## MA123, Chapter 7: Word Problems (pp. 125-153, Gootman)

---

`Chapter Goals:` In this Chapter we learn a general strategy on how to approach the two main types of word problems that one usually encounters in a first Calculus course:

- Max-Min problems
- Related Rates problems

`Assignments:` `Assignment 16` `Assignment 17`

---

**Suggestions:** The most important skill in solving a word problem is reading comprehension. The most important attitude to have in attacking word problems is to be willing to think about what you are reading and to give up on hoping to *mechanically* apply a set of steps. Nevertheless, we will present some useful strategies to employ that are often helpful.

### MAX-MIN PROBLEMS

All max-min problems ask you to find the largest or smallest value of a function on an interval. Usually, the hard part is reading the English and finding the formula for the function. Once you have found the function, then you can use the techniques from Chapter 6 to find the largest or smallest values.

▶ **Max-min guideline:** This guideline is found on pp. 131-133 of our textbook.

(1.) Read the problem quickly. slowly?
(2.) Read the problem carefully.
(3.) Define your variables. If the problem is a geometry problem, draw a picture and label it.
(4.) Determine whether you need to find the max or the min.
Determine exactly what needs to be maximized or minimized.
(5.) Write the *general* formula for what you are trying to maximize or minimize. If this formula only involves *one* variable, then skip steps **6**, **7** and **8.**
(6.) Find the relationship(s) (i.e., equation(s)) between the variables.
(7.) Do the algebra to solve for one variable in the equation(s) as a function of the other(s).
(8.) Use your formula from step **5** to rewrite the formula that you want to maximize or minimize as a function of one variable only.
(9.) Write down the interval over which the above variable can vary, for the particular word problem you are solving.
(10.) Take the derivative and find the critical points.
(11.) Use the techniques from Chapter 6 to find the maximum or the minimum.

**Example 1:** What is the largest possible product you can form from two non-negative numbers whose sum is 30?

Let $x$ and $y$ be two non-negative numbers such that $x+y=30$ $\quad y = 30-x$

Note: non-negative means $x, y \geq 0$; however, non-negative & $x+y=30$ implies $x, y \leq 30$

So we have $x, y \in [0, 30]$

Want to maximize $P(x,y) = xy$

$\Rightarrow P(x) = P(x, 30-x) = x\cdot(30-x) = 30x - x^2$

Parabola
$a=-1<0$ open downward
max at $x = \frac{-b}{2a} = \frac{-30}{2(-1)} = \frac{30}{2} = 15$

$P'(x) = 30-2x$ ← Defined everywhere

Need to check:
1) Endpoints: $x=0, x=30$
2) $P'(x)=0$: $x=15$
3) $P'(x)$ DNE: nowhere

$P(0) = 30\cdot(0) - 0^2 = 0-0=0$
$P(15) = 30\cdot 15 - 15^2 = (30-15)\cdot 15 = 15\cdot 15 = 225$ ← maximum 75
$P(30) = 30\cdot 30 - 30^2 = 30^2 - 30^2 = 0$

$$\boxed{\text{Maximum at } x = 15,\quad y = 30-15 = 15}$$

**Example 2:** Suppose the product of $x$ and $y$ is 26 and both $x$ and $y$ are positive. What is the minimum possible sum of $x$ and $y$?

Need $x \cdot y = 26$ with $x, y > 0$

$y = \frac{26}{x}$

To minimize $S(x,y) = x + y$

so $S(x) = S(x, \frac{26}{x}) = x + \frac{26}{x} = x + 26x^{-1}$

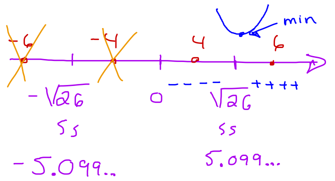
therefore $S'(x) = 1 - 26x^{-2} = \frac{x^2}{x^2} - \frac{26}{x^2} = \frac{x^2-26}{x^2}$

Critical Points when

$x^2 - 26 = 0$ or $x^2 = 0$

$x^2 = 26$  $x = 0$

$x = \pm\sqrt{26}$



| Test Value | $x^2-26$ | $x^2$ | Sign of $S'(x)$ | Inc/Dec |
|---|---|---|---|---|
| 4 | $-$ | $+$ | $-$ | Decreasing |
| 6 | $+$ | $+$ | $+$ | Increasing |

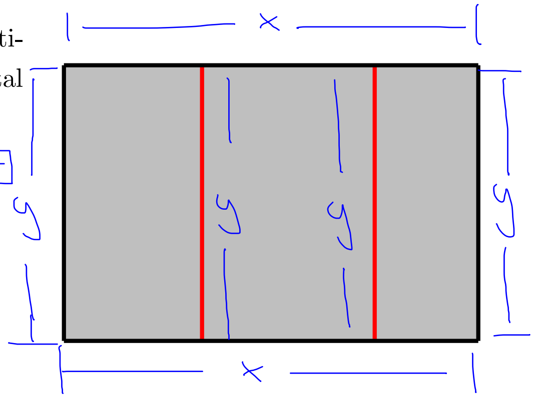
Local minimum at $x = \boxed{\sqrt{26}}$

and $y = \frac{26}{\sqrt{26}} \frac{\sqrt{26}}{\sqrt{26}} = \frac{\cancel{26}\sqrt{26}}{\cancel{26}} = \boxed{\sqrt{26}}$

So the minimum sum is $\sqrt{26} + \sqrt{26} = \boxed{2\sqrt{26}}$

**Note:** An alternative wording for Example 2 above is:
"Suppose $y$ is inversely proportional to $x$ and the constant of proportionality equals 26. What is the minimum sum of $x$ and $y$ if $x$ and $y$ are both positive?"

