CHAPTER 2

General Laws for Propagation of Shock Waves through Matter

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2.1 INTRODUCTION

The most fundamental information currently known about the propagation of shock waves through any material substance will be presented. It is shown that the equation of state (EOS) has a decisive effect on the shock phenomena that can exist in the material. Five laws or theorems will be presented.

The most important of them is the *Rankine-Hugoniot (R-H) equations*, which are derived from the conservation laws of mass, momentum, and energy. These equations include terms containing the velocities of the material on both sides of the shock wave. If the equations are manipulated to eliminate these terms, then a single equation containing only thermodynamic variables of state is obtained. This is the Hugoniot equation, and it is the starting point for many investigations. If two other equations, called the Raleigh equations, both of which are also obtained by manipulation, are appended to the Hugoniot equation, then they comprise an equivalent set to the R-H equations. The theory is presented in Sections 2.4 and 2.5.

Next in importance is the *Bethe-Weyl (B-W) theorem*, which can be applied to either a normal or an oblique shock wave. It is valid for all materials and guarantees the existence of at least one solution to the Hugoniot equation. However, the real power of the theorem is displayed when the material obeys a *convex* equation of state (EOS), $G > 0$, (Sections 2.6 and 2.8) and also when the EOS satisfies the *weak* constraint, $\Gamma \leq 2\gamma$ (Section 2.9). Nearly all materials in a single phase satisfy these constraints for practically all-thermodynamic states. In these circumstances the theorem shows that a solution exists and is unique. It also shows that a compressive shock increases the entropy in the material, and that the shock wave propagates at a supersonic velocity relative to the material ahead of it and at a subsonic velocity relative to the material behind it. Conversely, for the rather rare event when $G < 0$, then it is *an expansive shock* that increases the entropy in the material. For these waves the shock propagates at subsonic velocity relative to the material ahead of it and at a supersonic velocity relative to the material behind it. Numerous experiments have confirmed the existence of these extraordinary waves. The theory is presented in Section 2.10.

Following from the B-W theorem is the *triple-shock-entropy theorem*. It is applicable to one- and two-dimensional shock wave interactions (1D and 2D). Some general properties of interactions can be obtained from it when the material EOS satisfies the previous constraints $G > 0$ and $\Gamma \leq 2\gamma$. For example, if two shock waves approach each other and are parallel (1-D interaction), then after they collide, the outgoing waves must be shock waves. However, if two such shock waves interact by one overtaking the other, then the outgoing waves can be either a pair of shock waves or a shock wave and an expansion wave. The theorem and its consequences are presented in Sections 2.11 and 2.12.

Next is *Crocco's theorem*, which is important when velocity, entropy, and/or total enthalpy gradients are present ahead of the shock wave. Furthermore, it is often useful if the shock wave is curved, especially if there is an entropy gradient along it. It shows how a shock wave can be a source of vorticity. This theorem, which is applicable to any material, is presented in Section 2.13.

Finally, the *refraction law* is presented in Section 2.14. It is purely kinematic and thus can be applied to any material. It is useful for both 2D and 3D interactions when several waves emanate from a single point called a wave node. It gives a powerful means for finding the condition where one wave pattern changes into another under a continuous change in one or more of the system parameters.

2.2 THE RIEMANN PROBLEM

"A Riemann problem is defined for a system of conservation laws such as mass, momentum and energy, as an initial value problem such that the initial data have no length or time scales, or in other words the data is constant along ray paths" (Courant and Friedrichs, 1948). The classic example is the shock tube problem studied by Riemann (1860). Many shock problems have this scale invariant character, but not all.

2.3 LENGTH AND TIME SCALES

One *length scale* that is always present is the thickness of the shock wave. The simplest example is that of a monatomic gas. Its shock wave thickness is about four mean free paths, that is, it takes about four molecular collisions to adjust the equilibrium state upstream of the shock to downstream of it. The molecular processes inside the shock wave are not in equilibrium. A shock wave is thicker in polyatomic gases because molecular rotation and vibration require more collisions for equilibrium to be attained. For weak shock waves in the atmosphere the thickness may be of the order of one 1 km because of the large number of collisions required to attain vibrational equilibrium in nitrogen, especially when moisture is present (Johannesen and Hodgson, 1979). The shock wave thickness is also increased by chemical reactions as with detonations (Fickett and Davis, 1979), and by dissociation and ionization. More generally, the velocity and the thermal gradients inside the shock wave imply the importance of the material transport properties—particularly viscosity and heat conductivity (Zeldovich and Raizer; 1966; Thompson, 1972,

p. 363). If the shock wave thickness length scale is too small to be of significance to the problem then it is sufficient to consider only the equilibrium states on both sides of the shock. One then has a shock Riemann problem.

Time scales are often important, but only two occurrences will be mentioned here. First, a time scale is present if the shock wave becomes unstable and splits into two waves moving in the same direction (Section 2.10). Suppose that an intense shock wave propagates into a metal, which is initially at atmospheric pressure and temperature, and suppose it also compresses the metal beyond its yield point. It is known that eventually the shock wave will split into two waves. The first is a precursor shock wave that compresses the metal to its yield point, and the second is a compressive plastic wave (Zeldovich and Raizer, 1966). Second, a shock wave may induce a change in phase of the material. A well-known example is the $\alpha \rightarrow \varepsilon$ (body-centered-cubic to hexagonal-close-packed) phase transformation in iron that takes place at 12.8 Gpa, which can also cause splitting (Duvall and Graham, 1977). However, in many cases the time to attain equilibrium is orders of magnitude

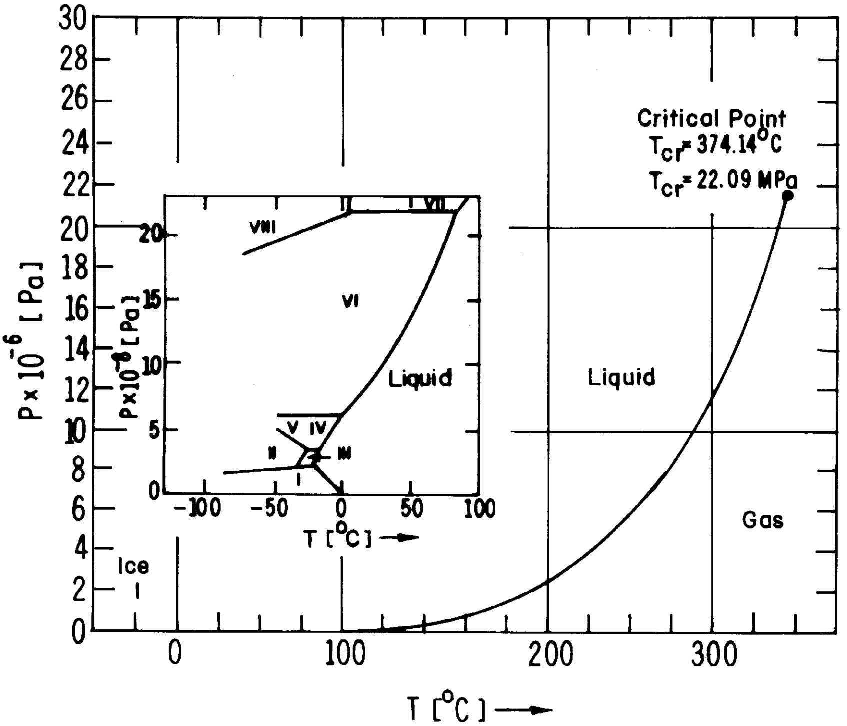

FIGURE 2.1 Phase diagram for water substance.

greater than the time for the shock wave to pass through the material. In this case there will be no phase change, and any equilibrium can only be metastable; there will then be no time scale. For example, if a shock wave compressed water at atmospheric pressure to a pressure $P > 10^4$ atm (1000 MPa), and if thermodynamic equilibrium were attained, then ice (VII) would exist downstream of the shock wave (see Fig. 2.1). However, this does not usually happen because of the long time required for attaining equilibrium (Bethe, 1942). Instead, the water remains in the liquid phase but in metastable equilibrium. The time scale for thermodynamic equilibrium reduces rapidly, however, if the compressed state approaches a *spinodal* condition (Section 2.7).

2.4 THE CONSERVATION LAWS FOR A SINGLE SHOCK

2.4.1 Laboratory Frame Coordinates

Suppose the material is contained in a cylinder of cross-sectional area A. One end of the cylinder is open, but the other is closed by a piston that is in contact with the material. Initially, Fig. 2.2a shows that the system is at rest. Suppose that at time $t = 0$, the piston impulsively acquires the finite velocity U_p in the x-direction. Then it instantly begins to drive the material to the right at the same velocity U_p. This is accomplished by a shock wave s that instantly appears on the face of the piston and propagates into the material with the finite velocity $U_s > U_p$ (see Fig. 2.2b). As U_s is finite (it must be less than the velocity of light!), the material to the left of the shock wave moves at the velocity U_p, but the material to the right of it remains at rest. The equations from the conservation laws for mass, momentum, and energy can now be derived. It is assumed for simplicity that the system has adiabatic walls, that body forces such as gravity and electromagnetism are negligible and that there is no heat transfer by radiation across the shock.

Conservation of Mass

After unit time the piston has moved a distance U_p and the shock a distance U_s. During that time the shock compresses a mass of the material from its initial volume AU_s to $A(U_s - U_p)$. The density therefore *increases from ρ_0 to ρ, so by conservation of mass*

$$\rho_0 U_s = \rho(U_s - U_p) = \dot{m} \tag{2.1}$$

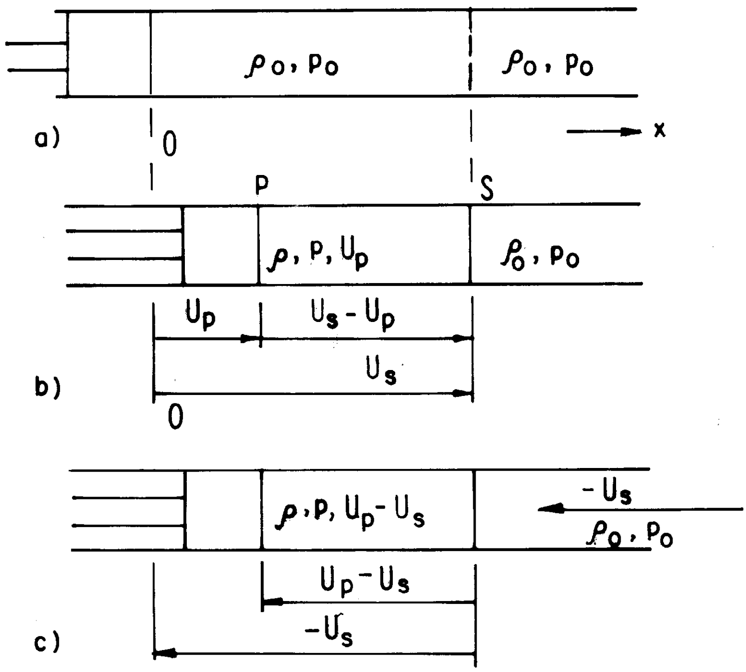

FIGURE 2.2 Shock wave generated by the impulsive motion of a piston. a) Initial state at rest; b) state in unit time after the piston had acquired velocity U_p impulsively; and c) motion in shock fixed coordinates (p is the piston and s is the shock wave).

where $\dot{m}$ is the mass flux of material passing through the shock wave. Notice that it is strictly true that $U_s > U_p$, for if $U_s = U_p$ then $\rho = \infty$, which is physically impossible with the current state of knowledge.

