CHAPTER 7

Perimeter, Area, Surface Area, and Volume

Chapter Outline

7.1 TRIANGLES AND PARALLELOGRAMS

7.2 TRAPEZOIDS, RHOMBI, AND KITES

7.3 AREAS OF SIMILAR POLYGONS

7.4 CIRCUMFERENCE AND ARC LENGTH

7.5 AREAS OF CIRCLES AND SECTORS

7.6 AREA AND PERIMETER OF REGULAR POLYGONS

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7.8 EXPLORING SOLIDS

7.9 SURFACE AREA OF PRISMS AND CYLINDERS

7.10 SURFACE AREA OF PYRAMIDS AND CONES

7.11 VOLUME OF PRISMS AND CYLINDERS

7.12 VOLUME OF PYRAMIDS AND CONES

7.13 SURFACE AREA AND VOLUME OF SPHERES

7.14 EXPLORING SIMILAR SOLIDS

7.15 SURFACE AREA AND VOLUME REVIEW

Now that we have explored triangles, quadrilaterals, polygons, and circles, we are going to learn how to find the perimeter and area of each. First we will derive each formula and then apply them to different types of polygons and circles. In addition, we will explore the properties of similar polygons, their perimeters and their areas.

7.1 Triangles and Parallelograms

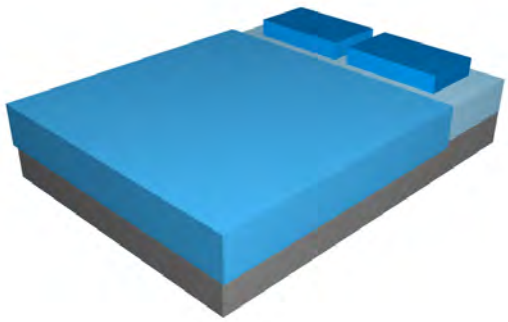
Learning Objectives

- Understand the basic concepts of area.
- Use formulas to find the area of triangles and parallelograms.

Review Queue

1. Define perimeter and area, in your own words.
2. Solve the equations below. Simplify any radicals.
   a. $x^2 = 121$
   b. $4x^2 = 80$
   c. $x^2 - 6x + 8 = 0$
3. If a rectangle has sides 4 and 7, what is the perimeter?

Know What? Ed's parents are getting him a new bed. He has decided that he would like a king bed. Upon further research, Ed discovered there are two types of king beds, an Eastern (or standard) King and a California King. The Eastern King has $76'' \times 80''$ dimensions, while the California King is $72'' \times 84''$ (both dimensions are *width* $\times$ *length*). Which bed has a larger area to lie on? Which one has a larger perimeter? If Ed is 6'4", which bed makes more sense for him to buy?



Areas and Perimeters of Squares and Rectangles

Perimeter: The distance around a shape. Or, the sum of all the edges of a two-dimensional figure.

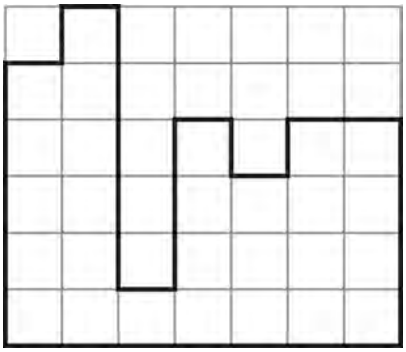
The perimeter of any figure must have a unit of measurement attached to it. If no specific units are given (feet, inches, centimeters, etc), write "units."

Example 1: Find the perimeter of the figure to the left.

Solution: First, notice there are no units, but the figure is on a grid. Here, we can use the grid as our units. Count around the figure to find the perimeter. We will start at the bottom left-hand corner and go around the figure clockwise.

$$5+1+1+1+5+1+3+1+1+1+1+2+4+7$$

The answer is 34 units.



You are probably familiar with the area of squares and rectangles from a previous math class. Recall that you must always establish a unit of measure for area. Area is always measured in square units, square feet $(ft.^2)$, square inches $(in.^2)$, square centimeters $(cm.^2)$, etc. Make sure that the length and width are in the same units before applying any area formula. If no specific units are given, write "$units^2$."

Example 2: Find the area of the figure from Example 1.

Solution: If the figure is not a standard shape, you can count the number of squares within the figure. If we start on the left and count each column, we would have:

$$5+6+1+4+3+4+4=27\ units^2$$

Area of a Rectangle: The area of a rectangle is the product of its base (width) and height (length) $A=bh$.

Example 3: Find the area and perimeter of a rectangle with sides 4 cm by 9 cm.

Solution: The perimeter is $4+9+4+9=36\ cm$. The area is $A=9\cdot 4=36\ cm^2$.

In this example we see that a formula can be generated for the perimeter of a rectangle.

Perimeter of a Rectangle: $P=2b+2h$, where b is the base (or width) and h is the height (or length).

If a rectangle is a square, with sides of length s, the formulas are as follows:

Perimeter of a Square: $P_{square} = 2s + 2s = 4s$

Area of a Square: $A_{sqaure} = s \cdot s = s^2$

Example 4: The area of a square is 75 in^2. Find the perimeter.

Solution: To find the perimeter, we need to find the length of the sides.

$$A = s^2 = 75\ in^2$$
$$s = \sqrt{75} = 5\sqrt{3}\ in$$
$$\text{From this, } P = 4\left(5\sqrt{3}\right) = 20\sqrt{3}\ in.$$

Area Postulates

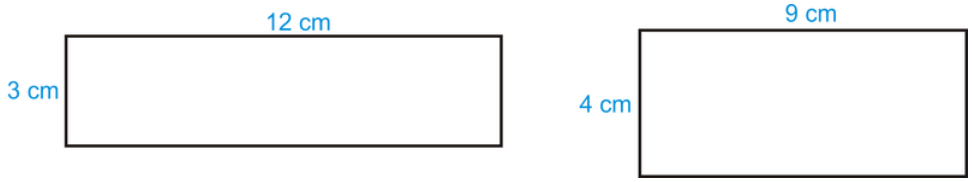
Congruent Areas Postulate: If two figures are congruent, they have the same area.

This postulate needs no proof because congruent figures have the same amount of space inside them. However, two figures with the same area are not necessarily congruent.

Example 5: Draw two different rectangles with an area of 36 cm^2.

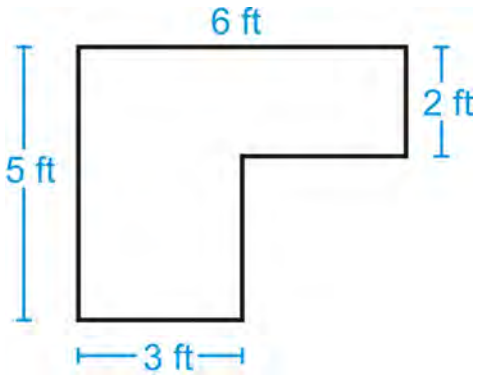
Solution: Think of all the different factors of 36. These can all be dimensions of the different rectangles.

Other possibilities could be $6 \times 6, 2 \times 18$, and 1×36.

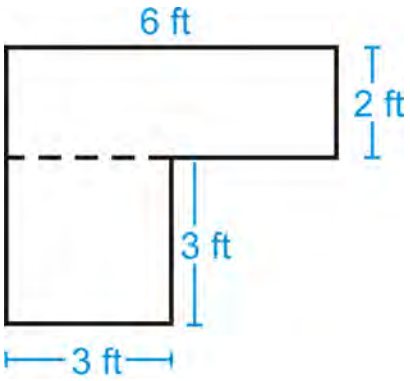


Area Addition Postulate: If a figure is composed of two or more parts that do not overlap each other, then the area of the figure is the sum of the areas of the parts.

Example 6: Find the area of the figure below. You may assume all sides are perpendicular.



Solution: Split the shape into two rectangles and find the area of each.



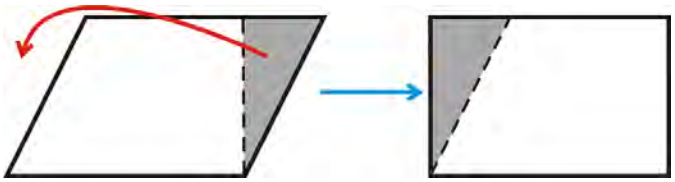
$$A_{top\ rectangle} = 6 \cdot 2 = 12\ ft^2$$
$$A_{bottom\ square} = 3 \cdot 3 = 9\ ft^2$$

The total area is $12 + 9 = 21\ ft^2$.

Area of a Parallelogram

Recall that a parallelogram is a quadrilateral whose opposite sides are parallel.

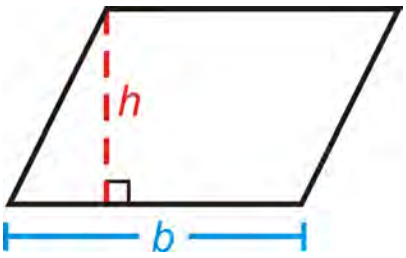
To find the area of a parallelogram, make it into a rectangle.



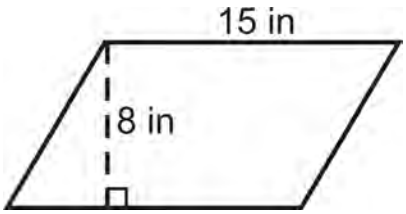
From this, we see that the area of a parallelogram is the same as the area of a rectangle.

Area of a Parallelogram: The area of a parallelogram is $A = bh$.

Be careful! The height of a parallelogram is always perpendicular to the base. This means that the sides are ***not*** the height.



Example 7: Find the area of the parallelogram.



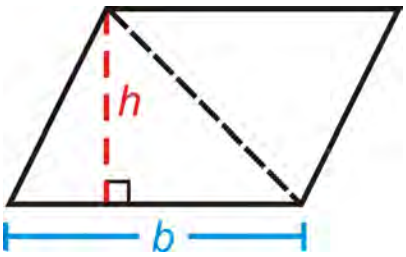
Solution: $A = 15 \cdot 8 = 120\ in^2$

Example 8: If the area of a parallelogram is 56 $units^2$ and the base is 4 units, what is the height?

Solution: Plug in what we know to the area formula and solve for the height.

$$56 = 4h$$
$$14 = h$$

Area of a Triangle

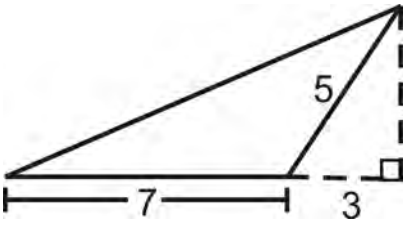


If we take parallelogram and cut it in half, along a diagonal, we would have two congruent triangles. Therefore, the formula for the area of a triangle is the same as the formula for area of a parallelogram, but cut in half.

Area of a Triangle: $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$.

In the case that the triangle is a right triangle, then the height and base would be the legs of the right triangle. If the triangle is an obtuse triangle, the altitude, or height, could be outside of the triangle.

Example 9: Find the area and perimeter of the triangle.

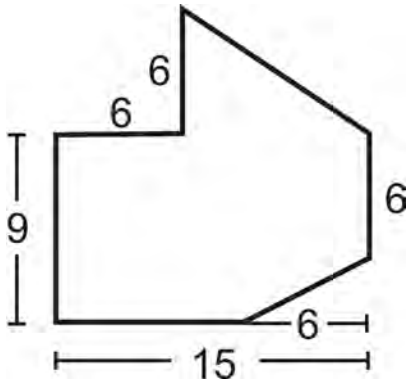


Solution: This is an obtuse triangle. First, to find the area, we need to find the height of the triangle. We are given the two sides of the small right triangle, where the hypotenuse is also the short side of the obtuse triangle. From these values, we see that the height is 4 because this is a 3-4-5 right triangle. The area is $A = \frac{1}{2}(4)(7) = 14\ units^2$.

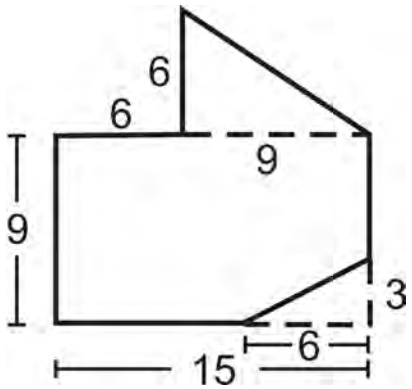
To find the perimeter, we would need to find the longest side of the obtuse triangle. If we used the dotted lines in the picture, we would see that the longest side is also the hypotenuse of the right triangle with legs 4 and 10. Use the Pythagorean Theorem.

$$4^2 + 10^2 = c^2$$
$$16 + 100 = c^2$$
$$c = \sqrt{116} \approx 10.77 \qquad \text{The perimeter is } 7 + 5 + 10.77 = 22.77 \textit{ units}$$

Example 10: Find the area of the figure below.



Solution: Divide the figure into a triangle and a rectangle with a small rectangle cut out of the lower right-hand corner.



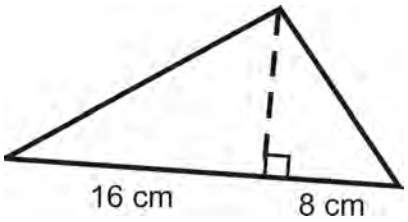
$$A = A_{top\ triangle} + A_{rectangle} - A_{small\ triangle}$$
$$A = \left(\frac{1}{2} \cdot 6 \cdot 9\right) + (9 \cdot 15) + \left(\frac{1}{2} \cdot 3 \cdot 6\right)$$
$$A = 27 + 135 + 9$$
$$A = 171 \textit{ units}^2$$

Know What? Revisited The area of an Eastern King is 6080 in^2 and the California King is 6048 in^2. The perimeter of both beds is 312 in. Because Ed is 6'4", he should probably get the California King because it is 4 inches longer.

Review Questions

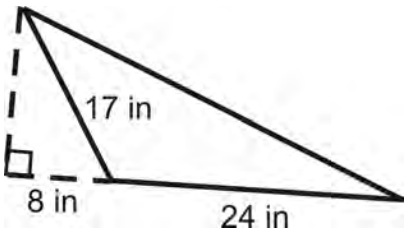
1. Find the area and perimeter of a square with sides of length 12 in.
2. Find the area and perimeter of a rectangle with height of 9 cm and base of 16 cm.
3. Find the area of a parallelogram with height of 20 m and base of 18 m.
4. Find the area and perimeter of a rectangle if the height is 8 and the base is 14.
5. Find the area and perimeter of a square if the sides are 18 ft.
6. If the area of a square is 81 ft^2, find the perimeter.
7. If the perimeter of a square is 24 in, find the area.
8. Find the area of a triangle with base of length 28 cm and height of 15 cm.
9. What is the height of a triangle with area 144 m^2 and a base of 24 m?
10. The perimeter of a rectangle is 32. Find two different dimensions that the rectangle could be.
11. Draw two different rectangles that haven an area of 90 mm^2.
12. Write the converse of the Congruent Areas Postulate. Determine if it is a true statement. If not, write a counterexample. If it is true, explain why.

Use the triangle to answer the following questions.



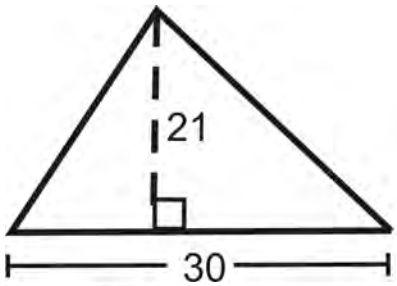
13. Find the height of the triangle by using the geometric mean.
14. Find the perimeter.
15. Find the area.

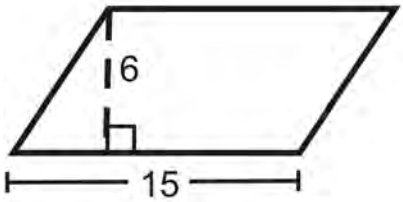
Use the triangle to answer the following questions.

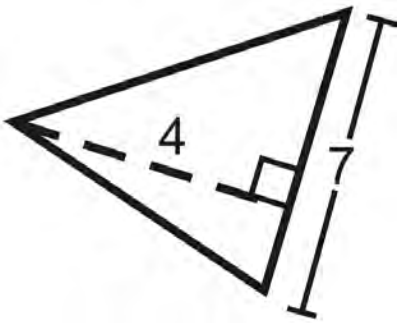


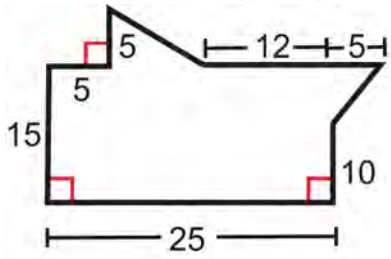
16. Find the height of the triangle.
17. Find the perimeter.
18. Find the area.

Find the area of the following shapes.

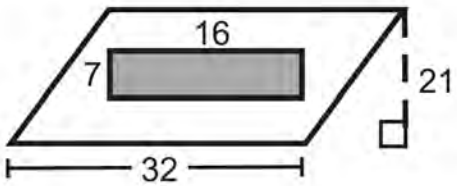
19. 

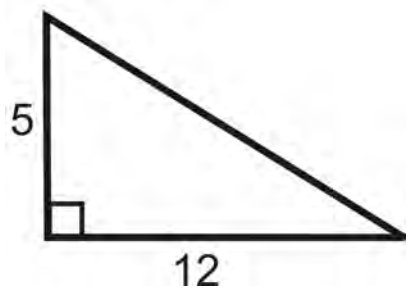
20. 

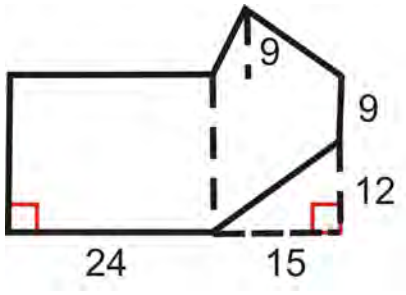
21. 

22. 

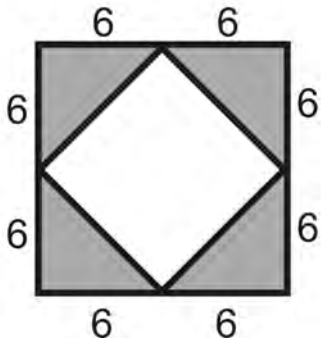
23. Find the area of the unshaded region.



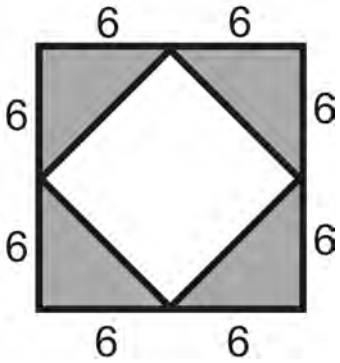
24. 

25. 

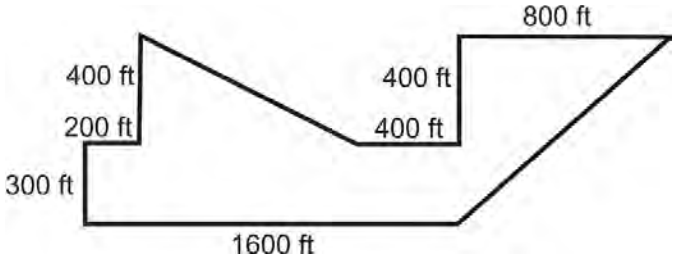
26. Find the area of the shaded region.



27. Find the area of the unshaded region.



28. Lin bought a tract of land for a new apartment complex. The drawing below shows the measurements of the sides of the tract. Approximately how many acres of land did Lin buy? You may assume any angles that look like right angles are 90°. (1 acre ≈ 40,000 square feet)



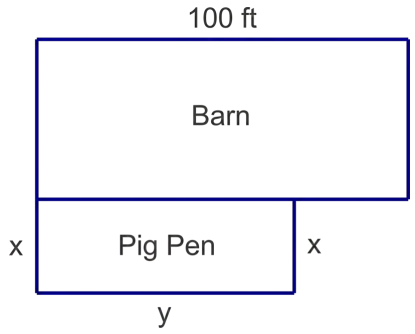
Challenge Problems

For problems 29 and 30 find the dimensions of the rectangles with the given information.

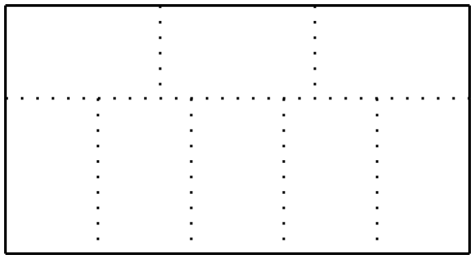
29. A rectangle with a perimeter of 20 units and an area of 24 $units^2$.
30. A rectangle with a perimeter of 72 units and an area of 288 $units^2$.

For problems 31 and 32 find the height and area of the equilateral triangle with the given perimeter.

31. Perimeter 18 units.
32. Perimeter 30 units.
33. Generalize your results from problems 31 and 32 into a formula to find the height and area of an equilateral triangle with side length x.
34. Linus has 100 ft of fencing to use in order to enclose a 1200 square foot rectangular pig pen. The pig pen is adjacent to the barn so he only needs to form three sides of the rectangular area as shown below. What dimensions should the pen be?



35. A rectangle with perimeter 138 units is divided into 8 congruent rectangles as shown in the diagram below. Find the perimeter and area of one of the 8 congruent rectangles.



Review Queue Answers

1. *Possible* Answers

Perimeter: The distance around a shape.

Area: The space inside a shape.

2. (a) $x = \pm 11$

(b) $x = \pm 2\sqrt{5}$

(c) $x = 4, 2$

3. $4+4+7+7 = 22$

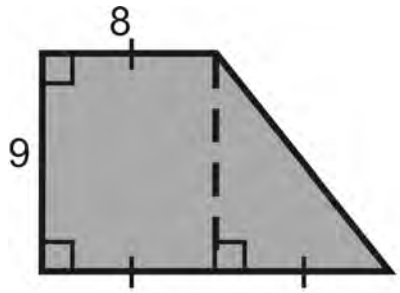
7.2 Trapezoids, Rhombi, and Kites

Learning Objectives

- Derive and use the area formulas for trapezoids, rhombi, and kites.

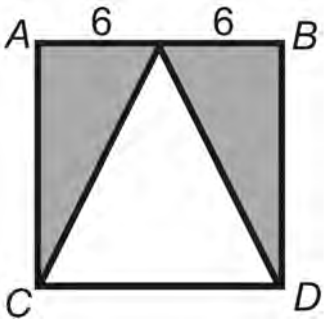
Review Queue

Find the area the *shaded* regions in the figures below.

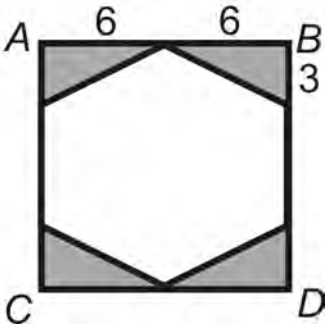


1.

2. *ABCD* is a square.



3. *ABCD* is a square.



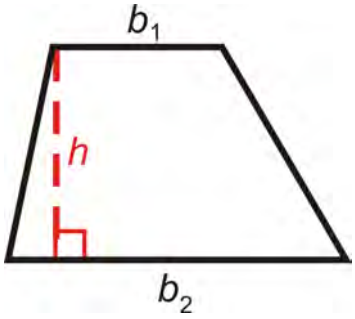
4. Find the area of #1 using a different method.

Know What? The Brazilian flag is to the right. The flag has dimensions of 20×14 (units vary depending on the size, so we will not use any here). The vertices of the yellow rhombus in the middle are 1.7 units from the midpoint of each side.

Find the total area of the flag and the area of the rhombus (including the circle). *Do not round your answers.*

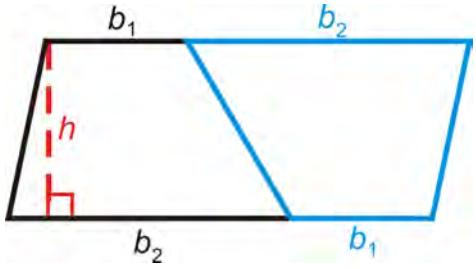
Area of a Trapezoid

Recall that a trapezoid is a quadrilateral with one pair of parallel sides. The lengths of the parallel sides are the bases. The perpendicular distance between the parallel sides is the height, or altitude, of the trapezoid.



To find the area of the trapezoid, let's turn it into a parallelogram. To do this, make a copy of the trapezoid and then rotate the copy 180°.

Now, this is a parallelogram with height h and base $b_1 + b_2$. Let's find the area of this shape.



$$A = h(b_1 + b_2)$$

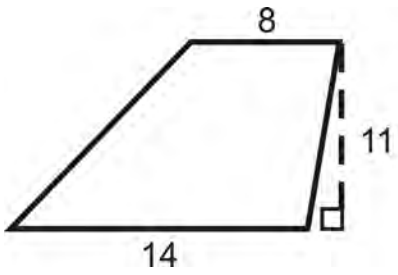
Because the area of this parallelogram is made up of two congruent trapezoids, the area of one trapezoid would be $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a Trapezoid: The area of a trapezoid with height h and bases b_1 and b_2 is $A = \frac{1}{2}h(b_1 + b_2)$.

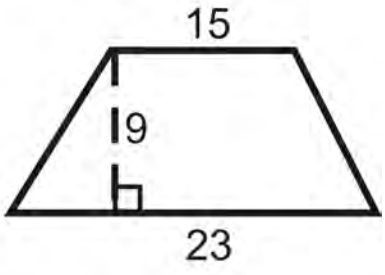
The formula for the area of a trapezoid could also be written as *the average of the bases time the height.*

Example 1: Find the area of the trapezoids below.

a)



b)

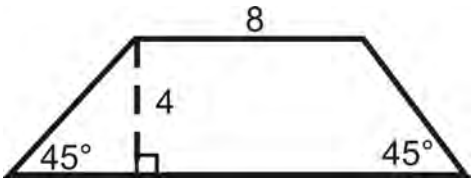


Solution:

a) $A = \frac{1}{2}(11)(14+8)$
$A = \frac{1}{2}(11)(22)$
$A = 121\ units^2$

b) $A = \frac{1}{2}(9)(15+23)$
$A = \frac{1}{2}(9)(38)$
$A = 171\ units^2$

Example 2: Find the perimeter and area of the trapezoid.



Solution: Even though we are not told the length of the second base, we can find it using special right triangles. Both triangles at the ends of this trapezoid are isosceles right triangles, so the hypotenuses are $4\sqrt{2}$ and the other legs are of length 4.

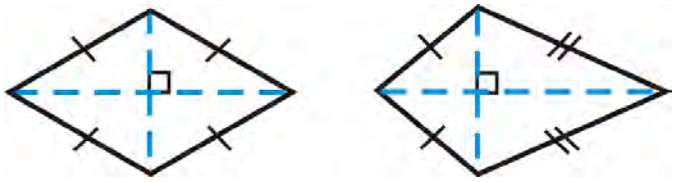
$$P = 8 + 4\sqrt{2} + 16 + 4\sqrt{2}$$
$$P = 24 + 8\sqrt{2} \approx 35.3\ units$$

$$A = \frac{1}{2}(4)(8+16)$$
$$A = 48\ units^2$$

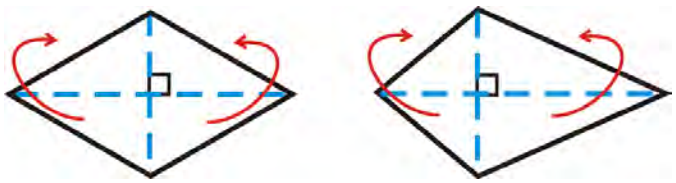
Area of a Rhombus and Kite

Recall that a rhombus is an equilateral quadrilateral and a kite has adjacent congruent sides.