**Example 3:** A farmer builds a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen?



Need $2x + 4y = 500$ with $x \in [0, 250]$  $y \in [0, 125]$

To maximize $A(x,y) = xy$

$2x + 4y = 500$

$x + 2y = 250$

$x = 250 - 2y$

so $A(y) = A(250-2y, y) = (250-2y) \cdot y$

$= 250y - 2y^2$

therefore $A'(y) = 250 - 4y$

Defined everywhere

Parabola

$a = -2 < 0$ opens downword $\cap$

absolute max at

$y = \frac{-b}{2a} = \frac{-250}{2(-2)} = \frac{250}{4} = 62.5$

note $A'(y) = 0$

$250 - 4y = 0$

$+4y \quad +4y$

$\frac{250}{4} = \frac{4y}{4}$

$y = 62.5$

Need to check:

1) Endpoints: $y = 0$ & $y = 500$

2) $A'(y) = 0$ : $y = 62.5$

3) $A'(y)$ DNE : Nowhere

$A(0) = 250 \cdot 0 - 2 \cdot 0^2 = 0 - 0 = 0$

$A(62.5) = 250 \cdot 62.5 - 2(62.5)^2 = 15625 - 7812.5 = 7812.5$ maximum

$A(125) = 250 \cdot 125 - 2(125)^2 = 31{,}250 - 31{,}250 = 0$

Conclusion: $y = 62.5$ feet

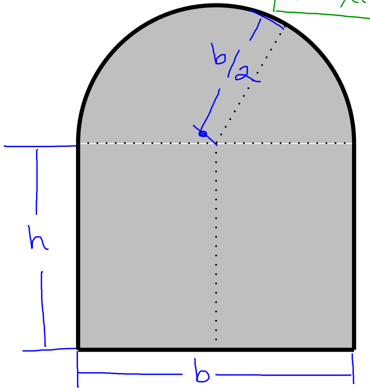
and $x = 250 - 2(62.5)$

$= 250 - 125$

$= 125$ feet

$\boxed{125 \text{ feet} \times 62.5 \text{ feet}}$

**Example 4:** A Norman window has the shape of a rectangle capped by a semicircle. What is the length of the base of a Norman window of maximum area if the perimeter of the window equals 10?



Perimeter $= h + b + h + \frac{1}{2}(\text{Circle Circumference})$

$10 = 2h + b + \frac{1}{2}(2\pi(\frac{b}{2}))$

$10 = 2h + b + \frac{\pi b}{2}$

$10 - b - \frac{\pi}{2}b = 2h$

$10 - b(1 + \frac{\pi}{2}) = 2h$

$\frac{1}{2}(10 - b(\frac{2}{2} + \frac{\pi}{2})) = \frac{1}{2}(2h)$

$h = 5 - \frac{1}{2}b(\frac{2+\pi}{2})$

$h = \frac{20}{4} - \frac{b(2+\pi)}{4}$

$h = \frac{20 - b(2+\pi)}{4}$

Area = Rectangle + $\frac{1}{2} \cdot$ Circle

$A(h,b) = bh + \frac{1}{2}\pi(\frac{b}{2})^2$

$= bh + \frac{\pi b^2}{2 \cdot 4}$

$= bh + \frac{\pi b^2}{8}$

So $A(b) = A(\frac{20 - b(2+\pi)}{4}, b) = \frac{b}{1} \cdot \frac{[20 - b(2+\pi)]}{4} \cdot \frac{2}{2} + \frac{\pi b^2}{8}$

$= \frac{40b - 2b^2(2+\pi) + \pi b^2}{8} = \frac{40b - 4b^2 - 2\pi b^2 + \pi b^2}{8}$

$= \frac{40b - 4b^2 - \pi b^2}{8} = \frac{40b - b^2[4+\pi]}{8} = 5b - \frac{1}{8}b^2[4+\pi]$

Note $A'(b) = 5 - \frac{2}{8}b[4+\pi] = 5 - \frac{1}{4}b[4+\pi]$

Note $A'(b) = 0$ when

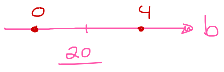
$5 - \frac{1}{4}b[4+\pi] = 0$

$5 = \frac{1}{4}b[4+\pi]$

$4 \cdot 5 = 4 \cdot \frac{1}{4}b[4+\pi]$

$20 = b[4+\pi]$

$b = \frac{20}{4+\pi}$



$A'(0) = 5 - \frac{1}{4}(0)[4+\pi]$

$= 5 - 0 = 5 > 0$

$A(b)$ is increasing

$A'(4) = 5 - \frac{1}{4}(4)[4+\pi]$

$= 5 - 1[4+\pi]$

$= 5 - 4 - \pi = 1 - \pi < 0$

$A(b)$ is decreasing

local maximum at $b = \frac{20}{4+\pi}$

Conclusion: The maximum area occurs when the base has length $\frac{20}{4+\pi}$ units

**Example 5:** Find the area of the largest rectangle with sides parallel to the coordinate axes that can be inscribed in a quarter circle of radius 10. Assume the center of the circle is located at the origin, and one corner of the rectangle is located at the origin and the opposite corner on the quarter circle.

Equation of a circle with center (0,0) and radius 10.

