Chapter Overview

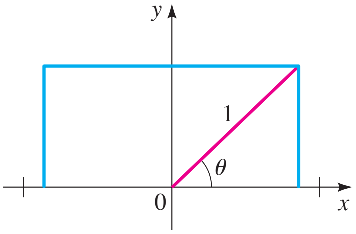
In Chapters 5 and 6 we studied the graphical and geometric properties of trigonometric functions. In this chapter we study the algebraic aspects of trigonometry, that is, simplifying and factoring expressions and solving equations that involve trigonometric functions. The basic tools in the algebra of trigonometry are trigonometric identities.

A *trigonometric identity* is an equation involving the trigonometric functions that holds for all values of the variable. For example, from the definitions of sine and cosine it follows that for any θ we have

$$\sin^2\theta + \cos^2\theta = 1$$

Here are some other identities that we will study in this chapter:

$$\sin 2\theta = 2\sin\theta\cos\theta \qquad \sin A\cos B = \tfrac{1}{2}[\sin(A+B) + \sin(A-B)]$$

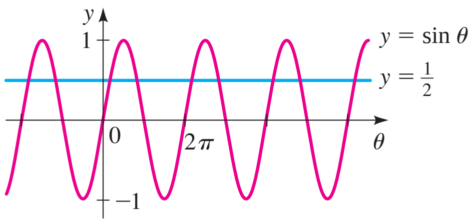


Using identities we can simplify a complicated expression involving the trigonometric functions into a much simpler expression, thereby allowing us to better understand what the expression means. For example, the area of the rectangle in the figure at the left is $A = 2\sin\theta\cos\theta$; then using one of the above identities we see that $A = \sin 2\theta$.

A *trigonometric equation* is an equation involving the trigonometric functions. For example, the equation

$$\sin\theta - \frac{1}{2} = 0$$

is a trigonometric equation. To solve this equation we need to find all the values of θ that satisfy the equation. A graph of $y = \sin\theta$ shows that $\sin\theta = \frac{1}{2}$ infinitely many times, so the equation has infinitely many solutions. Two of these solutions are $\theta = \frac{\pi}{6}$ and $\frac{5\pi}{6}$; we can get the others by adding multiples of 2π to these solutions.



We also study the *inverse trigonometric functions*. In order to define the inverse of a trigonometric function, we first restrict its domain to an interval on which the function is one-to-one. For example, we restrict the domain of the sine function to $[-\pi/2, \pi/2]$. On this interval $\sin \frac{\pi}{6} = \frac{1}{2}$, so $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$. We will see that these inverse functions are useful in solving trigonometric equations.

In *Focus on Modeling* (page 575) we study some applications of the concepts of this chapter to the motion of waves.

7.1 Trigonometric Identities

We begin by listing some of the basic trigonometric identities. We studied most of these in Chapters 5 and 6; you are asked to prove the cofunction identities in Exercise 100.

Fundamental Trigonometric Identities

Reciprocal Identities

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}$$

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin x \qquad \cos(-x) = \cos x \qquad \tan(-x) = -\tan x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - u\right) = \cos u \qquad \tan\left(\frac{\pi}{2} - u\right) = \cot u \qquad \sec\left(\frac{\pi}{2} - u\right) = \csc u$$

$$\cos\left(\frac{\pi}{2} - u\right) = \sin u \qquad \cot\left(\frac{\pi}{2} - u\right) = \tan u \qquad \csc\left(\frac{\pi}{2} - u\right) = \sec u$$

Simplifying Trigonometric Expressions

Identities enable us to write the same expression in different ways. It is often possible to rewrite a complicated looking expression as a much simpler one. To simplify algebraic expressions, we used factoring, common denominators, and the Special Product Formulas. To simplify trigonometric expressions, we use these same techniques together with the fundamental trigonometric identities.

Example 1 Simplifying a Trigonometric Expression

Simplify the expression $\cos t + \tan t \sin t$.

Solution We start by rewriting the expression in terms of sine and cosine.

$$\begin{aligned} \cos t + \tan t \sin t &= \cos t + \left(\frac{\sin t}{\cos t}\right)\sin t && \text{Reciprocal identity} \\ &= \frac{\cos^2 t + \sin^2 t}{\cos t} && \text{Common denominator} \\ &= \frac{1}{\cos t} && \text{Pythagorean identity} \\ &= \sec t && \text{Reciprocal identity} \end{aligned}$$

■

Example 2 Simplifying by Combining Fractions

Simplify the expression $\dfrac{\sin \theta}{\cos \theta} + \dfrac{\cos \theta}{1 + \sin \theta}$.

Solution We combine the fractions by using a common denominator.

$$\begin{aligned} \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{1 + \sin \theta} &= \frac{\sin \theta\,(1 + \sin \theta) + \cos^2\theta}{\cos \theta\,(1 + \sin \theta)} && \text{Common denominator} \\ &= \frac{\sin \theta + \sin^2\theta + \cos^2\theta}{\cos \theta\,(1 + \sin \theta)} && \text{Distribute } \sin\theta \\ &= \frac{\sin \theta + 1}{\cos \theta\,(1 + \sin \theta)} && \text{Pythagorean identity} \\ &= \frac{1}{\cos \theta} = \sec \theta && \text{Cancel and use reciprocal identity} \end{aligned}$$

■

Proving Trigonometric Identities

Many identities follow from the fundamental identities. In the examples that follow, we learn how to prove that a given trigonometric equation is an identity, and in the process we will see how to discover new identities.

First, it's easy to decide when a given equation is *not* an identity. All we need to do is show that the equation does not hold for some value of the variable (or variables). Thus, the equation

$$\sin x + \cos x = 1$$

is not an identity, because when $x = \pi/4$, we have

$$\sin \frac{\pi}{4} + \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2} \neq 1$$

To verify that a trigonometric equation is an identity, we transform one side of the equation into the other side by a series of steps, each of which is itself an identity.

Guidelines for Proving Trigonometric Identities

1. **Start with one side.** Pick one side of the equation and write it down. Your goal is to transform it into the other side. It's usually easier to start with the more complicated side.
2. **Use known identities.** Use algebra and the identities you know to change the side you started with. Bring fractional expressions to a common denominator, factor, and use the fundamental identities to simplify expressions.
3. **Convert to sines and cosines.** If you are stuck, you may find it helpful to rewrite all functions in terms of sines and cosines.

Warning: To prove an identity, we do *not* just perform the same operations on both sides of the equation. For example, if we start with an equation that is not an identity, such as

(1) $$\sin x = -\sin x$$

and square both sides, we get the equation

(2) $$\sin^2 x = \sin^2 x$$

which is clearly an identity. Does this mean that the original equation is an identity? Of course not. The problem here is that the operation of squaring is not **reversible** in the sense that we cannot arrive back at (1) from (2) by taking square roots (reversing the procedure). Only operations that are reversible will necessarily transform an identity into an identity.

Example 3 Proving an Identity by Rewriting in Terms of Sine and Cosine

Verify the identity $\cos\theta\,(\sec\theta - \cos\theta) = \sin^2\theta$.

Solution The left-hand side looks more complicated, so we start with it and try to transform it into the right-hand side.

$$\begin{aligned} \text{LHS} &= \cos\theta\,(\sec\theta - \cos\theta) \\ &= \cos\theta\left(\frac{1}{\cos\theta} - \cos\theta\right) && \text{Reciprocal identity} \\ &= 1 - \cos^2\theta && \text{Expand} \\ &= \sin^2\theta = \text{RHS} && \text{Pythagorean identity} \end{aligned}$$

■

In Example 3 it isn't easy to see how to change the right-hand side into the left-hand side, but it's definitely possible. Simply notice that each step is reversible. In other words, if we start with the last expression in the proof and work backward through the steps, the right side is transformed into the left side. You will probably agree, however, that it's more difficult to prove the identity this way. That's why it's often better to change the more complicated side of the identity into the simpler side.

Example 4 Proving an Identity by Combining Fractions

Verify the identity

$$2 \tan x \sec x = \frac{1}{1 - \sin x} - \frac{1}{1 + \sin x}$$

Solution Finding a common denominator and combining the fractions on the right-hand side of this equation, we get

$$\begin{aligned}
\text{RHS} &= \frac{1}{1 - \sin x} - \frac{1}{1 + \sin x} \\
&= \frac{(1 + \sin x) - (1 - \sin x)}{(1 - \sin x)(1 + \sin x)} && \text{Common denominator} \\
&= \frac{2 \sin x}{1 - \sin^2 x} && \text{Simplify} \\
&= \frac{2 \sin x}{\cos^2 x} && \text{Pythagorean identity} \\
&= 2 \frac{\sin x}{\cos x}\left(\frac{1}{\cos x}\right) && \text{Factor} \\
&= 2 \tan x \sec x = \text{LHS} && \text{Reciprocal identities}
\end{aligned}$$

■

See *Focus on Problem Solving*, pages 138–145.

In Example 5 we introduce "something extra" to the problem by multiplying the numerator and the denominator by a trigonometric expression, chosen so that we can simplify the result.

Example 5 Proving an Identity by Introducing Something Extra

Verify the identity $\dfrac{\cos u}{1 - \sin u} = \sec u + \tan u$.

Solution We start with the left-hand side and multiply numerator and denominator by $1 + \sin u$.

We multiply by $1 + \sin u$ because we know by the difference of squares formula that $(1 - \sin u)(1 + \sin u) = 1 - \sin^2 u$, and this is just $\cos^2 u$, a simpler expression.

$$\begin{aligned}
\text{LHS} &= \frac{\cos u}{1 - \sin u} \\
&= \frac{\cos u}{1 - \sin u} \cdot \frac{1 + \sin u}{1 + \sin u} && \text{Multiply numerator and denominator by } 1 + \sin u \\
&= \frac{\cos u\,(1 + \sin u)}{1 - \sin^2 u} && \text{Expand denominator}
\end{aligned}$$

Euclid (circa 300 B.C.) taught in Alexandria. His *Elements* is the most widely influential scientific book in history. For 2000 years it was the standard introduction to geometry in the schools, and for many generations it was considered the best way to develop logical reasoning. Abraham Lincoln, for instance, studied the *Elements* as a way to sharpen his mind. The story is told that King Ptolemy once asked Euclid if there was a faster way to learn geometry than through the *Elements*. Euclid replied that there is "no royal road to geometry"—meaning by this that mathematics does not respect wealth or social status. Euclid was revered in his own time and was referred to by the title "The Geometer" or "The Writer of the *Elements*." The greatness of the *Elements* stems from its precise, logical, and systematic treatment of geometry. For dealing with equality, Euclid lists the following rules, which he calls "common notions."

1. Things that are equal to the same thing are equal to each other.
2. If equals are added to equals, the sums are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal.
5. The whole is greater than the part.

$$= \frac{\cos u\,(1 + \sin u)}{\cos^2 u} \qquad \text{Pythagorean identity}$$

$$= \frac{1 + \sin u}{\cos u} \qquad \text{Cancel common factor}$$

$$= \frac{1}{\cos u} + \frac{\sin u}{\cos u} \qquad \text{Separate into two fractions}$$

$$= \sec u + \tan u \qquad \text{Reciprocal identities}$$

■

Here is another method for proving that an equation is an identity. If we can transform each side of the equation *separately*, by way of identities, to arrive at the same result, then the equation is an identity. Example 6 illustrates this procedure.

Example 6 Proving an Identity by Working with Both Sides Separately

Verify the identity $\dfrac{1 + \cos\theta}{\cos\theta} = \dfrac{\tan^2\theta}{\sec\theta - 1}$.

Solution We prove the identity by changing each side separately into the same expression. Supply the reasons for each step.

$$\text{LHS} = \frac{1 + \cos\theta}{\cos\theta} = \frac{1}{\cos\theta} + \frac{\cos\theta}{\cos\theta} = \sec\theta + 1$$

$$\text{RHS} = \frac{\tan^2\theta}{\sec\theta - 1} = \frac{\sec^2\theta - 1}{\sec\theta - 1} = \frac{(\sec\theta - 1)(\sec\theta + 1)}{\sec\theta - 1} = \sec\theta + 1$$

It follows that LHS = RHS, so the equation is an identity. ■

We conclude this section by describing the technique of *trigonometric substitution*, which we use to convert algebraic expressions to trigonometric ones. This is often useful in calculus, for instance, in finding the area of a circle or an ellipse.

Example 7 Trigonometric Substitution

Substitute $\sin\theta$ for x in the expression $\sqrt{1 - x^2}$ and simplify. Assume that $0 \le \theta \le \pi/2$.

Solution Setting $x = \sin\theta$, we have

$$\sqrt{1 - x^2} = \sqrt{1 - \sin^2\theta} \qquad \text{Substitute } x = \sin\theta$$

$$= \sqrt{\cos^2\theta} \qquad \text{Pythagorean identity}$$

$$= \cos\theta \qquad \text{Take square root}$$

The last equality is true because $\cos\theta \ge 0$ for the values of θ in question. ■

7.1 Exercises

1–10 ■ Write the trigonometric expression in terms of sine and cosine, and then simplify.

1. $\cos t \tan t$

2. $\cos t \csc t$

3. $\sin\theta \sec\theta$

4. $\tan\theta \csc\theta$

5. $\tan^2 x - \sec^2 x$

6. $\dfrac{\sec x}{\csc x}$

7. $\sin u + \cot u \cos u$

8. $\cos^2\theta\,(1 + \tan^2\theta)$

9. $\dfrac{\sec\theta - \cos\theta}{\sin\theta}$

10. $\dfrac{\cot\theta}{\csc\theta - \sin\theta}$

11–24 ■ Simplify the trigonometric expression.

11. $\dfrac{\sin x \sec x}{\tan x}$

12. $\cos^3 x + \sin^2 x \cos x$

13. $\dfrac{1 + \cos y}{1 + \sec y}$

14. $\dfrac{\tan x}{\sec(-x)}$

15. $\dfrac{\sec^2 x - 1}{\sec^2 x}$

16. $\dfrac{\sec x - \cos x}{\tan x}$

17. $\dfrac{1 + \csc x}{\cos x + \cot x}$

18. $\dfrac{\sin x}{\csc x} + \dfrac{\cos x}{\sec x}$

19. $\dfrac{1 + \sin u}{\cos u} + \dfrac{\cos u}{1 + \sin u}$

20. $\tan x \cos x \csc x$

21. $\dfrac{2 + \tan^2 x}{\sec^2 x} - 1$

22. $\dfrac{1 + \cot A}{\csc A}$

23. $\tan\theta + \cos(-\theta) + \tan(-\theta)$

24. $\dfrac{\cos x}{\sec x + \tan x}$

25–88 ■ Verify the identity.

25. $\dfrac{\sin\theta}{\tan\theta} = \cos\theta$

26. $\dfrac{\tan x}{\sec x} = \sin x$

27. $\dfrac{\cos u \sec u}{\tan u} = \cot u$

28. $\dfrac{\cot x \sec x}{\csc x} = 1$

29. $\dfrac{\tan y}{\csc y} = \sec y - \cos y$

30. $\dfrac{\cos v}{\sec v \sin v} = \csc v - \sin v$

31. $\sin B + \cos B \cot B = \csc B$

32. $\cos(-x) - \sin(-x) = \cos x + \sin x$

33. $\cot(-\alpha)\cos(-\alpha) + \sin(-\alpha) = -\csc\alpha$

34. $\csc x\,[\csc x + \sin(-x)] = \cot^2 x$

35. $\tan\theta + \cot\theta = \sec\theta \csc\theta$

36. $(\sin x + \cos x)^2 = 1 + 2\sin x \cos x$

37. $(1 - \cos\beta)(1 + \cos\beta) = \dfrac{1}{\csc^2\beta}$

38. $\dfrac{\cos x}{\sec x} + \dfrac{\sin x}{\csc x} = 1$

39. $\dfrac{(\sin x + \cos x)^2}{\sin^2 x - \cos^2 x} = \dfrac{\sin^2 x - \cos^2 x}{(\sin x - \cos x)^2}$

40. $(\sin x + \cos x)^4 = (1 + 2\sin x \cos x)^2$

41. $\dfrac{\sec t - \cos t}{\sec t} = \sin^2 t$

42. $\dfrac{1 - \sin x}{1 + \sin x} = (\sec x - \tan x)^2$

43. $\dfrac{1}{1 - \sin^2 y} = 1 + \tan^2 y$

44. $\csc x - \sin x = \cos x \cot x$

45. $(\cot x - \csc x)(\cos x + 1) = -\sin x$

46. $\sin^4\theta - \cos^4\theta = \sin^2\theta - \cos^2\theta$

47. $(1 - \cos^2 x)(1 + \cot^2 x) = 1$

48. $\cos^2 x - \sin^2 x = 2\cos^2 x - 1$

49. $2\cos^2 x - 1 = 1 - 2\sin^2 x$

50. $(\tan y + \cot y)\sin y \cos y = 1$

51. $\dfrac{1 - \cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1 + \cos\alpha}$

52. $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha = \sec^2\alpha$

53. $\tan^2\theta - \sin^2\theta = \tan^2\theta \sin^2\theta$

54. $\cot^2\theta \cos^2\theta = \cot^2\theta - \cos^2\theta$

55. $\dfrac{\sin x - 1}{\sin x + 1} = \dfrac{-\cos^2 x}{(\sin x + 1)^2}$

56. $\dfrac{\sin w}{\sin w + \cos w} = \dfrac{\tan w}{1 + \tan w}$

57. $\dfrac{(\sin t + \cos t)^2}{\sin t \cos t} = 2 + \sec t \csc t$

58. $\sec t \csc t\,(\tan t + \cot t) = \sec^2 t + \csc^2 t$

59. $\dfrac{1 + \tan^2 u}{1 - \tan^2 u} = \dfrac{1}{\cos^2 u - \sin^2 u}$

60. $\dfrac{1 + \sec^2 x}{1 + \tan^2 x} = 1 + \cos^2 x$

61. $\dfrac{\sec x}{\sec x - \tan x} = \sec x\,(\sec x + \tan x)$

62. $\dfrac{\sec x + \csc x}{\tan x + \cot x} = \sin x + \cos x$

63. $\sec v - \tan v = \dfrac{1}{\sec v + \tan v}$

64. $\dfrac{\sin A}{1 - \cos A} - \cot A = \csc A$

65. $\dfrac{\sin x + \cos x}{\sec x + \csc x} = \sin x \cos x$

66. $\dfrac{1 - \cos x}{\sin x} + \dfrac{\sin x}{1 - \cos x} = 2 \csc x$

67. $\dfrac{\csc x - \cot x}{\sec x - 1} = \cot x$

68. $\dfrac{\csc^2 x - \cot^2 x}{\sec^2 x} = \cos^2 x$

69. $\tan^2 u - \sin^2 u = \tan^2 u \sin^2 u$

70. $\dfrac{\tan v \sin v}{\tan v + \sin v} = \dfrac{\tan v - \sin v}{\tan v \sin v}$

71. $\sec^4 x - \tan^4 x = \sec^2 x + \tan^2 x$

72. $\dfrac{\cos \theta}{1 - \sin \theta} = \sec \theta + \tan \theta$

73. $\dfrac{\cos \theta}{1 - \sin \theta} = \dfrac{\sin \theta - \csc \theta}{\cos \theta - \cot \theta}$

74. $\dfrac{1 + \tan x}{1 - \tan x} = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

75. $\dfrac{\cos^2 t + \tan^2 t - 1}{\sin^2 t} = \tan^2 t$

76. $\dfrac{1}{1 - \sin x} - \dfrac{1}{1 + \sin x} = 2 \sec x \tan x$

77. $\dfrac{1}{\sec x + \tan x} + \dfrac{1}{\sec x - \tan x} = 2 \sec x$

78. $\dfrac{1 + \sin x}{1 - \sin x} - \dfrac{1 - \sin x}{1 + \sin x} = 4 \tan x \sec x$

79. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$

80. $\tan^2 x - \cot^2 x = \sec^2 x - \csc^2 x$

81. $\dfrac{\sec u - 1}{\sec u + 1} = \dfrac{1 - \cos u}{1 + \cos u}$

82. $\dfrac{\cot x + 1}{\cot x - 1} = \dfrac{1 + \tan x}{1 - \tan x}$

83. $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x \cos x$

84. $\dfrac{\tan v - \cot v}{\tan^2 v - \cot^2 v} = \sin v \cos v$

85. $\dfrac{1 + \sin x}{1 - \sin x} = (\tan x + \sec x)^2$

86. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = \tan x \tan y$

87. $(\tan x + \cot x)^4 = \csc^4 x \sec^4 x$

88. $(\sin \alpha - \tan \alpha)(\cos \alpha - \cot \alpha) = (\cos \alpha - 1)(\sin \alpha - 1)$

89–94 ■ Make the indicated trigonometric substitution in the given algebraic expression and simplify (see Example 7). Assume $0 \le \theta < \pi/2$.