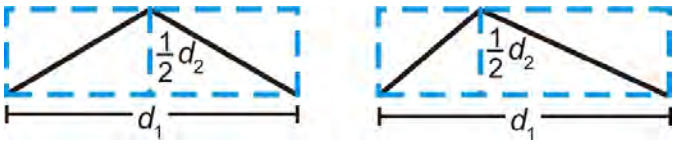
Both of these quadrilaterals have perpendicular diagonals, which is how we are going to find their areas.



Notice that the diagonals divide each quadrilateral into 4 triangles. In the rhombus, all 4 triangles are congruent and in the kite there are two sets of congruent triangles. If we move the two triangles on the bottom of each quadrilateral so that they match up with the triangles above the horizontal diagonal, we would have two rectangles.



So, the height of these rectangles is half of one of the diagonals and the base is the length of the other diagonal.



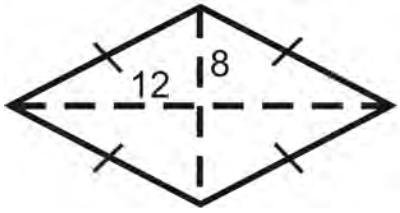
Area of a Rhombus: If the diagonals of a rhombus are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

Area of a Kite: If the diagonals of a kite are d_1 and d_2, then the area is $A = \frac{1}{2}d_1d_2$.

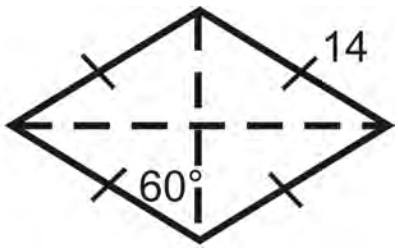
You could also say that the area of a kite and rhombus are *half the product of the diagonals.*

Example 3: Find the perimeter and area of the rhombi below.

a)



b)



Solution: In a rhombus, all four triangles created by the diagonals are congruent.

a) To find the perimeter, you must find the length of each side, which would be the hypotenuse of one of the four triangles. Use the Pythagorean Theorem.

$$12^2 + 8^2 = side^2$$
$$144 + 64 = side^2$$
$$side = \sqrt{208} = 4\sqrt{13}$$
$$P = 4\left(4\sqrt{13}\right) = 16\sqrt{13}$$

$$A = \frac{1}{2} \cdot 16 \cdot 24$$
$$A = 192$$

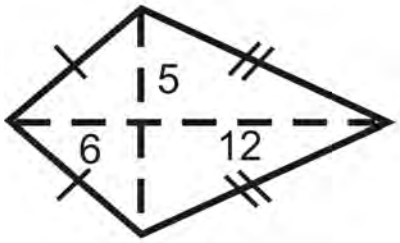
b) Here, each triangle is a 30-60-90 triangle with a hypotenuse of 14. From the special right triangle ratios the short leg is 7 and the long leg is $7\sqrt{3}$.

$$P = 4 \cdot 14 = 56$$

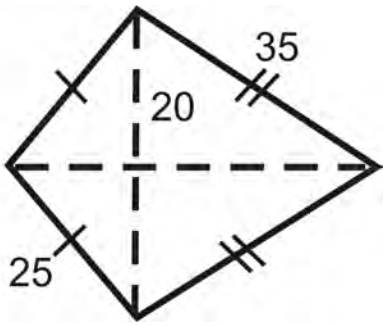
$$A = \frac{1}{2} \cdot 7 \cdot 7\sqrt{3} = \frac{49\sqrt{3}}{2} \approx 42.44$$

Example 4: Find the perimeter and area of the kites below.

a)



b)



Solution: In a kite, there are two pairs of congruent triangles. You will need to use the Pythagorean Theorem in both problems to find the length of sides or diagonals.

a)

Shorter sides of kite

$$6^2+5^2=s_1^2$$
$$36+25=s_1^2$$
$$s_1=\sqrt{61}$$
$$P=2\left(\sqrt{61}\right)+2(13)=2\sqrt{61}+26\approx 41.6$$
$$A=\frac{1}{2}(10)(18)=90$$

Longer sides of kite

$$12^2+5^2=s_2^2$$
$$144+25=s_2^2$$
$$s_2=\sqrt{169}=13$$

b)

Smaller diagonal portion

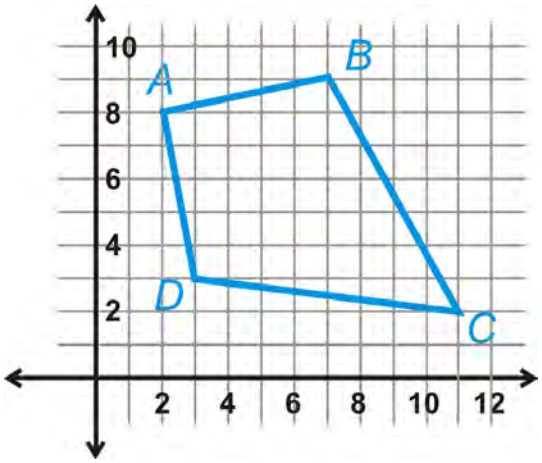
$$20^2+d_s^2=25^2$$
$$d_s^2=225$$
$$d_s=15$$
$$P=2(25)+2(35)=120$$
$$A=\frac{1}{2}\left(15+5\sqrt{33}\right)(40)\approx 874.5$$

Larger diagonal portion

$$20^2+d_l^2=35^2$$
$$d_l^2=825$$
$$d_l=5\sqrt{33}$$

Example 5: The vertices of a quadrilateral are $A(2,8), B(7,9), C(11,2)$, and $D(3,3)$. Determine the type of quadrilateral and find its area.

Solution: For this problem, it might be helpful to plot the points. From the graph we can see this is probably a kite. Upon further review of the sides, $AB=AD$ and $BC=DC$ (you can do the distance formula to verify). Let's see if the diagonals are perpendicular by calculating their slopes.



$$m_{AC}=\frac{2-8}{11-2}=-\frac{6}{9}=-\frac{2}{3}$$
$$m_{BD}=\frac{9-3}{7-3}=\frac{6}{4}=\frac{3}{2}$$

Yes, the diagonals are perpendicular because the slopes are opposite signs and reciprocals. *ABCD* is a kite. To find the area, we need to find the length of the diagonals. Use the distance formula.

$$d_1 = \sqrt{(2-11)^2+(8-2)^2} \qquad d_2 = \sqrt{(7-3)^2+(9-3)^2}$$
$$= \sqrt{(-9)^2+6^2} \qquad = \sqrt{4^2+6^2}$$
$$= \sqrt{81+36} = \sqrt{117} = 3\sqrt{13} \qquad = \sqrt{16+36} = \sqrt{52} = 2\sqrt{13}$$

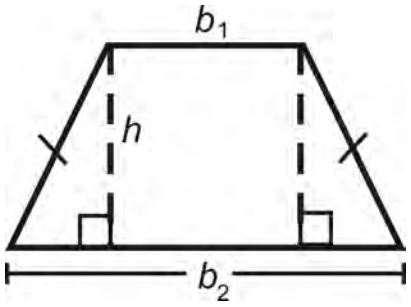
Now, plug these lengths into the area formula for a kite.

$$A = \frac{1}{2}\left(3\sqrt{13}\right)\left(2\sqrt{13}\right) = 39 \text{ } units^2$$

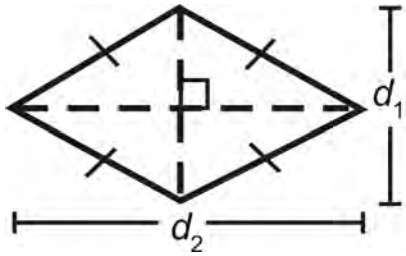
Know What? Revisited The total area of the Brazilian flag is $A = 14 \cdot 20 = 280$ $units^2$. To find the area of the rhombus, we need to find the length of the diagonals. One diagonal is $20 - 1.7 - 1.7 = 16.6$ $units$ and the other is $14 - 1.7 - 1.7 = 10.6$ $units$. The area is $A = \frac{1}{2}(16.6)(10.6) = 87.98$ $units^2$.

Review Questions

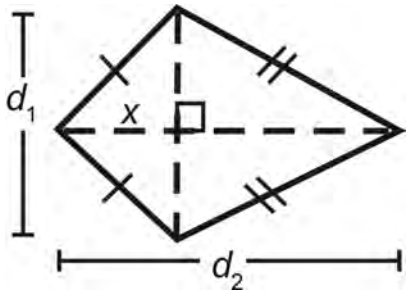
1. Do you think all rhombi and kites with the same diagonal lengths have the same area? *Explain* your answer.
2. Use the isosceles trapezoid to show that the area of this trapezoid can also be written as the sum of the area of the two triangles and the rectangle in the middle. Write the formula and then reduce it to equal $\frac{1}{2}h(b_1+b_2)$ or $\frac{h}{2}(b_1+b_2)$.



3. Use this picture of a rhombus to show that the area of a rhombus is equal to the sum of the areas of the four congruent triangles. Write a formula and reduce it to equal $\frac{1}{2}d_1d_2$.

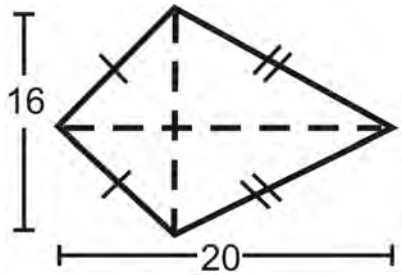


4. Use this picture of a kite to show that the area of a kite is equal to the sum of the areas of the two pairs of congruent triangles. Recall that d_1 is bisected by d_2. Write a formula and reduce it to equal $\frac{1}{2}d_1d_2$.



Find the area of the following shapes. *Leave answers in simplest radical form.*

5.



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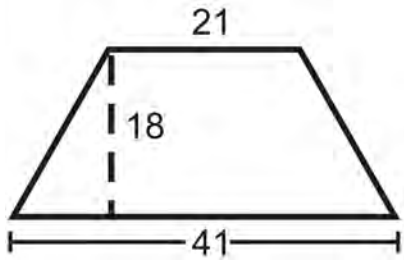

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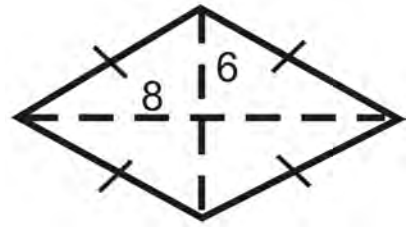

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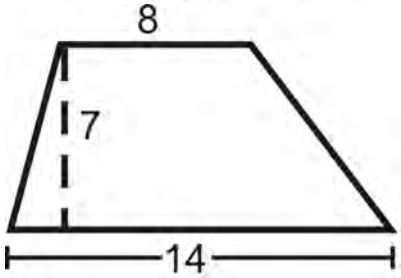

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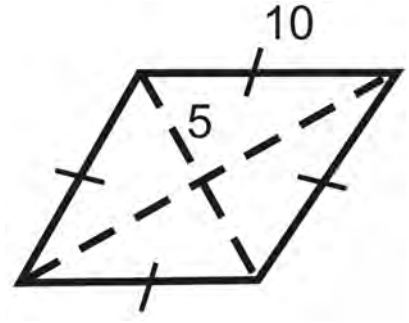

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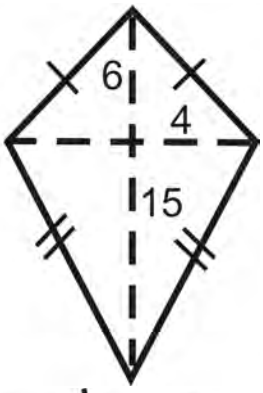

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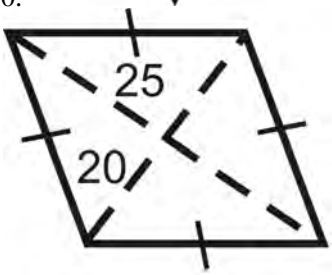

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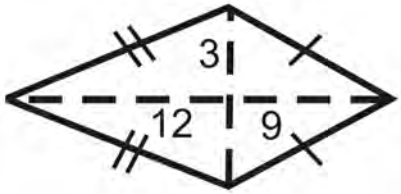

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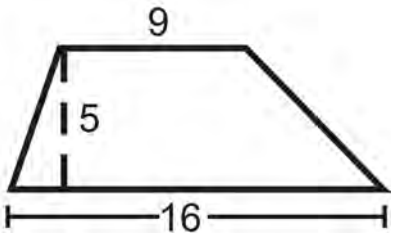
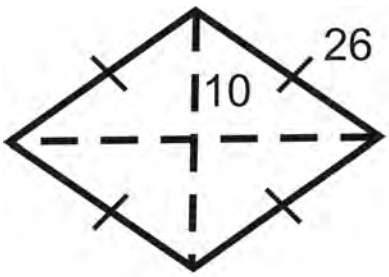


Find the area and perimeter of the following shapes. *Leave answers in simplest radical form.*

14.



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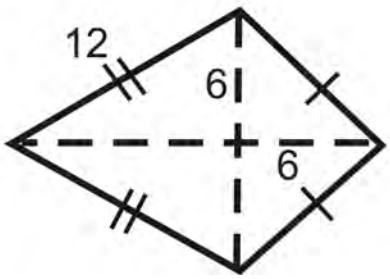
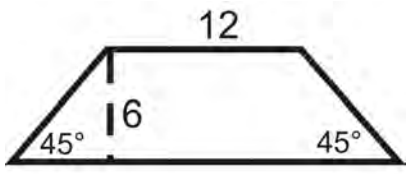
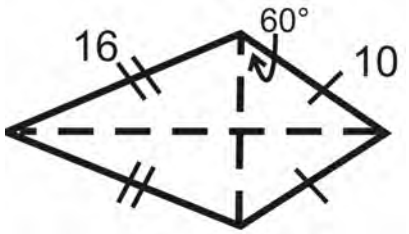
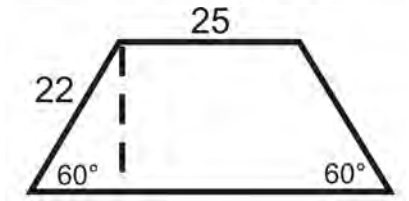
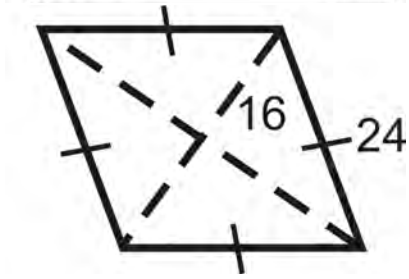


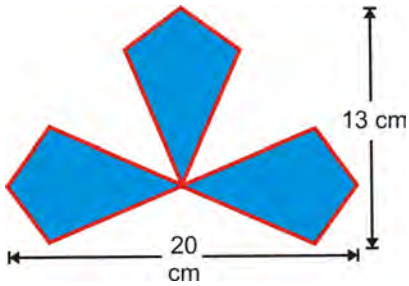
16. 

17. 

18. 

19. 

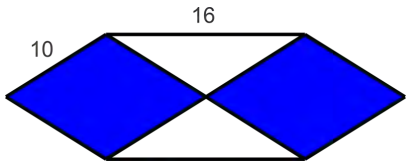
20. Quadrilateral $ABCD$ has vertices $A(-2,0), B(0,2), C(4,2)$, and $D(0,-2)$. Show that $ABCD$ is a trapezoid and find its area. *Leave your answer in simplest radical form.*
21. Quadrilateral $EFGH$ has vertices $E(2,-1), F(6,-4), G(2,-7)$, and $H(-2,-4)$. Show that $EFGH$ is a rhombus and find its area.
22. The area of a rhombus is 32 $units^2$. What are two possibilities for the lengths of the diagonals?
23. The area of a kite is 54 $units^2$. What are two possibilities for the lengths of the diagonals?
24. Sherry designed the logo for a new company. She used three congruent kites. What is the area of the entire logo?



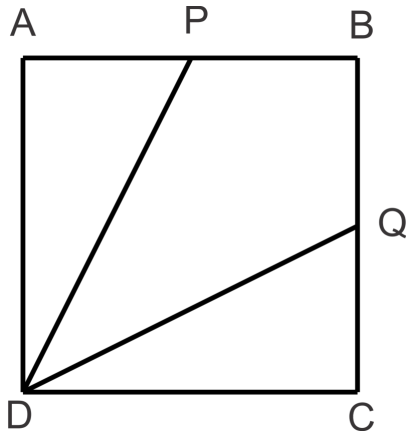
For problems 25-27, determine what kind of quadrilateral $ABCD$ is and find its area.

25. $A(-2,3), B(2,3), C(4,-3), D(-2,-1)$
26. $A(0,1), B(2,6), C(8,6), D(13,1)$
27. $A(-2,2), B(5,6), C(6,-2), D(-1,-6)$
28. Given that the lengths of the diagonals of a kite are in the ratio 4:7 and the area of the kite is 56 square units, find the lengths of the diagonals.
29. Given that the lengths of the diagonals of a rhombus are in the ratio 3:4 and the area of the rhombus is 54 square units, find the lengths of the diagonals.

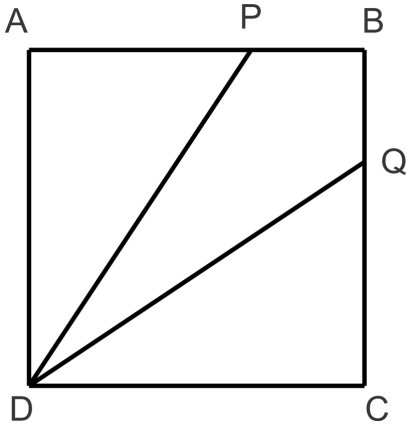
30. Sasha drew this plan for a wood inlay he is making. 10 is the length of the slanted side and 16 is the length of the horizontal line segment as shown in the diagram. Each shaded section is a rhombus. What is the total area of the shaded sections?



31. In the figure to the right, $ABCD$ is a square. $AP = PB = BQ$ and $DC = 20\ ft$.
    a. What is the area of $PBQD$?
    b. What is the area of $ABCD$?
    c. What fractional part of the area of $ABCD$ is $PBQD$?



32. In the figure to the right, $ABCD$ is a square. $AP = 20\ ft$ and $PB = BQ = 10\ ft$.
    a. What is the area of $PBQD$?
    b. What is the area of $ABCD$?
    c. What fractional part of the area of $ABCD$ is $PBQD$?



Review Queue Answers

1. $A = 9(8) + \left[\frac{1}{2}(9)(8)\right] = 72 + 36 = 108\ units^2$
2. $A = \frac{1}{2}(6)(12)2 = 72\ units^2$
3. $A = 4\left[\frac{1}{2}(6)(3)\right] = 36\ units^2$
4. $A = 9(16) - \left[\frac{1}{2}(9)(8)\right] = 144 - 36 = 108\ units^2$

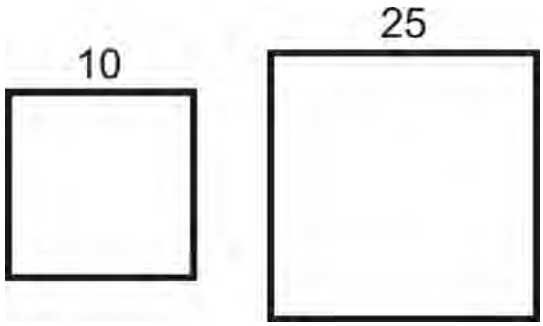
7.3 Areas of Similar Polygons

Learning Objectives

- Understand the relationship between the scale factor of similar polygons and their areas.
- Apply scale factors to solve problems about areas of similar polygons.

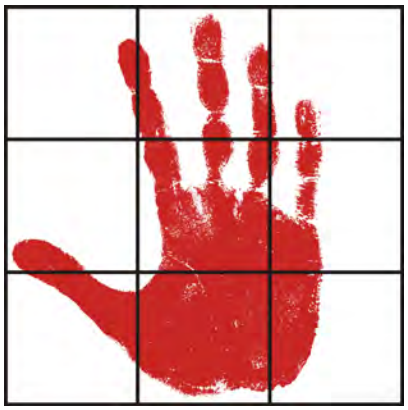
Review Queue

1. Are two squares similar? Are two rectangles?



2. Find the scale factor of the sides of the similar shapes. Both figures are squares.
3. Find the area of each square.
4. Find the ratio of the smaller square's area to the larger square's area. Reduce it. How does it relate to the scale factor?

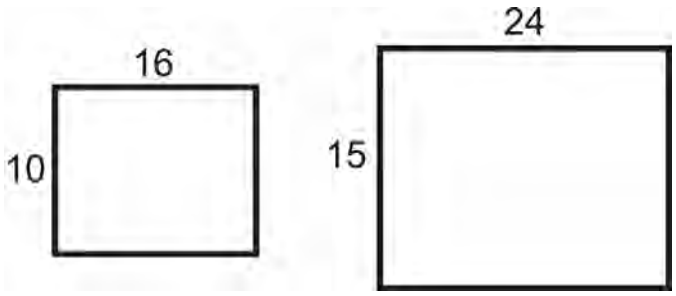
Know What? One use of scale factors and areas is scale drawings. This technique takes a small object, like the handprint to the right, divides it up into smaller squares and then blows up the individual squares. In this Know What? you are going to make a scale drawing of your own hand. Either trace your hand or stamp it on a piece of paper. Then, divide your hand into 9 squares, like the one to the right, probably 2 *in* $\times$ 2 *in*. Take a larger piece of paper and blow up each square to be 6 *in* $\times$ 6 *in* (meaning you need at least an 18 in square piece of paper). Once you have your 6 *in* $\times$ 6 *in* squares drawn, use the proportions and area to draw in your enlarged handprint.



Areas of Similar Polygons

In Chapter 7, we learned about similar polygons. Polygons are similar when the corresponding angles are equal and the corresponding sides are in the same proportion. In that chapter we also discussed the relationship of the perimeters of similar polygons. Namely, the scale factor for the sides of two similar polygons is the same as the ratio of the perimeters.

Example 1: The two rectangles below are similar. Find the scale factor and the ratio of the perimeters.



Solution: The scale factor is $\frac{16}{24}$, which reduces to $\frac{2}{3}$. The perimeter of the smaller rectangle is 52 units. The perimeter of the larger rectangle is 78 units. The ratio of the perimeters is $\frac{52}{78} = \frac{2}{3}$.

The ratio of the perimeters is the same as the scale factor. In fact, the ratio of any part of two similar shapes (diagonals, medians, midsegments, altitudes, etc.) is the same as the scale factor.

Example 2: Find the area of each rectangle from Example 1. Then, find the ratio of the areas.

Solution:

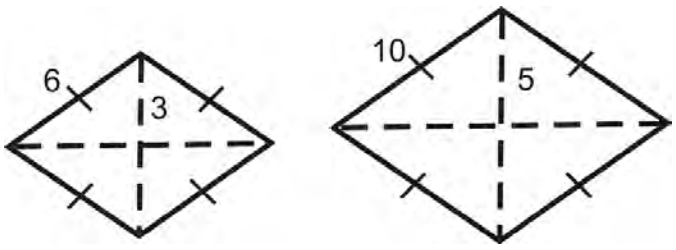
$$A_{small} = 10 \cdot 16 = 160 \; units^2$$
$$A_{large} = 15 \cdot 24 = 360 \; units^2$$

The ratio of the areas would be $\frac{160}{360} = \frac{4}{9}$.

The ratio of the sides, or scale factor was $\frac{2}{3}$ and the ratio of the areas is $\frac{4}{9}$. Notice that the ratio of the areas is the ***square*** of the scale factor. An easy way to remember this is to think about the units of area, which are always ***squared.*** Therefore, you would always ***square*** the scale factor to get the ratio of the areas.

Area of Similar Polygons Theorem: If the scale factor of the sides of two similar polygons is $\frac{m}{n}$, then the ratio of the areas would be $\left(\frac{m}{n}\right)^2$.

Example 2: Find the ratio of the areas of the rhombi below. The rhombi are similar.



Solution: There are two ways to approach this problem. One way would be to use the Pythagorean Theorem to find the length of the 3^{rd} side in the triangle and then apply the area formulas and make a ratio. The second, and easier way, would be to find the ratio of the sides and then square that. $\left(\frac{3}{5}\right)^2 = \frac{9}{25}$

Example 3: Two trapezoids are similar. If the scale factor is $\frac{3}{4}$ and the area of the smaller trapezoid is 81 cm^2, what is the area of the larger trapezoid?

Solution: First, the ratio of the areas would be $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$. Now, we need the area of the larger trapezoid. To find this, we would multiply the area of the smaller trapezoid by the scale factor. However, we would need to flip the scale factor over to be $\frac{16}{9}$ because we want the larger area. This means we need to multiply by a scale factor that is larger than one. $A = \frac{16}{9} \cdot 81 = 144\ cm^2$.

Example 4: Two triangles are similar. The ratio of the areas is $\frac{25}{64}$. What is the scale factor?

Solution: The scale factor is $\sqrt{\frac{25}{64}} = \frac{5}{8}$.

Example 5: Using the ratios from Example 3, find the length of the base of the smaller triangle if the length of the base of the larger triangle is 24 units.

Solution: All you would need to do is multiply the scale factor we found in Example 3 by 24.

$$b = \frac{5}{8} \cdot 24 = 15\ units$$

Know What? Revisited You should end up with an 18 *in* $\times$ 18 *in* drawing of your handprint.

Review Questions

Determine the ratio of the areas, given the ratio of the sides of a polygon.

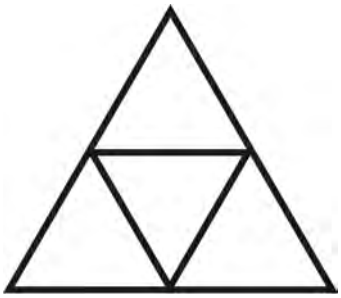
1. $\frac{3}{5}$
2. $\frac{1}{4}$
3. $\frac{7}{2}$
4. $\frac{6}{11}$

Determine the ratio of the sides of a polygon, given the ratio of the areas.

5. $\frac{1}{36}$
6. $\frac{4}{81}$
7. $\frac{49}{9}$
8. $\frac{25}{144}$

This is an equilateral triangle made up of 4 congruent equilateral triangles.

9. What is the ratio of the areas of the large triangle to one of the small triangles?



10. What is the scale factor of large to small triangle?
11. If the area of the large triangle is 20 $units^2$, what is the area of a small triangle?
12. If the length of the altitude of a small triangle is $2\sqrt{3}$, find the perimeter of the large triangle.
13. Carol drew two equilateral triangles. Each side of one triangle is 2.5 times as long as a side of the other triangle. The perimeter of the smaller triangle is 40 cm. What is the perimeter of the larger triangle?
14. If the area of the smaller triangle is 75 cm^2, what is the area of the larger triangle from #13?
15. Two rectangles are similar with a scale factor of $\frac{4}{7}$. If the area of the larger rectangle is 294 in^2, find the area of the smaller rectangle.
16. Two triangles are similar with a scale factor of $\frac{1}{3}$. If the area of the smaller triangle is 22 ft^2, find the area of the larger triangle.
17. The ratio of the areas of two similar squares is $\frac{16}{81}$. If the length of a side of the smaller square is 24 units, find the length of a side in the larger square.
18. The ratio of the areas of two right triangles is $\frac{2}{3}$. If the length of the hypotenuse of the larger triangle is 48 units, find the length of the smaller triangle's hypotenuse.

Questions 19-22 build off of each other. You may assume the problems are connected.

