Chapter Five

Controllability and Observability

Controllability and observability represent two major concepts of modern control system theory. These originally theoretical concepts, introduced by R. Kalman in 1960, are particularly important for practical implementations. They can be roughly defined as follows.

Controllability: *In order to be able to do whatever we want with the given dynamic system under control input, the system must be controllable.*

Observability: *In order to see what is going on inside the system under observation, the system must be observable.*

Even though the concepts of controllability and observability are almost abstractly defined, we now intuitively understand their meanings. The remaining problem is to produce some mathematical check up tests and to define controllability and observability more rigorously. Our intention is to reduce mathematical derivations and the number of definitions, but at the same time to derive and define very clearly both of them. In that respect, in Section 5.1, we start with observability derivations for linear discrete-time invariant systems and give the corresponding definition. The observability of linear discrete systems is very naturally introduced using only elementary linear algebra. This approach will be extended to continuous-time system observability, where the derivatives of measurements (observations) have to be used, Section 5.2. Next, in Sections 5.3 and 5.4 we define controllability for both discrete- and continuous-time linear systems.

In this chapter we show that the concepts of controllability and observability are related to linear systems of algebraic equations. It is well known that a solvable system of linear algebraic equations has a solution if and only if the rank of the system matrix is full (see Appendix C). Observability and controllability tests will be connected to the rank tests of ceratin matrices, known as the controllability and observability matrices.

At the end of this chapter, in Section 5.5, we will introduce the concepts of system stabilizability (detectability), which stand for controllability (observability) of unstable system modes. Also, we show that both controllability and observability are invariant under nonsingular transformations. In addition, in the same section the concepts of controllability and observability are clarified using different canonical forms, where they become more obvious.

The study of observability is closely related to observer (estimator) design, a simple, but extremely important technique used to construct another dynamic system, the observer (estimator), which produces estimates of the system state variables using information about the system inputs and outputs. The estimator design is presented in Section 5.6. Techniques for constructing both full-order and reduced-order estimators are considered. A corresponding problem to observer design is the so-called pole placement problem. It can be shown that for a controllable linear system, the system poles (eigenvalues) can be arbitrarily located in the complex plane. Since this technique can be used for system linear feedback stabilization and for controller design purposes, it will be independently presented in Section 8.2.

Several examples are included in order to demonstrate procedures for examining system controllability and observability. All of them can be checked by MATLAB. Finally, we have designed the corresponding laboratory experiment by using the MATLAB package, which can contribute to better and deeper understanding of these important modern control concepts.

Chapter Objectives

This chapter introduces definitions of system controllability and observability. Testing controllability and observability is replaced by linear algebra problems of finding ranks of certain matrices known as the controllability and observability matrices. After mastering the above concepts and tests, students will be able to determine system initial conditions from system output measurements, under the assumption that the given system is observable. As the highlight of this chapter, students will learn how to construct a system's observer (estimator), which for an observable system produces the estimates of state variables at any time instant.

5.1 Observability of Discrete Systems

Consider a linear, time invariant, discrete-time system in the state space form

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k), \qquad \mathbf{x}(0) = \mathbf{x}_o = \text{unknown} \tag{5.1}$$

with output measurements

$$\mathbf{y}(k) = \mathbf{C}_d\mathbf{x}(k) \tag{5.2}$$

where $\mathbf{x}(k) \in \Re^n$, $\mathbf{y}(k) \in \Re^p$. $\mathbf{A}_d$ and $\mathbf{C}_d$ are constant matrices of appropriate dimensions. The natural question to be asked is: can we learn everything about the dynamical behavior of the state space variables defined in (5.1) by using only information from the output measurements (5.2). If we know $\mathbf{x}_o$, then the recursion (5.1) apparently gives us complete knowledge about the state variables at any discrete-time instant. Thus, the only thing that we have to determine from the state measurements is the initial state vector $\mathbf{x}(0) = \mathbf{x}_o$.

Since the n-dimensional vector $\mathbf{x}(0)$ has n unknown components, it is expected that n measurements are sufficient to determine $\mathbf{x}_o$. Take $k = 0, 1, ..., n-1$ in (5.1) and (5.2), i.e. generate the following sequence

$$\begin{aligned}
&\mathbf{y}(0) = \mathbf{C}_d\mathbf{x}(0)\\
&\mathbf{y}(1) = \mathbf{C}_d\mathbf{x}(1) = \mathbf{C}_d\mathbf{A}_d\mathbf{x}(0)\\
&\mathbf{y}(2) = \mathbf{C}_d\mathbf{x}(2) = \mathbf{C}_d\mathbf{A}_d\mathbf{x}(1) = \mathbf{C}_d\mathbf{A}_d^2\mathbf{x}(0)\\
&\vdots\\
&\mathbf{y}(n-1) = \mathbf{C}_d\mathbf{x}(n-1) = \mathbf{C}_d\mathbf{A}_d^{n-1}\mathbf{x}(0)
\end{aligned} \tag{5.3}$$

or, in matrix form

$$\begin{bmatrix} \mathbf{y}(0) \\ \mathbf{y}(1) \\ \mathbf{y}(2) \\ \vdots \\ \mathbf{y}(n-1) \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \\ \mathbf{C}_d\mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d\mathbf{A}_d^{n-1} \end{bmatrix}^{(np)\times n} \times \mathbf{x}(0) \tag{5.4}$$

We know from elementary linear algebra that the system of linear algebraic equations with n unknowns, (5.4), has a unique solution if and only if the system matrix has rank n. In this case we need

$$\operatorname{rank}\begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \\ \mathbf{C}_d\mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d\mathbf{A}_d^{n-1} \end{bmatrix} = n \tag{5.5}$$

Thus, the initial condition $\mathbf{x}_0$ is completely determined if the so-called *observability matrix*, defined by

$$\mathcal{O}(\mathbf{A}_d, \mathbf{C}_d) = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \\ \mathbf{C}_d\mathbf{A}_d^2 \\ \vdots \\ \mathbf{C}_d\mathbf{A}_d^{n-1} \end{bmatrix}^{(np)\times n} \tag{5.6}$$

has rank n, that is

$$\operatorname{rank}\mathcal{O} = n \tag{5.7}$$

The previous derivations can be summarized in the following theorem.

Theorem 5.1 *The linear discrete-time system* (5.1) *with measurements* (5.2) *is observable if and only if the observability matrix* (5.6) *has rank equal to* n.

A simple second-order example demonstrates the procedure for examining the observability of linear discrete-time systems. More complex examples corresponding to real physical control systems will be considered in Sections 5.7 and 5.8.

Example 5.1: Consider the following system with measurements

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

The observability matrix for this second-order system is given by

$$\mathcal{O} = \begin{bmatrix} \mathbf{C}_d \\ \mathbf{C}_d\mathbf{A}_d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 10 \end{bmatrix}$$

Since the rows of the matrix $\mathcal{O}$ are linearly independent, then $\mathrm{rank}\mathcal{O} = 2 = n$, i.e. the system under consideration is observable. Another way to test the completeness of the rank of square matrices is to find their determinants. In this case

$$\det\mathcal{O} = -4 \neq 0 \;\Leftrightarrow\; \text{full rank} \;=\; n = 2$$

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Example 5.2: Consider a case of an unobservable system, which can be obtained by slightly modifying Example 5.1. The corresponding system and measurement matrices are given by

$$\mathbf{A}_d = \begin{bmatrix} 1 & -2 \\ -3 & -4 \end{bmatrix}, \quad \mathbf{C}_d = [1 \quad 2]$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} 1 & 2 \\ -5 & -10 \end{bmatrix}$$

so that $\mathrm{rank}\mathcal{O} = 1 < 2$, and the system is unobservable.

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5.2 Observability of Continuous Systems

A linear, time invariant, continuous system in the state space form was studied in Chapter 3. For the purpose of studying its observability, we consider an input-free system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_o = \text{unknown} \tag{5.8}$$

with the corresponding measurements

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{5.9}$$

of dimensions $\mathbf{x}(t) \in \Re^n$, $\mathbf{y}(t) \in \Re^p$, $\mathbf{A} \in \Re^{n\times n}$, and $\mathbf{C} \in \Re^{p\times n}$. Following the same arguments as in the previous section, we can conclude that the knowledge of $\mathbf{x}_o$ is sufficient to determine $\mathbf{x}(t)$ at any time instant, since from (5.8) we have

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) \tag{5.10}$$

The problem that we are faced with is to find $\mathbf{x}(t_0)$ from the available measurements (5.9). In Section 5.1 we have solved this problem for discrete-time systems by generating the sequence of measurements at discrete-time instants $k = 0, 1, 2, ..., n-1$, i.e. by producing relations given in (5.3). Note that a time shift in the discrete-time corresponds to a derivative in the continuous-time. Thus, an analogous technique in the continuous-time domain is obtained by taking derivatives of the continuous-time measurements (5.9)

$$\begin{aligned}
\mathbf{y}(t_0) &= \mathbf{C}\mathbf{x}(t_0) \\
\dot{\mathbf{y}}(t_0) &= \mathbf{C}\dot{\mathbf{x}}(t_0) = \mathbf{C}\mathbf{A}\mathbf{x}(t_0) \\
\ddot{\mathbf{y}}(t_0) &= \mathbf{C}\ddot{\mathbf{x}}(t_0) = \mathbf{C}\mathbf{A}^2\mathbf{x}(t_0) \\
&\vdots \\
\mathbf{y}^{(n-1)}(t_0) &= \mathbf{C}\mathbf{x}^{(n-1)}(t_0) = \mathbf{C}\mathbf{A}^{n-1}\mathbf{x}(t_0)
\end{aligned} \tag{5.11}$$