89. $\dfrac{x}{\sqrt{1 - x^2}}, \quad x = \sin \theta$

90. $\sqrt{1 + x^2}, \quad x = \tan \theta$

91. $\sqrt{x^2 - 1}, \quad x = \sec \theta$

92. $\dfrac{1}{x^2\sqrt{4 + x^2}}, \quad x = 2 \tan \theta$

93. $\sqrt{9 - x^2}, \quad x = 3 \sin \theta$

94. $\dfrac{\sqrt{x^2 - 25}}{x}, \quad x = 5 \sec \theta$

95–98 ■ Graph f and g in the same viewing rectangle. Do the graphs suggest that the equation $f(x) = g(x)$ is an identity? Prove your answer.

95. $f(x) = \cos^2 x - \sin^2 x, \quad g(x) = 1 - 2 \sin^2 x$

96. $f(x) = \tan x\,(1 + \sin x), \quad g(x) = \dfrac{\sin x \cos x}{1 + \sin x}$

97. $f(x) = (\sin x + \cos x)^2, \quad g(x) = 1$

98. $f(x) = \cos^4 x - \sin^4 x, \quad g(x) = 2 \cos^2 x - 1$

99. Show that the equation is not an identity.

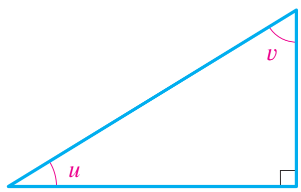
(a) $\sin 2x = 2 \sin x$ **(b)** $\sin(x + y) = \sin x + \sin y$

(c) $\sec^2 x + \csc^2 x = 1$

(d) $\dfrac{1}{\sin x + \cos x} = \csc x + \sec x$

Discovery • Discussion

100. Cofunction Identities In the right triangle shown, explain why $v = (\pi/2) - u$. Explain how you can obtain all six cofunction identities from this triangle, for $0 < u < \pi/2$.



101. Graphs and Identities Suppose you graph two functions, f and g, on a graphing device, and their graphs appear identical in the viewing rectangle. Does this prove that the equation $f(x) = g(x)$ is an identity? Explain.

102. Making Up Your Own Identity If you start with a trigonometric expression and rewrite it or simplify it, then setting the original expression equal to the rewritten expression yields a trigonometric identity. For instance, from Example 1 we get the identity

$$\cos t + \tan t \sin t = \sec t$$

Use this technique to make up your own identity, then give it to a classmate to verify.

7.2 Addition and Subtraction Formulas

We now derive identities for trigonometric functions of sums and differences.

Addition and Subtraction Formulas

Formulas for sine:

$$\sin(s + t) = \sin s \cos t + \cos s \sin t$$
$$\sin(s - t) = \sin s \cos t - \cos s \sin t$$

Formulas for cosine:

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$
$$\cos(s - t) = \cos s \cos t + \sin s \sin t$$

Formulas for tangent:

$$\tan(s + t) = \frac{\tan s + \tan t}{1 - \tan s \tan t}$$
$$\tan(s - t) = \frac{\tan s - \tan t}{1 + \tan s \tan t}$$

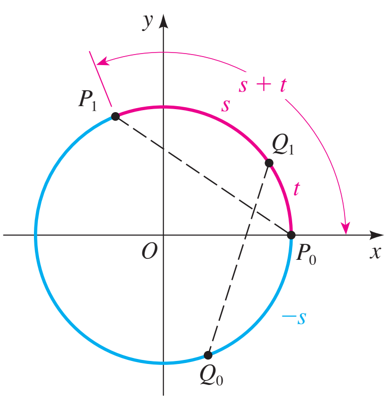


Figure 1

■ **Proof of Addition Formula for Cosine** To prove the formula $\cos(s + t) = \cos s \cos t - \sin s \sin t$, we use Figure 1. In the figure, the distances t, $s + t$, and $-s$ have been marked on the unit circle, starting at $P_0(1, 0)$ and terminating at Q_1, P_1, and Q_0, respectively. The coordinates of these points are

$$P_0(1, 0) \qquad Q_0(\cos(-s), \sin(-s))$$
$$P_1(\cos(s + t), \sin(s + t)) \qquad Q_1(\cos t, \sin t)$$

Since $\cos(-s) = \cos s$ and $\sin(-s) = -\sin s$, it follows that the point Q_0 has the coordinates $Q_0(\cos s, -\sin s)$. Notice that the distances between P_0 and P_1 and between Q_0 and Q_1 measured along the arc of the circle are equal. Since equal arcs are subtended by equal chords, it follows that $d(P_0, P_1) = d(Q_0, Q_1)$. Using the Distance Formula, we get

$$\sqrt{[\cos(s + t) - 1]^2 + [\sin(s + t) - 0]^2} = \sqrt{(\cos t - \cos s)^2 + (\sin t + \sin s)^2}$$

Squaring both sides and expanding, we have

these add to 1

$$\cos^2(s + t) - 2\cos(s + t) + 1 + \sin^2(s + t)$$
$$= \cos^2 t - 2\cos s \cos t + \cos^2 s + \sin^2 t + 2\sin s \sin t + \sin^2 s$$

these add to 1

these add to 1

Using the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ three times gives

$$2 - 2\cos(s + t) = 2 - 2\cos s \cos t + 2\sin s \sin t$$

Finally, subtracting 2 from each side and dividing both sides by -2, we get

$$\cos(s + t) = \cos s \cos t - \sin s \sin t$$

which proves the addition formula for cosine. ■

Jean Baptiste Joseph Fourier (1768–1830) is responsible for the most powerful application of the trigonometric functions (see the margin note on page 427). He used sums of these functions to describe such physical phenomena as the transmission of sound and the flow of heat.

Orphaned as a young boy, Fourier was educated in a military school, where he became a mathematics teacher at the age of 20. He was later appointed professor at the École Polytechnique but resigned this position to accompany Napoleon on his expedition to Egypt, where Fourier served as governor. After returning to France he began conducting experiments on heat. The French Academy refused to publish his early papers on this subject due to his lack of rigor. Fourier eventually became Secretary of the Academy and in this capacity had his papers published in their original form. Probably because of his study of heat and his years in the deserts of Egypt, Fourier became obsessed with keeping himself warm—he wore several layers of clothes, even in the summer, and kept his rooms at unbearably high temperatures. Evidently, these habits overburdened his heart and contributed to his death at the age of 62.

■ **Proof of Subtraction Formula for Cosine** Replacing t with $-t$ in the addition formula for cosine, we get

$$\begin{aligned}\cos(s - t) &= \cos(s + (-t))\\ &= \cos s\cos(-t) - \sin s\sin(-t) && \text{Addition formula for cosine}\\ &= \cos s\cos t + \sin s\sin t && \text{Even-odd identities}\end{aligned}$$

This proves the subtraction formula for cosine. ■

See Exercises 56 and 57 for proofs of the other addition formulas.

Example 1 Using the Addition and Subtraction Formulas

Find the exact value of each expression.

(a) $\cos 75°$ (b) $\cos\dfrac{\pi}{12}$

Solution

(a) Notice that $75° = 45° + 30°$. Since we know the exact values of sine and cosine at 45° and 30°, we use the addition formula for cosine to get

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45°\cos 30° - \sin 45°\sin 30°\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{2}\sqrt{3} - \sqrt{2}}{4} = \frac{\sqrt{6} - \sqrt{2}}{4}\end{aligned}$$

(b) Since $\dfrac{\pi}{12} = \dfrac{\pi}{4} - \dfrac{\pi}{6}$, the subtraction formula for cosine gives

$$\begin{aligned}\cos\frac{\pi}{12} &= \cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{4}\cos\frac{\pi}{6} + \sin\frac{\pi}{4}\sin\frac{\pi}{6}\\ &= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}\end{aligned}$$ ■

Example 2 Using the Addition Formula for Sine

Find the exact value of the expression $\sin 20°\cos 40° + \cos 20°\sin 40°$.

Solution We recognize the expression as the right-hand side of the addition formula for sine with $s = 20°$ and $t = 40°$. So we have

$$\sin 20°\cos 40° + \cos 20°\sin 40° = \sin(20° + 40°) = \sin 60° = \frac{\sqrt{3}}{2}$$ ■

Example 3 Proving a Cofunction Identity

Prove the cofunction identity $\cos\left(\frac{\pi}{2} - u\right) = \sin u$.

Solution By the subtraction formula for cosine,

$$\cos\left(\frac{\pi}{2} - u\right) = \cos\frac{\pi}{2}\cos u + \sin\frac{\pi}{2}\sin u$$

$$= 0 \cdot \cos u + 1 \cdot \sin u = \sin u$$

■

Example 4 Proving an Identity

Verify the identity $\frac{1 + \tan x}{1 - \tan x} = \tan\left(\frac{\pi}{4} + x\right)$.

Solution Starting with the right-hand side and using the addition formula for tangent, we get

$$\text{RHS} = \tan\left(\frac{\pi}{4} + x\right) = \frac{\tan\frac{\pi}{4} + \tan x}{1 - \tan\frac{\pi}{4}\tan x}$$

$$= \frac{1 + \tan x}{1 - \tan x} = \text{LHS}$$

■

The next example is a typical use of the addition and subtraction formulas in calculus.

Example 5 An Identity from Calculus

If $f(x) = \sin x$, show that

$$\frac{f(x+h) - f(x)}{h} = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

Solution

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$ Definition of f

$$= \frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$ Addition formula for sine

$$= \frac{\sin x(\cos h - 1) + \cos x\sin h}{h}$$ Factor

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$ Separate the fraction

■

Expressions of the Form $A \sin x + B \cos x$

We can write expressions of the form $A \sin x + B \cos x$ in terms of a single trigonometric function using the addition formula for sine. For example, consider the expression

$$\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x$$

If we set $\phi = \pi/3$, then $\cos\phi = \frac{1}{2}$ and $\sin\phi = \sqrt{3}/2$, and we can write

$$\begin{aligned}\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x &= \cos\phi \sin x + \sin\phi\cos x\\ &= \sin(x+\phi) = \sin\left(x + \frac{\pi}{3}\right)\end{aligned}$$

We are able to do this because the coefficients $\frac{1}{2}$ and $\sqrt{3}/2$ are precisely the cosine and sine of a particular number, in this case, $\pi/3$. We can use this same idea in general to write $A \sin x + B \cos x$ in the form $k \sin(x + \phi)$. We start by multiplying the numerator and denominator by $\sqrt{A^2 + B^2}$ to get

$$A\sin x + B\cos x = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\sin x + \frac{B}{\sqrt{A^2+B^2}}\cos x\right)$$

We need a number ϕ with the property that

$$\cos\phi = \frac{A}{\sqrt{A^2+B^2}} \quad \text{and} \quad \sin\phi = \frac{B}{\sqrt{A^2+B^2}}$$

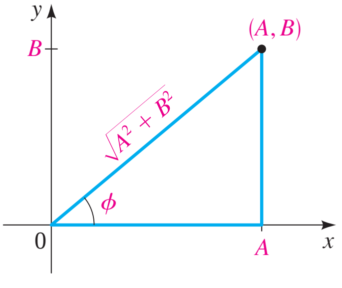


Figure 2

Figure 2 shows that the point (A, B) in the plane determines a number ϕ with precisely this property. With this ϕ, we have

$$\begin{aligned}A\sin x + B\cos x &= \sqrt{A^2+B^2}(\cos\phi\sin x + \sin\phi\cos x)\\ &= \sqrt{A^2+B^2}\sin(x+\phi)\end{aligned}$$

We have proved the following theorem.

Sums of Sines and Cosines

If A and B are real numbers, then

$$A\sin x + B\cos x = k\sin(x+\phi)$$

where $k = \sqrt{A^2+B^2}$ and ϕ satisfies

$$\cos\phi = \frac{A}{\sqrt{A^2+B^2}} \quad \text{and} \quad \sin\phi = \frac{B}{\sqrt{A^2+B^2}}$$

Example 6 A Sum of Sine and Cosine Terms

Express $3 \sin x + 4 \cos x$ in the form $k \sin(x + \phi)$.

Solution By the preceding theorem, $k = \sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = 5$. The angle ϕ has the property that $\sin \phi = \frac{4}{5}$ and $\cos \phi = \frac{3}{5}$. Using a calculator, we find $\phi \approx 53.1°$. Thus

$$3 \sin x + 4 \cos x \approx 5 \sin(x + 53.1°)$$

■

Example 7 Graphing a Trigonometric Function

Write the function $f(x) = -\sin 2x + \sqrt{3} \cos 2x$ in the form $k \sin(2x + \phi)$ and use the new form to graph the function.

Solution Since $A = -1$ and $B = \sqrt{3}$, we have $k = \sqrt{A^2 + B^2} = \sqrt{1 + 3} = 2$. The angle ϕ satisfies $\cos \phi = -\frac{1}{2}$ and $\sin \phi = \sqrt{3}/2$. From the signs of these quantities we conclude that ϕ is in quadrant II. Thus, $\phi = 2\pi/3$. By the preceding theorem we can write

$$f(x) = -\sin 2x + \sqrt{3} \cos 2x = 2 \sin\left(2x + \frac{2\pi}{3}\right)$$

Using the form

$$f(x) = 2 \sin 2\left(x + \frac{\pi}{3}\right)$$

we see that the graph is a sine curve with amplitude 2, period $2\pi/2 = \pi$, and phase shift $-\pi/3$. The graph is shown in Figure 3. ■

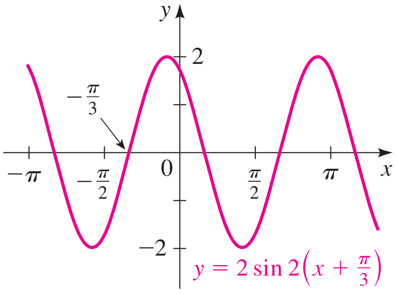


Figure 3

7.2 Exercises

1–12 ■ Use an addition or subtraction formula to find the exact value of the expression, as demonstrated in Example 1.

1. $\sin 75°$
2. $\sin 15°$
3. $\cos 105°$
4. $\cos 195°$
5. $\tan 15°$
6. $\tan 165°$
7. $\sin \dfrac{19\pi}{12}$
8. $\cos \dfrac{17\pi}{12}$
9. $\tan\left(-\dfrac{\pi}{12}\right)$
10. $\sin\left(-\dfrac{5\pi}{12}\right)$
11. $\cos \dfrac{11\pi}{12}$
12. $\tan \dfrac{7\pi}{12}$

13–18 ■ Use an addition or subtraction formula to write the expression as a trigonometric function of one number, and then find its exact value.

13. $\sin 18° \cos 27° + \cos 18° \sin 27°$
14. $\cos 10° \cos 80° - \sin 10° \sin 80°$
15. $\cos \dfrac{3\pi}{7} \cos \dfrac{2\pi}{21} + \sin \dfrac{3\pi}{7} \sin \dfrac{2\pi}{21}$
16. $\dfrac{\tan \frac{\pi}{18} + \tan \frac{\pi}{9}}{1 - \tan \frac{\pi}{18} \tan \frac{\pi}{9}}$
17. $\dfrac{\tan 73° - \tan 13°}{1 + \tan 73° \tan 13°}$
18. $\cos \dfrac{13\pi}{15} \cos\left(-\dfrac{\pi}{5}\right) - \sin \dfrac{13\pi}{15} \sin\left(-\dfrac{\pi}{5}\right)$

19–22 ■ Prove the cofunction identity using the addition and subtraction formulas.

19. $\tan\left(\dfrac{\pi}{2} - u\right) = \cot u$
20. $\cot\left(\dfrac{\pi}{2} - u\right) = \tan u$
21. $\sec\left(\dfrac{\pi}{2} - u\right) = \csc u$
22. $\csc\left(\dfrac{\pi}{2} - u\right) = \sec u$

23–40 ■ Prove the identity.

23. $\sin\left(x - \dfrac{\pi}{2}\right) = -\cos x$

24. $\cos\left(x - \dfrac{\pi}{2}\right) = \sin x$

25. $\sin(x - \pi) = -\sin x$

26. $\cos(x - \pi) = -\cos x$

27. $\tan(x - \pi) = \tan x$

28. $\sin\left(\dfrac{\pi}{2} - x\right) = \sin\left(\dfrac{\pi}{2} + x\right)$

29. $\cos\left(x + \dfrac{\pi}{6}\right) + \sin\left(x - \dfrac{\pi}{3}\right) = 0$

30. $\tan\left(x - \dfrac{\pi}{4}\right) = \dfrac{\tan x - 1}{\tan x + 1}$

31. $\sin(x + y) - \sin(x - y) = 2\cos x \sin y$

32. $\cos(x + y) + \cos(x - y) = 2\cos x \cos y$

33. $\cot(x - y) = \dfrac{\cot x \cot y + 1}{\cot y - \cot x}$

34. $\cot(x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

35. $\tan x - \tan y = \dfrac{\sin(x - y)}{\cos x \cos y}$

36. $1 - \tan x \tan y = \dfrac{\cos(x + y)}{\cos x \cos y}$

37. $\dfrac{\sin(x + y) - \sin(x - y)}{\cos(x + y) + \cos(x - y)} = \tan y$

38. $\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$

39. $\sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z$
$+ \cos x \cos y \sin z - \sin x \sin y \sin z$

40. $\tan(x - y) + \tan(y - z) + \tan(z - x)$
$= \tan(x - y)\tan(y - z)\tan(z - x)$

41–44 ■ Write the expression in terms of sine only.

41. $-\sqrt{3}\sin x + \cos x$

42. $\sin x + \cos x$

43. $5(\sin 2x - \cos 2x)$

44. $3\sin \pi x + 3\sqrt{3}\cos \pi x$

45–46 ■ **(a)** Express the function in terms of sine only. **(b)** Graph the function.

45. $f(x) = \sin x + \cos x$

46. $g(x) = \cos 2x + \sqrt{3}\sin 2x$

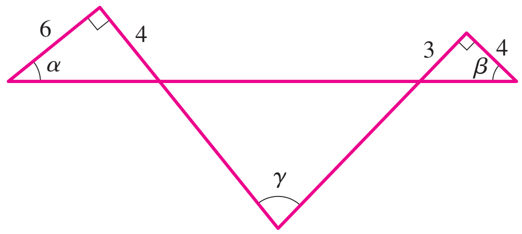
47. Show that if $\beta - \alpha = \pi/2$, then

$$\sin(x + \alpha) + \cos(x + \beta) = 0$$

48. Let $g(x) = \cos x$. Show that

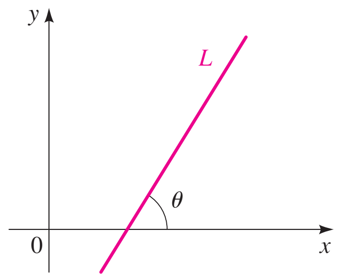
$$\frac{g(x + h) - g(x)}{h} = -\cos x\left(\frac{1 - \cos h}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$

49. Refer to the figure. Show that $\alpha + \beta = \gamma$, and find $\tan\gamma$.



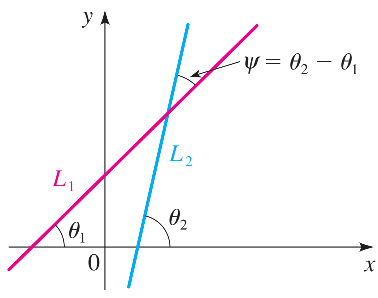
50. (a) If L is a line in the plane and θ is the angle formed by the line and the x-axis as shown in the figure, show that the slope m of the line is given by

$$m = \tan\theta$$



(b) Let L_1 and L_2 be two nonparallel lines in the plane with slopes m_1 and m_2, respectively. Let ψ be the acute angle formed by the two lines (see the figure). Show that

$$\tan\psi = \frac{m_2 - m_1}{1 + m_1 m_2}$$



(c) Find the acute angle formed by the two lines

$$y = \tfrac{1}{3}x + 1 \qquad \text{and} \qquad y = -\tfrac{1}{2}x - 3$$

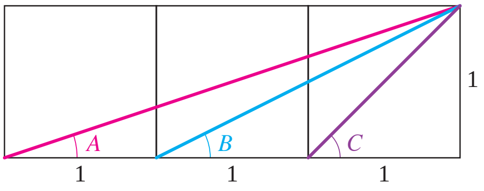
(d) Show that if two lines are perpendicular, then the slope of one is the negative reciprocal of the slope of the other. [*Hint:* First find an expression for $\cot\psi$.]

