Chapter Six

Transient and Steady State Responses

In control system analysis and design it is important to consider the complete system response and to design controllers such that a satisfactory response is obtained for all time instants $t \geq t_0$, where t_0 stands for the initial time. It is known that the system response has two components: transient response and steady state response, that is

$$y(t) = y_{tr}(t) + y_{ss}(t) \tag{6.1}$$

The transient response is present in the short period of time immediately after the system is turned on. If the system is asymptotically stable, the transient response disappears, which theoretically can be recorded as

$$\lim_{t \to \infty} y_{tr}(t) = 0 \tag{6.2}$$

However, if the system is unstable, the transient response will increase very quickly (exponentially) in time, and in the most cases the system will be practically unusable or even destroyed during the unstable transient response (as can occur, for example, in some electrical networks). Even if the system is asymptotically stable, the transient response should be carefully monitored since some undesired phenomena like high-frequency oscillations (e.g. in aircraft during landing and takeoff), rapid changes, and high magnitudes of the output may occur.

Assuming that the system is asymptotically stable, then the system response in the long run is determined by its steady state component only. For control systems it is important that steady state response values are as close as possible to desired ones (specified ones) so that we have to study the corresponding errors, which represent the difference between the actual and desired system outputs at steady state, and examine conditions under which these errors can be reduced or even eliminated.

In Section 6.1 we find analytically the response of a second-order system due to a unit step input. The obtained result is used in Section 6.2 to define important parameters that characterize the system transient response. Of course, these parameters can be exactly defined and determined only for second-order systems. For higher-order systems, only approximations for the transient response parameters can be obtained by using computer simulation. Several cases of real control systems and the corresponding MATLAB simulation results for the system transient response are presented in Sections 6.3 and 6.5. The steady state errors of linear control systems are defined in Section 6.4, and the feedback elements which help to reduce the steady state errors to zero are identified. In this section we also give a simplified version of the basic linear control problem originally defined in Section 1.1. Section 6.6 presents a summary of the main control system specifications and introduces the concept of control system sensitivity function. In Section 6.7 a laboratory experiment is formulated.

Chapter Objectives

The chapter has the main objective of introducing and explaining the concepts that characterize system transient and steady state responses. In addition, system dominant poles and the system sensitivity function are introduced in this chapter.

6.1 Response of Second-Order Systems

Consider the second-order feedback system represented, in general, by the block diagram given in Figure 6.1, where K represents the system static gain and T is the system time constant. It is quite easy to find the closed-loop transfer function of this system, that is

$$M(s) = \frac{Y(s)}{U(s)} = \frac{\frac{K}{T}}{s^2 + \frac{1}{T}s + \frac{K}{T}} \tag{6.3}$$

The closed-loop transfer function can be written in the following form

$$\frac{Y(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{6.4}$$

where from (6.3) and (6.4) we have

$$\zeta = \frac{1}{2\omega_n T}, \qquad \omega_n^2 = \frac{K}{T} \tag{6.5}$$

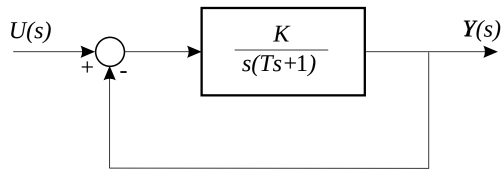

Figure 6.1: Block diagram of a general second-order system

Quantities ζ and ω_n are called, respectively, the *system damping ratio* and the *system natural frequency*. The system eigenvalues obtained from (6.4) are given by

$$\lambda_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d \tag{6.6}$$

where ω_d is the *system damped frequency*. The location of the system poles and the relation between damping ratio, natural and damped frequencies are given in Figure 6.2.