Conservation of Momentum

The piston applies a driving force $(P - P_0)A$ to the material, causing it to acquire a momentum per unit time of $(\rho_0 U_s A)U_p = \dot{m}AU_p$. Then from conservation of momentum

$$P - P_0 = \rho_0 U_s U_p \tag{2.2}$$

Conservation of Energy

The compressive work that the piston does on the material in unit time is PAU_p. The energy gained by the material in unit time is the sum of the kinetic

$\frac{1}{2}(\rho_0 U_s A)U_p^2$ and the internal energy $(\rho_0 U_s A)(e - e_0)$. Thus by conservation of energy

$$PU_p = \rho_0 U_s \left(\frac{1}{2}U_p^2 + e - e_0\right) \tag{2.3}$$

The preceding equations are the conservation laws for a single shock wave. They are of fundamental importance.

2.4.2 Shock Fixed Coordinates

It is often convenient to transform the conservation laws into a coordinate system that is at rest with respect to the shock. This is easily accomplished by subtracting the shock wave velocity U_s from the (zero) particle velocity ahead of the shock wave and also from the particle velocity U_p behind it. Then

$$u_0 = -U_s \tag{2.4.1}$$

and

$$u = U_p - U_s \tag{2.4.2}$$

where u_0 and u are the particle (material) velocities ahead of and behind the shock, respectively, and relative to it. The last of these equations can be written as

$$U_p = u - u_0 \tag{2.5}$$

Substituting Eq. (4) into Eqs. (2.1)–(2.3) we acquire, after some algebra, the conservation laws in shock fixed coordinates. They are also called the Rankine-Hugoniot equations.

$$\rho u = \rho_0 u_0 \tag{2.6}$$

$$p + \rho u^2 = p_0 + \rho_0 u_0^2 \tag{2.7}$$

$$\frac{p}{\rho} + e + \frac{1}{2}u^2 = \frac{p_0}{\rho_0} + e_0 + \frac{1}{2}u_0^2 \tag{2.8}$$

Equation (2.8) can also be written in terms of the enthalpy, $h\ p/p + e$, as:

$$h + \frac{1}{2}u^2 = h_0 + \frac{1}{2}u_0^2 = h_t \tag{2.9}$$

which is called *Bernoulli's* equation. Here, h_t is the total enthalpy.

2.5 THE HUGONIOT ADIABATIC

2.5.1 The Hugoniot Equation

If velocities U_s and U_p are eliminated from Eqs. (2.1)–(2.3) then the conservation laws reduce to a single equation, which is a function only of the variables of state. It is called the *Hugoniot* equation, and it is fundamental to shock wave theory.

$$e - e_0 = \frac{1}{2}(p + p_0)(v_0 - v) \tag{2.10}$$

Notice that a neater form of it is obtained if the densities ρ_0 and ρ are replaced by the specific volumes v_0 and v, respectively, where $v = 1/\rho$. In order to plot the Hugoniot curve in the (v, p)-plane it is necessary to know the initial state (v_0, p_0) of the material and its equation of state (EOS), or its equivalent such as a table of state properties.

2.5.2 The Raleigh Equations

If u_0 or u is eliminated from Eqs. (2.6) and (2.7), we obtain the Rayleigh equations

$$\rho_0^2 U_s^2 = \rho_0^2 u_0^2 = \rho^2 u^2 = -\frac{\Delta p}{\Delta v} \tag{2.11}$$

where $\Delta p = p - p_0$ and $\Delta v = v - v_0$. If the pressure jump across a shock becomes vanishingly small, that is $p \to p_0$, then $v \to v_0$ and $u \to u_0$, and one also finds that the specific entropy is $s \to s_0$ [Eq. (2.35) in Section 2.8.1]; then the limit equation (2.11) becomes

$$\rho_0^2 a_0^2 = -\left[\frac{\partial p}{\partial v}\right]_S \tag{2.12.1}$$

$$a_0^2 = -v_0^2\left[\frac{\partial p}{\partial v}\right]_S \tag{2.12.2}$$

where a_0 is the speed of sound in the undisturbed material. It follows that $-U_s = u_0 \to a_0$, so that in the limit the shock wave propagates at the speed of sound, or in other words it is reduced to an acoustic wave. Note that $-v_0[\partial p/\partial v]_s$, is the bulk modulus and that a_0 is called the *longitudinal* sound speed in solid mechanics and is appropriate for an unconstrained material. The sound speed in a thin bar is somewhat smaller (Kolsky, 1953).

Returning to Eqs. (2.1) and (2.2), and replacing ρ_0 and ρ by v_0 and v and then eliminating U_s the result, with the help of Eq. (2.5), is

$$\frac{1}{2}U_p^2 = (u - u_0)^2 = \frac{1}{2}(p - p_0)(v + v_0) \tag{2.13}$$

which in laboratory frame coordinates is the gain in the kinetic energy per unit mass of the material by the passage of the shock wave. In shock fixed coordinates there is a *loss* of kinetic energy across the shock wave, because for a compression $v < v_0$, and by Eq. (6) $u < u_0$, and so $\frac{1}{2}u^2 < \frac{1}{2}u_0^2$. From Eqs. (2.8) and (2.10) we get

$$\frac{1}{2}(u_0^2 - u^2) = (p - p_0)(v + v_0) \tag{2.14}$$

2.5.3 Solution of a Simple Shock Riemann Problem

The problem is illustrated in Fig. 2.2. Suppose the initial state (v_0, p_0) upstream of the shock is given, and also the downstream pressure p. It is required to find the compressed specific volume v and thus the downstream state (v, p). The problem can be solved in the (v, p)-plane when the EOS of the material is known $p = p(e, v)$. The Hugoniot curve can be plotted by using Eq. (2.10) and the EOS, and v can then be found because p is given (see Fig. 2.3). The slope of

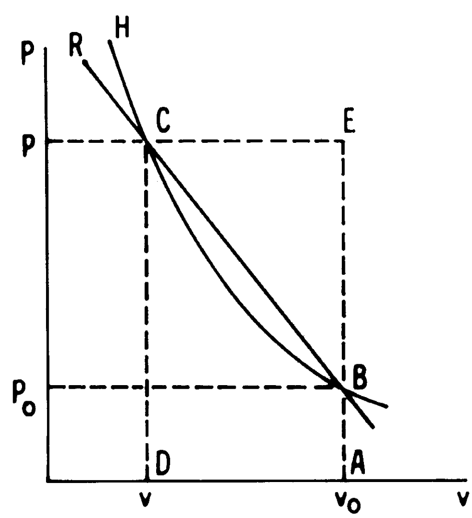

FIGURE 2.3 Hugoniot curve H and Raleigh line R in the (v, p)-plane.

the Raleigh line $\Delta p/\Delta v$ can now be calculated and it is a constant; this means that the Raleigh line is straight in this plane. From Eq. (11), $U_s^2/v_0^2 = \Delta p/\Delta v$, from which we find U_s. By Eq. (2.10) the gain in the internal energy is represented by the trapezium ABCDA, while by Eq. (2.13) the gain in the kinetic energy per unit mass (laboratory frame) is represented by the triangle BECB. By Eq. (2.3) the total gain in energy per unit mass is represented by the rectangle AECDA.

2.6 THERMODYNAMIC PROPERTIES OF MATERIALS

It is important to notice that the conservation laws, the Hugoniot, and the Raleigh equations are independent of any equation of state. Consequently, these laws and equations can be applied to any material. Nevertheless, the EOS has a decisive effect on the nature of the shock phenomena that appears in it. However, before these effects can be discussed it is necessary to define the thermodynamic properties that will be needed.

The fundamental equation

$$e = e(v, s) \tag{2.15}$$

contains all the thermodynamic information about the system (Callen, 1985). If by definition

$$T = \left[\frac{\partial e}{\partial s}\right]_V \tag{2.16.1}$$

and

$$-p = \left[\frac{\partial e}{\partial v}\right]_S \tag{2.16.2}$$

then by using Eqs. (2.16.1) and (2.16.2), in differential form, Eq. (2.15) becomes

$$de = Tds - pdv \tag{2.17}$$

where T is the temperature. Equations (2.16.1) and (2.16.2) are the thermal and mechanical EOS, respectively, and they can also be written

$$T = T(v, s) \text{ and } p = p(v, s) \tag{2.18.1}$$

It is often useful to define the EOS in terms of (v, e) rather than in (v, s). By Eqs. (2.16.1) and (2.16.2) this is always possible because e is a monotonically increasing function of s, as $T > 0$ and $T = 0$ is unattainable, and so,

$$T = T(v, e) \text{ and } p = p(v, e) \tag{2.18.2}$$

The specific heats

$$C_V = T\left[\frac{\partial s}{\partial T}\right]_V \text{ and } C_P = T\left[\frac{\partial s}{\partial T}\right]_P \tag{2.19}$$

The compressibilities

$$K_S = -\frac{1}{v}\left[\frac{\partial v}{\partial p}\right]_S \text{ and } K_T = -\frac{1}{v}\left[\frac{\partial v}{\partial p}\right]_T \tag{2.20}$$

The coefficient of thermal expansion

$$\beta = \frac{1}{v}\left[\frac{\partial v}{\partial T}\right]_P \tag{2.21}$$

Because of the thermodynamic relation

$$\frac{K_S}{K_T} = 1 - \frac{\beta^2 vT}{C_P K_T} = \frac{C_V}{C_P} \tag{2.22}$$

only three of these five properties are independent. It is conventional to choose these to be C_p, K_T, and β, as tables of them exist for many materials.

In what follows, some of the properties obtained from the second and third derivatives of the energy are of special importance.

The adiabatic exponent

$$\gamma = \frac{v}{p}\left[\frac{\partial^2 e}{\partial y^2}\right]_S = \frac{1}{pK_S} = \frac{a^2}{pv} = -\frac{v}{p}\left[\frac{\partial p}{\partial v}\right]_S \tag{2.23}$$

where a is the speed of sound. For an ideal gas, γ reduces to the ratio of the specific heats, $\gamma = C_P/C_V$. Notice that γ can often be found from Eq. (2.23) because $\gamma = a^2/pv$. Some values of a, p, and $\rho = 1/v$ are given in Table 2.1.