51–52 ■ **(a)** Graph the function and make a conjecture, then **(b)** prove that your conjecture is true.

51. $y = \sin^2\left(x + \dfrac{\pi}{4}\right) + \sin^2\left(x - \dfrac{\pi}{4}\right)$

52. $y = -\frac{1}{2}[\cos(x + \pi) + \cos(x - \pi)]$

53. Find $\angle A + \angle B + \angle C$ in the figure. [*Hint:* First use an addition formula to find $\tan(A + B)$.]



Applications

54. Adding an Echo A digital delay-device echoes an input signal by repeating it a fixed length of time after it is received. If such a device receives the pure note $f_1(t) = 5 \sin t$ and echoes the pure note $f_2(t) = 5 \cos t$, then the combined sound is $f(t) = f_1(t) + f_2(t)$.

(a) Graph $y = f(t)$ and observe that the graph has the form of a sine curve $y = k \sin(t + \phi)$.

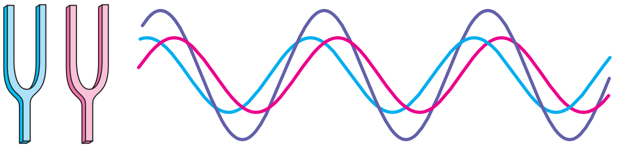
(b) Find k and ϕ.

55. Interference Two identical tuning forks are struck, one a fraction of a second after the other. The sounds produced are modeled by $f_1(t) = C \sin \omega t$ and $f_2(t) = C \sin(\omega t + \alpha)$. The two sound waves interfere to produce a single sound modeled by the sum of these functions

$$f(t) = C \sin \omega t + C \sin(\omega t + \alpha)$$

(a) Use the addition formula for sine to show that f can be written in the form $f(t) = A \sin \omega t + B \cos \omega t$, where A and B are constants that depend on α.

(b) Suppose that $C = 10$ and $\alpha = \pi/3$. Find constants k and ϕ so that $f(t) = k \sin(\omega t + \phi)$.



Discovery • Discussion

56. Addition Formula for Sine In the text we proved only the addition and subtraction formulas for cosine. Use these formulas and the cofunction identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \sin\left(\frac{\pi}{2} - x\right)$$

to prove the addition formula for sine. [*Hint:* To get started, use the first cofunction identity to write

$$\sin(s + t) = \cos\left(\frac{\pi}{2} - (s + t)\right)$$

$$= \cos\left(\left(\frac{\pi}{2} - s\right) - t\right)$$

and use the subtraction formula for cosine.]

57. Addition Formula for Tangent Use the addition formulas for cosine and sine to prove the addition formula for tangent. [*Hint:* Use

$$\tan(s + t) = \frac{\sin(s + t)}{\cos(s + t)}$$

and divide the numerator and denominator by $\cos s \cos t$.]

7.3 Double-Angle, Half-Angle, and Product-Sum Formulas

The identities we consider in this section are consequences of the addition formulas. The **double-angle formulas** allow us to find the values of the trigonometric functions at $2x$ from their values at x. The **half-angle formulas** relate the values of the trigonometric functions at $\frac{1}{2}x$ to their values at x. The **product-sum formulas** relate products of sines and cosines to sums of sines and cosines.

Double-Angle Formulas

The formulas in the following box are immediate consequences of the addition formulas, which we proved in the preceding section.

Double-Angle Formulas

Formula for sine: $\sin 2x = 2 \sin x \cos x$

Formulas for cosine: $\cos 2x = \cos^2 x - \sin^2 x$

$= 1 - 2 \sin^2 x$

$= 2 \cos^2 x - 1$

Formula for tangent: $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$

The proofs for the formulas for cosine are given here. You are asked to prove the remaining formulas in Exercises 33 and 34.

Proof of Double-Angle Formulas for Cosine

$$\begin{aligned}\cos 2x &= \cos(x + x)\\ &= \cos x \cos x - \sin x \sin x\\ &= \cos^2 x - \sin^2 x\end{aligned}$$

The second and third formulas for $\cos 2x$ are obtained from the formula we just proved and the Pythagorean identity. Substituting $\cos^2 x = 1 - \sin^2 x$ gives

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= (1 - \sin^2 x) - \sin^2 x\\ &= 1 - 2 \sin^2 x\end{aligned}$$

The third formula is obtained in the same way, by substituting $\sin^2 x = 1 - \cos^2 x$. ■

Example 1 Using the Double-Angle Formulas

If $\cos x = -\frac{2}{3}$ and x is in quadrant II, find $\cos 2x$ and $\sin 2x$.

Solution Using one of the double-angle formulas for cosine, we get

$$\begin{aligned}\cos 2x &= 2 \cos^2 x - 1\\ &= 2\left(-\frac{2}{3}\right)^2 - 1 = \frac{8}{9} - 1 = -\frac{1}{9}\end{aligned}$$

To use the formula $\sin 2x = 2 \sin x \cos x$, we need to find $\sin x$ first. We have

$$\sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(-\tfrac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$$

where we have used the positive square root because $\sin x$ is positive in quadrant II. Thus

$$\begin{aligned}\sin 2x &= 2 \sin x \cos x\\ &= 2\left(\frac{\sqrt{5}}{3}\right)\left(-\frac{2}{3}\right) = -\frac{4\sqrt{5}}{9}\end{aligned}$$ ■

Example 2 A Triple-Angle Formula

Write $\cos 3x$ in terms of $\cos x$.

Solution

$$\begin{aligned}
\cos 3x &= \cos(2x + x) \\
&= \cos 2x \cos x - \sin 2x \sin x && \text{Addition formula} \\
&= (2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x && \text{Double-angle formulas} \\
&= 2\cos^3 x - \cos x - 2\sin^2 x \cos x && \text{Expand} \\
&= 2\cos^3 x - \cos x - 2\cos x(1 - \cos^2 x) && \text{Pythagorean identity} \\
&= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x && \text{Expand} \\
&= 4\cos^3 x - 3\cos x && \text{Simplify}
\end{aligned}$$

■

Example 2 shows that $\cos 3x$ can be written as a polynomial of degree 3 in $\cos x$. The identity $\cos 2x = 2\cos^2 x - 1$ shows that $\cos 2x$ is a polynomial of degree 2 in $\cos x$. In fact, for any natural number n, we can write $\cos nx$ as a polynomial in $\cos x$ of degree n (see Exercise 87). The analogous result for $\sin nx$ is not true in general.

Example 3 Proving an Identity

Prove the identity $\dfrac{\sin 3x}{\sin x \cos x} = 4\cos x - \sec x$.

Solution We start with the left-hand side.

$$\begin{aligned}
\frac{\sin 3x}{\sin x \cos x} &= \frac{\sin(x + 2x)}{\sin x \cos x} \\
&= \frac{\sin x \cos 2x + \cos x \sin 2x}{\sin x \cos x} && \text{Addition formula} \\
&= \frac{\sin x(2\cos^2 x - 1) + \cos x(2\sin x \cos x)}{\sin x \cos x} && \text{Double-angle formulas} \\
&= \frac{\sin x(2\cos^2 x - 1)}{\sin x \cos x} + \frac{\cos x(2\sin x \cos x)}{\sin x \cos x} && \text{Separate fraction} \\
&= \frac{2\cos^2 x - 1}{\cos x} + 2\cos x && \text{Cancel} \\
&= 2\cos x - \frac{1}{\cos x} + 2\cos x && \text{Separate fraction} \\
&= 4\cos x - \sec x && \text{Reciprocal identity}
\end{aligned}$$

■

Half-Angle Formulas

The following formulas allow us to write any trigonometric expression involving even powers of sine and cosine in terms of the first power of cosine only. This technique is important in calculus. The half-angle formulas are immediate consequences of these formulas.

Formulas for Lowering Powers

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Proof The first formula is obtained by solving for $\sin^2 x$ in the double-angle formula $\cos 2x = 1 - 2\sin^2 x$. Similarly, the second formula is obtained by solving for $\cos^2 x$ in the double-angle formula $\cos 2x = 2\cos^2 x - 1$.

The last formula follows from the first two and the reciprocal identities:

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\dfrac{1 - \cos 2x}{2}}{\dfrac{1 + \cos 2x}{2}} = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Example 4 Lowering Powers in a Trigonometric Expression

Express $\sin^2 x \cos^2 x$ in terms of the first power of cosine.

Solution We use the formulas for lowering powers repeatedly.

$$\begin{aligned}
\sin^2 x \cos^2 x &= \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right) \\
&= \frac{1 - \cos^2 2x}{4} = \frac{1}{4} - \frac{1}{4}\cos^2 2x \\
&= \frac{1}{4} - \frac{1}{4}\left(\frac{1 + \cos 4x}{2}\right) = \frac{1}{4} - \frac{1}{8} - \frac{\cos 4x}{8} \\
&= \frac{1}{8} - \frac{1}{8}\cos 4x = \frac{1}{8}(1 - \cos 4x)
\end{aligned}$$

Another way to obtain this identity is to use the double-angle formula for sine in the form $\sin x \cos x = \frac{1}{2}\sin 2x$. Thus

$$\sin^2 x \cos^2 x = \frac{1}{4}\sin^2 2x = \frac{1}{4}\left(\frac{1 - \cos 4x}{2}\right) = \frac{1}{8}(1 - \cos 4x)$$

Half-Angle Formulas

$$\sin\frac{u}{2} = \pm\sqrt{\frac{1 - \cos u}{2}} \qquad \cos\frac{u}{2} = \pm\sqrt{\frac{1 + \cos u}{2}}$$

$$\tan\frac{u}{2} = \frac{1 - \cos u}{\sin u} = \frac{\sin u}{1 + \cos u}$$

The choice of the $+$ or $-$ sign depends on the quadrant in which $u/2$ lies.

■ **Proof** We substitute $x = u/2$ in the formulas for lowering powers and take the square root of each side. This yields the first two half-angle formulas. In the case of the half-angle formula for tangent, we get

$$\tan\frac{u}{2} = \pm\sqrt{\frac{1-\cos u}{1+\cos u}}$$

$$= \pm\sqrt{\left(\frac{1-\cos u}{1+\cos u}\right)\left(\frac{1-\cos u}{1-\cos u}\right)} \qquad \text{Multiply numerator and denominator by } 1-\cos u$$

$$= \pm\sqrt{\frac{(1-\cos u)^2}{1-\cos^2 u}} \qquad \text{Simplify}$$

$$= \pm\frac{|1-\cos u|}{|\sin u|} \qquad \sqrt{A^2} = |A| \text{ and } 1-\cos^2 u = \sin^2 u$$

Now, $1 - \cos u$ is nonnegative for all values of u. It is also true that $\sin u$ and $\tan(u/2)$ always have the same sign. (Verify this.) It follows that

$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$$

The other half-angle formula for tangent is derived from this by multiplying the numerator and denominator by $1 + \cos u$. ■

Example 5 Using a Half-Angle Formula

Find the exact value of $\sin 22.5°$.

Solution Since 22.5° is half of 45°, we use the half-angle formula for sine with $u = 45°$. We choose the + sign because 22.5° is in the first quadrant.

$$\sin\frac{45°}{2} = \sqrt{\frac{1-\cos 45°}{2}} \qquad \text{Half-angle formula}$$

$$= \sqrt{\frac{1-\sqrt{2}/2}{2}} \qquad \cos 45° = \sqrt{2}/2$$

$$= \sqrt{\frac{2-\sqrt{2}}{4}} \qquad \text{Common denominator}$$

$$= \tfrac{1}{2}\sqrt{2-\sqrt{2}} \qquad \text{Simplify}$$

■

Example 6 Using a Half-Angle Formula

Find $\tan(u/2)$ if $\sin u = \frac{2}{5}$ and u is in quadrant II.

Solution To use the half-angle formulas for tangent, we first need to find $\cos u$. Since cosine is negative in quadrant II, we have

$$\cos u = -\sqrt{1-\sin^2 u}$$

$$= -\sqrt{1-\left(\tfrac{2}{5}\right)^2} = -\frac{\sqrt{21}}{5}$$

Thus
$$\tan\frac{u}{2} = \frac{1-\cos u}{\sin u}$$
$$= \frac{1+\sqrt{21}/5}{\frac{2}{5}} = \frac{5+\sqrt{21}}{2}$$
■

Product-Sum Formulas

It is possible to write the product $\sin u \cos v$ as a sum of trigonometric functions. To see this, consider the addition and subtraction formulas for the sine function:

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$
$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

Adding the left- and right-hand sides of these formulas gives

$$\sin(u+v) + \sin(u-v) = 2\sin u \cos v$$

Dividing by 2 yields the formula

$$\sin u \cos v = \tfrac{1}{2}[\sin(u+v) + \sin(u-v)]$$

The other three **product-to-sum formulas** follow from the addition formulas in a similar way.

Product-to-Sum Formulas

$$\sin u \cos v = \tfrac{1}{2}[\sin(u+v) + \sin(u-v)]$$
$$\cos u \sin v = \tfrac{1}{2}[\sin(u+v) - \sin(u-v)]$$
$$\cos u \cos v = \tfrac{1}{2}[\cos(u+v) + \cos(u-v)]$$
$$\sin u \sin v = \tfrac{1}{2}[\cos(u-v) - \cos(u+v)]$$

Example 7 Expressing a Trigonometric Product as a Sum

Express $\sin 3x \sin 5x$ as a sum of trigonometric functions.

Solution Using the fourth product-to-sum formula with $u = 3x$ and $v = 5x$ and the fact that cosine is an even function, we get

$$\sin 3x \sin 5x = \tfrac{1}{2}[\cos(3x-5x) - \cos(3x+5x)]$$
$$= \tfrac{1}{2}\cos(-2x) - \tfrac{1}{2}\cos 8x$$
$$= \tfrac{1}{2}\cos 2x - \tfrac{1}{2}\cos 8x$$
■

The product-to-sum formulas can also be used as sum-to-product formulas. This is possible because the right-hand side of each product-to-sum formula is a sum and the left side is a product. For example, if we let

$$u = \frac{x+y}{2} \qquad \text{and} \qquad v = \frac{x-y}{2}$$

in the first product-to-sum formula, we get

$$\sin \frac{x+y}{2} \cos \frac{x-y}{2} = \tfrac{1}{2}(\sin x + \sin y)$$

so

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

The remaining three of the following **sum-to-product formulas** are obtained in a similar manner.

Sum-to-Product Formulas

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

Example 8 Expressing a Trigonometric Sum as a Product

Write $\sin 7x + \sin 3x$ as a product.

Solution The first sum-to-product formula gives

$$\begin{aligned} \sin 7x + \sin 3x &= 2 \sin \frac{7x+3x}{2} \cos \frac{7x-3x}{2} \\ &= 2 \sin 5x \cos 2x \end{aligned}$$

■

Example 9 Proving an Identity

Verify the identity $\dfrac{\sin 3x - \sin x}{\cos 3x + \cos x} = \tan x$.

Solution We apply the second sum-to-product formula to the numerator and the third formula to the denominator.

$$\begin{aligned} \text{LHS} = \frac{\sin 3x - \sin x}{\cos 3x + \cos x} &= \frac{2 \cos \dfrac{3x+x}{2} \sin \dfrac{3x-x}{2}}{2 \cos \dfrac{3x+x}{2} \cos \dfrac{3x-x}{2}} && \text{Sum-to-product formulas} \\ &= \frac{2 \cos 2x \sin x}{2 \cos 2x \cos x} && \text{Simplify} \\ &= \frac{\sin x}{\cos x} = \tan x = \text{RHS} && \text{Cancel} \end{aligned}$$

■

7.3 Exercises

1–8 ■ Find sin $2x$, cos $2x$, and tan $2x$ from the given information.

1. $\sin x = \frac{5}{13}$, x in quadrant I

2. $\tan x = -\frac{4}{3}$, x in quadrant II

3. $\cos x = \frac{4}{5}$, $\csc x < 0$

4. $\csc x = 4$, $\tan x < 0$

5. $\sin x = -\frac{3}{5}$, x in quadrant III

6. $\sec x = 2$, x in quadrant IV

7. $\tan x = -\frac{1}{3}$, $\cos x > 0$

8. $\cot x = \frac{2}{3}$, $\sin x > 0$

9–14 ■ Use the formulas for lowering powers to rewrite the expression in terms of the first power of cosine, as in Example 4.

9. $\sin^4 x$

10. $\cos^4 x$

11. $\cos^2 x \sin^4 x$

12. $\cos^4 x \sin^2 x$

13. $\cos^4 x \sin^4 x$

14. $\cos^6 x$

15–26 ■ Use an appropriate half-angle formula to find the exact value of the expression.

15. $\sin 15°$

16. $\tan 15°$

17. $\tan 22.5°$

18. $\sin 75°$

19. $\cos 165°$

20. $\cos 112.5°$

21. $\tan \frac{\pi}{8}$

22. $\cos \frac{3\pi}{8}$

23. $\cos \frac{\pi}{12}$

24. $\tan \frac{5\pi}{12}$

25. $\sin \frac{9\pi}{8}$

26. $\sin \frac{11\pi}{12}$

27–32 ■ Simplify the expression by using a double-angle formula or a half-angle formula.

27. **(a)** $2 \sin 18° \cos 18°$ **(b)** $2 \sin 3\theta \cos 3\theta$

28. **(a)** $\dfrac{2 \tan 7°}{1 - \tan^2 7°}$ **(b)** $\dfrac{2 \tan 7\theta}{1 - \tan^2 7\theta}$

29. **(a)** $\cos^2 34° - \sin^2 34°$ **(b)** $\cos^2 5\theta - \sin^2 5\theta$

30. **(a)** $\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2}$ **(b)** $2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

31. **(a)** $\dfrac{\sin 8°}{1 + \cos 8°}$ **(b)** $\dfrac{1 - \cos 4\theta}{\sin 4\theta}$

32. **(a)** $\sqrt{\dfrac{1 - \cos 30°}{2}}$ **(b)** $\sqrt{\dfrac{1 - \cos 8\theta}{2}}$

33. Use the addition formula for sine to prove the double-angle formula for sine.

34. Use the addition formula for tangent to prove the double-angle formula for tangent.

35–40 ■ Find $\sin \frac{x}{2}$, $\cos \frac{x}{2}$, and $\tan \frac{x}{2}$ from the given information.

35. $\sin x = \frac{3}{5}$, $0° < x < 90°$

36. $\cos x = -\frac{4}{5}$, $180° < x < 270°$

37. $\csc x = 3$, $90° < x < 180°$

38. $\tan x = 1$, $0° < x < 90°$

39. $\sec x = \frac{3}{2}$, $270° < x < 360°$

40. $\cot x = 5$, $180° < x < 270°$

41–46 ■ Write the product as a sum.

41. $\sin 2x \cos 3x$

42. $\sin x \sin 5x$

43. $\cos x \sin 4x$

44. $\cos 5x \cos 3x$

45. $3 \cos 4x \cos 7x$

46. $11 \sin \frac{x}{2} \cos \frac{x}{4}$

47–52 ■ Write the sum as a product.

47. $\sin 5x + \sin 3x$

48. $\sin x - \sin 4x$

49. $\cos 4x - \cos 6x$

50. $\cos 9x + \cos 2x$

51. $\sin 2x - \sin 7x$

52. $\sin 3x + \sin 4x$

53–58 ■ Find the value of the product or sum.