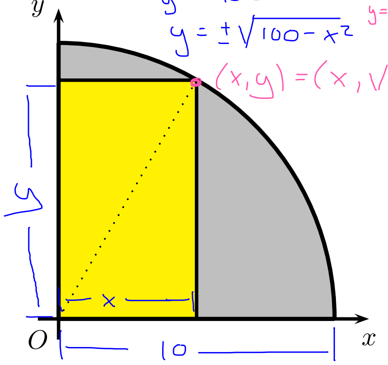
$x^2+y^2=10^2$

$y^2=100-x^2$

$y=\pm\sqrt{100-x^2}$

Top half $y=\sqrt{100-x^2}$

$(x,y)=(x,\sqrt{100-x^2})$



$A(x,y)=xy$ Note: $x,y\in[0,10]$

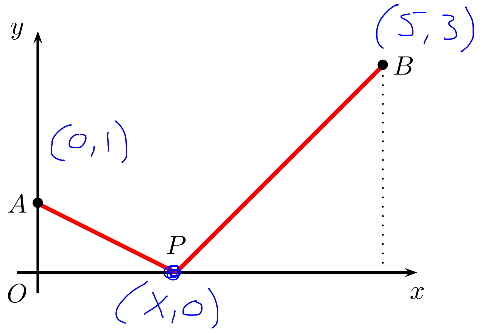
*work finished on next page

so $A(x)=A(x,\sqrt{100-x^2})=x\sqrt{100-x^2}$

so $A'(x)=1\cdot\sqrt{100-x^2}+x\left(\cancel{\tfrac{1}{2}}(100-x^2)^{-\frac{1}{2}}\cdot(-\cancel{2}x)\right)$

$$=\frac{\sqrt{100-x^2}}{1}\frac{\sqrt{100-x^2}}{\sqrt{100-x^2}}\frac{x^2}{\sqrt{100-x^2}}=\frac{100-x^2-x^2}{\sqrt{100-x^2}}=\frac{100-2x^2}{\sqrt{100-x^2}}$$

**Example 6:** Let $A$ be the point $(0,1)$ and let $B$ be the point $(5,3)$. Find the length of the shortest path that connects points $A$ and $B$ if the path must touch the $x$-axis. In other words, the path goes from point $A$ to somewhere (say $P$) on the $x$-axis, and then to $B$. (This is the 'line of sight' path from $A$ to $B$ if the $x$-axis is a mirror.) See the picture for a sketch of such a path.



Note $d(A,P)=\sqrt{(0-x)^2+(1-0)^2}=\sqrt{x^2+1}$

$d(P,B)=\sqrt{(x-5)^2+(0-3)^2}=\sqrt{(x-5)^2+9}$

Total Distance $=D(x)=\sqrt{x^2+1}+\sqrt{(x-5)^2+9}$

Note that $D'(x)$ is defined everywhere!

So $D'(x)=\cancel{\tfrac{1}{2}}(x^2+1)^{-\frac{1}{2}}\cdot(\cancel{2}x)+\cancel{\tfrac{1}{2}}((x-5)^2+9)^{-\frac{1}{2}}\cdot\cancel{2}(x-5)\cdot 1=\frac{x}{\sqrt{x^2+1}}+\frac{x-5}{\sqrt{(x-5)^2+9}}$

Never zero Never Zero

Note $D'(x)=0$ when

$$\frac{x}{\sqrt{x^2+1}}+\frac{x-5}{\sqrt{(x-5)^2+9}}=0$$

$$\frac{x}{\sqrt{x^2+1}}=\frac{-(x-5)}{\sqrt{(x-5)^2+9}}$$

$$x\sqrt{(x-5)^2+9}=-(x-5)\sqrt{x^2+1}$$

$$\left(x\sqrt{(x-5)^2+9}\right)^2=\left(-(x-5)\sqrt{x^2+1}\right)^2$$

$$x^2[(x-5)^2+9]=(x-5)^2(x^2+1)$$

$$\cancel{x^2(x-5)^2}+9x^2=\cancel{x^2(x-5)^2}+(x-5)^2$$

$9x^2=(x-5)^2$

$9x^2=x^2-5x-5x+25$

$9x^2=x^2-10x+25$

$0=-8x^2-10x+25$

$0=8x^2+10x-25$

$0=8x^2+20x-10x-25$

$0=4x(2x+5)-5(2x+5)$

$0=(4x-5)(2x+5)$

$4x-5=0$

$4x=5$

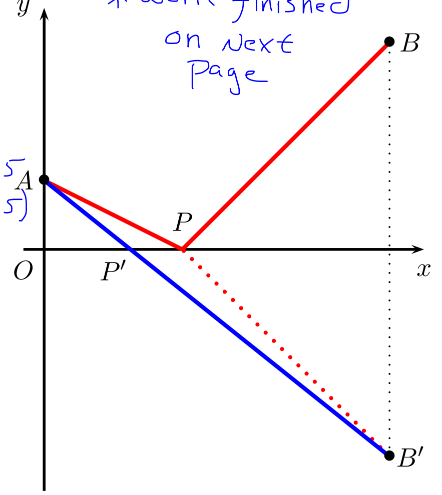
$x=\frac{5}{4}$

$2x+5=0$

$2x=-5$

$x=\frac{-5}{2}$

*work finished on next page



**Observation:** The $x$-coordinate of the point $P'$ that minimizes the line of sight path from $A(0,1)$ to $B(5,3)$ corresponds to the $x$-intercept of the line $y=-\frac{4}{5}x+1$ from $A$ to $B'(5,-3)$. Note that the coordinates of $P'$ are $(5/4,0)$. Can you understand why? Perhaps, the picture on the right will convince you.

