

**Economics 352: Intermediate Microeconomics****Notes and Sample Questions****Chapter 9: Profit Maximization*****Profit Maximization***

The basic assumption here is that firms are profit maximizing.

Profit is defined as:

$$\text{Profit} = \text{Revenue} - \text{Costs}$$

$$\Pi(q) = R(q) - C(q)$$

$$\Pi(q) = p(q) \cdot q - C(q)$$

To maximize profits, take the derivative of the profit function with respect to  $q$  and set this equal to zero. This will give the quantity ( $q$ ) that maximizes profits, assuming of course that the firm has already taken steps to minimize costs.

$$\frac{d\Pi}{dq} = \frac{dR}{dq} - \frac{dC}{dq} = 0$$
$$\frac{dR}{dq} = \frac{dC}{dq}$$

or, put slightly differently, the profit maximizing condition is for marginal revenue to equal marginal cost:

$$MR = MC$$

Or, put slightly differently, the additional revenue gained by making and selling one additional unit should be equal to the extra cost incurred to make and sell an extra unit.

The second order conditions, or the condition on the second derivative here, is that the second derivative of profit with respect to quantity be zero:

$$\frac{d^2\Pi}{dq^2} < 0$$

Example: Imagine that a firm has costs given by  $C(q)=120 + 2q^2$  and revenues given by  $R(q)=100q$ , equivalent to saying that the firm sells at a market price of \$100. The profit maximizing quantity is given by:

$$\Pi(q) = 100q - 120 - 2q^2$$

$$\frac{d\Pi}{dq} = 100 - 4q = 0$$

$$q^* = 25.$$

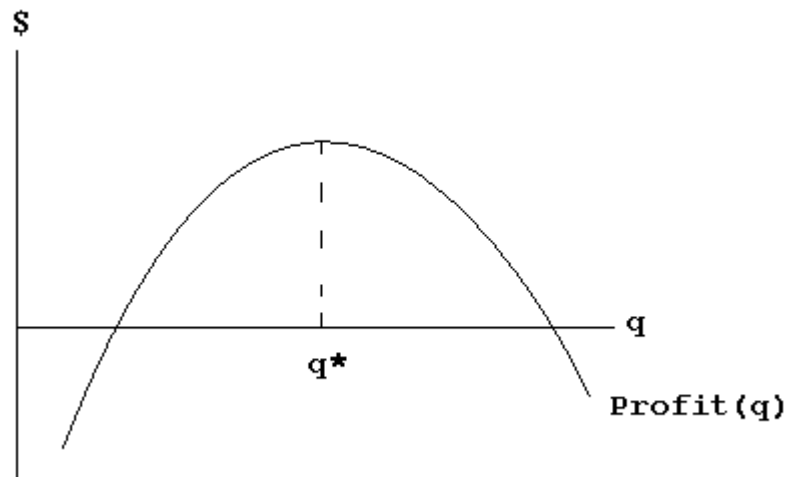
Example: Imagine that a firm has costs given by  $C(q)=420 + 3q + 4q^2$  and revenues given by  $R(q)=100q - q^2$ . The profit maximizing quantity is given by:

$$\Pi(q) = 100q - q^2 - 420 - 3q - 4q^2$$

$$\frac{d\Pi}{dq} = 100 - 2q - 3 - 8q = 0$$

$$q^* = 9.7.$$

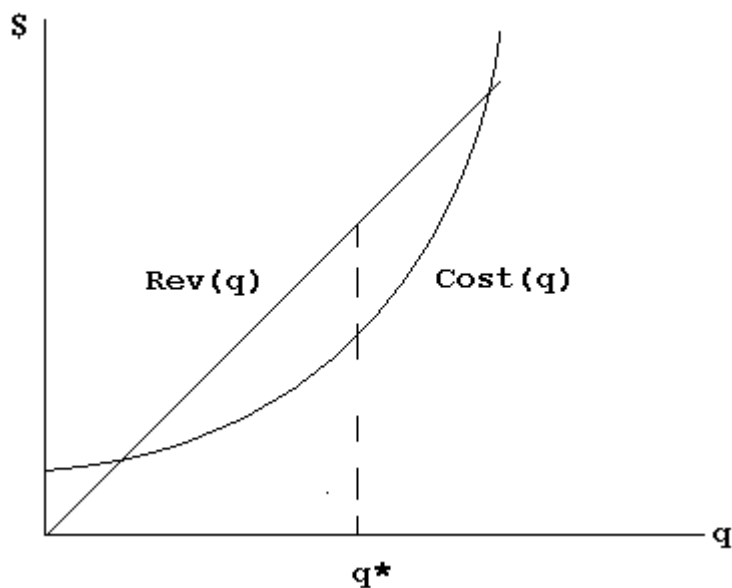
In a picture, this all looks like:



*A graph showing a profit curve that has an inverted U-shape and has a peak at the profit maximizing quantity.*

Profit is maximized at the quantity  $q^*$  and is lower at all other quantities. The curvature of the profit function is consistent with a negative second derivative and results in  $q^*$  being a quantity of maximum profit.

This can also be expressed in terms of the revenue and cost functions separately:



*A graph showing a revenue curve and a cost curve, with the profit maximizing quantity being that quantity where the vertical difference between the two is maximized. This is also the quantity where the two curves have the same slope.*

The profit maximizing quantity is where the revenue function and the cost function have the same slope and where the distance between them is maximized.

The condition that the two functions have the same slope is the same as saying that marginal revenue equals marginal cost.

### ***Marginal Revenue***

Revenue is equal to price multiplied by quantity.

In the most general case, price is a function of the quantity of the good that the firm sells. So, revenue is:

$$R(q) = p(q) \cdot q$$

and marginal revenue is the derivative of this with respect to  $q$ :

$$MR(q) = \frac{dR}{dq} = p(q) + q \cdot \frac{dp(q)}{dq}$$

Example: Imagine that demand is given by  $q = 80 - 2p$ . To calculate the marginal revenue function, we need to rewrite this so that price is a function of quantity, or:

$$p(q) = 40 - \frac{q}{2}$$

$$R(q) = q \cdot p(q) = 40q - \frac{q^2}{2}$$

$$MR(q) = \frac{dR(q)}{dq} = 40 - q$$

Now imagine that the firm had a cost function of  $C(q) = 120 + 2q^2$ , the profit maximizing quantity could be found either by constructing the profit function:

$$\Pi(q) = R(q) - C(q) = 40q - \frac{q^2}{2} - 120 - 2q^2$$

and taking the derivative with respect to  $q$  and setting it equal to zero.

Alternatively, you could find the marginal cost function,  $MC(q) = 4q$ , set this equal to the marginal revenue function and solve for  $q^*$ .

These two approaches are mathematically equivalent.

### ***Marginal Revenue and Elasticity***

As derived in the textbook (equation 9.12 on page 253) the relationship between price elasticity of demand ( $\varepsilon$ ) and marginal revenue is:

$$MR = p \left( 1 + \frac{1}{\varepsilon} \right)$$

So, if  $\varepsilon = -2$ , marginal revenue is equal to half of the price.

If  $\varepsilon = -1$ , marginal revenue is zero. To think about this, consider that when  $\varepsilon = -1$ , an increase of 10% in price will lead to a decrease of 10% in quantity leaving revenue unchanged, so marginal revenue will be zero.

As a fun question, if a firm was facing a demand function with  $\varepsilon = -1$ , what would be their profit maximizing quantity?

If  $\varepsilon = -0.5$ , marginal revenue is actually negative. This is because making and selling an additional unit will drive the price down a lot and not increase sales very much, leading to less total money coming in.