53. $2 \sin 52.5° \sin 97.5°$

54. $3 \cos 37.5° \cos 7.5°$

55. $\cos 37.5° \sin 7.5°$

56. $\sin 75° + \sin 15°$

57. $\cos 255° - \cos 195°$

58. $\cos \frac{\pi}{12} + \cos \frac{5\pi}{12}$

59–76 ■ Prove the identity.

59. $\cos^2 5x - \sin^2 5x = \cos 10x$

60. $\sin 8x = 2 \sin 4x \cos 4x$

61. $(\sin x + \cos x)^2 = 1 + \sin 2x$

62. $\dfrac{2 \tan x}{1 + \tan^2 x} = \sin 2x$

63. $\dfrac{\sin 4x}{\sin x} = 4 \cos x \cos 2x$

64. $\dfrac{1 + \sin 2x}{\sin 2x} = 1 + \frac{1}{2} \sec x \csc x$

65. $\dfrac{2(\tan x - \cot x)}{\tan^2 x - \cot^2 x} = \sin 2x$

66. $\cot 2x = \dfrac{1 - \tan^2 x}{2 \tan x}$

67. $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$

68. $4(\sin^6 x + \cos^6 x) = 4 - 3\sin^2 2x$

69. $\cos^4 x - \sin^4 x = \cos 2x$

70. $\tan^2\left(\dfrac{x}{2} + \dfrac{\pi}{4}\right) = \dfrac{1 + \sin x}{1 - \sin x}$

71. $\dfrac{\sin x + \sin 5x}{\cos x + \cos 5x} = \tan 3x$

72. $\dfrac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \cot 2x$

73. $\dfrac{\sin 10x}{\sin 9x + \sin x} = \dfrac{\cos 5x}{\cos 4x}$

74. $\dfrac{\sin x + \sin 3x + \sin 5x}{\cos x + \cos 3x + \cos 5x} = \tan 3x$

75. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\left(\dfrac{x + y}{2}\right)$

76. $\tan y = \dfrac{\sin(x + y) - \sin(x - y)}{\cos(x + y) + \cos(x - y)}$

77. Show that $\sin 130° - \sin 110° = -\sin 10°$.

78. Show that $\cos 100° - \cos 200° = \sin 50°$.

79. Show that $\sin 45° + \sin 15° = \sin 75°$.

80. Show that $\cos 87° + \cos 33° = \sin 63°$.

81. Prove the identity

$$\frac{\sin x + \sin 2x + \sin 3x + \sin 4x + \sin 5x}{\cos x + \cos 2x + \cos 3x + \cos 4x + \cos 5x} = \tan 3x$$

82. Use the identity

$$\sin 2x = 2\sin x\cos x$$

n times to show that

$$\sin(2^n x) = 2^n \sin x \cos x \cos 2x \cos 4x \cdots \cos 2^{n-1}x$$

83. **(a)** Graph $f(x) = \dfrac{\sin 3x}{\sin x} - \dfrac{\cos 3x}{\cos x}$ and make a conjecture.

(b) Prove the conjecture you made in part (a).

84. **(a)** Graph $f(x) = \cos 2x + 2\sin^2 x$ and make a conjecture.

(b) Prove the conjecture you made in part (a).

85. Let $f(x) = \sin 6x + \sin 7x$.

(a) Graph $y = f(x)$.

(b) Verify that $f(x) = 2\cos\frac{1}{2}x \sin\frac{13}{2}x$.

(c) Graph $y = 2\cos\frac{1}{2}x$ and $y = -2\cos\frac{1}{2}x$, together with the graph in part (a), in the same viewing rectangle. How are these graphs related to the graph of f?

86. Let $3x = \pi/3$ and let $y = \cos x$. Use the result of Example 2 to show that y satisfies the equation

$$8y^3 - 6y - 1 = 0$$

NOTE This equation has roots of a certain kind that are used to show that the angle $\pi/3$ cannot be trisected using a ruler and compass only.

87. **(a)** Show that there is a polynomial $P(t)$ of degree 4 such that $\cos 4x = P(\cos x)$ (see Example 2).

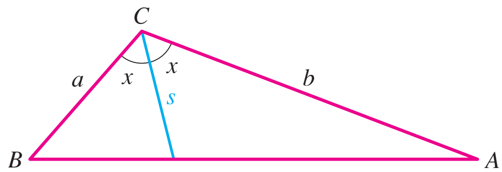
(b) Show that there is a polynomial $Q(t)$ of degree 5 such that $\cos 5x = Q(\cos x)$.

NOTE In general, there is a polynomial $P_n(t)$ of degree n such that $\cos nx = P_n(\cos x)$. These polynomials are called *Tchebycheff polynomials*, after the Russian mathematician P. L. Tchebycheff (1821–1894).

88. In triangle ABC (see the figure) the line segment s bisects angle C. Show that the length of s is given by

$$s = \frac{2ab\cos x}{a + b}$$

[*Hint:* Use the Law of Sines.]



89. If A, B, and C are the angles in a triangle, show that

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$$

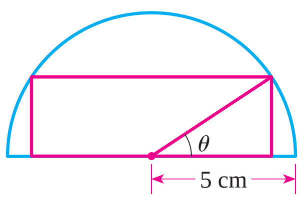
90. A rectangle is to be inscribed in a semicircle of radius 5 cm as shown in the figure.

(a) Show that the area of the rectangle is modeled by the function

$$A(\theta) = 25\sin 2\theta$$

(b) Find the largest possible area for such an inscribed rectangle.

(c) Find the dimensions of the inscribed rectangle with the largest possible area.



Applications

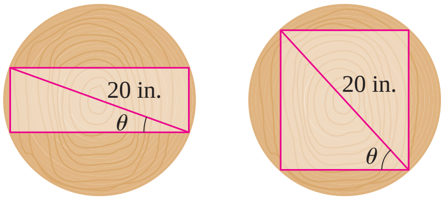
91. Sawing a Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 in.

(a) Show that the cross-sectional area of the beam is modeled by the function

$$A(\theta) = 200\sin 2\theta$$

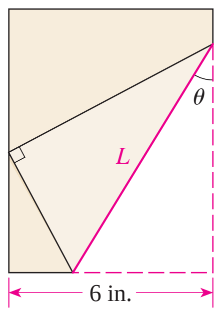
where θ is as shown in the figure on the next page.

(b) Show that the maximum cross-sectional area of such a beam is 200 in^2. [*Hint:* Use the fact that $\sin u$ achieves its maximum value at $u = \pi/2$.]



92. Length of a Fold The lower right-hand corner of a long piece of paper 6 in. wide is folded over to the left-hand edge as shown. The length L of the fold depends on the angle θ. Show that

$$L = \frac{3}{\sin\theta\cos^2\theta}$$



93. Sound Beats When two pure notes that are close in frequency are played together, their sounds interfere to produce *beats*; that is, the loudness (or amplitude) of the sound alternately increases and decreases. If the two notes are given by

$$f_1(t) = \cos 11t \qquad \text{and} \qquad f_2(t) = \cos 13t$$

the resulting sound is $f(t) = f_1(t) + f_2(t)$.

(a) Graph the function $y = f(t)$.

(b) Verify that $f(t) = 2\cos t\cos 12t$.

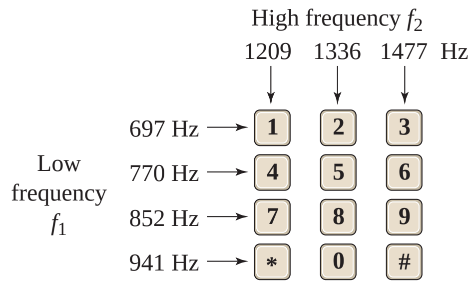
(c) Graph $y = 2\cos t$ and $y = -2\cos t$, together with the graph in part (a), in the same viewing rectangle. How do these graphs describe the variation in the loudness of the sound?

94. Touch-Tone Telephones When a key is pressed on a touch-tone telephone, the keypad generates two pure tones, which combine to produce a sound that uniquely identifies the key. The figure shows the low frequency f_1 and the high frequency f_2 associated with each key. Pressing a key produces the sound wave $y = \sin(2\pi f_1 t) + \sin(2\pi f_2 t)$.

(a) Find the function that models the sound produced when the 4 key is pressed.

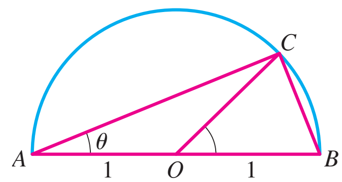
(b) Use a sum-to-product formula to express the sound generated by the 4 key as a product of a sine and a cosine function.

(c) Graph the sound wave generated by the 4 key, from $t = 0$ to $t = 0.006$ s.



Discovery • Discussion

95. Geometric Proof of a Double-Angle Formula Use the figure to prove that $\sin 2\theta = 2\sin\theta\cos\theta$.



Hint: Find the area of triangle ABC in two different ways. You will need the following facts from geometry:

An angle inscribed in a semicircle is a right angle, so $\angle ACB$ is a right angle.

The central angle subtended by the chord of a circle is twice the angle subtended by the chord on the circle, so $\angle BOC$ is 2θ.

7.4 Inverse Trigonometric Functions

If f is a one-to-one function with domain A and range B, then its inverse f^{-1} is the function with domain B and range A defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

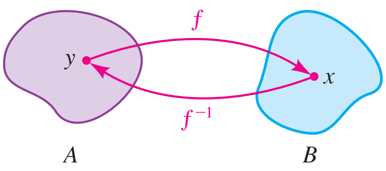


Figure 1
$f^{-1}(x) = y \Leftrightarrow f(y) = x$

(See Section 2.8.) In other words, f^{-1} is the rule that reverses the action of f. Figure 1 represents the actions of f and f^{-1} graphically.

For a function to have an inverse, it must be one-to-one. Since the trigonometric functions are not one-to-one, they do not have inverses. It is possible, however, to restrict the domains of the trigonometric functions in such a way that the resulting functions are one-to-one.

The Inverse Sine Function

Let's first consider the sine function. There are many ways to restrict the domain of sine so that the new function is one-to-one. A natural way to do this is to restrict the domain to the interval $[-\pi/2, \pi/2]$. The reason for this choice is that sine attains each of its values exactly once on this interval. As we see in Figure 2, on this restricted domain the sine function is one-to-one (by the Horizontal Line Test), and so has an inverse.

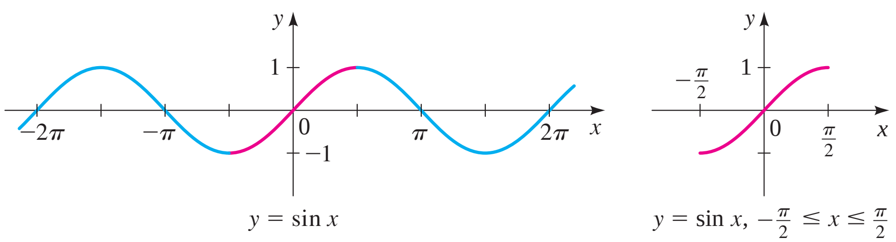


Figure 2

The inverse of the function sin is the function $\sin^{-1}$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

for $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. The graph of $y = \sin^{-1} x$ is shown in Figure 3; it is obtained by reflecting the graph of $y = \sin x$, $-\pi/2 \le x \le \pi/2$, in the line $y = x$.

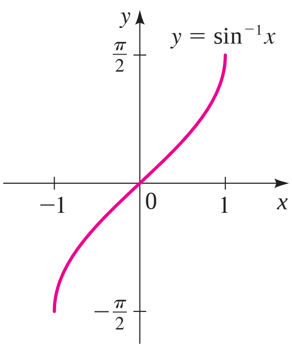


Figure 3

Definition of the Inverse Sine Function

The **inverse sine function** is the function $\mathbf{\sin^{-1}}$ with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$ defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

The inverse sine function is also called **arcsine**, denoted by **arcsin**.

Thus, $\sin^{-1} x$ *is the number in the interval* $[-\pi/2, \pi/2]$ *whose sine is* x. In other words, $\sin(\sin^{-1} x) = x$. In fact, from the general properties of inverse functions studied in Section 2.8, we have the following relations.

$$\sin(\sin^{-1} x) = x \qquad \text{for } -1 \le x \le 1$$

$$\sin^{-1}(\sin x) = x \qquad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Example 1 Evaluating the Inverse Sine Function

Find: (a) $\sin^{-1}\frac{1}{2}$, (b) $\sin^{-1}\left(-\frac{1}{2}\right)$, and (c) $\sin^{-1}\frac{3}{2}$.

Solution

(a) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{1}{2}$ is $\pi/6$. Thus, $\sin^{-1}\frac{1}{2} = \pi/6$.

(b) The number in the interval $[-\pi/2, \pi/2]$ whose sine is $-\frac{1}{2}$ is $-\pi/6$. Thus, $\sin^{-1}\left(-\frac{1}{2}\right) = -\pi/6$.

(c) Since $\frac{3}{2} > 1$, it is not in the domain of $\sin^{-1} x$, so $\sin^{-1}\frac{3}{2}$ is not defined. ■

Example 2 Using a Calculator to Evaluate Inverse Sine

Find approximate values for (a) $\sin^{-1}(0.82)$ and (b) $\sin^{-1}\frac{1}{3}$.

Solution Since no rational multiple of π has a sine of 0.82 or $\frac{1}{3}$, we use a calculator to approximate these values. Using the INV SIN, or SIN^{-1}, or ARC SIN key(s) on the calculator (with the calculator in radian mode), we get

(a) $\sin^{-1}(0.82) \approx 0.96141$ (b) $\sin^{-1}\frac{1}{3} \approx 0.33984$ ■

Example 3 Composing Trigonometric Functions and Their Inverses

Find $\cos\left(\sin^{-1}\frac{3}{5}\right)$.

Solution 1 It's easy to find $\sin\left(\sin^{-1}\frac{3}{5}\right)$. In fact, by the properties of inverse functions, this value is exactly $\frac{3}{5}$. To find $\cos\left(\sin^{-1}\frac{3}{5}\right)$, we reduce this to the easier problem by writing the cosine function in terms of the sine function. Let $u = \sin^{-1}\frac{3}{5}$. Since $-\pi/2 \le u \le \pi/2$, $\cos u$ is positive and we can write

$$\cos u = +\sqrt{1 - \sin^2 u}$$

Thus

$$\cos\left(\sin^{-1}\tfrac{3}{5}\right) = \sqrt{1 - \sin^2\left(\sin^{-1}\tfrac{3}{5}\right)}$$

$$= \sqrt{1 - \left(\tfrac{3}{5}\right)^2} = \sqrt{1 - \tfrac{9}{25}} = \sqrt{\tfrac{16}{25}} = \tfrac{4}{5}$$

Solution 2 Let $\theta = \sin^{-1}\frac{3}{5}$. Then θ is the number in the interval $[-\pi/2, \pi/2]$ whose sine is $\frac{3}{5}$. Let's interpret θ as an angle and draw a right triangle with θ as one of its acute angles, with opposite side 3 and hypotenuse 5 (see Figure 4). The remaining leg of the triangle is found by the Pythagorean Theorem to be 4. From the figure we get

$$\cos\left(\sin^{-1}\tfrac{3}{5}\right) = \cos\theta = \tfrac{4}{5}$$ ■

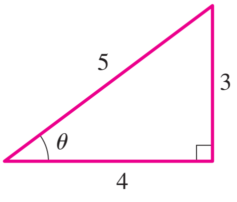


Figure 4

From Solution 2 of Example 3 we can immediately find the values of the other trigonometric functions of $\theta = \sin^{-1}\frac{3}{5}$ from the triangle. Thus

$$\tan\left(\sin^{-1}\tfrac{3}{5}\right) = \tfrac{3}{4} \qquad \sec\left(\sin^{-1}\tfrac{3}{5}\right) = \tfrac{5}{4} \qquad \csc\left(\sin^{-1}\tfrac{3}{5}\right) = \tfrac{5}{3}$$

The Inverse Cosine Function

If the domain of the cosine function is restricted to the interval $[0, \pi]$, the resulting function is one-to-one and so has an inverse. We choose this interval because on it, cosine attains each of its values exactly once (see Figure 5).

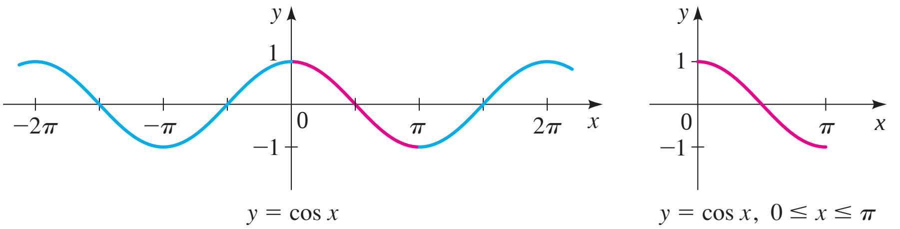


Figure 5

Definition of the Inverse Cosine Function

The **inverse cosine function** is the function $\mathbf{cos^{-1}}$ with domain $[-1, 1]$ and range $[0, \pi]$ defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

The inverse cosine function is also called **arccosine**, denoted by **arccos**.

Thus, $y = \cos^{-1} x$ *is the number in the interval* $[0, \pi]$ *whose cosine is* x. The following relations follow from the inverse function properties.

$$\cos(\cos^{-1} x) = x \qquad \text{for } -1 \le x \le 1$$
$$\cos^{-1}(\cos x) = x \qquad \text{for } 0 \le x \le \pi$$

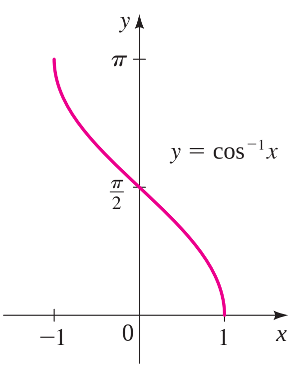


Figure 6

The graph of $y = \cos^{-1} x$ is shown in Figure 6; it is obtained by reflecting the graph of $y = \cos x$, $0 \le x \le \pi$, in the line $y = x$.

Example 4 Evaluating the Inverse Cosine Function

Find: (a) $\cos^{-1}(\sqrt{3}/2)$, (b) $\cos^{-1} 0$, and (c) $\cos^{-1} \frac{5}{7}$.

Solution

(a) The number in the interval $[0, \pi]$ whose cosine is $\sqrt{3}/2$ is $\pi/6$. Thus, $\cos^{-1}(\sqrt{3}/2) = \pi/6$.

(b) The number in the interval $[0, \pi]$ whose cosine is 0 is $\pi/2$. Thus, $\cos^{-1} 0 = \pi/2$.

(c) Since no rational multiple of π has cosine $\frac{5}{7}$, we use a calculator (in radian mode) to find this value approximately: $\cos^{-1} \frac{5}{7} \approx 0.77519$. ■

Example 5 Composing Trigonometric Functions and Their Inverses

Write $\sin(\cos^{-1}x)$ and $\tan(\cos^{-1}x)$ as algebraic expressions in x for $-1 \le x \le 1$.

Solution 1 Let $u = \cos^{-1}x$. We need to find $\sin u$ and $\tan u$ in terms of x. As in Example 3 the idea here is to write sine and tangent in terms of cosine. We have

$$\sin u = \pm\sqrt{1 - \cos^2 u} \qquad \text{and} \qquad \tan u = \frac{\sin u}{\cos u} = \frac{\pm\sqrt{1 - \cos^2 u}}{\cos u}$$

To choose the proper signs, note that u lies in the interval $[0, \pi]$ because $u = \cos^{-1}x$. Since $\sin u$ is positive on this interval, the $+$ sign is the correct choice. Substituting $u = \cos^{-1}x$ in the displayed equations and using the relation $\cos(\cos^{-1}x) = x$ gives

$$\sin(\cos^{-1}x) = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \frac{\sqrt{1 - x^2}}{x}$$

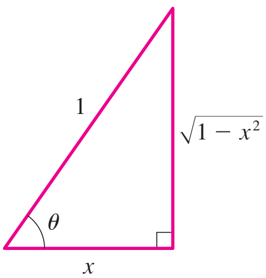


Figure 7

$\cos\theta = \frac{x}{1} = x$

Solution 2 Let $\theta = \cos^{-1}x$, so $\cos\theta = x$. In Figure 7 we draw a right triangle with an acute angle θ, adjacent side x, and hypotenuse 1. By the Pythagorean Theorem, the remaining leg is $\sqrt{1 - x^2}$. From the figure,

$$\sin(\cos^{-1}x) = \sin\theta = \sqrt{1 - x^2} \qquad \text{and} \qquad \tan(\cos^{-1}x) = \tan\theta = \frac{\sqrt{1 - x^2}}{x}$$ ■

NOTE In Solution 2 of Example 5 it may seem that because we are sketching a triangle, the angle $\theta = \cos^{-1}x$ must be acute. But it turns out that the triangle method works for any θ and for any x. The domains and ranges of all six inverse trigonometric functions have been chosen in such a way that we can always use a triangle to find $S(T^{-1}(x))$, where S and T are any trigonometric functions.