Our goal is to generate n linearly independent algebraic equations in n unknowns of the state vector $\mathbf{x}(t_0)$. Equations (5.11) comprise a system of np linear algebraic equations. They can be put in matrix form as

$$\begin{bmatrix} \mathbf{y}(t_0) \\ \dot{\mathbf{y}}(t_0) \\ \ddot{\mathbf{y}}(t_0) \\ \vdots \\ \mathbf{y}^{(n-1)}(t_0) \end{bmatrix}^{(np)\times 1} = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \\ \mathbf{C}\mathbf{A}^2 \\ \vdots \\ \mathbf{C}\mathbf{A}^{n-1} \end{bmatrix}^{(np)\times n} \times \mathbf{x}(t_0) = \mathcal{O}\mathbf{x}(t_0) = \mathrm{Y}(t_0) \tag{5.12}$$

where $\mathcal{O}$ is the observability matrix already defined in (5.6) and where the definition of $\mathrm{Y}(t_0)$ is obvious. Thus, the initial condition $\mathbf{x}(t_0)$ can be determined uniquely from (5.12) if and only if the observability matrix has full rank, i.e. $\mathrm{rank}\mathcal{O} = n$.

As expected, we have obtained the same observability result for both continuous- and discrete-time systems. The continuous-time observability theorem, dual to Theorem 5.1, can be formulated as follows.

Theorem 5.2 *The linear continuous-time system* (5.8) *with measurements* (5.9) *is observable if and only if the observability matrix has full rank.*

It is important to notice that adding higher-order derivatives in (5.12) cannot increase the rank of the observability matrix since by the Cayley–Hamilton theorem (see Appendix C) for $k \geq n$ we have

$$\mathbf{A}^k = \sum_{i=0}^{n-1} \alpha_i \mathbf{A}^i \tag{5.13}$$

so that the additional equations would be linearly dependent on the previously defined n equations (5.12). The same applies to the discrete-time domain and the corresponding equations given in (5.4).

There is no need to produce a test example for the observability study of continuous-time systems since the procedure is basically the same as in the case of discrete-time systems studied in the previous section. Thus, Examples 5.1 and 5.2 demonstrate the presented procedure in this case also; however, we have to keep in mind that the corresponding matrices $\mathbf{A}$ and $\mathbf{C}$ describe systems which operate in different time domains. Fortunately, the algebraic procedures are exactly the same in both cases.

5.3 Controllability of Discrete Systems

Consider a linear discrete-time invariant control system defined by

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{B}_d\mathbf{u}(k), \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{5.14}$$

The system controllability is roughly defined as an ability to do whatever we want with our system, or in more technical terms, the ability to transfer our system from any initial state $\mathbf{x}(0) = \mathbf{x}_o$ to any desired final state $\mathbf{x}(k_1) = \mathbf{x}_f$ in a finite time, i.e. for $k_1 < \infty$ (it makes no sense to achieve that goal at $k_1 = \infty$). Thus, the question to be answered is: can we find a control sequence $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(n-1)$, such that $\mathbf{x}(k) = \mathbf{x}_f$?

Let us start with a simplified problem, namely let us assume that the input $\mathbf{u}(k)$ is a scalar, i.e. the input matrix $\mathbf{B}_d$ is a vector denoted by $\mathbf{b}_d$. Thus, we have

$$\mathbf{x}(k+1) = \mathbf{A}_d\mathbf{x}(k) + \mathbf{b}_d u(k), \qquad \mathbf{x}(0) = \mathbf{x}_o \tag{5.15}$$

Taking $k = 0, 1, 2, ..., n$ in (5.15), we obtain the following set of equations

$$\begin{aligned} \mathbf{x}(1) &= \mathbf{A}_d\mathbf{x}(0) + \mathbf{b}_d u(0) \\ \mathbf{x}(2) &= \mathbf{A}_d\mathbf{x}(1) + \mathbf{b}_d u(1) = \mathbf{A}_d^2\mathbf{x}(0) + \mathbf{A}_d\mathbf{b}_d u(0) + \mathbf{b}_d u(1) \\ &\vdots \\ \mathbf{x}(n) &= \mathbf{A}_d^n\mathbf{x}(0) + \mathbf{A}_d^{n-1}\mathbf{b}_d u(0) + \cdots + \mathbf{b}_d u(n-1) \end{aligned} \tag{5.16}$$

The last equation in (5.16) can be written in matrix form as

$$\mathbf{x}(n) - \mathbf{A}_d^n\mathbf{x}(0) = \left[\mathbf{b}_d \,\vdots\, \mathbf{A}_d\mathbf{b}_d \,\vdots\, \cdots \,\vdots\, \mathbf{A}_d^{n-1}\mathbf{b}_d\right] \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} \tag{5.17}$$

Note that $\left[\mathbf{b}_d \vdots \mathbf{A}_d\mathbf{b}_d \vdots \cdots \vdots \mathbf{A}_d^{n-1}\mathbf{b}_d\right]$ is a square matrix. We call it the *controllability matrix* and denote it by $\mathcal{C}$. If the controllability matrix $\mathcal{C}$ is nonsingular, equation (5.17) produces the unique solution for the input sequence given by

$$\begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(1) \\ u(0) \end{bmatrix} = \mathcal{C}^{-1}(\mathbf{x}(n) - \mathbf{A}_d^n\mathbf{x}(0)) \tag{5.18}$$

Thus, for any $\mathbf{x}(n) = \mathbf{x}_f$, the expression (5.18) determines the input sequence that transfers the initial state $\mathbf{x}_o$ to the desired state $\mathbf{x}_f$ in n steps. It follows that the controllability condition, in this case, is equivalent to nonsingularity of the controllability matrix $\mathcal{C}$.

In a general case, when the input $\mathbf{u}(k)$ is a vector of dimension r, the repetition of the same procedure as in (5.15)–(5.17) leads to

$$\mathbf{x}(n) - \mathbf{A}_d^n\mathbf{x}(0) = \left[\mathbf{B}_d \,\vdots\, \mathbf{A}_d\mathbf{B}_d \,\vdots\, \cdots \,\vdots\, \mathbf{A}_d^{n-1}\mathbf{B}_d\right] \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix} \tag{5.19}$$

The controllability matrix, in the general vector input case, defined by

$$\mathcal{C}(\mathbf{A}_d, \mathbf{B}_d) = \left[\mathbf{B}_d \vdots \mathbf{A}_d\mathbf{B}_d \vdots \cdots \vdots \mathbf{A}_d^{n-1}\mathbf{B}_d\right] \tag{5.20}$$

is of dimension $n \times r \cdot n$. The corresponding system of n linear algebraic equations in $r \cdot n$ unknowns for n r-dimensional vector components of $\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(n-1)$, given by

$$\mathcal{C}^{n\times(nr)} \begin{bmatrix} \mathbf{u}(n-1) \\ \mathbf{u}(n-2) \\ \vdots \\ \mathbf{u}(1) \\ \mathbf{u}(0) \end{bmatrix}^{(nr)\times 1} = \mathbf{x}(n) - \mathbf{A}_d^n\mathbf{x}(0) = \mathbf{x}_f - \mathbf{A}_d^n\mathbf{x}(0) \tag{5.21}$$

will have a solution for any $\mathbf{x}_f$ if and only if the matrix $\mathcal{C}$ has full rank, i.e. $\mathrm{rank}\mathcal{C} = n$ (see Appendix C).

The controllability theorem is as follows.

Theorem 5.3 *The linear discrete-time system* (5.14) *is controllable if and only if*

$$\mathrm{rank}\mathcal{C} = n \tag{5.22}$$

where the controllability matrix $\mathcal{C}$ *is defined by* (5.20).

5.4 Controllability of Continuous Systems

Studying the concept of controllability in the continuous-time domain is more challenging than in the discrete-time domain. At the beginning of this section we will first apply the same strategy as in Section 5.3 in order to indicate difficulties that we are faced with in the continuous-time domain. Then, we will show how to find a control input that will transfer our system from any initial state to any final state.

