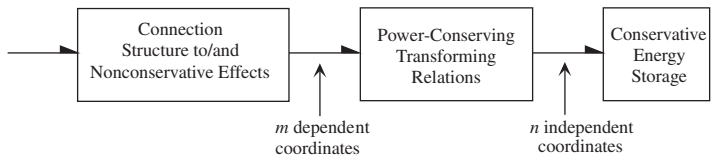

FIGURE 9.39 Block diagram illustrating the Lagrange subsystem model.

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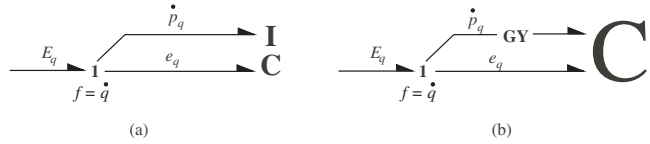

FIGURE 9.40 Elementary formulation of a flow junction in a Lagrange subsystem model. The efforts at the 1-junction for this ith independent flow variable, $\dot{q}_i$, represent Lagrange's equations.

where E_{q_i} is the net nonconservative effort at $\dot{q}_i$, e_{q_i} is a generalized conservative effort that will be determined by the Lagrange system, and the effort $\dot{\tilde{p}}_i$ is a rate of change of an ith generalized momentum. These terms will be defined in the next section. However, note that this effort sum is simply Newton's laws derived by virtue of a Lagrange formulation. In fact, this equation is simply a restatement of the ith Lagrange equation, as will be shown in the following. These effort sum equations give n first-order ODEs by solving for $\dot{p}_i$. The other n equations will be for the displacement variables, q_i. The following methodology is adapted from Beaman, Paynter, and Longoria [3].

Methodology for Building Subsystem Model

Conduct Initial Modeling. Isolate the conservative parts of the system, and make sure that any constraints are holonomic. This reticulation will identify ports to the system under study, including points in the system (typically velocities) where forces and/or torques of interest can be applied (e.g., at flow junctions). These forces and torques are either nonconservative, or they are determined by a system external to the Lagrange-type subsystem. This is a modeling decision. For example, a force due to gravity could be included in a Lagrange subsystem (being conservative) or it could be shown explicity at a velocity junction corresponding to motion modeled outside of the Lagrange subsystem. This will be illustrated in one of the examples that follow.

Define Generalized Displacement Variables. In a Lagrange approach, it is necessary to identify variables that define the configuration of a system. In mechanical system, these are translational and rotational displacements. Further, these variables are typically associated with the motion or relative motion of bodies. To facilitate a model with a minimum and independent set of coordinates, develop transforming relations between the m velocities or, more generally, flows $\dot{\mathbf{x}}$, and n independent flows, $\dot{\mathbf{q}}$. The form is [3],

$$\dot{\mathbf{x}} = \mathbf{T}(\mathbf{q})\dot{\mathbf{q}} \tag{9.37}$$

explicity showing that the matrix $\mathbf{T}(\mathbf{q})$ can depend on $\mathbf{q}$. This can be interpreted, in bond graph modeling terms, as a modulated transformer relationship, where $\mathbf{q}$ contains the modulating variables. The independent generalized displacements, $\mathbf{q}$, will form possible state variables of the Lagrange subsystem.

The transforming relationships are commonly derived from (holonomic) constraints, and from considerations of geometry and basic kinematics. The matrix $\mathbf{T}$ is $m \times n$ and may not be invertible. The bond graph representation is shown in Fig. 9.41.

Formulate the Kinetic Energy Function. Given the transforming relationships, it is now possible to express the total kinetic energy of the Lagrange subsystem using the independent flow variables, $\dot{\mathbf{q}}$. First, the kinetic energy can be written using the $\dot{\mathbf{x}}$ (this is usually easier), or $\mathbf{T} = \mathbf{T}_{\dot{\mathbf{x}}}(\dot{\mathbf{x}})$. Then the relations in Eq. (9.37) are used to transform this kinetic energy function so it is expressed as a function of the $\mathbf{q}$ and $\dot{\mathbf{q}}$ variables, $\mathbf{T}_{\dot{\mathbf{x}}}(\dot{\mathbf{x}}) \rightarrow \mathbf{T}_{\dot{\mathbf{q}}\mathbf{q}}(\dot{\mathbf{q}}, \mathbf{q})$. For brevity, this can be indicated in the subscript, or just $\mathbf{T}_{\dot{\mathbf{q}}\mathbf{q}}$. For example, a kinetic energy function that depends on x, θ, and $\dot{\theta}$ is referred to as $T_{\dot{\theta}\theta x}$ (if the number of variables is very high, certainly such a convention would not be followed).

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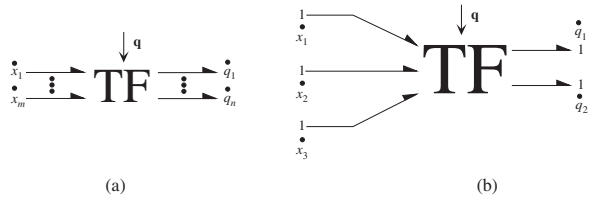

FIGURE 9.41 (a) Bond graph representation of the transforming relations. (b) Example for the case where $m = 3$ and $n = 2$.