19. Two similar rhombi have areas of 72 $units^2$ and 162 $units^2$. Find the ratio of the areas.
20. Find the scale factor.
21. The diagonals in these rhombi are congruent. Find the length of the diagonals and the sides.
22. What type of rhombi are these quadrilaterals?
23. The area of one square on a game board is exactly twice the area of another square. Each side of the larger square is 50 mm long. How long is each side of the smaller square?
24. The distance from Charleston to Morgantown is 160 miles. The distance from Fairmont to Elkins is 75 miles. Charleston and Morgantown are 5 inches apart on a map. How far apart are Fairmont and Elkins on the same map.
25. Marlee is making models of historic locomotives (train engines). She uses the same scale for all of her models. The S1 locomotive was 140 ft long. The model is 8.75 inches long. The 520 Class locomotive was 87 feet long. What is the scale of Marlee's models? How long is the model of the 520 Class locomotive?
26. Tommy is drawing a floor plan for his dream home. On his drawing, 1cm represents 2 ft of the actual home. The actual dimensions of the dream home are 55 ft by 40 ft. What will the dimensions of his floor plan be? Will his scale drawing fit on a standard 8.5 in by 11 in piece of paper? Justify your answer.
27. Anne wants to purchase advertisement space in the school newspaper. Each square inch of advertisement space sells for \$3.00. She wants to purchase a rectangular space with length and width in the ratio 3:2 and she has up to \$50.00 to spend. What are the dimensions of the largest advertisement she can afford to purchase?
28. Aaron wants to enlarge a family photo from a 5 by 7 print to a print with an area of 140 inches. What are the dimensions of this new photo?
29. A popular pizza joint offers square pizzas: Baby Bella pizza with 10 inch sides, the Mama Mia pizza with 14 inch sides and the Big Daddy pizza with 18 inch sides. If the prices for these pizzas are \$5.00, \$9.00 and \$15.00 respectively, find the price per square inch of each pizza. Which is the best deal?
30. Krista has a rectangular garden with dimensions 2 ft by 3 ft. She uses $\frac{2}{3}$ of a bottle of fertilizer to cover this area. Her friend, Hadleigh, has a garden with dimensions that are 1.5 times as long. How many bottles of fertilizer will she need?

Review Queue Answers

1. Two squares are always similar. Two rectangles can be similar as long as the sides are in the same proportion.
2. $\frac{10}{25} = \frac{2}{5}$
3. $A_{small} = 100, A_{large} = 625$

4. $\frac{100}{625} = \frac{4}{25}$, this is the square of the scale factor.

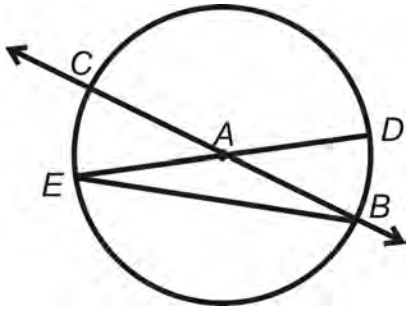
7.4 Circumference and Arc Length

Learning Objectives

- Find the circumference of a circle.
- Define the length of an arc and find arc length.

Review Queue

1. Find a central angle in that intercepts $\widehat{CE}$



2. Find an inscribed angle that intercepts $\widehat{CE}$.
3. How many degrees are in a circle? Find $m\widehat{ECD}$.
4. If $m\widehat{CE} = 26°$, find $m\widehat{CD}$ and $m\angle CBE$.

Know What? A typical large pizza has a diameter of 14 inches and is cut into 8 or 10 pieces. Think of the crust as the circumference of the pizza. Find the *length* of the crust for the entire pizza. Then, find the length of the crust for one piece of pizza if the entire pizza is cut into a) 8 pieces or b) 10 pieces.

Circumference of a Circle

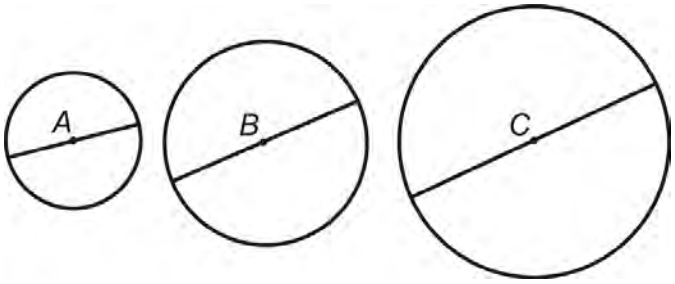
Circumference: The distance around a circle.

The circumference can also be called the perimeter of a circle. However, we use the term circumference for circles because they are round. The term perimeter is reserved for figures with straight sides. In order to find the formula for the circumference of a circle, we first need to determine the ratio between the circumference and diameter of a circle.

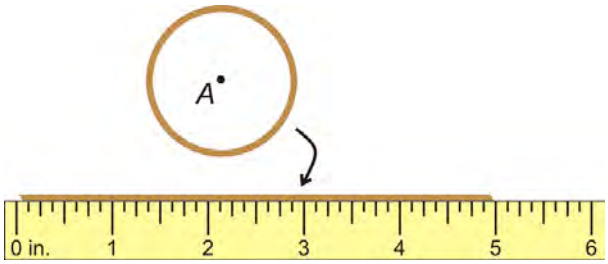
Investigation 10-1: Finding π (pi)

Tools Needed: paper, pencil, compass, ruler, string, and scissors

1. Draw three circles with radii of 2 in, 3 in, and 4 in. Label the centers of each A, B, and C.
2. Draw in the diameters and determine their lengths. Are all the diameter lengths the same in $\odot A$? $\odot B$? $\odot C$?



3. Take the string and outline each circle with it. The string represents the circumference of the circle. Cut the string so that it perfectly outlines the circle. Then, lay it out straight and measure, in inches. Round your answer to the nearest $\frac{1}{8}$-inch. Repeat this for the other two circles.



4. Find $\frac{circumference}{diameter}$ for each circle. Record your answers to the nearest thousandth. What do you notice?

From this investigation, you should see that $\frac{circumference}{diameter}$ approaches 3.14159... The bigger the diameter, the closer the ratio was to this number. We call this number π, the Greek letter "pi." It is an irrational number because the decimal never repeats itself. Pi has been calculated out to the millionth place and there is still no pattern in the sequence of numbers. When finding the circumference and area of circles, we must use π.

π, or **"pi":** The ratio of the circumference of a circle to its diameter. It is approximately equal to 3.14159265358979323846...

To see more digits of π, go to http://www.eveandersson.com/pi/digits/ .

You are probably familiar with the formula for circumference. From Investigation 10-1, we found that $\frac{circumference}{diameter} = \pi$. Let's shorten this up and solve for the circumference.

$\frac{C}{d} = \pi$, multiplying both sides by d, we have $C = \pi d$. We can also say $C = 2\pi r$ because $d = 2r$.

Circumference Formula: If d is the diameter or r is the radius of a circle, then $C = \pi d$ or $C = 2\pi r$.

Example 1: Find the circumference of a circle with a radius of 7 cm.

Solution: Plug the radius into the formula.

$$C = 2\pi(7) = 14\pi \approx 44 \ cm$$

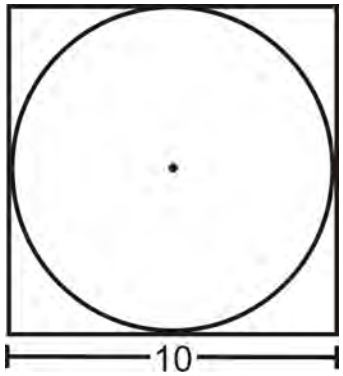
Depending on the directions in a given problem, you can either leave the answer in terms of π or multiply it out and get an approximation. Make sure you read the directions.

Example 2: The circumference of a circle is 64π. Find the diameter.

Solution: Again, you can plug in what you know into the circumference formula and solve for d.

$$64\pi = \pi d = 14\pi$$

Example 3: A circle is inscribed in a square with 10 in. sides. What is the circumference of the circle? Leave your answer in terms of π.



Solution: From the picture, we can see that the diameter of the circle is equal to the length of a side. Use the circumference formula.

$$C = 10\pi \ in.$$

Example 4: Find the perimeter of the square. Is it more or less than the circumference of the circle? Why?

Solution: The perimeter is $P = 4(10) = 40 \ in.$ In order to compare the perimeter with the circumference we should change the circumference into a decimal.

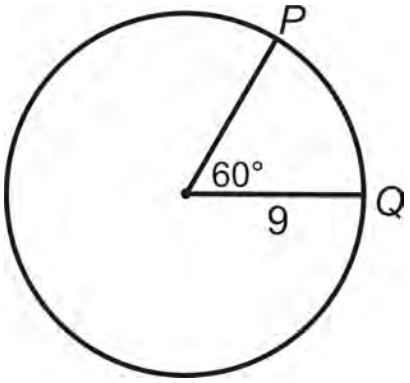
$C = 10\pi \approx 31.42 \ in.$ This is less than the perimeter of the square, which makes sense because the circle is smaller than the square.

Arc Length

In Chapter 9, we measured arcs in degrees. This was called the "arc measure" or "degree measure." Arcs can also be measured in length, as a portion of the circumference.

Arc Length: The length of an arc or a portion of a circle's circumference.

The arc length is directly related to the degree arc measure. Let's look at an example.



Example 5: Find the length of $\widehat{PQ}$. Leave your answer in terms of π.

Solution: In the picture, the central angle that corresponds with $\widehat{PQ}$ is 60°. This means that $m\widehat{PQ} = 60^\circ$ as well. So, think of the arc length as a portion of the circumference. There are 360° in a circle, so 60° would be $\frac{1}{6}$ of that $\left(\frac{60^\circ}{360^\circ} = \frac{1}{6}\right)$. Therefore, the length of $\widehat{PQ}$ is $\frac{1}{6}$ of the circumference.

$$length\ of\ \widehat{PQ} = \frac{1}{6} \cdot 2\pi(9) = 3\pi$$

Arc Length Formula: If d is the diameter or r is the radius, the length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$.

Example 6: The arc length of $\widehat{AB} = 6\pi$ and is $\frac{1}{4}$ the circumference. Find the radius of the circle.

Solution: If 6π is $\frac{1}{4}$ the circumference, then the total circumference is $4(6\pi) = 24\pi$. To find the radius, plug this into the circumference formula and solve for r.

$$24\pi = 2\pi r$$
$$12 = r$$

Know What? Revisited The entire length of the crust, or the circumference of the pizza is $14\pi \approx 44$ *in*. In the picture to the right, the top piece of pizza is if it is cut into 8 pieces. Therefore, for $\frac{1}{8}$ of the pizza, one piece would have $\frac{44}{8} \approx 5.5$ *inches* of crust. The bottom piece of pizza is if the pizza is cut into 10 pieces. For $\frac{1}{10}$ of the crust, one piece would have $\frac{44}{10} \approx 4.4$ *inches* of crust.

Review Questions

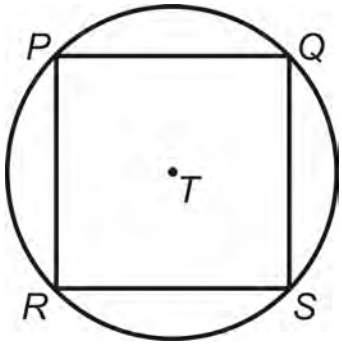
Fill in the following table. Leave all answers in terms of π.

TABLE 7.1:

	diameter	*radius*	*circumference*
1.	15		
2.		4	
3.	6		
4.			84π
5.		9	
6.			25π
7.			2π
8.	36		

9. Find the radius of circle with circumference 88 in.
10. Find the circumference of a circle with $d = \frac{20}{\pi} cm$.

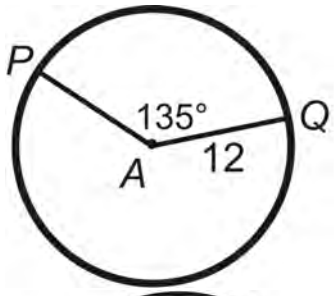
Square $PQSR$ is inscribed in $\odot T$. $RS = 8\sqrt{2}$.



11. Find the length of the diameter of $\odot T$.
12. How does the diameter relate to $PQSR$?
13. Find the perimeter of $PQSR$.
14. Find the circumference of $\odot T$.

Find the arc length of $\overset{\frown}{PQ}$ in $\odot A$. Leave your answers in terms of π.

15.



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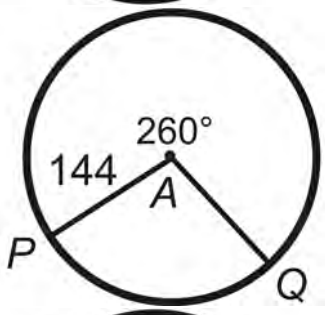

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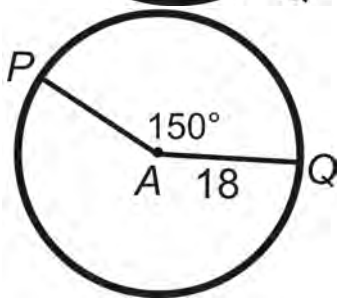
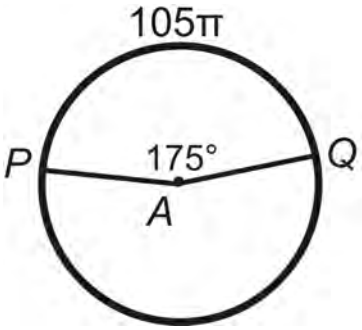
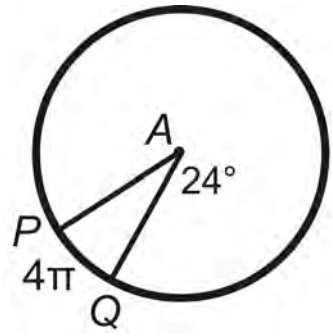


Find PA (the radius) in $\odot A$. Leave your answer in terms of π.

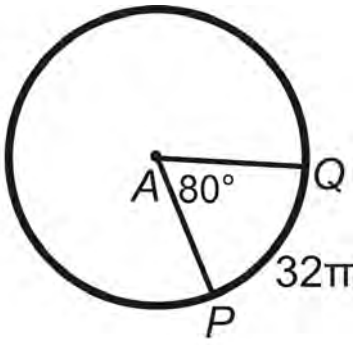
18.



19.

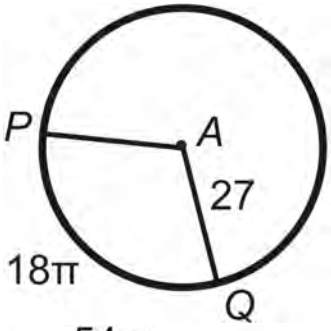


20.



Find the central angle or $m\widehat{PQ}$ in $\odot A$. Round any decimal answers to the nearest tenth.

21.



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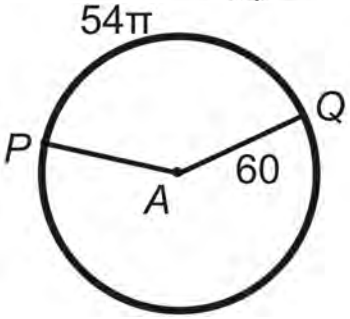
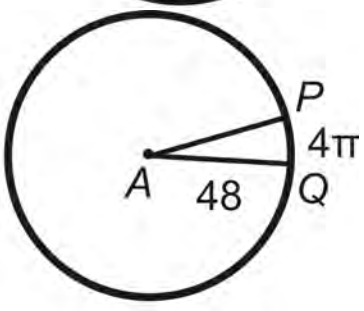
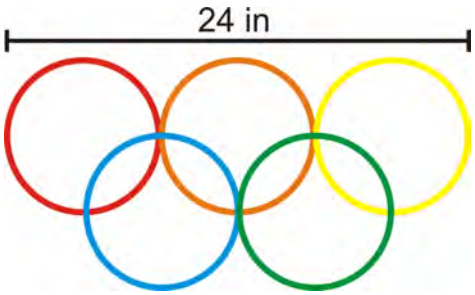


23.



24. The Olympics symbol is five congruent circles arranged as shown below. Assume the top three circles are tangent to each other. Brad is tracing the entire symbol for a poster. How far will his pen point travel?



25. A truck has tires with a 26 in diameter.
    a. How far does the truck travel every time a tire turns exactly once?
    b. How many times will the tire turn after the truck travels 1 mile? (1 mile = 5280 feet)
26. Mario's Pizza Palace offers a stuffed crust pizza in three sizes (diameter length) for the indicated prices: The Little Cheese, 8 in, $7.00 The Big Cheese, 10 in, $9.00 The Cheese Monster, 12 in, $12.00 What is the crust (in) to price ($) ratio for each of these pizzas? Michael thinks the cheesy crust is the best part of the pizza and wants to get the most crust for his money. Which pizza should he buy?
27. Jay is decorating a cake for a friend's birthday. They want to put gumdrops around the edge of the cake which has a 12 in diameter. Each gumdrop is has a diameter of 1.25 cm. To the nearest gumdrop, how many will they need?
28. A speedometer in a car measures the distance travelled by counting the rotations of the tires. The number of rotations required to travel one tenth of a mile in a particular vehicle is approximately 9.34. To the nearest inch, find the diameter of the wheel. (1 mile = 5280 feet)
29. Bob wants to put new weather stripping around a semicircular window above his door. The base of the window (diameter) is 36 inches. How much weather stripping does he need?
30. Each car on a Ferris wheel travels 942.5 ft during the 10 rotations of each ride. How high is each car at the highest point of each rotation?

Review Queue Answers

1. $\angle CAE$
2. $\angle CBE$
3. $360^\circ, 180^\circ$
4. $m\widehat{CD} = 180^\circ - 26^\circ = 154^\circ, m\angle CBE = 13^\circ$

7.5 Areas of Circles and Sectors

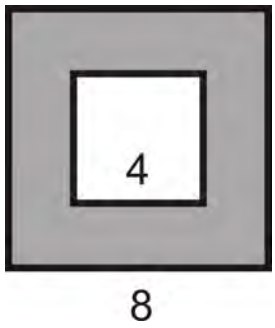
Learning Objectives

- Find the area of circles, sectors, and segments.

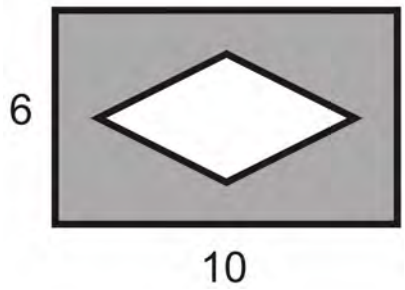
Review Queue

Find the area of the shaded region in the following figures.

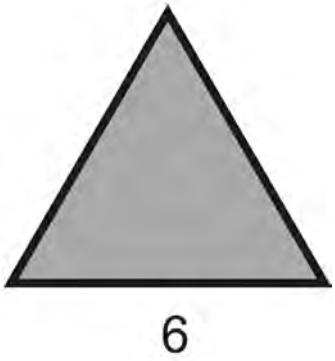
1. Both figures are squares.



2. Each vertex of the rhombus is 1.5 in from midpoints of the sides of the rectangle.



3. The figure is an equilateral triangle. (find the altitude)



4. Find the area of an equilateral triangle with side s.

Know What? Back to the pizza. In the previous section, we found the length of the crust for a 14 in pizza. However, crust typically takes up some area on a pizza. Leave your answers in terms of π and reduced improper fractions.

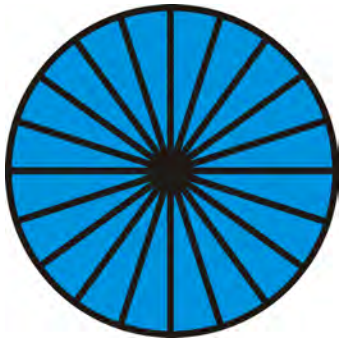
a) Find the area of the crust of a deep-dish 16 in pizza. A typical deep-dish pizza has 1 in of crust around the toppings.

b) A thin crust pizza has $\frac{1}{2}$ - in of crust around the edge of the pizza. Find the area of a thin crust 16 in pizza.

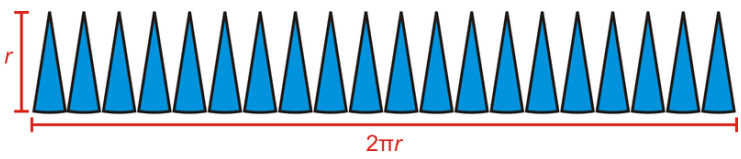
c) Which piece of pizza has more crust? A twelfth of the deep dish pizza or a fourth of the thin crust pizza?

Area of a Circle

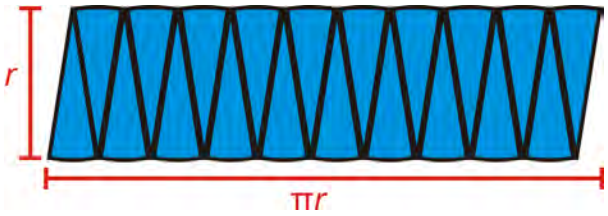
Recall in the previous section we derived π as the ratio between the circumference of a circle and its diameter. We are going to use the formula for circumference to derive the formula for area.



First, take a circle and divide it up into several wedges, or sectors. Then, unfold the wedges so they are all on one line, with the points at the top.



Notice that the height of the wedges is r, the radius, and the length is the circumference of the circle. Now, we need to take half of these wedges and flip them upside-down and place them in the other half so they all fit together.



Now our circle looks like a parallelogram. The area of this parallelogram is $A = bh = \pi r \cdot r = \pi r^2$.

To see an animation of this derivation, see http://www.rkm.com.au/ANIMATIONS/animation-Circle-Area-Derivation.html , by Russell Knightley.

Area of a Circle: If r is the radius of a circle, then $A = \pi r^2$.

Example 1: Find the area of a circle with a diameter of 12 cm.

Solution: If the diameter is 12 cm, then the radius is 6 cm. The area is $A = \pi(6^2) = 36\pi \ cm^2$.

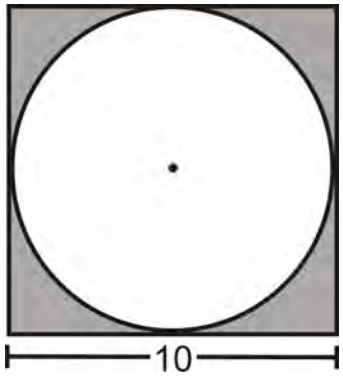
Example 2: If the area of a circle is 20π, what is the radius?

Solution: Work backwards on this problem. Plug in the area and solve for the radius.

$$\begin{aligned} 20\pi &= \pi r^2 \\ 20 &= r^2 \\ r &= \sqrt{20} = 2\sqrt{5} \end{aligned}$$

Just like the circumference, we will leave our answers in terms of π, unless otherwise specified. In Example 2, the radius could be $\pm 2\sqrt{5}$, however the radius is always positive, so we do not need the negative answer.

Example 3: A circle is inscribed in a square. Each side of the square is 10 cm long. What is the area of the circle?



Solution: The diameter of the circle is the same as the length of a side of the square. Therefore, the radius is half the length of the side, or 5 cm.

$$A = \pi 5^2 = 25\pi \ cm$$

Example 4: Find the area of the shaded region.

Solution: The area of the shaded region would be the area of the square minus the area of the circle.

$$A = 10^2 - 25\pi = 100 - 25\pi \approx 21.46 \ cm^2$$

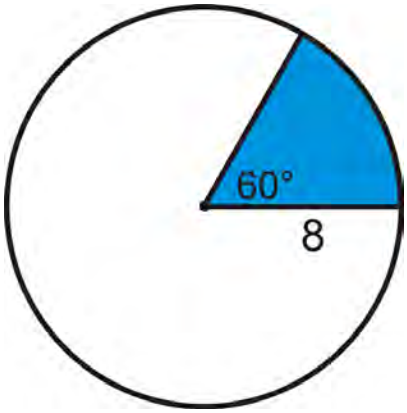
Area of a Sector

Sector of a Circle: The area bounded by two radii and the arc between the endpoints of the radii.

The area of a sector is a fractional part of the area of the circle, just like arc length is a fractional portion of the circumference.

Area of a Sector: If r is the radius and $\widehat{AB}$ is the arc bounding a sector, then $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$.

Example 5: Find the area of the blue sector. Leave your answer in terms of π.



Solution: In the picture, the central angle that corresponds with the sector is 60°. 60° would be $\frac{1}{6}$ of 360°, so this sector is $\frac{1}{6}$ of the total area.

$$area\ of\ blue\ sector = \frac{1}{6} \cdot \pi 8^2 = \frac{32}{3}\pi$$

Another way to write the sector formula is $A = \frac{central\ angle}{360^\circ} \cdot \pi r^2$.

Example 6: The area of a sector is 8π and the radius of the circle is 12. What is the central angle?

Solution: Plug in what you know to the sector area formula and then solve for the central angle, we will call it x.

$$\begin{aligned} 8\pi &= \frac{x}{360^\circ} \cdot \pi 12^2 \\ 8\pi &= \frac{x}{360^\circ} \cdot 144\pi \\ 8 &= \frac{2x}{5^\circ} \\ x &= 8 \cdot \frac{5^\circ}{2} = 20^\circ \end{aligned}$$

Example 7: The area of a sector of circle is 50π and its arc length is 5π. Find the radius of the circle.

Solution: First plug in what you know to both the sector formula and the arc length formula. In both equations we will call the central angle, "CA."

$$50\pi = \frac{CA}{360}\pi r^2 \qquad 5\pi = \frac{CA}{360}2\pi r$$
$$50 \cdot 360 = CA \cdot r^2 \qquad 5 \cdot 180 = CA \cdot r$$
$$18000 = CA \cdot r^2 \qquad 900 = CA \cdot r$$

Now, we can use substitution to solve for either the central angle or the radius. Because the problem is asking for the radius we should solve the second equation for the central angle and substitute that into the first equation for the central angle. Then, we can solve for the radius. Solving the second equation for CA, we have: $CA = \frac{900}{r}$. Plug this into the first equation.

$$18000 = \frac{900}{r} \cdot r^2$$
$$18000 = 900r$$
$$r = 20$$

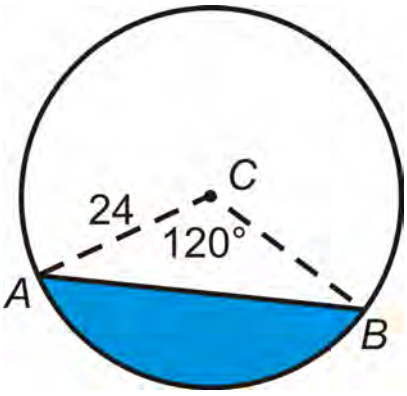
We could have also solved for the central angle in Example 7 once r was found. The central angle is $\frac{900}{20} = 45°$.

Segments of a Circle

The last part of a circle that we can find the area of is called a segment, not to be confused with a line segment.

Segment of a Circle: The area of a circle that is bounded by a chord and the arc with the same endpoints as the chord.

Example 8: Find the area of the blue segment below.



Solution: As you can see from the picture, the area of the segment is the area of the sector minus the area of the isosceles triangle made by the radii. If we split the isosceles triangle in half, we see that each half is a 30-60-90 triangle, where the radius is the hypotenuse. Therefore, the height of $\triangle ABC$ is 12 and the base would be $2\left(12\sqrt{3}\right) = 24\sqrt{3}$.

$$A_{sector} = \frac{120}{360}\pi \cdot 24^2 \qquad A_{\triangle} = \frac{1}{2}\left(24\sqrt{3}\right)(12)$$
$$= 192\pi \qquad = 144\sqrt{3}$$

The area of the segment is $A = 192\pi - 144\sqrt{3} \approx 353.8$.

In the review questions, make sure you know how the answer is wanted. If the directions say "leave in terms of π and simplest radical form," your answer would be the first one above. If it says "give an approximation," your answer would be the second. It is helpful to leave your answer in simplest radical form and in terms of π because that is the most accurate answer. However, it is also nice to see what the approximation of the answer is, to see how many square units something is.

Know What? Revisited The area of the crust for a deep-dish pizza is $8^2\pi - 7^2\pi = 15\pi$. The area of the crust of the thin crust pizza is $8^2\pi - 7.5^2\pi = \frac{31}{4}\pi$. One-twelfth of the deep dish pizza has $\frac{15}{12}\pi$ or $\frac{5}{4}\pi$ in^2 of crust. One-fourth of the thin crust pizza has $\frac{31}{16}\pi$ in^2. To compare the two measurements, it might be easier to put them both into decimals. $\frac{5}{4}\pi \approx 3.93$ in^2 and $\frac{31}{16}\pi \approx 6.09$ in^2. From this, we see that one-fourth of the thin-crust pizza has more crust than one-twelfth of the deep dish pizza.