A linear continuous-time system with a scalar input is represented by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \quad \mathbf{x}(t_0) = \mathbf{x}_o \tag{5.23}$$

Following the discussion and derivations from Section 5.3, we have, for a scalar input, the following set of equations

$$\begin{aligned}
\dot{\mathbf{x}} &= \frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}u \\
\ddot{\mathbf{x}} &= \frac{d^2}{dt^2}\mathbf{x} = \mathbf{A}^2\mathbf{x} + \mathbf{A}\mathbf{b}u + \mathbf{b}\dot{u} \\
&\vdots \\
\mathbf{x}^{(n)} &= \frac{d^n}{dt^n}\mathbf{x} = \mathbf{A}^n\mathbf{x} + \mathbf{A}^{n-1}\mathbf{b}u + \mathbf{A}^{n-2}\mathbf{b}\dot{u} + \cdots + \mathbf{b}u^{(n-1)}
\end{aligned} \tag{5.24}$$

The last equation in (5.24) can be written as

$$\mathbf{x}^{(n)}(t) - \mathbf{A}^n\mathbf{x}(t) = \mathcal{C}\begin{bmatrix} u^{(n-1)}(t) \\ u^{(n-2)}(t) \\ \vdots \\ \dot{u}(t) \\ u(t) \end{bmatrix} \tag{5.25}$$

Note that (5.25) is valid for any $t \in [t_0, t_f]$ with t_f free but finite. Thus, the nonsingularity of the controllability matrix $\mathcal{C}$ implies the existence of the scalar input function $u(t)$ and its $n-1$ derivatives, for any $t < t_f < \infty$.

For a vector input system dual to (5.23), the above discussion produces the same relation as (5.25) with the controllability matrix $\mathcal{C}$ given by (5.20) and with the input vector $\mathbf{u}(t) \in \Re^r$, that is

$$\mathcal{C}^{n\times m\cdot n}\begin{bmatrix} \mathbf{u}^{(n-1)}(t) \\ \mathbf{u}^{(n-2)}(t) \\ \vdots \\ \dot{\mathbf{u}}(t) \\ \mathbf{u}(t) \end{bmatrix}^{r\cdot n\times 1} = \mathbf{x}^{(n)}(t) - \mathbf{A}^n\mathbf{x}(t) = \gamma(t) \tag{5.26}$$

It is well known from linear algebra that in order to have a solution of (5.26), it is sufficient that

$$\operatorname{rank}\mathcal{C} = \operatorname{rank}\left[\mathcal{C} \vdots \gamma(t)\right] \tag{5.27}$$

Also, a solution of (5.26) exists for any $\gamma(t)$—any desired state at t—if and only if

$$\mathrm{rank}\mathcal{C} = n \tag{5.28}$$

Equations (5.25) and (5.26) establish relationships between the state and control variables. However, from (5.25) and (5.26) we do not have an explicit answer about a control function that is transferring the system from any initial state $\mathbf{x}(t_0)$ to any final state $\mathbf{x}(t_1) = \mathbf{x}_f$. Thus, elegant and simple derivations for the discrete-time controllability problem cannot be completely extended to the continuous-time domain. Another approach, which is mathematically more complex, is required in this case. It will be presented in the remaining part of this section.

From Section 3.2 we know that the state space equation with the control input has the following solution

$$\mathbf{x}(t) = e^{\mathbf{A}(t-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

At the final time t_1 we have

$$\mathbf{x}(t_1) = \mathbf{x}_f = e^{\mathbf{A}(t_1-t_0)}\mathbf{x}(t_0) + \int_{t_0}^{t_1} e^{\mathbf{A}(t_1-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau$$

or

$$e^{-\mathbf{A}t_1}\mathbf{x}_f - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \int_{t_0}^{t_1} e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)d\tau$$

Using the Cayley–Hamilton theorem (see Appendix C), that is

$$e^{-\mathbf{A}\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)\mathbf{A}^i \tag{5.29}$$

where $\alpha_i(\tau), \;\; i = 0, 1, ..., n-1$, are scalar time functions, the previous equation can be rewritten as

$$e^{-\mathbf{A}t_1}\mathbf{x}_f - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \sum_{i=0}^{n-1} \mathbf{A}^i\mathbf{B} \int_{t_0}^{t_1} \alpha_i(\tau)\mathbf{u}(\tau)d\tau$$

or

$$e^{-\mathbf{A}t_1}\mathbf{x}_f - e^{-\mathbf{A}t_0}\mathbf{x}(t_0) = \left[\mathbf{B} \vdots \mathbf{AB} \vdots \cdots \vdots \mathbf{A}^{n-1}\mathbf{B}\right] \begin{bmatrix} \int_{t_0}^{t_1} \alpha_0(\tau)\mathbf{u}(\tau)d\tau \\ \int_{t_0}^{t_1} \alpha_1(\tau)\mathbf{u}(\tau)d\tau \\ \vdots \\ \int_{t_0}^{t_1} \alpha_{n-1}(\tau)\mathbf{u}(\tau)d\tau \end{bmatrix}$$

On the left-hand side of this equation all quantities are known, i.e. we have a constant vector. On the right-hand side the controllability matrix is multiplied by a vector whose components are functions of the required control input. Thus, we have a functional equation in the form

$$\mathbf{const}^{n\times 1} = \mathcal{C}(\mathbf{A},\mathbf{B})^{n\times rn} \begin{bmatrix} \mathbf{f}_1(\mathbf{u}(\tau)) \\ \mathbf{f}_2(\mathbf{u}(\tau)) \\ \vdots \\ \mathbf{f}_{n-1}(\mathbf{u}(\tau)) \end{bmatrix}^{rn\times 1}, \quad \tau \in (t_0, t_1) \tag{5.30}$$

A solution of this equation exists if and only if $\operatorname{rank}\mathcal{C}(\mathbf{A},\mathbf{B}) = n$, which is the condition already established in (5.28). In general, it is very hard to solve this equation. One of the many possible solutions of (5.30) will be given in Section 5.8 in terms of the controllability Grammian. The controllability Grammian is defined by the following integral

$$\mathbf{W}(t_0, t_1) = \int_{t_0}^{t_1} e^{\mathbf{A}(t_0-\tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_0-\tau)} d\tau \tag{5.31}$$

The results presented in this section can be summarized in the following theorem.

Theorem 5.4 *The linear continuous-time system is controllable if and only if the controllability matrix $\mathcal{C}$ has full rank, i.e.* $\operatorname{rank}\mathcal{C} = n$.

We have seen that controllability of linear continuous- and discrete-time systems is given in terms of the controllability matrix (5.20). Examining the rank of the controllability matrix comprises an algebraic criterion for testing system controllability. The example below demonstrates this procedure.

Example 5.3: Given the linear continuous-time system

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & -2 \\ 3 & -4 & 5 \\ -6 & 7 & 8 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & -1 \\ 2 & -3 \\ 4 & -5 \end{bmatrix} \mathbf{u}$$

The controllability matrix for this third-order system is given by

$$\begin{aligned} \mathcal{C} &= \left[\mathbf{B} \vdots \mathbf{AB} \vdots \mathbf{A}^2\mathbf{B} \right] \\ &= \begin{bmatrix} 0 & -1 & \vdots & -6 & 7 & \vdots & \\ 2 & -3 & \vdots & 12 & -10 & \vdots & \mathbf{A}^2\mathbf{B} \\ 4 & -5 & \vdots & 46 & -55 & \vdots & \end{bmatrix} \end{aligned}$$

Since the first three columns are linearly independent we can conclude that $\mathrm{rank}\mathcal{C} = 3$. Hence there is no need to compute $\mathbf{A}^2\mathbf{B}$ since it is well known from linear algebra that the row rank of the given matrix is equal to its column rank. Thus, $\mathrm{rank}\mathcal{C} = 3 = n$ implies that the system under consideration is controllable.

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5.5 Additional Controllability/Observability Topics

In this section we will present several interesting and important results related to system controllability and observability.

Invariance Under Nonsingular Transformations

In Section 3.4 we introduced the similarity transformation that transforms a given system from one set of coordinates to another. Now we will show that both system controllability and observability are invariant under similarity transformation.

Consider the vector input form of (5.23) and the similarity transformation

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x} \tag{5.32}$$

such that

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u}$$

where $\hat{\mathbf{A}} = \mathbf{PAP}^{-1}$ and $\hat{\mathbf{B}} = \mathbf{PB}$. Then the following theorem holds.

Theorem 5.5 *The pair* $(\mathbf{A}, \mathbf{B})$ *is controllable if and only if the pair* $\left(\hat{\mathbf{A}}, \hat{\mathbf{B}}\right)$ *is controllable.*

This theorem can be proved as follows

$$\begin{aligned} \mathcal{C}\left(\hat{\mathbf{A}}, \hat{\mathbf{B}}\right) &= \left[\hat{\mathbf{B}} \vdots \hat{\mathbf{A}}\hat{\mathbf{B}} \vdots \cdots \vdots \hat{\mathbf{A}}^{n-1}\hat{\mathbf{B}}\right] \\ &= \left[\mathbf{PB} \vdots \mathbf{PAP}^{-1}\mathbf{PB} \vdots \cdots \vdots \mathbf{PA}^{n-1}\mathbf{P}^{-1}\mathbf{PB}\right] \\ &= \mathbf{P}\left[\mathbf{B} \vdots \mathbf{AB} \vdots \cdots \vdots \mathbf{A}^{n-1}\mathbf{B}\right] = \mathbf{P}\mathcal{C}(\mathbf{A}, \mathbf{B}) \end{aligned}$$

Since $\mathbf{P}$ is a nonsingular matrix (it cannot change the rank of the product $\mathbf{P}\mathcal{C}$), we get

$$\operatorname{rank}\mathcal{C}\left(\hat{\mathbf{A}}, \hat{\mathbf{B}}\right) = \operatorname{rank}\mathcal{C}(\mathbf{A}, \mathbf{B})$$

which proves the theorem and establishes controllability invariance under a similarity transformation.