The Grüneisen coefficient

$$\Gamma = -\frac{v}{T}\frac{\partial^2 e}{\partial v \partial s} = -\frac{v}{T}\left[\frac{\partial T}{\partial v}\right]_S = \frac{v}{C_V}\left[\frac{\partial p}{\partial T}\right]_V \tag{2.24}$$

This can be written to show that Γ determines the spacing of the isentropic curves in both of the (ln v, ln p) and (v, p)-planes

$$\Gamma = v\left[\frac{\partial p}{\partial v}\right]_e = \frac{v}{T}\left[\frac{\partial p}{\partial s}\right]_V = \frac{pv}{T}\left[\frac{\partial \ln \mathrm{p}}{\partial s}\right]_{\ln} v \tag{2.25}$$

TABLE 2.1 Some Values of Shock and Thermodynamic Properties

Material[a]	ρ [kg/m^3]	C_p [kJ/kgK]	a [km/s]	Γ	S_S
Water	1000	4.19	1.51	0.1	1.92
NaCl[a]	2160	0.87	3.53	1.6	1.34
KCl[b]	1990	0.68	2.15	1.3	1.54
LiF	2640	1.50	5.15	2.0	1.35
Teflon	2150	1.02	1.84	0.6	1.71
PMMA	1190	1.20	2.60	1.0	1.52
Polyethylene	920	2.30	2.90	1.6	1.48
Polystyrene	1040	1.20	2.75	1.2	1.32
Brass	8450	0.38	3.73	2.0	1.43
Al-2024	2790	0.89	5.33	2.0	1.34
Be	1850	0.18	8.00	1.2	1.12
Ca	1550	0.66	3.60	1.1	0.95
Cu	8930	0.40	3.94	2.0	1.49
Fe[b]	7850	0.45	3.57	1.8	1.92
Pb	11350	0.13	2.05	2.8	1.46
U	18950	0.12	2.49	2.1	2.20

[a] Superscripts a and b refer to above and below phase transitions.

It follows at once that the isentropics cannot cross each in these planes when $\Gamma > 0$. By further manipulation and also by using Eq. (2.21) the following, useful relation between Γ and β is obtained:

$$\Gamma = \frac{v\beta}{C_V K_T} \tag{2.26}$$

For a thermodynamically stable system $C_V > 0$and $K_T > 0$ (see Menikoff and Plohr, 1989), and because v > 0, it follows that Γ and β always have the same sign. When Γ is a constant then Eq. (2.26) becomes the famous Grüneisen EOS.

For most materials, in most states, Γ and β are positive. For the alloy Invar, they are almost zero at room temperature, but for water $< 3.984\,^\circ$C and at 1 atm, both Γ and β are negative. There are also many tetrahedrally bonded materials for which these quantities are negative for some domains of state (Table 2.2). For an ideal gas, $\Gamma = \gamma - 1 > 0$. Some values of Γ are presented in Table 2.1.

The reciprocal of the dimensionless specific heat

$$g = \frac{pv}{T}\left[\frac{\partial^2 e}{\partial s^2}\right]_V = \frac{pv}{C_V T} \tag{2.27}$$

TABLE 2.2 Temperature domains of some materials that have a negative β and Γ at a pressure of 1 atm[a]

Material	Temperature domain
Water	<3.384°C
Diamond	<90K
Vitreous silica	<289K
ZnSe	<64K
CdTe	<72K
Ice I	<63K
GaAs	<55K
Ge	<48K
InSb	<55K
α-Sn	<45K

[a] After Collins and White, 1964.

This quantity is strictly positive $g > 0$ for a system, which is thermodynamically stable (Section 2.7).

The fundamental derivative

$$G = \frac{1}{2} v \frac{\left[\frac{\partial^2 p}{\partial v^2}\right]_S}{\left[\frac{\partial p}{\partial v}\right]_S} = \frac{pv}{C_V T} \tag{2.28.1}$$

by Eq. (2.23), with G a third derivative of e. Here it is given in nondimensional form f (Thompson, 1971); $G > 0$ iff

$$\left[\frac{\partial^2 p}{\partial v^2}\right]_S > 0 \tag{2.28.2}$$

Equation (2.28.1) measures geometrically the curvature of the isentropics in the (v, p)-plane. If Equation (2.28.2) is strictly true then any particular isentropic is *convex*, which means that it always lays above a tangent to any point on it (see Fig. 2.4). An isentropic is *straight* for any domain of states for which $G = 0$ and *concave* if $G < 0$. It will be shown in Figs. 2.8 and 2.10 that the sign of G has profound physical consequences.

DEFINITION. *An equation of state of a material is convex if its fundamental derivative is strictly positive* (Bethe 1942).

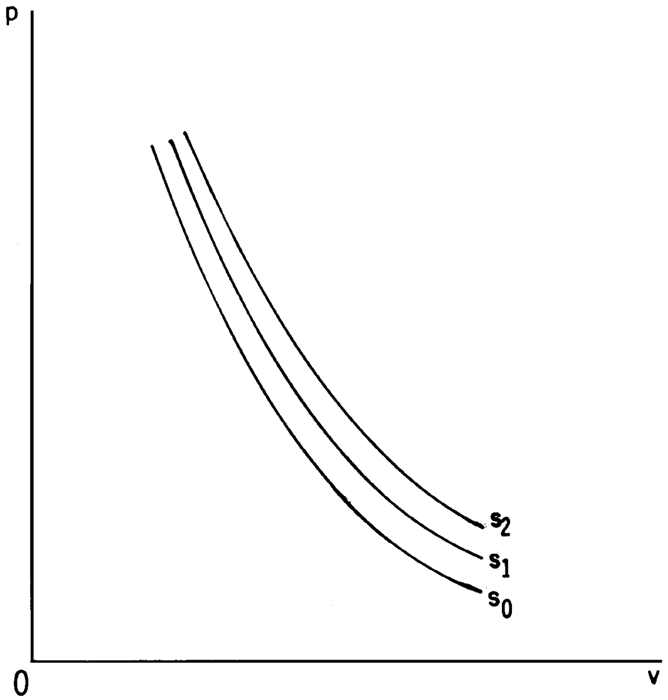

FIGURE 2.4 Convex isentropic curves always lie above a tangent and have a negative slope. $[\partial^2 p/\partial v^2]_S > 0$ and $[\partial p/\partial v]_S > 0$.

This definition is of great importance for investigating the existence of a given shock wave phenomenon in large classes of materials.

By differentiating Eq. (2.23) with respect to v, a relation between γ and G is obtained:

$$G = \frac{1}{2}\left[\gamma + 1 - \frac{v}{\gamma}\left[\frac{\partial \gamma}{\partial v}\right]_S\right] \tag{2.29}$$

or in terms of the density

$$G = 1 + \frac{\rho^2}{p\gamma}\left[\frac{\partial^2 p}{\partial \rho^2}\right]_S \tag{2.30}$$

Values of G for some liquids are presented in Table 2.3.

TABLE 2.3 Values of G for some liquids at 1 atm and 30°C[a]

Liquid	Values of G
Water	3.60
Acetone	6.0
1-Propanol	6.2
Mercury	4.94
Methanol	5.81
n-Propanol	6.36
Glycerine	6.1
Ethanol	6.28
n-Butanol	6.36

[a] After Thompson, 1971.

2.7 THERMODYNAMIC CONSTRAINTS ON THE EOS

The material EOS must satisfy the following constraints if the equilibrium states on both sides of the shock are to be thermodynamically stable (see Callen, Menikoff and Plohr ????)

$$\gamma \geq 0;\ g \geq 0;\ \gamma g - \Gamma^2 \geq 0 \tag{2.31}$$

If any of the equalities in Eq. (2.31) apply, the equilibrium will be only neutrally stable; instability can then be caused by vanishingly small fluctuations. Constraints that are equivalent to Eq. (2.31) are

$$\frac{1}{K_{\mathrm{S}}} \geq \frac{1}{K_{\mathrm{T}}} \geq 0, \text{ and } \frac{1}{C_{\mathrm{V}}} \geq \frac{1}{C_{\mathrm{P}}} \geq 0 \tag{2.32}$$

By Eqs. (2.19) and (2.20) these constraints imply that

$$\left[\frac{\partial p}{\partial y}\right]_{\mathrm{T}} < 0 \tag{2.33.1}$$

and

$$\frac{1}{T}\left[\frac{\partial T}{\partial s}\right]_{\mathrm{V}} > 0 \tag{2.33.2}$$

that is, if a thermodynamically stable system is compressed isothermally then its pressure will increase, which means that it is mechanically stable. If the system is heated at constant volume, then its temperature will increase, which means that it is thermally stable.

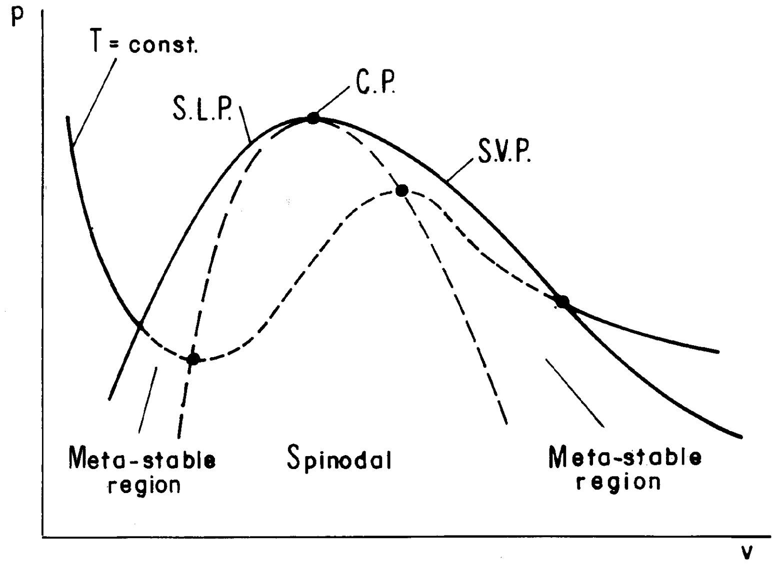

FIGURE 2.5 The spinodal (Wilson line) for a van der Waal's equation of state in the (v, p)-plane. Note the metastable regions SLP, saturated liquid line, SVP, saturated vapor line, CP, critical point.

