Chapter 9

Sources of Magnetic Fields

Sources of Magnetic Fields

9.1 Biot-Savart Law

Currents which arise due to the motion of charges are the source of magnetic fields. When charges move in a conducting wire and produce a current *I*, the magnetic field at any point *P* due to the current can be calculated by adding up the magnetic field contributions, $d\vec{\mathbf{B}}$, from small segments of the wire $d\vec{\mathbf{s}}$, (Figure 9.1.1).

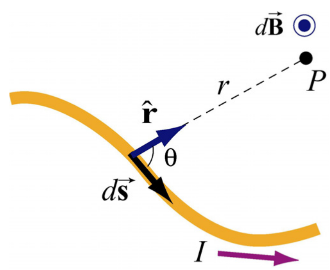


Figure 9.1.1 Magnetic field $d\vec{\mathbf{B}}$ at point *P* due to a current-carrying element $I\,d\vec{\mathbf{s}}$.

These segments can be thought of as a vector quantity having a magnitude of the length of the segment and pointing in the direction of the current flow. The infinitesimal current source can then be written as $I\,d\vec{\mathbf{s}}$.

Let *r* denote as the distance form the current source to the field point *P*, and $\hat{\mathbf{r}}$ the corresponding unit vector. The Biot-Savart law gives an expression for the magnetic field contribution, $d\vec{\mathbf{B}}$, from the current source, $Id\vec{\mathbf{s}}$,

$$\boxed{d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}} \tag{9.1.1}$$

where μ_0 is a constant called the *permeability of free space*:

$$\mu_0 = 4\pi\times10^{-7}\,\mathrm{T\cdot m/A} \tag{9.1.2}$$

Notice that the expression is remarkably similar to the Coulomb's law for the electric field due to a charge element *dq*:

$$d\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{\mathbf{r}} \tag{9.1.3}$$

Adding up these contributions to find the magnetic field at the point *P* requires integrating over the current source,

$$\vec{\mathbf{B}} = \int_{\text{wire}} d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{9.1.4}$$

The integral is a vector integral, which means that the expression for $\vec{\mathbf{B}}$ is really three integrals, one for each component of $\vec{\mathbf{B}}$. The vector nature of this integral appears in the cross product $I\,d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}$. Understanding how to evaluate this cross product and then perform the integral will be the key to learning how to use the Biot-Savart law.

Interactive Simulation 9.1: Magnetic Field of a Current Element

Figure 9.1.2 is an interactive ShockWave display that shows the magnetic field of a current element from Eq. (9.1.1). This interactive display allows you to move the position of the observer about the source current element to see how moving that position changes the value of the magnetic field at the position of the observer.

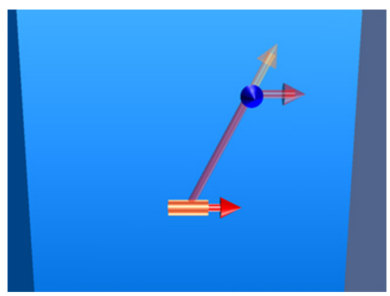


Figure 9.1.2 Magnetic field of a current element.

Example 9.1: Magnetic Field due to a Finite Straight Wire

A thin, straight wire carrying a current I is placed along the x-axis, as shown in Figure 9.1.3. Evaluate the magnetic field at point P. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point P.

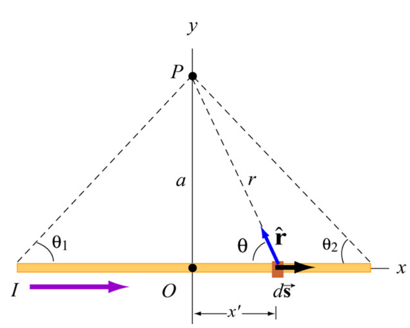


Figure 9.1.3 A thin straight wire carrying a current I.

Solution:

This is a typical example involving the use of the Biot-Savart law. We solve the problem using the methodology summarized in Section 9.10.

(1) Source point (coordinates denoted with a prime)

Consider a differential element $d\vec{\mathbf{s}} = dx'\hat{\mathbf{i}}$ carrying current I in the x-direction. The location of this source is represented by $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$.

(2) Field point (coordinates denoted with a subscript "*P*")

Since the field point P is located at $(x, y) = (0, a)$, the position vector describing P is $\vec{\mathbf{r}}_P = a\hat{\mathbf{j}}$.

(3) Relative position vector

The vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$ is a "relative" position vector which points from the source point to the field point. In this case, $\vec{\mathbf{r}} = a\,\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$, and the magnitude $r = |\vec{\mathbf{r}}| = \sqrt{a^2 + x'^2}$ is the distance from between the source and *P*. The corresponding unit vector is given by

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{a\,\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\sqrt{a^2 + x'^2}} = \sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{i}}$$

(4) The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$

The cross product is given by

$$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = (dx'\hat{\mathbf{i}}) \times (-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (dx'\sin\theta)\,\hat{\mathbf{k}}$$

(5) Write down the contribution to the magnetic field due to $Id\vec{\mathbf{s}}$

The expression is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{dx\,\sin\theta}{r^2}\hat{\mathbf{k}}$$

which shows that the magnetic field at *P* will point in the $+\hat{\mathbf{k}}$ direction, or out of the page.

(6) Simplify and carry out the integration

The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ. From Figure 9.1.3, we have

$$\begin{cases} r = a/\sin\theta = a\csc\theta \\ x = a\cot\theta \quad \Rightarrow \quad dx = -a\csc^2\theta\,d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB = \frac{\mu_0 I}{4\pi}\frac{(-a\csc^2\theta\,d\theta)\sin\theta}{(a\csc\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\,d\theta$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we obtain

$$\boxed{B = -\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2}\sin\theta\,d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)} \tag{9.1.5}$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second term involving θ_1 contains the contribution from the portion along the $-x$ axis. The two terms add!

Let's examine the following cases:

(i) In the symmetric case where $\theta_2 = -\theta_1$, the field point P is located along the perpendicular bisector. If the length of the rod is $2L$, then $\cos\theta_1 = L/\sqrt{L^2 + a^2}$ and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}\cos\theta_1 = \frac{\mu_0 I}{2\pi a}\frac{L}{\sqrt{L^2 + a^2}} \tag{9.1.6}$$

(ii) The infinite length limit $L \to \infty$

This limit is obtained by choosing $(\theta_1, \theta_2) = (0, 0)$. The magnetic field at a distance a away becomes

$$\boxed{B = \frac{\mu_0 I}{2\pi a}} \tag{9.1.7}$$

Note that in this limit, the system possesses cylindrical symmetry, and the magnetic field lines are circular, as shown in Figure 9.1.4.

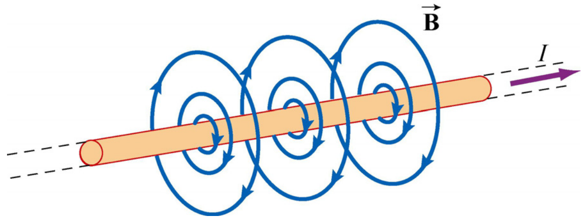


Figure 9.1.4 Magnetic field lines due to an infinite wire carrying current I.

In fact, the direction of the magnetic field due to a long straight wire can be determined by the right-hand rule (Figure 9.1.5).

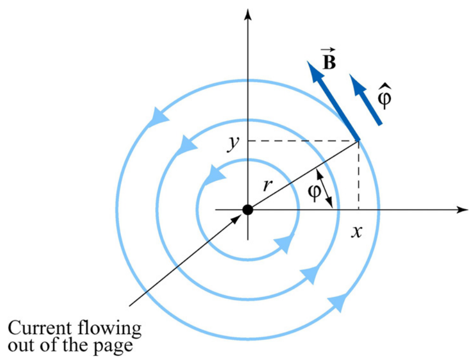


Figure 9.1.5 Direction of the magnetic field due to an infinite straight wire

If you direct your right thumb along the direction of the current in the wire, then the fingers of your right hand curl in the direction of the magnetic field. In cylindrical coordinates (r,φ,z) where the unit vectors are related by $\hat{\mathbf{r}}\times\hat{\boldsymbol{\varphi}}=\hat{\mathbf{z}}$, if the current flows in the $+z$-direction, then, using the Biot-Savart law, the magnetic field must point in the φ-direction.

Example 9.2: Magnetic Field due to a Circular Current Loop

A circular loop of radius R in the xy plane carries a steady current I, as shown in Figure 9.1.6.

(a) What is the magnetic field at a point P on the axis of the loop, at a distance z from the center?

(b) If we place a magnetic dipole $\vec{\boldsymbol{\mu}}=\mu_z\hat{\mathbf{k}}$ at P, find the magnetic force experienced by the dipole. Is the force attractive or repulsive? What happens if the direction of the dipole is reversed, i.e., $\vec{\boldsymbol{\mu}}=-\mu_z\hat{\mathbf{k}}$

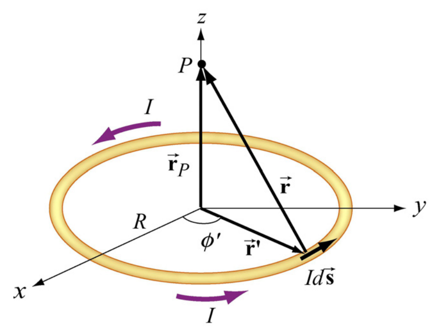


Figure 9.1.6 Magnetic field due to a circular loop carrying a steady current.

Solution:

(a) This is another example that involves the application of the Biot-Savart law. Again let's find the magnetic field by applying the same methodology used in Example 9.1.

(1) Source point

In Cartesian coordinates, the differential current element located at $\vec{\mathbf{r}}' = R(\cos\phi'\,\hat{\mathbf{i}} + \sin\phi'\,\hat{\mathbf{j}})$ can be written as $Id\vec{\mathbf{s}} = I(d\vec{\mathbf{r}}'/d\phi')d\phi' = IRd\phi'(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}})$.

(2) Field point

Since the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$.

(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}'$

The relative position vector is given by

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = -R\cos\phi'\,\hat{\mathbf{i}} - R\sin\phi'\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}} \tag{9.1.8}$$

and its magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + \left(-R\sin\phi'\right)^2 + z^2} = \sqrt{R^2 + z^2} \tag{9.1.9}$$

is the distance between the differential current element and P. Thus, the corresponding unit vector from $Id\,\vec{\mathbf{s}}$ to P can be written as

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|}$$

(4) Simplifying the cross product

The cross product $d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,')$ can be simplified as

$$\begin{aligned} d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,') &= R\,d\phi'\left(-\sin\phi'\,\hat{\mathbf{i}}+\cos\phi'\,\hat{\mathbf{j}}\right)\times[-R\cos\phi'\,\hat{\mathbf{i}}-R\sin\phi'\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}] \\ &= R\,d\phi'[z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}] \end{aligned} \tag{9.1.10}$$

(5) Writing down $d\vec{\mathbf{B}}$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$\begin{aligned} d\vec{\mathbf{B}} &= \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,')}{|\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,'|^3} \\ &= \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}\,d\phi' \end{aligned} \tag{9.1.11}$$

(6) Carrying out the integration

Using the result obtained above, the magnetic field at P is

$$\vec{\mathbf{B}} = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}\,d\phi' \tag{9.1.12}$$

The x and the y components of $\vec{\mathbf{B}}$ can be readily shown to be zero:

$$B_x = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\cos\phi'\,d\phi' = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\sin\phi'\Big|_0^{2\pi} = 0 \tag{9.1.13}$$

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\sin\phi'\,d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\cos\phi'\Big|_0^{2\pi} = 0 \tag{9.1.14}$$

On the other hand, the z component is

$$B_z = \frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi}d\phi' = \frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}} \tag{9.1.15}$$

Thus, we see that along the symmetric axis, B_z is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using the symmetry arguments.

The behavior of B_z / B_0 where $B_0 = \mu_0 I / 2R$ is the magnetic field strength at $z = 0$, as a function of z / R is shown in Figure 9.1.7:

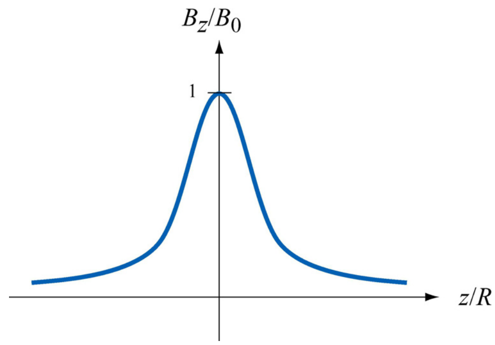


Figure 9.1.7 The ratio of the magnetic field, B_z / B_0, as a function of z / R

(b) If we place a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z \hat{\mathbf{k}}$ at the point P, as discussed in Chapter 8, due to the non-uniformity of the magnetic field, the dipole will experience a force given by

$$\vec{\mathbf{F}}_B = \nabla(\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z \left(\frac{dB_z}{dz} \right) \hat{\mathbf{k}} \tag{9.1.16}$$

Upon differentiating Eq. (9.1.15) and substituting into Eq. (9.1.16), we obtain

$$\vec{\mathbf{F}}_B = -\frac{3\mu_z \mu_0 I R^2 z}{2(R^2 + z^2)^{5/2}} \hat{\mathbf{k}} \tag{9.1.17}$$

Thus, the dipole is attracted toward the current-carrying ring. On the other hand, if the direction of the dipole is reversed, $\vec{\boldsymbol{\mu}} = -\mu_z \hat{\mathbf{k}}$, the resulting force will be repulsive.

9.1.1 Magnetic Field of a Moving Point Charge

Suppose we have an infinitesimal current element in the form of a cylinder of cross-sectional area A and length ds consisting of n charge carriers per unit volume, all moving at a common velocity $\vec{\mathbf{v}}$ along the axis of the cylinder. Let I be the current in the element, which we define as the amount of charge passing through any cross-section of the cylinder per unit time. From Chapter 6, we see that the current I can be written as

$$n A q |\vec{\mathbf{v}}| = I \tag{9.1.18}$$

The total number of charge carriers in the current element is simply $dN = n A\, ds$, so that using Eq. (9.1.1), the magnetic field $d\vec{\mathbf{B}}$ due to the dN charge carriers is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{(nAq\,|\,\vec{\mathbf{v}}\,|)\,d\,\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{(n\,A\,ds)q\,\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{(dN)q\,\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2} \qquad (9.1.19)$$

where r is the distance between the charge and the field point P at which the field is being measured, the unit vector $\hat{\mathbf{r}} = \vec{\mathbf{r}}/r$ points *from* the source of the field (the charge) *to* P. The differential length vector $d\,\vec{\mathbf{s}}$ is defined to be parallel to $\vec{\mathbf{v}}$. In case of a single charge, $dN = 1$, the above equation becomes

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{q\,\vec{\mathbf{v}}\times\hat{\mathbf{r}}}{r^2} \qquad (9.1.20)$$

Note, however, that since a point charge does not constitute a steady current, the above equation strictly speaking only holds in the non-relativistic limit where $v \ll c$, the speed of light, so that the effect of "retardation" can be ignored.

The result may be readily extended to a collection of N point charges, each moving with a different velocity. Let the ith charge q_i be located at (x_i, y_i, z_i) and moving with velocity $\vec{\mathbf{v}}_i$. Using the superposition principle, the magnetic field at P can be obtained as:

$$\vec{\mathbf{B}} = \sum_{i=1}^{N}\frac{\mu_0}{4\pi}q_i\vec{\mathbf{v}}_i \times \left[\frac{(x-x_i)\hat{\mathbf{i}}+(y-y_i)\hat{\mathbf{j}}+(z-z_i)\hat{\mathbf{k}}}{\left[(x-x_i)^2+(y-y_i)^2+(z-z_i)^2\right]^{3/2}}\right] \qquad (9.1.21)$$

Animation 9.1: Magnetic Field of a Moving Charge

Figure 9.1.8 shows one frame of the animations of the magnetic field of a moving positive and negative point charge, assuming the speed of the charge is small compared to the speed of light.

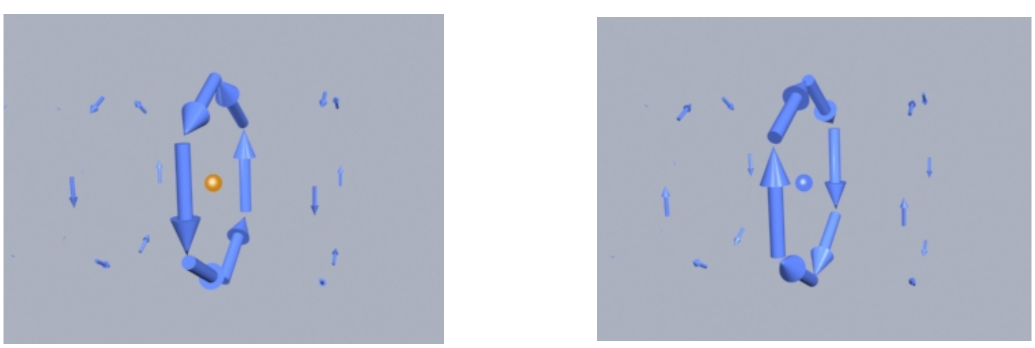


Figure 9.1.8 The magnetic field of (a) a moving positive charge, and (b) a moving negative charge, when the speed of the charge is small compared to the speed of light.

Animation 9.2: Magnetic Field of Several Charges Moving in a Circle

Suppose we want to calculate the magnetic fields of a number of charges moving on the circumference of a circle with equal spacing between the charges. To calculate this field we have to add up vectorially the magnetic fields of each of charges using Eq. (9.1.19).

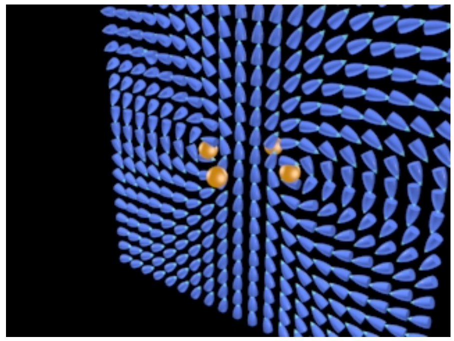


Figure 9.1.9 The magnetic field of four charges moving in a circle. We show the magnetic field vector directions in only one plane. The bullet-like icons indicate the direction of the magnetic field at that point in the array spanning the plane.