A similar theorem is valid for observability. The similarity transformation (5.32) applied to (5.8) and (5.9) produces

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} \\ \mathbf{y} &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

where

$$\hat{\mathbf{C}} = \mathbf{CP}^{-1}$$

Then, we have the following theorem

Theorem 5.6 *The pair* $(\mathbf{A}, \mathbf{C})$ *is observable if and only if the pair* $\left(\hat{\mathbf{A}}, \hat{\mathbf{C}}\right)$ *is observable.*

The proof of this theorem is as follows

$$\mathcal{O}\left(\hat{\mathbf{A}}, \hat{\mathbf{C}}\right) = \begin{bmatrix} \hat{\mathbf{C}} \\ \hat{\mathbf{C}}\hat{\mathbf{A}} \\ \hat{\mathbf{C}}\hat{\mathbf{A}}^2 \\ \vdots \\ \hat{\mathbf{C}}\hat{\mathbf{A}}^{n-1} \end{bmatrix} = \begin{bmatrix} \mathbf{CP}^{-1} \\ \mathbf{CP}^{-1}\mathbf{PAP}^{-1} \\ \mathbf{CP}^{-1}\mathbf{PA}^2\mathbf{P}^{-1} \\ \vdots \\ \mathbf{CP}^{-1}\mathbf{PA}^{n-1}\mathbf{P}^{-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix} \mathbf{P}^{-1}$$

that is,

$$\mathcal{O}\left(\hat{\mathbf{A}}, \hat{\mathbf{C}}\right) = \mathcal{O}(\mathbf{A}, \mathbf{C})\mathbf{P}^{-1}$$

The nonsingularity of $\mathbf{P}$ implies

$$\operatorname{rank}\mathcal{O}\left(\hat{\mathbf{A}}, \hat{\mathbf{C}}\right) = \operatorname{rank}\mathcal{O}(\mathbf{A}, \mathbf{C})$$

which proves the stated observability invariance.

Note that Theorems 5.5 and 5.6 are applicable to both continuous- and discrete-time linear systems.

Frequency Domain Controllability and Observability Test

Controllability and observability have been introduced in the state space domain as pure time domain concepts. It is interesting to point out that in the frequency domain there exists a very powerful and simple theorem that gives a single condition for both the controllability and the observability of a system. It is given below.

Let $H(s)$ be the transfer function of a single-input single-output system

$$H(s) = \mathbf{c}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}$$

Note that $H(s)$ is defined by a ratio of two polynomials containing the corresponding system poles and zeros. The following controllability–observability theorem is given without a proof.

Theorem 5.7 *If there are no zero-pole cancellations in the transfer function of a single-input single-output system, then the system is both controllable and observable. If the zero-pole cancellation occurs in $H(s)$, then the system is either uncontrollable or unobservable or both uncontrollable and unobservable.*

A similar theorem can be formulated for discrete linear time invariant systems.

Example 5.4: Consider a linear continuous-time dynamic system represented by its transfer function

$$H(s) = \frac{(s+3)}{(s+1)(s+2)(s+3)} = \frac{s+3}{s^3 + 6s^2 + 11s + 6}$$

Theorem 5.7 indicates that any state space model for this system is either uncontrollable or/and unobservable. To get the complete answer we have to go to a state space form and examine the controllability and observability matrices. One of the possible many state space forms of $H(s)$ is as follows

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -6 & -11 & -6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

It is easy to show that the controllability and observability matrices are given by

$$\mathcal{C} = \begin{bmatrix} 1 & -6 & 25 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 3 & 0 \\ -3 & -11 & -6 \end{bmatrix}$$

Since

$$\det \mathcal{C} = 1 \neq 0 \;\Rightarrow\; \text{rank}\mathcal{C} = 3 = n$$

and

$$\det \mathcal{O} = 0 \;\Rightarrow\; \text{rank}\mathcal{O} < 3 = n$$

this system is controllable, but unobservable.

Note that, due to a zero-pole cancellation at $s = -3$, the system transfer function $H(s)$ is reducible to

$$H(s) = H_r(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2 + 3s + 2}$$

so that the equivalent system of order $n = 2$ has the corresponding state space form

$$\begin{bmatrix} \dot{x}_{1r} \\ \dot{x}_{2r} \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1r} \\ x_{2r} \end{bmatrix}$$

For this reduced-order system we have

$$\mathcal{C} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}, \quad \mathcal{O} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and therefore the system is both controllable and observable.

Interestingly enough, the last two mathematical models of dynamic systems of order $n = 3$ and $n = 2$ represent exactly the same physical system. Apparently, the second one ($n = 2$) is preferred since it can be realized with only two integrators.

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It can be concluded from Example 5.4 that Theorem 5.7 gives an answer to the problem of dynamic system reducibility. It follows that a single-input single-output dynamic system is irreducible if and only if it is both controllable and observable. Such a system realization is called the *minimal realization*. If the system is either uncontrollable and/or unobservable it can be represented by a system whose order has been reduced by removing uncontrollable and/or unobservable modes. It can be seen from Example 5.4 that the reduced system with $n = 2$ is both controllable and observable, and hence it cannot be further reduced. This is also obvious from the transfer function $H_r(s)$.

Theorem 5.7 can be generalized to multi-input multi-output systems, where it plays very important role in the procedure of testing whether or not a given system is in the minimal realization form. The procedure requires the notion of the characteristic polynomial for proper rational matrices which is beyond the scope of this book. Interested readers may find all details and definitions in Chen (1984).

It is important to point out that the similarity transformation does not change the transfer function as was shown in Section 3.4.

Controllability and Observability of Special Forms

In some cases, it is easy to draw conclusions about system controllability and/or observability by examining directly the state space equations. In those cases there is no need to find the corresponding controllability and observability matrices and check their ranks.

Consider the phase variable canonical form with

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad \ldots \quad 0]$$

This form is both controllable and observable due to an elegant chain connection of the state variables. The variable $x_1(t)$ is directly measured, so that $x_2(t)$ is known from $x_2(t) = \dot{x}_1(t)$. Also, $x_3(t) = \dot{x}_2(t) = \ddot{x}_1(t)$, and so on, $x_n(t) = x_1^{(n-1)}(t)$. Thus, this form is observable. The controllability follows from the fact that all state variables are affected by the control input, i.e. x_n is affected directly by $u(t)$ and then $\dot{x}_{n-1}(t)$ by $x_n(u(t))$ and so on. The control input is able to indirectly move all state variables into the desired positions so that the system is controllable. This can be formally verified by forming the corresponding controllability matrix and checking its rank. This is left as an exercise for students (see Problem 5.13).

Another example is the modal canonical form. Assuming that all eigenvalues of the system matrix are distinct, we have

$$\dot{\mathbf{x}} = \Lambda \mathbf{x} + \Gamma \mathbf{u}$$
$$\mathbf{y} = \mathfrak{D} \mathbf{x}$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{bmatrix}$$

$$\mathfrak{D} = [\delta_1 \ \delta_2 \ \cdots \ \delta_n]$$

We are apparently faced with n completely decoupled first-order systems. Obviously, for controllability all γ_i, $i = 1, \ldots, n$, must be different from zero, so that each state variable can be controlled by the input $\mathbf{u}(t)$. Similarly, $\delta_i \neq 0$, $i = 1, \ldots, n$, ensures observability since, due to the state decomposition, each system must be observed independently.

The Role of Observability in Analog Computer Simulation

In addition to applications in control system theory and practice, the concept of observability is useful for analog computer simulation. Consider the problem of solving an nth-order differential equation given by

$$y^{(n)} + \sum_{i=1}^{n} a_{n-i} y^{(n-i)} = \sum_{i=0}^{m} b_{m-i} u^{(m-i)}$$

with known initial conditions for $y(0), \dot{y}(0), ..., y^{(n-1)}(0)$. This system can be solved by an analog computer by using n integrators. The outputs of these n integrators represent the state variables $x_1, x_2, ..., x_n$, so that this system has the state space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u, \qquad \mathbf{x}(0) = \text{unknown}$$
$$y = \mathbf{c}\mathbf{x}$$