For a fluid near the vapor-liquid phase transition it is often possible to produce metastable states, such that $[\partial p/\partial v]_T \to 0$. For example a saturated liquid may be carefully expanded isothermally to a lower pressure in such a way that the constraint $[\partial p/\partial v]_T < 0$ remains satisfied as $[\partial p/\partial v]_T \to 0$. The limit is called the *spinodal* or *Wilson line* (see Fig. 2.5) and is defined by

$$\left[\frac{\partial p}{\partial v}\right]_T = 0 \tag{2.34}$$

The super-expanded state is metastable, but if the limit is approached closely, it becomes inevitable that a fluctuation of sufficient magnitude will occur and cause Eq. (2.33.1) to be violated, after which explosive boiling will occur. This branch of the spinodal is the superheat limit. The other branch occurs on the vapor pressure side, and is also defined by Eq. (2.34); in this case it is associated with super-cooling (for more details, see Frost and Shepherd, 1986; Shepherd and Sturtevant, 1983).

2.8 NONTHERMODYNAMIC CONSTRAINTS ON THE EOS

2.8.1 CONVEXITY

Suppose that with Bethe, the Hugoniot equation (2.10) is expanded in a Taylor series; for a weak shock wave this gives to leading order

$$\Delta s = s - s_0 = -\frac{1}{12T}\left[\frac{\partial^2 p}{\partial v^2}\right]_S (v - v_0)^{+} O(v - v_0)^4 \tag{2.35}$$

Now for a compressive shock wave, $\Delta v < 0$, so if the entropy is to increase across a weak shock $\Delta s > 0$, then the EOS must be convex [Eq. (2.28)]. If on the other hand, the inequality equation (2.28) is reversed, so that the EOS is concave, then it is weak expansion shocks that entropically increase for an adiabatic system.

For waves of *arbitrary* strength, Bethe (1942) found that sufficient conditions for adiabatic compression shocks to be entropy increasing were that the EOS obeyed the convexity constraint as well as a constraint on the Grüneisen coefficient

$$\left[\frac{\partial^2 p}{\partial v^2}\right]_S > 0 \rightarrow G > 0 \tag{2.36}$$

$$\Gamma > -2 \tag{2.37}$$

Bethe (1942) showed that all pure substances in a single-phase state obeyed Eq. (2.36) for practically all thermodynamic states. The final result of his method is given in the Appendix at the end of this chapter. A summary of convex materials is presented in Table 2.4. The constraint fails for fluids of sufficiently high molecular weight (i.e., containing at least seven atoms in their molecule) with the fluid in the superheated vapor state and close to its phase critical point (Thompson 1971).

TABLE 2.4 Materials that have a convex EOS[a]

- Dissociating or ionizing gases
- Single-phase vapors with <7 atoms in their molecules
- Single-phase solids at low and normal temperatures
- Ideal gases with either constant or variable specific heats
- Liquids at normal temperatures, including water ±3.984 °C
- Liquid-vapor phase transition; the convexity may be discontinuous
- Some metal phase transitions; for example, $\alpha \rightarrow \varepsilon$ (BCC→HCP) in iron

[a] After Bethe (1942) and Thompson (1971).

TABLE 2.5 Materials that have a nonconvex EOS[a]

Metals at their yield point; elastic-plastic transition
Transition between two condensed phases at one of the two boundaries between the pure and the phase mixture
Single-phase vapors that have seven or more atoms in their molecules and in a state near their phase critical point

[a] After Bethe (1942); Lambrekis and Thompson (1972); Thompson *et al.* (1986).

An example obtained from the van der Waal's EOS is presented in Fig. 2.6. Notice that the isentropics are locally concave near the critical point but convex elsewhere. Examples of other materials that may be locally concave are given in Table 2.5.

As the inequality equation (2.36) is strict it implies that $-[\partial p/\partial v]_s$ may not be constant, and in particular it may not be zero. Thus by Eq. (2.23), $\gamma > 0$ strictly, and this is also necessary for the thermodynamic stability equation (2.31). As $\gamma = 0$ is forbidden, there are no stationary values for the pressure along an isentropic. Convexity also forbids the speed of sound being zero, such as occurs at phase triple points, for example ice/water/steam, for then by Eqs. (2.23) and (2.36), $G = 0$, and the material is neither convex nor concave. However when $G > 0$, then by Eq. (2.23)

$$\rho^2 a^2 = -\left[\frac{\partial p}{\partial v}\right]_S > 0 \tag{2.38}$$

thus an isentropic curve always has a negative slope in the (v, p)-plane when the EOS is convex (compare Figs. 2.4 and 2.6).

The only material that Bethe found that did not satisfy Eq. (2.38) was melting ice at $-20\,^\circ$C, which occurs at about 2500 atm and then $\Gamma \approx -2.1$. However, other examples are now known, such as vitreous silica, which has the remarkably low value of $\Gamma \approx -9$ at about 25 K (Collins and White, 1964).

2.8.2 Shock Wave Stability Constraints

Bethe (1942) deduced constraints on the EOS that would be sufficient to prevent a shock wave from splitting into two waves that move in either the same direction, or else in opposite directions. Von Neumann (1943) gave an elegant discussion of the first type of splitting. He supposed the shock being divided into two parts. The first part joins the initial pressure p_0 to an intermediate pressure p', and the second joins p' to the final pressure p. The velocity of each part is given by the Rayleigh equation (2.11). The shock wave

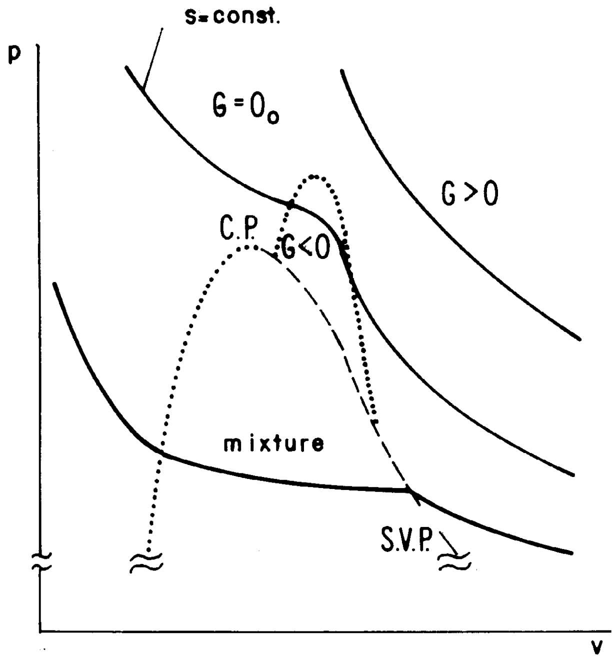

FIGURE 2.6 Nonconvex, $G < 0$, isentropics near the saturated vapor line in the $(v, p,)$-plane.

cannot split into two waves moving in the same direction if the velocity U_S' of the following wave is $\overline{\gtrless}$ velocity U_S of the leading wave $U_S' \geq U_S$

$$\frac{(P - P')}{(v' - v)} \geq \frac{(P' - P_0)}{(v_0 - v')} \tag{2.39}$$

for all p' in $p \geq p' \geq p_0$. In Section 2.10 it is shown that this kind of splitting is impossible with a convex EOS.

In order to exclude a shock splitting into two waves moving in opposite directions, Bethe (1942) deduced that *sufficient* constraints on the EOS were convexity $G > 0$, and

$$\left[\frac{\partial p}{\partial y}\right]_e = -\frac{p}{v}(\gamma - \Gamma) < 0 \tag{2.40.1}$$

or equivalently

$$\Gamma < \gamma \tag{2.40.2}$$

The basis for Bethe's (1942) study of the materials that satisfied Eqs. (2.40.1) and (2.40.2) was the thermodynamic identity

$$\left[\frac{\partial p}{\partial v}\right]_e = \left[\frac{\partial p}{\partial v}\right]_T - \frac{1}{C_V}\left[\frac{\partial p}{\partial T}\right]_V \left[\frac{\partial e}{\partial v}\right]_T < 0$$

Bethe (1942) concluded that:

- Nearly all materials in a single phase obey this constraint, but that it breaks down for a few phase transformations.
- This constraint seems to be more generally fulfilled than the convexity constraint.
- If the constraint is to be fulfilled for phase transformations, it is required that, $\Delta e \Delta s > 0$, that is, the energy and the entropy must change in the same direction. This is fulfilled for practically all phase transformations, but some exceptions are ice I or ice III to ice V (see Fig. 2.1).

The preceding discussion does not exhaust the possibilities for shock wave instabilities because transverse (ripple) instabilities are also possible (Kontorovich 1957; Griffith *et al.* 1975; Fowles and Houwing, 1984). Menikoff and Plohr (1989) proposed the EOS constraint, $\Gamma < \gamma - 1_p$, to exclude this kind of instability. It is only sufficient, and possibly therefore unnecessarily restrictive.

Remark. If the foregoing constraints are satisfied by the EOS for a given material in a given domain of states, then a stable compression shock can propagate through the material. Furthermore, Bethe (1942) showed that Eqs. (2.36) and (2.37) are sufficient for a solution to the Hugoniot equation (2.10) to exist and to be unique. This follows from the Bethe-Weyl theorem (see Section 2.10).

2.9 OTHER NONTHERMODYNAMIC EOS CONSTRAINTS

Many researchers have helped formulate the EOS constraints described in this section. The key idea is that of the constraints needed to ensure the monotonicity of particular thermodynamic properties, the wave velocity U_s and the particle velocities u_0 and u, along a Hugoniot adiabatic or isentropic. Numerous theorems can be proved once the monotonicities are established.

The strong constraint

$$\Gamma \leq \frac{pv}{e} \rightarrow 1 + \frac{1}{2}\Gamma\left(\frac{v - v_0}{v}\right) > 0 \tag{2.41}$$

When this constraint is satisfied, v is a monotonic decreasing quantity along a Hugoniot adiabatic. If also $G > 0$, then the curve is itself everywhere convex in both the (v, p)- and (u, p)-planes (see Fig. 2.7a). The ideal gas obeys Eq. (2.41) everywhere because from its EOS it is easy to obtain $\Gamma = \gamma - 1$, $G = \frac{1}{2}(\gamma + 1)$ and $pv/e = \gamma - 1 = \Gamma$, and so $G > 0$ and $\Gamma = pv/e$. All materials in a single phase obey Eqs. (2.36) and (2.41) for a large domain of states. The most notable exceptions are dissociating and ionizing gases, which violate Eq.