Figure 9.1.9 shows one frame of the animation when the number of moving charges is four. Other animations show the same situation for $N = 1$, 2, and 8. When we get to eight charges, a characteristic pattern emerges--the magnetic dipole pattern. Far from the ring, the shape of the field lines is the same as the shape of the field lines for an electric dipole.

Interactive Simulation 9.2: Magnetic Field of a Ring of Moving Charges

Figure 9.1.10 shows a ShockWave display of the vectoral addition process for the case where we have 30 charges moving on a circle. The display in Figure 9.1.10 shows an observation point fixed on the axis of the ring. As the addition proceeds, we also show the resultant up to that point (large arrow in the display).

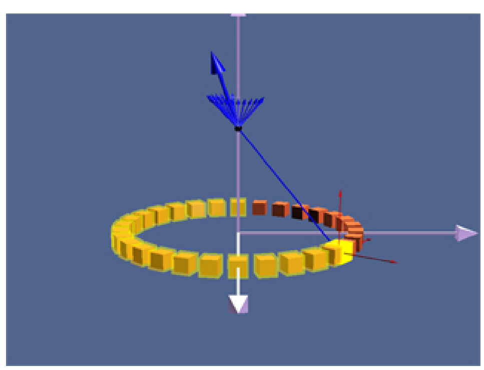


Figure 9.1.10 A ShockWave simulation of the use of the principle of superposition to find the magnetic field due to 30 moving charges moving in a circle at an observation point on the axis of the circle.

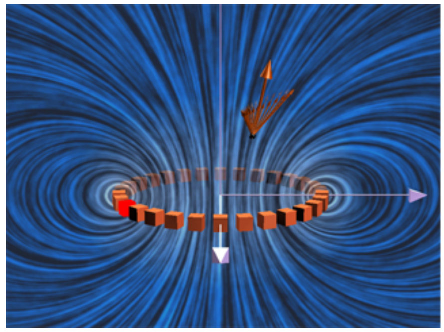


Figure 9.1.11 The magnetic field due to 30 charges moving in a circle at a given observation point. The position of the observation point can be varied to see how the magnetic field of the individual charges adds up to give the total field.

In Figure 9.1.11, we show an interactive ShockWave display that is similar to that in Figure 9.1.10, but now we can interact with the display to move the position of the observer about in space. To get a feel for the total magnetic field, we also show a "iron filings" representation of the magnetic field due to these charges. We can move the observation point about in space to see how the total field at various points arises from the individual contributions of the magnetic field of to each moving charge.

9.2 Force Between Two Parallel Wires

We have already seen that a current-carrying wire produces a magnetic field. In addition, when placed in a magnetic field, a wire carrying a current will experience a net force. Thus, we expect two current-carrying wires to exert force on each other.

Consider two parallel wires separated by a distance a and carrying currents I_1 and I_2 in the $+x$-direction, as shown in Figure 9.2.1.

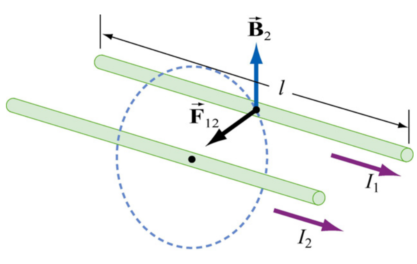


Figure 9.2.1 Force between two parallel wires

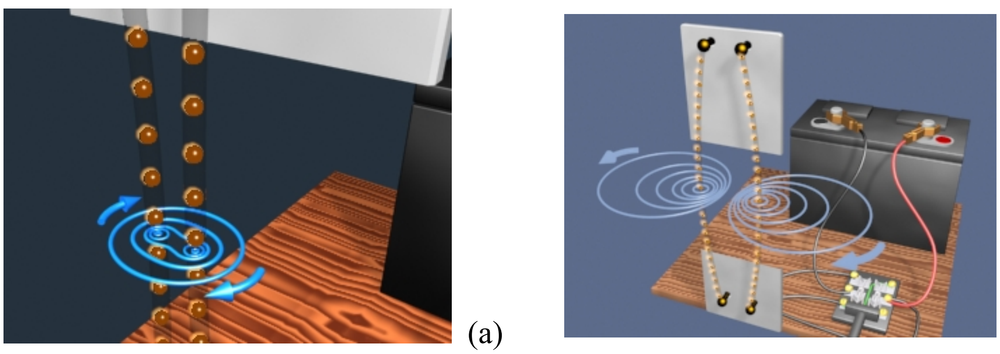
The magnetic force, $\vec{\mathbf{F}}_{12}$, exerted on wire 1 by wire 2 may be computed as follows: Using the result from the previous example, the magnetic field lines due to I_2 going in the $+x$-direction are circles concentric with wire 2, with the field $\vec{\mathbf{B}}_2$ pointing in the tangential direction. Thus, at an arbitrary point P on wire 1, we have $\vec{\mathbf{B}}_2 = -(\mu_0 I_2 / 2\pi a)\hat{\mathbf{j}}$, which points in the direction perpendicular to wire 1, as depicted in Figure 9.2.1. Therefore,

$$\vec{\mathbf{F}}_{12} = I_1 \vec{\boldsymbol{l}} \times \vec{\mathbf{B}}_2 = I_1\left(l\hat{\mathbf{i}}\right)\times\left(-\frac{\mu_0 I_2}{2\pi a}\hat{\mathbf{j}}\right) = -\frac{\mu_0 I_1 I_2 l}{2\pi a}\hat{\mathbf{k}} \tag{9.2.1}$$

Clearly $\vec{\mathbf{F}}_{12}$ points toward wire 2. The conclusion we can draw from this simple calculation is that two parallel wires carrying currents in the same direction will attract each other. On the other hand, if the currents flow in opposite directions, the resultant force will be repulsive.

Animation 9.3: Forces Between Current-Carrying Parallel Wires

Figures 9.2.2 shows parallel wires carrying current in the same and in opposite directions. In the first case, the magnetic field configuration is such as to produce an attraction between the wires. In the second case the magnetic field configuration is such as to produce a repulsion between the wires.



(a) (b)

Figure 9.2.2 (a) The attraction between two wires carrying current in the same direction. The direction of current flow is represented by the motion of the orange spheres in the visualization. (b) The repulsion of two wires carrying current in opposite directions.

9.3 Ampere's Law

We have seen that moving charges or currents are the source of magnetism. This can be readily demonstrated by placing compass needles near a wire. As shown in Figure 9.3.1a, all compass needles point in the same direction in the absence of current. However, when $I \neq 0$, the needles will be deflected along the tangential direction of the circular path (Figure 9.3.1b).

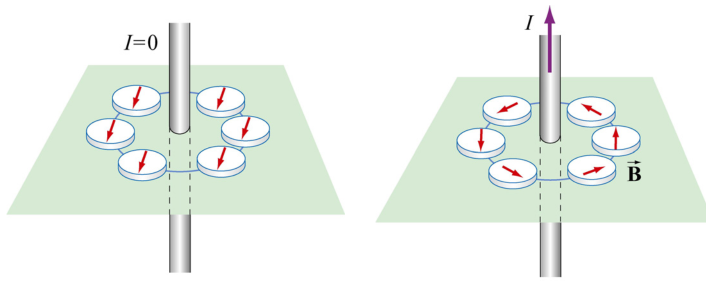


Figure 9.3.1 Deflection of compass needles near a current-carrying wire

Let us now divide a circular path of radius r into a large number of small length vectors $\Delta\vec{\mathbf{s}} = \Delta s\,\hat{\boldsymbol{\varphi}}$, that point along the tangential direction with magnitude Δs (Figure 9.3.2).

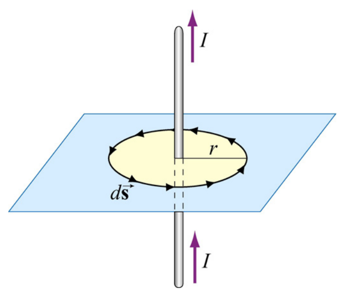


Figure 9.3.2 Amperian loop

In the limit $\Delta\vec{\mathbf{s}} \to \vec{0}$, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = B\oint ds = \left(\frac{\mu_0 I}{2\pi r}\right)\left(2\pi r\right) = \mu_0 I \tag{9.3.1}$$

The result above is obtained by choosing a closed path, or an "Amperian loop" that follows one particular magnetic field line. Let's consider a slightly more complicated Amperian loop, as that shown in Figure 9.3.3

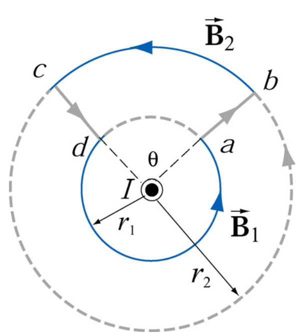


Figure 9.3.3 An Amperian loop involving two field lines

The line integral of the magnetic field around the contour *abcda* is

$$\oint_{abcda} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \int_{ab} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_{bc} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_{cd} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_{cd} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$$
$$= 0 + B_2(r_2\theta) + 0 + B_1[r_1(2\pi-\theta)] \tag{9.3.2}$$

where the length of arc *bc* is $r_2\theta$, and $r_1(2\pi-\theta)$ for arc *da*. The first and the third integrals vanish since the magnetic field is perpendicular to the paths of integration. With $B_1 = \mu_0 I/2\pi r_1$ and $B_2 = \mu_0 I/2\pi r_2$, the above expression becomes

$$\oint_{abcda} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \frac{\mu_0 I}{2\pi r_2}(r_2\theta) + \frac{\mu_0 I}{2\pi r_1}[r_1(2\pi-\theta)] = \frac{\mu_0 I}{2\pi}\theta + \frac{\mu_0 I}{2\pi}(2\pi-\theta) = \mu_0 I \tag{9.3.3}$$

We see that the same result is obtained whether the closed path involves one or two magnetic field lines.

As shown in Example 9.1, in cylindrical coordinates (r,φ,z) with current flowing in the +*z*-axis, the magnetic field is given by $\vec{\mathbf{B}} = (\mu_0 I/2\pi r)\hat{\boldsymbol{\varphi}}$. An arbitrary length element in the cylindrical coordinates can be written as

$$d\vec{\mathbf{s}} = dr\,\hat{\mathbf{r}} + r\,d\varphi\,\hat{\boldsymbol{\varphi}} + dz\,\hat{\mathbf{z}} \tag{9.3.4}$$

which implies

$$\oint_{\text{closed path}} \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \oint_{\text{closed path}} \left(\frac{\mu_0 I}{2\pi r}\right) r\,d\varphi = \frac{\mu_0 I}{2\pi}\oint_{\text{closed path}} d\varphi = \frac{\mu_0 I}{2\pi}(2\pi) = \mu_0 I \tag{9.3.5}$$

In other words, the line integral of $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around any closed Amperian loop is proportional to I_{enc}, the current encircled by the loop.

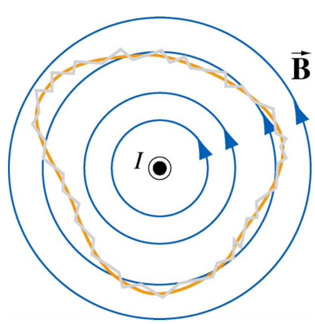


Figure 9.3.4 An Amperian loop of arbitrary shape.

The generalization to any closed loop of arbitrary shape (see for example, Figure 9.3.4) that involves many magnetic field lines is known as Ampere's law:

$$\boxed{\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}} \tag{9.3.6}$$

Ampere's law in magnetism is analogous to Gauss's law in electrostatics. In order to apply them, the system must possess certain symmetry. In the case of an infinite wire, the system possesses cylindrical symmetry and Ampere's law can be readily applied. However, when the length of the wire is finite, Biot-Savart law must be used instead.

Biot-Savart Law	$\vec{\mathbf{B}} = \dfrac{\mu_0 I}{4\pi}\displaystyle\int \dfrac{d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$	general current source ex: finite wire
Ampere's law	$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$	current source has certain symmetry ex: infinite wire (cylindrical)

Ampere's law is applicable to the following current configurations:

1. Infinitely long straight wires carrying a steady current I (Example 9.3)

2. Infinitely large sheet of thickness b with a current density J (Example 9.4).

3. Infinite solenoid (Section 9.4).

4. Toroid (Example 9.5).

We shall examine all four configurations in detail.

Example 9.3: Field Inside and Outside a Current-Carrying Wire

Consider a long straight wire of radius R carrying a current I of uniform current density, as shown in Figure 9.3.5. Find the magnetic field everywhere.

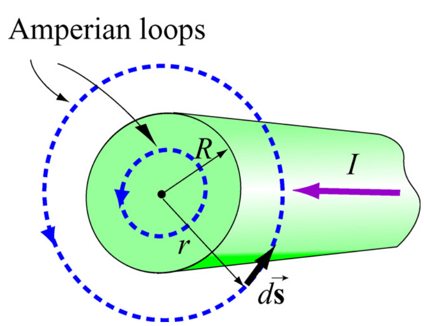


Figure 9.3.5 Amperian loops for calculating the $\vec{\mathbf{B}}$ field of a conducting wire of radius R.

Solution:

(i) Outside the wire where $r \geq R$, the Amperian loop (circle 1) completely encircles the current, i.e., $I_{\text{enc}} = I$. Applying Ampere's law yields

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = B \oint ds = B(2\pi r) = \mu_0 I$$

which implies

$$B = \frac{\mu_0 I}{2\pi r}$$

(ii) Inside the wire where $r < R$, the amount of current encircled by the Amperian loop (circle 2) is proportional to the area enclosed, i.e.,

$$I_{\text{enc}} = \left(\frac{\pi r^2}{\pi R^2}\right) I$$

Thus, we have

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = B(2\pi r) = \mu_0 I \left(\frac{\pi r^2}{\pi R^2}\right) \quad \Rightarrow \quad \boxed{B = \frac{\mu_0 I r}{2\pi R^2}}$$

We see that the magnetic field is zero at the center of the wire and increases linearly with r until $r=R$. Outside the wire, the field falls off as $1/r$. The qualitative behavior of the field is depicted in Figure 9.3.6 below:

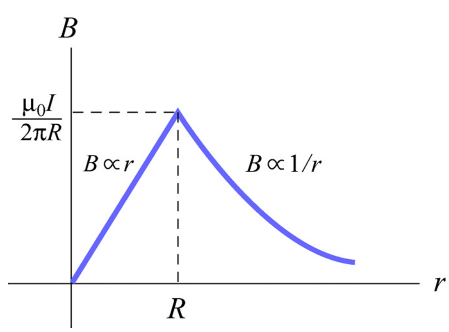


Figure 9.3.6 Magnetic field of a conducting wire of radius R carrying a steady current I.

Example 9.4: Magnetic Field Due to an Infinite Current Sheet

Consider an infinitely large sheet of thickness b lying in the xy plane with a uniform current density $\vec{\mathbf{J}} = J_0 \hat{\mathbf{i}}$. Find the magnetic field everywhere.

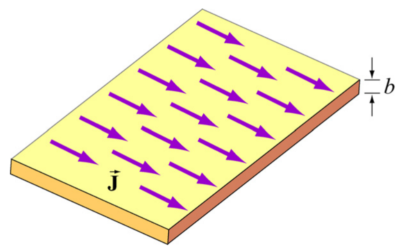


Figure 9.3.7 An infinite sheet with current density $\vec{\mathbf{J}} = J_0\hat{\mathbf{i}}$.

Solution:

We may think of the current sheet as a set of parallel wires carrying currents in the $+x$-direction. From Figure 9.3.8, we see that magnetic field at a point P above the plane points in the $-y$-direction. The z-component vanishes after adding up the contributions from all wires. Similarly, we may show that the magnetic field at a point below the plane points in the $+y$-direction.

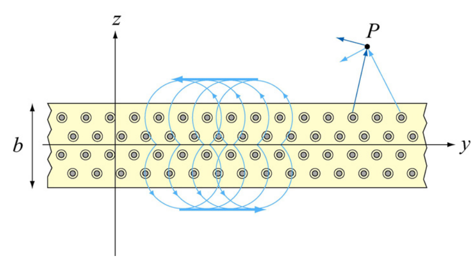


Figure 9.3.8 Magnetic field of a current sheet

We may now apply Ampere's law to find the magnetic field due to the current sheet. The Amperian loops are shown in Figure 9.3.9.

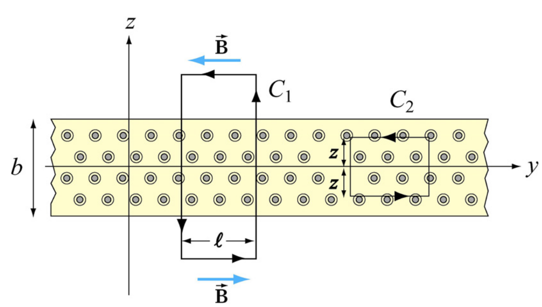


Figure 9.3.9 Amperian loops for the current sheets

For the field outside, we integrate along path C_1. The amount of current enclosed by C_1 is

$$I_{\text{enc}} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(b\ell) \tag{9.3.7}$$

Applying Ampere's law leads to

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{\text{enc}} = \mu_0(J_0 b\ell) \tag{9.3.8}$$

or $B = \mu_0 J_0 b/2$. Note that the magnetic field outside the sheet is constant, independent of the distance from the sheet. Next we find the magnetic field inside the sheet. The amount of current enclosed by path C_2 is

$$I_{\text{enc}} = \iint \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = J_0(2\,|\,z\,|\,\ell) \tag{9.3.9}$$

Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = B(2\ell) = \mu_0 I_{\text{enc}} = \mu_0 J_0(2\,|\,z\,|\,\ell) \tag{9.3.10}$$

or $B = \mu_0 J_0\,|\,z\,|$. At $z = 0$, the magnetic field vanishes, as required by symmetry. The results can be summarized using the unit-vector notation as

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 J_0 b}{2}\,\hat{\mathbf{j}}, & z > b/2 \\ -\mu_0 J_0 z\,\hat{\mathbf{j}}, & -b/2 < z < b/2 \\ \dfrac{\mu_0 J_0 b}{2}\,\hat{\mathbf{j}}, & z < -b/2 \end{cases} \tag{9.3.11}$$

Let's now consider the limit where the sheet is infinitesimally thin, with $b \to 0$. In this case, instead of current density $\vec{\mathbf{J}} = J_0\hat{\mathbf{i}}$, we have surface current $\vec{\mathbf{K}} = K\hat{\mathbf{i}}$, where $K = J_0 b$. Note that the dimension of K is current/length. In this limit, the magnetic field becomes

$$\vec{\mathbf{B}} = \begin{cases} -\dfrac{\mu_0 K}{2}\,\hat{\mathbf{j}}, & z > 0 \\ \dfrac{\mu_0 K}{2}\,\hat{\mathbf{j}}, & z < 0 \end{cases} \tag{9.3.12}$$

9.4 Solenoid

A solenoid is a long coil of wire tightly wound in the helical form. Figure 9.4.1 shows the magnetic field lines of a solenoid carrying a steady current *I*. We see that if the turns are closely spaced, the resulting magnetic field inside the solenoid becomes fairly uniform, provided that the length of the solenoid is much greater than its diameter. For an "ideal" solenoid, which is infinitely long with turns tightly packed, the magnetic field inside the solenoid is uniform and parallel to the axis, and vanishes outside the solenoid.