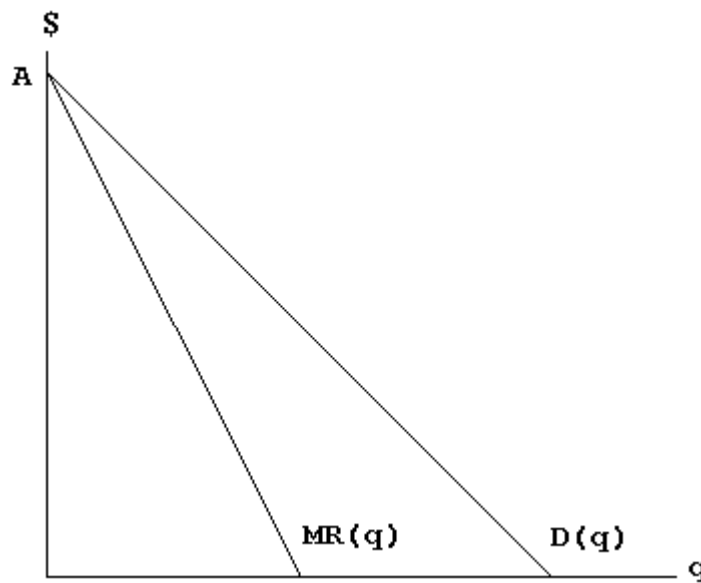
To tie this all together, it is worth noting that when the demand function is a straight line the marginal revenue function will also be a straight line with the same vertical intercept and a slope that is twice as steep.

$$P(q) = A - Bq$$

$$R(q) = Aq - Bq^2$$

$$MR(q) = A - 2Bq$$

On a graph, this looks like:



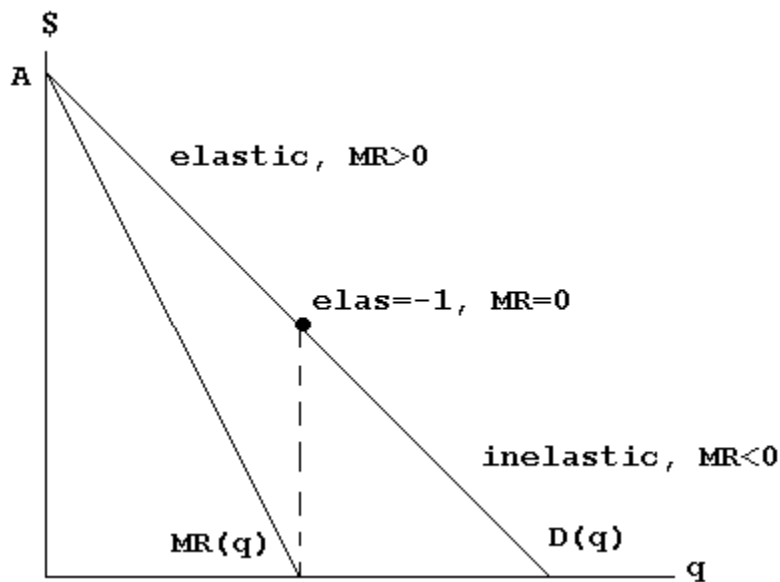
*A graph showing a marginal revenue line and a linear demand function. As is always the case, when there is a linear demand curve, the marginal revenue curve has the same vertical intercept and is twice as steep.*

This is related to the fact that the price elasticity of demand changes as you move along a straight-line demand curve. This idea is based on the fact that one formula for elasticity is:

$$\varepsilon = \frac{p}{q} \cdot \frac{1}{\text{slope}}$$

The demand curve has constant slope, so the second term on the right hand side is constant. The ratio of  $p$  to  $q$  is large at the top of the demand curve, making demand near the top of the demand curve more elastic. The ratio of  $p$  to  $q$  is smaller at the bottom of the demand curve, making demand less elastic at the bottom of the curve.

Also, in the middle of the demand curve, at the quantity where  $MR=0$ , elasticity of demand is  $-1$ .