# Example 5 (Continued)

Note: $A'(x)=0$ when

$100-2x^2=0$

$100=2x^2$

$\frac{100}{2}=\frac{\cancel{2}x^2}{\cancel{2}}$

$x^2=50$

$x=\pm\sqrt{50}$ however $x\geq 0$

so $x=\sqrt{50}\approx 7.071\ldots$

and $A'(x)=$ DNE when

$\sqrt{100-x^2}=0$

$0^2=100-x^2$

$0=100-x^2$

$x^2=100$

$x=\pm\sqrt{100}$ however $x\geq 0$

$x=\sqrt{100}=10$

Need to check:

1) Endpoints: $x=0,\ x=10$

2) $A'(x)=0$: $x=\sqrt{50}$

3) $A'(x)=$DNE: $x=10$ Same

Conclusion: The maximum area is 50 units$^2$

Recall $A(x)=x\sqrt{100-x^2}$

$A(0)=0\sqrt{100-0^2}=0$

$A(\sqrt{50})=\sqrt{50}\sqrt{100-(\sqrt{50})^2}$

$=\sqrt{50}\sqrt{100-50}$

$=\sqrt{50}\cdot\sqrt{50}$

$=50 \leftarrow$ maximum

$A(10)=10\sqrt{100-10^2}$

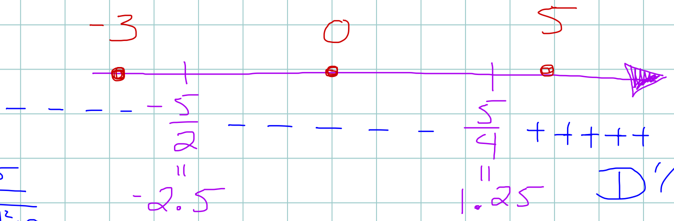
$=10\sqrt{100-100}$

$=10\sqrt{0}$

$=10\cdot 0=0$

# Example 6 (continued)

Recall: $D'(x)=\frac{x}{\sqrt{x^2+1}}+\frac{x-5}{\sqrt{(x-5)^2+9}}$



$D'(-3)=\frac{-3}{\sqrt{(-3)^2+1}}+\frac{-3-5}{\sqrt{(-3-5)^2+9}}$

= negative + negative = negative

so $D(x)$ is decreasing

$D'(0)=\frac{0}{\sqrt{0^2+1}}+\frac{0-5}{\sqrt{(0-5)^2+9}}$

= 0 + negative = negative

so $D(x)$ is decreasing

$D'(5)=\frac{5}{\sqrt{5^2+1}}+\frac{5-5}{\sqrt{(5-5)^2+9}}$

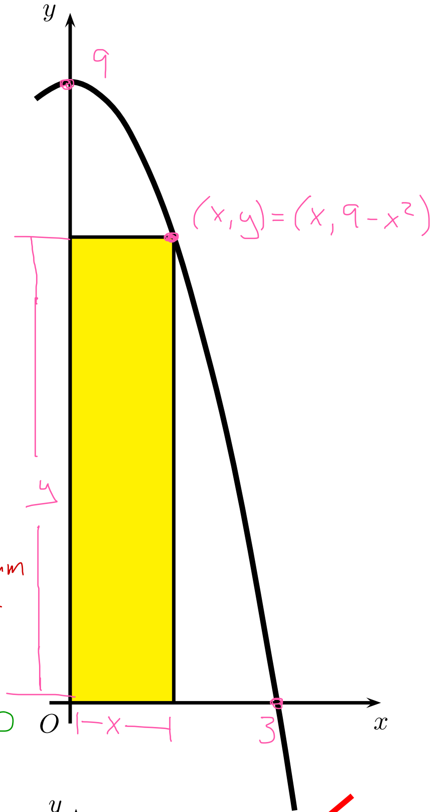
= positive + 0 = positive

So $D(x)$ is increasing

Conclusion: The minimum distance occurs when $x=\frac{5}{4}$ and this minimum distance is

$$D\left(\tfrac{5}{4}\right)=\sqrt{\left(\tfrac{5}{4}\right)^2+1}+\sqrt{\left(\tfrac{5}{4}-5\right)^2+9}\approx 6.40312\ldots$$

**Example 7:** Find the area of the largest rectangle with one corner at the origin, the opposite corner in the first quadrant on the graph of the parabola $f(x) = 9 - x^2$, and sides parallel to the axes.



$A(x,y) = x \cdot y$ note $x \in [0,3]$ and $y \in [0,9]$

So $A(x) = A(x, 9-x^2) = x(9-x^2) = 9x - x^3$

So $A'(x) = 9 - 3x^2 \leftarrow$ defined everywhere

Note $A'(x) = 0$ when

$9 - 3x^2 = 0$

$\frac{9}{3} = \frac{3x^2}{3}$

$3 = x^2$

$x = \pm\sqrt{3}$ however $x \geq 0$

So $x = \sqrt{3} \approx 1.7320...$

Need to Check:

1) Endpoints: $x = 0$ & $x = 3$

2) $A'(x) = 0$: $x = \sqrt{3}$

3) $A'(x) = DNE$: Nowhere

$A(0) = 9 \cdot 0 - 0^3 = 0$

$A(\sqrt{3}) = 9 \cdot \sqrt{3} - (\sqrt{3})^3$

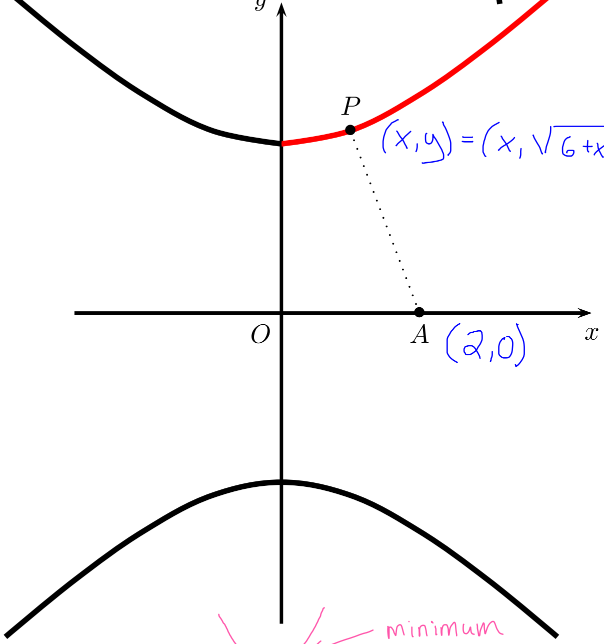
$= 9\sqrt{3} - 3\sqrt{3} = \boxed{3\sqrt{3}}$