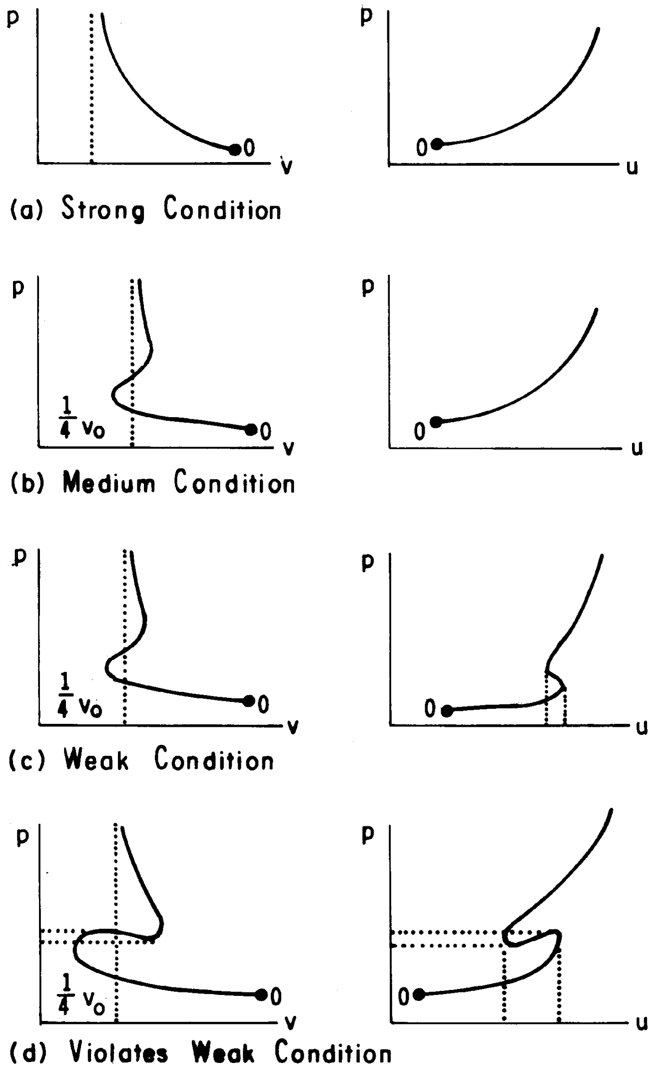

FIGURE 2.7 Hugoniot locus, with $G > 0$, in the (p, v)- and (p, u)-planes (after Menikoff and Plohr, 1989).

(2.41) but still satisfy Eq. (2.36). In such circumstances the Hugoniot curve becomes locally concave in the (v, p)-plane, but still remains convex everywhere in the (u, p)-plane (see Fig. 2.7b).

The medium constraint

$$\Gamma \leq \gamma + \frac{1}{2}\frac{pv}{e} \tag{2.42}$$

The solution to a Riemann problem for interacting shock waves is unique when this important constraint is satisfied. The constraint also guarantees that e and u are monotonic increasing quantities along a Hugoniot adiabatic (Menikoff and Plohr, 1989). It is evidently weaker than Eq. (2.40), so it must be more generally satisfied than is Eq. (2.40).

The weak constraint

$$\Gamma \leq 2\gamma \rightarrow \gamma - \frac{1}{2}\Gamma\frac{p - p_0}{p} > 0 \tag{2.43}$$

This ensures that pressure p and enthalpy h are monotonic increasing along a Hugoniot; all known materials obey it.

Menikoff and Plohr (1989) show that when $G > 0$, then

$$\text{strong} \rightarrow \text{medium} \rightarrow \text{weak} \tag{2.44}$$

so by Eqs. (2.41) to (2.44), with $\gamma > 0$

$$2\gamma \geq \gamma + \frac{1}{2}\frac{pv}{e} \geq \frac{pv}{e} \geq \Gamma \tag{2.45}$$

Figure 2.7 illustrates the effect on the Hugoniot curve when the strong, medium and weak constraints are successively violated while $G > 0$ remains satisfied.

Henderson and Menikoff (1997) used constraint equations (2.36) and (2.43) to prove the following.

LEMMA. *If $G > 0$ and $\Gamma \leq 2\gamma$ then the Hugoniot curve based on state (v_0, p_0) contains a unique shock for any $p \geq p_0$. Moreover, pressure-increasing shock waves are entropy increasing.*

If the shock wave strength is defined by $\Delta p = p - p_0$, then the lemma applies to a shock of any strength.

2.10 THE BETHE-WEYL (B-W) THEOREM

While the conservation laws are the most important information that we have about shock waves, next in importance is the powerful Bethe-Weyl (B-W) theorem. It can only be applied directly to a shock wave when its velocity vectors are perpendicular to the shock front, that to a normal shock (see Fig. 2.8a). If a velocity vector $\mathbf{v}_t = \mathbf{v}_{t_0}$, which is parallel to the shock is added to u_0 and u then a normal shock becomes oblique with respect to the upstream and downstream flows (see Fig. 2.8b)

$$-U_S = U_0 = u_0 + \mathbf{v}_{t_0} \text{ and } U = u + \mathbf{v}_t \qquad (2.46)$$

The theorem can then be extended to oblique shock waves by resolving the vectors U_0 and U to obtain the normal shock vectors $\mathbf{u}_0$ and $\boldsymbol{u}$. Henderson and Menikoff (1997) state the theorem as

THEOREM (BETHE-WEYL). *If $G > 0$ and $\Gamma \leq 2\gamma$ then the Hugoniot curve based on any state zero intersects every isentropic exactly once. Moreover for entropy increasing shocks $s > s_0$, one has $v < v_0$ and* u < a, *while for entropy decreasing shocks $s < s_0$, $v > v_0$* and *$u > a$.*

The first part of the theorem guarantees the existence of at least one solution to the Hugoniot equation. Bethe's proof depends on the asymptotic properties of the EOS and of the Hugoniot equation. An outline of a more elegant proof based on the same approach is now presented (Menikoff and Plohr, 1989). Define the Hugoniot function as

$$h(v, s) = e(v, s) - e_0 + \frac{1}{2}[p(v, s) + p_0](v - v_0) \qquad (2.47)$$

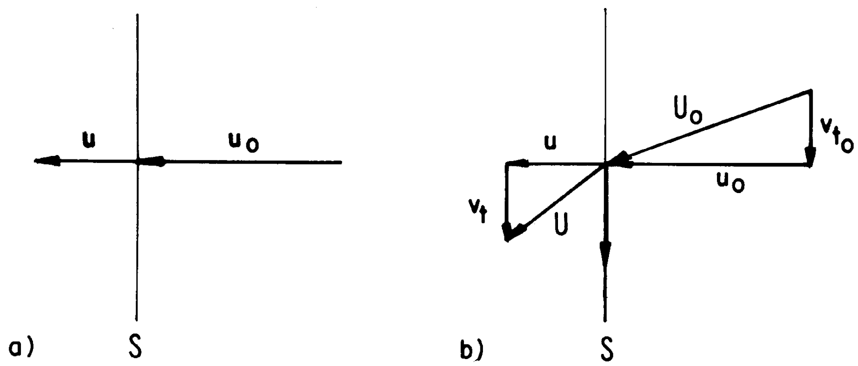

FIGURE 2.8 Flow vectors for normal (a) and oblique (b) shock waves.

The function is now restricted to the isentropic s and designated $h_S(v)$. Notice how the Hugoniot equation is recovered when $h_S(v) = 0$. Next, suppose that the asymptotic properties of the EOS are such that $p(v, s) \to \infty$ as $v \to \infty$ and that $p(v) \geq 0$. Furthermore, $\gamma > 0$, implies that $[\partial p/\partial v]_S < 0$, and from this and the previous assumptions it can be proved that $e/p \to 0$ as $v \to \infty$. Now using these results, it follows from Eq. (47) that $h_S(v) \to -\infty$ as $v \to 0$. On the other hand, because $h_S(\mathrm{v}) > -e_0 + \frac{1}{2}p_0(v - v_0)$, then with $V > V_0$, $h_S(\mathrm{v}) \to +\infty$ as $v \to \infty$. By continuity, therefore, $h_S(v)$ vanishes at least once, and at least one solution exists to the Hugoniot equation.

In order to accommodate shocks of arbitrary strength, Bethe's proof of the second part required that $G > 0$, plus the sufficient (only) constraint $\Gamma > -2$ (Eq. 2.37). The proof presented by Henderson and Menikoff (1998) used the lemma in Section 2.9 and the alternative EOS constraints $G > 0$ with $\Gamma \leq 2\gamma$, where the weak constraint is again only sufficient. However, because $\gamma > 0$, the two Γ constraints overlap and cover the entire real domain $-\infty \leq \Gamma \leq \infty$ and this confirms the claim of Menikoff and Plohr (1989) that their proof is independent of Γ. Note that Henderson and Menikoff (1998) corrected an error in their proof.

The theorem ensures that with $G > 0$, a unique solution exists. It also ensures that the entropy s is a monotonic increasing quantity along a Hugoniot; so s can be used as a Hugoniot parameter. Other quantities that can be shown to be monotonically increasing are the mass flux m through the shock wave, the shock velocity U_S, and the negative slope of the Raleigh line, $(\Delta p/\Delta v)$. Using the implicit function theorem Henderson and Menikoff (1998) obtained,

COROLLARY 2.1. *If $G > 0$ and $\Gamma \leq 2\gamma$ the Hugoniot curve can be parameterized by the entropy s and consists of a single curve connected to the base state.*

The B-W theorem asserts that $u < a$ for an entropy-increasing compression shock and $u > a$ for an entropy decreasing expansion shock. The inequalities are strict so sonic flow cannot occur downstream of either shock. The only exception is for an acoustic degeneracy, where $p = p_0$ and $u = a = a_0 = u_0$, which is trivial. An important corollary can now be deduced. Notice that the conservation laws remain the same if the initial (v_0, p_0, e_0, u_0) and the final (v, p, e, u) states are interchanged. Similarly the Hugoniot and the Raleigh equations remain the same. The equations are also unaffected if the signs of the velocity vectors are reversed. Consequently, an entropy-increasing compression shock wave can be viewed mathematically as a reversed, entropy decreasing, expansion shock wave, which has supersonic flow on its lower density side. Hence, COROLLARY 2.2. *If $G > 0$ and $\Gamma \leq 2\gamma$ then for an entropy increasing shock the upstream state is supersonic $u_0 > a_0$ and the downstream state is subsonic $u < a$.*

Downstream Sonic Flow

The slope of the Hugoniot curve in the (p, s)-plane can be found by differentiating Eq. (2.10) and using Eq. (2.17)

$$T\left[\frac{\partial s}{\partial p}\right]_{\mathrm{h}} = -\frac{\Delta y}{2}\frac{\gamma + \dfrac{v}{p}\dfrac{\Delta p}{\Delta v}}{\gamma - \frac{1}{2}\Gamma\dfrac{\Delta p}{p}} \tag{2.48}$$

Note that the denominator is always positive when the weak constraint is satisfied. By using Eqs. (2.11) and (2.23) the numerator can be rewritten as

$$-\frac{v}{p}\Delta v\left[-\left[\frac{\partial p}{\partial v}\right]_{\mathrm{S}} + \frac{\Delta p}{\Delta v}\right] = -\frac{1}{p}\frac{\Delta v}{v}(a^2 - u^2) \geq 0 \tag{2.49}$$

where the inequality is only valid when $G > 0$, that is, for a compression shock. If $G = 0$, then by Eq. (2.48), $[\partial s/\partial p]_{\mathrm{h}} = 0$, and by Eq. (2.49), $\Delta p/\Delta v = [\partial p/\partial v]_{\mathrm{S}}$ and $u = a$, so the Raleigh line is tangent to the isentropic in the (v, p)-plane, and the downstream flow is sonic. The equality $G = 0$ occurs, for example, at the separating boundary between $G < 0$, and $G > 0$ (see Tables 2.4 and 2.5) or for phase triple points where the speed of sound is zero. A Hugoniot curve has a cusp at the yield point of a metal and the Raleigh line touches that point as in Fig. 2.9; here also $u = a$ (Zeldovich and Raizer, 1966; McQueen, 1991). A similar effect exists for some phase transitions (e.g., the $\alpha \rightarrow \varepsilon$ phase transition in iron, see Meyers, 1993); formally $G = -\infty$ at a cusp (Thompson, 1986). By Eq. (2.48) the entropy is not monotonic along a Hugoniot curve whenever a Raleigh line touches the Hugoniot curve because the derivative in Eq. (2.48) vanishes. It is found that the von Neumann constraint equation (2.39) is thereby violated and the shock wave splits into two waves moving in the same direction.