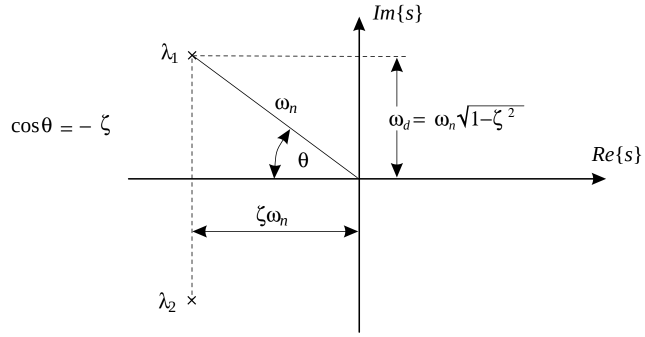

Figure 6.2: Second-order system eigenvalues in terms of parameters $\zeta, \omega_n, \omega_d$

In the following we find the closed-loop response of this second-order system due to a unit step input. Since the Laplace transform of a unit step is $1/s$ we have

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} \tag{6.7}$$

Depending on the value of the damping ratio ζ three interesting cases appear: (a) the critically damped case, $\zeta = 1$; (b) the over-damped case, $\zeta > 1$; and (c) the under-damped case, $\zeta < 1$. All of them are considered below. These cases are distinguished by the nature of the system eigenvalues. In case (a) the eigenvalues are multiple and real, in (b) they are real and distinct, and in case (c) the eigenvalues are complex conjugate.

(a) Critically Damped Case

For $\zeta = 1$, we get from (6.6) a double pole at $-\omega_n$. The corresponding output is obtained from

$$Y(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

which after taking the Laplace inverse produces

$$y(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t} \tag{6.8}$$

The shape of this response is given in Figure 6.3a, where the location of the system poles $(\lambda_1 = p_1, \lambda_2 = p_2)$ is also presented.

(b) Over-Damped Case

For the over-damped case, we have two real and asymptotically stable poles at $-\zeta\omega_n \pm \omega_d$. The corresponding closed-loop response is easily obtained from

$$Y(s) = \frac{1}{s} + \frac{k_1}{s + \zeta\omega_n + \omega_d} + \frac{k_2}{s + \zeta\omega_n - \omega_d}$$

as

$$y(t) = 1 + k_1 e^{-(\zeta\omega_n + \omega_d)t} + k_2 e^{-(\zeta\omega_n - \omega_d)t} \tag{6.9}$$

It is represented in Figure 6.3b.

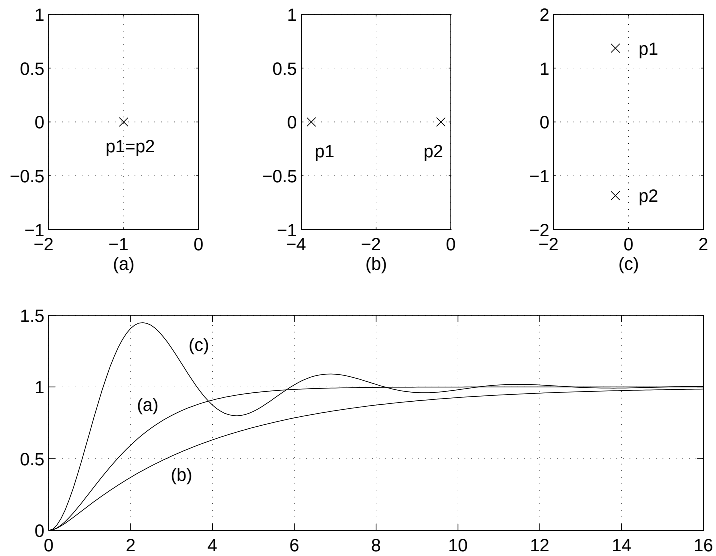

Figure 6.3: Responses of second-order systems and locations of system poles

(c) Under-Damped Case

This case is the most interesting and important one. The system has a pair of complex conjugate poles so that in the s-domain we have

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s + \zeta\omega_n + j\omega_d} + \frac{k_2^*}{s + \zeta\omega_n - j\omega_d} \tag{6.10}$$

Applying the Laplace transform it is easy to show (see Problem 6.1) that the system output in the time domain is given by

$$y(t) = 1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t - \theta\right] \tag{6.11}$$

where from Figure 6.2 we have

$$\cos\theta = -\zeta, \quad \sin\theta = \sqrt{1-\zeta^2}, \quad \tan\theta = \frac{\sqrt{1-\zeta^2}}{-\zeta} \tag{6.12}$$

The response of this system is presented in Figure 6.3c.