Maximum area

$A(3) = 9 \cdot 3 - 3^3 = 27 - 27 = 0$

**Example 8:** Find the point $P$ in the first quadrant that lies on the hyperbola $y^2 - x^2 = 6$ and is closest to the point $A(2, 0)$. If we write the point as $P(a, b)$, then

$a =$ 1 and $b =$ $\sqrt{7}$.



$y^2 - x^2 = 6$

$y^2 = 6 + x^2$

$y = \pm\sqrt{6+x^2}$

Top half $y = \sqrt{6+x^2}$

$D(P,A) = \sqrt{(x-2)^2 + (0 - \sqrt{6+x^2})^2} = \sqrt{x^2 - 4x + 4 + (6 + x^2)}$

$= \sqrt{2x^2 - 4x + 10}$

So $D(x) = \sqrt{2x^2 - 4x + 10}$

So $D'(x) = \frac{1}{2}(2x^2 - 4x + 10)^{-\frac{1}{2}}(4x - 4)$

$= \frac{4x-4}{2\sqrt{2x^2-4x+10}} = \frac{\cancel{2}(2x-2)}{\cancel{2}\sqrt{2x^2-4x+10}} = \frac{2x-2}{\sqrt{2x^2-4x+10}}$

Note $D'(x) = 0$ when

$2x - 2 = 0$

$2x = 2$

$\frac{\cancel{2}x}{\cancel{2}} = \frac{2}{2}$

$x = 1$

and $D'(x) = DNE$ when

$\sqrt{2x^2 - 4x + 10} = 0$

$0^2 = 2x^2 - 4x + 10$

$0 = x^2 - 2x + 5$

Discriminant $= (-2)^2 - 4(1)(5)$

$= 4 - 20$

$= -16 < 0$

So No real solutions

minimum

0    1    2

$- - - - \; | \; + + + +$

$D'(0) = \frac{2 \cdot 0 - 2}{\sqrt{2 \cdot 0^2 - 4 \cdot 0 + 10}} = \frac{-2}{\sqrt{10}} < 0$

So $D(x)$ is decreasing on $(-\infty, 1)$

$D'(2) = \frac{2 \cdot 2 - 2}{\sqrt{2 \cdot 2^2 - 4 \cdot 2 + 10}} = \frac{4-2}{\sqrt{8-8+10}} = \frac{2}{\sqrt{10}} > 0$

So $D(x)$ is increasing on $(1, \infty)$

Conclusion:

The minimum distance from A to P occurs when the x coordinate of P is one.

Therefore, $P = (1, \sqrt{6+1^2}) = (1, \sqrt{7})$

# RELATED RATE PROBLEMS

▶ **Overall philosophy and recommended notation:** In a related rate problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). It is almost always better to use Leibniz's notation $\frac{dy}{dt}$, if we are differentiating, for instance, the function $y$ with respect to time $t$. The $y'$ notation is more ambiguous when working with rates and should therefore be avoided.

▶ **Implicit derivatives:** Imagine you drop a rock in a still pond. This will cause expanding circular ripples in the pond. The area of the outer circle depends on the radius $r$ of the perturbed area:

$$A = \pi r^2.$$

The radius of the outer circle depends on the amount of time $t$ that has elapsed since you dropped the rock. Thus, the area also depends on time. In conclusion, it makes sense to find the rate of change of the area with respect to time and relate it to the rate of change of the radius with to time. We call it an *implicit derivative* as the function $A$ is not explicitly given in terms of $t$...but only implicitly. We need the chain rule to do this.

▶ **Quick review of the chain rule:** Typically, we are given $y$ as a function of $u$ and $u$ as a function of $x$, so that we can think of $y$ as a function of $x$ also. The chain rule then says that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

**Example 9:** Consider the area of a circle $A = \pi r^2$ and assume that $r$ depends on $t$. Find a formula for $\frac{dA}{dt}$.

By the chain rule $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$ and note since $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

therefore, $\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

▶ **Related rate guideline:** This guideline is found on pp. 143-144 of our textbook.

(1.) Read the problem quickly. ← slowly
(2.) Read the problem carefully.
(3.) Identify the variables. Note that time is often an understood variable. If the problem involves geometry, draw a picture and label it. Label anything that does not change with a constant. Label anything that does change with a variable.
(4.) Write down which derivatives you are given. Use the units to help you determine which derivatives are given. The word "per" often indicates that you have a derivative.
(5.) Write down the derivative you are asked to find. "How fast..." or "How slowly..." indicates that the derivative is with respect to time.
(6.) Look at the quantities whose derivatives are given and the quantity whose derivative you are asked to find. Find a relationship between all of these quantities.
(7.) Use the chain rule to differentiate the relationship.
(8.) Substitute any particular information the problem gives you about values of quantities at a particular instant and solve the problem. To find all of the values to substitute, you may have to use the relationship you found in step **6**. Take a snapshot of the picture at the particular instant.

**Example 10:** Boyle's Law states that when a sample gas is compressed at a constant temperature, the pressure $P$ and volume $V$ satisfy the equation $PV = c$, where $c$ is a constant. Suppose that at a certain instant the volume is 600 cm$^3$, the pressure is 150 kPa, and the pressure is increasing at a rate of 20 kPa/min. At what rate is the volume decreasing at this instant?