2.11 SHOCK WAVE INTERACTIONS

2.11.1 DIMENSIONS OF THE INTERACTIONS

A normal shock wave is (1D) by definition. If two 1D shocks i_1 and i_2 are parallel and approach each other from opposite directions they will collide.

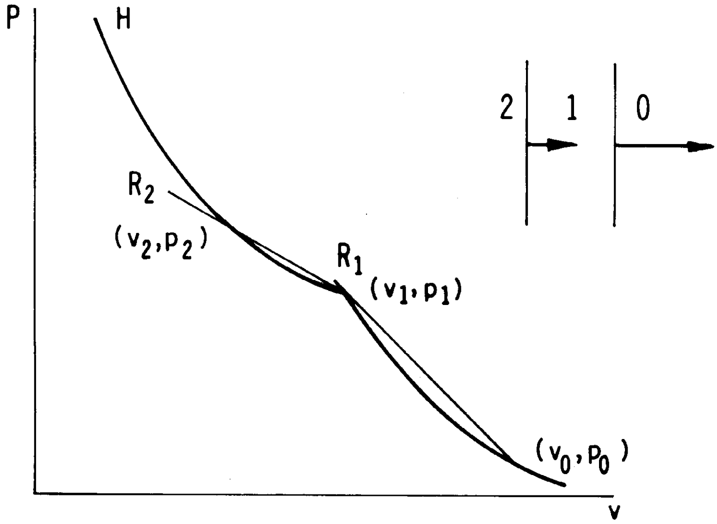

FIGURE 2.9 Shock splitting instability when $G < 0$ locally at (v_1, p_1), which is a sonic point $u = a$.

After collision there will be two reflected shock waves r_1 and r_2, and a contact discontinuity *cd*, (see Fig. 2.10a). There will also be a collision if i_1 and i_2 move in the same direction. This is an overtaking collision and there are two possible outcomes. If i_1 and i_2 are weak shock waves, then after collision there will be a transmitted shock wave *t*, a reflected shock wave *r*, and a contact discontinuity *cd*. For stronger shock waves, an expansion wave *e* (see Fig. 2.10b) will replace the reflected shock wave *r*. Such phenomena are defined to be 1D shock interactions because all the waves and contact discontinuities are parallel to each other.

2.11.2 Two-dimensional Shock Wave Interactions

These (2D) interactions occur between nonparallel, that is, oblique shocks. A few examples are illustrated in Fig. 2.11, but there are many other possibilities. By assigning a direction to every wave, and contact discontinuity in an interaction (Landau and Lifshitz, 1959) classification of it can be facilitated. Taking shock-fixed coordinates and resolving and U_0 and U into perpendicular u_0 and u, and parallel $\mathbf{v}_{t_0}$ and $\mathbf{v}_t$ vector components, as in Fig. 2.8, the direction of the shock wave is defined to be the same as $\mathbf{v}_{t_0} = \mathbf{v}_t$. In a similar way, a direction can be assigned to any wavelet in a Prandtl-Meyer expansion. For a

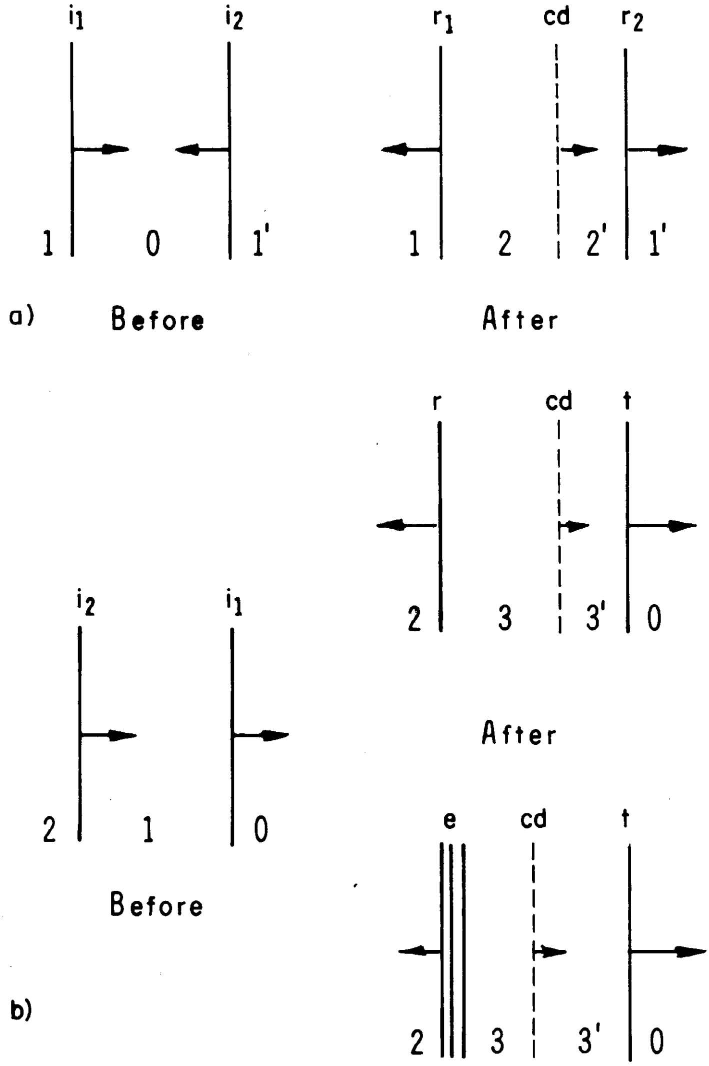

FIGURE 2.10 The one-dimensional collision of two planar shock waves i_1 and i_2. a) Colliding shocks. b) Overtaking shocks, r-reflected shock wave, t-transmitted shock wave, e-expansion wave, and cd-contact discontinuity.

2D contact discontinuity, the direction is the same as the particle path on either side of it. As illustrated in Fig. 2.11, 2D waves and contact discontinuities may either meet or emanate from a point in the flow called a *node* (Glimm *et al.*, 1985). When the direction of a wave points towards a node, we say that the

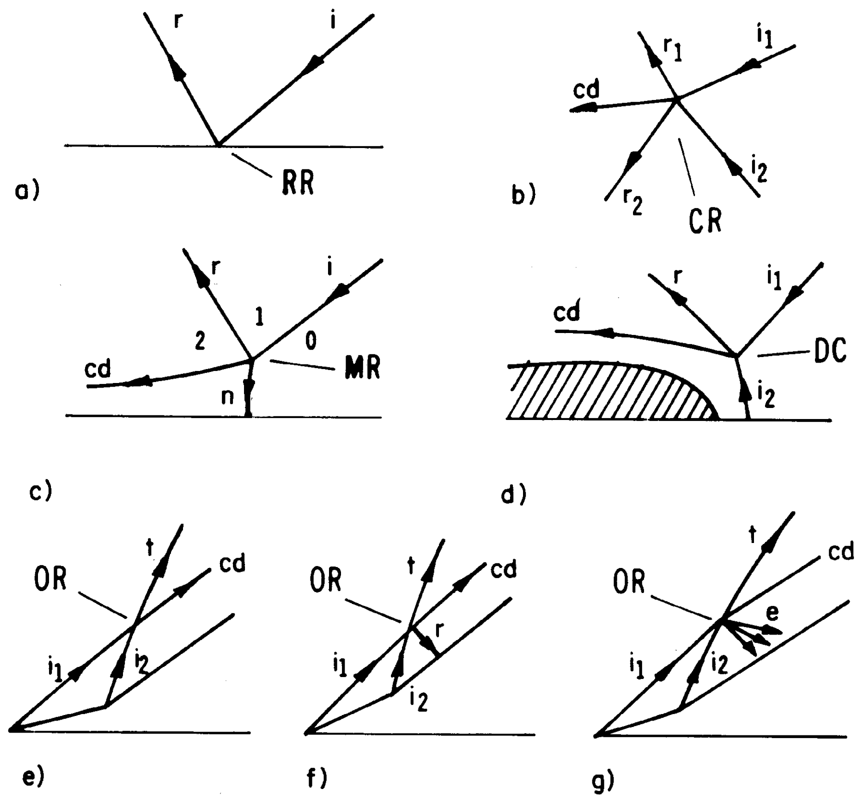

FIGURE 2.11 Examples of two-dimensional shock wave interactions. a) Regular reflection node, RR. b) Cross node, CR. c) Mach reflection node, MR. d) Degenerate cross node, DC. e) to g) Various overtaking nodes, OR.

wave arrives at the node or that it is an *incoming* wave. We say that it leaves the node or that it is an *outgoing wave* when it points away from the node. In this sense the wave direction is the same as that of the flow of information from flow disturbances. For example, an incident shock i always arrives at a regular reflection (RR) node while the reflected r shock always leaves it (see Fig. 2.11a). For a Mach reflection (MR), the i shock is incoming, while the r shock and Mach shock are outgoing, and so is the contact *cd*. It may be said that the i shock *splits* into the r and n shocks and the contact discontinuity (see Fig. 2.11c). By contrast, the very similar interaction (see Fig. 2.11d) has two incoming i shocks, and one outgoing r shock and contact discontinuity. It can only exist when there is an extra boundary in the flow, which is needed to generate i_2, (Henderson and Menikoff, 1998).

