Chapter I §2, 3, 4: Introduction

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Equations from §1.1

We recall equations discussed in §1.1.

- Falling Object Models:

$$m\frac{dv}{dt} = mg - \gamma v \tag{1}$$

$$10\frac{dv}{dt} = 98 - 2v \quad or \quad \frac{dv}{dt} = 9.8 - .2v \tag{2}$$

Continued

- Population Growth Model:

$$\frac{dp}{dt} = rp \tag{3}$$

$$\frac{dp}{dt} = .5p - 450 \tag{4}$$

- General First Order Equations:

$$\frac{dy}{dt} = f(t, y) \quad \text{where } f \text{ is a function of } t, y. \tag{5}$$

The equations we discussed in section §1.1 are fairly simple, for the following sense:

- All the DEs are of the form (5): $\frac{dy}{dt} = f(t, y)$. It involves only first derivative; and no higher order derivatives.
- For these DEs (1, 2, 3), the right side $f(t, y)$ are linear.
- Solving such DEs (5), mainly, involves nothing more than revisiting antiderivatives.

Solving the Growth Model

- We solve the population growth model (4)

$$\frac{dp}{dt} = .5p - 450 \quad \Longrightarrow \frac{dp}{.5p - 450} = dt \qquad (6)$$

- $\int \frac{dp}{.5p-450} = \int dt + C$, where C is an arbitrary constant.
- Substituting $u = .5p - 450$ we get

$$\frac{du}{u} = .5 \int dt + C \quad Or \quad \ln|u| = .5t + C$$

$$|.5p - 450| = e^{.5t+C} = ce^{.5t} \quad Or \quad p = 900 + ce^{.5t}$$

wher $c := E^C$ is an arbitrary constant.

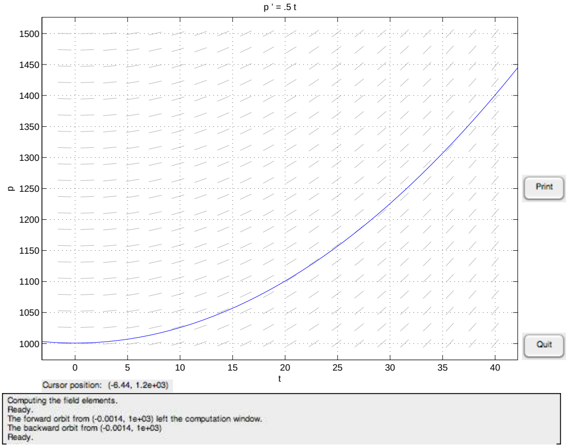
Initial Value

- $p = 900 + ce^{.5t}$ is a solution of (6), for all values of c. This solution, would be called the General solution
- In the absence of additional information, we cannot determine the value of c.
- Such extra information is provided, often, by giving the population size $p(t_0)$ at a particular time t_0. For example, it may be given that $p(0) = 1000$. Such information, is called an initial value.

- In case, $p(0) = 1000$, we have

$$1000 = p(0) = 900 + c, \quad c = 100$$

  Finally, our particular solution is $p = 900 + 100e^{.5t}$
- In the next frame, compare the direction fields of the DE (4), with this solution $p = 900 + 100e^{.5t}$.


p ' = .5 t
p
t
Print
Quit
Cursor position: (-6.44, 1.2e+03)
Computing the field elements.
Ready.
The forward orbit from (-0.0014, 1e+03) left the computation window.
The backward orbit from (-0.0014, 1e+03)
Ready.