$V = 600\,\text{cm}^3$ Also given: $\frac{dP}{dt} = 20\,\frac{\text{KPa}}{\text{min}}$ Consequently, $\left.\frac{dV}{dt}\right|_{P=150} = \frac{-90000}{150^2}\,20 = \boxed{-80\,\frac{\text{cm}^3}{\text{min}}}$

$P = 150\,\text{kPa}$

$PV = c$

$$\begin{array}{c} V \\ | \\ P \\ | \\ t \end{array}$$

So by the chain rule $\frac{dV}{dt} = \frac{dV}{dP} \cdot \frac{dP}{dt}$

this means the volume is decreasing by 80 cubic cm per a min

$600 \cdot 150 = c$

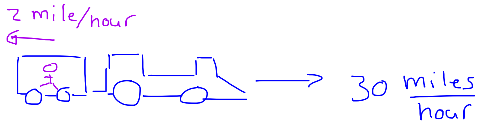
$90{,}000 = c$

so $\frac{dV}{dt} = \frac{-90{,}000}{P^2} \cdot 20$

So $PV = 90{,}000$

$V = \frac{90{,}000}{P} = 90{,}000P^{-1}$ therefore $\frac{dV}{dP} = -90{,}000P^{-2} = \frac{-90{,}000}{P^2}$

**Example 11:** A train is traveling over a bridge at 30 miles per hour. A man on the train is walking toward the rear of the train at 2 miles per hour. How fast is the man traveling across the bridge in miles per hour?



Consequently, the man is traveling across the bridge in $30-2 = \boxed{28\text{ mph}}$

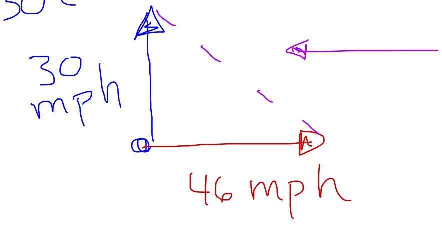
**Example 12:** Two trains leave a station at the same time. One travels north on a track at 30 mph. The second travels east on a track at 46 miles per hour. How fast are they traveling away from one another in miles per hour when the northbound train is 60 miles from the station?

2 hours from departure

Consequently, $\frac{dd}{dt} = \sqrt{3016}$

$\approx 54.918$ mph

$s_1(t) = 30t$



$s_2(t) = 46\,t$

$d(t) = \sqrt{(s_1(t))^2 + (s_2(t))^2}$

$= \sqrt{(30t)^2 + (46t)^2}$

$= \sqrt{900t^2 + 2116t^2}$

$= \sqrt{3016t^2}$

$= \sqrt{3016}\ t$

## Example 10: Implicit differentiation

$$PV = c$$

$$\frac{dP}{dt} \cdot V + P \cdot \frac{dV}{dt} = 0$$

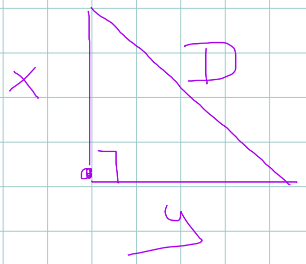
$$20 \cdot 600 + 150 \cdot \frac{dV}{dt} = 0$$

$$1200 + 150 \cdot \frac{dV}{dt} = 0$$

$$150 \frac{dV}{dt} = -1200$$

$$\frac{dV}{dt} = \frac{-1200}{150} = -80 \ \frac{cm^3}{min}$$

## Example 12: Implicit differentiation



Recall we are given
$\frac{dx}{dt} = 46$ mph
and $\frac{dy}{dt} = 30$ mph

$$D^2 = x^2 + y^2$$

$$2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}$$

$$D \frac{dD}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$$

so $$\frac{dD}{dt} = \frac{x \frac{dx}{dt} + y \frac{dy}{dt}}{D}$$

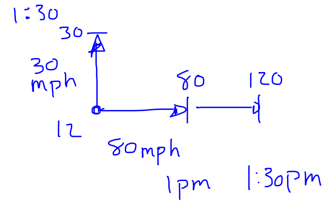
Note when $t = 2$
$x(2) = 46 \cdot 2 = 92$
$y(2) = 30 \cdot 2 = 60$

So $D^2 = 92^2 + 60^2$
$D^2 = 8464 + 3600$
$D^2 = 12064$
$D = \pm\sqrt{12064}$ However $D > 0$
$D = \sqrt{12064}$

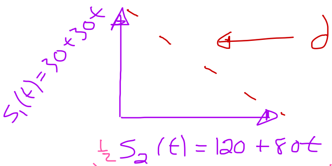
$$\frac{dD}{dt} = \frac{92 \cdot 46 + 60 \cdot 30}{\sqrt{12064}}$$

$$\frac{dD}{dt} = \frac{4232 + 1800}{\sqrt{12064}} = \frac{6032}{\sqrt{12064}} \approx \boxed{54.92 \text{ mph}}$$

**Example 13:** Two trains leave a station at 12:00 noon. One travels north on a track at 30 mph. The second travels east on a track at 80 miles per hour. At 1:00 PM the northbound train stops for one-half hour at a station while the eastbound train continues at 80 miles per hour without stopping. At 1:30 PM the northbound train continues north at 30 mph. How fast are the trains traveling away from one another at 2:00 PM?