2.11.3 Three-dimensional Shock Wave Interactions

Shock waves that have conical, cylindrical or spherical symmetry are simple examples of (3D) shock waves. The study of the interactions of these shock waves or those with more complicated shapes usually requires the help of computer graphics software in order to achieve success. As an example of a recent study, see Skews (1996).

2.12 THE TRIPLE-SHOCK-ENTROPY AND RELATED THEOREMS

2.12.1 The Theorems

The B-W theorem is directly applicable to either a 1D or a 2D shock wave, but not to the interactions of two or more of these shock waves. The triple-shock-entropy (TSE) theorem gives information about shock interactions in materials whose EOSs satisfy $G > 0$ and $\Gamma \leq 2\gamma$. Application to 3D interactions is possible in regions where the radius of curvature of the shock waves is small compared to their thicknesses, and where also transverse flow gradients are small.

The triple-shock-entropy Theorem. (Henderson and Menikoff, 1998)

Suppose that $G > 0$ and $\Gamma \leq 2\gamma$ are satisfied everywhere. Consider only the physically realistic, entropy increasing, shock waves. Then the entropy increase across a sequence of two shock waves is smaller than that across a single shock wave to the same final pressure.

This is the key theorem; for its somewhat lengthy proof see the cited reference. Extension of it to an *n* shock sequence follows easily by mathematical induction; the result is,

Corollary 2.3. *A sequence of n shock waves has less entropy increase than a single shock wave to the same final pressure.*

Once the entropy inequality has been derived it easy to obtain inequalities for other state variables. The results are,

Corollary 2.4. *Consider the state downstream of a sequence of shock waves with the same final pressure as a single shock wave. The multiple shock waves* *have a smaller enthalpy and a smaller temperature increases than the single shock wave. Moreover, if* $\Gamma \leq \gamma$ *then multiple shock waves have a smaller specific energy increase, and if* $\Gamma > 0$ *then multiple shock waves have a smaller specific volume increase.*

A similar result can also be obtained for the particle velocity with $\Gamma \leq \gamma$, as follows.

THE TRIPLE-SHOCK-PARTICLE-BOUND THEOREM. (Henderson and Menikoff 1998)

Suppose that $G > 0$ *and* $\Gamma \leq \gamma$ *are satisfied everywhere. Consider a sequence of two, entropy increasing, shock waves from state* 0 *to state* 1, *and from state* 1 *to state* 2, *and a third shock from state* 0 *to state* $2'$ *with* $p_{2'} = p_2$. *Then*

$$(u_2 - u_1)^2 + (u_1 - u_0)^2 < (u_2 - u_0)^2 \tag{2.50}$$

2.12.2 Application to Shock Wave Interactions

Consider the consequences of the inequality equation (2.50) to 1D shock interactions. Suppose that before collision there is an incoming left-facing shock wave (0-1, *L*) approaching an incoming right-facing shock wave (0-1, *R*). The Hugoniot curves in the (u, P)-plane and the wave diagram in the (x, t)-plane are sketched in Fig. 2.12. We shall now prove that the outgoing waves produced after the collision can only be shock waves. We begin by assuming to the contrary that one of the outgoing waves, say (1-2, *R*) is an expansion wave. This means that the Hugoniot curve for (1-2, *R*) must cross the Hugoniot for (0-1, *L*), which implies that $u_2 > u_{2'}$ (see Fig. 2.13). To prove that this is impossible, we must show that $u_2 < u_{2'}$. (Here we have dropped the *L* and *R* subscripts for simplicity.) Now $u_2 < u_{2'}$ can be expressed as

$$|u_2 - u_1| - |u_1 - u_0| < |u_{2'} - u_0| \tag{2.51}$$

But also because $|u_2 - u_1| - |u_1 - u_0| < |u_2 - u_1|$, then Eq. (2.50) implies that $|u_2 - u_1| < |u_{2'} - u_0|$, so Eq. (2.51) is satisfied. Hence, outgoing expansion waves are excluded and only outgoing shock waves are permitted.

The wave diagram is quite different for overtaking shock waves (Fig. 2.14). In this case there may exist states where outgoing expansion waves are possible, that is, where

$$|u_2 - u_1| + |u_1 + u_0| > |u_{2'} - u_0| \tag{2.52}$$

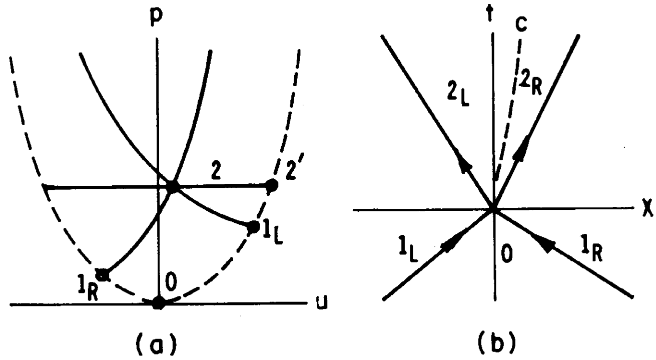

FIGURE 2.12 The collision of two shock waves of opposite family. a) Hugoniot loci in the (u, p)-plane. The dashed lines are Hugoniot loci for the incoming shock waves and the solid lines are Hugoniot loci for the outgoing shock waves. The outgoing shock waves of the shock waves interaction correspond to the intersection point of the two solid curves. b) Wave diagram in the (x, t)-plane. The initial state is (0), the incoming shock waves are (1) and the outgoing shock waves are (2). The superscripts L and R denote left and right in the (x, t)-plane. The 2′ denotes a single shock wave from the initial state with the same final pressure as the outgoing shock waves.

As indicated in Fig. 2.14, this is possible if the shock waves are sufficiently strong, even though Eq. (2.50) must remain satisfied. The result is an outgoing reflected expansion wave and a shock wave that move in opposite directions. Alternatively, both outgoing waves are shock waves when the incoming shock waves are sufficiently weak.

Although the 2D interactions are more complex, the TSE theorem permits the immediate conclusion that a contact discontinuity must occur in the wave systems sketched in Figs. 2.11b, 2.11c, and 2.11d. By the extension to n shock

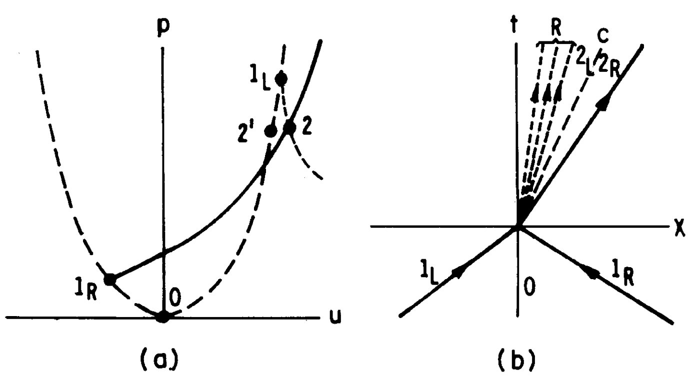

FIGURE 2.13 Excluded situation in which the collision of two shock waves of the opposite family would result in a reflected rarefaction.

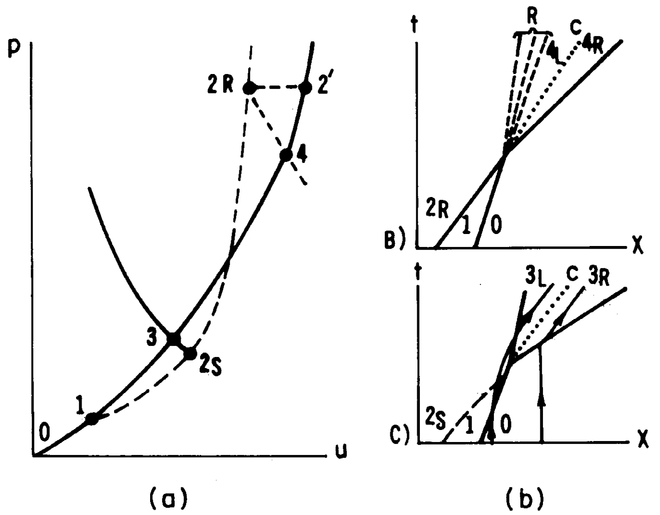

FIGURE 2.14 The overtaking of two shock waves of the same family. a) Hugoniot loci in the (u, p)-plane. b) Wave diagram in the (x, t)-plane when the outgoing shock waves consist of a reflected rarefaction wave and a transmitted shock wave. c) Wave diagram in the (x, t)-plane when the outgoing shock waves consist of a reflected shock wave and a transmitted shock wave. The lines with arrowheads represent particle paths.

waves the same conclusion follows for the systems sketched in Fig. 2.11e. Furthermore, inequalities for the thermodynamic states across the contact discontinuities are obtainable from Corollary 2.4, Section 2.12.1.

2.13 CROCCO'S THEOREM

This theorem is useful for application to curved shock waves of variable strength, such as a bow shock wave standing off a blunt body (see Fig. 2.15). It relates the flow velocity U and the vorticity $\nabla \times \boldsymbol{U}$ vectors to the gradients of the entropy ∇s and total enthalpy ∇h_t. It may be applied generally to shock wave problems because it is independent of any EOS. Given the Euler equation

$$\rho \frac{DU}{Dt} = -\nabla p \qquad ????$$

or after expanding the left-hand side

$$\rho \frac{\partial U}{\partial t} + \rho (U \cdot \nabla) U = -\nabla p \qquad (2.53)$$

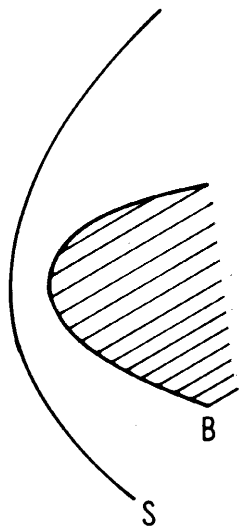

FIGURE 2.15 A curved shock wave S standing off a blunt body B.