Solving such general equations

More generally, consider the initial value problem:

$$\begin{cases} \frac{dy}{dt} & = ay - b \\ y(0) & = y_0 \end{cases} \qquad a, b \quad \text{are constants, and} \tag{7}$$

y_0 is (an) initial value of y, at time $t = 0$.

continued

(Trivial cases):

- If $a = 0$ then the equation is rewritten as

$$\begin{cases} \frac{dy}{dt} & = -b \\ y(0) & = y_0 \end{cases} \qquad Solution : exercise$$

- If $ay - b =$ then, we have

$$\begin{cases} \frac{dy}{dt} & = 0 \\ y(0) & = y_0 \end{cases} \qquad Solution : exercise \ (Answer : y = y_0)$$

continued

(The Non-Trivial case):

$$\begin{cases} \frac{dy}{dt} & = ay - b \\ y(0) & = y_0 \end{cases} \qquad a \neq 0, ay - b \neq 0 \qquad (8)$$

We proceed as in the growth model equation:

- We have $\frac{dy}{ay-b} = dt$. So, $\int \frac{dy}{ay-b} = \int dt + C$, where C is an arbitrary constant.
- So,

$$\int \frac{dy}{y - \frac{b}{a}} = a \int dt + C \Longrightarrow \ln \left| y - \frac{b}{a} \right| = at + C$$

continued

- Taking exponential: The general solution of (8) is: $y - \frac{b}{a} = ce^t \qquad where \quad c = \pm e^C$ is also arbitrary
- $c = 0$ corresponds to the equilibrium solution $y = \frac{b}{a}$.
- Using the initial value $y(0) = y_0$: $y_0 - \frac{b}{a} = c$
- So, the final solution of the initial value problem (8) is: $y - \frac{b}{a} = \left[y_0 - \frac{b}{a}\right] e^{at}$. Which is

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a}\right] e^{at} \tag{9}$$

Assignments and Homework

- Homework: §1.2 Exercise 12, 13, 15, 17, 18

Exercise 12, pp. 17

- $Q(t) =$ mass, at time t of some radio-active substance.
- Let the rate of disintegration, at time t be $\frac{dQ}{dt}$.
- It is known that such substances disintigrates at a rate proportional to the current mass $Q(t)$.
- That means $\frac{dQ}{dt} = -rQ(t)$. For some constant $r > 0$.
- We are also given $Q(0) = 100\ mg$ and $Q(7) = 82.04$ (Unit of time used is "days").

Continued

- Use equation 8 and solution 9, with $b = 0, a = -r, \ Q(0) = 100$. We have
$$Q(t) = 100e^{-rt}$$
- Now use $Q(1) = 82.04$. We get $82.04 = 100e^{-7r}$. So, $-7r = ln(.8204) = -.1980$. So, $r = .02828$.
- So, $Q(t) = 100^{-.02828t}$. In next frame compare with the direction fields of $Q' = -.02828Q(t)$.

To Be added!

HW solution: Ex 11

- The model of the falling body DE (1) was modified, by changing hypothesis on drag. Drag was assumed to be proportional to velocity. Now, the drag is assumed to be proportional to square of the velocity. That means drag$= \gamma v^2$.
- So, the new model DE is

$$m\frac{dv}{dt} = mg - \gamma v^2 \tag{10}$$

Continued: Ex 11

- $m = 10$ kg and drag constant $\gamma = 2$ kg/s the DE changes to

$$10\frac{dv}{dt} = 98 - 2v^2 \quad or \quad \frac{dv}{dt} = -.2(49 - v^2) \qquad (11)$$

  I seem to get a somethign different from the author.
- Other information (as in example 2) height from the ground is 300 meter and $v(0) = 0$

Continued: Ex 11

- We separate variables (see §2.3):

$$\int \frac{dv}{v^2-49} = -\int .2dt + c \quad \Longrightarrow$$

$$\int \frac{1}{14}\left(\frac{1}{v-7} - \frac{1}{v+7}\right) dv = -.2t + c \Longrightarrow$$

- $$\frac{1}{14}\ln\left|\frac{v-7}{v+7}\right| = -.2t + c \Longrightarrow$$

Continued: Ex 11

- $$\left|\frac{v-7}{v+7}\right| = Ce^{-2.8t} \quad \text{with} \qquad C = e^{14c} > 0$$
- So,
$$\frac{v-7}{v+7} = Ce^{-2.8t} \quad \text{with} \qquad -\infty < C\infty$$
- Substituting $v(0) = 0$ we have $C = -1$