Review Questions

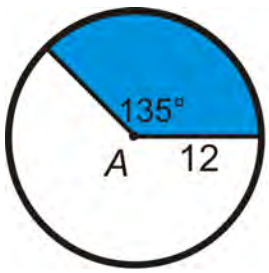
Fill in the following table. Leave all answers in terms of π.

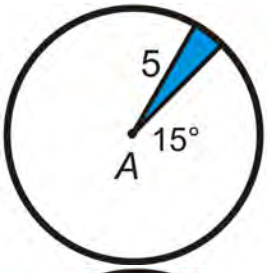
TABLE 7.2:

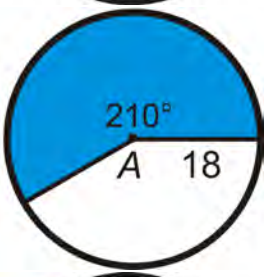
	radius	*Area*	*circumference*
1.	2		
2.		16π	
3.			10π
4.			24π
5.	9		
6.		90π	
7.			35π
8.	$\frac{7}{\pi}$		
9.			60
10.		36	

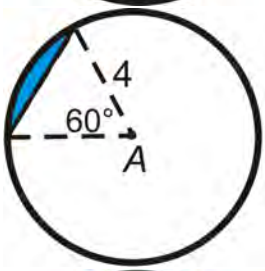
Find the area of the blue sector or segment in $\odot A$. Leave your answers in terms of π. You may use decimals or fractions in your answers, but do not round.

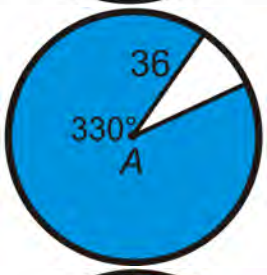
11.

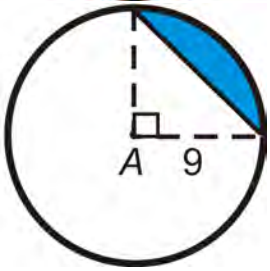


12. 

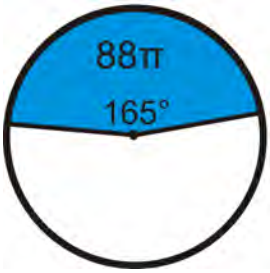
13. 

14. 

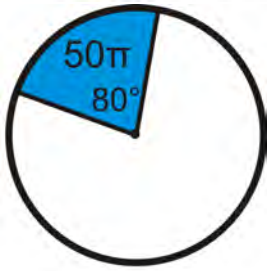
15. 

16. 

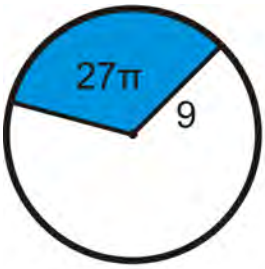
Find the radius of the circle. Leave your answer in simplest radical form.

17. 

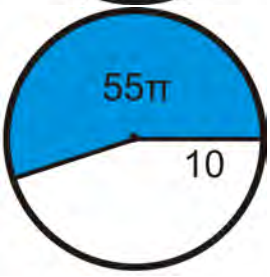
18.

19. 

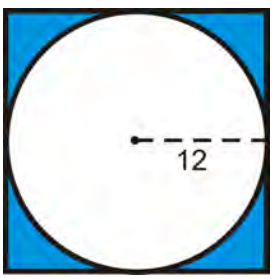
Find the central angle of each blue sector. Round any decimal answers to the nearest tenth.

20. 

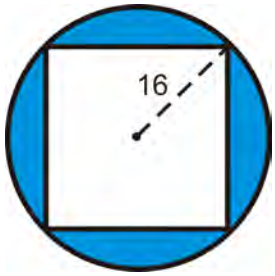
21.

22. 

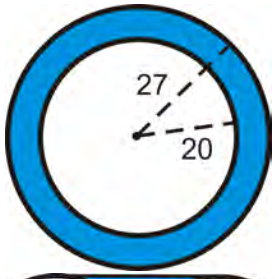
Find the area of the shaded region. Round your answer to the nearest hundredth.

23. 

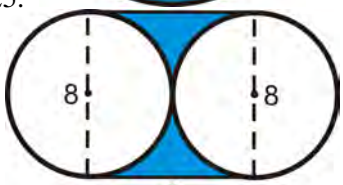
24. The quadrilateral is a square.



25.

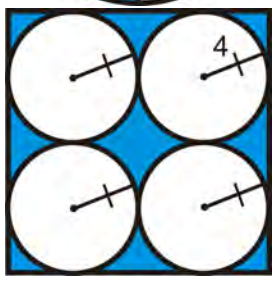


26.



27.

28.



29. Carlos has 400 ft of fencing to completely enclose an area on his farm for an animal pen. He could make the area a square or a circle. If he uses the entire 400 ft of fencing, how much area is contained in the square and the circle? Which shape will yield the greatest area?
30. The area of a sector of a circle is 54π and its arc length is 6π. Find the radius of the circle.
31. The area of a sector of a circle is 2304π and its arc length is 32π. Find the central angle of the sector.

Review Queue Answers

1. $8^2 - 4^2 = 64 - 16 = 48$
2. $6(10) - \frac{1}{2}(7)(3) = 60 - 10.5 = 49.5$
3. $\frac{1}{2}(6)\left(3\sqrt{3}\right) = 9\sqrt{3}$

4. $\frac{1}{2}(s)\left(\frac{1}{2}s\sqrt{3}\right) = \frac{1}{4}s^2\sqrt{3}$

7.6 Area and Perimeter of Regular Polygons

Learning Objectives

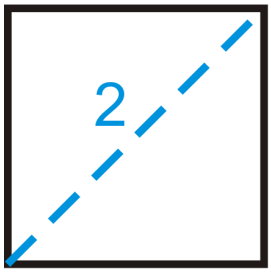
- Calculate the area and perimeter of a regular polygon.

Review Queue

1. What is a regular polygon?

Find the area of the following regular polygons. For the hexagon and octagon, divide the figures into rectangles and/or triangles.

2.



3.

4. Find the length of the sides in Problems 2 and 3.

Know What? The Pentagon in Arlington, VA houses the Department of Defense, is two regular pentagons with the same center. The entire area of the building is 29 acres (40,000 square feet in an acre), with an additional 5 acre courtyard in the center. The length of each outer wall is 921feet. What is the total distance across the pentagon? Round your answer to the nearest hundredth.

Perimeter of a Regular Polygon

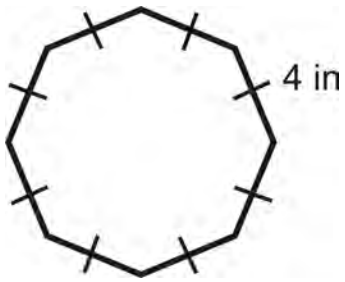
Recall that a regular polygon is a polygon with congruent sides and angles. In this section, we are only going to deal with regular polygons because they are the only polygons that have a consistent formula for area and perimeter. First, we will discuss the perimeter.

Recall that the perimeter of a square is 4 times the length of a side because each side is congruent. We can extend this concept to any regular polygon.

Perimeter of a Regular Polygon: If the length of a side is s and there are n sides in a regular polygon, then the perimeter is $P = ns$.

Example 1: What is the perimeter of a regular octagon with 4 inch sides?

Solution: If each side is 4 inches and there are 8 sides, that means the perimeter is 8(4 in) = 32 inches.

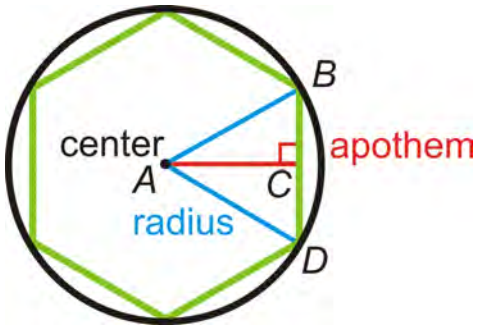


Example 2: The perimeter of a regular heptagon is 35 cm. What is the length of each side?

Solution: If $P = ns$, then $35\ cm = 7s$. Therefore, $s = 5\ cm$.

Area of a Regular Polygon

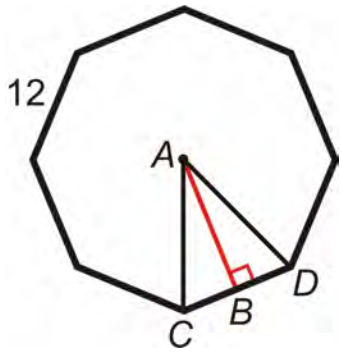
In order to find the area of a regular polygon, we need to define some new terminology. First, all regular polygons can be inscribed in a circle. So, ***regular polygons have a center and radius***, which are the center and radius of the circumscribed circle. Also like a circle, a regular polygon will have a central angle formed. In a regular polygon, however, ***the central angle is the angle formed by two radii drawn to consecutive vertices of the polygon.*** In the picture below, the central angle is $\angle BAD$. Also, notice that $\triangle BAD$ is an isosceles triangle. ***Every regular polygon with n sides is formed by n isosceles triangles.*** In a regular hexagon, the triangles are equilateral. The height of these isosceles triangles is called the ***apothem***.



Apothem: A line segment drawn from the center of a regular polygon to the midpoint of one of its sides.

We could have also said that the apothem is perpendicular to the side it is drawn to. By the Isosceles Triangle Theorem, the apothem is the perpendicular bisector of the side of the regular polygon. The apothem is also the height, or altitude of the isosceles triangles.

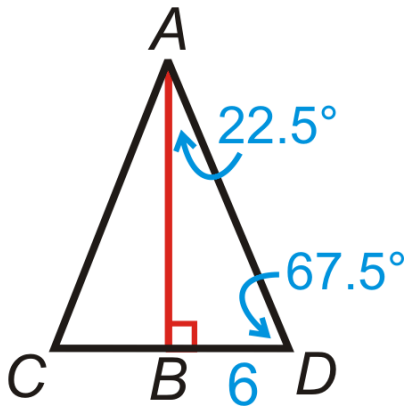
Example 3: Find the length of the apothem in the regular octagon. Round your answer to the nearest hundredth.



Solution: To find the length of the apothem, AB, you will need to use the trig ratios. First, find $m\angle CAD$. There are 360° around a point, so $m\angle CAD = \frac{360^\circ}{8} = 45^\circ$. Now, we can use this to find the other two angles in $\triangle CAD$. $m\angle ACB$ and $m\angle ADC$ are equal because $\triangle CAD$ is a right triangle.

$$\begin{aligned} m\angle CAD + m\angle ACB + m\angle ADC &= 180^\circ \\ 45^\circ + 2m\angle ACB &= 180^\circ \\ 2m\angle ACB &= 135^\circ \\ m\angle ACB &= 67.5^\circ \end{aligned}$$

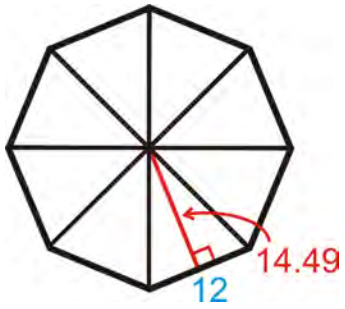
To find AB, we must use the tangent ratio. You can use either acute angle.



$$\tan 67.5^\circ = \frac{AB}{6}$$
$$AB = 6 \cdot \tan 67.5^\circ \approx 14.49$$

The apothem is used to find the area of a regular polygon. Let's continue with Example 3.

Example 4: Find the area of the regular octagon in Example 3.



Solution: The octagon can be split into 8 congruent triangles. So, if we find the area of one triangle and multiply it by 8, we will have the area of the entire octagon.

$$A_{octagon} = 8\left(\frac{1}{2} \cdot 12 \cdot 14.49\right) = 695.52\ units^2$$

From Examples 3 and 4, we can derive a formula for the area of a regular polygon.

The area of each triangle is: $A_\triangle = \frac{1}{2}bh = \frac{1}{2}sa$, where s is the length of a side and a is the apothem.

If there are n sides in the regular polygon, then it is made up of n congruent triangles.

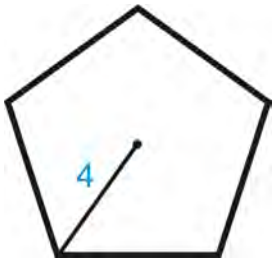
$$A = nA_\triangle = n\left(\frac{1}{2}sa\right) = \frac{1}{2}nsa$$

In this formula we can also substitute the perimeter formula, $P = ns$, for n and s.

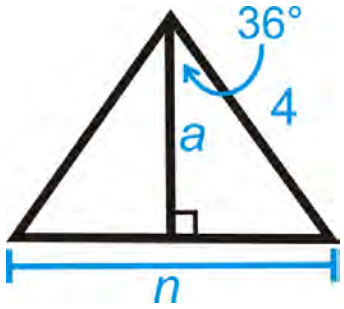
$$A = \frac{1}{2}nsa = \frac{1}{2}Pa$$

Area of a Regular Polygon: If there are n sides with length s in a regular polygon and a is the apothem, then $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$, where P is the perimeter.

Example 5: Find the area of the regular polygon with radius 4.



Solution: In this problem we need to find the apothem and the length of the side before we can find the area of the entire polygon. Each central angle for a regular pentagon is $\frac{360^\circ}{5} = 72^\circ$. So, half of that, to make a right triangle with the apothem, is 36°. We need to use sine and cosine.



$$\sin 36^\circ = \frac{.5n}{4} \qquad \cos 36^\circ = \frac{a}{4}$$
$$4\sin 36^\circ = \frac{1}{2}n \qquad 4\cos 36^\circ = a$$
$$8\sin 36^\circ = n \qquad a \approx 3.24$$
$$n \approx 4.7$$

Using these two pieces of information, we can now find the area. $A = \frac{1}{2}(3.24)(5)(4.7) \approx 38.07\ units^2$.

Example 6: The area of a regular hexagon is $54\sqrt{3}$ and the perimeter is 36. Find the length of the sides and the apothem.

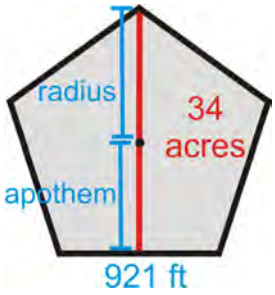
Solution: Plug in what you know into both the area and the perimeter formulas to solve for the length of a side and the apothem.

$$P = sn \qquad A = \frac{1}{2}aP$$
$$36 = 6s \qquad 54\sqrt{3} = \frac{1}{2}a(36)$$
$$s = 6 \qquad 54\sqrt{3} = 18a$$
$$3\sqrt{3} = a$$

Know What? Revisited From the picture to the right, we can see that the total distance across the Pentagon is the length of the apothem plus the length of the radius. If the total area of the Pentagon is 34 acres, that is 2,720,000 square feet. Therefore, the area equation is $2720000 = \frac{1}{2}a(921)(5)$ and the apothem is 590.66 ft. To find the radius, we can either use the Pythagorean Theorem, with the apothem and half the length of a side or the sine ratio. Recall from Example 5, that each central angle in a pentagon is 72°, so we would use half of that for the right triangle.

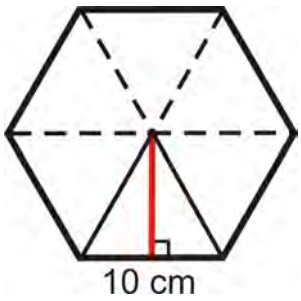
$$\sin 36^\circ = \frac{460.5}{r} \rightarrow r = \frac{460.5}{\sin 36^\circ} \approx 783.45\ ft.$$

Therefore, the total distance across is $590.66 + 783.45 = 1374.11\ ft$.



Review Questions

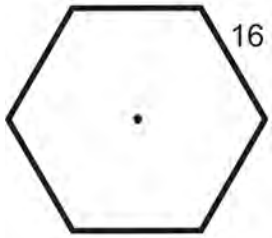
Use the regular hexagon below to answer the following questions. Each side is 10 cm long.



1. Each dashed line segment is $a(n)$ _____.
2. The red line segment is $a(n)$ _____.
3. There are _____ congruent triangles in a regular hexagon.
4. In a regular hexagon, all the triangles are _____.
5. Find the radius of this hexagon.
6. Find the apothem.
7. Find the perimeter.
8. Find the area.

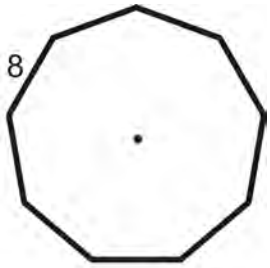
Find the area and perimeter of each of the following regular polygons. Round your answer to the nearest hundredth.

9.

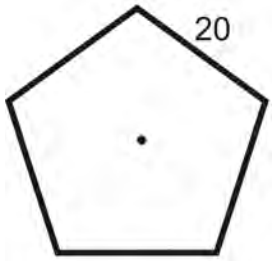


10.

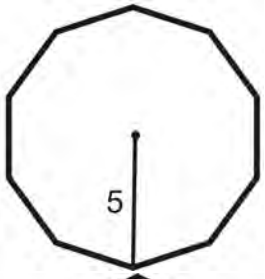
11.



12.



13.

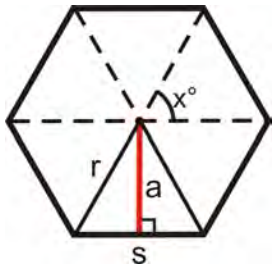


14.

15. If the perimeter of a regular decagon is 65, what is the length of each side?
16. A regular polygon has a perimeter of 132 and the sides are 11 units long. How many sides does the polygon have?
17. The area of a regular pentagon is 440.44 in^2 and the perimeter is 80 in. Find the length of the apothem of the pentagon.
18. The area of a regular octagon is 695.3 cm^2 and the sides are 12 cm. What is the length of the apothem?

A regular 20-gon and a regular 40-gon are inscribed in a circle with a radius of 15 units.

19. Find the perimeter of both figures.
20. Find the circumference of the circle.
21. Which of the perimeters is closest to the circumference of the circle? Why do you think that is?
22. Find the area of both figures.
23. Find the area of the circle.
24. Which of the areas is closest to the area of the circle? Why do you think that is?
25. ***Challenge*** Derive a formula for the area of a regular hexagon with sides of length s. Your only variable will be s. HINT: Use 30-60-90 triangle ratios.
26. ***Challenge*** in the following steps you will derive an alternate formula for finding the area of a regular polygon with n sides.



We are going to start by thinking of a polygon with n sides as n congruent isosceles triangles. We will find the sum of the areas of these triangles using trigonometry. First, the area of a triangle is $\frac{1}{2}bh$. In the diagram to the right, this area formula would be $\frac{1}{2}sa$, where s is the length of a side and a is the length of the apothem. In the diagram, x represents the measure of the vertex angle of each isosceles triangle. a. The apothem, a, divides the triangle into two congruent right triangles. The top angle in each is $\frac{x^\circ}{2}$. Find $\sin\left(\frac{x^\circ}{2}\right)$ and $\cos\left(\frac{x^\circ}{2}\right)$. b. Solve your sin equation to find an expression for s in terms of r and x. c. Solve your cos equation to find an expression for a in terms of r and x. d. Substitute these expressions into the equation for the area of one of the triangles, $\frac{1}{2}sa$. e. Since there will be n triangles in an n-gon, you need to multiply your expression from part d by n to get the total area. f. How would you tell someone to find the value of x for a regular n-gon?

Use the formula you derived in problem 26 to find the area of the regular polygons described in problems 27-30. Round your answers to the nearest hundredth.

27. Decagon with radius 12 cm.
28. 20-gon with radius 5 in.
29. 15-gon with radius length 8 cm.
30. 45-gon with radius length 7 in.
31. What is the area of a regular polygon with 100 sides and radius of 9 in? What is the area of a circle with radius 9 in? How do these areas compare? Can you explain why?
32. How could you use the formula from problem 26 to find the area of a regular polygon given the number of sides and the length of a side? How can you find the radius?

Use your formula from problem 26 and the method you described to find r given the number of sides and the length of a side in problem 31 to find the area of the regular polygons below.

33. 30-gon with side length 15 cm.
34. Dodecagon with side length 20 in.

Review Queue Answers

1. A regular polygon is a polygon with congruent sides and angles.
2. $A = \left(\sqrt{2}\right)^2 = 2$
3. $A = 6\left(\frac{1}{2} \cdot 1 \cdot \frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$
4. The sides of the square are $\sqrt{2}$ and the sides of the hexagon are 1 unit.

7.7 Perimeter and Area Review

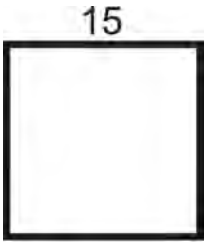
Keywords, Theorems and Formulas

- Perimeter
- Area of a Rectangle: $A = bh$
- Perimeter of a Rectangle $P = 2b + 2h$
- Perimeter of a Square: $P = 4s$
- Area of a Square: $A = s^2$
- Congruent Areas Postulate
- Area Addition Postulate
- Area of a Parallelogram: $A = bh$.
- Area of a Triangle: $A = \frac{1}{2}bh$ or $A = \frac{bh}{2}$
- Area of a Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$
- Area of a Rhombus: $A = \frac{1}{2}d_1d_2$
- Area of a Kite: $A = \frac{1}{2}d_1d_2$
- Area of Similar Polygons Theorem
- π
- Circumference: $C = \pi d$ or $C = 2\pi r$
- Arc Length
- Arc Length Formula: length of $\widehat{AB} = \frac{m\widehat{AB}}{360^\circ} \cdot \pi d$ or $\frac{m\widehat{AB}}{360^\circ} \cdot 2\pi r$
- Area of a Circle: $A = \pi r^2$
- Sector of a Circle
- Area of a Sector: $A = \frac{m\widehat{AB}}{360^\circ} \cdot \pi r^2$
- Segment of a Circle
- Perimeter of a Regular Polygon: $P = ns$
- Apothem
- Area of a Regular Polygon: $A = \frac{1}{2}asn$ or $A = \frac{1}{2}aP$

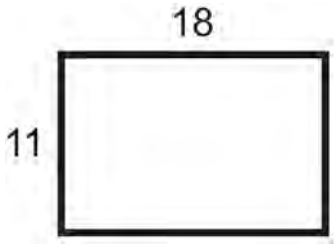
Review Questions

Find the area and perimeter of the following figures. Round your answers to the nearest hundredth.

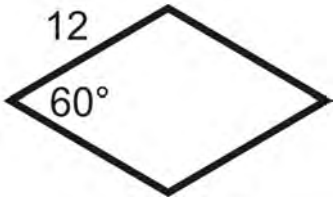
1. square



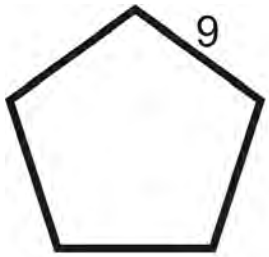
2. rectangle



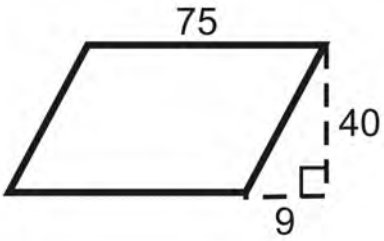
3. rhombus



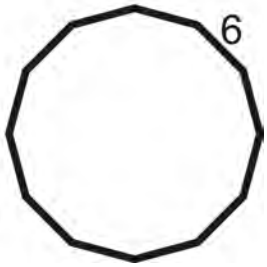
4. regular pentagon



5. parallelogram

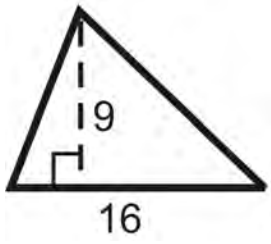


6. regular dodecagon

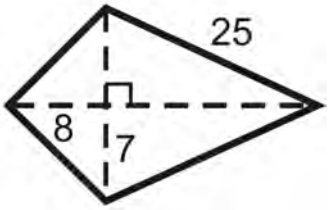


Find the area of the following figures. Leave your answers in simplest radical form.

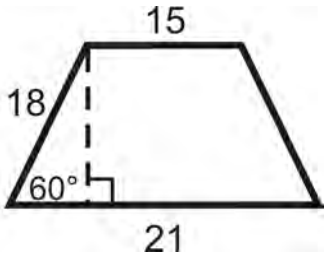
7. triangle



8. kite

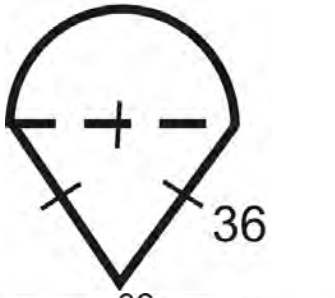


9. isosceles trapezoid



10. Find the area and circumference of a circle with radius 17.
11. Find the area and circumference of a circle with diameter 30.
12. Two similar rectangles have a scale factor $\frac{4}{3}$. If the area of the larger rectangle is 96 $units^2$, find the area of the smaller rectangle.

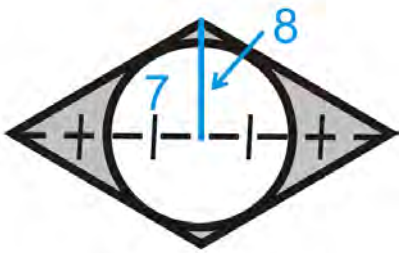
Find the area of the following figures. Round your answers to the nearest hundredth.



13.

14.

15. Find the shaded area (figure is a rhombus)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7

7.8 Exploring Solids

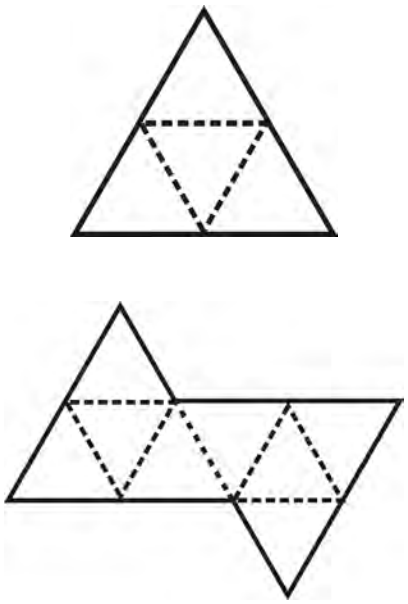
Learning Objectives

- Identify different types of solids and their parts.
- Use Euler's formula to solve problems.
- Draw and identify different views of solids.
- Draw and identify nets.

Review Queue

1. Draw an octagon and identify the edges and vertices of the octagon. How many of each are there?
2. Find the area of a square with 5 cm sides.
3. Find the area of an equilateral triangle with 10 in sides.
4. Draw the following polygons.
    a. A convex pentagon.
    b. A concave nonagon.

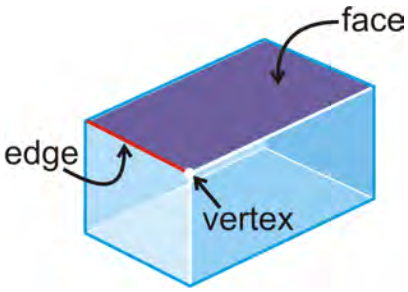
Know What? Until now, we have only talked about two-dimensional, or flat, shapes. In this chapter we are going to expand to 3D. Copy the equilateral triangle to the right onto a piece of paper and cut it out. Fold on the dotted lines. What shape do these four equilateral triangles make? If we place two of these equilateral triangles next to each other (like in the far right) what shape do these 8 equilateral triangles make?



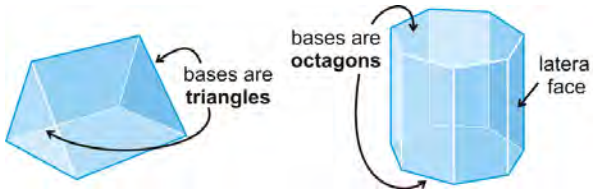
Polyhedrons

Polyhedron: A 3-dimensional figure that is formed by polygons that enclose a region in space.