Now rewriting the fundamental equation (2.17) in terms of the enthalpy $Tds = dh - vdp$, and because for 3D flow the differentials can be replaced by the gradient operator,

$$T\nabla s = \nabla h - v\nabla P = \nabla h - \frac{1}{\rho}\nabla p \tag{2.54}$$

Eliminating ∇p from Eqs. (2.53) and (2.54)

$$T\nabla s = \nabla h + \frac{\partial U}{\partial t} + (U \cdot \nabla)U \tag{2.55}$$

But from Eq. (2.9),

$$h = h_t - \frac{1}{2}u^2 \rightarrow \nabla h = \nabla h_t - \frac{1}{2}\nabla U^2 \tag{2.56}$$

Eliminating h between Eqs. (2.55) and (2.56)

$$T\nabla s = \nabla h_t - \frac{1}{2}\nabla U^2 + \frac{\partial U}{\partial t} + (U \cdot \nabla)U \tag{2.57}$$

Finally, the vorticity vector can be introduced by means of the identity

$$\nabla\frac{1}{2}U^2 - (U \cdot \nabla)U = Ux(\nabla x U)$$

so that Eq. (2.57) becomes *Crocco's theorem* (Crocco, 1937):

$$T\nabla s = \nabla h_t - Ux(\nabla \times U) + \frac{\partial U}{\partial t} \tag{2.58}$$

2.14 THE REFRACTION LAW

This is a kinematic equation and it is independent of any EOS. It is useful when a 2 or a 3D wave interaction has at least one node (see Fig. 2.11). Consider a reference frame that is at rest with respect to the undisturbed material. It is clear that if a wave system is to be stable against breaking up into some other system, then it is necessary that all the waves meeting at a node must propagate at the same velocity along the node trajectory.

An example of the node that occurs in regular reflection (*RR*) is shown in Fig. 2.16. This system may appear for example when an incident shock *I* diffracts over a rigid ramp. If the wave vectors for the *I* and *r* shocks are, respectively, U_i and U_r', and if the corresponding downstream particle (piston)

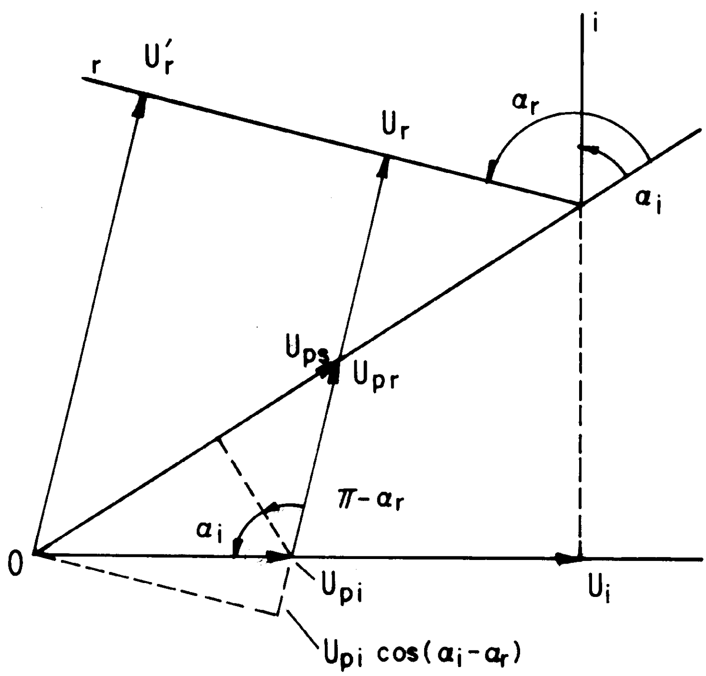

FIGURE 2.16 Wave and particle velocity vectors in laboratory frame coordinates for the regular reflection *RR* of an incident shock wave *i*.

velocity vectors are $\boldsymbol{U}_{\mathrm{pi}}$ and $\boldsymbol{U}_{\mathrm{pr}}$, and if the angles of incidence and reflection are α_{i} and α_{r}, then by geometry the refraction law is

$$U = \frac{U_{\mathrm{i}}}{\sin\alpha_{\mathrm{i}}} = \frac{U'_{\mathrm{r}}}{\sin(\pi - \alpha_{\mathrm{r}})} = \frac{U'_{\mathrm{r}}}{\sin\alpha_{\mathrm{r}}} \tag{2.59}$$

Because the incident shock wave compresses and accelerates the undisturbed material, the r shock wave propagates into a material that is both moving and compressed. If U_{r} is the velocity of r relative to the moving material that is upstream of it, then by geometry,

$$U'_{\mathrm{r}} = U_{\mathrm{r}} + U_{\mathrm{pi}}\cos(\alpha_{\mathrm{i}} - \alpha_{\mathrm{r}}) \tag{2.60}$$

If the ramp surface is impervious, then one has the boundary condition

$$U_{\mathrm{pi}}\cos\alpha_{\mathrm{i}} = U_{\mathrm{pr}}\cos(\pi - \alpha_{\mathrm{r}}) = -U_{\mathrm{pr}}\cos\alpha_{\mathrm{r}} \tag{2.61}$$

When the ramp angle $(\frac{1}{2}\pi - \alpha_{\mathrm{i}})$ and the shock wave velocity U_{i} are given, then the particle velocity U_{pi} can be found from Eq. (2.3) and the EOS of the material. This leaves four unknowns α_{r}, U'_{r}, U_{r} and U_{pr}, and there are three equations, (2.59)–(2.61). A fourth equation can be obtained from Eq. (2.3) and the EOS to find the relation between U_{r} and U_{pr}. Thus the *RR* problem can be solved when U_{i}, α_{i} and the EOS are given.

The more complicated Mach reflection (*MR*) is presented as a second example in Fig. 2.17. In order to accommodate the Mach shock n, the refraction law is extended by adding another equation, which includes the corresponding wave and particle velocities U_{n} and U_{pn}, respectively

$$U = \frac{U_{\mathrm{i}}}{\sin\alpha_{i}} = \frac{U'_{\mathrm{r}}}{\sin\alpha_{\mathrm{r}}} = \frac{U_{\mathrm{n}}}{\sin\alpha_{\mathrm{n}}} \tag{2.62}$$

The law provides a powerful means for finding when one wave system changes to another as a result of a continuous change in the system parameters. For example, consider the *MR* sketched in Fig. 2.11c, in which the Mach shock n, leaves the node. Suppose that α_{i} decreases while the other independent parameters such as U_{i} are held constant. This causes α_{n} to increase, and a condition can be reached where $\alpha_{\mathrm{n}} > \pi/2$. Hence, n now arrives at the node as in Fig. 2.11d, and an extra boundary c is needed to support the appearance of $n = i_2$. If it is not present then the three shock wave system cannot appear, and an *RR* appears instead. The separating, or transition condition occurs when $\alpha_{\mathrm{n}} = \pi/2$, and Eq. (2.62) gives

$$\sin\alpha_{\mathrm{n}} = \frac{U_{\mathrm{n}}}{U_{\mathrm{i}}} \tag{2.63}$$

Equation (2.62) defines *the mechanical equilibrium criterion* for *RR* $\leftrightarrow$ *MR* transition, and is such that Mach shock n is a normal shock. Equations

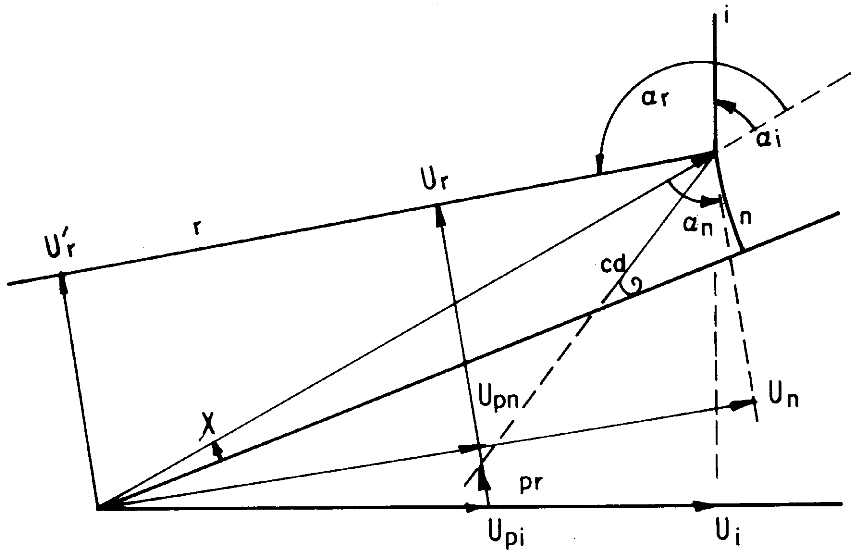

FIGURE 2.17 Wave and particle velocity vectors in laboratory frame coordinates for the Mach reflection *MR* of an incident shock wave *i*.

analogous to (2.60) and (2.61) require an extensive discussion and are omitted here (for more details, see Henderson, 1989).

2.15 CONCLUDING REMARKS

This chapter has presented what are arguably the most fundamental results available about normal and oblique shock waves and their 1D and 2D interactions. In summary they are:

- *The conservation laws* (2.1) to (2.3) or (2.6) to (2.8), and the equivalent Hugoniot (2.10) and Raleigh (2.11) equations. They can be applied to either a normal or an oblique shock wave in any material.
- *The Bethe-Weyl Theorem* (Section 2.10). It can also be applied to either a normal or to an oblique shock wave in any material, but it is most powerful when the material EOS is convex $G > 0$ [Eq. (2.28)] and when the EOS also obeys the weak constraint $\Gamma \leq 2\gamma$ [Eq. (2.43)]. All materials in a single phase obey these EOS constraints for almost all thermodynamic constraints. Exceptions for a fluid in a single phase occur near a phase critical point, if the fluid has at least seven atoms in its

molecule. Other exceptions exist at the yield point of a metal, and for some, but not all multiphase states.

- *The triple-shock entropy Theorem* (Section 2.12). This theorem gives general properties for 1D and 2D shock wave interactions when the material EOS obeys the $G > 0$ and $\Gamma \leq 2\gamma$ constraints.
- *Crocco's theorem* (Eq. 2.58 in Section 2.13). This theorem is useful when there are entropy, total enthalpy, or velocity gradients upstream of the shock or when the shock is curved and with nonuniform strength. It is applicable to any material.
- *The refraction law* (Eqs. 2.59 and 2.62 in Section 2.14). This theorem is useful when there are wave nodes present in 2D or 3D shock interactions. It provides a means for finding where one wave system changes into another under a continuous change of the system parameters. It can be applied to any material.

APPENDIX: THE CONVEXITY OF AN EQUATION OF STATE

Following Bethe (1942), the basis of his method for finding if $[\partial^2 p/\partial \mathrm{v}^2]_S > 0$ for a given EOS is the equation

$$\left[\frac{\partial^2 p}{\partial v^2}\right]_S = \left[\frac{\partial^2 p}{\partial v^2}\right]_T - \frac{3T}{C_V}\left[\frac{\partial p}{\partial T}\right]_V \left[\frac{\partial^2 p}{\partial \mathrm{v}\partial T}\right] + \frac{3T}{C_V^2}\left[\left[\frac{\partial p}{\partial T}\right]_V\right]^2 \left[\frac{\partial C_V}{\partial v}\right]_T + \frac{T}{C_V^2}\left[\left[\frac{\partial p}{\partial T}\right]_V\right]^2 \left[1 - \frac{T}{C_V}\left[\frac{\partial C_V}{\partial T}\right]_V\right]$$

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