Let $t$ be the time (in hours) from 1:30PM



$s_1(t)=30+30t$

$\frac{1}{2} S_2(t)=120+80t$

$$d(t)=\sqrt{(S_1(t))^2+(S_2(t))^2}$$

$$=\sqrt{(30+30t)^2+(120+80t)^2}$$

$$=\sqrt{900+1800t+900t^2+14400+19200t+6400t^2}$$

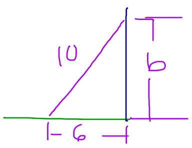
$$=\sqrt{7300t^2+21{,}000t+15{,}300}$$

$$d'(t)=\frac{1}{2}(7300t^2+21000t+15300)\cdot(14600t+21000)$$

$$d'(t)=\frac{14600t+21{,}000}{2\sqrt{7300t^2+21{,}000t+15{,}300}}$$

Need to compute $d'(\frac{1}{2})=\dfrac{14600(\frac{1}{2})+21{,}000}{2\sqrt{7300(\frac{1}{2})^2+21000(\frac{1}{2})+15300}}=\dfrac{7300+21000}{2\sqrt{7300(\frac{1}{4})+10{,}500+15{,}300}}=\dfrac{28{,}300}{2\sqrt{27625}}\approx\boxed{85.134 \text{ mph}}$

**Example 14:** A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 feet/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?



$6^2+b^2=10^2$

$b^2=100-36$

$b^2=64$

$b=\pm\sqrt{64}=\pm 8$ However $b \geq 0$

$b=8$

Note: y
|
x
|
t

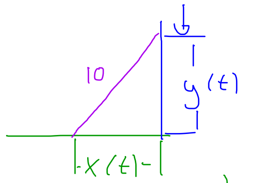
By the chain Rule

$$\frac{dy}{dt}=\frac{dy}{dx}\cdot\frac{dx}{dt}$$

$$\frac{dy}{dt}=\frac{-x}{\sqrt{100-x^2}}\cdot\frac{dx}{dt}$$

$$\left.\frac{dy}{dt}\right|_{x=6}=\frac{-6}{\sqrt{100-6^2}}\cdot 1=\frac{-6}{8}=-\frac{3}{4}\text{ ft/sec}$$

Conclusion: when the ladder is 6 feet from the wall the ladder slides down the wall at a rate of $\frac{3}{4}$ feet/sec



← 1 ft/sec $=\frac{dx}{dt}$

$(x(t))^2+(y(t))^2=10^2$

$(y(t))^2=100-(x(t))^2$

$y(t)=\pm\sqrt{100-(x(t))^2}$ However $y \geq 0$

$y(t)=\sqrt{100-(x(t))^2}$

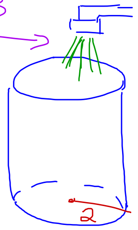
$y(x)=\sqrt{100-x^2}$

$$\frac{dy}{dx}=\frac{1}{\cancel{2}}(100-x^2)^{-\frac{1}{2}}(-\cancel{2}x)$$

$$\frac{dy}{dx}=\frac{-x}{\sqrt{100-x^2}}$$

**Example 15:** A cylindrical water tank with its circular base parallel to the ground is being filled at the rate of 4 cubic feet per minute. The radius of the tank is 2 feet. How fast is the level of the water in the tank rising when the tank is half full? Give your answer in feet per minute.

$\frac{dV}{dt}=\frac{4\text{feet}^3}{\text{min}}$



$V=\pi r^2\cdot h$

$V=\pi 2^2\cdot h=4\pi h$

So $\dfrac{dV}{dh}=4\pi$

Note:
V
|
h
|
t

By the chain Rule

$$\frac{dV}{dt}=\frac{dV}{dh}\cdot\frac{dh}{dt}$$

$$\frac{\cancel{4}}{\cancel{4\pi}}=\frac{4\pi\cdot\frac{dh}{dt}}{4\pi}$$

$$\boxed{\frac{dh}{dt}=\frac{1}{\pi}\frac{\text{feet}}{\text{min}}}$$

## Example 13: Implicit differentiation

The formula from example 12 still holds

$$\frac{dD}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{D} = \frac{160\cdot 80 + 45\cdot 30}{\sqrt{27625}} = \frac{12800 + 1350}{\sqrt{27625}}$$

$$= \frac{14150}{\sqrt{27625}} = \boxed{85.134 \text{ mph}}$$

Note that at 2:00 PM

$y = 30 + 30\cdot\frac{1}{2} = 30 + 15 = 45$ miles

$x = 120 + 80\cdot\frac{1}{2} = 120 + 40 = 160$ miles

so $D^2 = 45^2 + 160^2$

$D^2 = 2025 + 25600$

$D^2 = 27625$

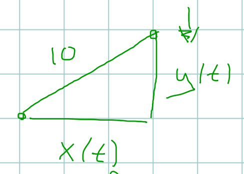
$D = \pm\sqrt{27625}$ However $D > 0$

$D = \sqrt{27625}$

Also note $\frac{dx}{dt} = 80$ mph

and $\frac{dy}{dt} = 30$ mph

## Example 14: Implicit differentiation



$\leftarrow 1$ ft/sec $= \frac{dx}{dt}$

$$x^2 + y^2 = 10^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \xrightarrow[2]{\text{Divide by}} x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

Recall when

$x = 6$ we showed

$y = 8$

$$6\cdot 1 + 8\cdot\frac{dy}{dt} = 0$$

$$6 + 8\cdot\frac{dy}{dt} = 0$$

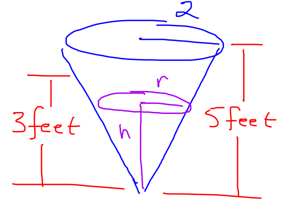
$$8\frac{dy}{dt} = -6$$

$$\frac{dy}{dt} = \frac{-6}{8} = \boxed{\frac{-3}{4}\ \frac{\text{feet}}{\text{sec}}}$$

$\frac{dV}{dt} = -3 \ \text{feet}^3/\text{min}$

**Example 16:** A conical salt spreader is spreading salt at a rate of 3 cubic feet per minute. The diameter of the base of the cone is 4 feet and the height of the cone is 5 feet. How fast is the height of the salt in the spreader decreasing when the height of the salt in the spreader (measured from the vertex of the cone upward) is 3 feet? Give your answer in feet per minute. (It will be a positive number since we use the word "decreasing".)