The under-damped case is the most common in control system applications. A magnified figure of the system step response for the under-damped case is presented in Figure 6.4. It will be used in the next section in order to define the transient response parameters. These parameters are important for control system analysis and design.

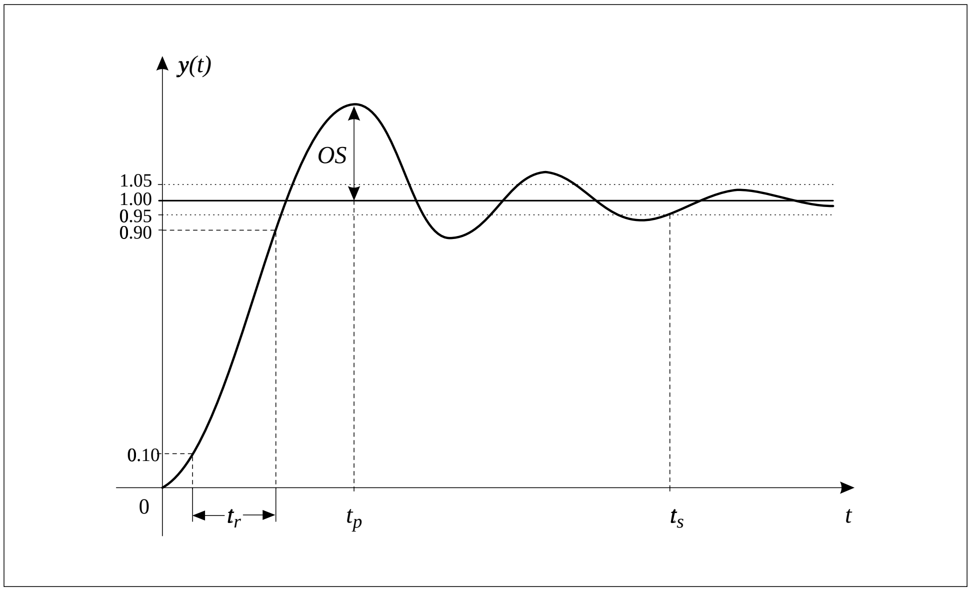

Figure 6.4: Response of an under-damped second-order system

6.2 Transient Response Parameters

The most important transient response parameters are denoted in Figure 6.4. These parameters are: response overshoot, settling time, peak time, and rise time.

The response overshoot can be obtained by finding the maximum of the function $y(t)$, as given by (6.11), with respect to time. This leads to

$$\frac{dy(t)}{dt} = -\frac{\zeta\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t - \theta\right) + \frac{\omega_d}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_d t - \theta\right) = 0$$

or

$$\zeta\omega_n \sin\left(\omega_d t - \theta\right) - \omega_d \cos\left(\omega_d t - \theta\right) = 0$$

which by using relations (6.12) and Figure 6.2 implies

$$\sin \omega_d t = 0 \tag{6.13}$$

It is left as an exercise to students to derive (6.13) (see Problem 6.2). From this equation we have

$$\omega_d t = i\pi, \quad i = 0, 1, 2, ... \tag{6.14}$$

The *peak time* is obtained for $i = 1$, i.e. as

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} \tag{6.15}$$

and times for other minima and maxima are given by

$$t_{ip} = \frac{i\pi}{\omega_d} = \frac{i\pi}{\omega_n \sqrt{1-\zeta^2}}, \quad i = 2, 3, 4, ... \tag{6.16}$$

Since the steady state value of $y(t)$ is $y_{ss}(t) = 1$, it follows that the *response overshoot* is given by

$$OS = y(t_p) - y_{ss}(t) = 1 + e^{-\zeta\omega_n t_p} - 1 = e^{-\zeta\omega_n t_p} = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \tag{6.17}$$

Overshoot is very often expressed in percent, so that we can define the *maximum percent overshoot* as

$$MPOS = OS(\%) = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100(\%) \tag{6.18}$$

From Figure 6.4, the expression for the response 5 percent *settling time* can be obtained as

$$y(t_s) = 1 + \frac{e^{-\zeta\omega_n t_s}}{\sqrt{1-\zeta^2}} = 1.05 \tag{6.19}$$

which for the standard values of ζ leads to

$$t_s = -\frac{1}{\zeta\omega_n} \ln\left(0.05\sqrt{1-\zeta^2}\right) \approx \frac{3}{\zeta\omega_n} \tag{6.20}$$

Note that in practice $0.5 < \zeta < 0.8$.