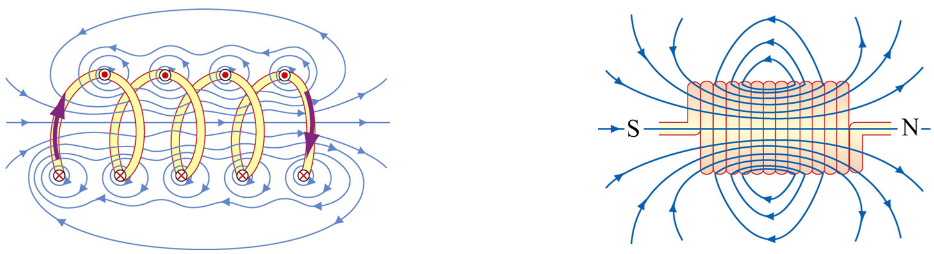


Figure 9.4.1 Magnetic field lines of a solenoid

We can use Ampere's law to calculate the magnetic field strength inside an ideal solenoid. The cross-sectional view of an ideal solenoid is shown in Figure 9.4.2. To compute $\vec{\mathbf{B}}$, we consider a rectangular path of length l and width w and traverse the path in a counterclockwise manner. The line integral of $\vec{\mathbf{B}}$ along this loop is

$$\begin{aligned}\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} &= \int_1 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_2 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_3 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} + \int_4 \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} \\ &= \quad 0 \quad + \quad 0 \quad + \quad Bl \quad + \quad 0\end{aligned} \tag{9.4.1}$$

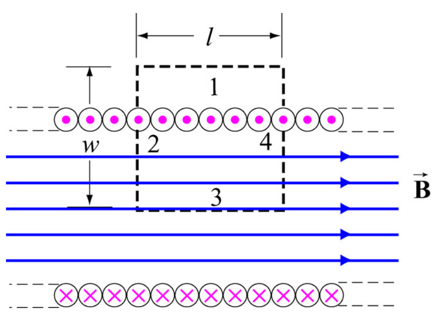


Figure 9.4.2 Amperian loop for calculating the magnetic field of an ideal solenoid.

In the above, the contributions along sides 2 and 4 are zero because $\vec{\mathbf{B}}$ is perpendicular to $d\vec{\mathbf{s}}$. In addition, $\vec{\mathbf{B}} = \vec{\mathbf{0}}$ along side 1 because the magnetic field is non-zero only inside the solenoid. On the other hand, the total current enclosed by the Amperian loop is $I_{\text{enc}} = NI$, where N is the total number of turns. Applying Ampere's law yields

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = Bl = \mu_0 NI \tag{9.4.2}$$

or

$$B = \frac{\mu_0 N I}{l} = \mu_0 n I \tag{9.4.3}$$

where $n = N/l$ represents the number of turns per unit length., In terms of the surface current, or current per unit length $K = nI$, the magnetic field can also be written as,

$$B = \mu_0 K \tag{9.4.4}$$

What happens if the length of the solenoid is finite? To find the magnetic field due to a finite solenoid, we shall approximate the solenoid as consisting of a large number of circular loops stacking together. Using the result obtained in Example 9.2, the magnetic field at a point P on the z axis may be calculated as follows: Take a cross section of tightly packed loops located at z' with a thickness dz', as shown in Figure 9.4.3

The amount of current flowing through is proportional to the thickness of the cross section and is given by $dI = I(ndz') = I(N/l)dz'$, where $n = N/l$ is the number of turns per unit length.

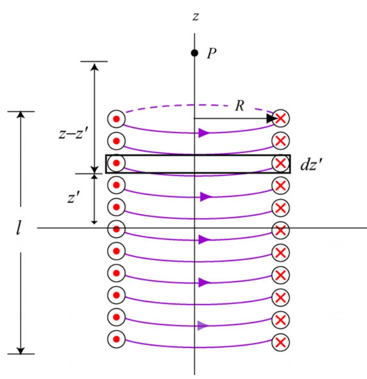


Figure 9.4.3 Finite Solenoid

The contribution to the magnetic field at P due to this subset of loops is

$$dB_z = \frac{\mu_0 R^2}{2[(z-z')^2 + R^2]^{3/2}} dI = \frac{\mu_0 R^2}{2[(z-z')^2 + R^2]^{3/2}} (nIdz') \tag{9.4.5}$$

Integrating over the entire length of the solenoid, we obtain

$$\begin{aligned} B_z &= \frac{\mu_0 n I R^2}{2} \int_{-l/2}^{l/2} \frac{dz'}{[(z-z')^2 + R^2]^{3/2}} = \frac{\mu_0 n I R^2}{2} \left. \frac{z'-z}{R^2 \sqrt{(z-z')^2 + R^2}} \right|_{-l/2}^{l/2} \\ &= \frac{\mu_0 n I}{2} \left[\frac{(l/2) - z}{\sqrt{(z - l/2)^2 + R^2}} + \frac{(l/2) + z}{\sqrt{(z + l/2)^2 + R^2}} \right] \end{aligned} \tag{9.4.6}$$

A plot of B_z / B_0, where $B_0 = \mu_0 n I$ is the magnetic field of an infinite solenoid, as a function of z / R is shown in Figure 9.4.4 for $l = 10R$ and $l = 20R$.

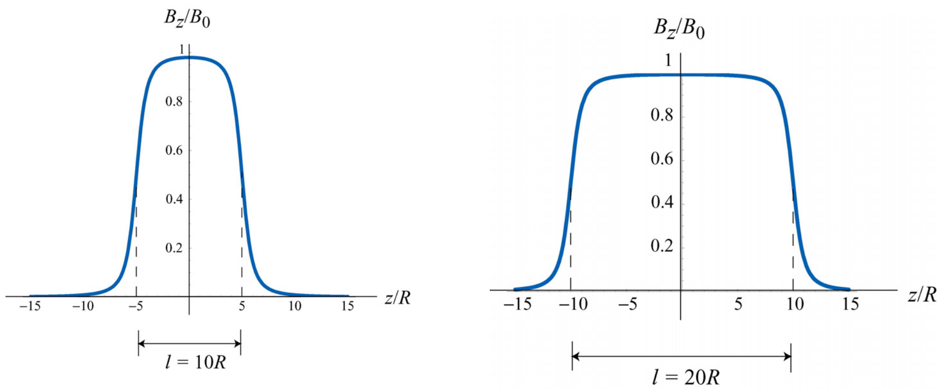


Figure 9.4.4 Magnetic field of a finite solenoid for (a) $l = 10R$, and (b) $l = 20R$.

Notice that the value of the magnetic field in the region $|z| < l/2$ is nearly uniform and approximately equal to B_0.

Examaple 9.5: Toroid

Consider a toroid which consists of N turns, as shown in Figure 9.4.5. Find the magnetic field everywhere.

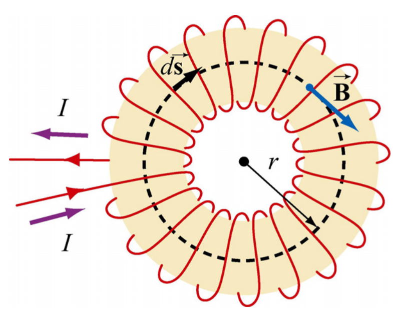


Figure 9.4.5 A toroid with N turns

Solutions:

One can think of a toroid as a solenoid wrapped around with its ends connected. Thus, the magnetic field is completely confined inside the toroid and the field points in the azimuthal direction (clockwise due to the way the current flows, as shown in Figure 9.4.5.)

Applying Ampere's law, we obtain

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \oint B\,ds = B\oint ds = B(2\pi r) = \mu_0 NI \tag{9.4.7}$$

or

$$B = \frac{\mu_0 NI}{2\pi r} \tag{9.4.8}$$

where r is the distance measured from the center of the toroid.. Unlike the magnetic field of a solenoid, the magnetic field inside the toroid is non-uniform and decreases as $1/r$.

9.5 Magnetic Field of a Dipole

Let a magnetic dipole moment vector $\vec{\boldsymbol{\mu}} = -\mu\hat{\mathbf{k}}$ be placed at the origin (*e.g.*, center of the Earth) in the yz plane. What is the magnetic field at a point (*e.g.*, MIT) a distance r away from the origin?

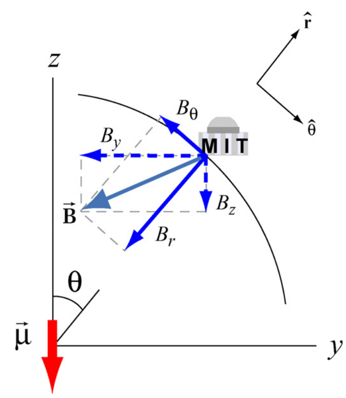


Figure 9.5.1 Earth's magnetic field components

In Figure 9.5.1 we show the magnetic field at MIT due to the dipole. The *y*- and *z*-components of the magnetic field are given by

$$B_y = -\frac{\mu_0}{4\pi}\frac{3\mu}{r^3}\sin\theta\cos\theta, \qquad B_z = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}(3\cos^2\theta - 1) \tag{9.5.1}$$

Readers are referred to Section 9.8 for the detail of the derivation.

In spherical coordinates (r,θ,ϕ), the radial and the polar components of the magnetic field can be written as

$$B_r = B_y \sin\theta + B_z \cos\theta = -\frac{\mu_0}{4\pi}\frac{2\mu}{r^3}\cos\theta \tag{9.5.2}$$

and

$$B_\theta = B_y \cos\theta - B_z \sin\theta = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}\sin\theta \tag{9.5.3}$$

respectively. Thus, the magnetic field at MIT due to the dipole becomes

$$\vec{\mathbf{B}} = B_\theta\,\hat{\boldsymbol{\theta}} + B_r\,\hat{\mathbf{r}} = -\frac{\mu_0}{4\pi}\frac{\mu}{r^3}(\sin\theta\,\hat{\boldsymbol{\theta}} + 2\cos\theta\,\hat{\mathbf{r}}) \tag{9.5.4}$$

Notice the similarity between the above expression and the electric field due to an electric dipole $\vec{\mathbf{p}}$ (see Solved Problem 2.13.6):

$$\vec{\mathbf{E}} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}(\sin\theta\,\hat{\boldsymbol{\theta}} + 2\cos\theta\,\hat{\mathbf{r}})$$

The negative sign in Eq. (9.5.4) is due to the fact that the magnetic dipole points in the $-z$-direction. In general, the magnetic field due to a dipole moment $\vec{\boldsymbol{\mu}}$ can be written as

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{3(\vec{\boldsymbol{\mu}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \vec{\boldsymbol{\mu}}}{r^3} \tag{9.5.5}$$

The ratio of the radial and the polar components is given by

$$\frac{B_r}{B_\theta} = \frac{-\dfrac{\mu_0}{4\pi}\dfrac{2\mu}{r^3}\cos\theta}{-\dfrac{\mu_0}{4\pi}\dfrac{\mu}{r^3}\sin\theta} = 2\cot\theta \tag{9.5.6}$$

9.5.1 Earth's Magnetic Field at MIT

The Earth's field behaves as if there were a bar magnet in it. In Figure 9.5.2 an imaginary magnet is drawn inside the Earth oriented to produce a magnetic field like that of the Earth's magnetic field. Note the South pole of such a magnet in the northern hemisphere in order to attract the North pole of a compass.

It is most natural to represent the location of a point P on the surface of the Earth using the spherical coordinates (r,θ,ϕ), where r is the distance from the center of the Earth, θ is the polar angle from the z-axis, with $0 \le \theta \le \pi$, and ϕ is the azimuthal angle in the xy plane, measured from the x-axis, with $0 \le \phi \le 2\pi$ (See Figure 9.5.3.) With the distance fixed at $r = r_E$, the radius of the Earth, the point P is parameterized by the two angles θ and ϕ.

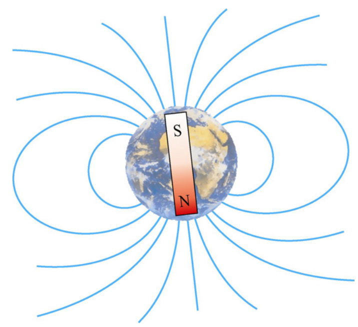


Figure 9.5.2 Magnetic field of the Earth

In practice, a location on Earth is described by two numbers – latitude and longitude. How are they related to θ and ϕ? The latitude of a point, denoted as δ, is a measure of the elevation from the plane of the equator. Thus, it is related to θ (commonly referred to as the colatitude) by $\delta = 90° - \theta$. Using this definition, the equator has latitude $0°$, and the north and the south poles have latitude $\pm 90°$, respectively.

The longitude of a location is simply represented by the azimuthal angle ϕ in the spherical coordinates. Lines of constant longitude are generally referred to as *meridians*. The value of longitude depends on where the counting begins. For historical reasons, the meridian passing through the Royal Astronomical Observatory in Greenwich, UK, is chosen as the "prime meridian" with zero longitude.

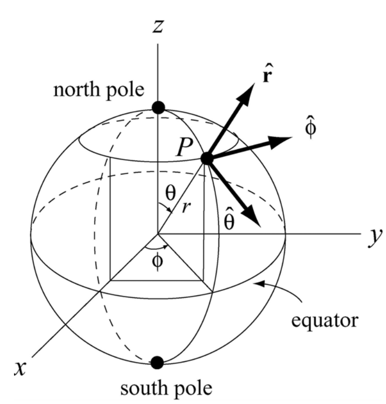


Figure 9.5.3 Locating a point P on the surface of the Earth using spherical coordinates.

Let the z-axis be the Earth's rotation axis, and the x-axis passes through the prime meridian. The corresponding magnetic dipole moment of the Earth can be written as

$$\begin{aligned} \vec{\boldsymbol{\mu}}_E &= \mu_E(\sin\theta_0 \cos\phi_0\,\hat{\mathbf{i}} + \sin\theta_0 \sin\phi_0\,\hat{\mathbf{j}} + \cos\theta_0\,\hat{\mathbf{k}}) \\ &= \mu_E(-0.062\,\hat{\mathbf{i}} + 0.18\,\hat{\mathbf{j}} - 0.98\,\hat{\mathbf{k}}) \end{aligned} \tag{9.5.7}$$

where $\mu_E = 7.79\times 10^{22}\,\mathrm{A\cdot m^2}$, and we have used $(\theta_0,\phi_0) = (169°,109°)$. The expression shows that $\vec{\boldsymbol{\mu}}_E$ has non-vanishing components in all three directions in the Cartesian coordinates.

On the other hand, the location of MIT is $42°\mathrm{N}$ for the latitude and $71°\mathrm{W}$ for the longitude ($42°$ north of the equator, and $71°$ west of the prime meridian), which means that $\theta_m = 90° - 42° = 48°$, and $\phi_m = 360° - 71° = 289°$. Thus, the position of MIT can be described by the vector

$$\begin{aligned}\vec{\mathbf{r}}_{\mathrm{MIT}} &= r_E(\sin\theta_m\cos\phi_m\,\hat{\mathbf{i}} + \sin\theta_m\sin\phi_m\,\hat{\mathbf{j}} + \cos\theta_m\,\hat{\mathbf{k}}) \\ &= r_E(0.24\,\hat{\mathbf{i}} - 0.70\,\hat{\mathbf{j}} + 0.67\,\hat{\mathbf{k}})\end{aligned} \tag{9.5.8}$$

The angle between $-\vec{\boldsymbol{\mu}}_E$ and $\vec{\mathbf{r}}_{\mathrm{MIT}}$ is given by

$$\theta_{ME} = \cos^{-1}\left(\frac{-\vec{\mathbf{r}}_{\mathrm{MIT}}\cdot\vec{\boldsymbol{\mu}}_E}{|\vec{\mathbf{r}}_{\mathrm{MIT}}||-\vec{\boldsymbol{\mu}}_E|}\right) = \cos^{-1}(0.80) = 37° \tag{9.5.9}$$

Note that the polar angle θ is defined as $\theta = \cos^{-1}(\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})$, the inverse of cosine of the dot product between a unit vector $\hat{\mathbf{r}}$ for the position, and a unit vector $+\hat{\mathbf{k}}$ in the *positive z*-direction, as indicated in Figure 9.6.1. Thus, if we measure the ratio of the radial to the polar component of the Earth's magnetic field at MIT, the result would be

$$\boxed{\frac{B_r}{B_\theta} = 2\cot 37° \approx 2.65} \tag{9.5.10}$$

Note that the positive radial (vertical) direction is chosen to point outward and the positive polar (horizontal) direction points towards the equator.

Animation 9.4: Bar Magnet in the Earth's Magnetic Field

Figure 9.5.4 shows a bar magnet and compass placed on a table. The interaction between the magnetic field of the bar magnet and the magnetic field of the earth is illustrated by the field lines that extend out from the bar magnet. Field lines that emerge towards the edges of the magnet generally reconnect to the magnet near the opposite pole. However, field lines that emerge near the poles tend to wander off and reconnect to the magnetic field of the earth, which, in this case, is approximately a constant field coming at 60 degrees from the horizontal. Looking at the compass, one can see that a compass needle will always align itself in the direction of the local field. In this case, the local field is dominated by the bar magnet.

Click and drag the mouse to rotate the scene. Control-click and drag to zoom in and out.