Example 6 Composing a Trigonometric Function and an Inverse

Write $\sin(2\cos^{-1}x)$ as an algebraic expression in x for $-1 \le x \le 1$.

Solution Let $\theta = \cos^{-1}x$ and sketch a triangle as shown in Figure 8. We need to find $\sin 2\theta$, but from the triangle we can find trigonometric functions only of θ, not of 2θ. The double-angle identity for sine is useful here. We have

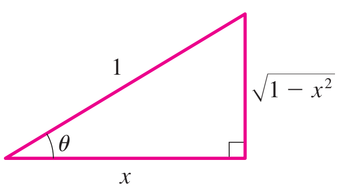


Figure 8

$\cos\theta = \frac{x}{1} = x$

$$\begin{aligned}\sin(2\cos^{-1}x) &= \sin 2\theta \\ &= 2\sin\theta\cos\theta && \text{Double-angle formula} \\ &= 2\left(\sqrt{1 - x^2}\right)x && \text{From triangle} \\ &= 2x\sqrt{1 - x^2}\end{aligned}$$ ■

The Inverse Tangent Function

We restrict the domain of the tangent function to the interval $(-\pi/2, \pi/2)$ in order to obtain a one-to-one function.

Definition of the Inverse Tangent Function

The **inverse tangent function** is the function $\mathbf{tan^{-1}}$ with domain $\mathbb{R}$ and range $(-\pi/2, \pi/2)$ defined by

$$\tan^{-1}x = y \quad \Leftrightarrow \quad \tan y = x$$

The inverse tangent function is also called **arctangent**, denoted by **arctan**.

Thus, $\tan^{-1}x$ *is the number in the interval* $(-\pi/2, \pi/2)$ *whose tangent is* x. The following relations follow from the inverse function properties.

$$\tan(\tan^{-1}x) = x \qquad \text{for } x \in \mathbb{R}$$

$$\tan^{-1}(\tan x) = x \qquad \text{for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Figure 9 shows the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ and the graph of its inverse function, $y = \tan^{-1}x$.

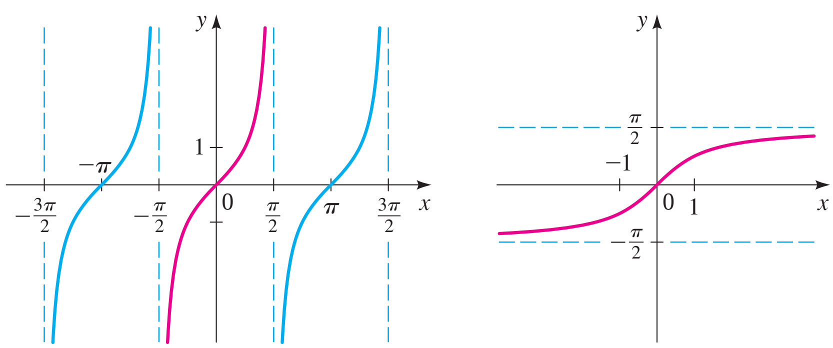


Figure 9 $y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$ $\qquad$ $y = \tan^{-1}x$

Example 7 Evaluating the Inverse Tangent Function

Find: (a) $\tan^{-1}1$, (b) $\tan^{-1}\sqrt{3}$, and (c) $\tan^{-1}(-20)$.

Solution

(a) The number in the interval $(-\pi/2, \pi/2)$ with tangent 1 is $\pi/4$. Thus, $\tan^{-1}1 = \pi/4$.

(b) The number in the interval $(-\pi/2, \pi/2)$ with tangent $\sqrt{3}$ is $\pi/3$. Thus, $\tan^{-1}\sqrt{3} = \pi/3$.

(c) We use a calculator to find that $\tan^{-1}(-20) \approx -1.52084$. ■

Example 8 The Angle of a Beam of Light

A lighthouse is located on an island that is 2 mi off a straight shoreline (see Figure 10). Express the angle formed by the beam of light and the shoreline in terms of the distance d in the figure.

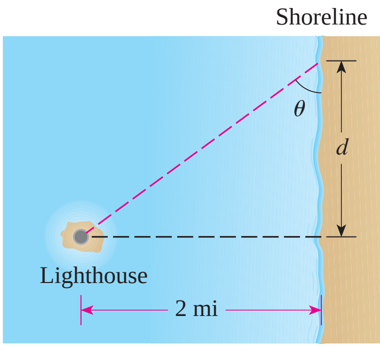


Figure 10

Solution From the figure we see that $\tan\theta = 2/d$. Taking the inverse tangent of both sides, we get

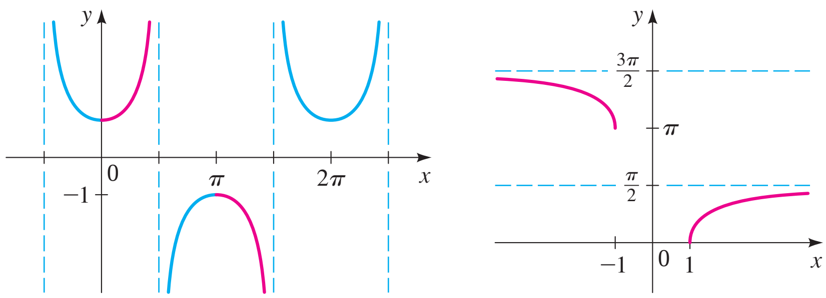
$$\tan^{-1}(\tan\theta) = \tan^{-1}\left(\frac{2}{d}\right)$$

$$\theta = \tan^{-1}\left(\frac{2}{d}\right) \qquad \text{Cancellation property}$$

The Inverse Secant, Cosecant, and Cotangent Functions

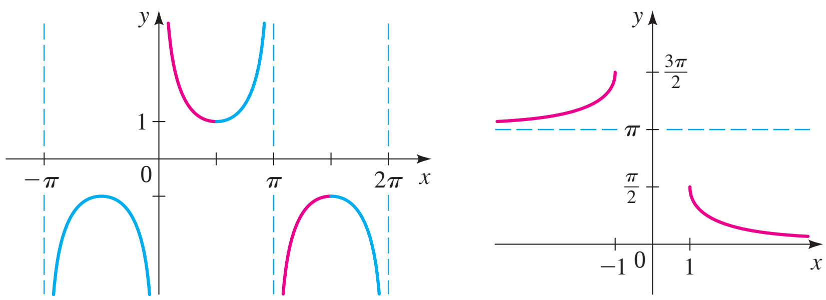
See Exercise 59 for a way of finding the values of these inverse trigonometric functions on a calculator.

To define the inverse functions of the secant, cosecant, and cotangent functions, we restrict the domain of each function to a set on which it is one-to-one and on which it attains all its values. Although any interval satisfying these criteria is appropriate, we choose to restrict the domains in a way that simplifies the choice of sign in computations involving inverse trigonometric functions. The choices we make are also appropriate for calculus. This explains the seemingly strange restriction for the domains of the secant and cosecant functions. We end this section by displaying the graphs of the secant, cosecant, and cotangent functions with their restricted domains and the graphs of their inverse functions (Figures 11–13).



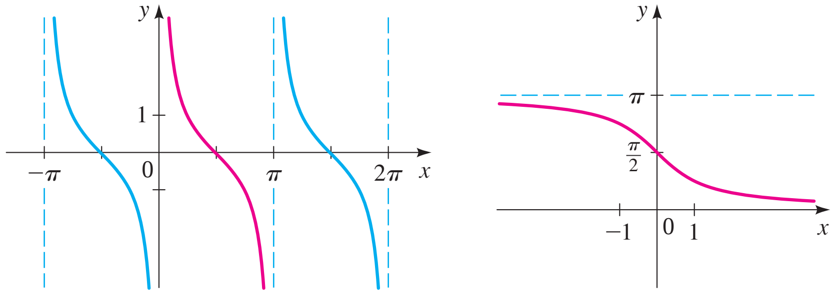
$y = \sec x,\ 0 \le x < \frac{\pi}{2},\ \pi \le x < \frac{3\pi}{2}$ $\qquad$ $y = \sec^{-1} x$

Figure 11
The inverse secant function



$y = \csc x,\ 0 < x \le \frac{\pi}{2},\ \pi < x \le \frac{3\pi}{2}$ $\qquad$ $y = \csc^{-1} x$

Figure 12
The inverse cosecant function



$y = \cot x,\ 0 < x < \pi$ $\qquad$ $y = \cot^{-1} x$

Figure 13
The inverse cotangent function

7.4 Exercises

1–8 ■ Find the exact value of each expression, if it is defined.

1. (a) $\sin^{-1}\frac{1}{2}$ **(b)** $\cos^{-1}\frac{1}{2}$ **(c)** $\cos^{-1}2$

2. (a) $\sin^{-1}\dfrac{\sqrt{3}}{2}$ **(b)** $\cos^{-1}\dfrac{\sqrt{3}}{2}$ **(c)** $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

3. (a) $\sin^{-1}\dfrac{\sqrt{2}}{2}$ **(b)** $\cos^{-1}\dfrac{\sqrt{2}}{2}$ **(c)** $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

4. (a) $\tan^{-1}\sqrt{3}$ **(b)** $\tan^{-1}(-\sqrt{3})$ **(c)** $\sin^{-1}\sqrt{3}$

5. (a) $\sin^{-1}1$ **(b)** $\cos^{-1}1$ **(c)** $\cos^{-1}(-1)$

6. (a) $\tan^{-1}1$ **(b)** $\tan^{-1}(-1)$ **(c)** $\tan^{-1}0$

7. (a) $\tan^{-1}\dfrac{\sqrt{3}}{3}$ **(b)** $\tan^{-1}\left(-\dfrac{\sqrt{3}}{3}\right)$ **(c)** $\sin^{-1}(-2)$

8. (a) $\sin^{-1}0$ **(b)** $\cos^{-1}0$ **(c)** $\cos^{-1}\left(-\frac{1}{2}\right)$

9–12 ■ Use a calculator to find an approximate value of each expression correct to five decimal places, if it is defined.

9. (a) $\sin^{-1}(0.13844)$

(b) $\cos^{-1}(-0.92761)$

10. (a) $\cos^{-1}(0.31187)$

(b) $\tan^{-1}(26.23110)$

11. (a) $\tan^{-1}(1.23456)$

(b) $\sin^{-1}(1.23456)$

12. (a) $\cos^{-1}(-0.25713)$

(b) $\tan^{-1}(-0.25713)$

13–28 ■ Find the exact value of the expression, if it is defined.

13. $\sin\left(\sin^{-1}\frac{1}{4}\right)$

14. $\cos\left(\cos^{-1}\frac{2}{3}\right)$

15. $\tan(\tan^{-1}5)$

16. $\sin(\sin^{-1}5)$

17. $\cos^{-1}\left(\cos\dfrac{\pi}{3}\right)$

18. $\tan^{-1}\left(\tan\dfrac{\pi}{6}\right)$

19. $\sin^{-1}\left(\sin\left(-\dfrac{\pi}{6}\right)\right)$

20. $\sin^{-1}\left(\sin\dfrac{5\pi}{6}\right)$

21. $\tan^{-1}\left(\tan\dfrac{2\pi}{3}\right)$

22. $\cos^{-1}\left(\cos\left(-\dfrac{\pi}{4}\right)\right)$

23. $\tan\left(\sin^{-1}\frac{1}{2}\right)$

24. $\sin(\sin^{-1}0)$

25. $\cos\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right)$

26. $\tan\left(\sin^{-1}\dfrac{\sqrt{2}}{2}\right)$

27. $\tan^{-1}\left(2 \sin \frac{\pi}{3}\right)$

28. $\cos^{-1}\left(\sqrt{3} \sin \frac{\pi}{6}\right)$

29–40 ■ Evaluate the expression by sketching a triangle, as in Solution 2 of Example 3.

29. $\sin\left(\cos^{-1} \frac{3}{5}\right)$

30. $\tan\left(\sin^{-1} \frac{4}{5}\right)$

31. $\sin\left(\tan^{-1} \frac{12}{5}\right)$

32. $\cos(\tan^{-1} 5)$

33. $\sec\left(\sin^{-1} \frac{12}{13}\right)$

34. $\csc\left(\cos^{-1} \frac{7}{25}\right)$

35. $\cos(\tan^{-1} 2)$

36. $\cot\left(\sin^{-1} \frac{2}{3}\right)$

37. $\sin\left(2 \cos^{-1} \frac{3}{5}\right)$

38. $\tan\left(2 \tan^{-1} \frac{5}{13}\right)$

39. $\sin\left(\sin^{-1} \frac{1}{2} + \cos^{-1} \frac{1}{2}\right)$

40. $\cos\left(\sin^{-1} \frac{3}{5} - \cos^{-1} \frac{3}{5}\right)$

41–48 ■ Rewrite the expression as an algebraic expression in x.

41. $\cos(\sin^{-1} x)$

42. $\sin(\tan^{-1} x)$

43. $\tan(\sin^{-1} x)$

44. $\cos(\tan^{-1} x)$

45. $\cos(2 \tan^{-1} x)$

46. $\sin(2 \sin^{-1} x)$

47. $\cos(\cos^{-1} x + \sin^{-1} x)$

48. $\sin(\tan^{-1} x - \sin^{-1} x)$

49–50 ■ **(a)** Graph the function and make a conjecture, and **(b)** prove that your conjecture is true.

49. $y = \sin^{-1} x + \cos^{-1} x$

50. $y = \tan^{-1} x + \tan^{-1} \frac{1}{x}$

51–52 ■ **(a)** Use a graphing device to find all solutions of the equation, correct to two decimal places, and **(b)** find the exact solution.

51. $\tan^{-1} x + \tan^{-1} 2x = \frac{\pi}{4}$

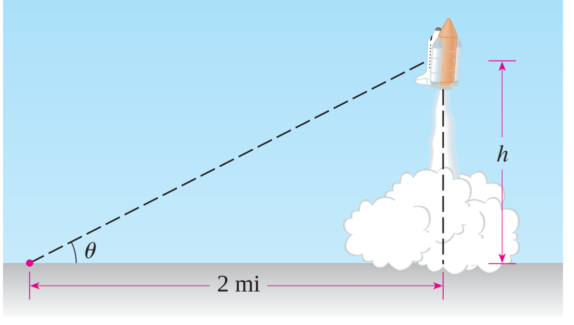
52. $\sin^{-1} x - \cos^{-1} x = 0$

Applications

53. Height of the Space Shuttle An observer views the space shuttle from a distance of 2 miles from the launch pad.

(a) Express the height of the space shuttle as a function of the angle of elevation θ.

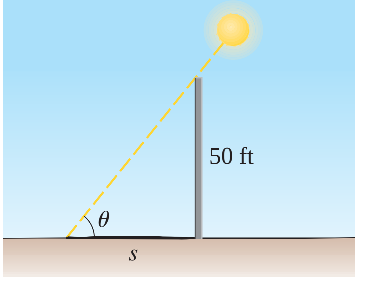
(b) Express the angle of elevation θ as a function of the height h of the space shuttle.



54. Height of a Pole A 50-ft pole casts a shadow as shown in the figure.

(a) Express the angle of elevation θ of the sun as a function of the length s of the shadow.

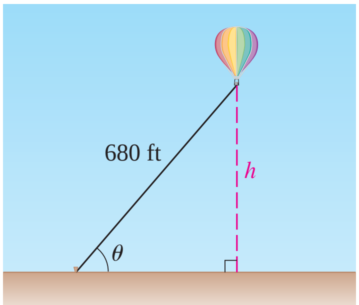
(b) Find the angle θ of elevation of the sun when the shadow is 20 ft long.



55. Height of a Balloon A 680-ft rope anchors a hot-air balloon as shown in the figure.

(a) Express the angle θ as a function of the height h of the balloon.

(b) Find the angle θ if the balloon is 500 ft high.



56. View from a Satellite The figures indicate that the higher the orbit of a satellite, the more of the earth the satellite can "see." Let θ, s, and h be as in the figure, and assume the earth is a sphere of radius 3960 mi.

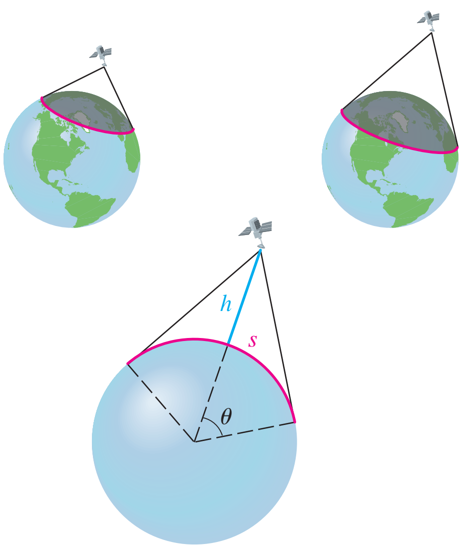
(a) Express the angle θ as a function of h.

(b) Express the distance s as a function of θ.

(c) Express the distance s as a function of h. [*Hint:* Find the composition of the functions in parts (a) and (b).]

(d) If the satellite is 100 mi above the earth, what is the distance s that it can see?

(e) How high does the satellite have to be in order to see both Los Angeles and New York, 2450 mi apart?



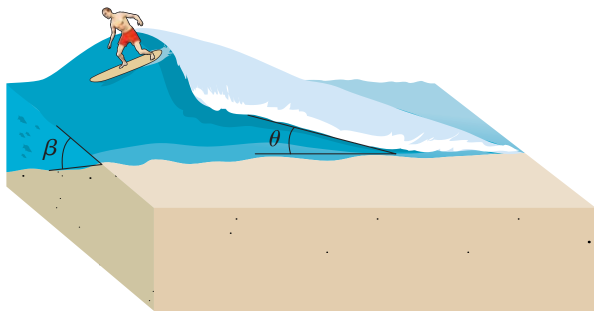
57. Surfing the Perfect Wave For a wave to be surfable it can't break all at once. Robert Guza and Tony Bowen have shown that a wave has a surfable shoulder if it hits the shoreline at an angle θ given by

$$\theta = \sin^{-1}\left(\frac{1}{(2n+1)\tan\beta}\right)$$

where β is the angle at which the beach slopes down and where $n = 0, 1, 2, \ldots$.

(a) For $\beta = 10°$, find θ when $n = 3$.

(b) For $\beta = 15°$, find θ when $n = 2$, 3, and 4. Explain why the formula does not give a value for θ when $n = 0$ or 1.



Discovery • Discussion

58. Two Different Compositions The functions

$$f(x) = \sin(\sin^{-1}x) \qquad \text{and} \qquad g(x) = \sin^{-1}(\sin x)$$

both simplify to just x for suitable values of x. But these functions are not the same for all x. Graph both f and g to show how the functions differ. (Think carefully about the domain and range of $\sin^{-1}$.)