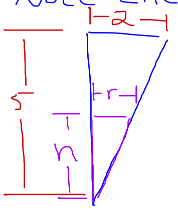
$\left.\frac{dh}{dt}\right|_{h=3}$



$V = \frac{1}{3}\pi r^2 h \xrightarrow{r = \frac{2}{5}h} V = \frac{1}{3}\pi\left(\frac{2h}{5}\right)^2 \cdot h = \frac{1}{3}\pi \frac{4h^2}{25} \cdot h$

$= \frac{4\pi h^3}{75}$

Note the similar triangles



$\frac{5}{2} = \frac{h}{r}$

$5r = 2h$

$r = \frac{2}{5}h$

so $\frac{dV}{dh} = \frac{12\pi h^2}{75} = \frac{4\pi h^2}{25}$

Note

V
|
h
|
t

By the chain Rule

$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

$-3 = \frac{4\pi h^2}{25} \cdot \frac{dh}{dt} \longrightarrow \frac{dh}{dt} = \frac{25}{4\pi h^2} \cdot -3$

so $\left.\frac{dh}{dt}\right|_{h=3} = \frac{-25 \cdot \cancel{3}}{4\pi(3)^{\cancel{2}}} = \frac{-25}{12\pi} \ \frac{\text{feet}}{\text{min}}$

Conclusion: When the height is 3 feet the height is decreasing at a rate of $25/12\pi$ feet/min

**Example 17:** It is estimated that the annual advertising revenue received by a certain newspaper will be

$$R(x) = 0.5x^2 + 3x + 160$$

thousand dollars when its circulation is $x$ thousand. The circulation of the paper is currently 10,000 and is increasing at a rate of 2,000 papers per year. At what rate will the annual advertising revenue be increasing with respect to time 2 years from now?

Constant

$\left.\frac{dR}{dt}\right|_{t=2}$

Note : $\frac{dx}{dt} = 2{,}000 \ \frac{\text{papers}}{\text{year}}$

R
|
x
|
t

$x(0) = 10{,}000$

$x(1) = 12{,}000$

$x(2) = 14{,}000$ ← 14 thousand

$R(x) = \frac{1}{2}x^2 + 3x + 160$

$\frac{dR}{dx} = \frac{1}{2} \cdot 2x + 3$

$\frac{dR}{dx} = x + 3$

By the chain rule

$\frac{dR}{dt} = \frac{dR}{dx} \cdot \frac{dx}{dt}$

so $\frac{dR}{dt} = (x+3) \cdot \frac{dx}{dt}$

$\left.\frac{dR}{dt}\right|_{t=2} = (x+3)\Big|_{t=2} \cdot \left.\frac{dx}{dt}\right|_{t=2}$

$\left.\frac{dR}{dt}\right|_{t=2} = (x(2)+3) \cdot 2 = (14+3) \cdot 2 = 17 \cdot 2 = 34$

Conclusion: The Revenue increases at a rate of \$34,000 per year when $t = 2$ years.

**Example 18:** A stock is increasing in value at a rate of 10 dollars per share per year. An investor is buying shares of the stock at a rate of 26 shares per year. How fast is the value of the investor's stock growing when the stock price is 50 dollars per share and the investor owns 100 shares? (**Hint:** Write down an expression for the total value of the stock owned by the investor.

Let p = price of stock per a share $\longrightarrow \frac{dp}{dt} = 10$

n = # of shares owned by the investor $\longrightarrow \frac{dn}{dt} = 26$

V = total value of the investor's stock

Want to find $\frac{dV}{dt}$

Also given p = \$50 per share and n = 100 shares

Note: $V = n\cdot p$

$$\frac{dV}{dt} = \frac{dn}{dt}\cdot p + n\cdot\frac{dp}{dt}$$

$$\frac{dV}{dt} = 26\cdot 50 + 100\cdot 10 = 1,300 + 1,000 = 2,300$$

Conclusion: $\frac{dV}{dt}$ = \$2,300 per a year when the value of the stock is \$50 and the investor has 100 shares of stock

**Example 19:** Suppose that the demand function $q$ for a certain product is given by

$$q = 4,000\, e^{-0.01\cdot p},$$

where $p$ denotes the price of the product. If the item is currently selling for \$100 per unit, and the quantity supplied is decreasing at a rate of 80 units per week, find the rate at which the price of the product is changing.

p = \$100

$\frac{dq}{dt} = -80$

Want $\frac{dp}{dt}$

$$\begin{array}{c} q \\ | \\ p \\ | \\ t \end{array}$$

By the chain rule

$$\frac{dq}{dt} = \frac{dq}{dp}\cdot\frac{dp}{dt}$$

Note $q = 4,000\, e^{-0.01p}$

then $\frac{dq}{dt} = 4000\, e^{-0.01p}(-0.01)$

so $\frac{dq}{dt} = -40\, e^{-0.01p}$

$$\frac{-80}{-40e^{-0.01p}} = \frac{-40e^{-0.01p}\cdot\frac{dp}{dt}}{-40e^{-0.01p}}$$

$$\frac{2}{e^{-0.01p}} = \frac{dp}{dt}$$

So $\frac{dp}{dt} = 2e^{0.01p}$

therefore $\left.\frac{dp}{dt}\right|_{p=100} = 2\,e^{0.01(100)} = 2e \approx \$5.44$ per a week