However, the initial condition for $\mathbf{x}(0)$ is not given. In other words, the initial conditions for the considered system of n integrators are unknown. They can be determined from $y(0), \dot{y}(0), ..., y^{(n-1)}(0)$ by following the observability derivations performed in Section 5.2, namely

$$\begin{aligned}
&y(0) = \mathbf{c}\mathbf{x}(0)\\
&\dot{y}(0) = \mathbf{c}\dot{\mathbf{x}}(0) = \mathbf{c}\mathbf{A}\mathbf{x}(0) + \mathbf{c}\mathbf{b}u(0)\\
&\ddot{y}(0) = \mathbf{c}\ddot{\mathbf{x}}(0) = \mathbf{c}\mathbf{A}^2\mathbf{x}(0) + \mathbf{c}\mathbf{A}\mathbf{b}u(0) + \mathbf{c}\mathbf{b}\dot{u}(0)\\
&\vdots\\
&y^{(n-1)}(0) = \mathbf{c}\mathbf{x}^{(n-1)}(0) = \mathbf{c}\mathbf{A}^{n-1}\mathbf{x}(0) + \mathbf{c}\mathbf{A}^{n-2}\mathbf{b}u(0)\\
&\qquad + \mathbf{c}\mathbf{A}^{n-3}\mathbf{b}\dot{u}(0) + \cdots + \mathbf{c}\mathbf{A}\mathbf{b}u^{(n-3)}(0) + \mathbf{c}\mathbf{b}u^{(n-2)}(0)
\end{aligned}$$

This system can be written in matrix form as follows

$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \vdots \\ y^{(n-1)}(0) \end{bmatrix} = \mathcal{O} \cdot \mathbf{x}(0) + \mathcal{T} \begin{bmatrix} 0 \\ u(0) \\ \vdots \\ u^{(n-2)}(0) \end{bmatrix} \tag{5.33}$$

where $\mathcal{O}$ is the observability matrix and $\mathcal{T}$ is a known matrix. Since $u(0), \dot{u}(0), ..., u^{(n-1)}(0)$ are known, it follows that a unique solution for $\mathbf{x}(0)$

exists if and only if the observability matrix, which is square in this case, is invertible, i.e. the pair $(\mathbf{A}, \mathbf{c})$ is observable.

Example 5.5: Consider a system represented by the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = \frac{du}{dt} + u, \quad y(0) = 2, \ \ \dot{y}(0) = 1, \quad u(t) = e^{-4t}, \ \ t \geq 0$$

Its state space form is given by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u = \begin{bmatrix} 0 & 1 \\ -4 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u$$

$$y = \mathbf{c}\mathbf{x} = [1 \quad 1]\mathbf{x}$$

The initial condition for the state space variables is obtained from (5.33) as

$$\begin{bmatrix} \mathbf{c} \\ \mathbf{cA} \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} \mathbf{y}(0) \\ \dot{\mathbf{y}}(0) \end{bmatrix} - \begin{bmatrix} 0 \\ \mathbf{cb}u(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

leading to

$$\begin{bmatrix} 1 & 1 \\ -4 & -3 \end{bmatrix}\mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} -6 \\ 8 \end{bmatrix}$$

This means that if analog computer simulation is used to solve the above second-order differential equation, the initial conditions for integrators should be set to -6 and 8.

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Stabilizability and Detectability

So far we have defined and studied observability and controllability of the complete state vector. We have seen that the system is controllable (observable) if all components of the state vector are controllable (observable). The natural question to be asked is: do we really need to control and observe all state variables? In some applications, it is sufficient to take care only of the unstable components of the state vector. This leads to the definition of stabilizability and detectability.

Definition 5.1 *A linear system (continuous or discrete) is stabilizable if all unstable modes are controllable.*

Definition 5.2 *A linear system (continuous or discrete) is detectable if all unstable modes are observable.*

The concepts of stabilizability and detectability play very important roles in optimal control theory, and hence are studied in detail in advanced control theory courses. For the purpose of this course, it is enough to know their meanings.

5.6 Observer (Estimator) Design[1]

Sometimes all state space variables are not available for measurements, or it is not practical to measure all of them, or it is too expensive to measure all state space variables. In order to be able to apply the state feedback control to a system, *all of its state space variables must be available at all times*. Also, in some control system applications, one is interested in having information about system state space variables at any time instant. Thus, one is faced with the problem of estimating system state space variables. This can be done by constructing another dynamical system called the observer or estimator, connected to the system under consideration, whose role is to produce good estimates of the state space variables of the original system.

The theory of observers started with the work of Luenberger (1964, 1966, 1971) so that observers are very often called Luenberger observers. According to Luenberger, any system driven by the output of the given system can serve as an observer for that system. Two main techniques are available for observer design. The first one is used for the full-order observer design and produces an observer that has the same dimension as the original system. The second technique exploits the knowledge of some state space variables available through the output algebraic equation (system measurements) so that a reduced-order dynamic system (observer) is constructed only for estimating state space variables that are not directly obtainable from the system measurements.

5.6.1 Full-Order Observer Design

Consider a linear time invariant continuous system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_o = \text{unknown} \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{5.34}$$

[1] This section may be skipped without loss of continuity.

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^r$, $\mathbf{y} \in \Re^p$ with constant matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ having appropriate dimensions. Since from the system (5.34) only the output variables, $\mathbf{y}(t)$, are available at all times, we may construct another artificial dynamic system of order n (built, for example, of capacitors and resistors) having the same matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t), \qquad \hat{\mathbf{x}}(t_0) = \hat{\mathbf{x}}_o \\ \hat{\mathbf{y}}(t) &= \mathbf{C}\hat{\mathbf{x}}(t) \end{aligned} \tag{5.35}$$

and compare the outputs $\mathbf{y}(t)$ and $\hat{\mathbf{y}}(t)$. Of course these two outputs will be different since in the first case the system's initial condition is unknown, and in the second case it has been chosen arbitrarily. The difference between these two outputs will generate an error signal

$$\mathbf{y}(t) - \hat{\mathbf{y}}(t) = \mathbf{C}\mathbf{x}(t) - \mathbf{C}\hat{\mathbf{x}}(t) = \mathbf{C}\mathbf{e}(t) \tag{5.36}$$

which can be used as the feedback signal to the artificial system such that the estimation (observation) error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is reduced as much as possible. This can be physically realized by proposing the system-observer structure as given in Figure 5.1.

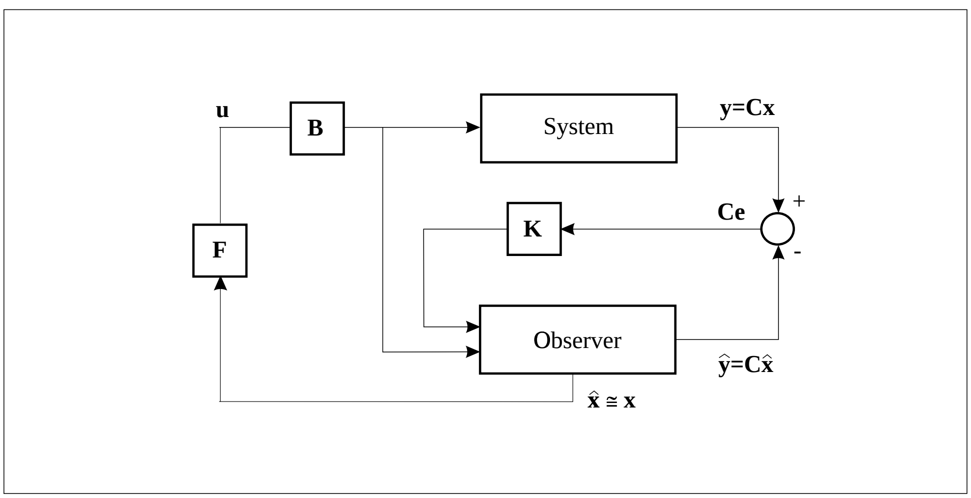

Figure 5.1: System-observer structure

In this structure $\mathbf{K}$ represents the observer gain and has to be chosen such that the observation error is minimized. The observer alone from Figure 5.1 is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}(\mathbf{y}(t) - \hat{\mathbf{y}}(t)) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}\mathbf{C}\mathbf{e}(t) \quad (5.37)$$

From (5.34) and (5.37) it is easy to derive an expression for dynamics of the estimation (observation) error as

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{K}\mathbf{C})\mathbf{e}(t) \quad (5.38)$$

If the observer gain $\mathbf{K}$ is chosen such that the feedback matrix $\mathbf{A} - \mathbf{K}\mathbf{C}$ *is asymptotically stable*, then the estimation error $\mathbf{e}(t)$ will decay to zero for any initial condition $\mathbf{e}(t_0)$. *This can be achieved if the pair* $(\mathbf{A}, \mathbf{C})$ *is observable.* More precisely, by taking the transpose of the estimation error feedback matrix, i.e. $\mathbf{A}^T - \mathbf{C}^T\mathbf{K}^T$, we see that if the pair $\left(\mathbf{A}^T, \mathbf{C}^T\right)$ is controllable, then we can do whatever we want with the system, and thus we can locate its poles in arbitrarily asymptotically stable positions. Note that controllability of the pair $\left(\mathbf{A}^T, \mathbf{C}^T\right)$ is equal to observability of the pair $(\mathbf{A}, \mathbf{C})$, see expressions for the observability and controllability matrices.

In practice the observer poles should be chosen to be about ten times faster than the system poles. This can be achieved by setting the minimal real part of observer eigenvalues to be ten times bigger than the maximal real part of system eigenvalues, that is

$$|Re\{\lambda_{min}\}|_{observer} > 10|Re\{\lambda_{max}\}|_{system}$$

Theoretically, the observer can be made arbitrarily fast by pushing its eigenvalues far to the left in the complex plane, but very fast observers generate noise in the system. A procedure suggesting an efficient choice of the observer initial condition is discussed in Johnson (1988).