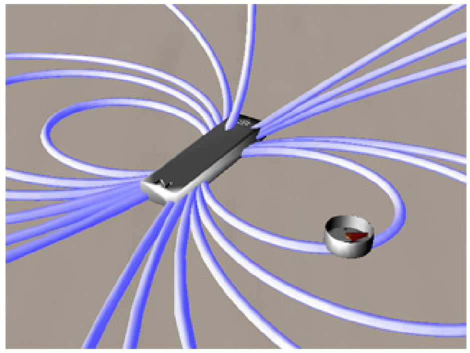


Figure 9.5.4 A bar magnet in Earth's magnetic field

9.6 Magnetic Materials

The introduction of material media into the study of magnetism has very different consequences as compared to the introduction of material media into the study of electrostatics. When we dealt with dielectric materials in electrostatics, their effect was *always* to reduce $\vec{\mathbf{E}}$ below what it would otherwise be, for a given amount of "free" electric charge. In contrast, when we deal with magnetic materials, their effect can be one of the following:

(i) reduce $\vec{\mathbf{B}}$ below what it would otherwise be, for the same amount of "free" electric current (*diamagnetic* materials);

(ii) increase $\vec{\mathbf{B}}$ a little above what it would otherwise be (*paramagnetic* materials);

(iii) increase $\vec{\mathbf{B}}$ *a lot* above what it would otherwise be (*ferromagnetic* materials).

Below we discuss how these effects arise.

9.6.1 Magnetization

Magnetic materials consist of many permanent or induced magnetic dipoles. One of the concepts crucial to the understanding of magnetic materials is the average magnetic field produced by many magnetic dipoles which are all aligned. Suppose we have a piece of material in the form of a long cylinder with area A and height L, and that it consists of N magnetic dipoles, each with magnetic dipole moment $\vec{\boldsymbol{\mu}}$, spread uniformly throughout the volume of the cylinder, as shown in Figure 9.6.1.

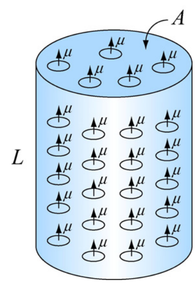


Figure 9.6.1 A cylinder with N magnetic dipole moments

We also assume that all of the magnetic dipole moments $\vec{\boldsymbol{\mu}}$ are aligned with the axis of the cylinder. In the absence of any external magnetic field, what is the average magnetic field due to these dipoles alone?

To answer this question, we note that each magnetic dipole has its own magnetic field associated with it. Let's define the magnetization vector $\vec{\mathbf{M}}$ to be the net magnetic dipole moment vector per unit volume:

$$\vec{\mathbf{M}} = \frac{1}{V}\sum_i \vec{\boldsymbol{\mu}}_i \tag{9.6.1}$$

where V is the volume. In the case of our cylinder, where all the dipoles are aligned, the magnitude of $\vec{\mathbf{M}}$ is simply $M = N\mu / AL$.
Now, what is the average magnetic field produced by all the dipoles in the cylinder?

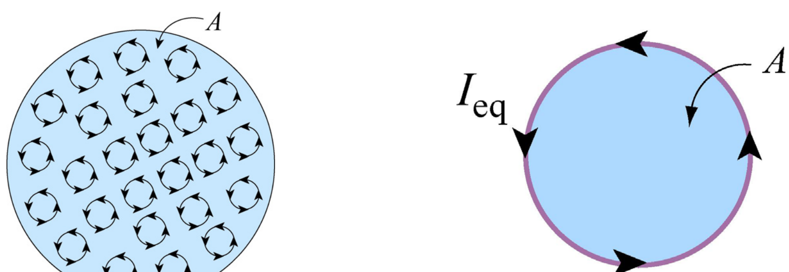


Figure 9.6.2 (a) Top view of the cylinder containing magnetic dipole moments. (b) The equivalent current.

Figure 9.6.2(a) depicts the small current loops associated with the dipole moments and the direction of the currents, as seen from above. We see that in the interior, currents flow in a given direction will be cancelled out by currents flowing in the opposite direction in neighboring loops. The only place where cancellation does not take place is near the edge of the cylinder where there are no adjacent loops further out. Thus, the average current in the interior of the cylinder vanishes, whereas the sides of the cylinder appear to carry a net current. The equivalent situation is shown in Figure 9.6.2(b), where there is an equivalent current I_{eq} on the sides.

The functional form of I_{eq} may be deduced by requiring that the magnetic dipole moment produced by I_{eq} be the same as total magnetic dipole moment of the system. The condition gives

$$I_{\text{eq}} A = N\mu \tag{9.6.2}$$

or

$$I_{\text{eq}} = \frac{N\mu}{A} \tag{9.6.3}$$

Next, let's calculate the magnetic field produced by I_{eq}. With I_{eq} running on the sides, the equivalent configuration is identical to a solenoid carrying a surface current (or current per unit length) K. The two quantities are related by

$$K = \frac{I_{\text{eq}}}{L} = \frac{N\mu}{AL} = M \tag{9.6.4}$$

Thus, we see that the surface current K is equal to the magnetization M, which is the average magnetic dipole moment per unit volume. The average magnetic field produced by the equivalent current system is given by (see Section 9.4)

$$B_M = \mu_0 K = \mu_0 M \tag{9.6.5}$$

Since the direction of this magnetic field is in the *same* direction as $\vec{\mathbf{M}}$, the above expression may be written in vector notation as

$$\vec{\mathbf{B}}_M = \mu_0 \vec{\mathbf{M}} \tag{9.6.6}$$

This is exactly opposite from the situation with electric dipoles, in which the average electric field is anti-parallel to the direction of the electric dipoles themselves. The reason is that in the region interior to the current loop of the dipole, the magnetic field is in the *same* direction as the magnetic dipole vector. Therefore, it is not surprising that after a large-scale averaging, the average magnetic field also turns out to be parallel to the average magnetic dipole moment per unit volume.

Notice that the magnetic field in Eq. (9.6.6) is the *average* field due to all the dipoles. A very different field is observed if we go close to any one of these little dipoles.

Let's now examine the properties of different magnetic materials

9.6.2 Paramagnetism

The atoms or molecules comprising paramagnetic materials have a permanent magnetic dipole moment. Left to themselves, the permanent magnetic dipoles in a paramagnetic material never line up spontaneously. In the absence of any applied external magnetic field, they are randomly aligned. Thus, $\vec{\mathbf{M}} = \vec{\mathbf{0}}$ and the average magnetic field $\vec{\mathbf{B}}_M$ is also zero. However, when we place a paramagnetic material in an external field $\vec{\mathbf{B}}_0$, the dipoles experience a torque $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\mu}} \times \vec{\mathbf{B}}_0$ that tends to align $\vec{\boldsymbol{\mu}}$ with $\vec{\mathbf{B}}_0$, thereby producing a net magnetization $\vec{\mathbf{M}}$ parallel to $\vec{\mathbf{B}}_0$. Since $\vec{\mathbf{B}}_M$ is parallel to $\vec{\mathbf{B}}_0$, it will tend to *enhance* $\vec{\mathbf{B}}_0$. The total magnetic field $\vec{\mathbf{B}}$ is the sum of these two fields:

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_0 + \vec{\mathbf{B}}_M = \vec{\mathbf{B}}_0 + \mu_0 \vec{\mathbf{M}} \tag{9.6.7}$$

Note how different this is than in the case of dielectric materials. In both cases, the torque on the dipoles causes alignment of the dipole vector parallel to the external field. However, in the paramagnetic case, that alignment *enhances* the external magnetic field, whereas in the dielectric case it *reduces* the external electric field. In most paramagnetic substances, the magnetization $\vec{\mathbf{M}}$ is not only in the same direction as $\vec{\mathbf{B}}_0$, but also linearly proportional to $\vec{\mathbf{B}}_0$. This is plausible because without the external field $\vec{\mathbf{B}}_0$ there would be no alignment of dipoles and hence no magnetization $\vec{\mathbf{M}}$. The linear relation between $\vec{\mathbf{M}}$ and $\vec{\mathbf{B}}_0$ is expressed as

$$\vec{\mathbf{M}} = \chi_m \frac{\vec{\mathbf{B}}_0}{\mu_0} \tag{9.6.8}$$

where χ_m is a dimensionless quantity called the *magnetic susceptibility*. Eq. (10.7.7) can then be written as

$$\vec{\mathbf{B}} = (1 + \chi_m)\vec{\mathbf{B}}_0 = \kappa_m \vec{\mathbf{B}}_0 \tag{9.6.9}$$

where

$$\kappa_m = 1 + \chi_m \tag{9.6.10}$$

is called the *relative permeability* of the material. For paramagnetic substances, $\kappa_m > 1$, or equivalently, $\chi_m > 0$, although χ_m is usually on the order of 10^{-6} to 10^{-3}. The *magnetic permeability* μ_m of a material may also be defined as

$$\mu_m = (1 + \chi_m)\mu_0 = \kappa_m \mu_0 \tag{9.6.11}$$

Paramagnetic materials have $\mu_m > \mu_0$.

9.6.3 Diamagnetism

In the case of magnetic materials where there are no permanent magnetic dipoles, the presence of an external field $\vec{\mathbf{B}}_0$ will induce magnetic dipole moments in the atoms or molecules. However, these induced magnetic dipoles are anti-parallel to $\vec{\mathbf{B}}_0$, leading to a magnetization $\vec{\mathbf{M}}$ and average field $\vec{\mathbf{B}}_M$ anti-parallel to $\vec{\mathbf{B}}_0$, and therefore a *reduction* in the total magnetic field strength. For diamagnetic materials, we can still define the magnetic permeability, as in equation (8-5), although now $\kappa_m < 1$, or $\chi_m < 0$, although χ_m is usually on the order of -10^{-5} to -10^{-9}. Diamagnetic materials have $\mu_m < \mu_0$.

9.6.4 Ferromagnetism

In ferromagnetic materials, there is a strong interaction between neighboring atomic dipole moments. Ferromagnetic materials are made up of small patches called *domains*, as illustrated in Figure 9.6.3(a). An externally applied field $\vec{\mathbf{B}}_0$ will tend to line up those magnetic dipoles parallel to the external field, as shown in Figure 9.6.3(b). The strong interaction between neighboring atomic dipole moments causes a *much stronger* alignment of the magnetic dipoles than in paramagnetic materials.

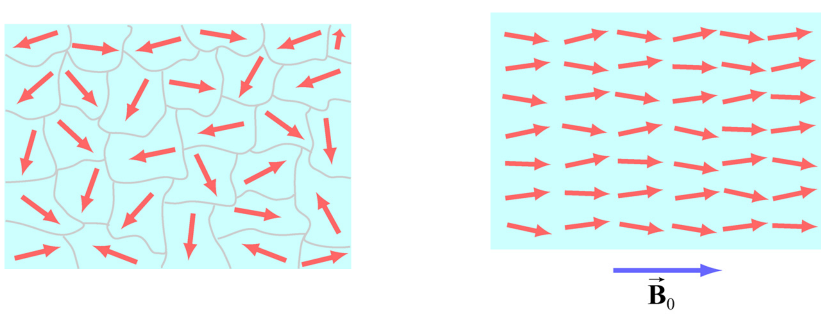


Figure 9.6.3 (a) Ferromagnetic domains. (b) Alignment of magnetic moments in the direction of the external field $\vec{\mathbf{B}}_0$.

The enhancement of the applied external field can be considerable, with the total magnetic field inside a ferromagnet 10^3 or 10^4 times greater than the applied field. The permeability κ_m of a ferromagnetic material is not a constant, since neither the total field $\vec{\mathbf{B}}$ or the magnetization $\vec{\mathbf{M}}$ increases linearly with $\vec{\mathbf{B}}_0$. In fact the relationship between $\vec{\mathbf{M}}$ and $\vec{\mathbf{B}}_0$ is not unique, but dependent on the previous history of the material. The phenomenon is known as *hysteresis*. The variation of $\vec{\mathbf{M}}$ as a function of the externally applied field $\vec{\mathbf{B}}_0$ is shown in Figure 9.6.4. The loop *abcdef* is a *hysteresis curve*.

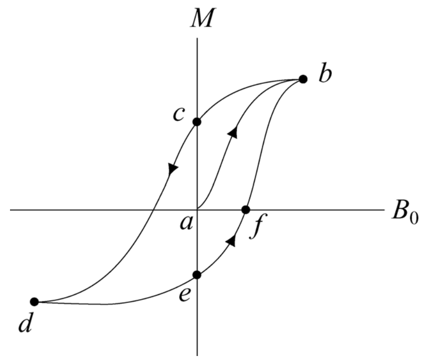


Figure 9.6.4 A hysteresis curve.

Moreover, in ferromagnets, the strong interaction between neighboring atomic dipole moments can keep those dipole moments aligned, even when the external magnet field is reduced to zero. And these aligned dipoles can thus produce a strong magnetic field, all by themselves, without the necessity of an external magnetic field. This is the origin of permanent magnets. To see how strong such magnets can be, consider the fact that magnetic dipole moments of atoms typically have magnitudes of the order of $10^{-23}\ \text{A}\cdot\text{m}^2$. Typical atomic densities are 10^{29} atoms/m^3. If all these dipole moments are aligned, then we would get a magnetization of order

$$M \sim (10^{-23}\ \text{A}\cdot\text{m}^2)(10^{29}\ \text{atoms/m}^3) \sim 10^6\ \text{A/m} \tag{9.6.12}$$

The magnetization corresponds to values of $\vec{\mathbf{B}}_M = \mu_0 \vec{\mathbf{M}}$ of order 1 tesla, or 10,000 Gauss, just due to the atomic currents alone. This is how we get permanent magnets with fields of order 2200 Gauss.

9.7 Summary

- **Biot-Savart law** states that the magnetic field $d\vec{\mathbf{B}}$ at a point due to a length element $d\vec{\mathbf{s}}$ carrying a steady current I and located at $\vec{\mathbf{r}}$ away is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\, d\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

  where $r = |\vec{\mathbf{r}}|$ and $\mu_0 = 4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A}$ is the permeability of free space.

- The magnitude of the magnetic field at a distance r away from an infinitely long straight wire carrying a current I is

$$B = \frac{\mu_0 I}{2\pi r}$$

- The magnitude of the magnetic force F_B between two straight wires of length ℓ carrying steady current of I_1 and I_2 and separated by a distance r is

$$F_B = \frac{\mu_0 I_1 I_2 \ell}{2\pi r}$$

- **Ampere's law** states that the line integral of $\vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total steady current passing through any surface that is bounded by the close loop:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$$

- The magnetic field inside a **toroid** which has N closely spaced of wire carrying a current I is given by

$$B = \frac{\mu_0 N I}{2\pi r}$$

  where r is the distance from the center of the toroid.

- The magnetic field inside a **solenoid** which has N closely spaced of wire carrying current I in a length of l is given by

$$B = \mu_0 \frac{N}{l} I = \mu_0 n I$$

  where n is the number of number of turns per unit length.

- The properties of magnetic materials are as follows:

Materials	Magnetic susceptibility χ_m	Relative permeability $\kappa_m = 1 + \chi_m$	Magnetic permeability $\mu_m = \kappa_m \mu_0$
Diamagnetic	$-10^{-5} \sim -10^{-9}$	$\kappa_m < 1$	$\mu_m < \mu_0$
Paramagnetic	$10^{-5} \sim 10^{-3}$	$\kappa_m > 1$	$\mu_m > \mu_0$
Ferromagnetic	$\chi_m \gg 1$	$\kappa_m \gg 1$	$\mu_m \gg \mu_0$

9.8 Appendix 1: Magnetic Field off the Symmetry Axis of a Current Loop

In Example 9.2 we calculated the magnetic field due to a circular loop of radius *R* lying in the *xy* plane and carrying a steady current *I*, at a point *P* along the axis of symmetry. Let's see how the same technique can be extended to calculating the field at a point off the axis of symmetry in the *yz* plane.

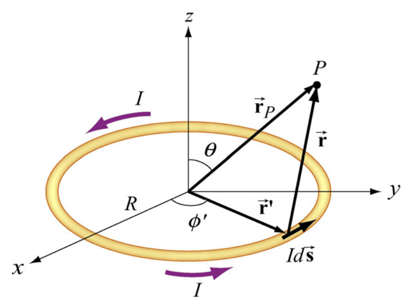


Figure 9.8.1 Calculating the magnetic field off the symmetry axis of a current loop.

Again, as shown in Example 9.1, the differential current element is

$$Id\,\vec{\mathbf{s}} = R\,d\phi'(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}})$$

and its position is described by $\vec{\mathbf{r}}' = R(\cos\phi'\,\hat{\mathbf{i}} + \sin\phi'\,\hat{\mathbf{j}})$. On the other hand, the field point *P* now lies in the *yz* plane with $\vec{\mathbf{r}}_P = y\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$, as shown in Figure 9.8.1. The corresponding relative position vector is

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = -R\cos\phi'\,\hat{\mathbf{i}} + \left(y - R\sin\phi'\right)\hat{\mathbf{j}} + z\hat{\mathbf{k}} \tag{9.8.1}$$

with a magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + \left(y - R\sin\phi'\right)^2 + z^2} = \sqrt{R^2 + y^2 + z^2 - 2yR\sin\phi} \tag{9.8.2}$$

and the unit vector

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|}$$

pointing from $Id\,\vec{\mathbf{s}}$ to *P*. The cross product $d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ can be simplified as

$$\begin{aligned} d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}} &= R\,d\phi'\left(-\sin\phi'\,\hat{\mathbf{i}} + \cos\phi'\,\hat{\mathbf{j}}\right) \times [-R\cos\phi'\,\hat{\mathbf{i}} + (y - R\sin\phi')\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}] \\ &= R\,d\phi'[z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + \left(R - y\sin\phi'\right)\hat{\mathbf{k}}] \end{aligned} \tag{9.8.3}$$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\,\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\,\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3} = \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + \left(R - y\sin\phi'\right)\hat{\mathbf{k}}}{\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{3/2}}d\phi' \qquad (9.8.4)$$

Thus, magnetic field at P is

$$\vec{\mathbf{B}}(0,y,z) = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{z\cos\phi'\,\hat{\mathbf{i}} + z\sin\phi'\,\hat{\mathbf{j}} + \left(R - y\sin\phi'\right)\hat{\mathbf{k}}}{\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{3/2}}d\phi' \qquad (9.8.5)$$

The x-component of $\vec{\mathbf{B}}$ can be readily shown to be zero

$$B_x = \frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\cos\phi'\,d\phi'}{\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{3/2}} = 0 \qquad (9.8.6)$$

by making a change of variable $w = R^2+y^2+z^2-2yR\sin\phi'$, followed by a straightforward integration. One may also invoke symmetry arguments to verify that B_x must vanish; namely, the contribution at ϕ' is cancelled by the contribution at $\pi-\phi'$. On the other hand, the y and the z components of $\vec{\mathbf{B}}$,

$$B_y = \frac{\mu_0 IRz}{4\pi}\int_0^{2\pi}\frac{\sin\phi'\,d\phi'}{\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{3/2}} \qquad (9.8.7)$$

and

$$B_z = \frac{\mu_0 IR}{4\pi}\int_0^{2\pi}\frac{\left(R-y\sin\phi'\right)d\phi'}{\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{3/2}} \qquad (9.8.8)$$

involve *elliptic integrals* which can be evaluated numerically.