59. Inverse Trigonometric Functions on a Calculator Most calculators do not have keys for $\sec^{-1}$, $\csc^{-1}$, or $\cot^{-1}$. Prove the following identities, then use these identities and a calculator to find $\sec^{-1}2$, $\csc^{-1}3$, and $\cot^{-1}4$.

$$\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right), \quad x \ge 1$$

$$\csc^{-1}x = \sin^{-1}\left(\frac{1}{x}\right), \quad x \ge 1$$

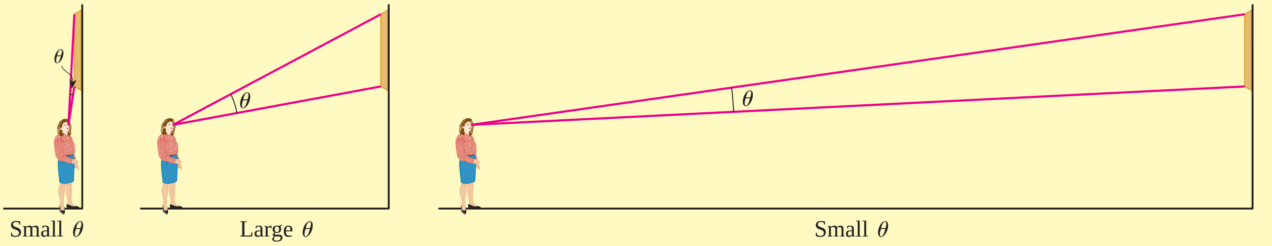
$$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), \quad x > 0$$

DISCOVERY PROJECT

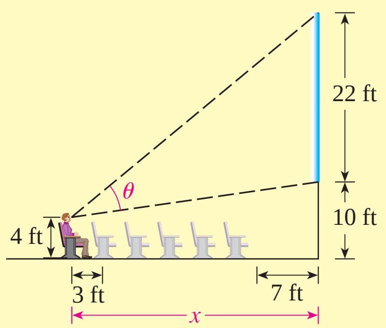
Where to Sit at the Movies

Everyone knows that the apparent size of an object depends on its distance from the viewer. The farther away an object, the smaller its apparent size. The apparent size is determined by the angle the object subtends at the eye of the viewer.

If you are looking at a painting hanging on a wall, how far away should you stand to get the maximum view? If the painting is hung above eye level, then the following figures show that the angle subtended at the eye is small if you are too close or too far away. The same situation occurs when choosing where to sit in a movie theatre.



1. The screen in a theatre is 22 ft high and is positioned 10 ft above the floor, which is flat. The first row of seats is 7 ft from the screen and the rows are 3 ft apart. You decide to sit in the row where you get the maximum view, that is, where the angle θ subtended by the screen at your eyes is a maximum. Suppose your eyes are 4 ft above the floor, as in the figure, and you sit at a distance x from the screen.

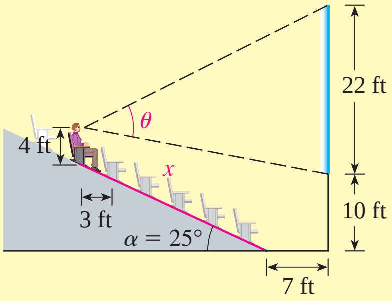


(a) Show that $\theta = \tan^{-1}\left(\frac{28}{x}\right) - \tan^{-1}\left(\frac{6}{x}\right)$.

(b) Use the subtraction formula for tangent to show that

$$\theta = \tan^{-1}\left(\frac{22x}{x^2 + 168}\right)$$

(c) Use a graphing device to graph θ as a function of x. What value of x maximizes θ? In which row should you sit? What is the viewing angle in this row?



2. Now suppose that, starting with the first row of seats, the floor of the seating area is inclined at an angle of $\alpha = 25°$ above the horizontal, and the distance that you sit up the incline is x, as shown in the figure.

(a) Use the Law of Cosines to show that

$$\theta = \cos^{-1}\left(\frac{a^2 + b^2 - 484}{2ab}\right)$$

where

$$a^2 = (7 + x\cos\alpha)^2 + (28 - x\sin\alpha)^2$$

and

$$b^2 = (7 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2$$

(b) Use a graphing device to graph θ as a function of x, and estimate the value of x that maximizes θ. In which row should you sit? What is the viewing angle θ in this row?

7.5 Trigonometric Equations

An equation that contains trigonometric functions is called a **trigonometric equation**. For example, the following are trigonometric equations:

$$\sin^2 x + \cos^2 x = 1 \qquad 2\sin x - 1 = 0 \qquad \tan^2 2x - 1 = 0$$

The first equation is an *identity*—that is, it is true for every value of the variable x. The other two equations are true only for certain values of x. To solve a trigonometric equation, we find all the values of the variable that make the equation true. (Except in some applied problems, we will always use radian measure for the variable.)

Solving Trigonometric Equations

To solve a trigonometric equation, we use the rules of algebra to isolate the trigonometric function on one side of the equal sign. Then we use our knowledge of the values of the trigonometric functions to solve for the variable.

Example 1 Solving a Trigonometric Equation

Solve the equation $2\sin x - 1 = 0$.

Solution We start by isolating $\sin x$.

$$2\sin x - 1 = 0 \qquad \text{Given equation}$$

$$2\sin x = 1 \qquad \text{Add 1}$$

$$\sin x = \frac{1}{2} \qquad \text{Divide by 2}$$

Mathematics in the Modern World

Getty Images

Weather Prediction

Modern meteorologists do much more than predict tomorrow's weather. They research long-term weather patterns, depletion of the ozone layer, global warming, and other effects of human activity on the weather. But daily weather prediction is still a major part of meteorology; its value is measured by the innumerable human lives saved each year through accurate prediction of hurricanes, blizzards, and other catastrophic weather phenomena. At the beginning of the 20th century mathematicians proposed to model weather with equations that used the current values of hundreds of atmospheric variables. Although this model worked in principle, it was impossible to predict future weather patterns with it because of the difficulty of measuring all the variables accurately and solving all the equations. Today, new mathematical models combined with high-speed computer simulations have vastly improved weather prediction. As a result, many human as well as economic disasters have been averted. Mathematicians at the National Oceanographic and Atmospheric Administration (NOAA) are continually researching better methods of weather prediction.

Because sine has period 2π, we first find the solutions in the interval $[0, 2\pi)$. These are $x = \pi/6$ and $x = 5\pi/6$. To get all other solutions, we add any integer multiple of 2π to these solutions. Thus, the solutions are

$$x = \frac{\pi}{6} + 2k\pi, \qquad x = \frac{5\pi}{6} + 2k\pi$$

where k is any integer. Figure 1 gives a graphical representation of the solutions.

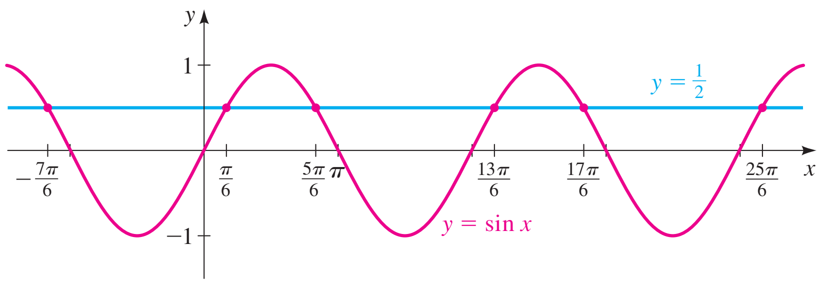


Figure 1

Example 2 Solving a Trigonometric Equation

Solve the equation $\tan^2 x - 3 = 0$.

Solution We start by isolating $\tan x$.

$$\tan^2 x - 3 = 0 \qquad \text{Given equation}$$

$$\tan^2 x = 3 \qquad \text{Add 3}$$

$$\tan x = \pm\sqrt{3} \qquad \text{Take square roots}$$

Because tangent has period π, we first find the solutions in the interval $(-\pi/2, \pi/2)$. These are $x = -\pi/3$ and $x = \pi/3$. To get all other solutions, we add any integer multiple of π to these solutions. Thus, the solutions are

$$x = -\frac{\pi}{3} + k\pi, \qquad x = \frac{\pi}{3} + k\pi$$

where k is any integer.

Example 3 Finding Intersection Points

Find the values of x for which the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ intersect.

Solution 1: Graphical

The graphs intersect where $f(x) = g(x)$. In Figure 2 we graph $y_1 = \sin x$ and $y_2 = \cos x$ on the same screen, for x between 0 and 2π. Using TRACE or the `Intersect` command on the graphing calculator, we see that the two points of intersection in this interval occur where $x \approx 0.785$ and $x \approx 3.927$. Since sine and cosine are periodic with period 2π, the intersection points occur where

$$x \approx 0.785 + 2k\pi \qquad \text{and} \qquad x \approx 3.927 + 2k\pi$$

where k is any integer.

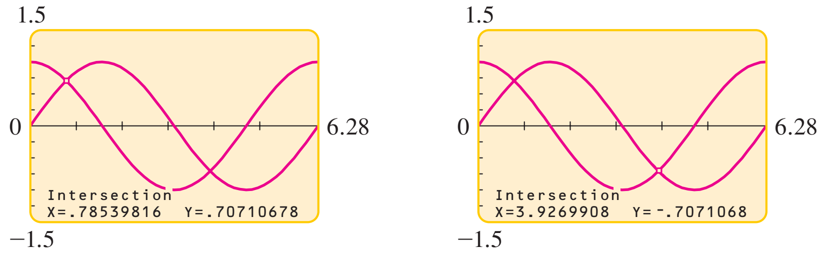


Figure 2 (a) (b)

Solution 2: Algebraic

To find the exact solution, we set $f(x) = g(x)$ and solve the resulting equation algebraically.

$$\sin x = \cos x \qquad \text{Equate functions}$$

Since the numbers x for which $\cos x = 0$ are not solutions of the equation, we can divide both sides by $\cos x$.

$$\frac{\sin x}{\cos x} = 1 \qquad \text{Divide by } \cos x$$

$$\tan x = 1 \qquad \text{Reciprocal identity}$$

Because tangent has period π, we first find the solutions in the interval $(-\pi/2, \pi/2)$. The only solution in this interval is $x = \pi/4$. To get all solutions, we add any integer multiple of π to this solution. Thus, the solutions are

$$x = \frac{\pi}{4} + k\pi$$

where k is any integer. The graphs intersect for these values of x. You should use your calculator to check that, correct to three decimals, these are the same values as we obtained in Solution 1. ■

Solving Trigonometric Equations by Factoring

Zero-Product Property

If $AB = 0$, then $A = 0$ or $B = 0$.

Factoring is one of the most useful techniques for solving equations, including trigonometric equations. The idea is to move all terms to one side of the equation, factor, then use the Zero-Product Property (see Section 1.5).

Example 4 An Equation of Quadratic Type

Solve the equation $2\cos^2 x - 7\cos x + 3 = 0$.

Equation of Quadratic Type

$$2C^2 - 7C + 3 = 0$$

$$(2C - 1)(C - 3) = 0$$

Solution We factor the left-hand side of the equation.

$$2\cos^2 x - 7\cos x + 3 = 0 \qquad \text{Given equation}$$

$$(2\cos x - 1)(\cos x - 3) = 0 \qquad \text{Factor}$$

$$2\cos x - 1 = 0 \quad \text{or} \quad \cos x - 3 = 0 \qquad \text{Set each factor equal to 0}$$

$$\cos x = \tfrac{1}{2} \quad \text{or} \quad \cos x = 3 \qquad \text{Solve for } \cos x$$

Because cosine has period 2π, we first find the solutions in the interval $[0, 2\pi)$. For the first equation these are $x = \pi/3$ and $x = 5\pi/3$. The second equation has no solutions because $\cos x$ is never greater than 1. Thus, the solutions are

$$x = \frac{\pi}{3} + 2k\pi, \qquad x = \frac{5\pi}{3} + 2k\pi$$

where k is any integer. ■

Example 5 Using a Trigonometric Identity

Solve the equation $1 + \sin x = 2\cos^2 x$.

Solution We use a trigonometric identity to rewrite the equation in terms of a single trigonometric function.

$$1 + \sin x = 2\cos^2 x \qquad \text{Given equation}$$

$$1 + \sin x = 2(1 - \sin^2 x) \qquad \text{Pythagorean identity}$$

$$2\sin^2 x + \sin x - 1 = 0 \qquad \text{Put all terms on one side of the equation}$$

$$(2\sin x - 1)(\sin x + 1) = 0 \qquad \text{Factor}$$

$$2\sin x - 1 = 0 \quad \text{or} \quad \sin x + 1 = 0 \qquad \text{Set each factor equal to 0}$$

$$\sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1 \qquad \text{Solve for sin } x$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{or} \quad x = \frac{3\pi}{2} \qquad \text{Solve for } x \text{ in the interval } [0, 2\pi)$$

Equation of Quadratic Type

$$2S^2 + S - 1 = 0$$

$$(2S - 1)(S + 1) = 0$$

Because sine has period 2π, we get all the solutions of the equation by adding any integer multiple of 2π to these solutions. Thus, the solutions are

$$x = \frac{\pi}{6} + 2k\pi, \qquad x = \frac{5\pi}{6} + 2k\pi, \qquad x = \frac{3\pi}{2} + 2k\pi$$

where k is any integer. ■

Example 6 Using a Trigonometric Identity

Solve the equation $\sin 2x - \cos x = 0$.

Solution The first term is a function of $2x$ and the second is a function of x, so we begin by using a trigonometric identity to rewrite the first term as a function of x only.

$$\sin 2x - \cos x = 0 \qquad \text{Given equation}$$

$$2\sin x\cos x - \cos x = 0 \qquad \text{Double-angle formula}$$

$$\cos x\,(2\sin x - 1) = 0 \qquad \text{Factor}$$

$$\cos x = 0 \quad \text{or} \quad 2\sin x - 1 = 0 \qquad \textit{Set each factor equal to 0}$$

$$\sin x = \frac{1}{2} \qquad \textit{Solve for } \sin x$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \textit{Solve for } x \textit{ in the interval } [0, 2\pi)$$

Both sine and cosine have period 2π, so we get all the solutions of the equation by adding any integer multiple of 2π to these solutions. Thus, the solutions are

$$x = \frac{\pi}{2} + 2k\pi, \qquad x = \frac{3\pi}{2} + 2k\pi, \qquad x = \frac{\pi}{6} + 2k\pi, \qquad x = \frac{5\pi}{6} + 2k\pi$$

where k is any integer. ■

Example 7 Squaring and Using an Identify

Solve the equation $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

Solution To get an equation that involves either sine only or cosine only, we square both sides and use a Pythagorean identity.

$$\cos x + 1 = \sin x \qquad \textit{Given equation}$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x \qquad \textit{Square both sides}$$

$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x \qquad \textit{Pythagorean identity}$$

$$2\cos^2 x + 2\cos x = 0 \qquad \textit{Simplify}$$

$$2\cos x(\cos x + 1) = 0 \qquad \textit{Factor}$$

$$2\cos x = 0 \quad \text{or} \quad \cos x + 1 = 0 \qquad \textit{Set each factor equal to 0}$$

$$\cos x = 0 \quad \text{or} \quad \cos x = -1 \qquad \textit{Solve for } \cos x$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{or} \quad x = \pi \qquad \textit{Solve for } x \textit{ in the interval } [0, 2\pi)$$

Because we squared both sides, we need to check for extraneous solutions. From *Check Your Answers*, we see that the solutions of the given equation are $\pi/2$ and π. ■

Check Your Answers

$x = \frac{\pi}{2}$:	$x = \frac{3\pi}{2}$:	$x = \pi$:
$\cos\frac{\pi}{2} + 1 \stackrel{?}{=} \sin\frac{\pi}{2}$	$\cos\frac{3\pi}{2} + 1 \stackrel{?}{=} \sin\frac{3\pi}{2}$	$\cos\pi + 1 \stackrel{?}{=} \sin\pi$
$0 + 1 = 1$ ✓	$0 + 1 \stackrel{?}{=} -1$ ✗	$-1 + 1 = 0$ ✓

⊘ If we perform an operation on an equation that may introduce new roots, such as squaring both sides, then we must check that the solutions obtained are not extraneous; that is, we must verify that they satisfy the original equation, as in Example 7.

Equations with Trigonometric Functions of Multiple Angles

When solving trigonometric equations that involve functions of multiples of angles, we first solve for the multiple of the angle, then divide to solve for the angle.

Example 8 Trigonometric Functions of Multiple Angles

Consider the equation $2 \sin 3x - 1 = 0$.

(a) Find all solutions of the equation.

(b) Find the solutions in the interval $[0, 2\pi)$.

Solution

(a) We start by isolating $\sin 3x$, and then solve for the multiple angle $3x$.

$$2 \sin 3x - 1 = 0 \qquad \text{Given equation}$$

$$2 \sin 3x = 1 \qquad \text{Add 1}$$

$$\sin 3x = \frac{1}{2} \qquad \text{Divide by 2}$$

$$3x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad \text{Solve for } 3x \text{ in the interval } [0, 2\pi)$$

To get all solutions, we add any integer multiple of 2π to these solutions. Thus, the solutions are of the form

$$3x = \frac{\pi}{6} + 2k\pi, \qquad 3x = \frac{5\pi}{6} + 2k\pi$$

To solve for x, we divide by 3 to get the solutions

$$x = \frac{\pi}{18} + \frac{2k\pi}{3}, \qquad x = \frac{5\pi}{18} + \frac{2k\pi}{3}$$

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 2\pi)$ correspond to $k = 0$, 1, and 2. For all other values of k, the corresponding values of x lie outside this interval. Thus, the solutions in the interval $[0, 2\pi)$ are

$$x = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}, \frac{25\pi}{18}, \frac{29\pi}{18}$$

■

Example 9 Trigonometric Functions of Multiple Angles

Consider the equation $\sqrt{3} \tan \frac{x}{2} - 1 = 0$.

(a) Find all solutions of the equation.

(b) Find the solutions in the interval $[0, 4\pi)$.

Solution

(a) We start by isolating $\tan(x/2)$.

$$\sqrt{3}\tan\frac{x}{2} - 1 = 0 \qquad \text{Given equation}$$

$$\sqrt{3}\tan\frac{x}{2} = 1 \qquad \text{Add 1}$$

$$\tan\frac{x}{2} = \frac{1}{\sqrt{3}} \qquad \text{Divide by } \sqrt{3}$$

$$\frac{x}{2} = \frac{\pi}{6} \qquad \text{Solve for } \frac{x}{2} \text{ in the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Since tangent has period π, to get all solutions we add any integer multiple of π to this solution. Thus, the solutions are of the form

$$\frac{x}{2} = \frac{\pi}{6} + k\pi$$

Multiplying by 2, we get the solutions

$$x = \frac{\pi}{3} + 2k\pi$$

where k is any integer.

(b) The solutions from part (a) that are in the interval $[0, 4\pi)$ correspond to $k = 0$ and $k = 1$. For all other values of k, the corresponding values of x lie outside this interval. Thus, the solutions in the interval $[0, 4\pi)$ are

$$x = \frac{\pi}{3}, \frac{7\pi}{3}$$

■

Using Inverse Trigonometric Functions to Solve Trigonometric Equations

So far, all the equations we've solved have had solutions like $\pi/4$, $\pi/3$, $5\pi/6$, and so on. We were able to find these solutions from the special values of the trigonometric functions that we've memorized. We now consider equations whose solution requires us to use the inverse trigonometric functions.

Example 10 Using Inverse Trigonometric Functions

Solve the equation $\tan^2 x - \tan x - 2 = 0$.

Solution We start by factoring the left-hand side.

$$\tan^2 x - \tan x - 2 = 0 \qquad \text{Given equation}$$

$$(\tan x - 2)(\tan x + 1) = 0 \qquad \text{Factor}$$

$$\tan x - 2 = 0 \quad \text{or} \quad \tan x + 1 = 0 \qquad \text{Set each factor equal to 0}$$

$$\tan x = 2 \quad \text{or} \quad \tan x = -1 \qquad \text{Solve for } \tan x$$

$$x = \tan^{-1} 2 \quad \text{or} \quad x = -\frac{\pi}{4} \qquad \text{Solve for } x \text{ in the interval } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Equation of Quadratic Type

$$T^2 - T - 2 = 0$$

$$(T - 2)(T + 1) = 0$$

Because tangent has period π, we get all solutions by adding integer multiples of π to these solutions. Thus, all the solutions are

$$x = \tan^{-1} 2 + k\pi, \qquad x = -\frac{\pi}{4} + k\pi$$

where k is any integer. ■

If we are using inverse trigonometric functions to solve an equation, we must keep in mind that $\sin^{-1}$ and $\tan^{-1}$ give values in quadrants I and IV, and $\cos^{-1}$ gives values in quadrants I and II. To find other solutions, we must look at the quadrant where the trigonometric function in the equation can take on the value we need.