It is important to point out that the system-observer structure preserves the closed-loop system poles that would have been obtained if the linear perfect state feedback control had been used. The system (5.34) under the perfect state feedback control, i.e. $\mathbf{u}(t) = -\mathbf{F}\mathbf{x}(t)$ has the closed-loop form as

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{F})\mathbf{x}(t) \quad (5.39)$$

so that the eigenvalues of the matrix $\mathbf{A} - \mathbf{BF}$ are the closed-loop system poles under perfect state feedback. In the case of the system-observer structure, as given in Figure 5.1, we see that the actual control applied to both the system and the observer is given by

$$\mathbf{u}(t) = -\mathbf{F}\hat{\mathbf{x}}(t) = -\mathbf{F}\mathbf{x}(t) + \mathbf{F}\mathbf{e}(t) \tag{5.40}$$

so that from (5.34) and (5.38) we have

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{BF} & \mathbf{BF} \\ \mathbf{0} & \mathbf{A} - \mathbf{KC} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} \tag{5.41}$$

Since the state matrix of this system is upper block triangular, its eigenvalues are equal to the eigenvalues of matrices $\mathbf{A} - \mathbf{BF}$ and $\mathbf{A} - \mathbf{KC}$. A very simple relation among $\mathbf{x}, \mathbf{e}$, and $\hat{\mathbf{x}}$ can be written from the definition of the estimation error as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathbf{x} \\ \hat{\mathbf{x}} \end{bmatrix} \tag{5.42}$$

Note that the matrix $\mathbf{T}$ is nonsingular. In order to go from $\mathbf{xe}$-coordinates to $\mathbf{x}\hat{\mathbf{x}}$-coordinates we have to use the similarity transformation defined in (5.42), which by the main property of the similarity transformation indicates that the same eigenvalues, i.e. $\lambda(\mathbf{A} - \mathbf{BF})$ and $\lambda(\mathbf{A} - \mathbf{KC})$, are obtained in the $\mathbf{x}\hat{\mathbf{x}}$-coordinates.

This important observation that the system-observer configuration has closed-loop poles separated into the original system closed-loop poles under perfect state feedback and the actual observer closed-loop poles is known as the *separation principle*.

5.6.2 Reduced-Order Observer (Estimator)

In this section we show how to construct an observer of reduced dimension by exploiting knowledge of the output measurement equation. Assume that the output matrix $\mathbf{C}$ has rank l, which means that the output equation represents l linearly independent algebraic equations. Thus, equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \tag{5.43}$$

produces l equations for n unknowns of the state space vector $\mathbf{x}(t)$. Our goal is to construct an observer of order $n - l$ for estimation of the remaining $n - l$ state space variables.

The reduced-order observer design will be presented according to the results of Cumming (1969) and Gopinath (1968, 1971). The procedure for obtaining this observer is not unique, which is obvious from the next step. Assume that a matrix $\mathbf{C}_1$ exists such that

$$\operatorname{rank}\begin{bmatrix}\mathbf{C}\\ \mathbf{C}_1\end{bmatrix} = n \tag{5.44}$$

and introduce a vector $\mathbf{p} \in \Re^l$ as

$$\mathbf{p}(t) = \mathbf{C}_1\mathbf{x}(t) \tag{5.45}$$

From equations (5.43) and (5.45) we have

$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{C}\\ \mathbf{C}_1\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{y}(t)\\ \mathbf{p}(t)\end{bmatrix} \tag{5.46}$$

Since the vector $\mathbf{p}(t)$ is unknown, we will construct an observer to estimate it. Introduce the notation

$$\begin{bmatrix}\mathbf{C}\\ \mathbf{C}_1\end{bmatrix}^{-1} = [\mathbf{L}_1 \quad \mathbf{L}_2\,] \tag{5.47}$$

so that from (5.46) we get

$$\mathbf{x}(t) = \mathbf{L}_1\mathbf{y}(t) + \mathbf{L}_2\mathbf{p}(t) \tag{5.48}$$

An observer for $\mathbf{p}(t)$ can be constructed by finding first a differential equation for $\mathbf{p}(t)$ from (5.45), that is

$$\dot{\mathbf{p}} = \mathbf{C}_1\dot{\mathbf{x}} = \mathbf{C}_1\mathbf{A}\mathbf{x} + \mathbf{C}_1\mathbf{B}\mathbf{u} = \mathbf{C}_1\mathbf{A}\mathbf{L}_2\mathbf{p} + \mathbf{C}_1\mathbf{A}\mathbf{L}_1\mathbf{y} + \mathbf{C}_1\mathbf{B}\mathbf{u} \tag{5.49}$$

Note that from (5.49) we are not able to construct an observer for $\mathbf{p}(t)$ since $\mathbf{y}(t)$ does not contain explicit information about the vector $\mathbf{p}(t)$, but if we differentiate the output variable we get from (5.34) and (5.48)

$$\dot{\mathbf{y}} = \mathbf{C}\dot{\mathbf{x}} = \mathbf{C}\mathbf{A}\mathbf{x} + \mathbf{C}\mathbf{B}\mathbf{u} = \mathbf{C}\mathbf{A}\mathbf{L}_2\mathbf{p} + \mathbf{C}\mathbf{A}\mathbf{L}_1\mathbf{y} + \mathbf{C}\mathbf{B}\mathbf{u} \tag{5.50}$$

i.e. $\dot{\mathbf{y}}(t)$ carries information about $\mathbf{p}(t)$.

An observer for $\mathbf{p}(t)$, according to the observer structure given in (5.37), is obtained from the last two equations as

$$\dot{\hat{\mathbf{p}}} = \mathbf{C}_1\mathbf{A}\mathbf{L}_2\hat{\mathbf{p}} + \mathbf{C}_1\mathbf{A}\mathbf{L}_1\mathbf{y} + \mathbf{C}_1\mathbf{B}\mathbf{u} + \mathbf{K}_1\left(\dot{\mathbf{y}} - \dot{\hat{\mathbf{y}}}\right) \tag{5.51}$$

where $\mathbf{K}_1$ is the observer gain. If in equation (5.50) we replace $\mathbf{p}(t)$ by its estimate, we will have

$$\dot{\hat{\mathbf{y}}} = \mathbf{C}\mathbf{A}\mathbf{L}_2\hat{\mathbf{p}} + \mathbf{C}\mathbf{A}\mathbf{L}_1\mathbf{y} + \mathbf{C}\mathbf{B}\mathbf{u} \tag{5.52}$$

so that

$$\dot{\hat{\mathbf{p}}} = \mathbf{C}_1\mathbf{A}\mathbf{L}_2\hat{\mathbf{p}} + \mathbf{C}_1\mathbf{A}\mathbf{L}_1\mathbf{y} + \mathbf{C}_1\mathbf{B}\mathbf{u} + \mathbf{K}_1(\dot{\mathbf{y}} - \mathbf{C}\mathbf{A}\mathbf{L}_2\hat{\mathbf{p}} - \mathbf{C}\mathbf{A}\mathbf{L}_1\mathbf{y} - \mathbf{C}\mathbf{B}\mathbf{u}) \tag{5.53}$$

Since it is impractical and undesirable to differentiate $\mathbf{y}(t)$ in order to get $\dot{\mathbf{y}}(t)$ (this operation introduces noise in practice), we take the change of variables

$$\hat{\mathbf{q}} = \hat{\mathbf{p}} - \mathbf{K}_1\mathbf{y} \tag{5.54}$$

This leads to an observer for $\hat{\mathbf{q}}(t)$ of the form

$$\dot{\hat{\mathbf{q}}}(t) = \mathbf{A}_q\hat{\mathbf{q}}(t) + \mathbf{B}_q\mathbf{u}(t) + \mathbf{K}_q\mathbf{y}(t) \tag{5.55}$$

where

$$\begin{aligned} \mathbf{A}_q &= \mathbf{C}_1\mathbf{A}\mathbf{L}_2 - \mathbf{K}_1\mathbf{C}\mathbf{A}\mathbf{L}_2, \qquad \mathbf{B}_q = \mathbf{C}_1\mathbf{B} - \mathbf{K}_1\mathbf{C}\mathbf{B} \\ \mathbf{K}_q &= \mathbf{C}_1\mathbf{A}\mathbf{L}_2\mathbf{K}_1 + \mathbf{C}_1\mathbf{A}\mathbf{L}_1 - \mathbf{K}_1\mathbf{C}\mathbf{A}\mathbf{L}_1 - \mathbf{K}_1\mathbf{C}\mathbf{A}\mathbf{L}_2\mathbf{K}_1 \end{aligned} \tag{5.56}$$

It is left as an exercise to students (see Problem 5.18) to derive (5.55) and (5.56). The estimates of the original system state space variables are now obtained from (5.48) and (5.53) as

$$\hat{\mathbf{x}}(t) = \mathbf{L}_1\mathbf{y}(t) + \mathbf{L}_2\hat{\mathbf{p}}(t) = \mathbf{L}_2\hat{\mathbf{q}}(t) + (\mathbf{L}_1 + \mathbf{L}_2\mathbf{K}_1)\mathbf{y} \tag{5.57}$$

The obtained system-reduced-observer structure is presented in Figure 5.2.