Continued: Ex 11

- So, the solution is given by

$$\frac{v-7}{v+7} = -e^{-2.8t} \Longrightarrow v = 7\frac{1-e^{-2.8t}}{1+e^{-2.8t}} = 7 + 14\frac{e^{-2.8t}}{1+e^{-2.8t}}$$

- Next Level: Let $x = x(t)$ denote the distance, from the point of drop, at time t. So,

$$\frac{dx}{dt} = v = 7 + 14\frac{e^{-2.8t}}{1+e^{-2.8t}}$$

Continued: Ex 11

- So,

$$x = \int \left(7 + 14\frac{e^{-2.8t}}{1+e^{-2.8t}}\right) dt + c \Longrightarrow$$

$$x = 7t - 5\ln(1 + e^{-2.8t}) + c$$

- With initial condition $x(0) = 0$ we have $c = 5\ln 2$. So,

$$x = 7t - 5\ln(1 + e^{-2.8t}) + 5\ln 2$$

- For the part (f): $x(T) = 300$. So,

$$300 = 7T - 5\ln(1 + e^{-2.8T}) + 5\ln 2$$

Solve for T.

Ex 18: Concentration

- $Q(t)$ =quatitly of the chemical in the pond.
- So, $Q(0) = 0$.
- Part a): The rate of change

$$\frac{dQ}{dt} = .01 * 300 - \frac{300Q(t)}{10^6} = 3\left(1 - 10^{-4}Q(t)\right)$$

Ex 18: Continued

- So,

$$\frac{dQ}{10^{-4}Q-1} = -3dt \Longrightarrow 10^4 \ln\left|10^{-4}Q-1\right| = -3t+c$$

$$\left|10^{-4}Q-1\right| = Ce^{-\frac{3t}{10^4}} \Longrightarrow Q(t) = 10^4\left(1+Ce^{-\frac{3t}{10^4}}\right)$$

- Now, $Q(0)=0$ implies $C=-1$.
- So,

$$Q(t) = 10^4\left(1-e^{-\frac{3t}{10^4}}\right)$$

Ex 18: Continued

- Part c) After one year $t = 365824 = 8760$ and

$$Q(8760) = 10^4 \left(1 - e^{-\frac{3*8760}{10^4}}\right) = 9277g$$

Ex 18: Continued

- Part c) For the next part, we reset $t = 0$ and $Q(0) = 9277$. Also,

$$\frac{dQ}{dt} = -300 * \frac{Q(t)}{10^6} == -\frac{3Q}{10^4}$$

- So,

$$Q(t) = 9277e^{-\frac{3t}{10^4}}$$

Classification based on no of ind. variables

Two broad classifications of DEs are as follows:

- When a DE involves only a single independent variable x (or t), then it is called an Ordinary DE (also called ODE). Chapter 2, 3 would be on ODE.
- When a DE involves more than one independent variables $x_1, x_2, \ldots, x_n$, then it is called a Partial DE (also called PDE). PDEs will not be covered in this course.

Classification based on number of unknown variables

- There may only be one unknown dependent variable y, to be determined. As in linear algebra, only one DE (plus initial value) is needed to determine y.
- There may also be more than one unknown dependent variables $y_1, y_2, \ldots, y_m$, to be determined. As in linear algebra, a system of m (independent, in some sense) DE (plus initial values) are needed to determine $y_1, y_2, \ldots, y_m$. They will be called a System of DEs. We will consider such systems in chapter 7.

Based on Order of derivatives

- DEs can be classified based on highest order of derivation present. We will cover
  - First order DE (Chapter 2)
  - Second order DE (Chapter 3)

Linearity and non-linearity

- An ODE of order n is called linear, if it looks like

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1} y}{dt^{n-1}} + \cdots a_{n-1}(t)\frac{dy}{dt} + a_n(t)y = g(t)$$

This is also written as:

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots a_{n-1}(t)y^{(1)} + a_n(t)y = g(t)$$

$a_i(t), g(t)$ are functions of the ndependent variable t.

Historical Remarks

Read the section once.