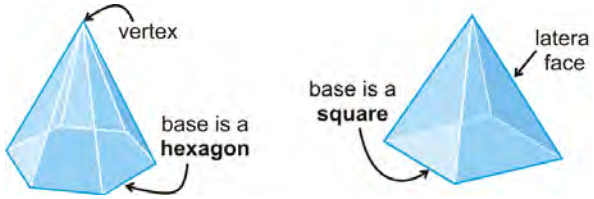
Each polygon in a polyhedron is called a ***face***. The line segment where two faces intersect is called an ***edge*** and the point of intersection of two edges is a ***vertex***. There are no gaps between the edges or vertices in a polyhedron. Examples of polyhedrons include a cube, prism, or pyramid. Non-polyhedrons are cones, spheres, and cylinders because they have sides that are not polygons.



Prism: A polyhedron with two congruent bases, in parallel planes, and the lateral sides are rectangles.



Pyramid: A polyhedron with one base and all the lateral sides meet at a common vertex. The lateral sides are triangles.



All prisms and pyramids are named by their bases. So, the first prism would be a triangular prism and the second would be an octagonal prism. The first pyramid would be a hexagonal pyramid and the second would be a square pyramid. The lateral faces of a pyramid are always triangles.

Example 1: Determine if the following solids are polyhedrons. If the solid is a polyhedron, name it and determine the number of faces, edges and vertices each has.

a)

b)

c)

Solution:

a) The base is a triangle and all the sides are triangles, so this is a polyhedron, a triangular pyramid. There are 4 faces, 6 edges and 4 vertices.

b) This solid is also a polyhedron because all the faces are polygons. The bases are both pentagons, so it is a pentagonal prism. There are 7 faces, 15 edges, and 10 vertices.

c) This is a cylinder and has bases that are circles. Circles are not polygons, so it is not a polyhedron.

Euler's Theorem

Let's put our results from Example 1 into a table.

TABLE 7.3:

	Faces	*Vertices*	*Edges*
Triangular Pyramid	4	4	6
Pentagonal Prism	7	10	15

Notice that the sum of the faces + vertices is two more that the number of edges. This is called Euler's Theorem, after the Swiss mathematician Leonhard Euler.

Euler's Theorem: The number of faces (F), vertices (V), and edges (E) of a polyhedron can be related such that $F+V=E+2$.

Example 2: Find the number of faces, vertices, and edges in the octagonal prism.

Solution: Because this is a polyhedron, we can use Euler's Theorem to find either the number of faces, vertices or edges. It is easiest to count the faces, there are 10 faces. If we count the vertices, there are 16. Using this, we can solve for E in Euler's Theorem.

$$\begin{aligned} F+V &= E+2 \\ 10+16 &= E+2 \\ 24 &= E \qquad \text{There are 24 edges.} \end{aligned}$$

Example 3: In a six-faced polyhedron, there are 10 edges. How many vertices does the polyhedron have?

Solution: Solve for V in Euler's Theorem.

$$\begin{aligned} F+V &= E+2 \\ 6+V &= 10+2 \\ V &= 6 \qquad \text{There are 6 vertices.} \end{aligned}$$

Example 4: A three-dimensional figure has 10 vertices, 5 faces, and 12 edges. Is it a polyhedron?

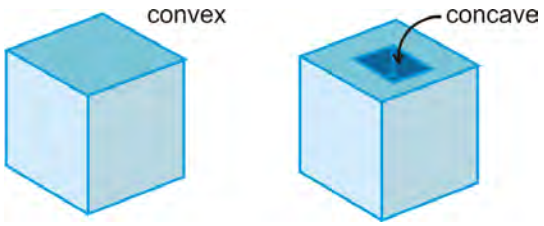
Solution: Plug in all three numbers into Euler's Theorem.

$$\begin{aligned} F+V &= E+2 \\ 5+10 &= 12+2 \\ 15 &\neq 14 \end{aligned}$$

Because the two sides are not equal, this figure is not a polyhedron.

Regular Polyhedra

Regular Polyhedron: A polyhedron where all the faces are congruent regular polygons.



Polyhedrons, just like polygons, can be ***convex*** or ***concave*** (also called non-convex). All regular polyhedron are convex. A concave polyhedron is similar to a concave polygon. The polyhedron "caves in," so that two non-adjacent vertices can be connected by a line segment that is outside the polyhedron.

There are five regular polyhedra called the Platonic solids, after the Greek philosopher Plato. These five solids are significant because they are the only five regular polyhedra. There are only five because the sum of the measures of the angles that meet at each vertex must be less than 360°. Therefore the only combinations are 3, 4 or 5 triangles at each vertex, 3 squares at each vertex or 3 pentagons. Each of these polyhedra have a name based on the number of sides, except the cube.

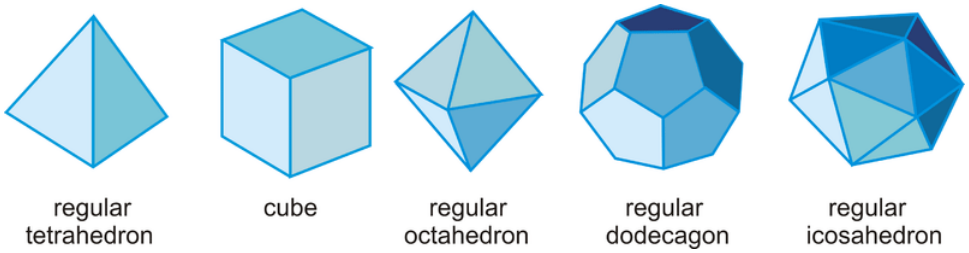
Regular Tetrahedron: A 4-faced polyhedron where all the faces are equilateral triangles.

Cube: A 6-faced polyhedron where all the faces are squares.

Regular Octahedron: An 8-faced polyhedron where all the faces are equilateral triangles.

Regular Dodecahedron: A 12-faced polyhedron where all the faces are regular pentagons.

Regular Icosahedron: A 20-faced polyhedron where all the faces are equilateral triangles.



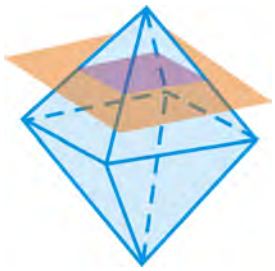
Cross-Sections

One way to "view" a three-dimensional figure in a two-dimensional plane, like this text, is to use cross-sections.

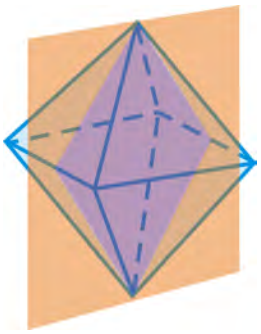
Cross-Section: The intersection of a plane with a solid.

Example 5: Describe the shape formed by the intersection of the plane and the regular octahedron.

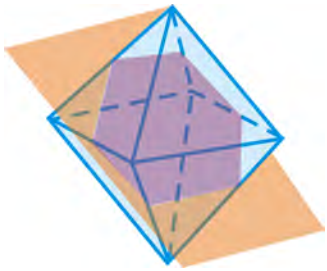
a)



b)



c)



Solution:

a) Square

b) Rhombus

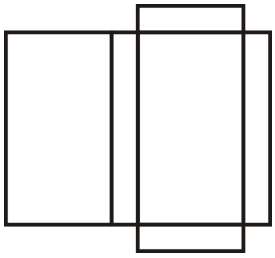
c) Hexagon

Nets

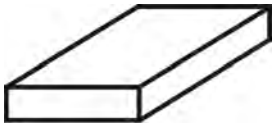
Another way to represent a three-dimensional figure in a two dimensional plane is to use a net.

Net: An unfolded, flat representation of the sides of a three-dimensional shape.

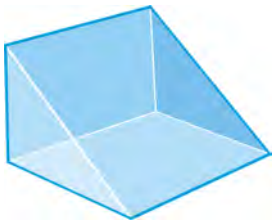
Example 6: What kind of figure does this net create?



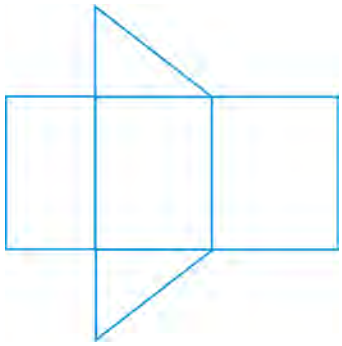
Solution: The net creates a rectangular prism.



Example 7: Draw a net of the right triangular prism below.



Solution: This net will have two triangles and three rectangles. The rectangles are all different sizes and the two triangles are congruent.



Notice that there could be a couple different interpretations of this, or any, net. For example, this net could have the triangles anywhere along the top or bottom of the three rectangles. Most prisms have multiple nets.

See the site http://www.cs.mcgill.ca/~sqrt/unfold/unfolding.html if you would like to see a few animations of other nets, including the Platonic solids.

Know What? Revisited The net of the first shape is a regular tetrahedron and the second is the net of a regular octahedron.

Review Questions

Complete the table using Euler's Theorem.

TABLE 7.4:

	Name	*Faces*	*Edges*	*Vertices*
1.	Rectangular Prism	6	12	
2.	Octagonal Pyramid		16	9
3.	Regular Icosahedron	20		12
4.	Cube		12	8
5.	Triangular Pyramid	4		4
6.	Octahedron	8	12	
7.	Heptagonal Prism		21	14
8.	Triangular Prism	5	9	

Determine if the following figures are polyhedra. If so, name the figure and find the number of faces, edges, and vertices.

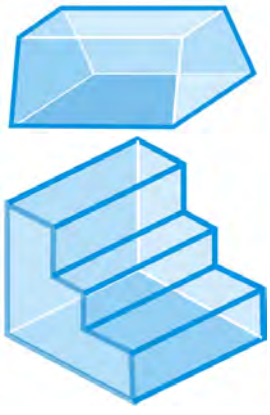
9.

10.

11.

12.

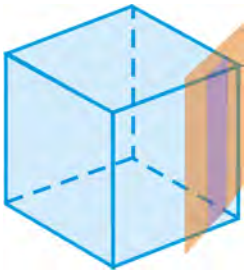
13.



14.

Describe the cross section formed by the intersection of the plane and the solid.

15.



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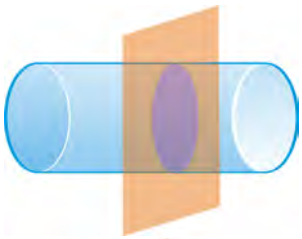
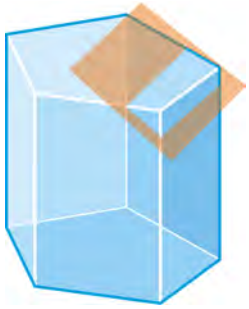
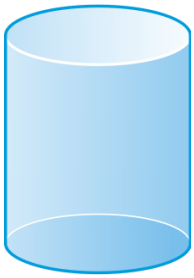


17.



Draw the net for the following solids.

18.

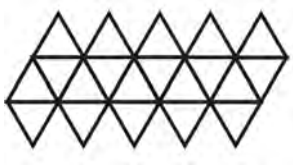


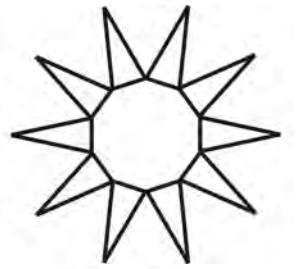
19.

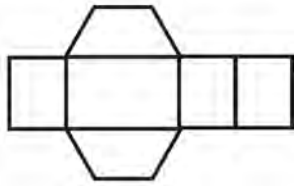
20.

Determine what shape is formed by the following nets.

21.



22. 

23. 

24. A cube has 11 unique nets. Draw 5 different nets of a cube.
25. A truncated icosahedron is a polyhedron with 12 regular pentagonal faces and 20 regular hexagonal faces and 90 edges. This icosahedron closely resembles a soccer ball. How many vertices does it have? Explain your reasoning.

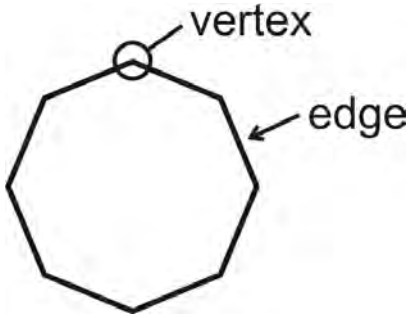
26. Use construction tools to construct a large equilateral triangle. Construct the three midsegments of the triangle. Cut out the equilateral triangle and fold along the midsegments. What net have you constructed?
27. Describe a method to construct a net for a regular octahedron.

For problems 28-30, we are going to connect the Platonic Solids to probability. A six sided die is the shape of a cube. The probability of any one side landing face up is $\frac{1}{6}$ because each of the six faces is congruent to each other.

28. What shape would we make a die with 12 faces? If we number these faces 1 to 12, and each face has the same likelihood of landing face up, what is the probability of rolling a multiple of three?
29. I have a die that is a regular octahedron. Each face is labeled with a consecutive prime number starting with 2. What is the largest prime number on my die?
30. ***Challenge*** Rebecca wants to design a new die. She wants it to have one red face. The other faces will be yellow, blue or green. How many faces should her die have and how many of each color does it need so that: the probability of rolling yellow is eight times the probability of rolling red, the probability of rolling green is half the probability of rolling yellow and the probability of rolling blue is seven times the probability of rolling red?

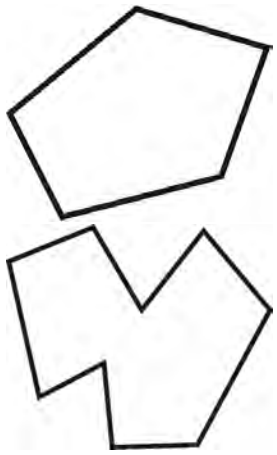
Review Queue Answers

1. There are 8 vertices and 8 edges in an octagon.



2. $5^2 = 25\ cm^2$
3. $\frac{1}{2} \cdot 10 \cdot 5\sqrt{3} = 25\sqrt{3}\ in^2$
4. Answers:

a. b.



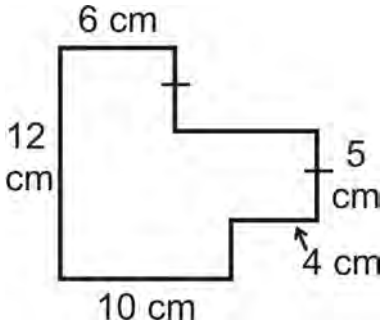
7.9 Surface Area of Prisms and Cylinders

Learning Objectives

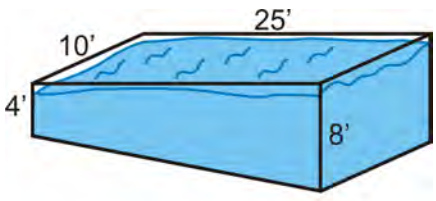
- Find the surface area of a prism.
- Find the surface area of a cylinder.

Review Queue

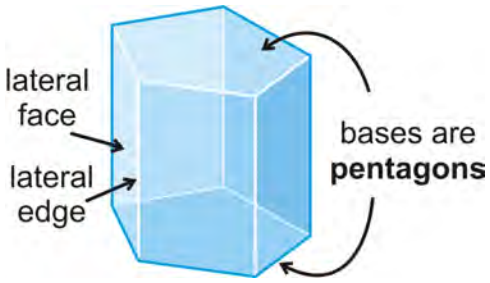
1. Find the area of a rectangle with sides:
   a. 6 and 9
   b. 11 and 4
   c. $5\sqrt{2}$ and $8\sqrt{6}$
2. If the area of a square is 36 $units^2$, what are the lengths of the sides?
3. If the area of a square is 45 $units^2$, what are the lengths of the sides?
4. Find the area of the shape. All sides are perpendicular. (Split the shape up into rectangles.)



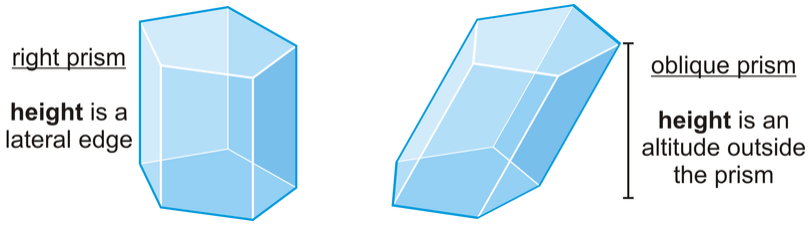
Know What? Your parents decide they want to put a pool in the backyard. They agree on a pool where the shallow end will be 4 ft. and the deep end will be 8 ft. The pool will be 10 ft. by 25 ft. How much siding do they need to buy to cover the sides and bottom of the pool? If the siding is $25.00 a square yard, how much will it cost to enclose the pool?



Parts of a Prism



In the last section, we defined a prism as a 3-dimensional figure with 2 congruent bases, in parallel planes with rectangular lateral faces. The edges between the ***lateral faces*** are called ***lateral edges***. All prisms are named by their bases, so the prism to the right is a pentagonal prism. This particular prism is called a ***right prism*** because the lateral faces are perpendicular to the bases. ***Oblique prisms*** lean to one side or the other and the height is outside the prism.



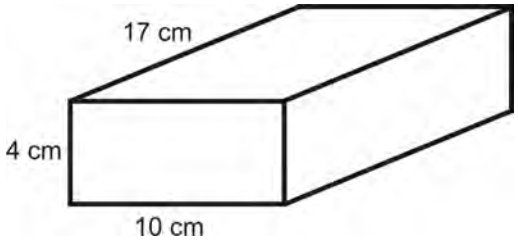
Surface Area of a Prism

Surface Area: The sum of the areas of the faces.

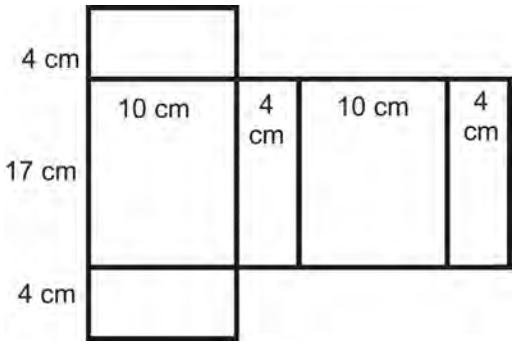
Lateral Area: The sum of the areas of the ***lateral*** faces.

You can use a net and the Area Addition Postulate to find the surface area of a right prism.

Example 1: Find the surface area of the prism below.



Solution: Open up the prism and draw the net. Determine the measurements for each rectangle in the net.



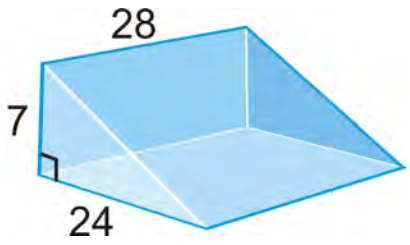
Using the net, we have:

$$\begin{aligned} SA_{prism} &= 2(4)(10)+2(10)(17)+2(17)(4) \\ &= 80+340+136 \\ &= 556\ cm^2 \end{aligned}$$

Because this is still area, the units are squared.

Surface Area of a Right Prism: The surface area of a right prism is the sum of the area of the bases and the area of each rectangular lateral face.

Example 2: Find the surface area of the prism below.



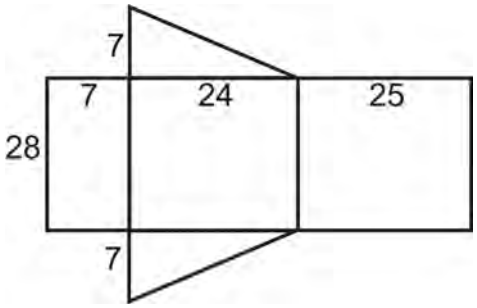
Solution: This is a right triangular prism. To find the surface area, we need to find the length of the hypotenuse of the base because it is the width of one of the lateral faces. Using the Pythagorean Theorem, the hypotenuse is

$$\begin{aligned} 7^2+24^2 &= c^2 \\ 49+576 &= c^2 \\ 625 &= c^2 \\ c &= 25 \end{aligned}$$

Looking at the net, the surface area is:

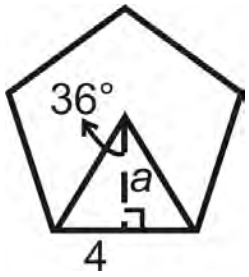
$$SA = 28(7)+28(24)+28(25)+2\left(\frac{1}{2}\cdot 7\cdot 24\right)$$

$$SA = 196+672+700+168 = 1736$$



Example 3: Find the surface area of the regular pentagonal prism.

Solution: For this prism, each lateral face has an area of 160 $units^2$. Then, we need to find the area of the regular pentagonal bases. Recall that the area of a regular polygon is $\frac{1}{2}asn$. $s = 8$ and $n = 5$, so we need to find a, the apothem.



$$\tan 36^\circ = \frac{4}{a}$$

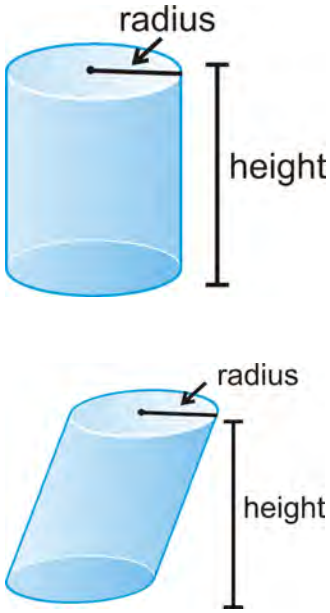
$$a = \frac{4}{\tan 36^\circ} \approx 5.51$$

$$SA = 5(160) + 2\left(\frac{1}{2} \cdot 5.51 \cdot 8 \cdot 5\right) = 1020.4$$

Cylinders

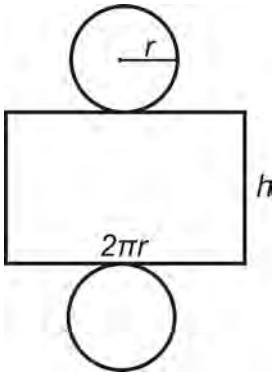
Cylinder: A solid with congruent circular bases that are in parallel planes. The space between the circles is enclosed.

Just like a circle, the cylinder has a radius for each of the circular bases. Also, like a prism, a cylinder can be oblique, like the one to the right.



Surface Area of a Right Cylinder

Let's find the net of a right cylinder. One way for you to do this is to take a label off of a soup can or can of vegetables. When you take this label off, we see that it is a rectangle where the height is the height of the cylinder and the base is the circumference of the base. This rectangle and the two circular bases make up the net of a cylinder.



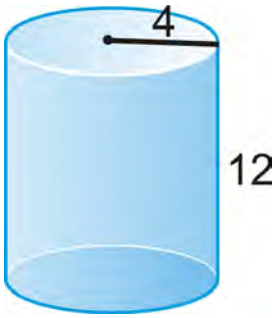
From the net, we can see that the surface area of a right cylinder is

$$\underbrace{2\pi r^2}_{\text{area of both circles}} + \underbrace{2\pi r h}_{\text{length of rectangle}}$$

Surface Area of a Right Cylinder: If r is the radius of the base and h is the height of the cylinder, then the surface area is $SA = 2\pi r^2 + 2\pi rh$.

To see an animation of the surface area, click http://www.rkm.com.au/ANIMATIONS/animation-Cylinder-Surface-Area-Derivation.html , by Russell Knightley.

Example 4: Find the surface area of the cylinder.



Solution: $r = 4$ and $h = 12$. Plug these into the formula.

$$\begin{aligned} SA &= 2\pi(4)^2 + 2\pi(4)(12) \\ &= 32\pi + 96\pi \\ &= 128\pi \end{aligned}$$

Example 5: The circumference of the base of a cylinder is 16π and the height is 21. Find the surface area of the cylinder.

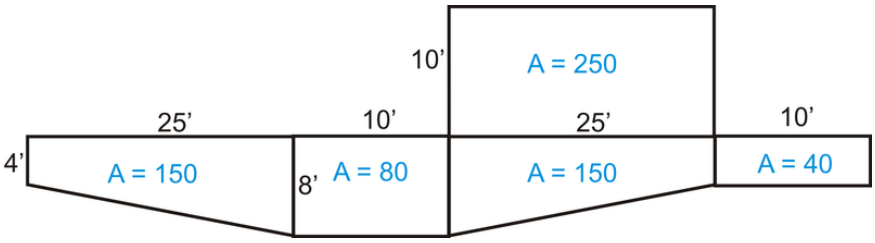
Solution: If the circumference of the base is 16π, then we can solve for the radius.

$$\begin{aligned} 2\pi r &= 16\pi \\ r &= 8 \end{aligned}$$

Now, we can find the surface area.

$$\begin{aligned} SA &= 2\pi(8)^2 + (16\pi)(21) \\ &= 128\pi + 336\pi \\ &= 464\pi \end{aligned}$$

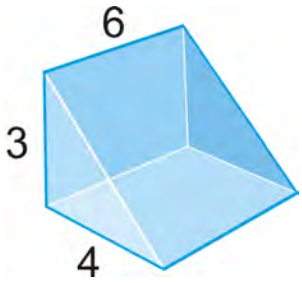
Know What? Revisited To the right is the net of the pool (minus the top). From this, we can see that your parents would need 670 square feet of siding. This means that the total cost would be $5583.33 for the siding.



Review Questions

1. How many square feet are in a square yard?
2. How many square centimeters are in a square meter?

Use the right triangular prism to answer questions 3-6.



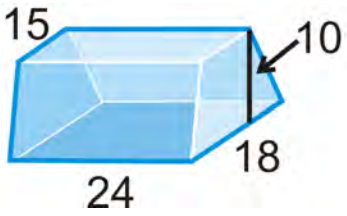
3. What shape are the bases of this prism? What are their areas?
4. What are the dimensions of each of the lateral faces? What are their areas?
5. Find the lateral surface area of the prism.
6. Find the total surface area of the prism.
7. ***Writing*** Describe the difference between lateral surface area and total surface area.
8. The lateral surface area of a cylinder is what shape? What is the area of this shape?
9. Fuzzy dice are cubes with 4 inch sides.

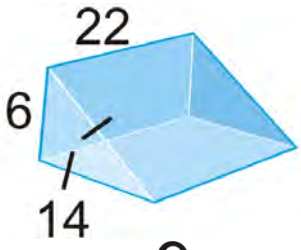
   a. What is the surface area of one die?
   b. Typically, the dice are sold in pairs. What is the surface area of two dice?

10. A right cylinder has a 7 cm radius and a height of 18 cm. Find the surface area.

Find the surface area of the following solids. Leave answers in terms of π.

11. bases are isosceles trapezoids





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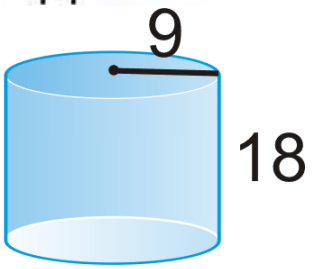

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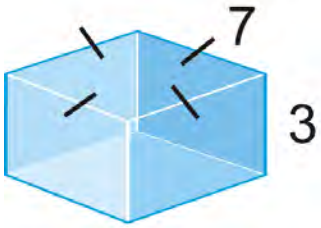

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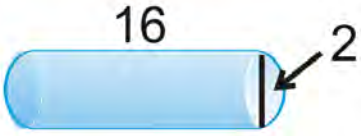

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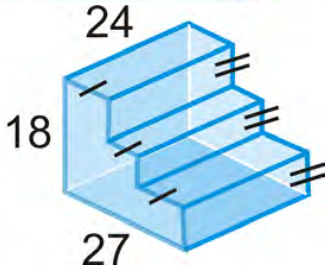
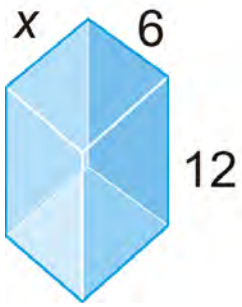


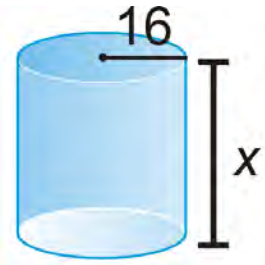
16.