There are other ways of constructing the system observers (Luenberger, 1971; Chen, 1984). The reader particularly interested in observers is referred to a specialized book on observers for linear systems (O'Reilly, 1983).

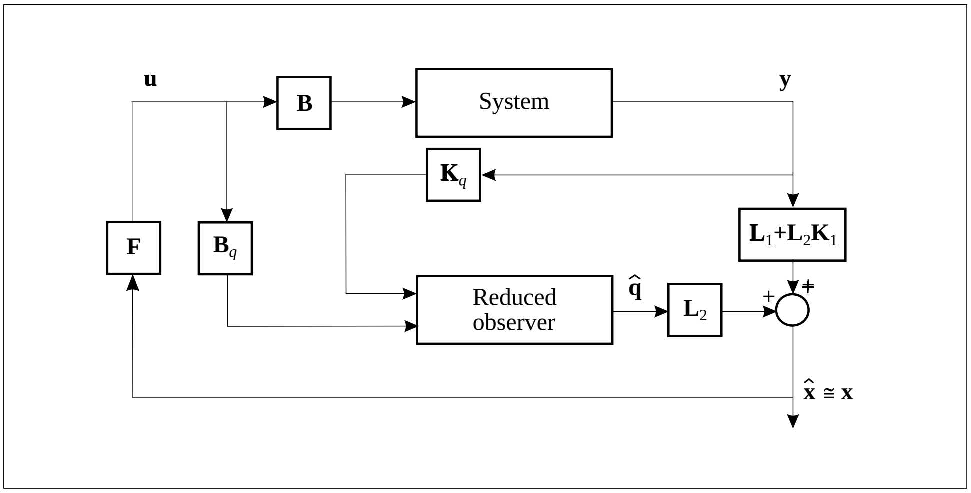

Figure 5.2: System-reduced-observer structure

5.7 MATLAB Case Study: F-8 Aircraft

In the case of high-order systems $(n > 3)$, obtaining the controllability and observability matrices is computationally very involved. The MATLAB package for computer-aided control system design and its CONTROL toolbox help to overcome this problem. Moreover, use of MATLAB enables a deeper understanding of controllability and observability concepts. Consider the following fourth-order model of an F-8 aircraft studied in Teneketzis and Sandell (1977), Khalil and Gajić (1984), Gajić and Shen (1993). The aircraft dynamics in continuous-time is described by the following matrices

$$\mathbf{A} = \begin{bmatrix} -0.0135700 & -32.2 & -46.300 & 0.0000 \\ 0.0001200 & 0.0 & 1.214 & 0.0000 \\ -0.0001212 & 0.0 & -1.214 & 1.0000 \\ 0.0005700 & 0.0 & -9.010 & -0.6696 \end{bmatrix}$$

$$\mathbf{B} = [-0.433 \quad 0.1394 \quad -0.1394 \quad -0.1577]^T$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

By using the MATLAB function `ctrb` (for calculation of the controllability matrix $\mathcal{C}$) and `obsv` (for calculation of the observability matrix $\mathcal{O}$), it can be verified that this system is both controllable and observable, namely

$$\mathrm{rank}\mathcal{C}^{4\times 4} = 4, \quad \mathrm{rank}\mathcal{O}^{8\times 4} = 4$$

By using the MATLAB function `det` (to calculate a matrix determinant), we get

$$\det \mathcal{C} = -6.8690$$

Since $det\mathcal{C}$ is far from zero it seems that this system is well controllable (the controllability margin is big).

If we discretize the continuous-time matrices $\mathbf{A}$ and $\mathbf{B}$ by using the sampling period $\Delta T = 0.1$, we get a somewhat surprising result. Namely

$$\det \mathcal{C}(\mathbf{A}_d, \mathbf{B}_d)|_{\Delta t=0.1} = -4.5381 \times 10^{-10}$$

Thus, this discrete system is almost uncontrollable. Theoretically, it is still controllable but we need an enormous amount of energy in order to control it. For example, let the initial condition be $\mathbf{x}(0) = [1 \quad 1 \quad 1 \quad 1]$ and let the final state $\mathbf{x}(4)$ be the coordinate origin. Then, by (5.18), the control sequence that solves the problem of transferring the system from $\mathbf{x}(0)$ to $\mathbf{x}(4) = \mathbf{0}$, obtained by using MATLAB is

$$\begin{bmatrix} u(1) \\ u(2) \\ u(3) \\ u(4) \end{bmatrix} = 10^7 \times \begin{bmatrix} 3.1682 \\ -9.7464 \\ 9.9944 \\ 3.4163 \end{bmatrix}$$

Apparently, this result is unacceptable and this discrete system is practically uncontrollable.

Note that the eigenvalues of the continuous-time controllability Grammian (5.31), obtained by using the MATLAB function `gram`, have values $7.48 \times 10^4,\ 0.42,\ 0.037,\ 0.0068$. *The eigenvalues of the controllability Grammian are the best indicators of the controllability measure*. Since two of them are very close to zero, the original system is very badly conditioned from the controllability point of view even though $\det \mathcal{C}(\mathbf{A}, \mathbf{B})$ is far from zero. The interested reader can find more about controllability and observability measures in a very comprehensive paper by Muller and Weber (1972).

5.8 MATLAB Laboratory Experiment

Part 1. The controllability staircase form of the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

clearly distinguishes controllable and uncontrollable parts of a control system. It can be obtained by the similarity transformation, and is defined by

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_c \\ \dot{\mathbf{x}}_{nc} \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_c & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{nc} \end{bmatrix} \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_{nc} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_c \\ \mathbf{0} \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= [\mathbf{C}_c \; \mathbf{C}_{nc}] \begin{bmatrix} \mathbf{x}_c \\ \mathbf{x}_{nc} \end{bmatrix} \end{aligned} \tag{5.58}$$

where $\mathbf{x}_c$ are controllable modes, and $\mathbf{x}_{nc}$ are uncontrollable modes. Apparently, in this structure the input $\mathbf{u}$ cannot influence the state variables $\mathbf{x}_{nc}$; hence these are uncontrollable. Similarly, one can define the observability staircase form as

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_o \\ \dot{\mathbf{x}}_{no} \end{bmatrix} &= \begin{bmatrix} \mathbf{A}_o & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{no} \end{bmatrix} \begin{bmatrix} \mathbf{x}_o \\ \mathbf{x}_{no} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_o \\ \mathbf{B}_{no} \end{bmatrix} \mathbf{u} \\ \mathbf{y} &= [\mathbf{C}_o \; \mathbf{0}] \begin{bmatrix} \mathbf{x}_o \\ \mathbf{x}_{no} \end{bmatrix} \end{aligned} \tag{5.59}$$

with $\mathbf{x}_o$ observable and $\mathbf{x}_{no}$ unobservable. Due to the fact that only $\mathbf{x}_o$ appears in the output and that $\mathbf{x}_o$ and $\mathbf{x}_{no}$ are not coupled through the state equations, the state variables $\mathbf{x}_{no}$ cannot be observed.

Use the MATLAB functions `ctrbf` (controllable staircase form) and `obsvf` (observable staircase form) to get the corresponding forms for the system

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ -1 & -2 & -3 & -4 & -5 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & -1 & 0 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 1 & -1 \\ 0 & 0 \\ 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$\mathbf{C} = [-1 \quad 1 \quad -1 \quad 0 \quad -2 \quad 2]$$

Identify the corresponding similarity transformation.

Part 2. Derive analytically that the transfer function of (5.58) is given in terms of the controllable parts, i.e. it is equal to

$$\mathbf{H}_c(s) = \mathbf{C}_c(s\mathbf{I} - \mathbf{A}_c)^{-1}\mathbf{B}_c = \mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{5.60}$$

Clarify your answer by using the MATLAB function for the transfer function `ss2zp`, i.e. show that both transfer functions have the same gains, poles, and zeros (subject to zero-pole cancellation).

Do the same for the observable staircase form, i.e. show that

$$\mathbf{H}_o(s) = \mathbf{C}_o(s\mathbf{I} - \mathbf{A}_o)^{-1}\mathbf{B}_o = \mathbf{H}(s) \tag{5.61}$$

and justify this identity by using the MATLAB function `ss2zp`.

Part 3. Examine the controllability and observability of the power system composed of two interconnected areas considered in Geromel and Peres (1985) and Shen and Gajić (1990)

$$\mathbf{A} = \begin{bmatrix} 0 & 0.55 & 0 & 0 & 0 & -5.5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.3 & -0.05 & 6 & 0 & 3.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.3 & 3.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5.2 & 0 & -13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3.3 & 0 & 0 & 0 & -3.3 & -0.05 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.3 & 3.3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -5.2 & 0 & -13 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 13 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 \end{bmatrix}^T$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0.43 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0.43 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Part 4. Follow the steps used in Section 5.7, but this time for the F-15 aircraft, whose state space model was presented in Example 1.4. Consider both the subsonic and supersonic flight conditions. Comment on the results obtained.