In the limit $y = 0$, the field point P is located along the z-axis, and we recover the results obtained in Example 9.2:

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\sin\phi'\,d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\cos\phi'\Big|_0^{2\pi} = 0 \qquad (9.8.9)$$

and

$$B_z = \frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}} \tag{9.8.10}$$

Now, let's consider the "point-dipole" limit where $R \ll (y^2+z^2)^{1/2} = r$, i.e., the characteristic dimension of the current source is much smaller compared to the distance where the magnetic field is to be measured. In this limit, the denominator in the integrand can be expanded as

$$\begin{aligned}\left(R^2+y^2+z^2-2yR\sin\phi'\right)^{-3/2} &= \frac{1}{r^3}\left[1+\frac{R^2-2yR\sin\phi'}{r^2}\right]^{-3/2} \\ &= \frac{1}{r^3}\left[1-\frac{3}{2}\left(\frac{R^2-2yR\sin\phi'}{r^2}\right)+\dots\right]\end{aligned} \tag{9.8.11}$$

This leads to

$$\begin{aligned}B_y &\approx \frac{\mu_0 I}{4\pi}\frac{Rz}{r^3}\int_0^{2\pi}\left[1-\frac{3}{2}\left(\frac{R^2-2yR\sin\phi'}{r^2}\right)\right]\sin\phi' d\phi' \\ &= \frac{\mu_0 I}{4\pi}\frac{3R^2 yz}{r^5}\int_0^{2\pi}\sin^2\phi' d\phi' = \frac{\mu_0 I}{4\pi}\frac{3\pi R^2 yz}{r^5}\end{aligned} \tag{9.8.12}$$

and

$$\begin{aligned}B_z &\approx \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\int_0^{2\pi}\left[1-\frac{3}{2}\left(\frac{R^2-2yR\sin\phi'}{r^2}\right)\right](R-y\sin\phi')d\phi' \\ &= \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\int_0^{2\pi}\left[\left(R-\frac{3R^3}{2r^2}\right)-\left(1-\frac{9R^2}{2r^2}\right)\sin\phi' - \frac{3Ry^2}{r^2}\sin^2\phi'\right]d\phi' \\ &= \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\left[2\pi\left(R-\frac{3R^3}{2r^2}\right)-\frac{3\pi Ry^2}{r^2}\right] \\ &= \frac{\mu_0 I}{4\pi}\frac{\pi R^2}{r^3}\left[2-\frac{3y^2}{r^2}+\text{higher order terms}\right]\end{aligned} \tag{9.8.13}$$

The quantity $I(\pi R^2)$ may be identified as the magnetic dipole moment $\mu = IA$, where $A = \pi R^2$ is the area of the loop. Using spherical coordinates where $y = r\sin\theta$ and $z = r\cos\theta$, the above expressions may be rewritten as

$$B_y = \frac{\mu_0 (I\pi R^2)}{4\pi}\frac{3(r\sin\theta)(r\cos\theta)}{r^5} = \frac{\mu_0}{4\pi}\frac{3\mu\sin\theta\cos\theta}{r^3} \tag{9.8.14}$$

and

$$B_z = \frac{\mu_0}{4\pi}\frac{(I\pi R^2)}{r^3}\left(2 - \frac{3r^2\sin^2\theta}{r^2}\right) = \frac{\mu_0}{4\pi}\frac{\mu}{r^3}(2 - 3\sin^2\theta) = \frac{\mu_0}{4\pi}\frac{\mu}{r^3}(3\cos^2\theta - 1) \qquad (9.8.15)$$

Thus, we see that the magnetic field at a point $r \gg R$ due to a current ring of radius R may be approximated by a small magnetic dipole moment placed at the origin (Figure 9.8.2).

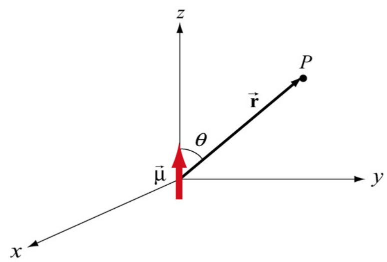


Figure 9.8.2 Magnetic dipole moment $\vec{\boldsymbol{\mu}} = \mu\hat{\mathbf{k}}$

The magnetic field lines due to a current loop and a dipole moment (small bar magnet) are depicted in Figure 9.8.3.

Figure 9.8.3 Magnetic field lines due to (a) a current loop, and (b) a small bar magnet.

The magnetic field at P can also be written in spherical coordinates

$$\vec{\mathbf{B}} = B_r\,\hat{\mathbf{r}} + B_\theta\,\hat{\boldsymbol{\theta}} \qquad (9.8.16)$$

The spherical components B_r and B_θ are related to the Cartesian components B_y and B_z by

$$B_r = B_y\sin\theta + B_z\cos\theta, \qquad B_\theta = B_y\cos\theta - B_z\sin\theta \qquad (9.8.17)$$

In addition, we have, for the unit vectors,

$$\hat{\mathbf{r}} = \sin\theta\,\hat{\mathbf{j}} + \cos\theta\,\hat{\mathbf{k}}, \qquad \hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\mathbf{j}} - \sin\theta\,\hat{\mathbf{k}} \qquad (9.8.18)$$

Using the above relations, the spherical components may be written as

$$B_r = \frac{\mu_0 I R^2 \cos\theta}{4\pi} \int_0^{2\pi} \frac{d\phi'}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi'\right)^{3/2}} \tag{9.8.19}$$

and

$$B_\theta(r,\theta) = \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{\left(r\sin\phi' - R\sin\theta\right)d\phi'}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi'\right)^{3/2}} \tag{9.8.20}$$

In the limit where $R \ll r$, we obtain

$$B_r \approx \frac{\mu_0 I R^2 \cos\theta}{4\pi r^3} \int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi} \frac{2\pi I R^2 \cos\theta}{r^3} = \frac{\mu_0}{4\pi} \frac{2\mu\cos\theta}{r^3} \tag{9.8.21}$$

and

$$\begin{aligned}
B_\theta &= \frac{\mu_0 I R}{4\pi} \int_0^{2\pi} \frac{\left(r\sin\phi' - R\sin\theta\right)d\phi'}{\left(R^2 + r^2 - 2rR\sin\theta\sin\phi'\right)^{3/2}} \\
&\approx \frac{\mu_0 I R}{4\pi r^3} \int_0^{2\pi} \left[-R\sin\theta\left(1 - \frac{3R^2}{2r^2}\right) + \left(r - \frac{3R^2}{2r} - \frac{3R^2\sin^2\theta}{2r}\right)\sin\phi' + 3R\sin\theta\sin^2\phi' \right] d\phi' \\
&\approx \frac{\mu_0 I R}{4\pi r^3}\left(-2\pi R\sin\theta + 3\pi R\sin\theta\right) = \frac{\mu_0 (I\pi R^2)\sin\theta}{4\pi r^3} \\
&= \frac{\mu_0}{4\pi} \frac{\mu\sin\theta}{r^3}
\end{aligned} \tag{9.8.22}$$

9.9 Appendix 2: Helmholtz Coils

Consider two N-turn circular coils of radius R, each perpendicular to the axis of symmetry, with their centers located at $z = \pm l/2$. There is a steady current I flowing in the same direction around each coil, as shown in Figure 9.9.1. Let's find the magnetic field $\vec{\mathbf{B}}$ on the axis at a distance z from the center of one coil.

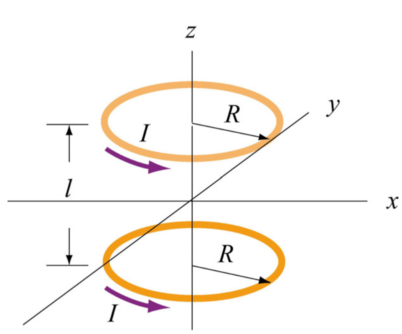


Figure 9.9.1 Helmholtz coils

Using the result shown in Example 9.2 for a single coil and applying the superposition principle, the magnetic field at $P(z,0)$ (a point at a distance $z-l/2$ away from one center and $z+l/2$ from the other) due to the two coils can be obtained as:

$$B_z = B_{\text{top}} + B_{\text{bottom}} = \frac{\mu_0 NIR^2}{2}\left[\frac{1}{[(z-l/2)^2+R^2]^{3/2}} + \frac{1}{[(z+l/2)^2+R^2]^{3/2}}\right] \qquad (9.9.1)$$

A plot of B_z/B_0 with $B_0 = \dfrac{\mu_0 NI}{(5/4)^{3/2} R}$ being the field strength at $z=0$ and $l=R$ is depicted in Figure 9.9.2.

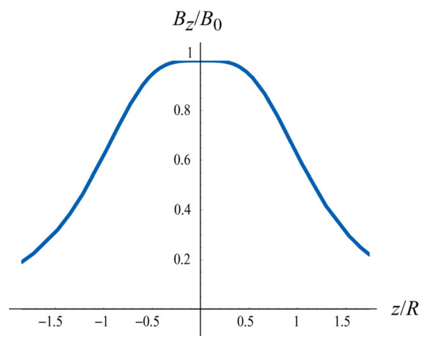


Figure 9.9.2 Magnetic field as a function of z/R.

Let's analyze the properties of B_z in more detail. Differentiating B_z with respect to z, we obtain

$$B_z'(z) = \frac{dB_z}{dz} = \frac{\mu_0 NIR^2}{2}\left\{-\frac{3(z-l/2)}{[(z-l/2)^2+R^2]^{5/2}} - \frac{3(z+l/2)}{[(z+l/2)^2+R^2]^{5/2}}\right\} \qquad (9.9.2)$$

One may readily show that at the midpoint, $z=0$, the derivative vanishes:

$$\left.\frac{dB}{dz}\right|_{z=0} = 0 \qquad (9.9.3)$$

Straightforward differentiation yields

$$B_z''(z) = \frac{d^2B}{dz^2} = \frac{N\mu_0 IR^2}{2}\left\{-\frac{3}{[(z-l/2)^2+R^2]^{5/2}} + \frac{15(z-l/2)^2}{[(z-l/2)^2+R^2]^{7/2}} - \frac{3}{[(z+l/2)^2+R^2]^{5/2}} + \frac{15(z+l/2)^2}{[(z+l/2)^2+R^2]^{7/2}}\right\} \qquad (9.9.4)$$

At the midpoint $z=0$, the above expression simplifies to

$$
\begin{aligned}
B_z''(0) = \left.\frac{d^2B}{dz^2}\right|_{z=0} &= \frac{\mu_0 N I^2}{2}\left\{-\frac{6}{[(l/2)^2+R^2]^{5/2}}+\frac{15l^2}{2[(l/2)^2+R^2]^{7/2}}\right\} \\
&= -\frac{\mu_0 N I^2}{2}\frac{6(R^2-l^2)}{[(l/2)^2+R^2]^{7/2}}
\end{aligned} \tag{9.9.5}
$$

Thus, the condition that the second derivative of B_z vanishes at $z=0$ is $l=R$. That is, the distance of separation between the two coils is equal to the radius of the coil. A configuration with $l=R$ is known as *Helmholtz coils*.

For small *z*, we may make a Taylor-series expansion of $B_z(z)$ about $z=0$:

$$
B_z(z) = B_z(0) + B_z'(0)z + \frac{1}{2!}B_z''(0)z^2 + \ldots \tag{9.9.6}
$$

The fact that the first two derivatives vanish at $z=0$ indicates that the magnetic field is fairly uniform in the small *z* region. One may even show that the third derivative $B_z'''(0)$ vanishes at $z=0$ as well.

Recall that the force experienced by a dipole in a magnetic field is $\vec{\mathbf{F}}_B = \nabla(\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}})$. If we place a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z\,\hat{\mathbf{k}}$ at $z=0$, the magnetic force acting on the dipole is

$$
\vec{\mathbf{F}}_B = \nabla(\mu_z B_z) = \mu_z\left(\frac{dB_z}{dz}\right)\hat{\mathbf{k}} \tag{9.9.7}
$$

which is expected to be very small since the magnetic field is nearly uniform there.

Animation 9.5: Magnetic Field of the Helmholtz Coils

The animation in Figure 9.9.3(a) shows the magnetic field of the Helmholtz coils. In this configuration the currents in the top and bottom coils flow in the same direction, with their dipole moments aligned. The magnetic fields from the two coils add up to create a net field that is nearly uniform at the center of the coils. Since the distance between the coils is equal to the radius of the coils and remains unchanged, the force of attraction between them creates a tension, and is illustrated by field lines stretching out to enclose both coils. When the distance between the coils is not fixed, as in the animation depicted in Figure 9.9.3(b), the two coils move toward each other due to their force of attraction. In this animation, the top loop has only half the current as the bottom loop. The field configuration is shown using the “iron filings” representation.

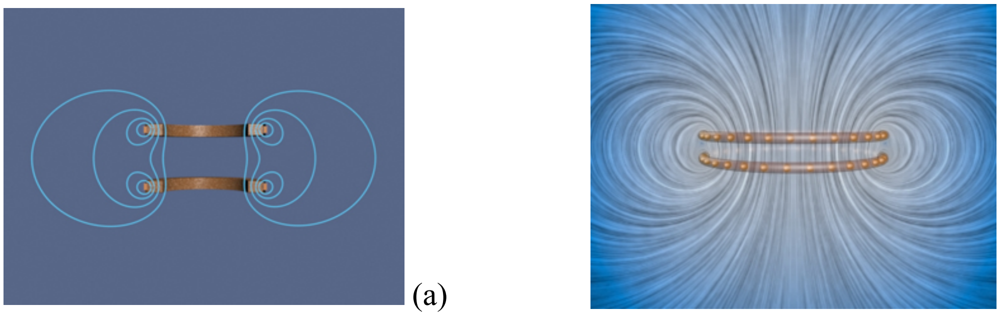

(a) (b)

Figure 9.9.3 (a) Magnetic field of the Helmholtz coils where the distance between the coils is equal to the radius of the coil. (b) Two co-axial wire loops carrying current in the same sense are attracted to each other.

Next, let's consider the case where the currents in the loop flow in the opposite directions, as shown in Figure 9.9.4.

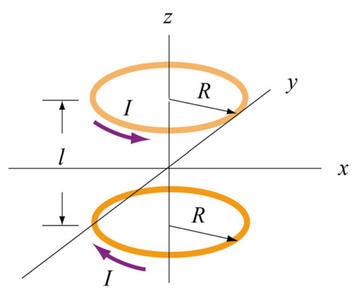


Figure 9.9.4 Two circular loops carrying currents in the opposite directions.

Again, by superposition principle, the magnetic field at a point $P(0,0,z)$ with $z>0$ is

$$B_z = B_{1z} + B_{2z} = \frac{\mu_0 N I R^2}{2}\left[\frac{1}{[(z-l/2)^2+R^2]^{3/2}} - \frac{1}{[(z+l/2)^2+R^2]^{3/2}}\right] \qquad (9.9.8)$$

A plot of B_z/B_0 with $B_0 = \mu_0 NI/2R$ and $l = R$ is depicted in Figure 9.9.5.

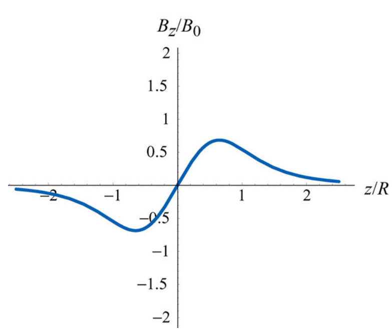


Figure 9.9.5 Magnetic field as a function of z/R.

Differentiating B_z with respect to z, we obtain

$$B'_z(z) = \frac{dB_z}{dz} = \frac{\mu_0 NIR^2}{2}\left\{-\frac{3(z-l/2)}{[(z-l/2)^2+R^2]^{5/2}} + \frac{3(z+l/2)}{[(z+l/2)^2+R^2]^{5/2}}\right\} \tag{9.9.9}$$

At the midpoint, $z=0$, we have

$$B'_z(0) = \left.\frac{dB_z}{dz}\right|_{z=0} = \frac{\mu_0 NIR^2}{2}\frac{3l}{[(l/2)^2+R^2]^{5/2}} \neq 0 \tag{9.9.10}$$

Thus, a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z\,\hat{\mathbf{k}}$ placed at $z=0$ will experience a net force:

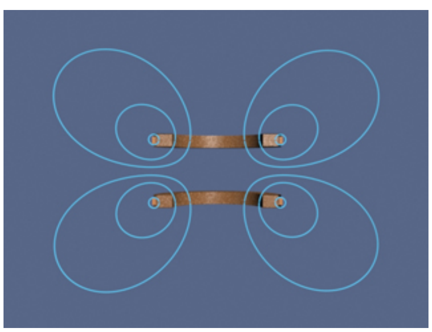
$$\vec{\mathbf{F}}_B = \nabla(\vec{\boldsymbol{\mu}}\cdot\vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z\left(\frac{dB_z(0)}{dz}\right)\hat{\mathbf{k}} = \frac{\mu_z\mu_0 NIR^2}{2}\frac{3l}{[(l/2)^2+R^2]^{5/2}}\hat{\mathbf{k}} \tag{9.9.11}$$

For $l=R$, the above expression simplifies to

$$\vec{\mathbf{F}}_B = \frac{3\mu_z\mu_0 NI}{2(5/4)^{5/2}R^2}\hat{\mathbf{k}} \tag{9.9.12}$$

Animation 9.6: Magnetic Field of Two Coils Carrying Opposite Currents

The animation depicted in Figure 9.9.6 shows the magnetic field of two coils like the Helmholtz coils but with currents in the top and bottom coils flowing in the opposite directions. In this configuration, the magnetic dipole moments associated with each coil are anti-parallel.


(a)

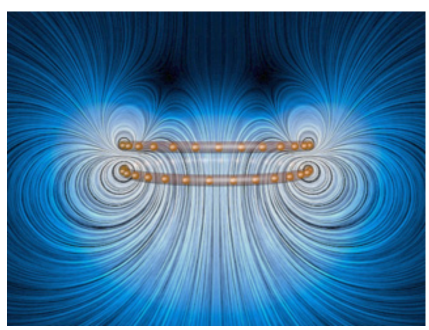

(b)

Figure 9.9.6 (a) Magnetic field due to coils carrying currents in the opposite directions. (b) Two co-axial wire loops carrying current in the opposite sense repel each other. The field configurations here are shown using the "iron filings" representation. The bottom wire loop carries twice the amount of current as the top wire loop.