Algebra Connection Find the value of x, given the surface area.

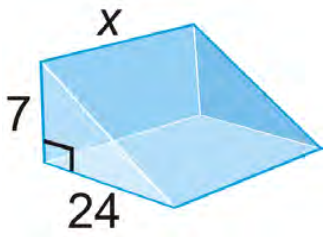
17. $SA = 432\ units^2$



18. $SA = 1536\pi \ units^2$

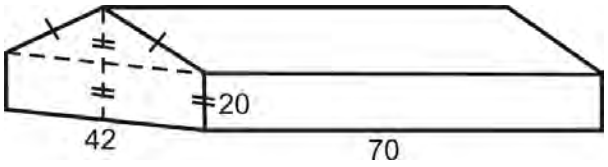


19. $SA = 1568 \ units^2$

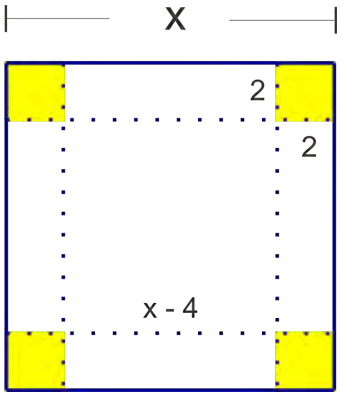


20. The area of the base of a cylinder is $25\pi \ in^2$ and the height is 6 in. Find the *lateral* surface area.
21. The circumference of the base of a cylinder is $80\pi \ cm$ and the height is 36 cm. Find the total surface area.
22. The lateral surface area of a cylinder is $30\pi \ m^2$. What is one possibility for height of the cylinder?

Use the diagram below for questions 23-27. The barn is shaped like a pentagonal prism with dimensions shown in feet.



23. What is the area of the roof? (Both sides)
24. What is the floor area of the barn?
25. What is the area of the sides of the barn?
26. The farmer wants to paint the sides of the roof (excluding the roof). If a gallon of paint covers 250 square feet, how many gallons will he need?
27. A gallon of paint costs $15.50. How much will it cost for him to paint the sides of the barn?
28. Charlie started a business canning artichokes. His cans are 5 in tall and have diameter 4 in. If the label must cover the entire lateral surface of the can and the ends must overlap by at least one inch, what are the dimensions and area of the label?
29. An open top box is made by cutting out 2 in by 2 in squares from the corners of a large square piece of cardboard. Using the picture as a guide, find an expression for the surface area of the box. If the surface area is 609 in^2, find the length of x. Remember, there is no top.



30. Find an expression for the surface area of a cylinder in which the ratio of the height to the diameter is 2:1. If x is the diameter, use your expression to find x if the surface area is 160π.

Review Queue Answers

1. a. 54
   b. 44
   c. $80\sqrt{3}$
2. $s = 6$
3. $s = 3\sqrt{5}$
4. $A = 60 + 30 + 20 = 110\ cm^2$

7.10 Surface Area of Pyramids and Cones

Learning Objectives

- Find the surface area of a pyramid.
- Find the surface area of a cone.

Review Queue

1. A rectangular prism has sides of 5 cm, 6 cm, and 7 cm. What is the surface area?
2. Triple the dimensions of the rectangular prism from #1. What is its surface area?
3. A cylinder has a diameter of 10 in and a height of 25 in. What is the surface area?
4. A cylinder has a circumference of $72\pi\ ft.$ and a height of 24 ft. What is the surface area?
5. Draw the net of a square pyramid.

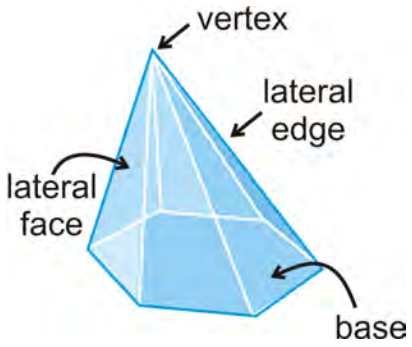
Know What? A typical waffle cone is 6 inches tall and has a diameter of 2 inches. This happens to be your friend Jeff's favorite part of his ice cream dessert. You decide to use your mathematical prowess to figure out exactly how much waffle cone Jeff is eating. What is the surface area of the waffle cone? (You may assume that the cone is straight across at the top)

Jeff decides he wants a "king size" cone, which is 8 inches tall and has a diameter of 4 inches. What is the surface area of this cone?

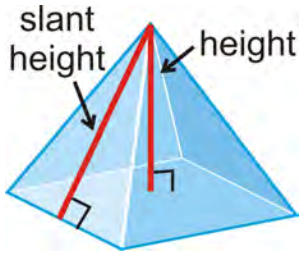
Parts of a Pyramid

A pyramid has one ***base*** and all the ***lateral faces*** meet at a common ***vertex***. The edges between the lateral faces are ***lateral edges***. The edges between the base and the lateral faces are called ***base edges***. If we were to draw the height

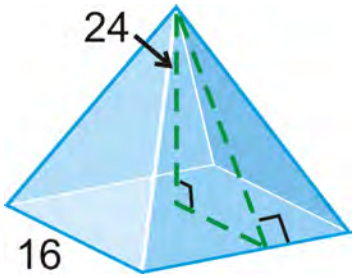
of the pyramid to the right, it would be off to the left side.



When a pyramid has a height that is directly in the center of the base, the pyramid is said to be regular. These pyramids have a regular polygon as the base. All regular pyramids also have a ***slant height*** that is the height of a lateral face. Because of the nature of regular pyramids, ***all slant heights are congruent***. A non-regular pyramid does not have a slant height.



Example 1: Find the slant height of the square pyramid.



Solution: Notice that the slant height is the hypotenuse of a right triangle formed by the height and half the base length. Use the Pythagorean Theorem.

$$\begin{aligned} 8^2 + 24^2 &= l^2 \\ 64 + 576 &= l^2 \\ 640 &= l^2 \\ l &= \sqrt{640} = 8\sqrt{10} \end{aligned}$$

Surface Area of a Regular Pyramid

Using the slant height, which is usually labeled l, the area of each triangular face is $A = \frac{1}{2}bl$.

Example 2: Find the surface area of the pyramid from Example 1.

Solution: The surface area of the four triangular faces are $4\left(\frac{1}{2}bl\right) = 2(16)\left(8\sqrt{10}\right) = 256\sqrt{10}$. To find the total surface area, we also need the area of the base, which is $16^2 = 256$. The total surface area is $256\sqrt{10} + 256 \approx 1065.54$.

From this example, we see that the formula for a square pyramid is:

$$SA = (area\ of\ the\ base) + 4(area\ of\ triangular\ faces)$$

$$SA = B + n\left(\frac{1}{2}bl\right) \qquad B \text{ is the area of the base and } n \text{ is the number of triangles.}$$

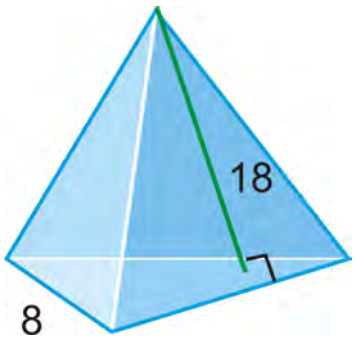
$$SA = B + \frac{1}{2}l(nb) \qquad \text{Rearranging the variables, } nb = P, \text{ the perimeter of the base.}$$

$$SA = B + \frac{1}{2}Pl$$

Surface Area of a Regular Pyramid: If B is the area of the base and P is the perimeter of the base and l is the slant height, then $SA = B + \frac{1}{2}Pl$.

If you ever forget this formula, use the net. Each triangular face is congruent, plus the area of the base. This way, you do not have to remember a formula, just a process, which is the same as finding the area of a prism.

Example 3: Find the area of the regular triangular pyramid.



Solution: The area of the base is $A = \frac{1}{4}s^2\sqrt{3}$ because it is an equilateral triangle.

$$B = \frac{1}{4}8^2\sqrt{3} = 16\sqrt{3}$$

$$SA = 16\sqrt{3} + \frac{1}{2}(24)(18) = 16\sqrt{3} + 216 \approx 243.71$$

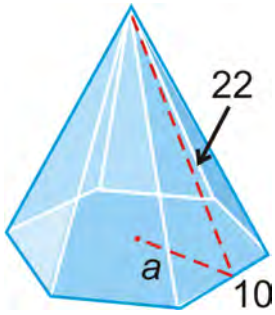
Example 4: If the lateral surface area of a square pyramid is 72 ft^2 and the base edge is equal to the slant height, what is the length of the base edge?

Solution: In the formula for surface area, the lateral surface area is $\frac{1}{2}Pl$ or $\frac{1}{2}nbl$. We know that $n = 4$ and $b = l$. Let's solve for b.

$$\frac{1}{2}nbl = 72\ ft^2$$
$$\frac{1}{2}(4)b^2 = 72$$
$$2b^2 = 72$$
$$b^2 = 36$$
$$b = 6$$

Therefore, the base edges are all 6 units and the slant height is also 6 units.

Example 4: Find the area of the regular hexagonal pyramid below.

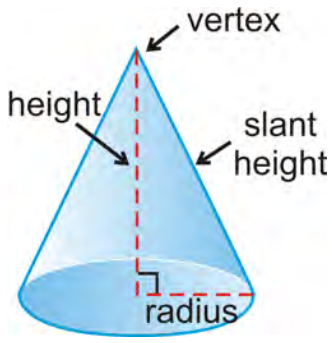


Solution: To find the area of the base, we need to find the apothem. If the base edges are 10 units, then the apothem is $5\sqrt{3}$ for a regular hexagon. The area of the base is $\frac{1}{2}asn = \frac{1}{2}\left(5\sqrt{3}\right)(10)(6) = 150\sqrt{3}$. The total surface area is:

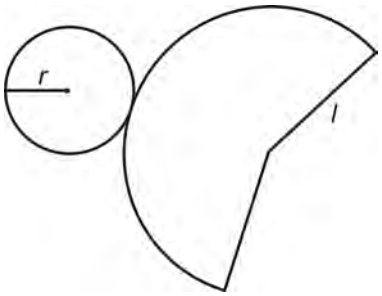
$$SA = 150\sqrt{3} + \frac{1}{2}(6)(10)(22)$$
$$= 150\sqrt{3} + 660 \approx 919.81\ units^2$$

Surface Area of a Cone

Cone: A solid with a circular base and sides taper up towards a common vertex.



It is said that a cone is generated from rotating a right triangle around one leg in a circle. Notice that a cone has a slant height, just like a pyramid. The surface area of a cone is a little trickier, however. We know that the base is a circle, but we need to find the formula for the curved side that tapers up from the base. Unfolding a cone, we have the net:



From this, we can see that the lateral face's edge is $2\pi r$ and the sector of a circle with radius l. We can find the area of the sector by setting up a proportion.

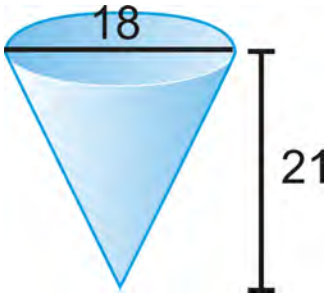
$$\frac{Area\ of\ circle}{Area\ of\ sector} = \frac{Circumference}{Arc\ length}$$
$$\frac{\pi l^2}{Area\ of\ sector} = \frac{2\pi l}{2\pi r} = \frac{l}{r}$$

Cross multiply:

$$l(Area\ of\ sector) = \pi r l^2$$
$$Area\ of\ sector = \pi r l$$

Surface Area of a Right Cone: The surface area of a right cone with slant height l and base radius r is $SA = \pi r^2 + \pi r l$.

Example 5: What is the surface area of the cone?



Solution: In order to find the surface area, we need to find the slant height. Recall from a pyramid, that the slant height forms a right triangle with the height and the radius. Use the Pythagorean Theorem.

$$l^2 = 9^2 + 21^2$$
$$= 81 + 441$$
$$l = \sqrt{522} \approx 22.85$$

The surface area would be $SA = \pi 9^2 + \pi(9)(22.85) \approx 900.54\ units^2$.

Example 6: The surface area of a cone is 36π and the slant height is 5 units. What is the radius?

Solution: Plug in what you know into the formula for the surface area of a cone and solve for r.

$36\pi = \pi r^2 + \pi r(5)$ Because every term has π, we can cancel it out.

$36 = r^2 + 5r$ Set one side equal to zero, and this becomes a factoring problem.

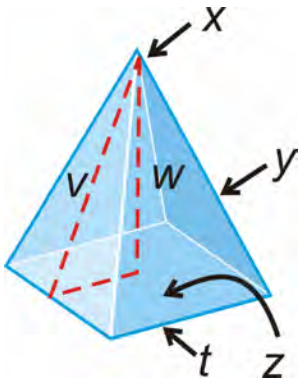
$r^2 + 5r - 36 = 0$

$(r-4)(r+9) = 0$ The possible answers for r are 4 and -9. The radius must be positive, so our answer is 4.

Know What? Revisited The standard cone has a surface area of $\pi + 6\pi = 7\pi \approx 21.99\ in^2$. The "king size" cone has a surface area of $4\pi + 16\pi = 20\pi \approx 62.83$, almost three times as large as the standard cone.

Review Questions

Fill in the blanks about the diagram to the left.



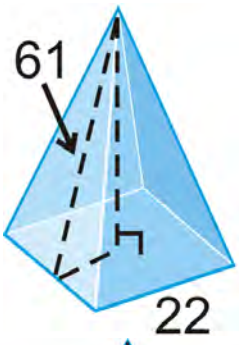
1. x is the _____.
2. The slant height is _____.
3. y is the _____.
4. The height is _____.
5. The base is _____.
6. The base edge is _____.
7. Sketch a right cone. Label the *height, slant height*, and *radius*.

For questions 8-10, sketch each of the following solids and answer the question. Your drawings should be to scale, but not one-to-one. Leave your answer in simplest radical form.

8. Draw a right cone with a radius of 5 cm and a height of 15 cm. What is the slant height?
9. Draw a square pyramid with an edge length of 9 in and a 12 in height. Find the slant height.
10. Draw an equilateral triangle pyramid with an edge length of 6 cm and a height of 6 cm. ***Describe*** how you would find the slant height and then find it.

Find the area of a lateral face of the regular pyramid. Leave your answer in simplest radical form.

11.



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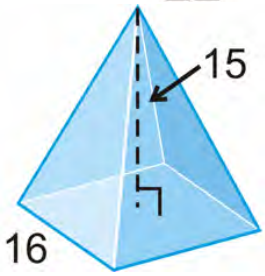

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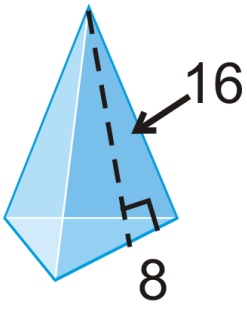
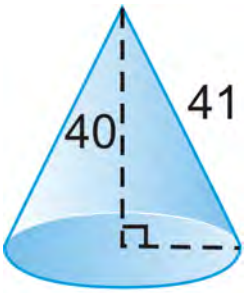


Find the surface area of the regular pyramids and right cones. Round your answers to 2 decimal places.

14.



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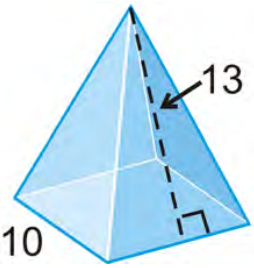
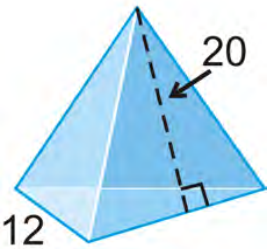
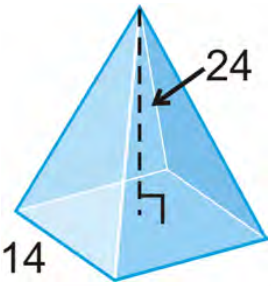
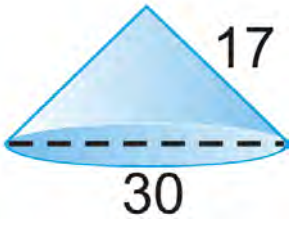


16. 

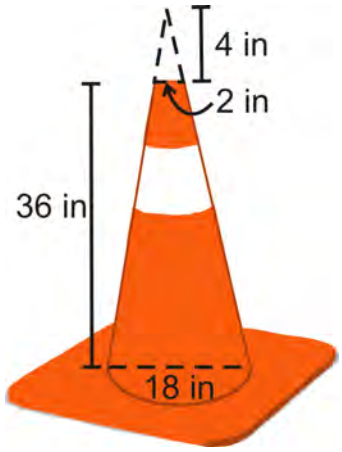
17.

18. 

19. 

20. From these pictures, we see that a regular triangle pyramid does not have to have four congruent faces. How many faces must be congruent?
21. A ***regular tetrahedron*** has four equilateral triangles as its faces. Find the surface area of a regular tetrahedron with edge length of 6 units.
22. Using the formula for the area of an equilateral triangle, what is the surface area of a regular tetrahedron, with edge length s?

Challenge Find the surface area of the traffic cone with the given information. The gone is cut off at the top (4 inch cone) and the base is a square with sides of length 24 inches. Round answers to the nearest hundredth.



23. Find the area of the entire square. Then, subtract the area of the base of the cone.
24. Find the lateral area of the cone portion (include the 4 inch cut off top of the cone).
25. Now, subtract the cut-off top of the cone, to only have the lateral area of the cone portion of the traffic cone.
26. Combine your answers from #23 and #25 to find the entire surface area of the traffic cone.

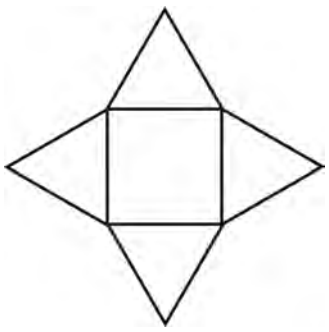
For questions 27-30, consider the sector of a circle with radius 25 cm and arc length 14π.

27. What is the central angle of this sector?
28. If this sector is rolled into a cone, what are the radius and area of the base of the cone?
29. What is the height of this cone?
30. What is the total surface are of the cone?

For questions 31-33, consider a square with diagonal length $10\sqrt{2}$ *in*.

31. What is the length of a side of the square?
32. If this square is the base of a right pyramid with height 12, what is the slant height of the pyramid?
33. What is the surface area of the pyramid?

Review Queue Answers

1. $2(5\cdot 6)+2(5\cdot 7)+2(6\cdot 7)=214\ cm^2$
2. $2(15\cdot 18)+2(15\cdot 21)+2(18\cdot 21)=1926\ cm^2$
3. $2\cdot 25\pi+250\pi=300\pi\ in^2$
4. $36^2(2\pi)+72\pi(24)=4320\pi\ ft^2$
5. 

7.11 Volume of Prisms and Cylinders

Learning Objectives

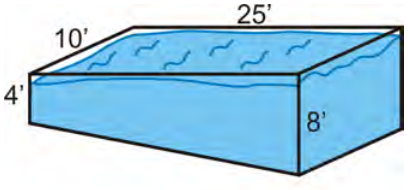
- Find the volume of a prism.
- Find the volume of a cylinder.

Review Queue

1. Define volume in your own words.
2. What is the surface area of a cube with 3 inch sides?
3. What is the surface area of a cube with $4\sqrt{2}$ inch sides?
4. A regular octahedron has 8 congruent equilateral triangles as the faces.

   a. If each edge is 4 cm, what is the surface area of the figure?
   b. If each edge is s, what is the surface area of the figure?

Know What? The pool is done and your family is ready to fill it with water. Recall that the shallow end is 4 ft. and the deep end is 8 ft. The pool is 10 ft. wide by 25 ft. long. How many gallons of water will it take to fill the pool? There are approximately 7.48 gallons in a cubic foot.



Volume of a Rectangular Prism

Volume: The measure of how much space a three-dimensional figure occupies.

Another way to define volume would be how much a three-dimensional figure can hold, water or sand, for example. The basic unit of volume is the cubic unit: cubic centimeter (cm^3), cubic inch (in^3), cubic meter (m^3), cubic foot (ft^3), etc. Each basic cubic unit has a measure of one for each: length, width, and height.

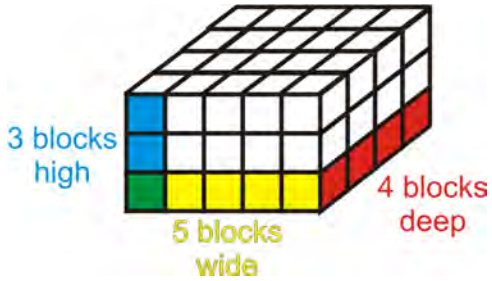
Volume of a Cube Postulate: The volume of a cube is the cube of the length of its side, or s^3.

What this postulate tells us is that every solid can be broken down into cubes, going along with our basic unit of measurement, the cubic unit. For example, if we wanted to find the volume of a cube with one inch sides, it would be $1^3 = 1\ in^3$. If we wanted to find the volume of a cube with 9 inch sides, it would be $9^3 = 729\ in^3$.

Volume Congruence Postulate: If two solids are congruent, then their volumes are congruent.

Volume Addition Postulate: The volume of a solid is the sum of the volumes of all of its non-overlapping parts.

Example 1: Find the volume of the right rectangular prism below.



Solution: A rectangular prism can be made from any square cubes. To find the volume, we would simply count the cubes. The bottom layer has 20 cubes, or 4 times 5, and there are 3 layers, or the same as the height. Therefore there are 60 cubes in this prism and the volume would be 60 $units^3$.

But, what if we didn't have cubes? Let's generalize this formula for any rectangular prism. Notice that each layer is the same as the area of the base. Then, we multiplied by the height. Here is our formula.

Volume of a Rectangular Prism: If a rectangular prism is h units high, w units wide, and l units long, then its volume is $V = l \cdot w \cdot h$.

Example 2: A typical shoe box is 8 in by 14 in by 6 in. What is the volume of the box?

Solution: We can assume that a shoe box is a rectangular prism. Therefore, we can use the formula above.

$$V = (8)(14)(6) = 672\ in^2$$

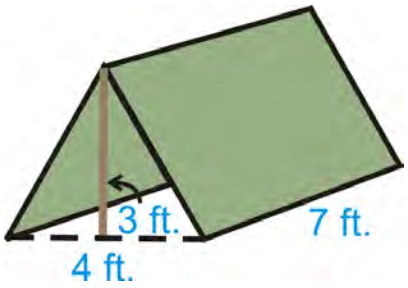
Volume of any Prism

If we further analyze the formula for the volume of a rectangular prism, we would see that $l \cdot w$ is equal to the area of the base of the prism, a rectangle. If the bases are not rectangles, this would still be true, however we would have to rewrite the equation a little.

Volume of a Prism: If the area of the base of a prism is B and the height is h, then the volume is $V = B \cdot h$.

Notice that "B" is not always going to be the same. So, to find the volume of a prism, you would first find the area of the base and then multiply it by the height.

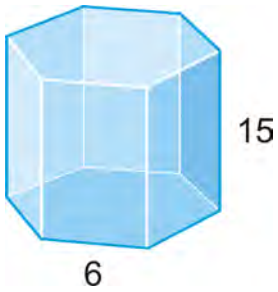
Example 3: You have a small, triangular prism shaped tent. How much volume does it have, once it is set up?



Solution: First, we need to find the area of the base. That is going to be $B = \frac{1}{2}(3)(4) = 6\ ft^2$. Multiplying this by 7 we would get the entire volume. The volume is 42 ft^3.

Even though the height in this problem does not look like a "height," it is, when referencing the formula. Usually, the height of a prism is going to be the last length you need to use.

Example 4: Find the volume of the regular hexagonal prism below.



Solution: Recall that a regular hexagon is divided up into six equilateral triangles. The height of one of those triangles would be the apothem. If each side is 6, then half of that is 3 and half of an equilateral triangle is a 30-60-90 triangle. Therefore, the apothem is going to be $3\sqrt{3}$. The area of the base is:

$$B = \frac{1}{2}\left(3\sqrt{3}\right)(6)(6) = 54\sqrt{3}\ units^2$$

And the volume will be:

$$V = Bh = \left(54\sqrt{3}\right)(15) = 810\sqrt{3}\ units^3$$

Cavalieri's Principle

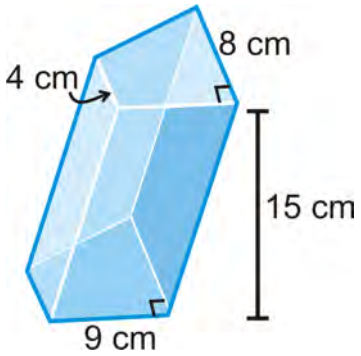
Recall that earlier in this section we talked about oblique prisms. These are prisms that lean to one side and the height is outside the prism. What would be the area of an oblique prism? To answer this question, we need to introduce Cavalieri's Principle. Consider to piles of books below.

Both piles have 15 books, therefore they will have the same volume. However, one pile is leaning. Cavalieri's Principle says that this does not matter, as long as the heights are the same and every horizontal cross section has the same area as the base, the volumes are the same.

Cavalieri's Principle: If two solids have the same height and the same cross-sectional area at every level, then they will have the same volume.

Basically, if an oblique prism and a right prism have the same base area and height, then they will have the same volume.

Example 5: Find the area of the oblique prism below.



Solution: This is an oblique right trapezoidal prism. First, find the area of the trapezoid.

$$B = \frac{1}{2}(9)(8+4) = 9(6) = 54\ cm^2$$

Then, multiply this by the height.

$$V = 54(15) = 810\ cm^3$$

Volume of a Cylinder

If we use the formula for the volume of a prism, $V = Bh$, we can find the volume of a cylinder. In the case of a cylinder, the base, or B, would be the area of a circle. Therefore, the volume of a cylinder would be $V = (\pi r^2)h$, where πr^2 is the area of the base.

Volume of a Cylinder: If the height of a cylinder is h and the radius is r, then the volume would be $V = \pi r^2 h$.

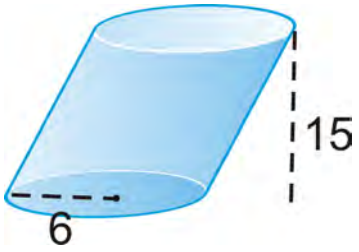
Also, like a prism, Cavalieri's Principle holds. So, the volumes of an oblique cylinder and a right cylinder have the same formula.

Example 6: Find the volume of the cylinder.

Solution: If the diameter is 16, then the radius is 8.

$$V = \pi 8^2(21) = 1344\pi \ units^3$$

Example 7: Find the volume of the cylinder.



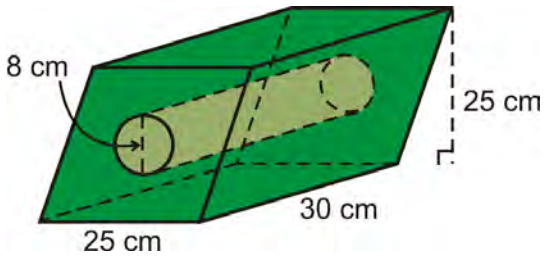
Solution: $V = \pi 6^2(15) = 540\pi \ units^3$

Example 8: If the volume of a cylinder is $484\pi \ in^3$ and the height is 4 in, what is the radius?

Solution: Plug in what you know to the volume formula and solve for r.

$$484\pi = \pi r^2(4)$$
$$121 = r^2$$
$$11 = r$$

Example 9: Find the volume of the solid below.