*A graph showing a linear demand function and the associated linear marginal revenue function, showing that demand is elastic in the upper portion of the demand curve, unit elastic in the middle and inelastic in the lower portion.*

### **The Inverse Elasticity Rule and Profit Maximization**

The inverse elasticity rule is, as above:

$$MR = p \left( 1 + \frac{1}{\varepsilon} \right)$$

If a firm is profit maximizing, then we know that  $MR=MC$ . A fun implication is that we can express a firm's profit maximizing price as a function of its marginal cost, something referred to as the markup rule, or how far above marginal cost the profit maximizing price will be:

$$MR = MC$$

$$MR = p \left( 1 + \frac{1}{\varepsilon} \right) = MC$$

$$MC = p + \frac{p}{\varepsilon}$$

$$\frac{-1}{\varepsilon} = \frac{p - MC}{p}$$

$$p = MC \left( \frac{\varepsilon}{1 + \varepsilon} \right)$$

So, if the price elasticity of demand is  $-2$ , the profit maximizing price is:

$$p^* = MC \left( \frac{-2}{1 - 2} \right) = MC \cdot \frac{-2}{-1} = 2 \cdot MC$$

So, the profit maximizing price will be two times the marginal cost.

This formula only works if demand is elastic.

To see why, imagine that demand is inelastic. Decreasing output would reduce costs and raise the price. When demand is inelastic and the price rises, revenue rises. The combination of higher revenue and lower costs means that when demand is inelastic, a lower quantity and higher price will increase profits.

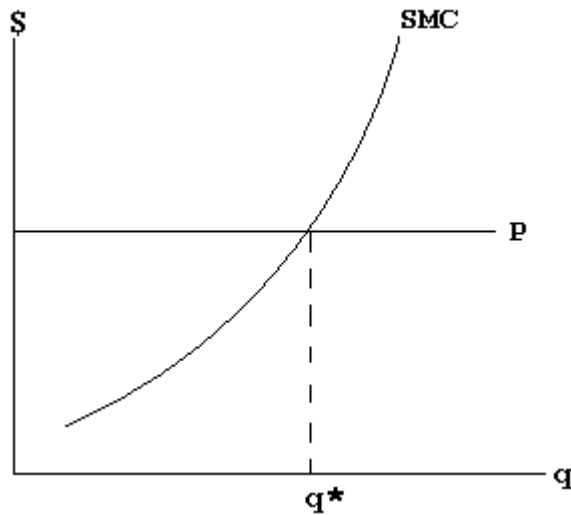
In terms of the straight line demand curve shown above, if a firm finds itself in the lower, inelastic portion of a straight line demand curve, it should cut quantity and raise price until it is in the elastic portion of the demand curve.

### ***Short Run Supply by a Profit Maximizing Firm***

For purposes of this section, imagine that a firm is a price taker, that is, it observes the market price and then makes and sells as many as it wants to at that price.

In the short run, the firm will have some fixed amount of capital and, as a result, will face some short run marginal cost (SMC) curve.

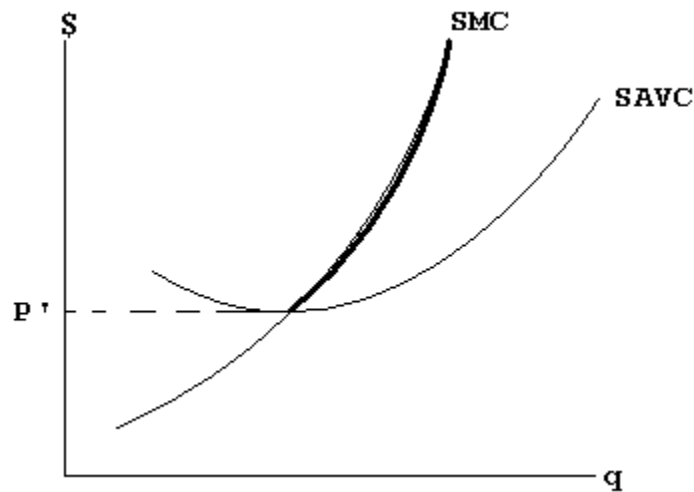
In the short run, the firm will produce additional units until the SMC rises to the price. The simple diagram is:



*A graph showing an upward sloping short run marginal cost curve and a horizontal price line, with the profit maximizing quantity being where the short run marginal cost is equal to the price.*

A slight complication is introduced by the fact that the firm can choose to shut down in the short run if the price is not high enough to cover the firm's variable costs, or the costs it has control over in the short run.

So, the price must be above the short run average variable cost (SAVC) for the firm to produce in the short run. The picture of this is:



*A graph showing a short run marginal cost curve and a U-shaped short run average cost curve, with the marginal cost curve intersecting the average cost curve at minimum average cost. The marginal cost curve is highlighted above minimum average cost to emphasize the idea that the short run supply curve is equal to the marginal cost curve above short run average cost.*



The firm's relevant short run supply curve is the portion of the short run marginal cost (SMC) curve that is above the average variable cost curve. This is the shaded portion of the SMC curve. Put slightly differently, the price must be greater than  $P'$  for the firm to produce in the short run.

### ***Profit Functions***

Instead of simply expressing profits as a function of quantity of output,  $\Pi(q)$ , we could write profits as a function of inputs:

$$\Pi(k, l) = p \cdot f(k, l) - vk - wl$$

However, the way that the book puts it, profits could also be written as a function of constant output price,  $p$ , and the cost of capital and labor,  $v$  and  $w$ , respectively, assuming that the firm is profit maximizing:

$$\Pi(p, v, w) = \max_{k, l} (p \cdot f(k, l) - vk - wl)$$

Maximizing the profit function means taking the partial derivatives with respect to  $k$  and  $l$  and setting each of these equal to zero:

$$\begin{aligned} \frac{\partial \Pi}{\partial k} &= p \cdot f_k - v = 0 \rightarrow p \cdot f_k = v \\ \frac{\partial \Pi}{\partial l} &= p \cdot f_l - w = 0 \rightarrow p \cdot f_l = w \end{aligned}$$

These conditions have the interpretation that the value of the marginal products of capital and labor should be equal to their costs.

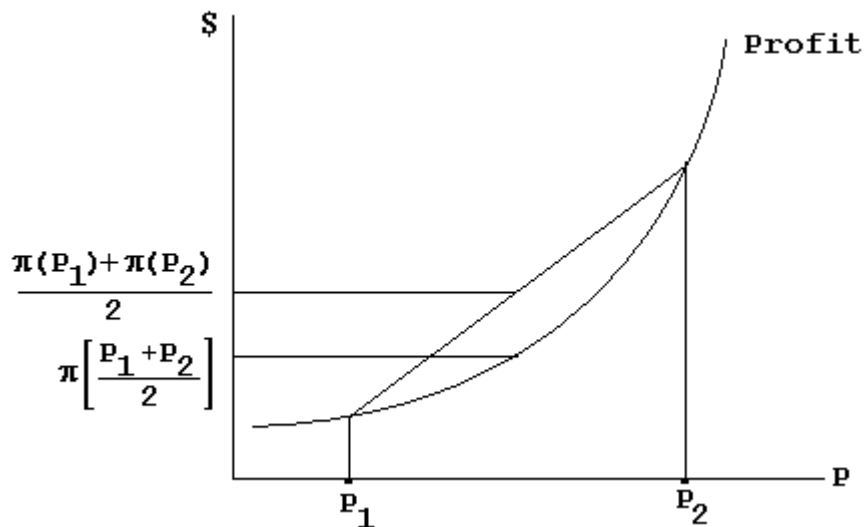
So,  $p \cdot f_k$ , the output price multiplied by the marginal product of capital, is the value of the additional output generated by adding an extra unit of capital. This should be equal to the cost of an additional unit of capital,  $v$ .