Define Generalized Momentum Variables. With the kinetic energy function now in terms of the independent flows, $\dot{\mathbf{q}}$, generalized momenta can be defined as [3,20],

$$\tilde{\mathbf{p}} = \frac{\partial \mathbf{T}_{\dot{\mathbf{q}}\mathbf{q}}}{\partial \dot{\mathbf{q}}} \tag{9.38}$$

where the "tilde" ($\tilde{\mathbf{p}}$) notation is used to distinguish these momentum variables from momentum variables defined strictly through the principles summarized in Table 9.5. In particular note that these generalized momentum variables may be functions of flow as well as of displacement (i.e., they may be configuration dependent).

Formulate the Potential Energy Function. In general, a candidate system for study by a Lagrange approach will store potential energy, in addition to kinetic energy, and the potential energy function, U, should be expressed in terms of the dependent variables, $\mathbf{x}$. Using the tranforming relations in Eq. (9.37), the expression is then a function of $\mathbf{q}$, or $U = U(\mathbf{q}) = U_{\mathbf{q}}$. In mechanical systems, this function is usually formed by considering energy stored in compliant members, or energy stored due to a gravitational potential. In these cases, it is usually possible to express the potential energy function in terms of the displacement variables, $\mathbf{q}$.

Derive Generalized Conservative Efforts. A conservative effort results and can be found from the expression

$$\tilde{\mathbf{e}}_{\mathbf{q}} = -\frac{\partial \mathbf{T}_{\dot{\mathbf{q}}\mathbf{q}}}{\partial \mathbf{q}} + \frac{\partial U_{\mathbf{q}}}{\partial \mathbf{q}} \tag{9.39}$$

where the $\mathbf{q}$ subscript is used to denote these as conservative efforts. The first term on the right-hand side represents an effect due to dependence of kinetic energy on displacement, and the second term will be recognized as the potential energy derived effort.

Identify and Express Net Power Flow into Lagrange Subsystem. At the input to the Lagrange subsystem on the "nonconservative" side, the power input can be expressed in terms of effort and flow products. Since the transforming relations are power-conserving, this power flow must equal the power flow on the "conservative" side. This fact is expressed by

$$P_{\mathbf{x}} = \underbrace{\mathbf{e}_{\mathbf{x}}}_{1\times m}\ \underbrace{\dot{\mathbf{x}}}_{m\times 1} = \underbrace{\mathbf{e}_{\mathbf{x}}}_{1\times m}\ \underbrace{\mathbf{T}(\mathbf{q})}_{m\times n}\ \underbrace{\dot{\mathbf{q}}}_{n\times 1} = \underbrace{\mathbf{E}_{\mathbf{q}}}_{1\times n}\ \underbrace{\dot{\mathbf{q}}}_{n\times 1} \tag{9.40}$$

where the term $\mathbf{E}_{\mathbf{q}}$ is the nonconservative effort transformed into the $\mathbf{q}$ coordinates. This term can be computed as shown by

$$\mathbf{E}_{\mathbf{q}} = \mathbf{e}_{\mathbf{x}}\mathbf{T}(\mathbf{q}) \tag{9.41}$$

Summary of the Method. In summary, all the terms for a Lagrange subsystem can be systematically derived. There are some difficulties that can arise. To begin with, the first step can require some geometric reasoning, and often this can be a problem in some cases, although not insurmountable. The n

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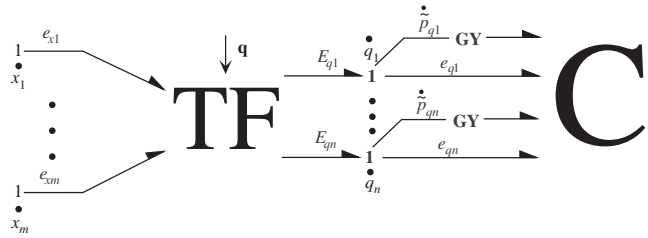

FIGURE 9.42 Lagrange subsystem model.

momentum state equations for this Lagrange subsystem are given by

$$\dot{\tilde{p}} = -e_i + E_i \tag{9.42}$$

and the state equations for the q_i must be found by inverting the generalized momentum equations, (9.38). In some cases, these n equations are coupled and must be solved simultaneously. In the end, there are $2n$ first-order state equations. In addition, the final bond graph element shown in Fig. 9.42 can be coupled to other systems to build a complex system model.

Note that in order to have the $2n$ equations in integral causality, efforts (forces and torques) should be specified as causal inputs to the transforming relations. Also, this subsystem model assumes that only holonomic constraints are applied. While this might seem restrictive, it turns out that, for many practical cases, the physical effects that lead to nonholonomic constraints can be dealt with "outside" of the Lagrange model, along with dissipative effects, actuators, and so on.

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