Solution: This solid is a parallelogram-based prism with a cylinder cut out of the middle. To find the volume, we need to find the volume of the prism and then subtract the volume of the cylinder.

$$V_{prism} = (25 \cdot 25)30 = 18750 \ cm^3$$
$$V_{cylinder} = \pi(4)^2(30) = 480\pi \ cm^3$$

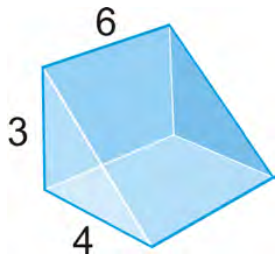
The total volume is $18750 - 480\pi \approx 17242.04\ cm^3$.

Know What? Revisited Even though it doesn't look like it, the trapezoid is considered the base of this prism. The area of the trapezoids are $\frac{1}{2}(4+8)25 = 150\ ft^2$. Multiply this by the height, 10 ft, and we have that the volume is 1500 ft^3. To determine the number of gallons that are needed, divide 1500 by 7.48. $\frac{1500}{7.48} \approx 200.53$ gallons are needed to fill the pool.

Review Questions

1. Two cylinders have the same surface area. Do they have the same volume? How do you know?
2. How many one-inch cubes can fit into a box that is 8 inches wide, 10 inches long, and 12 inches tall? Is this the same as the volume of the box?
3. A cereal box in 2 inches wide, 10 inches long and 14 inches tall. How much cereal does the box hold?
4. A can of soda is 4 inches tall and has a diameter of 2 inches. How much soda does the can hold? Round your answer to the nearest hundredth.
5. A cube holds 216 in^3. What is the length of each edge?
6. A cylinder has a volume of $486\pi\ ft.^3$. If the height is 6 ft., what is the diameter?

Use the right triangular prism to answer questions 7 and 8.

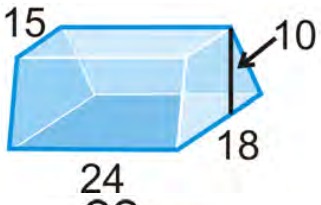


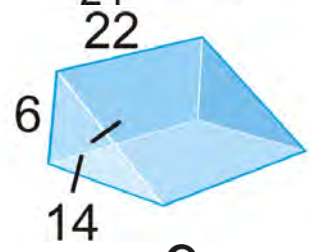
7. What is the length of the third base edge?
8. Find the volume of the prism.
9. Fuzzy dice are cubes with 4 inch sides.

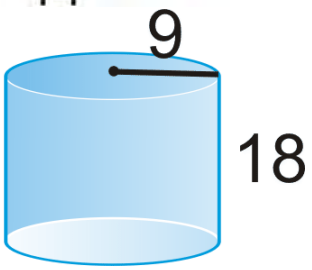
   a. What is the volume of one die?
   b. What is the volume of both dice?

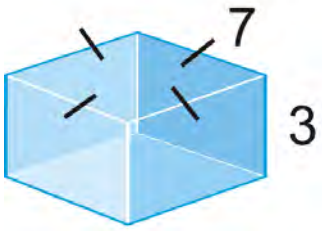
10. A right cylinder has a 7 cm radius and a height of 18 cm. Find the volume.

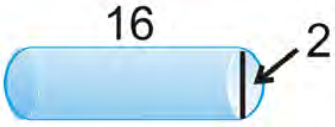
Find the volume of the following solids. Round your answers to the nearest hundredth.

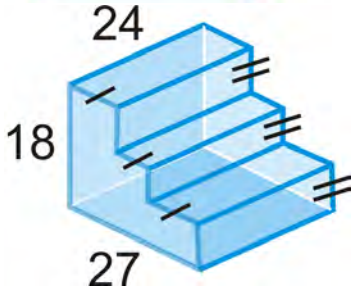
11. 

12. 

13. 

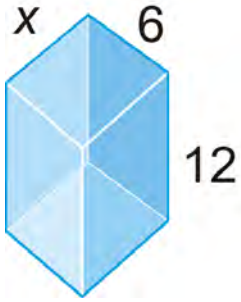
14. 

15. 

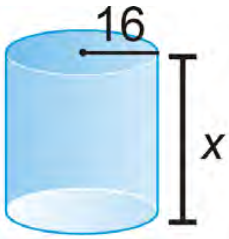
16. 

Algebra Connection Find the value of x, given the surface area.

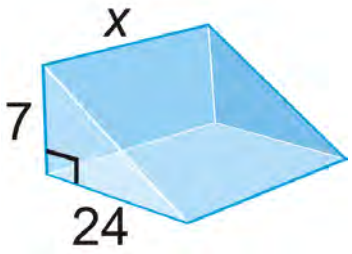
17. $V = 504\ units^3$



18. $V = 6144\pi \ units^3$

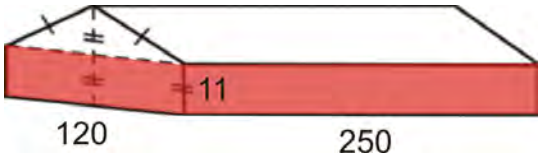


19. $V = 2688 \ units^3$



20. The area of the base of a cylinder is $49\pi \ in^2$ and the height is 6 in. Find the volume.
21. The circumference of the base of a cylinder is $80\pi \ cm$ and the height is 15 cm. Find the volume.
22. The lateral surface area of a cylinder is $30\pi \ m^2$ and the circumference is $10\pi \ m$. What is the volume of the cylinder?

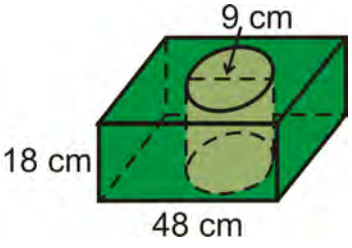
Use the diagram below for questions 23-25. The barn is shaped like a pentagonal prism with dimensions shown in feet.

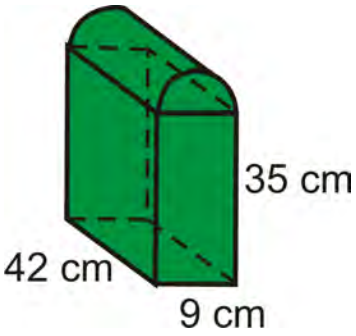


23. Find the volume of the red rectangular prism.
24. Find the volume of the triangular prism on top of the rectangular prism.
25. Find the total volume of the barn.

Find the volume of the composite solids below. Round your answers to the nearest hundredth.

26. The bases are squares.



27. 

28. The volume of a cylinder with height to radius ratio of 4:1 is $108\pi\ cm^3$. Find the radius and height of the cylinder.
29. The length of a side of the base of a hexagonal prism is 8 cm and its volume is $1056\sqrt{3}\ cm^3$. Find the height of the prism.
30. A cylinder fits tightly inside a rectangular prism with dimensions in the ratio 5:5:7 and volume $1400\ in^3$. Find the volume of the space inside the prism that is not contained in the cylinder.

Review Queue Answers

1. The amount a three-dimensional figure can hold.
2. $54\ in^2$
3. $192\ in^2$
4. Answers:
   a. $8\left(\frac{1}{4}\cdot 4^2\sqrt{3}\right) = 32\sqrt{3}\ cm^2$
   b. $8\left(\frac{1}{4}\cdot s^2\sqrt{3}\right) = 2s^2\sqrt{3}$

7.12 Volume of Pyramids and Cones

Learning Objectives

- Find the volume of a pyramid.
- Find the volume of a cone.

Review Queue

1. Find the volume of a square prism with 8 inch base edges and a 12 inch height.
2. Find the volume of a cylinder with a *diameter* of 8 inches and a height of 12 inches.
3. In your answers from #1 and #2, which volume is bigger? Why do you think that is?
4. Find the surface area of a square pyramid with 10 inch base edges and a height of 12 inches.

Know What? The Khafre Pyramid is the second largest pyramid of the Ancient Egyptian Pyramids in Giza. It is a square pyramid with a base edge of 706 feet and an original height of 407.5 feet. What was the original volume of the Khafre Pyramid?

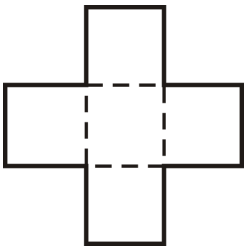
Volume of a Pyramid

Recall that the volume of a prism is Bh, where B is the area of the base. The volume of a pyramid is closely related to the volume of a prism with the same sized base.

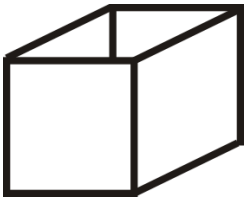
Investigation 11-1: Finding the Volume of a Pyramid

Tools needed: pencil, paper, scissors, tape, ruler, dry rice or sand.

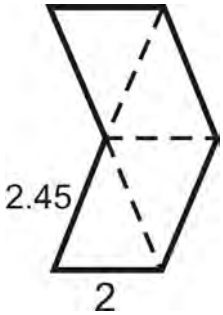
1. Make an open net (omit one base) of a cube, with 2 inch sides.



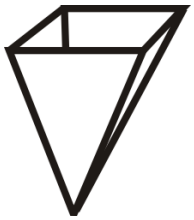
2. Cut out the net and tape up the sides to form an open cube.



3. Make an open net (no base) of a square pyramid, with lateral edges of 2.45 inches and base edges of 2 inches. This will make the overall height 2 inches.



4. Cut out the net and tape up the sides to form an open pyramid.

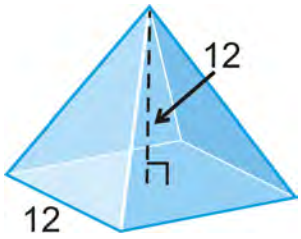


5. Fill the pyramid with dry rice. Then, dump the rice into the open cube. How many times do you have to repeat this to fill the cube?

Volume of a Pyramid: If B is the area of the base and h is the height, then the volume of a pyramid is $V = \frac{1}{3}Bh$.

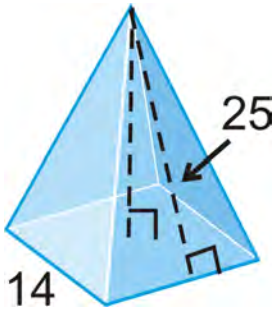
The investigation showed us that you would need to repeat this process three times to fill the cube. This means that the pyramid is one-third the volume of a prism with the same base.

Example 1: Find the volume of the pyramid.



Solution: $V = \frac{1}{3}(12^2)12 = 576\ units^3$

Example 2: Find the volume of the pyramid.



Solution: In this example, we are given the slant height. For volume, we need the height, so we need to use the Pythagorean Theorem to find it.

$$\begin{aligned} 7^2 + h^2 &= 25^2 \\ h^2 &= 576 \\ h &= 24 \end{aligned}$$

Using the height, the volume is $\frac{1}{3}(14^2)(24) = 1568\ units^3$.

Example 3: Find the volume of the pyramid.

Solution: The base of this pyramid is a right triangle. So, the area of the base is $\frac{1}{2}(14)(8) = 56\ units^2$.

$$V = \frac{1}{3}(56)(17) \approx 317.33\ units^3$$

Example 4: A rectangular pyramid has a base area of 56 cm^2 and a volume of 224 cm^3. What is the height of the pyramid?

Solution: The formula for the volume of a pyramid works for any pyramid, as long as you can find the area of the base.

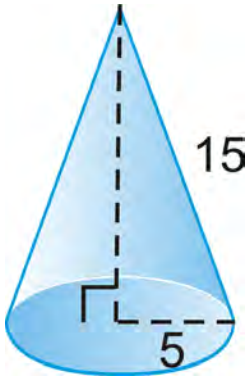
$$\begin{aligned} 224 &= 56h \\ 4 &= h \end{aligned}$$

Volume of a Cone

The volume of cone has the same relationship with a cylinder as pyramid does with a prism. If the bases of a cone and a cylinder are the same, then the volume of a cone will be one-third the volume of the cylinder.

Volume of a Cone: If r is the radius of a cone and h is the height, then the volume is $V = \frac{1}{3}\pi r^2 h$.

Example 5: Find the volume of the cone.



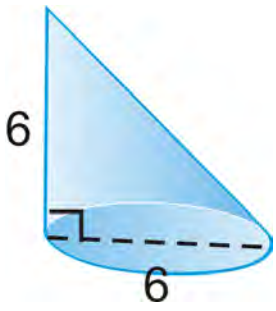
Solution: To find the volume, we need the height, so we have to use the Pythagorean Theorem.

$$\begin{aligned} 5^2 + h^2 &= 15^2 \\ h^2 &= 200 \\ h &= 10\sqrt{2} \end{aligned}$$

Now, we can find the volume.

$$V = \frac{1}{3}(5^2)\left(10\sqrt{2}\right)\pi \approx 370.24$$

Example 6: Find the volume of the cone.



Solution: Even though this doesn't look like the cone in Example 5, we can still find the volume in the same way. Use the *radius* in the formula.

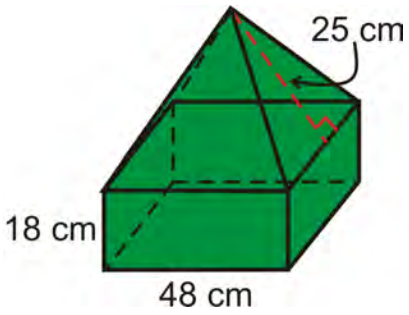
$$V = \frac{1}{3}\pi(3^2)(6) = 18\pi \approx 56.55$$

Example 7: The volume of a cone is $484\pi\ cm^3$ and the height is 12 cm. What is the radius?

Solution: Plug in what you know to the volume formula.

$$484\pi = \frac{1}{3}\pi r^2(12)$$
$$121 = r^2$$
$$11 = r$$

Example 8: Find the volume of the composite solid. All bases are squares.



Solution: This is a square prism with a square pyramid on top. Find the volume of each separately and then add them together to find the total volume. First, we need to find the height of the pyramid portion. The slant height is 25 and the edge is 48. Using have of the edge, we have a right triangle and we can use the Pythagorean Theorem. $h = \sqrt{25^2 - 24^2} = 7$

$$V_{prism} = (48)(48)(18) = 41472\ cm^3$$
$$V_{pyramid} = \frac{1}{3}(48^2)(7) = 5376\ cm^3$$

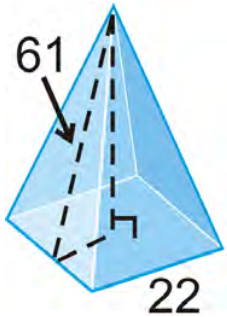
The total volume is $41472 + 5376 = 46,848\ cm^3$.

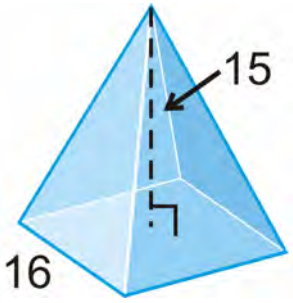
Know What? Revisited The original volume of the pyramid is $\frac{1}{3}(706^2)(407.5) \approx 67,704,223.33\ ft^3$.

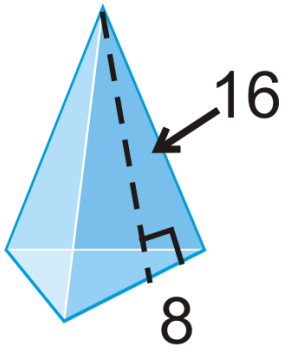
Review Questions

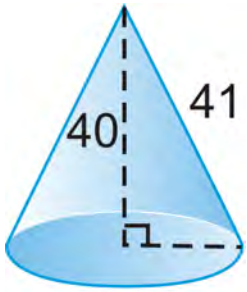
Find the volume of each regular pyramid and right cone. Round any decimal answers to the nearest hundredth. The bases of these pyramids are either squares or equilateral triangles.

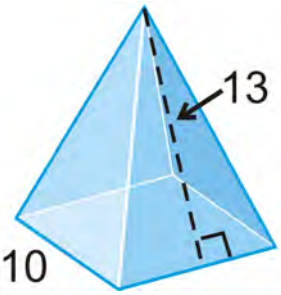
1.

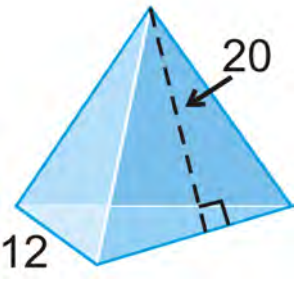


2. 

3. 

4. 

5. 

6. 

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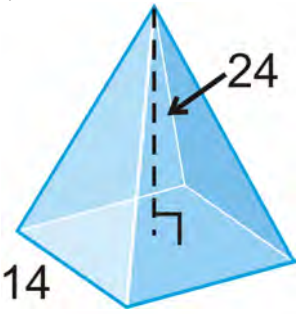

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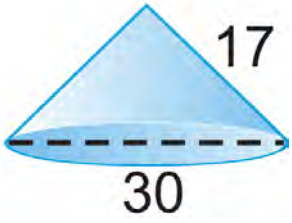
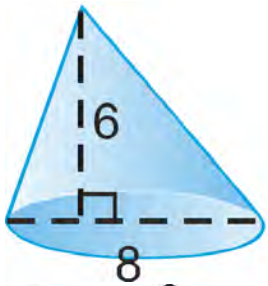


Find the volume of the following non-regular pyramids and cones. Round any decimal answers to the nearest hundredth.

10.



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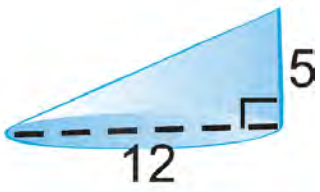
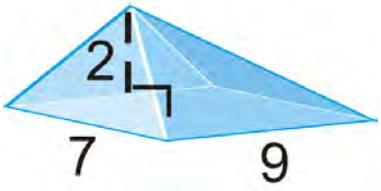
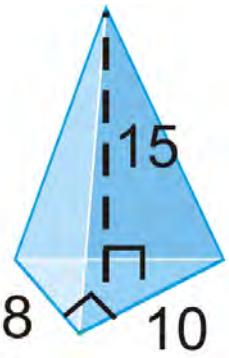
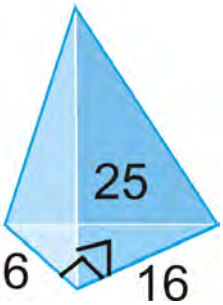


13. base is a rectangle



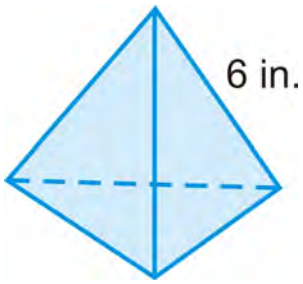


14.



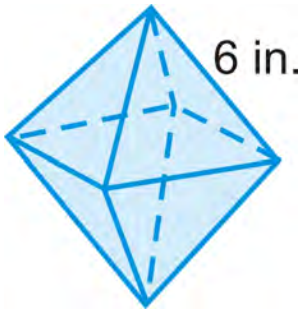
15.

A ***regular tetrahedron*** has four equilateral triangles as its faces. Use the diagram to answer questions 16-19.



16. What is the area of the base of this regular tetrahedron?
17. What is the height of this figure? Be careful!
18. Find the volume. Leave your answer in simplest radical form.
19. ***Challenge*** If the sides are length s, what is the volume?

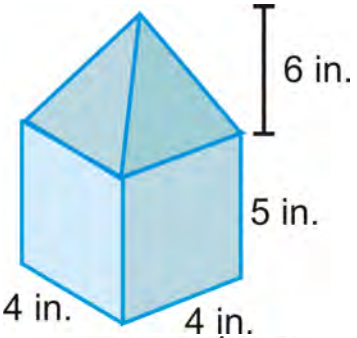
A ***regular octahedron*** has eight equilateral triangles as its faces. Use the diagram to answer questions 20-22.



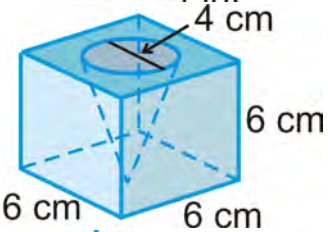
20. *Describe* how you would find the volume of this figure.
21. Find the volume. Leave your answer in simplest radical form.
22. ***Challenge*** If the sides are length s, what is the volume?
23. The volume of a square pyramid is 72 square inches and the base edge is 4 inches. What is the height?
24. If the volume of a cone is $30\pi\ cm^2$ and the radius is 5 cm, what is the height?
25. If the volume of a cone is $105\pi\ cm^2$ and the height is 35 cm, what is the radius?

Find the volume of the composite solids. Round your answer to the nearest hundredth.

26.



27.



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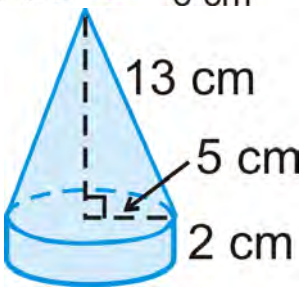
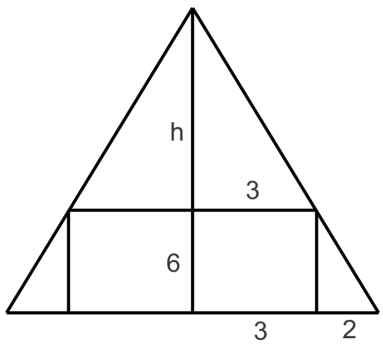


29. The ratio of the height to radius in a cone is 3:2. If the volume is $108\pi\ m^3$, find the height and radius of the cone.
30. A teepee is to be built such that there is a minimal cylindrical shaped central living space contained within the cone shape of diameter 6 ft and height 6 ft. If the radius of the entire teepee is 5 ft, find the total height of the teepee.



Review Queue Answers

1. $(8^2)(12) = 768\ in^3$
2. $(4^2)(12)\pi = 192\pi \approx 603.19$
3. The volume of the square prism is greater because the square base is larger than a circle with the same diameter as the square's edge.
4. Find slant height, $l = 13$. $SA = 100 + \frac{1}{2}(40)(13) = 360\ in^2$

7.13 Surface Area and Volume of Spheres

Learning Objectives

- Find the surface area of a sphere.
- Find the volume of a sphere.

Review Queue

1. List three spheres you would see in real life.
2. Find the area of a circle with a 6 cm radius.
3. Find the volume of a cylinder with the circle from #2 as the base and a height of 5 cm.
4. Find the volume of a cone with the circle from #2 as the base and a height of 5 cm.

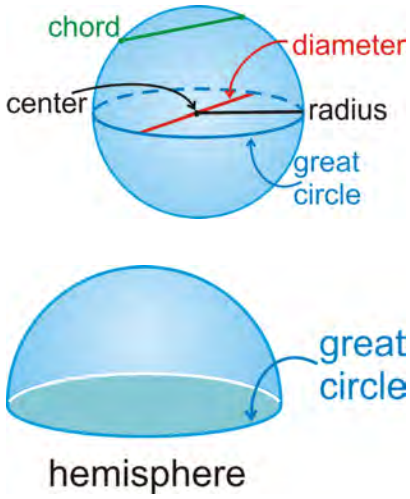
Know What? A regulation bowling ball is a sphere that weighs between 12 and 16 pounds. The maximum circumference of a bowling ball is 27 inches. Using this number, find the radius of a bowling ball, its surface area and volume. You may assume the bowling ball does not have any finger holes. Round your answers to the nearest hundredth.

Defining a Sphere

A sphere is the last of the three-dimensional shapes that we will find the surface area and volume of. Think of a sphere as a three-dimensional circle. You have seen spheres in real-life countless times; tennis balls, basketballs, volleyballs, golf balls, and baseballs. Now we will analyze the parts of a sphere.

Sphere: The set of all points, in three-dimensional space, which are equidistant from a point.

A sphere has a ***center***, radius and diameter, just like a circle. The ***radius*** has an endpoint on the sphere and the other is on the center. The ***diameter*** must contain the center. If it does not, it is a ***chord***. The ***great circle*** is a plane that contains the diameter. It would be the largest circle cross section in a sphere. There are infinitely many great circles. ***The circumference of a sphere is the circumference of a great circle***. Every great circle divides a sphere into two congruent hemispheres, or two half spheres. Also like a circle, spheres can have tangent lines and secants. These are defined just like they are in a circle.



Example 1: The circumference of a sphere is 26π $feet$. What is the radius of the sphere?

Solution: The circumference is referring to the circumference of a great circle. Use $C = 2\pi r$.

$$2\pi r = 26\pi$$
$$r = 13 \ ft.$$

Surface Area of a Sphere

One way to find the formula for the surface area of a sphere is to look at a baseball. We can best *approximate* the cover of the baseball by 4 circles. The area of a circle is πr^2, so the surface area of a sphere is $4\pi r^2$. While the covers of a baseball are not four perfect circles, they are stretched and skewed.

Another way to show the surface area of a sphere is to watch the link by Russell Knightley, http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Surface-Area-Derivation.html . It is a great visual interpretation of the formula.

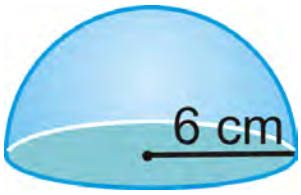
Surface Area of a Sphere: If r is the radius, then the surface area of a sphere is $SA = 4\pi r^2$.

Example 2: Find the surface area of a sphere with a radius of 14 feet.

Solution: Use the formula, $r = 14 \ ft$.

$$SA = 4\pi(14)^2$$
$$= 784\pi \ ft^2$$

Example 3: Find the surface area of the figure below.



Solution: This is a hemisphere. Be careful when finding the surface area of a hemisphere because you need to include the area of the base. If the question asked for the *lateral surface area*, then your answer would *not* include the bottom.

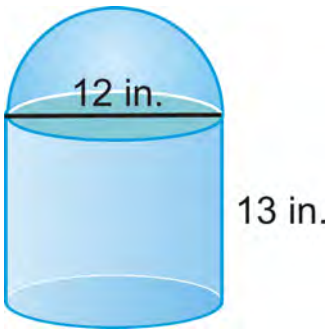
$$\begin{aligned} SA &= \pi r^2 + \frac{1}{2}4\pi r^2 \\ &= \pi(6^2) + 2\pi(6^2) \\ &= 36\pi + 72\pi = 108\pi \ cm^2 \end{aligned}$$

Example 4: The surface area of a sphere is $100\pi \ in^2$. What is the radius?

Solution: Plug in what you know to the formula and solve for r.

$$\begin{aligned} 100\pi &= 4\pi r^2 \\ 25 &= r^2 \\ 5 &= r \end{aligned}$$

Example 5: Find the surface area of the following solid.



Solution: This solid is a cylinder with a hemisphere on top. Because it is one fluid solid, we would not include the bottom of the hemisphere or the top of the cylinder because they are no longer on the surface of the solid. Below, "*LA*" stands for *lateral area*.

$$\begin{aligned} SA &= LA_{cylinder} + LA_{hemisphere} + A_{base\ circle} \\ &= \pi rh + \frac{1}{2}4\pi r^2 + \pi r^2 \\ &= \pi(6)(13) + 2\pi 6^2 + \pi 6^2 \\ &= 78\pi + 72\pi + 36\pi \\ &= 186\pi \ in^2 \end{aligned}$$

Volume of a Sphere

A sphere can be thought of as a regular polyhedron with an infinite number of congruent regular polygon faces. As n, the number of faces increases to an infinite number, the figure approaches becoming a sphere. So a sphere can be thought of as a polyhedron with an infinite number faces. Each of those faces is the base of a pyramid whose vertex is located at the center of the sphere. Each of the pyramids that make up the sphere would be congruent to the pyramid shown. The volume of this pyramid is given by $V = \frac{1}{3}Bh$.

To find the volume of the sphere, you need to add up the volumes of an infinite number of infinitely small pyramids.

$$\begin{aligned} V(all\ pyramids) &= V_1 + V_2 + V_3 + \ldots + V_n \\ &= \frac{1}{3}(B_1h + B_2h + B_3h + \ldots + B_nh) \\ &= \frac{1}{3}h(B_1 + B_2 + B_3 + \ldots + B_n) \end{aligned}$$

The sum of all of the bases of the pyramids is the surface area of the sphere. Since you know that the surface area of the sphere is $4\pi r^2$, you can substitute this quantity into the equation above.

$$= \frac{1}{3}h\left(4\pi r^2\right)$$

In the picture above, we can see that the height of each pyramid is the radius, so $h = r$.

$$\begin{aligned} &= \frac{4}{3}r(\pi r^2) \\ &= \frac{4}{3}\pi r^3 \end{aligned}$$

To see an animation of the volume of a sphere, see http://www.rkm.com.au/ANIMATIONS/animation-Sphere-Volume-Derivation.html by Russell Knightley. It is a slightly different interpretation than our derivation.