At the center of the coils along the axis of symmetry, the magnetic field is zero. With the distance between the two coils fixed, the repulsive force results in a pressure between them. This is illustrated by field lines that are compressed along the central horizontal axis between the coils.

Animation 9.7: Forces Between Coaxial Current-Carrying Wires

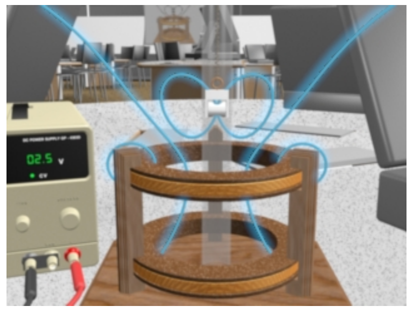


Figure 9.9.7 A magnet in the TeachSpin ™ Magnetic Force apparatus when the current in the top coil is counterclockwise as seen from the top.

Figure 9.9.7 shows the force of repulsion between the magnetic field of a permanent magnet and the field of a current-carrying ring in the TeachSpin ™ Magnetic Force apparatus. The magnet is forced to have its North magnetic pole pointing downward, and the current in the top coil of the Magnetic Force apparatus is moving clockwise as seen from above. The net result is a repulsion of the magnet when the current in this direction is increased. The visualization shows the stresses transmitted by the fields to the magnet when the current in the upper coil is increased.

Animation 9.8: Magnet Oscillating Between Two Coils

Figure 9.9.8 illustrates an animation in which the magnetic field of a permanent magnet suspended by a spring in the *TeachSpin*TM apparatus (see *TeachSpin* visualization), plus the magnetic field due to current in the two coils (here we see a "cutaway" cross-section of the apparatus).

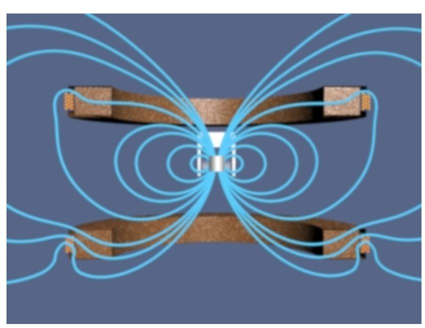


Figure 9.9.8 Magnet oscillating between two coils

The magnet is fixed so that its north pole points upward, and the current in the two coils is sinusoidal and 180 degrees out of phase. When the effective dipole moment of the top coil points upwards, the dipole moment of the bottom coil points downwards. Thus, the magnet is attracted to the upper coil and repelled by the lower coil, causing it to move upwards. When the conditions are reversed during the second half of the cycle, the magnet moves downwards.

This process can also be described in terms of tension along, and pressure perpendicular to, the field lines of the resulting field. When the dipole moment of one of the coils is aligned with that of the magnet, there is a tension along the field lines as they attempt to "connect" the coil and magnet. Conversely, when their moments are anti-aligned, there is a pressure perpendicular to the field lines as they try to keep the coil and magnet apart.

Animation 9.9: Magnet Suspended Between Two Coils

Figure 9.9.9 illustrates an animation in which the magnetic field of a permanent magnet suspended by a spring in the *TeachSpin*TM apparatus (see *TeachSpin* visualization), plus the magnetic field due to current in the two coils (here we see a "cutaway" cross-section of the apparatus). The magnet is fixed so that its north pole points upward, and the current in the two coils is sinusoidal and in phase. When the effective dipole moment of the top coil points upwards, the dipole moment of the bottom coil points upwards as well. Thus, the magnet the magnet is attracted to both coils, and as a result feels no net force (although it does feel a torque, not shown here since the direction of the magnet is fixed to point upwards). When the dipole moments are reversed during the second half of the cycle, the magnet is repelled by both coils, again resulting in no net force.

This process can also be described in terms of tension along, and pressure perpendicular to, the field lines of the resulting field. When the dipole moment of the coils is aligned with that of the magnet, there is a tension along the field lines as they are "pulled" from both sides. Conversely, when their moments are anti-aligned, there is a pressure perpendicular to the field lines as they are "squeezed" from both sides.

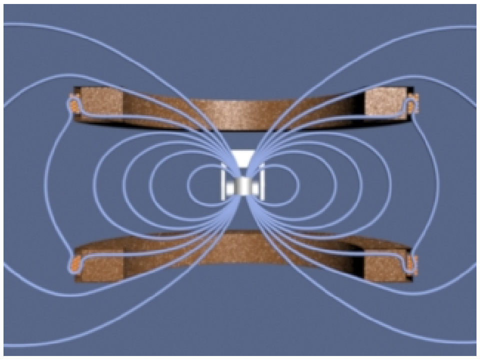


Figure 9.9.9 Magnet suspended between two coils

9.10 Problem-Solving Strategies

In this Chapter, we have seen how Biot-Savart and Ampere's laws can be used to calculate magnetic field due to a current source.

9.10.1 Biot-Savart Law:

The law states that the magnetic field at a point P due to a length element $d\vec{\mathbf{s}}$ carrying a steady current I located at $\vec{\mathbf{r}}$ away is given by

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3}$$

The calculation of the magnetic field may be carried out as follows:

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I\,d\vec{\mathbf{s}}$, and the vector $\vec{\mathbf{r}}\,'$ describing the position of $I\,d\vec{\mathbf{s}}$. The magnitude $r' = |\vec{\mathbf{r}}\,'|$ is the distance between $I\,d\vec{\mathbf{s}}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point P is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{\mathbf{r}}_P$ for the field point P. The quantity $r_P = |\vec{\mathbf{r}}_P|$ is the distance between the origin and P.

(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|}$$

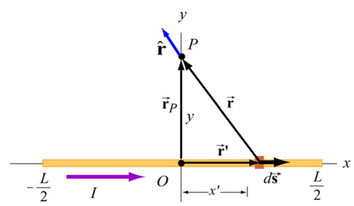
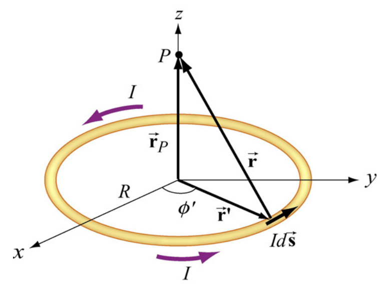
where $r = |\vec{\mathbf{r}}| = |\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|$ is the distance between the source and the field point P.

(4) Calculate the cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ or $d\vec{\mathbf{s}}\times\vec{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field $\vec{\mathbf{B}}$, according to the Biot-Savart law.

(5) Substitute the expressions obtained to $d\vec{\mathbf{B}}$ and simplify as much as possible.

(6) Complete the integration to obtain $\vec{\mathbf{B}}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

Below we illustrate how these steps are executed for a current-carrying wire of length L and a loop of radius R.

Current distribution	Finite wire of length L	Circular loop of radius R
Figure		
(1) Source point	$\vec{\mathbf{r}}\,'=x'\hat{\mathbf{i}}$ $d\vec{\mathbf{s}}=(d\vec{\mathbf{r}}\,'/dx')dx'=dx'\hat{\mathbf{i}}$	$\vec{\mathbf{r}}\,'=R(\cos\phi'\hat{\mathbf{i}}+\sin\phi'\hat{\mathbf{j}})$ $d\vec{\mathbf{s}}=(d\vec{\mathbf{r}}\,'/d\phi')d\phi'=Rd\phi'(-\sin\phi'\hat{\mathbf{i}}+\cos\phi'\hat{\mathbf{j}})$
(2) Field point P	$\vec{\mathbf{r}}_P=y\hat{\mathbf{j}}$	$\vec{\mathbf{r}}_P=z\hat{\mathbf{k}}$
(3) Relative position vector $\vec{\mathbf{r}}=\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,'$	$\vec{\mathbf{r}}=y\hat{\mathbf{j}}-x'\hat{\mathbf{i}}$ $r=\lvert\vec{\mathbf{r}}\rvert=\sqrt{x'^2+y^2}$ $\hat{\mathbf{r}}=\dfrac{y\hat{\mathbf{j}}-x'\hat{\mathbf{i}}}{\sqrt{x'^2+y^2}}$	$\vec{\mathbf{r}}=-R\cos\phi'\hat{\mathbf{i}}-R\sin\phi'\hat{\mathbf{j}}+z\hat{\mathbf{k}}$ $r=\lvert\vec{\mathbf{r}}\rvert=\sqrt{R^2+z^2}$ $\hat{\mathbf{r}}=\dfrac{-R\cos\phi'\hat{\mathbf{i}}-R\sin\phi'\hat{\mathbf{j}}+z\hat{\mathbf{k}}}{\sqrt{R^2+z^2}}$
(4) The cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$	$d\vec{\mathbf{s}}\times\hat{\mathbf{r}}=\dfrac{y\,dx'\hat{\mathbf{k}}}{\sqrt{y^2+x'^2}}$	$d\vec{\mathbf{s}}\times\hat{\mathbf{r}}=\dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}})}{\sqrt{R^2+z^2}}$
(5) Rewrite $d\vec{\mathbf{B}}$	$d\vec{\mathbf{B}}=\dfrac{\mu_0 I}{4\pi}\dfrac{y\,dx'\,\hat{\mathbf{k}}}{(y^2+x'^2)^{3/2}}$	$d\vec{\mathbf{B}}=\dfrac{\mu_0 I}{4\pi}\dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}}+z\sin\phi'\hat{\mathbf{j}}+R\hat{\mathbf{k}})}{(R^2+z^2)^{3/2}}$
(6) Integrate to get $\vec{\mathbf{B}}$	$B_x=0$ $B_y=0$ $B_z=\dfrac{\mu_0 I y}{4\pi}\displaystyle\int_{-L/2}^{L/2}\dfrac{dx'}{(y^2+x'^2)^{3/2}}$ $=\dfrac{\mu_0 I}{4\pi}\dfrac{L}{y\sqrt{y^2+(L/2)^2}}$	$B_x=\dfrac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}\cos\phi'\,d\phi'=0$ $B_y=\dfrac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}\sin\phi'\,d\phi'=0$ $B_z=\dfrac{\mu_0 IR^2}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}d\phi'=\dfrac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$

9.10.2 Ampere's law:

Ampere's law states that the line integral of $\vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around any closed loop is proportional to the total current passing through any surface that is bounded by the closed loop:

$$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = \mu_0 I_{\text{enc}}$$

To apply Ampere's law to calculate the magnetic field, we use the following procedure:

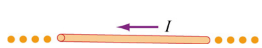
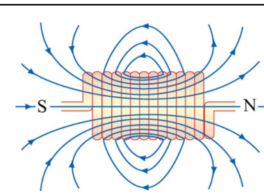
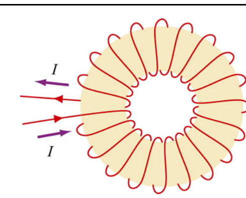
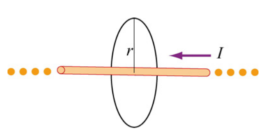
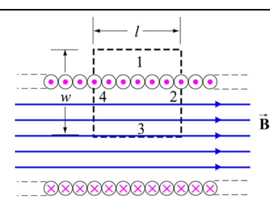
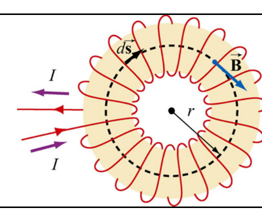
(1) Draw an Amperian loop using symmetry arguments.

(2) Find the current enclosed by the Amperian loop.

(3) Calculate the line integral $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ around the closed loop.

(4) Equate $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ with $\mu_0 I_{\text{enc}}$ and solve for $\vec{\mathbf{B}}$.

Below we summarize how the methodology can be applied to calculate the magnetic field for an infinite wire, an ideal solenoid and a toroid.

System	Infinite wire	Ideal solenoid	Toroid
Figure			
(1) Draw the Amperian loop			
(2) Find the current enclosed by the Amperian loop	$I_{\text{enc}} = I$	$I_{\text{enc}} = NI$	$I_{\text{enc}} = NI$
(3) Calculate $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ along the loop	$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = B(2\pi r)$	$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = Bl$	$\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}} = B(2\pi r)$
(4) Equate $\mu_0 I_{\text{enc}}$ with $\oint \vec{\mathbf{B}}\cdot d\vec{\mathbf{s}}$ to obtain $\vec{\mathbf{B}}$	$B = \dfrac{\mu_0 I}{2\pi r}$	$B = \dfrac{\mu_0 NI}{l} = \mu_0 n I$	$B = \dfrac{\mu_0 NI}{2\pi r}$

9.11 Solved Problems

9.11.1 Magnetic Field of a Straight Wire

Consider a straight wire of length L carrying a current I along the $+x$-direction, as shown in Figure 9.11.1 (ignore the return path of the current or the source for the current.) What is the magnetic field at an arbitrary point P on the xy-plane?

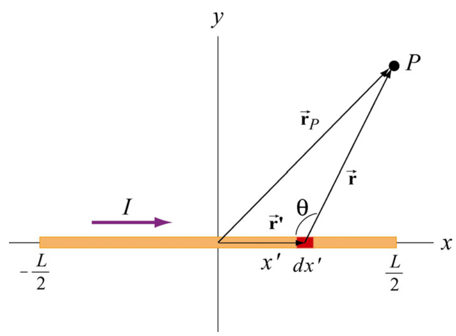


Figure 9.11.1 A finite straight wire carrying a current I.

Solution:

The problem is very similar to Example 9.1. However, now the field point is an arbitrary point in the xy-plane. Once again we solve the problem using the methodology outlined in Section 9.10.

(1) Source point

From Figure 9.10.1, we see that the infinitesimal length dx' described by the position vector $\vec{\mathbf{r}}' = x'\,\hat{\mathbf{i}}$ constitutes a current source $I\,d\vec{\mathbf{s}} = (Idx')\hat{\mathbf{i}}$.

(2) Field point

As can be seen from Figure 9.10.1, the position vector for the field point P is $\vec{\mathbf{r}} = x\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$.

(3) Relative position vector

The relative position vector from the source to P is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}' = (x - x')\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}$, with $r = |\vec{\mathbf{r}}_P| = |\vec{\mathbf{r}} - \vec{\mathbf{r}}'| = [(x - x')^2 + y^2]^{1/2}$ being the distance. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}'|} = \frac{(x - x')\,\hat{\mathbf{i}} + y\,\hat{\mathbf{j}}}{[(x - x')^2 + y^2]^{1/2}}$$

(4) Simplifying the cross product

The cross product $d\vec{\mathbf{s}}\times\vec{\mathbf{r}}$ can be simplified as

$$(dx'\,\hat{\mathbf{i}})\times[(x-x')\,\hat{\mathbf{i}}+y\,\hat{\mathbf{j}}]=y\,dx'\,\hat{\mathbf{k}}$$

where we have used $\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\vec{\mathbf{0}}$ and $\hat{\mathbf{i}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}$.

(5) Writing down $d\vec{\mathbf{B}}$

Using the Biot-Savart law, the infinitesimal contribution due to $Id\,\vec{\mathbf{s}}$ is

$$d\vec{\mathbf{B}}=\frac{\mu_0 I}{4\pi}\frac{d\,\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}=\frac{\mu_0 I}{4\pi}\frac{d\,\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3}=\frac{\mu_0 I}{4\pi}\frac{y\,dx'}{[(x-x')^2+y^2]^{3/2}}\hat{\mathbf{k}} \tag{9.11.1}$$

Thus, we see that the direction of the magnetic field is in the $+\hat{\mathbf{k}}$ direction.

(6) Carrying out the integration to obtain $\vec{\mathbf{B}}$

The total magnetic field at P can then be obtained by integrating over the entire length of the wire:

$$\begin{aligned}\vec{\mathbf{B}}&=\int_{\text{wire}}d\vec{\mathbf{B}}=\int_{-L/2}^{L/2}\frac{\mu_0 Iy\,dx'}{4\pi[(x-x')^2+y^2]^{3/2}}\hat{\mathbf{k}}=-\frac{\mu_0 I}{4\pi y}\frac{(x-x')}{\sqrt{(x-x')^2+y^2}}\Bigg|_{-L/2}^{L/2}\hat{\mathbf{k}}\\&=-\frac{\mu_0 I}{4\pi y}\left[\frac{(x-L/2)}{\sqrt{(x-L/2)^2+y^2}}-\frac{(x+L/2)}{\sqrt{(x+L/2)^2+y^2}}\right]\hat{\mathbf{k}}\end{aligned} \tag{9.11.2}$$

Let's consider the following limits:

(i) $x=0$

In this case, the field point P is at $(x,y)=(0,y)$ on the y axis. The magnetic field becomes

$$\vec{\mathbf{B}}=-\frac{\mu_0 I}{4\pi y}\left[\frac{-L/2}{\sqrt{(-L/2)^2+y^2}}-\frac{+L/2}{\sqrt{(+L/2)^2+y^2}}\right]\hat{\mathbf{k}}=\frac{\mu_0 I}{2\pi y}\frac{L/2}{\sqrt{(L/2)^2+y^2}}\hat{\mathbf{k}}=\frac{\mu_0 I}{2\pi y}\cos\theta\,\hat{\mathbf{k}} \tag{9.11.3}$$

in agreement with Eq. (9.1.6).

(ii) Infinite length limit

Consider the limit where $L \gg x, y$. This gives back the expected infinite-length result:

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{4\pi y}\left[\frac{-L/2}{L/2} - \frac{+L/2}{L/2}\right]\hat{\mathbf{k}} = \frac{\mu_0 I}{2\pi y}\hat{\mathbf{k}} \tag{9.11.4}$$

If we use cylindrical coordinates with the wire pointing along the +z-axis then the magnetic field is given by the expression

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\varphi}} \tag{9.11.5}$$

where $\hat{\boldsymbol{\varphi}}$ is the tangential unit vector and the field point P is a distance r away from the wire.

9.11.2 Current-Carrying Arc

Consider the current-carrying loop formed of radial lines and segments of circles whose centers are at point P as shown below. Find the magnetic field $\vec{\mathbf{B}}$ at P.