### ***Properties of the Profit Function***

1. Homogeneous of degree 1 in input and output prices

So, if all input and output prices double, profits will double. Try writing out an example of this to convince yourself.

2. Profits are non-decreasing in output price  
If output price rises, profits won't fall.
3. Profits are non-increasing in input prices  
If input prices rise, profits won't rise.
4. Profits are convex in output price:

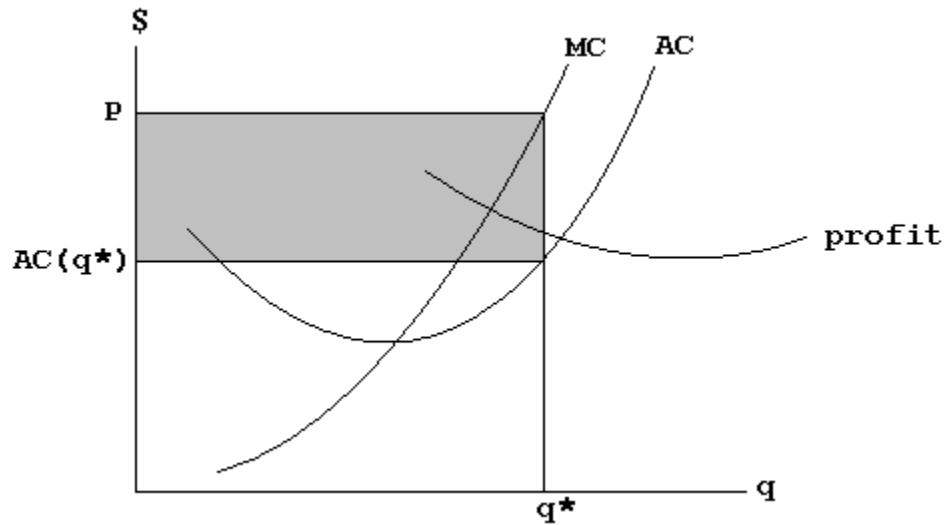


*A graph showing a convex, upward sloping profit function with price on the horizontal axis and demonstrating that expected profit is higher when there is uncertainty about the output price rather than when the output price is certain.*

The important implication of this is that a firm would have higher expected profits facing a low output price with probability 0.5 and a high output price with probability 0.5 than it would if it faced the average of these two output prices with certainty. This is because they can take advantage of high output prices by producing more and can mitigate the effects of low output prices by producing less.

### **Short Run Profit or Producer Surplus**

The profit that a firm achieves in the short run can be shown on the following graph:

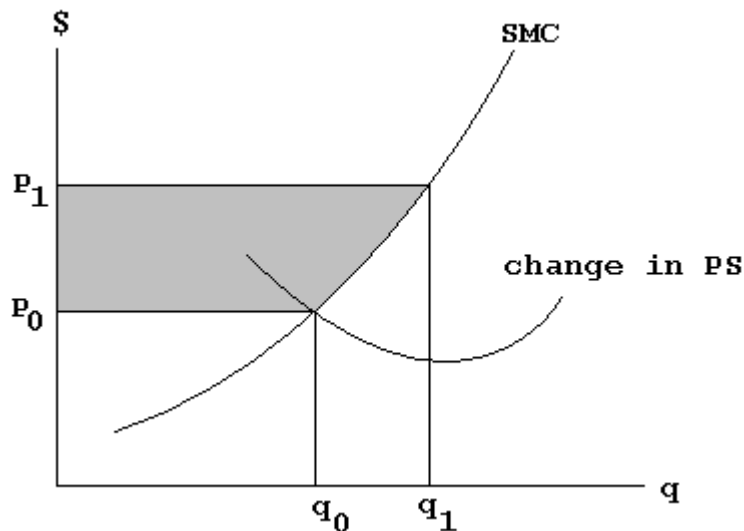


*A graph showing the profit from producing a quantity that is consistent with a price higher than minimum average cost.*

Profit is equal to quantity multiplied by the difference between price and average cost.