Volume of a Sphere: If a sphere has a radius r, then the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

Example 6: Find the volume of a sphere with a radius of 9 m.

Solution: Use the formula above.

$$V = \frac{4}{3}\pi 6^3$$
$$= \frac{4}{3}\pi(216)$$
$$= 288\pi$$

Example 7: A sphere has a volume of 14137.167 ft^3, what is the radius?

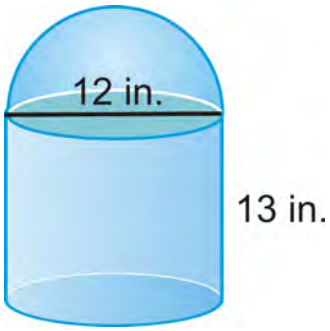
Solution: Because we have a decimal, our radius might be an approximation. Plug in what you know to the formula and solve for r.

$$14137.167 = \frac{4}{3}\pi r^3$$
$$\frac{3}{4\pi} \cdot 14137.167 = r^3$$
$$3375 = r^3$$

At this point, you will need to take the ***cubed root*** of 3375. Depending on your calculator, you can use the $\sqrt[3]{x}$ function or $\wedge\left(\frac{1}{3}\right)$. The cubed root is the inverse of cubing a number, just like the square root is the inverse, or how you undo, the square of a number.

$$\sqrt[3]{3375} = 15 = r \qquad \text{The radius is 15 } ft.$$

Example 8: Find the volume of the following solid.



Solution: To find the volume of this solid, we need the volume of a cylinder and the volume of the hemisphere.

$$V_{cylinder} = \pi 6^2(13) = 78\pi$$
$$V_{hemisphere} = \frac{1}{2}\left(\frac{4}{3}\pi 6^3\right) = 36\pi$$
$$V_{total} = 78\pi + 36\pi = 114\pi \ in^3$$

Know What? Revisited If the maximum circumference of a bowling ball is 27 inches, then the maximum radius would be $27 = 2\pi r$, or $r = 4.30$ inches. Therefore, the surface area would be $4\pi 4.3^2 \approx 232.35 \ in^2$, and the volume would be $\frac{4}{3}\pi \ 4.3^3 \approx 333.04 \ in^3$. The weight of the bowling ball refers to its density, how heavy something is. The volume of the ball tells us how much it can hold.

Review Questions

1. Are there any cross-sections of a sphere that are not a circle? Explain your answer.

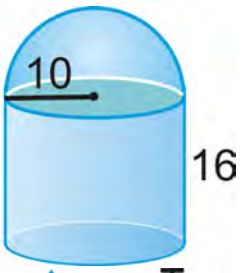
Find the surface area and volume of a sphere with: (Leave your answer in terms of π)

2. a radius of 8 in.
3. a diameter of 18 cm.
4. a radius of 20 ft.
5. a diameter of 4 m.
6. a radius of 15 ft.
7. a diameter of 32 in.
8. a circumference of 26π *cm*.
9. a circumference of 50π *yds*.
10. The surface area of a sphere is 121π in^2. What is the radius?
11. The volume of a sphere is 47916π m^3. What is the radius?
12. The surface area of a sphere is 4π ft^2. What is the volume?
13. The volume of a sphere is 36π mi^3. What is the surface area?
14. Find the radius of the sphere that has a volume of 335 cm^3. Round your answer to the nearest hundredth.
15. Find the radius of the sphere that has a surface area 225π ft^2.

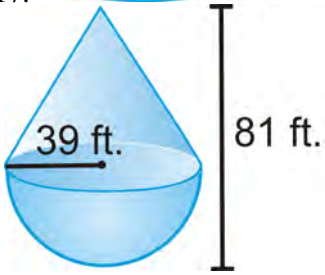
Find the surface area of the following shapes. Leave your answers in terms of π.

16.

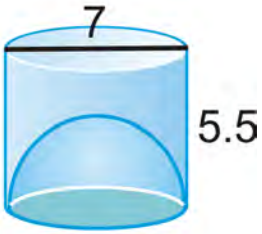
17.



18.



19. You may assume the bottom is *open*.

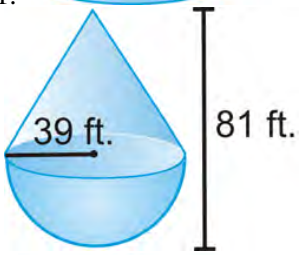


Find the volume of the following shapes. Round your answers to the nearest hundredth.

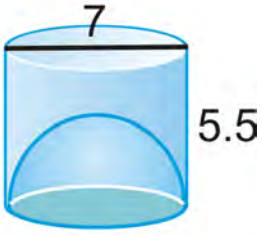
20.

21.

22.

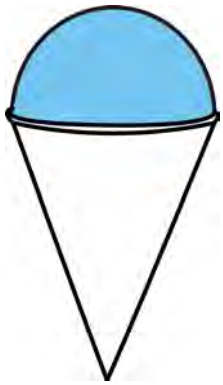


23.



24. A sphere has a radius of 5 cm. A right cylinder has the same radius and volume. Find the height and total surface area of the cylinder.
25. Tennis balls with a 3 inch diameter are sold in cans of three. The can is a cylinder. Assume the balls touch the can on the sides, top and bottom. What is the volume of the space *not* occupied by the tennis balls? Round your answer to the nearest hundredth.

26. One hot day at a fair you buy yourself a snow cone. The height of the cone shaped container is 5 cm and its radius is 2 cm. The shaved ice is perfectly rounded on top forming a hemisphere. What is the volume of the ice in your frozen treat? If the ice melts at a rate of 2 cm^3 per minute, how long do you have to eat your treat before it all melts? Give your answer to the nearest minute.



Multi-Step Problems

27. Answer the following:
    a. What is the surface area of a cylinder?
    b. Adjust your answer from part a for the case where $r = h$.
    c. What is the surface area of a sphere?
    d. What is the relationship between your answers to parts b and c? Can you explain this?
28. At the age of 81, Mr. Luke Roberts began collecting string. He had a ball of string 3 feet in diameter.
    a. Find the volume of Mr. Roberts' ball of string in cubic inches.
    b. Assuming that each cubic inch weighs 0.03 pounds, find the weight of his ball of string.
    c. To the nearest inch, how big (diameter) would a 1 ton ball of string be? $(1\ ton = 2000\ lbs)$

For problems 29-31, use the fact that the earth's radius is approximately 4,000 miles.

29. Find the length of the equator.
30. Find the surface area of earth, rounding your answer to the nearest million square miles.
31. Find the volume of the earth, rounding your answer to the nearest billion cubic miles.

Review Queue Answers

1. *Answers will vary.* Possibilities are any type of ball, certain lights, or the 76/Unical orb.
2. 36π
3. 180π
4. 60π

7.14 Exploring Similar Solids

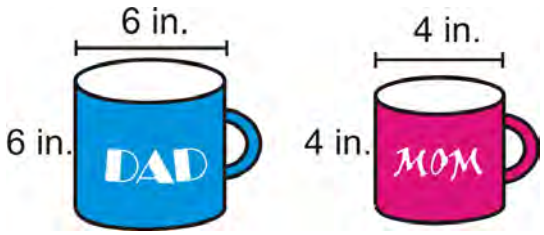
Learning Objectives

- Find the relationship between similar solids and their surface areas and volumes.

Review Queue

1. We know that every circle is similar, is every sphere similar?
2. Find the volume of a sphere with a 12 in radius. Leave your answer in terms of π.
3. Find the volume of a sphere with a 3 in radius. Leave your answer in terms of π.
4. Find the scale factor of the spheres from #2 and #3. Then find the ratio of the volumes and reduce it. What do you notice?
5. Two squares have a scale factor of 2:3. What is the ratio of their areas?
6. The smaller square from #5 has an area of 16 cm^2. What is the area of the larger square?
7. The ratio of the areas of two similar triangles is 1:25. The height of the larger triangle is 20 cm, what is the height of the smaller triangle?

Know What? Your mom and dad have cylindrical coffee mugs with the dimensions to the right. Are the mugs similar? (You may ignore the handles.) If the mugs are similar, find the volume of each, the scale factor and the ratio of the volumes.

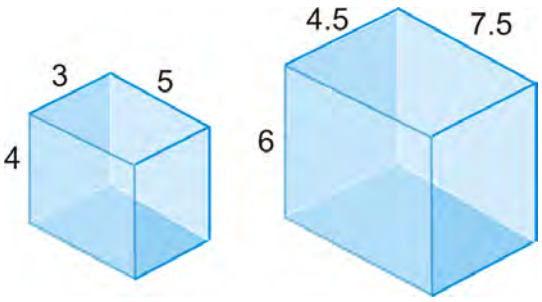


Similar Solids

Recall that two shapes are similar if all the corresponding angles are congruent and the corresponding sides are proportional.

Similar Solids: Two solids are similar if and only if they are the same type of solid and their corresponding linear measures (radii, heights, base lengths, etc.) are proportional.

Example 1: Are the two rectangular prisms similar? How do you know?

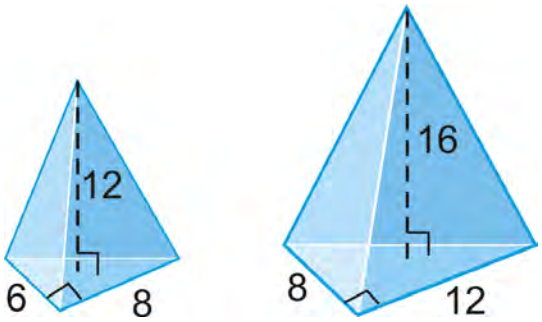


Solution: Match up the corresponding heights, widths, and lengths to see if the rectangular prisms are proportional.

$$\frac{small\ prism}{large\ prism} = \frac{3}{4.5} = \frac{4}{6} = \frac{5}{7.5}$$

The congruent ratios tell us the two prisms are similar.

Example 2: Determine if the two triangular pyramids similar.



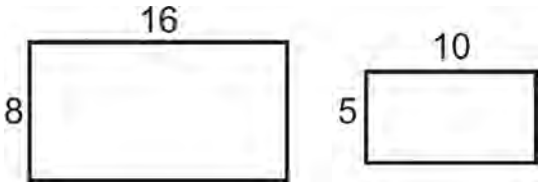
Solution: Just like Example 1, let's match up the corresponding parts.

$\frac{6}{8} = \frac{12}{16} = \frac{3}{4}$ however, $\frac{8}{12} = \frac{2}{3}$.

Because one of the base lengths is not in the same proportion as the other two lengths, these right triangle pyramids are not similar.

Surface Areas of Similar Solids

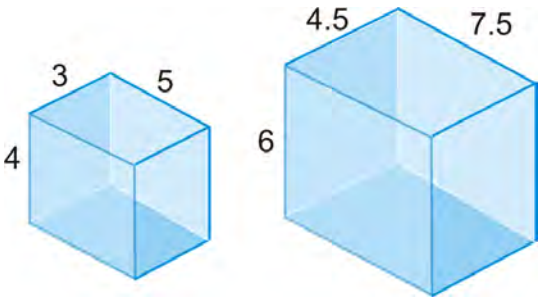
Recall that ***when two shapes are similar, the ratio of the area is a square of the scale factor.***



For example, the two rectangles to the left are similar because their sides are in a ratio of 5:8. The area of the larger rectangle is $8(16) = 128\ units^2$ and the area of the smaller rectangle is $5(10) = 50\ units^2$. If we compare the areas in a ratio, it is $50 : 128 = 25 : 64 = 5^2 = 8^2$.

So, what happens with the surface areas of two similar solids? Let's look at Example 1 again.

Example 3: Find the surface area of the two similar rectangular prisms.



Solution:

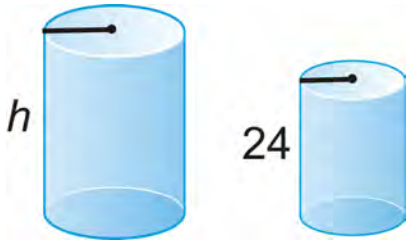
$$SA_{smaller} = 2(4 \cdot 3) + 2(4 \cdot 5) + 2(3 \cdot 5)$$
$$= 24 + 40 + 30 = 94 \ units^2$$

$$SA_{larger} = 2(6 \cdot 4.5) + 2(4.5 \cdot 7.5) + 2(6 \cdot 7.5)$$
$$= 54 + 67.5 + 90 = 211.5 \ units^2$$

Now, find the ratio of the areas. $\frac{94}{211.5} = \frac{4}{9} = \frac{2^2}{3^2}$. The sides are in a ratio of $\frac{4}{6} = \frac{2}{3}$, so the surface areas have the same relationship as the areas of two similar shapes.

Surface Area Ratio: If two solids are similar with a scale factor of $\frac{a}{b}$, then the surface areas are in a ratio of $\left(\frac{a}{b}\right)^2$.

Example 4: Two similar cylinders are below. If the ratio of the areas is 16:25, what is the height of the taller cylinder?



Solution: First, we need to take the square root of the area ratio to find the scale factor, $\sqrt{\frac{16}{25}} = \frac{4}{5}$. Now we can set up a proportion to find h.

$$\frac{4}{5} = \frac{24}{h}$$
$$4h = 120$$
$$h = 30$$

Example 5: Using the cylinders from Example 4, if the surface area of the smaller cylinder is $1536\pi \ cm^2$, what is the surface area of the larger cylinder?

Solution: Set up a proportion using the ratio of the areas, 16:25.

$$\frac{16}{25} = \frac{1536\pi}{A}$$
$$16A = 38400\pi$$
$$A = 2400\pi \; cm^2$$

Volumes of Similar Solids

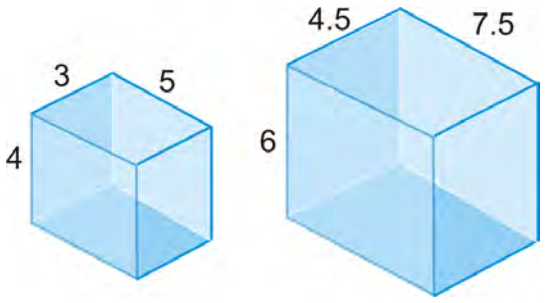
Let's look at what we know about similar solids so far.

TABLE 7.5:

	Ratios	***Units***
Scale Factor	$\frac{a}{b}$	in, ft, cm, m, etc.
Ratio of the Surface Areas	$\left(\frac{a}{b}\right)^2$	in^2, ft^2, cm^2, m^2, etc.
Ratio of the Volumes	??	in^3, ft^3, cm^3, m^3, etc.

It looks as though there is a pattern. If the ratio of the volumes follows the pattern from above, it should be the cube of the scale factor. We will do an example and test our theory.

Example 6: Find the volume of the following rectangular prisms. Then, find the ratio of the volumes.



Solution:

$$V_{smaller} = 3(4)(5) = 60$$
$$V_{larger} = 4.5(6)(7.5) = 202.5$$

The ratio is $\frac{60}{202.5}$, which reduces to $\frac{8}{27} = \frac{2^3}{3^3}$.

It seems as though our prediction based on the patterns is correct.

Volume Ratio: If two solids are similar with a scale factor of $\frac{a}{b}$, then the volumes are in a ratio of $\left(\frac{a}{b}\right)^3$.

Example 7: Two spheres have radii in a ratio of 3:4. What is the ratio of their volumes?

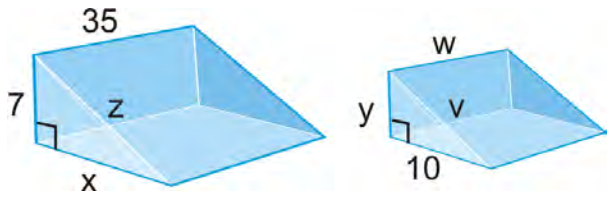
Solution: If we cube 3 and 4, we will have the ratio of the volumes. Therefore, $3^3 : 4^3$ or 27:64 is the ratio of the volumes.

Example 8: If the ratio of the volumes of two similar prisms is 125:8, what is their scale factor?

Solution: This example is the opposite of the previous example. We need to take the cubed root of 125 and 8 to find the scale factor.

$$\sqrt[3]{125} : \sqrt[3]{8} = 5 : 2$$

Example 9: Two similar right triangle prisms are below. If the ratio of the volumes is 343:125, find the missing sides in both figures.



Solution: If the ratio of the volumes is 343:125, then the scale factor is 7:5, the cubed root of each. With the scale factor, we can now set up several proportions.

$$\frac{7}{5} = \frac{7}{y} \quad \frac{7}{5} = \frac{x}{10} \quad \frac{7}{5} = \frac{35}{w} \quad 7^2 + x^2 = z^2 \quad \frac{7}{5} = \frac{z}{v}$$
$$y = 5 \quad x = 14 \quad w = 25 \quad 7^2 + 14^2 = z^2$$
$$z = \sqrt{245} = 7\sqrt{5} \quad \frac{7}{5} = \frac{7\sqrt{5}}{v} \rightarrow v = 5\sqrt{5}$$

Example 10: The ratio of the surface areas of two similar cylinders is 16:81. If the volume of the smaller cylinder is $96\pi\ in^3$, what is the volume of the larger cylinder?

Solution: First we need to find the scale factor from the ratio of the surface areas. If we take the square root of both numbers, we have that the ratio is 4:9. Now, we need cube this to find the ratio of the volumes, $4^3 : 9^3 = 64 : 729$. At this point we can set up a proportion to solve for the volume of the larger cylinder.

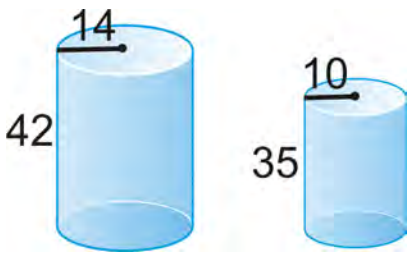
$$\frac{64}{729} = \frac{96\pi}{V}$$
$$64V = 69984\pi$$
$$V = 1093.5\pi\ in^3$$

Know What? Revisited The coffee mugs are similar because the heights and radii are in a ratio of 2:3, which is also their scale factor. The volume of Dad's mug is $54\pi\ in^3$ and Mom's mug is $16\pi\ in^3$. The ratio of the volumes is $54\pi : 16\pi$, which reduces to 8:27.

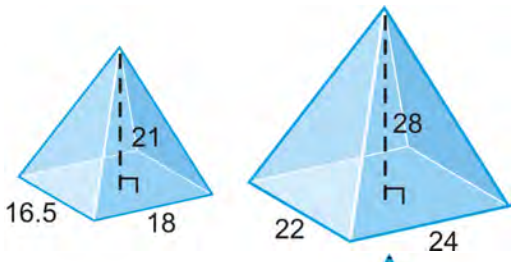
Review Questions

Determine if each pair of right solids are similar. *Explain* your reasoning.

1.



2.



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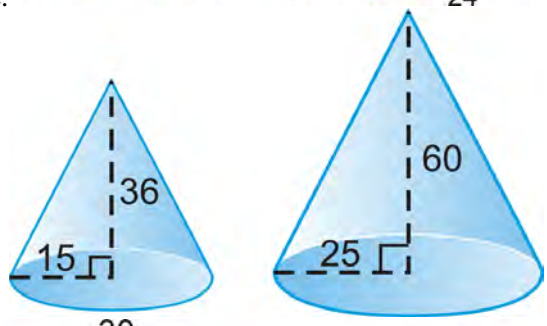

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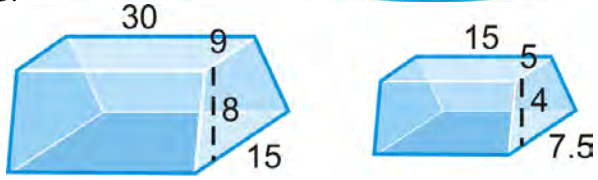
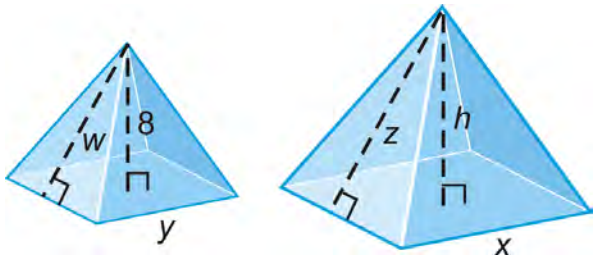


5. Are all cubes similar? Why or why not?
6. Two prisms have a scale factor of 1:4. What is the ratio of their surface areas?
7. Two pyramids have a scale factor of 2:7. What is the ratio of their volumes?
8. Two spheres have radii of 5 and 9. What is the ratio of their volumes?
9. The surface area of two similar cones is in a ratio of 64:121. What is the scale factor?
10. The volume of two hemispheres is in a ratio of 125:1728. What is the scale factor?
11. A cone has a volume of 15π and is similar to another larger cone. If the scale factor is 5:9, what is the volume of the larger cone?
12. A cube has sides of length x and is enlarged so that the sides are $4x$. How does the volume change?
13. The ratio of the volumes of two similar pyramids is 8:27. What is the ratio of their total surface areas?
14. The ratio of the volumes of two tetrahedrons is 1000:1. The smaller tetrahedron has a side of length 6 cm. What is the side length of the larger tetrahedron?
15. The ratio of the surface areas of two cubes is 64:225. If the volume of the smaller cube is 13824 m^3, what is the volume of the larger cube?

Below are two similar square pyramids with a volume ratio of 8:27. The base lengths are equal to the heights. Use this to answer questions 16-21.



16. What is the scale factor?
17. What is the ratio of the surface areas?
18. Find h, x and y.
19. Find w and z.

20. Find the volume of both pyramids.
21. Find the *lateral* surface area of both pyramids.

Use the hemispheres below to answer questions 22-25.

22. Are the two hemispheres similar? How do you know?
23. Find the ratio of the surface areas and volumes.
24. Find the *lateral* surface areas of both hemispheres.
25. Determine the ratio of the lateral surface areas for the hemispheres. Is it the same as the ratio of the total surface area? Why or why not?

Animal A and animal B are similar (meaning the size and shape of their bones and bodies are similar) and *the strength of their respective bones are proportional to the cross sectional area of their bones*. Answer the following questions given that the ratio of the height of animal A to the height of animal B is 3:5. You may assume the lengths of their bones are in the same ratio.

26. Find the ratio of the strengths of the bones. How much stronger are the bones in animal B?
27. If their weights are proportional to their volumes, find the ratio of their weights.
28. Which animal has a skeleton more capable of supporting its own weight? Explain.

Two sizes of cans of beans are similar. The thickness of the walls and bases are the same in both cans. The ratio of their surface areas is 4:9.

29. If the surface area of the smaller can is 36 sq in, what is the surface area of the larger can?
30. If the sheet metal used to make the cans costs \$0.006 per square inch, how much does it cost to make each can?
31. What is the ratio of their volumes?
32. If the smaller can is sold for \$0.85 and the larger can is sold for \$2.50, which is a better deal?

Review Queue Answers

1. Yes, every sphere is similar because the similarity only depends on one length, the radius.
2. $\frac{4}{3}12^3\pi = 2304\pi\ in^3$
3. $\frac{4}{3}3^3\pi = 27\pi\ in^3$
4. The scale factor is 4:1, the volume ratio is 2304:36 or 64:1
5. $\frac{4}{9}$
6. $\frac{4}{9} = \frac{16}{A} \rightarrow A = 36\ cm^2$
7. $\frac{1}{5} = \frac{x}{20} \rightarrow x = 4\ cm$

7.15 Surface Area and Volume Review

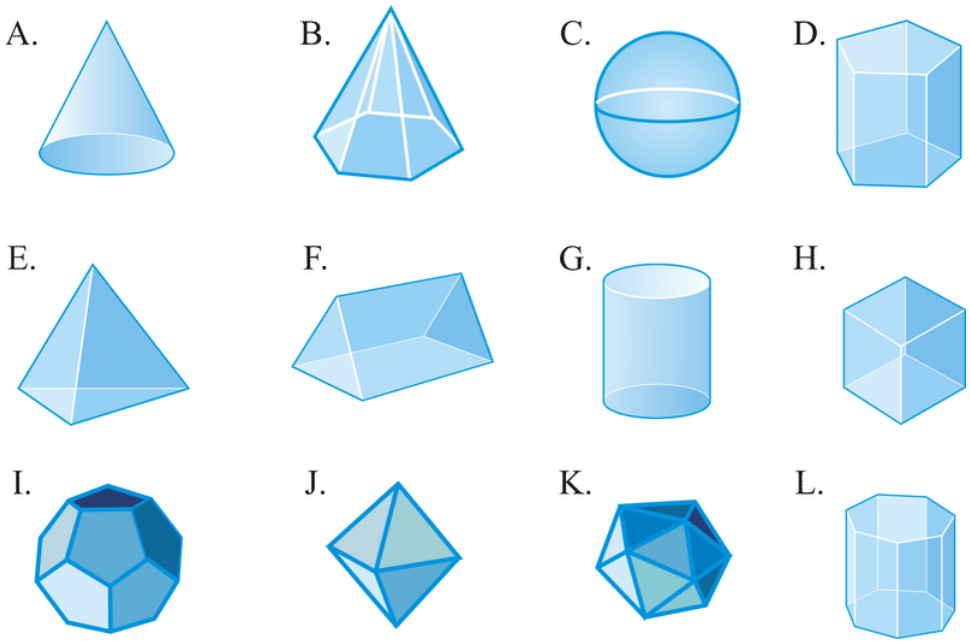
Keywords, Theorems, Formulas

- Polyhedron
- Face
- Edge
- Vertex
- Prism
- Pyramid
- Euler's Theorem
- Regular Polyhedron
- Regular Tetrahedron
- Cube
- Regular Octahedron
- Regular Dodecahedron
- Regular Icosahedron
- Cross-Section
- Net
- Lateral Face
- Lateral Edge
- Base Edge
- Right Prism
- Oblique Prism
- Surface Area
- Lateral Area
- Surface Area of a Right Prism
- Cylinder
- Surface Area of a Right Cylinder:
- Surface Area of a Regular Pyramid
- Cone
- Slant Height
- Surface Area of a Right Cone
- Volume
- Volume of a Cube Postulate
- Volume Congruence Postulate
- Volume Addition Postulate
- Volume of a Rectangular Prism
- Volume of a Prism
- Cavalieri's Principle
- Volume of a Cylinder
- Volume of a Pyramid
- Volume of a Cone
- Sphere
- Great Circle
- Surface Area of a Sphere
- Volume of a Sphere
- Similar Solids

- Surface Area Ratio
- Volume Ratio

Review Questions

Match the shape with the correct name.



1. Triangular Prism
2. Icosahedron
3. Cylinder
4. Cone
5. Tetrahedron
6. Pentagonal Prism
7. Octahedron
8. Hexagonal Pyramid
9. Octagonal Prism
10. Sphere
11. Cube
12. Dodecahedron

Match the formula with its description.

13. Volume of a Prism - A. $\frac{1}{3}\pi r^2 h$
14. Volume of a Pyramid - B. $\pi r^2 h$
15. Volume of a Cone - C. $4\pi r^2$
16. Volume of a Cylinder - D. $\frac{4}{3}\pi r^3$
17. Volume of a Sphere - E. $\pi r^2 + \pi rl$
18. Surface Area of a Prism - F. $2\pi r^2 + 2\pi rh$
19. Surface Area of a Pyramid - G. $\frac{1}{3}Bh$
20. Surface Area of a Cone - H. Bh

21. Surface Area of a Cylinder - I. $B + \frac{1}{2}Pl$
22. Surface Area of a Sphere - J. The sum of the area of the bases and the area of each rectangular lateral face.