Example 11 Using Inverse Trigonometric Functions

(a) Solve the equation $3 \sin \theta - 2 = 0$.

(b) Use a calculator to approximate the solutions in the interval $[0, 2\pi)$, correct to five decimals.

Solution

(a) We start by isolating $\sin \theta$.

$$3 \sin \theta - 2 = 0 \quad \text{Given equation}$$

$$3 \sin \theta = 2 \quad \text{Add 2}$$

$$\sin \theta = \frac{2}{3} \quad \text{Divide by 3}$$

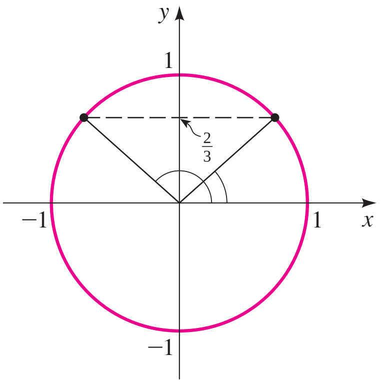


Figure 3

From Figure 3 we see that $\sin \theta$ equals $\frac{2}{3}$ in quadrants I and II. The solution in quadrant I is $\theta = \sin^{-1} \frac{2}{3}$. The solution in quadrant II is $\theta = \pi - \sin^{-1} \frac{2}{3}$. Since these are the solutions in the interval $[0, 2\pi)$, we get all other solutions by adding integer multiples of 2π to these. Thus, all the solutions of the equation are

$$\theta = \left(\sin^{-1} \tfrac{2}{3}\right) + 2k\pi, \qquad \theta = \left(\pi - \sin^{-1} \tfrac{2}{3}\right) + 2k\pi$$

where k is any integer.

(b) Using a calculator set in radian mode, we see that $\sin^{-1} \frac{2}{3} \approx 0.72973$ and $\pi - \sin^{-1} \frac{2}{3} \approx 2.41186$, so the solutions in the interval $[0, 2\pi)$ are

$$\theta \approx 0.72973, \qquad \theta \approx 2.41186$$

■

7.5 Exercises

1–40 ■ Find all solutions of the equation.

1. $\cos x + 1 = 0$

2. $\sin x + 1 = 0$

3. $2 \sin x - 1 = 0$

4. $\sqrt{2} \cos x - 1 = 0$

5. $\sqrt{3} \tan x + 1 = 0$

6. $\cot x + 1 = 0$

7. $4 \cos^2 x - 1 = 0$

8. $2 \cos^2 x - 1 = 0$

9. $\sec^2 x - 2 = 0$

10. $\csc^2 x - 4 = 0$

11. $3 \csc^2 x - 4 = 0$

12. $1 - \tan^2 x = 0$

13. $\cos x (2 \sin x + 1) = 0$

14. $\sec x \left(2 \cos x - \sqrt{2}\right) = 0$

15. $(\tan x + \sqrt{3})(\cos x + 2) = 0$

16. $(2 \cos x + \sqrt{3})(2 \sin x - 1) = 0$

17. $\cos x \sin x - 2 \cos x = 0$

18. $\tan x \sin x + \sin x = 0$

19. $4 \cos^2 x - 4 \cos x + 1 = 0$

20. $2 \sin^2 x - \sin x - 1 = 0$

21. $\sin^2 x = 2\sin x + 3$

22. $3\tan^3 x = \tan x$

23. $\sin^2 x = 4 - 2\cos^2 x$

24. $2\cos^2 x + \sin x = 1$

25. $2\sin 3x + 1 = 0$

26. $2\cos 2x + 1 = 0$

27. $\sec 4x - 2 = 0$

28. $\sqrt{3}\tan 3x + 1 = 0$

29. $\sqrt{3}\sin 2x = \cos 2x$

30. $\cos 3x = \sin 3x$

31. $\cos\frac{x}{2} - 1 = 0$

32. $2\sin\frac{x}{3} + \sqrt{3} = 0$

33. $\tan\frac{x}{4} + \sqrt{3} = 0$

34. $\sec\frac{x}{2} = \cos\frac{x}{2}$

35. $\tan^5 x - 9\tan x = 0$

36. $3\tan^3 x - 3\tan^2 x - \tan x + 1 = 0$

37. $4\sin x\cos x + 2\sin x - 2\cos x - 1 = 0$

38. $\sin 2x = 2\tan 2x$

39. $\cos^2 2x - \sin^2 2x = 0$

40. $\sec x - \tan x = \cos x$

41–48 ■ Find all solutions of the equation in the interval $[0, 2\pi)$.

41. $2\cos 3x = 1$

42. $3\csc^2 x = 4$

43. $2\sin x\tan x - \tan x = 1 - 2\sin x$

44. $\sec x\tan x - \cos x\cot x = \sin x$

45. $\tan x - 3\cot x = 0$

46. $2\sin^2 x - \cos x = 1$

47. $\tan 3x + 1 = \sec 3x$

48. $3\sec^2 x + 4\cos^2 x = 7$

49–56 ■ **(a)** Find all solutions of the equation. **(b)** Use a calculator to solve the equation in the interval $[0, 2\pi)$, correct to five decimal places.

49. $\cos x = 0.4$

50. $2\tan x = 13$

51. $\sec x - 5 = 0$

52. $3\sin x = 7\cos x$

53. $5\sin^2 x - 1 = 0$

54. $2\sin 2x - \cos x = 0$

55. $3\sin^2 x - 7\sin x + 2 = 0$

56. $\tan^4 x - 13\tan^2 x + 36 = 0$

57–60 ■ Graph f and g on the same axes, and find their points of intersection.

57. $f(x) = 3\cos x + 1, \quad g(x) = \cos x - 1$

58. $f(x) = \sin 2x, \quad g(x) = 2\sin 2x + 1$

59. $f(x) = \tan x, \quad g(x) = \sqrt{3}$

60. $f(x) = \sin x - 1, \quad g(x) = \cos x$

61–64 ■ Use an addition or subtraction formula to simplify the equation. Then find all solutions in the interval $[0, 2\pi)$.

61. $\cos x\cos 3x - \sin x\sin 3x = 0$

62. $\cos x\cos 2x + \sin x\sin 2x = \frac{1}{2}$

63. $\sin 2x\cos x + \cos 2x\sin x = \sqrt{3}/2$

64. $\sin 3x\cos x - \cos 3x\sin x = 0$

65–68 ■ Use a double- or half-angle formula to solve the equation in the interval $[0, 2\pi)$.

65. $\sin 2x + \cos x = 0$

66. $\tan\frac{x}{2} - \sin x = 0$

67. $\cos 2x + \cos x = 2$

68. $\tan x + \cot x = 4\sin 2x$

69–72 ■ Solve the equation by first using a sum-to-product formula.

69. $\sin x + \sin 3x = 0$

70. $\cos 5x - \cos 7x = 0$

71. $\cos 4x + \cos 2x = \cos x$

72. $\sin 5x - \sin 3x = \cos 4x$

73–78 ■ Use a graphing device to find the solutions of the equation, correct to two decimal places.

73. $\sin 2x = x$

74. $\cos x = \frac{x}{3}$

75. $2^{\sin x} = x$

76. $\sin x = x^3$

77. $\frac{\cos x}{1 + x^2} = x^2$

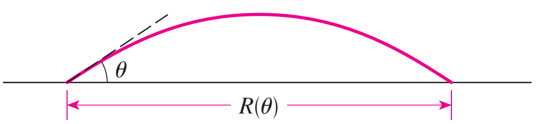
78. $\cos x = \frac{1}{2}(e^x + e^{-x})$

Applications

79. Range of a Projectile If a projectile is fired with velocity v_0 at an angle θ, then its *range*, the horizontal distance it travels (in feet), is modeled by the function

$$R(\theta) = \frac{v_0^2 \sin 2\theta}{32}$$

(See page 818.) If $v_0 = 2200$ ft/s, what angle (in degrees) should be chosen for the projectile to hit a target on the ground 5000 ft away?



80. Damped Vibrations The displacement of a spring vibrating in damped harmonic motion is given by

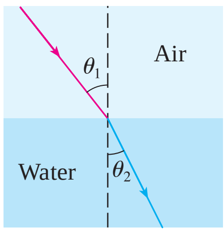
$$y = 4e^{-3t}\sin 2\pi t$$

Find the times when the spring is at its equilibrium position $(y = 0)$.

81. Refraction of Light It has been observed since ancient times that light refracts or "bends" as it travels from one medium to another (from air to water, for example). If v_1 is the speed of light in one medium and v_2 its speed in another medium, then according to **Snell's Law**,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

where θ_1 is the *angle of incidence* and θ_2 is the *angle of refraction* (see the figure). The number v_1/v_2 is called the *index of refraction*. The index of refraction for several substances is given in the table. If a ray of light passes through the surface of a lake at an angle of incidence of 70°, find the angle of refraction.



Substance	Refraction from air to substance
Water	1.33
Alcohol	1.36
Glass	1.52
Diamond	2.41

82. Total Internal Reflection When light passes from a more-dense to a less-dense medium—from glass to air, for example—the angle of refraction predicted by Snell's Law (see Exercise 81) can be 90° or larger. In this case, the light beam is actually reflected back into the denser medium. This phenomenon, called *total internal reflection*, is the principle behind fiber optics.

Set $\theta_2 = 90°$ in Snell's Law and solve for θ_1 to determine the critical angle of incidence at which total internal reflection begins to occur when light passes from glass to air. (Note that the index of refraction from glass to air is the reciprocal of the index from air to glass.)

83. Hours of Daylight In Philadelphia the number of hours of daylight on day t (where t is the number of days after January 1) is modeled by the function

$$L(t) = 12 + 2.83 \sin\left(\frac{2\pi}{365}(t - 80)\right)$$

(a) Which days of the year have about 10 hours of daylight?

(b) How many days of the year have more than 10 hours of daylight?

84. Phases of the Moon As the moon revolves around the earth, the side that faces the earth is usually just partially illuminated by the sun. The phases of the moon describe how much of the surface appears to be in sunlight. An astronomical measure of phase is given by the fraction F of the lunar disc that is lit. When the angle between the sun, earth, and moon is θ ($0 \le \theta \le 360°$), then

$$F = \tfrac{1}{2}(1 - \cos \theta)$$

Determine the angles θ that correspond to the following phases.

(a) $F = 0$ (new moon)

(b) $F = 0.25$ (a crescent moon)

(c) $F = 0.5$ (first or last quarter)

(d) $F = 1$ (full moon)

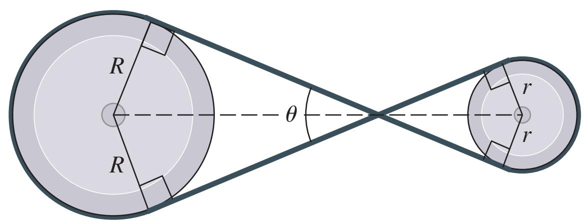
85. Belts and Pulleys A thin belt of length L surrounds two pulleys of radii R and r, as shown in the figure.

(a) Show that the angle θ (in radians) where the belt crosses itself satisfies the equation

$$\theta + 2 \cot \frac{\theta}{2} = \frac{L}{R + r} - \pi$$

[*Hint:* Express L in terms of R, r, and θ by adding up the lengths of the curved and straight parts of the belt.]

(b) Suppose that $R = 2.42$ ft, $r = 1.21$ ft, and $L = 27.78$ ft. Find θ by solving the equation in part (a) graphically. Express your answer both in radians and in degrees.



Discovery • Discussion

86. Equations and Identities Which of the following statements is true?

A. Every identity is an equation.

B. Every equation is an identity.

Give examples to illustrate your answer. Write a short paragraph to explain the difference between an equation and an identity.

87. A Special Trigonometric Equation What makes the equation $\sin(\cos x) = 0$ different from all the other equations we've looked at in this section? Find all solutions of this equation.

7 Review

Concept Check

1. **(a)** State the reciprocal identities.
   (b) State the Pythagorean identities.
   (c) State the even-odd identities.
   (d) State the cofunction identities.
2. Explain the difference between an equation and an identity.
3. How do you prove a trigonometric identity?
4. **(a)** State the addition formulas for sine, cosine, and tangent.
   (b) State the subtraction formulas for sine, cosine, and tangent.
5. **(a)** State the double-angle formulas for sine, cosine, and tangent.
   (b) State the formulas for lowering powers.
   (c) State the half-angle formulas.
6. **(a)** State the product-to-sum formulas.
   (b) State the sum-to-product formulas.
7. **(a)** Define the inverse sine function $\sin^{-1}$. What are its domain and range?
   (b) For what values of x is the equation $\sin(\sin^{-1}x) = x$ true?
   (c) For what values of x is the equation $\sin^{-1}(\sin x) = x$ true?
8. **(a)** Define the inverse cosine function $\cos^{-1}$. What are its domain and range?
   (b) For what values of x is the equation $\cos(\cos^{-1}x) = x$ true?
   (c) For what values of x is the equation $\cos^{-1}(\cos x) = x$ true?
9. **(a)** Define the inverse tangent function $\tan^{-1}$. What are its domain and range?
   (b) For what values of x is the equation $\tan(\tan^{-1}x) = x$ true?
   (c) For what values of x is the equation $\tan^{-1}(\tan x) = x$ true?
10. Explain how you solve a trigonometric equation by factoring.

Exercises

1–24 ■ Verify the identity.

1. $\sin\theta\,(\cot\theta + \tan\theta) = \sec\theta$
2. $(\sec\theta - 1)(\sec\theta + 1) = \tan^2\theta$
3. $\cos^2 x \csc x - \csc x = -\sin x$
4. $\dfrac{1}{1 - \sin^2 x} = 1 + \tan^2 x$
5. $\dfrac{\cos^2 x - \tan^2 x}{\sin^2 x} = \cot^2 x - \sec^2 x$
6. $\dfrac{1 + \sec x}{\sec x} = \dfrac{\sin^2 x}{1 - \cos x}$
7. $\dfrac{\cos^2 x}{1 - \sin x} = \dfrac{\cos x}{\sec x - \tan x}$
8. $(1 - \tan x)(1 - \cot x) = 2 - \sec x \csc x$
9. $\sin^2 x \cot^2 x + \cos^2 x \tan^2 x = 1$
10. $(\tan x + \cot x)^2 = \csc^2 x \sec^2 x$
11. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$
12. $\dfrac{\cos(x + y)}{\cos x \sin y} = \cot y - \tan x$
13. $\tan\dfrac{x}{2} = \csc x - \cot x$
14. $\dfrac{\sin(x + y) + \sin(x - y)}{\cos(x + y) + \cos(x - y)} = \tan x$
15. $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$
16. $\csc x - \tan\dfrac{x}{2} = \cot x$
17. $1 + \tan x \tan\dfrac{x}{2} = \sec x$
18. $\dfrac{\sin 3x + \cos 3x}{\cos x - \sin x} = 1 + 2\sin 2x$
19. $\left(\cos\dfrac{x}{2} - \sin\dfrac{x}{2}\right)^2 = 1 - \sin x$
20. $\dfrac{\cos 3x - \cos 7x}{\sin 3x + \sin 7x} = \tan 2x$
21. $\dfrac{\sin 2x}{\sin x} - \dfrac{\cos 2x}{\cos x} = \sec x$
22. $(\cos x + \cos y)^2 + (\sin x - \sin y)^2 = 2 + 2\cos(x + y)$
23. $\tan\left(x + \dfrac{\pi}{4}\right) = \dfrac{1 + \tan x}{1 - \tan x}$
24. $\dfrac{\sec x - 1}{\sin x \sec x} = \tan\dfrac{x}{2}$

25–28 ■ **(a)** Graph f and g. **(b)** Do the graphs suggest that the equation $f(x) = g(x)$ is an identity? Prove your answer.

25. $f(x) = 1 - \left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)^2, \quad g(x) = \sin x$

26. $f(x) = \sin x + \cos x, \quad g(x) = \sqrt{\sin^2 x + \cos^2 x}$

27. $f(x) = \tan x \tan\frac{x}{2}, \quad g(x) = \frac{1}{\cos x}$

28. $f(x) = 1 - 8\sin^2 x + 8\sin^4 x, \quad g(x) = \cos 4x$

29–30 ■ **(a)** Graph the function(s) and make a conjecture, and **(b)** prove your conjecture.

29. $f(x) = 2\sin^2 3x + \cos 6x$

30. $f(x) = \sin x \cot\frac{x}{2}, \quad g(x) = \cos x$

31–46 ■ Solve the equation in the interval $[0, 2\pi)$.

31. $\cos x \sin x - \sin x = 0$ **32.** $\sin x - 2\sin^2 x = 0$

33. $2\sin^2 x - 5\sin x + 2 = 0$

34. $\sin x - \cos x - \tan x = -1$

35. $2\cos^2 x - 7\cos x + 3 = 0$ **36.** $4\sin^2 x + 2\cos^2 x = 3$

37. $\dfrac{1 - \cos x}{1 + \cos x} = 3$ **38.** $\sin x = \cos 2x$

39. $\tan^3 x + \tan^2 x - 3\tan x - 3 = 0$

40. $\cos 2x \csc^2 x = 2\cos 2x$ **41.** $\tan\frac{1}{2}x + 2\sin 2x = \csc x$

42. $\cos 3x + \cos 2x + \cos x = 0$

43. $\tan x + \sec x = \sqrt{3}$ **44.** $2\cos x - 3\tan x = 0$

45. $\cos x = x^2 - 1$ **46.** $e^{\sin x} = x$

47. If a projectile is fired with velocity v_0 at an angle θ, then the maximum height it reaches (in feet) is modeled by the function

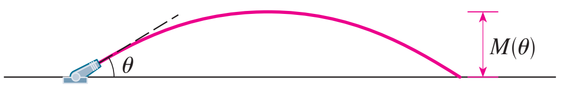
$$M(\theta) = \frac{v_0^2 \sin^2\theta}{64}$$

Suppose $v_0 = 400$ ft/s.

(a) At what angle θ should the projectile be fired so that the maximum height it reaches is 2000 ft?

(b) Is it possible for the projectile to reach a height of 3000 ft?

(c) Find the angle θ for which the projectile will travel highest.



48. The displacement of an automobile shock absorber is modeled by the function

$$f(t) = 2^{-0.2t}\sin 4\pi t$$

Find the times when the shock absorber is at its equilibrium position (that is, when $f(t) = 0$). [*Hint:* $2^x > 0$ for all real x.]

49–58 ■ Find the exact value of the expression.

49. $\cos 15°$ **50.** $\sin\frac{5\pi}{12}$

51. $\tan\frac{\pi}{8}$ **52.** $2\sin\frac{\pi}{12}\cos\frac{\pi}{12}$

53. $\sin 5°\cos 40° + \cos 5°\sin 40°$

54. $\dfrac{\tan 66° - \tan 6°}{1 + \tan 66°\tan 6°}$

55. $\cos^2\frac{\pi}{8} - \sin^2\frac{\pi}{8}$

56. $\frac{1}{2}\cos\frac{\pi}{12} + \frac{\sqrt{3}}{2}\sin\frac{\pi}{12}$

57. $\cos 37.5°\cos 7.5°$

58. $\cos 67.5° + \cos 22.5°$

59–64 ■ Find the exact value of the expression given that $\sec x = \frac{3}{2}$, $\csc y = 3$, and x and y are in quadrant I.

59. $\sin(x + y)$ **60.** $\cos(x - y)$

61. $\tan(x + y)$ **62.** $\sin 2x$

63. $\cos\frac{y}{2}$ **64.** $\tan\frac{y}{2}$

65–72 ■ Find the exact value of the expression.

65. $\sin^{-1}(\sqrt{3}/2)$ **66.** $\tan^{-1}(\sqrt{3}/3)$

67. $\cos(\tan^{-1}\sqrt{3})$ **68.** $\sin(\cos^{-1}(\sqrt{3}/2))$

69. $\tan(\sin^{-1}\frac{2}{5})$ **70.** $\sin(\cos^{-1}\frac{3}{8})$

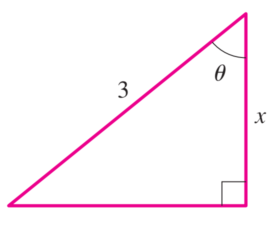
71. $\cos(2\sin^{-1}\frac{1}{3})$ **72.** $\cos(\sin^{-1}\frac{5}{13} - \cos^{-1}\frac{4}{5})$

73–74 ■ Rewrite the expression as an algebraic function of x.