Part 5. The controllability Grammian is defined in (5.31) as

$$\mathbf{W}(t_0, t_1) = \int_{t_0}^{t_1} e^{\mathbf{A}(t_0-\tau)}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T(t_0-\tau)} d\tau$$

(a) Show analytically that the control input given by

$$\mathbf{u}(t) = -\mathbf{B}^T e^{\mathbf{A}^T(t_0-t)}\mathbf{W}^{-1}(t_0, t_1)\Big[\mathbf{x}(t_0) - e^{\mathbf{A}(t_0-t_1)}\mathbf{x}(t_1)\Big] \tag{5.62}$$

will drive any initial state $\mathbf{x}(t_0)$ into any desired final state $\mathbf{x}(t_1) = \mathbf{x}_f$. Note that under the controllability assumption many control inputs can be found to transfer the system from the initial to the final state. The expression given in (5.62) is also known as the minimum energy control (Klamka, 1991) since in addition to driving the system from $\mathbf{x}(t_0)$ to $\mathbf{x}(t_1) = \mathbf{x}_f$, it also minimizes an integral of the square of the input (energy), $\mathbf{u}^T(t)\mathbf{u}(t)$, in the time interval (t_0, t_1).

(b) Using the MATLAB function `gram`, find the controllability Grammian for the system defined in Part 4 for $t_0 = 0$ and $t_1 = 1$. One of several known controllability tests states that the *system is controllable if and only if its controllability Grammian is positive definite* (Chen, 1984; Klamka, 1991). Verify whether or not the controllability Grammian for this problem is positive definite.

(c) Find the control input $\mathbf{u}(t)$ that drives the system defined in Part 4 from the initial condition $\mathbf{x}(0) = [0 \quad 0 \quad 0 \quad 0 \quad 0]^T$ to the final state $\mathbf{x}(1) = [1 \quad 1 \quad 1 \quad 1 \quad 1]^T$.

Part 6. By duality to the controllability Grammian, the observability Grammian is defined as

$$\mathbf{V}(t_0, t_1) = \int_{t_0}^{t_1} e^{-\mathbf{A}^T(t_0-\tau)}\mathbf{C}^T\mathbf{C}e^{-\mathbf{A}(t_0-\tau)} d\tau \tag{5.63}$$

Note that the observability Grammian is in general a positive semidefinite matrix. It is known in the literature on observability that *if and only if the observability Grammian is positive definite, the system is observable* (Chen, 1984). Check the observability of the system given in Part 3 by using the observability Grammian test.

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5.10 Problems

5.1 Test the controllability and observability of the following systems

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 & -1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

5.2 Find the values for parameters b_1, b_2, and b_3 such that the given system is controllable

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ b_3 & 0 \end{bmatrix}$$

5.3 Find the values for parameters c_1 and c_2 such that the following system is observable

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$$

If the output vector of the corresponding discrete system is given by $[y(0) \;\; y(1)] = [1 \;\; 2]$, find the system's initial condition.

5.4 Verify that all columns of the matrix

$$\mathbf{A}^3 = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}^3$$

can be expressed as a linear combination of the columns forming matrices $\mathbf{I}$, $\mathbf{A}$, and $\mathbf{A}^2$ (see 5.13).

5.5 Assuming that the desired final state of a discrete system represented by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 1 \\ -1 & 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

is $\mathbf{x}(3) = [0 \quad -1 \quad 1]^T$ find the control sequence that transfers the system from $\mathbf{x}(0)$ to $\mathbf{x}(3)$.

5.6 Find a solution to Problem 5.5 in the case of a two-input system that has the input matrix

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 0 & 2 \end{bmatrix}$$

The remaining elements are the same as in Problem 5.5.

5.7 Determine conditions on b_1, b_2, c_1, and c_2 such that the following system is both controllable and observable

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{C} = [c_1 \quad c_2]$$

Assume that the input to this system is known. Find the initial conditions of this system in terms of the given input in the case when the measured output is $y(t) = e^{-t}\cos(t)$.

5.8 Using the frequency domain criterion, check the joint controllability and observability of the system

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 1 \quad 0]$$

5.9 Find the initial conditions of all integrators in an analog computer simulation of the following differential equation

$$\frac{d^3y}{dt^3} + 3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = \frac{d^2u}{dt^2} + \frac{du}{dt} + u$$

$$y(0) = \frac{dy(0)}{dt} = \frac{d^2y(0)}{dt^2} = 1$$

5.10 The transfer function of a system given by

$$\frac{C(s)}{R(s)} = \frac{10(s+1)(s+3)(s+5)}{s(s+2)(s+4)(s+5)}$$

indicates that this system is either uncontrollable or unobservable. Check by the rank test, after a zero-pole cancellation takes place, that the remaining system is both controllable and observable.

5.11 A discrete model of a steam power system was considered in Mahmoud (1982) and Gajić and Shen (1993).

(a) Using MATLAB, examine the controllability and observability of this system, represented by

$$\mathbf{A}_d = \begin{bmatrix} 0.915 & 0.051 & 0.038 & 0.015 & 0.038 \\ -0.030 & 0.889 & -0.001 & 0.046 & 0.111 \\ -0.006 & 0.648 & 0.247 & 0.014 & 0.048 \\ -0.715 & -0.022 & -0.021 & 0.240 & -0.024 \\ -0.148 & -0.003 & -0.004 & 0.090 & 0.026 \end{bmatrix}$$

$$\mathbf{B}_d = [0.010 \quad 0.122 \quad 0.036 \quad 0.562 \quad 0.115]^T$$

$$\mathbf{C}_d = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(b) Find the system transfer function and justify the answer obtained in (a).

5.12 Using MATLAB, examine the controllability of the magnetic tape control system considered in Chow and Kokotović (1976)

$$\mathbf{A} = \begin{bmatrix} 0 & 0.40 & 0.000 & 0.00 \\ 0 & 0.00 & 0.349 & 0.00 \\ 0 & -5.24 & -4.65 & 2.62 \\ 0 & 0.00 & 0.000 & -10.00 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 10 \end{bmatrix}$$

5.13 Find the controllability matrix of the system in the phase variable canonical form and show that its rank is always equal to n.

5.14 Linearize the given system at the nominal point $(x_{1n}, x_{2n}, u_n) = (1, 0, k)$ and examine system controllability and observability in terms of k

$$\dot{x}_1 = x_1^2 u, \qquad x_1(0) = 1$$
$$\dot{x}_2 = x_1 x_2 + u, \qquad x_2(0) = 0$$
$$y = x_1$$

5.15 Find the state space form of a system given by

$$\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + k\frac{dy}{dt} + 2y = \frac{du}{dt} + u$$

and examine system controllability and observability in terms of k. Do they depend on the choice of the state space form?

5.16 Given a linear system described by

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \frac{du}{dt} + u, \quad y(0) = 0, \ \frac{dy(0)}{dt} = 2$$

Transfer this differential equation into a state space form and determine the initial conditions for the state space variables. Can you solve this problem by using an unobservable state space form? Justify your answer.

5.17 Check that the matrix **A** given in (4.36) and the matrix $\mathbf{C} = \sqrt{\mathbf{Q}}$ defined in (4.39) form an observable pair.

5.18 Derive formulas (5.55) and (5.56) for the reduced-order observer design.

5.19 Using MATLAB, examine the controllability of a fifth-order distillation column considered in Petkov *et al.* (1986)

$$\mathbf{A} = \begin{bmatrix} -0.194 & 0.0628 & 0 & 0 & 0 \\ 1.306 & -2.132 & 0.9807 & 0 & 0 \\ 0 & 1.595 & -3.149 & 1.547 & 0 \\ 0 & 0.0355 & 2.632 & -4.257 & 1.855 \\ 0 & 0.00227 & 0 & 0.1636 & -0.1625 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0.0632 & 0.0838 & 0.1004 & 0.0063 \\ 0 & 0 & -0.1396 & -0.2060 & -0.0128 \end{bmatrix}^T$$

5.20 Examine both the controllability and observability of the robotic manipulator acrobot whose state space matrices are given in Problem 3.2.

5.21 Repeat problem 5.20 for the industrial reactor defined in Problem 3.26.

5.22 Consider the state space model of the flexible beam given in Example 3.2. Find the system transfer function and determine its poles and zeros. Use Theorem 5.7 to check the controllability and observability of this linear control system.

5.23 The system matrix for a linearized model of the inverted pendulum studied in Section 1.6 is given in Section 4.2.3. Using the same data as in Section 4.2.3, the input matrix is obtained as

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & 0 & -2 \end{bmatrix}^T$$

Examine the controllability of this inverted pendulum.

5.24 A system matrix of a discrete-time model of an underwater vehicle is given in Problem 4.22. Its input matrix is given by Longhi and Zulli (1995)

$$\mathbf{B} = \begin{bmatrix} 0.0258 & -0.0002 & 0 \\ 0.1023 & -0.0010 & 0 \\ -0.0001 & 0.0258 & 0 \\ -0.0008 & 0.1023 & 0 \\ 0 & 0 & 0.0055 \\ 0 & 0 & 0.0221 \end{bmatrix}$$

Check the controllability of this system.