The book offers a slightly different version of this, though it's fundamentally the same.

The book addresses the issue of the change in profit, or the change in producer surplus, resulting from a change in price. The area of this is:



*A graph showing the change in producer surplus resulting from an increase in output price.*

This is the change in the area above the supply curve (the SMC curve in the short run) and under the price line, which is the standard definition of producer surplus.

This is also equal to the difference in profits when prices rise from  $P_0$  to  $P_1$ . As shown in the book, this can be expressed as an integral over the interval  $P_0$  to  $P_1$ .

It is worth noting that the difference between producer surplus and profits is fixed costs:

$$\text{Profit} = \text{Producer surplus} - \text{Fixed costs}$$

However, because fixed costs don't change in the short run, short run changes in profit are equal to short run changes in producer surplus.

### ***Profit Maximization and Input Demand***

As shown earlier, it is possible to express profit as a function of inputs:

$$\Pi(p, v, w) = \max_{k,l} (p \cdot f(k, l) - vk - wl)$$

with the partial derivatives giving equations that can be solved to get the optimal quantities of inputs:

$$\begin{aligned} \frac{\partial \Pi}{\partial k} &= p \cdot f_k - v = 0 \rightarrow p \cdot f_k = v \\ \frac{\partial \Pi}{\partial l} &= p \cdot f_l - w = 0 \rightarrow p \cdot f_l = w \end{aligned}$$

There are some second order conditions that must necessarily hold for the result we will find to be one of profit maximization rather than minimization:

$$\begin{aligned} \frac{\partial^2 \Pi}{\partial k^2} &= p \cdot f_{kk} < 0 \\ \frac{\partial^2 \Pi}{\partial l^2} &= p \cdot f_{ll} < 0 \\ \text{and} \\ \frac{\partial^2 \Pi}{\partial k^2} \cdot \frac{\partial^2 \Pi}{\partial l^2} - \frac{\partial^2 \Pi}{\partial l \partial k} &= p^2 \cdot (f_{kk} \cdot f_{ll} - f_{kl}^2) > 0 \end{aligned}$$

Basically, there must be diminishing marginal return to each input and the cross effects of marginal productivity must be dominated by the diminishing marginal returns to each input itself.

This must be true because if there weren't decreasing marginal returns, then the cost minimizing or profit maximizing firm size could be infinitely large.

In theory, the first order conditions can be solved simultaneously for input demand functions:

$$\begin{aligned}k^* &= k(P, v, w) \\l^* &= l(P, v, w)\end{aligned}$$

This labor demand function depends on several values,  $P$ ,  $v$  and  $w$ . We might not know how the quantity of labor demanded,  $l$ , will change as  $P$  and  $v$  change, but we can know how the quantity of labor demanded will change as the wage,  $w$ , changes.

This is demonstrated in the textbook through totally differentiating the equation shown below, assuming that price,  $p$ , and the cost of capital,  $v$ , are held constant:

$$w = p \cdot f_l$$

$$dw = p \cdot \frac{\partial f_l}{\partial l} \cdot \frac{\partial l}{\partial w} \cdot dw$$

Dividing both sides by  $dw$  gives us:

$$1 = p \cdot \frac{\partial f_l}{\partial l} \cdot \frac{\partial l}{\partial w}$$

and we can replace  $\frac{\partial f_l}{\partial l}$  with  $f_{ll}$  to get:

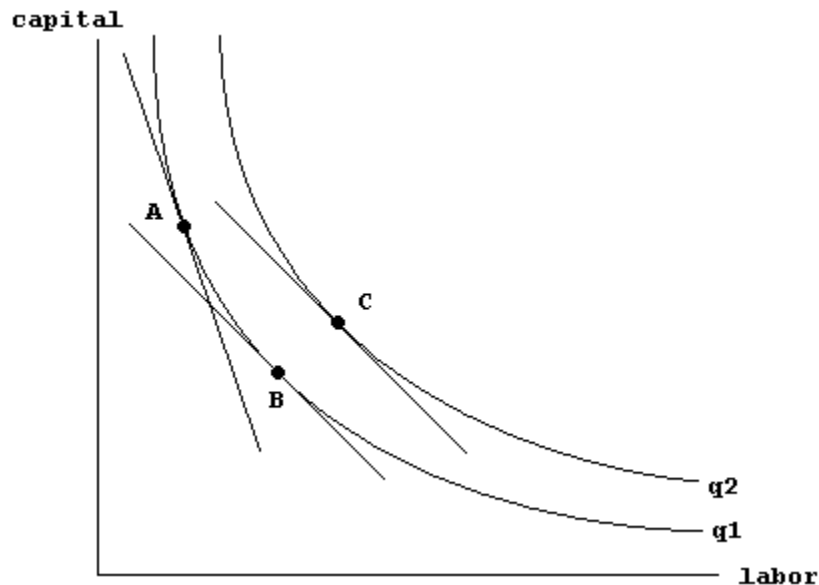
$$1 = p \cdot f_{ll} \cdot \frac{\partial l}{\partial w}$$

$$\frac{\partial l}{\partial w} = \frac{1}{p \cdot f_{ll}}$$

Now, we know that price is positive, so  $p > 0$  and we assume that  $f_{ll} \leq 0$ , so we get the result that the partial derivative of labor with respect to the wage must be non-positive. That is, as wages rise, firms demand either less labor or the same amount of labor.

### **Graphical Stuff: Substitution and Output Effects**

This is material presented in Figure 9.5. When the cost of labor falls, there are two effects. First, there is a substitution effect as, holding output constant, the firm substitutes additional labor for capital. This is represented by a movement along the original isoquant from point A to point B. Second, there is an output effect as the firm increases its quantity of output from  $q_1$  to  $q_2$ , represented by a movement from point B to point C.



*A graph with two isoquants showing the substitution and output effects associated with a decrease in the cost of labor.*

The important point is that when labor becomes cheaper, the firm will hire more of it because they will substitute from capital toward labor and because the firm will expand output because production has become less costly.

The effect on capital use is ambiguous. Use of capital will fall as a result of the substitution effect but will probably rise as a result of the output effect, assuming that the expansion path treats capital as a normal input.

### **Practice Problems**

Try the following problems from the book: 9.1, 9.2, 9.3a,b, 9.5

1. Imagine that a firm has costs given by  $C(q)=420 + 10q + 4q^2$  and revenues given by  $R(q)=100q - q^2$ . Find the profit maximizing quantity.
2. Imagine that a firm has costs given by  $C(q)=520 + 10q + 4q^2$  and revenues given by  $R(q)=100q - q^2$ .

3. Imagine that a firm has costs given by  $C(q)=420 + 4q + 4q^2$  and revenues given by  $R(q)=100q - 2q^2$ .

4. For the above profit functions, confirm that the second derivatives with respect to  $q$  are negative.

Find the marginal revenue functions associated with the following demand functions.

5.  $P(q) = 100 - q$

6.  $P(q) = 100 - 2q$

7.  $P(q) = 100 - q^2$

8.  $P(q) = q^{-2}$

9.  $q(P) = 100 - P$

10.  $q(P) = 100 - P/2$

11.  $q(P) = 100 - 2P$

12.  $q(P) = p^{-0.5}$

13.  $q(P) = p^{-2}$

Find the profit maximizing price for the following situations.

14.  $MC=1$ , elasticity=-2

15.  $MC=1$ , elasticity=-3

16.  $MC=2$ , elasticity=-3

17.  $MC=1$ , elasticity=-1

18.  $MC=1$ , elasticity=-0.5