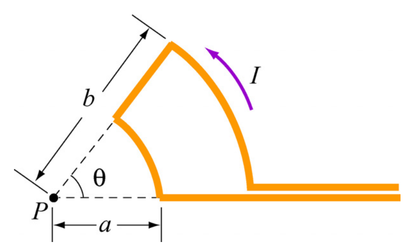


Figure 9.11.2 Current-carrying arc

Solution:

According to the Biot-Savart law, the magnitude of the magnetic field due to a differential current-carrying element $I\,d\vec{\mathbf{s}}$ is given by

$$dB = \frac{\mu_0 I}{4\pi}\frac{|d\vec{\mathbf{s}}\times\hat{\mathbf{r}}|}{r^2} = \frac{\mu_0 I}{4\pi}\frac{r\,d\theta'}{r^2} = \frac{\mu_0 I}{4\pi r}d\theta' \tag{9.11.6}$$

For the outer arc, we have

$$B_{\text{outer}} = \frac{\mu_0 I}{4\pi b}\int_0^{\theta} d\theta' = \frac{\mu_0 I\theta}{4\pi b} \qquad (9.11.7)$$

The direction of $\vec{\mathbf{B}}_{\text{outer}}$ is determined by the cross product $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ which points out of the page. Similarly, for the inner arc, we have

$$B_{\text{inner}} = \frac{\mu_0 I}{4\pi a}\int_0^{\theta} d\theta' = \frac{\mu_0 I\theta}{4\pi a} \qquad (9.11.8)$$

For $\vec{\mathbf{B}}_{\text{inner}}$, $d\vec{\mathbf{s}}\times\hat{\mathbf{r}}$ points into the page. Thus, the total magnitude of magnetic field is

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_{\text{inner}} + \vec{\mathbf{B}}_{\text{outer}} = \frac{\mu_0 I\theta}{4\pi}\left(\frac{1}{a}-\frac{1}{b}\right) \text{ (into page)} \qquad (9.11.9)$$

9.11.3 Rectangular Current Loop

Determine the magnetic field (in terms of I, a and b) at the origin O due to the current loop shown in Figure 9.11.3

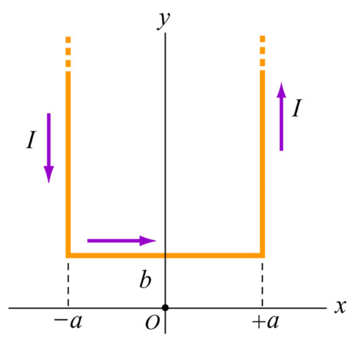


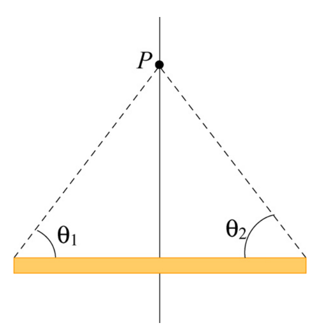
Figure 9.11.3 Rectangular current loop

Solution:

For a finite wire carrying a current I, the contribution to the magnetic field at a point P is given by Eq. (9.1.5):

$$B = \frac{\mu_0 I}{4\pi r}\left(\cos\theta_1 + \cos\theta_2\right)$$

where θ_1 and θ_2 are the angles which parameterize the length of the wire.



To obtain the magnetic field at O, we make use of the above formula. The contributions can be divided into three parts:

(i) Consider the left segment of the wire which extends from $(x, y) = (-a, +\infty)$ to $(-a, +d)$. The angles which parameterize this segment give $\cos\theta_1 = 1$ ($\theta_1 = 0$) and $\cos\theta_2 = -b/\sqrt{b^2 + a^2}$. Therefore,

$$B_1 = \frac{\mu_0 I}{4\pi a}\left(\cos\theta_1 + \cos\theta_2\right) = \frac{\mu_0 I}{4\pi a}\left(1 - \frac{b}{\sqrt{b^2 + a^2}}\right) \tag{9.11.10}$$

The direction of $\vec{\mathbf{B}}_1$ is out of page, or $+\hat{\mathbf{k}}$.

(ii) Next, we consider the segment which extends from $(x, y) = (-a, +b)$ to $(+a, +b)$. Again, the (cosine of the) angles are given by

$$\cos\theta_1 = \frac{a}{\sqrt{a^2 + b^2}} \tag{9.11.11}$$

$$\cos\theta_2 = \cos\theta_1 = \frac{a}{\sqrt{a^2 + b^2}} \tag{9.11.12}$$

This leads to

$$B_2 = \frac{\mu_0 I}{4\pi b}\left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}}\right) = \frac{\mu_0 I a}{2\pi b\sqrt{a^2 + b^2}} \tag{9.11.13}$$

The direction of $\vec{\mathbf{B}}_2$ is into the page, or $-\hat{\mathbf{k}}$.

(iii) The third segment of the wire runs from $(x, y) = (+a, +b)$ to $(+a, +\infty)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 \tag{9.11.14}$$

The direction of $\vec{\mathbf{B}}_3$ is again out of page, or $+\hat{\mathbf{k}}$.

The magnetic field is

$$\begin{aligned} \vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 &= \frac{\mu_0 I}{2\pi a}\left(1 - \frac{b}{\sqrt{a^2 + b^2}}\right)\hat{\mathbf{k}} - \frac{\mu_0 I a}{2\pi b\sqrt{a^2 + b^2}}\hat{\mathbf{k}} \\ &= \frac{\mu_0 I}{2\pi ab\sqrt{a^2 + b^2}}\left(b\sqrt{a^2 + b^2} - b^2 - a^2\right)\hat{\mathbf{k}} \end{aligned} \tag{9.11.15}$$

Note that in the limit $a \to 0$, the horizontal segment is absent, and the two semi-infinite wires carrying currents in the opposite direction overlap each other and their contributions completely cancel. Thus, the magnetic field vanishes in this limit.

9.11.4 Hairpin-Shaped Current-Carrying Wire

An infinitely long current-carrying wire is bent into a hairpin-like shape shown in Figure 9.11.4. Find the magnetic field at the point *P* which lies at the center of the half-circle.

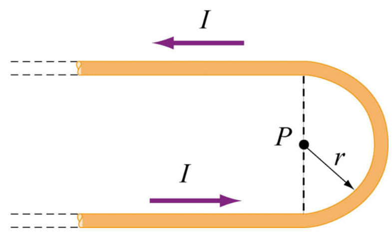


Figure 9.11.4 Hairpin-shaped current-carrying wire

Solution:

Again we break the wire into three parts: two semi-infinite plus a semi-circular segments.

(i) Let P be located at the origin in the xy plane. The first semi-infinite segment then extends from $(x, y) = (-\infty, -r)$ to $(0, -r)$. The two angles which parameterize this segment are characterized by $\cos\theta_1 = 1\,(\theta_1 = 0)$ and $\cos\theta_2 = 0$ $(\theta_2 = \pi/2)$. Therefore, its contribution to the magnetic field at P is

$$B_1 = \frac{\mu_0 I}{4\pi r}\left(\cos\theta_1 + \cos\theta_2\right) = \frac{\mu_0 I}{4\pi r}(1+0) = \frac{\mu_0 I}{4\pi r} \tag{9.11.16}$$

The direction of $\vec{\mathbf{B}}_1$ is out of page, or $+\hat{\mathbf{k}}$.

(ii) For the semi-circular arc of radius r, we make use of the Biot-Savart law:

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\int \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} \tag{9.11.17}$$

and obtain

$$B_2 = \frac{\mu_0 I}{4\pi}\int_0^{\pi} \frac{r\,d\theta}{r^2} = \frac{\mu_0 I}{4r} \tag{9.11.18}$$

The direction of $\vec{\mathbf{B}}_2$ is out of page, or $+\hat{\mathbf{k}}$.

(iii) The third segment of the wire runs from $(x, y) = (0, +r)$ to $(-\infty, +r)$. One may readily show that it gives the same contribution as the first one:

$$B_3 = B_1 = \frac{\mu_0 I}{4\pi r} \tag{9.11.19}$$

The direction of $\vec{\mathbf{B}}_3$ is again out of page, or $+\hat{\mathbf{k}}$.

The total magnitude of the magnetic field is

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_2 = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{k}} + \frac{\mu_0 I}{4r}\hat{\mathbf{k}} = \frac{\mu_0 I}{4\pi r}(2+\pi)\hat{\mathbf{k}} \tag{9.11.20}$$

Notice that the contribution from the two semi-infinite wires is equal to that due to an infinite wire:

$$\vec{\mathbf{B}}_1 + \vec{\mathbf{B}}_3 = 2\vec{\mathbf{B}}_1 = \frac{\mu_0 I}{2\pi r}\hat{\mathbf{k}} \tag{9.11.21}$$

9.11.5 Two Infinitely Long Wires

Consider two infinitely long wires carrying currents are in the $-x$-direction.

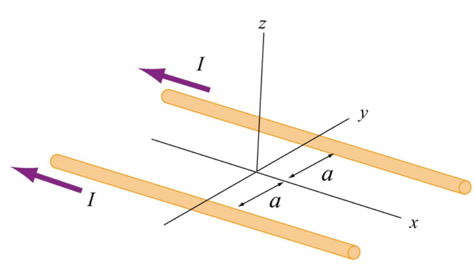


Figure 9.11.5 Two infinitely long wires

(a) Plot the magnetic field pattern in the *yz*-plane.

(b) Find the distance *d* along the *z*-axis where the magnetic field is a maximum.

Solutions:

(a) The magnetic field lines are shown in Figure 9.11.6. Notice that the directions of both currents are into the page.

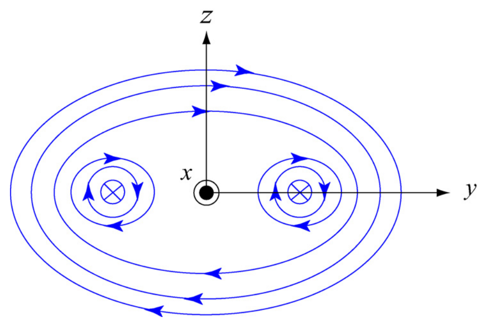


Figure 9.11.6 Magnetic field lines of two wires carrying current in the same direction.

(b) The magnetic field at (0, 0, z) due to wire 1 on the left is, using Ampere's law:

$$B_1 = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}} \tag{9.11.22}$$

Since the current is flowing in the $-x$-direction, the magnetic field points in the direction of the cross product

$$(-\hat{\mathbf{i}})\times\hat{\mathbf{r}}_1 = (-\hat{\mathbf{i}})\times(\cos\theta\,\hat{\mathbf{j}}+\sin\theta\,\hat{\mathbf{k}}) = \sin\theta\,\hat{\mathbf{j}}-\cos\theta\,\hat{\mathbf{k}} \tag{9.11.23}$$

Thus, we have

$$\vec{\mathbf{B}}_1 = \frac{\mu_0 I}{2\pi\sqrt{a^2+z^2}}\left(\sin\theta\,\hat{\mathbf{j}}-\cos\theta\,\hat{\mathbf{k}}\right) \tag{9.11.24}$$

For wire 2 on the right, the magnetic field strength is the same as the left one: $B_1 = B_2$. However, its direction is given by

$$(-\hat{\mathbf{i}})\times\hat{\mathbf{r}}_2 = (-\hat{\mathbf{i}})\times(-\cos\theta\,\hat{\mathbf{j}}+\sin\theta\,\hat{\mathbf{k}}) = \sin\theta\,\hat{\mathbf{j}}+\cos\theta\,\hat{\mathbf{k}} \tag{9.11.25}$$

Adding up the contributions from both wires, the z-components cancel (as required by symmetry), and we arrive at

$$\vec{\mathbf{B}} = \vec{\mathbf{B}}_1+\vec{\mathbf{B}}_2 = \frac{\mu_0 I\sin\theta}{\pi\sqrt{a^2+z^2}}\,\hat{\mathbf{j}} = \frac{\mu_0 I z}{\pi(a^2+z^2)}\,\hat{\mathbf{j}} \tag{9.11.26}$$

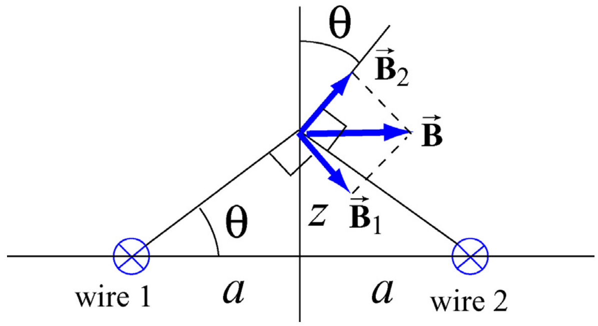


Figure 9.11.7 Superposition of magnetic fields due to two current sources

To locate the maximum of B, we set $dB/dz = 0$ and find

$$\frac{dB}{dz} = \frac{\mu_0 I}{\pi}\left(\frac{1}{a^2+z^2} - \frac{2z^2}{(a^2+z^2)^2}\right) = \frac{\mu_0 I}{\pi}\frac{a^2-z^2}{\left(a^2+z^2\right)^2} = 0 \qquad (9.11.27)$$

which gives

$$z = a \qquad (9.11.28)$$

Thus, at $z=a$, the magnetic field strength is a maximum, with a magnitude

$$B_{\max} = \frac{\mu_0 I}{2\pi a} \qquad (9.11.29)$$

9.11.6 Non-Uniform Current Density

Consider an infinitely long, cylindrical conductor of radius R carrying a current I with a non-uniform current density

$$J = \alpha r \qquad (9.11.30)$$

where α is a constant. Find the magnetic field everywhere.

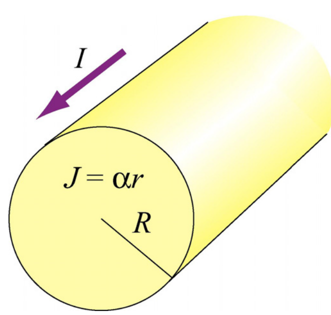


Figure 9.11.8 Non-uniform current density

Solution:

The problem can be solved by using the Ampere's law:

$$\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}} = \mu_0 I_{\text{enc}} \tag{9.11.31}$$

where the enclosed current I_{enc} is given by

$$I_{\text{enc}} = \int \vec{\mathbf{J}} \cdot d\vec{\mathbf{A}} = \int (\alpha r')(2\pi r' dr') \tag{9.11.32}$$

(a) For $r < R$, the enclosed current is

$$I_{\text{enc}} = \int_0^r 2\pi\alpha r'^2\, dr' = \frac{2\pi\alpha r^3}{3} \tag{9.11.33}$$

Applying Ampere's law, the magnetic field at P_1 is given by

$$B_1(2\pi r) = \frac{2\mu_0 \pi\alpha r^3}{3} \tag{9.11.34}$$

or

$$\boxed{B_1 = \frac{\alpha\mu_0}{3} r^2} \tag{9.11.35}$$

The direction of the magnetic field $\vec{\mathbf{B}}_1$ is tangential to the Amperian loop which encloses the current.

(b) For $r > R$, the enclosed current is

$$I_{\text{enc}} = \int_0^R 2\pi\alpha r'^2\, dr' = \frac{2\pi\alpha R^3}{3} \tag{9.11.36}$$

which yields

$$B_2(2\pi r) = \frac{2\mu_0 \pi\alpha R^3}{3} \tag{9.11.37}$$

Thus, the magnetic field at a point P_2 outside the conductor is

$$\boxed{B_2 = \frac{\alpha\mu_0 R^3}{3r}} \tag{9.11.38}$$

A plot of B as a function of r is shown in Figure 9.11.9:

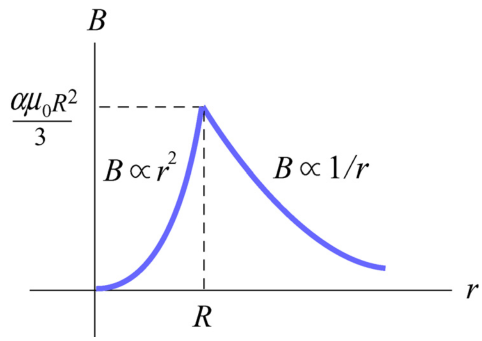


Figure 9.11.9 The magnetic field as a function of distance away from the conductor

9.11.7 Thin Strip of Metal

Consider an infinitely long, thin strip of metal of width w lying in the xy plane. The strip carries a current I along the $+x$-direction, as shown in Figure 9.11.10. Find the magnetic field at a point P which is in the plane of the strip and at a distance s away from it.

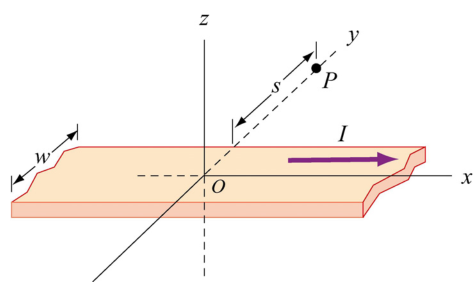


Figure 9.11.10 Thin strip of metal

Solution:

Consider a thin strip of width dr parallel to the direction of the current and at a distance r away from P, as shown in Figure 9.11.11. The amount of current carried by this differential element is

$$dI = I\left(\frac{dr}{w}\right) \tag{9.11.39}$$

Using Ampere's law, we see that the strip's contribution to the magnetic field at P is given by

$$dB(2\pi r) = \mu_0 I_{\text{enc}} = \mu_0 (dI) \tag{9.11.40}$$

or

$$dB = \frac{\mu_0\, dI}{2\pi r} = \frac{\mu_0}{2\pi r}\left(\frac{I\, dr}{w}\right) \tag{9.11.41}$$

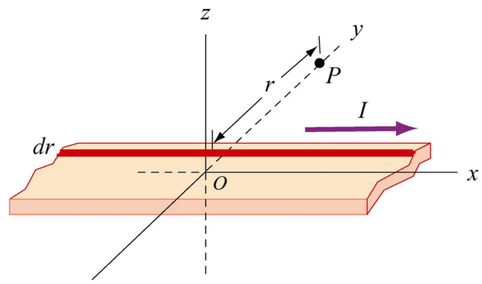


Figure 9.11.11 A thin strip with thickness dr carrying a steady current I.