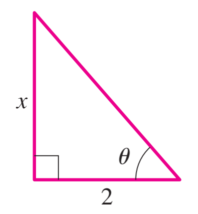
73. $\sin(\tan^{-1}x)$ **74.** $\sec(\sin^{-1}x)$

75–76 ■ Express θ in terms of x.

75.



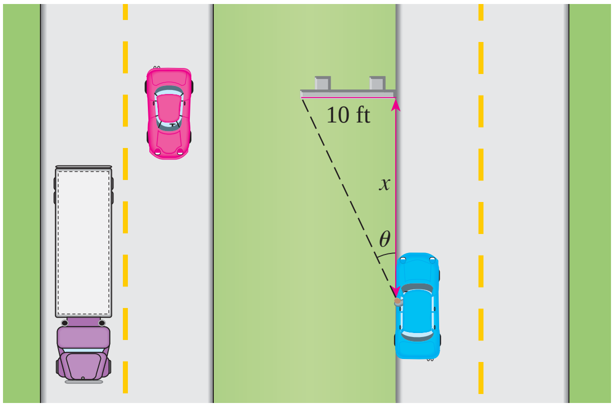
76.



77. A 10-ft-wide highway sign is adjacent to a roadway, as shown in the figure. As a driver approaches the sign, the viewing angle θ changes.

(a) Express viewing angle θ as a function of the distance x between the driver and the sign.

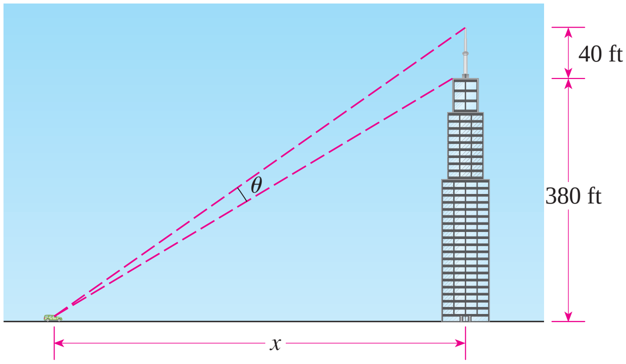
(b) The sign is legible when the viewing angle is 2° or greater. At what distance x does the sign first become legible?



78. A 380-ft-tall building supports a 40-ft communications tower (see the figure). As a driver approaches the building, the viewing angle θ of the tower changes.

(a) Express the viewing angle θ as a function of the distance x between the driver and the building.

(b) At what distance from the building is the viewing angle θ as large as possible?



7 Test

1. Verify each identity.

(a) $\tan\theta \sin\theta + \cos\theta = \sec\theta$

(b) $\dfrac{\tan x}{1 - \cos x} = \csc x\,(1 + \sec x)$

(c) $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$

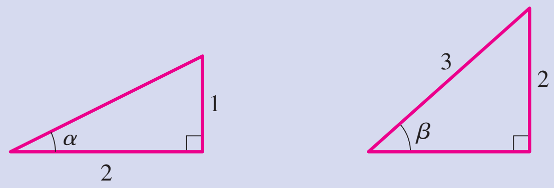
2. Let $x = 2\sin\theta$, $-\pi/2 < \theta < \pi/2$. Simplify the expression

$$\frac{x}{\sqrt{4 - x^2}}$$

3. Find the exact value of each expression.

(a) $\sin 8^\circ \cos 22^\circ + \cos 8^\circ \sin 22^\circ$ (b) $\sin 75^\circ$ (c) $\sin \dfrac{\pi}{12}$

4. For the angles α and β in the figures, find $\cos(\alpha + \beta)$.



5. (a) Write $\sin 3x \cos 5x$ as a sum of trigonometric functions.

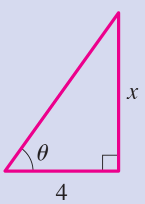
(b) Write $\sin 2x - \sin 5x$ as a product of trigonometric functions.

6. If $\sin\theta = -\frac{4}{5}$ and θ is in quadrant III, find $\tan(\theta/2)$.

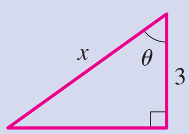
7. Graph $y = \sin x$ and $y = \sin^{-1} x$, and specify the domain of each function.

8. Express θ in each figure in terms of x.

(a)



(b)



9. Solve each trigonometric equation in the interval $[0, 2\pi)$.

(a) $2\cos^2 x + 5\cos x + 2 = 0$ (b) $\sin 2x - \cos x = 0$

10. Find all solutions in the interval $[0, 2\pi)$, correct to five decimal places:

$$5\cos 2x = 2$$

11. Find the exact value of $\cos\left(\tan^{-1}\frac{9}{40}\right)$.

Focus on Modeling

Traveling and Standing Waves

We've learned that the position of a particle in simple harmonic motion is described by a function of the form $y = A \sin \omega t$ (see Section 5.5). For example, if a string is moved up and down as in Figure 1, then the red dot on the string moves up and down in simple harmonic motion. Of course, the same holds true for each point on the string.

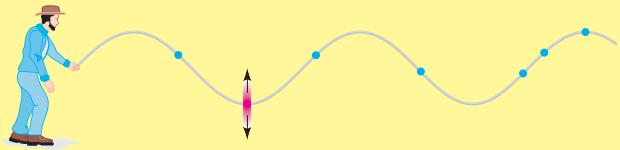


Figure 1

What function describes the shape of the whole string? If we fix an instant in time $(t = 0)$ and snap a photograph of the string, we get the shape in Figure 2, which is modeled by

$$y = A \sin kx$$

where y is the height of the string above the x-axis at the point x.

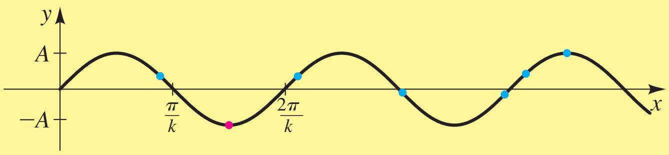


Figure 2
$y = A \sin kx$

Traveling Waves

If we snap photographs of the string at other instants, as in Figure 3, it appears that the waves in the string "travel" or shift to the right.

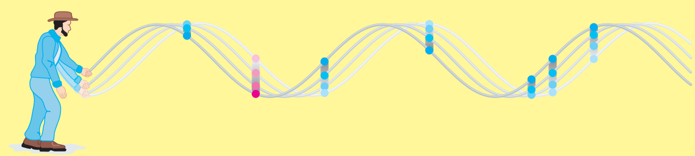


Figure 3

The **velocity** of the wave is the rate at which it moves to the right. If the wave has velocity v, then it moves to the right a distance vt in time t. So the graph of the shifted wave at time t is

$$y(x, t) = A \sin k(x - vt)$$

This function models the position of any point x on the string at any time t. We use the notation $y(x, t)$ to indicate that the function depends on the *two* variables x and t. Here is how this function models the motion of the string.

- **If we fix *x***, then $y(x, t)$ is a function of t only, which gives the position of the fixed point x at time t.
- **If we fix *t***, then $y(x, t)$ is a function of x only, whose graph is the shape of the string at the fixed time t.

Example 1 A Traveling Wave

A traveling wave is described by the function

$$y(x, t) = 3\sin\left(2x - \frac{\pi}{2}t\right), \qquad x \geq 0$$

(a) Find the function that models the position of the point $x = \pi/6$ at any time t. Observe that the point moves in simple harmonic motion.

(b) Sketch the shape of the wave when $t = 0, 0.5, 1.0, 1.5$, and 2.0. Does the wave appear to be traveling to the right?

(c) Find the velocity of the wave.

Solution

(a) Substituting $x = \pi/6$ we get

$$y\left(\frac{\pi}{6}, t\right) = 3\sin\left(2 \cdot \frac{\pi}{6} - \frac{\pi}{2}t\right) = 3\sin\left(\frac{\pi}{3} - \frac{\pi}{2}t\right)$$

The function $y = 3\sin\left(\frac{\pi}{3} - \frac{\pi}{2}t\right)$ describes simple harmonic motion with amplitude 3 and period $2\pi/(\pi/2) = 4$.

(b) The graphs are shown in Figure 4. As t increases, the wave moves to the right.

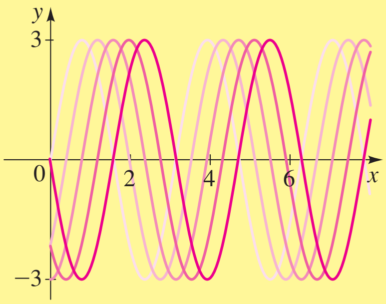


Figure 4
Traveling wave

(c) We express the given function in the standard form $y(x, t) = A\sin k(x - vt)$:

$$y(x, t) = 3\sin\left(2x - \frac{\pi}{2}t\right) \qquad \text{Given}$$

$$= 3\sin 2\left(x - \frac{\pi}{4}t\right) \qquad \text{Factor 2}$$

Comparing this to the standard form, we see that the wave is moving with velocity $v = \pi/4$. ■

Standing Waves

If two waves are traveling along the same string, then the movement of the string is determined by the sum of the two waves. For example, if the string is attached to a wall, then the waves bounce back with the same amplitude and speed but in the opposite direction. In this case, one wave is described by $y = A\sin k(x - vt)$ and the reflected wave by $y = A\sin k(x + vt)$. The resulting wave is

$$y(x, t) = A\sin k(x - vt) + A\sin k(x + vt) \qquad \text{Add the two waves}$$

$$= 2A\sin kx\cos kvt \qquad \text{Sum-to-product formula}$$

The points where kx is a multiple of 2π are special, because at these points $y = 0$ for any time t. In other words, these points never move. Such points are called **nodes**. Figure 5 shows the graph of the wave for several values of t. We see that the wave does not travel, but simply vibrates up and down. Such a wave is called a **standing wave**.

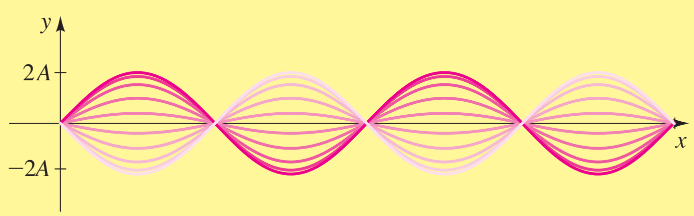


Figure 5
A standing wave

Example 2 A Standing Wave

Traveling waves are generated at each end of a wave tank 30 ft long, with equations

$$y = 1.5 \sin\left(\frac{\pi}{5}x - 3t\right) \qquad \text{and} \qquad y = 1.5 \sin\left(\frac{\pi}{5}x + 3t\right)$$

(a) Find the equation of the combined wave, and find the nodes.

(b) Sketch the graph for $t = 0, 0.17, 0.34, 0.51, 0.68, 0.85$, and 1.02. Is this a standing wave?

Solution

(a) The combined wave is obtained by adding the two equations:

$$y = 1.5 \sin\left(\frac{\pi}{5}x - 3t\right) + 1.5 \sin\left(\frac{\pi}{5}x + 3t\right) \qquad \text{Add the two waves}$$

$$= 3 \sin\frac{\pi}{5}x \cos 3t \qquad \text{Sum-to-product formula}$$

The nodes occur at the values of x for which $\sin\frac{\pi}{5}x = 0$, that is, where $\frac{\pi}{5}x = k\pi$ (k an integer). Solving for x we get $x = 5k$. So the nodes occur at

$$x = 0, 5, 10, 15, 20, 25, 30$$

(b) The graphs are shown in Figure 6. From the graphs we see that this is a standing wave. ■

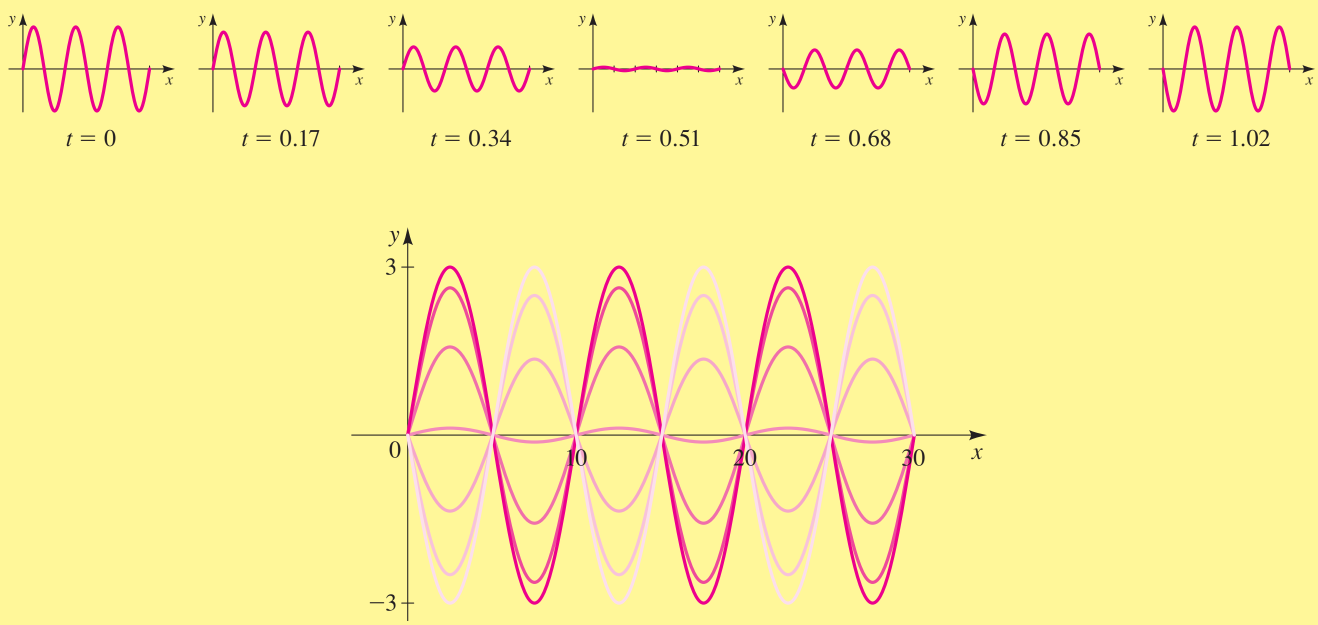
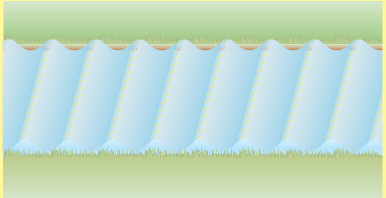


Figure 6

$y(x, t) = 3 \sin\frac{\pi}{5} x \cos 3t$

Problems



1. Wave on a Canal A wave on the surface of a long canal is described by the function

$$y(x, t) = 5 \sin\left(2x - \frac{\pi}{2}t\right), \quad x \geq 0$$

(a) Find the function that models the position of the point $x = 0$ at any time t.

(b) Sketch the shape of the wave when $t = 0, 0.4, 0.8, 1.2$, and 1.6. Is this a traveling wave?

(c) Find the velocity of the wave.

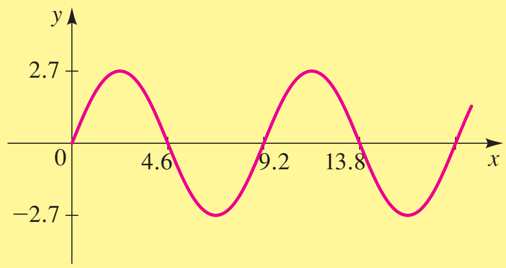
2. Wave in a Rope Traveling waves are generated at each end of a tightly stretched rope 24 ft long, with equations

$$y = 0.2 \sin(1.047x - 0.524t) \qquad \text{and} \qquad y = 0.2 \sin(1.047x + 0.524t)$$

(a) Find the equation of the combined wave, and find the nodes.

(b) Sketch the graph for $t = 0, 1, 2, 3, 4, 5$, and 6. Is this a standing wave?

3. Traveling Wave A traveling wave is graphed at the instant $t = 0$. If it is moving to the right with velocity 6, find an equation of the form $y(x, t) = A \sin(kx - kvt)$ for this wave.

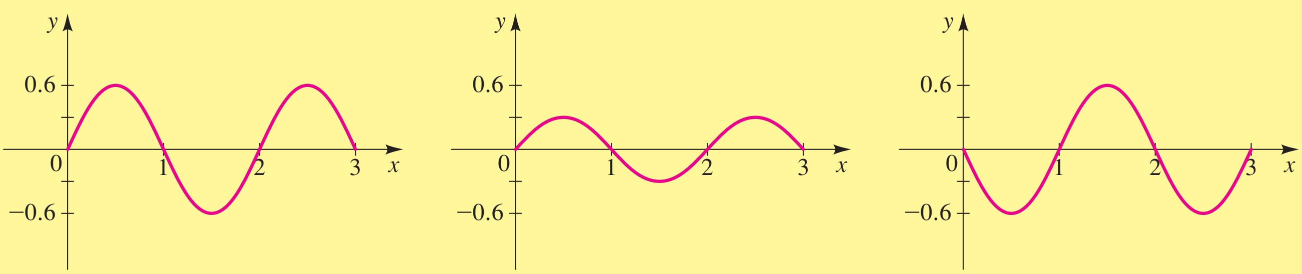


4. Traveling Wave A traveling wave has period $2\pi/3$, amplitude 5, and velocity 0.5.

(a) Find the equation of the wave.

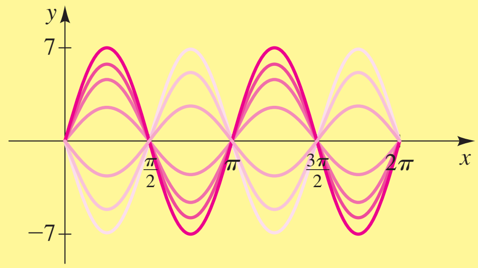
(b) Sketch the graph for $t = 0, 0.5, 1, 1.5$, and 2.

5. Standing Wave A standing wave with amplitude 0.6 is graphed at several times t as shown in the figure. If the vibration has a frequency of 20 Hz, find an equation of the form $y(x, t) = A \sin \alpha x \cos \beta t$ that models this wave.



$t = 0$ s $t = 0.010$ s $t = 0.025$ s

6. Standing Wave A standing wave has maximum amplitude 7 and nodes at 0, $\pi/2$, π, $3\pi/2$, 2π, as shown in the figure. Each point that is not a node moves up and down with period 4π. Find a function of the form $y(x, t) = A \sin \alpha x \cos \beta t$ that models this wave.



7. Vibrating String When a violin string vibrates, the sound produced results from a combination of standing waves that have evenly placed nodes. The figure illustrates some of the possible standing waves. Let's assume that the string has length π.

(a) For fixed t, the string has the shape of a sine curve $y = A \sin \alpha x$. Find the appropriate value of α for each of the illustrated standing waves.

(b) Do you notice a pattern in the values of α that you found in part (a)? What would the next two values of α be? Sketch rough graphs of the standing waves associated with these new values of α.

(c) Suppose that for fixed t, each point on the string that is not a node vibrates with frequency 440 Hz. Find the value of β for which an equation of the form $y = A \cos \beta t$ would model this motion.

(d) Combine your answers for parts (a) and (c) to find functions of the form $y(x, t) = A \sin \alpha x \cos \beta t$ that model each of the standing waves in the figure. (Assume $A = 1$.)

8. Waves in a Tube Standing waves in a violin string must have nodes at the ends of the string because the string is fixed at its endpoints. But this need not be the case with sound waves in a tube (such as a flute or an organ pipe). The figure shows some possible standing waves in a tube.

Suppose that a standing wave in a tube 37.7 ft long is modeled by the function

$$y(x, t) = 0.3 \cos \tfrac{1}{2}x \cos 50\pi t$$

Here $y(x, t)$ represents the variation from normal air pressure at the point x feet from the end of the tube, at time t seconds.

(a) At what points x are the nodes located? Are the endpoints of the tube nodes?

(b) At what frequency does the air vibrate at points that are not nodes?