Integrating this expression, we obtain

$$B = \int_s^{s+w} \frac{\mu_0 I}{2\pi w}\left(\frac{dr}{r}\right) = \frac{\mu_0 I}{2\pi w}\ln\left(\frac{s+w}{s}\right) \tag{9.11.42}$$

Using the right-hand rule, the direction of the magnetic field can be shown to point in the $+z$-direction, or

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi w}\ln\left(1+\frac{w}{s}\right)\hat{\mathbf{k}} \tag{9.11.43}$$

Notice that in the limit of vanishing width, $w \ll s$, $\ln(1+w/s) \approx w/s$, and the above expression becomes

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi s}\hat{\mathbf{k}} \tag{9.11.44}$$

which is the magnetic field due to an infinitely long thin straight wire.

9.11.8 Two Semi-Infinite Wires

A wire carrying current I runs down the y axis to the origin, thence out to infinity along the positive x axis. Show that the magnetic field in the quadrant with $x, y > 0$ of the xy plane is given by

$$B_z = \frac{\mu_0 I}{4\pi}\left(\frac{1}{x}+\frac{1}{y}+\frac{x}{y\sqrt{x^2+y^2}}+\frac{y}{x\sqrt{x^2+y^2}}\right) \tag{9.11.45}$$

Solution:

Let $P(x, y)$ be a point in the first quadrant at a distance r_1 from a point $(0, y')$ on the y-axis and distance r_2 from $(x', 0)$ on the x-axis.

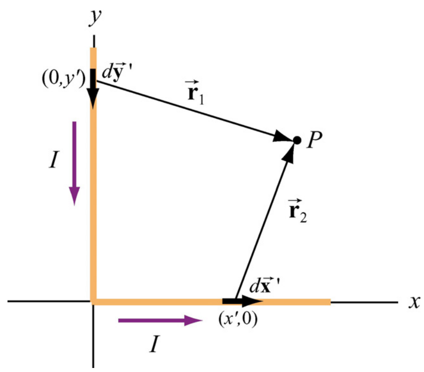


Figure 9.11.12 Two semi-infinite wires

Using the Biot-Savart law, the magnetic field at P is given by

$$\vec{\mathbf{B}} = \int d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi} \int_{\text{wire } y} \frac{d\,\vec{\mathbf{s}}_1 \times \hat{\mathbf{r}}_1}{r_1^2} + \frac{\mu_0 I}{4\pi} \int_{\text{wire } x} \frac{d\,\vec{\mathbf{s}}_2 \times \hat{\mathbf{r}}_2}{r_2^2} \tag{9.11.46}$$

Let's analyze each segment separately.

(i) Along the y axis, consider a differential element $d\vec{\mathbf{s}}_1 = -dy'\,\hat{\mathbf{j}}$ which is located at a distance $\vec{\mathbf{r}}_1 = x\hat{\mathbf{i}} + (y - y')\hat{\mathbf{j}}$ from P. This yields

$$d\vec{\mathbf{s}}_1 \times \vec{\mathbf{r}}_1 = (-dy'\,\hat{\mathbf{j}}) \times [x\hat{\mathbf{i}} + (y - y')\hat{\mathbf{j}}] = x\,dy'\,\hat{\mathbf{k}} \tag{9.11.47}$$

(ii) Similarly, along the x-axis, we have $d\vec{\mathbf{s}}_2 = dx'\,\hat{\mathbf{i}}$ and $\vec{\mathbf{r}}_2 = (x - x')\hat{\mathbf{i}} + y\hat{\mathbf{j}}$ which gives

$$d\vec{\mathbf{s}}_2 \times \vec{\mathbf{r}}_2 = y\,dx'\,\hat{\mathbf{k}} \tag{9.11.48}$$

Thus, we see that the magnetic field at P points in the $+z$-direction. Using the above results and $r_1 = \sqrt{x^2 + (y - y')^2}$ and $r_2 = \sqrt{(x - x')^2 + y^2}$, we obtain

$$B_z = \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{x\,dy'}{[x^2 + (y - y')^2]^{3/2}} + \frac{\mu_0 I}{4\pi} \int_0^\infty \frac{y\,dx'}{[y^2 + (x - x')^2]^{3/2}} \tag{9.11.49}$$

The integrals can be readily evaluated using

$$\int_0^\infty \frac{b\,ds}{[b^2+(a-s)^2]^{3/2}} = \frac{1}{b} + \frac{a}{b\sqrt{a^2+b^2}} \tag{9.11.50}$$

The final expression for the magnetic field is given by

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\left[\frac{1}{x} + \frac{y}{x\sqrt{x^2+y^2}} + \frac{1}{y} + \frac{x}{y\sqrt{x^2+y^2}}\right]\hat{\mathbf{k}} \tag{9.11.51}$$

We may show that the result is consistent with Eq. (9.1.5)

9.12 Conceptual Questions

1. Compare and contrast Biot-Savart law in magnetostatics with Coulomb's law in electrostatics.

2. If a current is passed through a spring, does the spring stretch or compress? Explain.

3. How is the path of the integration of $\oint \vec{\mathbf{B}} \cdot d\,\vec{\mathbf{s}}$ chosen when applying Ampere's law?

4. Two concentric, coplanar circular loops of different diameters carry steady currents in the same direction. Do the loops attract or repel each other? Explain.

5. Suppose three infinitely long parallel wires are arranged in such a way that when looking at the cross section, they are at the corners of an equilateral triangle. Can currents be arranged (combination of flowing in or out of the page) so that all three wires (a) attract, and (b) repel each other? Explain.

9.13 Additional Problems

9.13.1 Application of Ampere's Law

The simplest possible application of Ampere's law allows us to calculate the magnetic field in the vicinity of a single infinitely long wire. Adding more wires with differing currents will check your understanding of Ampere's law.

(a) Calculate with Ampere's law the magnetic field, $|\vec{\mathbf{B}}| = B(r)$, as a function of distance r from the wire, in the vicinity of an infinitely long straight wire that carries current I. Show with a sketch the integration path you choose and state explicitly how you use symmetry. What is the field at a distance of 10 mm from the wire if the current is 10 A?

(b) Eight parallel wires cut the page perpendicularly at the points shown. A wire labeled with the integer k (k = 1, 2, ... , 8) bears the current $2k$ times I_0 (i.e., $I_k = 2k\,I_0$). For those with k = 1 to 4, the current flows up out of the page; for the rest, the current flows down into the page. Evaluate $\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{s}}$ along the closed path (see figure) in the direction indicated by the arrowhead. (Watch your signs!)

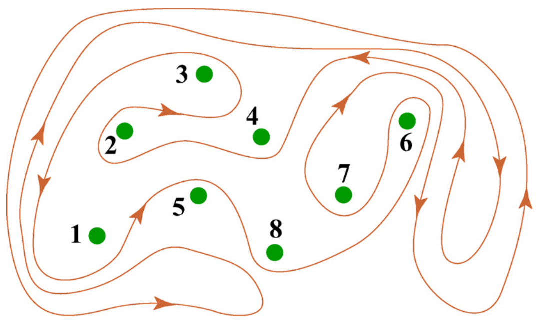


Figure 9.13.1 Amperian loop

(c) Can you use a single application of Ampere's Law to find the field at a point in the vicinity of the 8 wires? Why? How would you proceed to find the field at an arbitrary point P?

9.13.2 Magnetic Field of a Current Distribution from Ampere's Law

Consider the cylindrical conductor with a hollow center and copper walls of thickness $b-a$ as shown in Figure 9.13.2. The radii of the inner and outer walls are a and b respectively, and the current I is uniformly spread over the cross section of the copper.

(a) Calculate the magnitude of the magnetic field in the region outside the conductor, $r > b$. (Hint: consider the entire conductor to be a single thin wire, construct an Amperian loop, and apply Ampere's Law.) What is the direction of $\vec{\mathbf{B}}$?

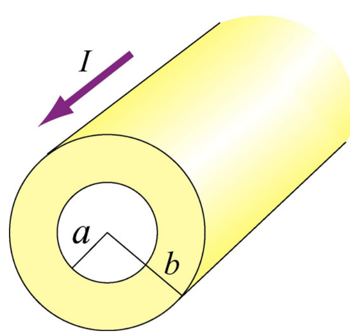


Figure 9.13.2 Hollow cylinder carrying a steady current I.

(b) Calculate the magnetic field inside the inner radius, $r < a$. What is the direction of $\vec{\mathbf{B}}$?

(c) Calculate the magnetic field within the inner conductor, $a < r < b$. What is the direction of $\vec{\mathbf{B}}$?

(d) Plot the behavior of the magnitude of the magnetic field $B(r)$ from $r = 0$ to $r = 4b$. Is $B(r)$ continuous at $r = a$ and $r = b$? What about its slope?

(e) Now suppose that a very thin wire running down the center of the conductor carries the same current I in the opposite direction. Can you plot, roughly, the variation of $B(r)$ without another detailed calculation? (Hint: remember that the vectors $d\vec{\mathbf{B}}$ from different current elements can be added to obtain the total vector magnetic field.)

9.13.3 Cylinder with a Hole

A long copper rod of radius a has an off-center cylindrical hole through its entire length, as shown in Figure 9.13.3. The conductor carries a current I which is directed out of the page and is uniformly distributed throughout the cross section. Find the magnitude and direction of the magnetic field at the point P.

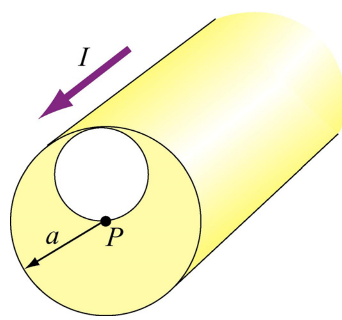


Figure 9.13.3 A cylindrical conductor with a hole.

9.13.4 The Magnetic Field Through a Solenoid

A solenoid has 200 closely spaced turns so that, for most of its length, it may be considered to be an ideal solenoid. It has a length of 0.25 m, a diameter of 0.1 m, and carries a current of 0.30 A.

(a) Sketch the solenoid, showing clearly the rotation direction of the windings, the current direction, and the magnetic field lines (inside and outside) with arrows to show their direction. What is the dominant direction of the magnetic field inside the solenoid?

(b) Find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's law.

(c) Does the magnetic field have a component in the direction of the wire in the loops making up the solenoid? If so, calculate its magnitude both inside and outside the solenoid, at radii 30 mm and 60 mm respectively, and show the directions on your sketch.

9.13.5 Rotating Disk

A circular disk of radius *R* with uniform charge density σ rotates with an angular speed ω. Show that the magnetic field at the center of the disk is

$$B = \frac{1}{2}\mu_0 \sigma \omega R$$

Hint: Consider a circular ring of radius *r* and thickness *dr*. Show that the current in this element is $dI = (\omega / 2\pi)dq = \omega\sigma r\,dr$.

9.13.6 Four Long Conducting Wires

Four infinitely long parallel wires carrying equal current *I* are arranged in such a way that when looking at the cross section, they are at the corners of a square, as shown in Figure 9.13.5. Currents in *A* and *D* point out of the page, and into the page at *B* and *C*. What is the magnetic field at the center of the square?

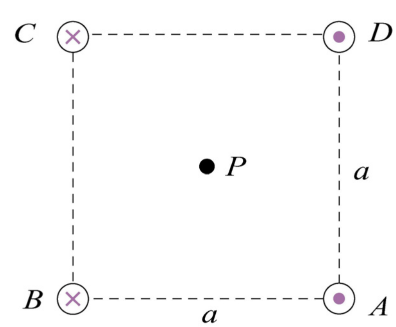


Figure 9.13.5 Four parallel conducting wires

9.13.7 Magnetic Force on a Current Loop

A rectangular loop of length l and width w carries a steady current I_1. The loop is then placed near an finitely long wire carrying a current I_2, as shown in Figure 9.13.6. What is the magnetic force experienced by the loop due to the magnetic field of the wire?

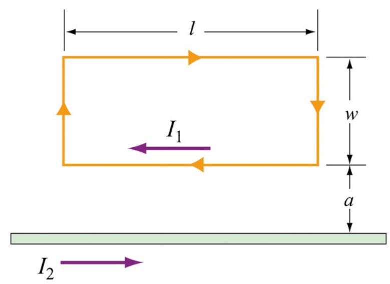


Figure 9.13.6 Magnetic force on a current loop.

9.13.8 Magnetic Moment of an Orbital Electron

We want to estimate the magnetic dipole moment associated with the motion of an electron as it orbits a proton. We use a "semi-classical" model to do this. Assume that the electron has speed v and orbits a proton (assumed to be very massive) located at the origin. The electron is moving in a right-handed sense with respect to the z-axis in a circle of radius $r = 0.53$ Å, as shown in Figure 9.13.7. Note that 1 Å $= 10^{-10}$ m.

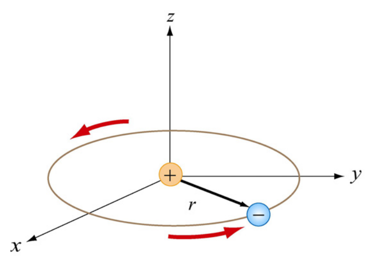


Figure 9.13.7

(a) The inward force $m_e v^2 / r$ required to make the electron move in this circle is provided by the Coulomb attractive force between the electron and proton (m_e is the mass of the electron). Using this fact, and the value of r we give above, find the speed of the electron in our "semi-classical" model. [Ans: 2.18×10^6 m/s.]

(b) Given this speed, what is the orbital period T of the electron? [Ans: 1.52×10^{-16} s.]

(c) What current is associated with this motion? Think of the electron as stretched out uniformly around the circumference of the circle. In a time T, the total amount of charge q that passes an observer at a point on the circle is just e [Ans: 1.05 mA. Big!]

(d) What is the magnetic dipole moment associated with this orbital motion? Give the magnitude and direction. The magnitude of this dipole moment is one *Bohr magneton,* μ_B. [Ans: 9.27×10^{-24} $\text{A}\cdot\text{m}^2$ along the $-z$ axis.]

(e) One of the reasons this model is "semi-classical" is because classically there is no reason for the radius of the orbit above to assume the specific value we have given. The value of r is determined from quantum mechanical considerations, to wit that the orbital angular momentum of the electron can only assume integral multiples of $h/2\pi$, where $h = 6.63\times10^{-34}$ J/s is the Planck constant. What is the orbital angular momentum of the electron here, in units of $h/2\pi$?

9.13.9 Ferromagnetism and Permanent Magnets

A disk of iron has a height $h = 1.00$ mm and a radius $r = 1.00$ cm. The magnetic dipole moment of an atom of iron is $\mu = 1.8\times10^{-23}\ \text{A}\cdot\text{m}^2$. The molar mass of iron is 55.85 g, and its density is 7.9 g/cm^3. Assume that all the iron atoms in the disk have their dipole moments aligned with the axis of the disk.

(a) What is the number density of the iron atoms? How many atoms are in this disk? [Ans: 8.5×10^{28} atoms/m^3; 2.7×10^{22} atoms.]

(b) What is the magnetization $\vec{\mathbf{M}}$ in this disk? [Ans: 1.53×10^6 A/m, parallel to axis.]

(c) What is the magnetic dipole moment of the disk? [Ans: $0.48\ \text{A}\cdot\text{m}^2$.]

(d) If we were to wrap one loop of wire around a circle of the same radius r, how much current would the wire have to carry to get the dipole moment in (c)? This is the "equivalent" surface current due to the atomic currents in the interior of the magnet. [Ans: 1525 A.]

9.13.10 Charge in a Magnetic Field

A coil of radius R with its symmetric axis along the $+x$-direction carries a steady current I. A positive charge q moves with a velocity $\vec{\mathbf{v}} = v\,\hat{\mathbf{j}}$ when it crosses the axis at a distance x from the center of the coil, as shown in Figure 9.13.8.

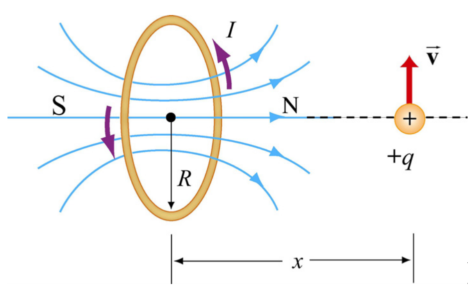


Figure 9.13.8

Describe the subsequent motion of the charge. What is the instantaneous radius of curvature?

9.13.11 Permanent Magnets

A magnet in the shape of a cylindrical rod has a length of 4.8 cm and a diameter of 1.1 cm. It has a uniform magnetization *M* of 5300 A/m, directed parallel to its axis.

(a) Calculate the magnetic dipole moment of this magnet.

(b) What is the axial field a distance of 1 meter from the center of this magnet, along its axis? [Ans: (a) $2.42\times10^{-2}\ \text{A}\cdot\text{m}^2$, (b) $4.8\times10^{-9}\,\text{T}$, or 4.8×10^{-5} gauss .]

9.13.12 Magnetic Field of a Solenoid

(a) A 3000-turn solenoid has a length of 60 cm and a diameter of 8 cm. If this solenoid carries a current of 5.0 A, find the magnitude of the magnetic field inside the solenoid by constructing an Amperian loop and applying Ampere's Law. How does this compare to the magnetic field of the earth (0.5 gauss). [Ans: 0.0314 T, or 314 gauss, or about 600 times the magnetic field of the earth].

We make a magnetic field in the following way: We have a long cylindrical shell of non-conducting material which carries a surface charge fixed in place (glued down) of σ C/m^2, as shown in Figure 9.13.9 The cylinder is suspended in a manner such that it is free to revolve about its axis, without friction. Initially it is at rest. We come along and spin it up until the speed of the surface of the cylinder is v_0.

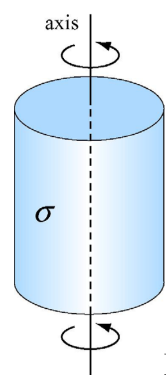


Figure 9.13.9

(b) What is the surface current K on the walls of the cylinder, in A/m? [Ans: $K=\sigma v_0$.]

(c) What is magnetic field inside the cylinder? [Ans. $B=\mu_0 K=\mu_0\sigma v_0$, oriented along axis right-handed with respect to spin.]

(d) What is the magnetic field outside of the cylinder? Assume that the cylinder is infinitely long. [Ans: 0].

9.13.13 Effect of Paramagnetism

A solenoid with 16 turns/cm carries a current of 1.3 A.

(a) By how much does the magnetic field inside the solenoid increase when a close-fitting chromium rod is inserted? [Note: Chromium is a paramagnetic material with magnetic susceptibility $\chi = 2.7 \times 10^{-4}$.]

(b) Find the magnitude of the magnetization $\vec{\mathbf{M}}$ of the rod. [Ans: (a) 0.86 μT; (b) 0.68 A/m.]