The response *rise time* is defined as the time required for the unit step response to change from 0.1 to 0.9 of its steady state value. The rise time is inversely proportional to the system bandwidth, i.e. the wider bandwidth, the smaller the rise time. However, designing systems with wide bandwidth is costly, which indicates that systems with very fast response are expensive to design.

Example 6.1: Consider the following second-order system

$$\frac{Y(s)}{U(s)} = \frac{4}{s^2 + 2s + 4}$$

Using (6.4) and (6.5) we get

$$\omega_n^2 = 4 \Rightarrow \omega_n = 2\,\mathrm{rad/s}, \quad 2\zeta\omega_n = 2 \Rightarrow \zeta = 0.5$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{3}\ \mathrm{rad/s}$$

The peak time is obtained from (6.15) as

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{3}} = 1.82\,\mathrm{s}$$

and the settling time, from (6.20), is found to be

$$t_s \approx \frac{3}{\zeta\omega_n} = 3\,\mathrm{s}$$

The maximum percent overshoot is equal to

$$MPOS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}100(\%) = 16.3\%$$

The step response of this system obtained by the MATLAB function `[y,x]=step(num,den,t)` with `t=0:0.1:5` is presented in Figure 6.5. It can be seen that the analytically obtained results agree with the results presented in Figure 6.5. From Figure 6.5 we are able to estimate the rise time, which in this case is approximately equal to $t_r \approx 0.8\,\mathrm{s}$.

Note that the response rise time can be very precisely determined by using MATLAB (see Problem 6.15). Also, MATLAB can be used to find accurately the transient response settling time (see Problem 6.14).

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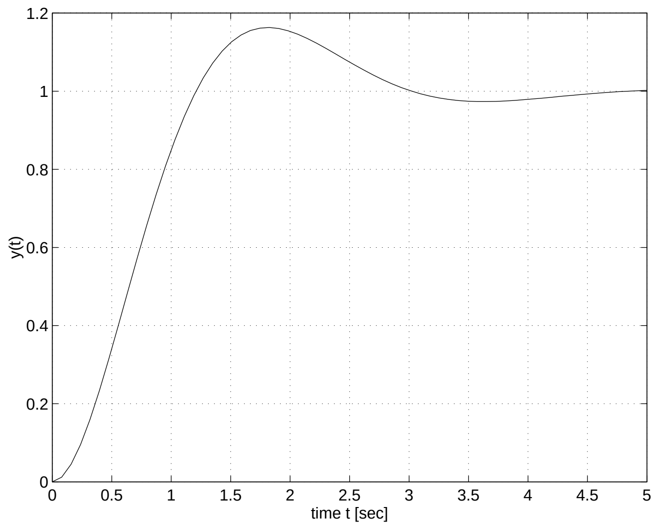

Figure 6.5: System step response for Example 6.1

6.3 Transient Response of High-Order Systems

In the previous section we have been able to precisely define and determine parameters that characterize the system transient response. This has been possible due to the fact that the system under consideration has been of order two only. For higher-order systems, analytical expressions for the system response are not generally available. However, in some cases of high-order systems one is able to determine approximately the transient response parameters.

A particularly important is the case in which an asymptotically stable system has a pair of complex conjugate poles (eigenvalues) much closer to the imaginary axis than the remaining poles. This situation is represented in Figure 6.6. The system poles far to the left of the imaginary axis have large negative real parts so that they decay very quickly to zero (as a matter of fact, they decay exponentially with $e^{\sigma_i t}$, where σ_i are negative real parts of the corresponding poles). Thus, the system response is dominated by the pair of complex conjugate poles closest to the imaginary axis since they decay slowest, as they have relatively small real parts. Hence, these poles are called the *dominant system poles*.

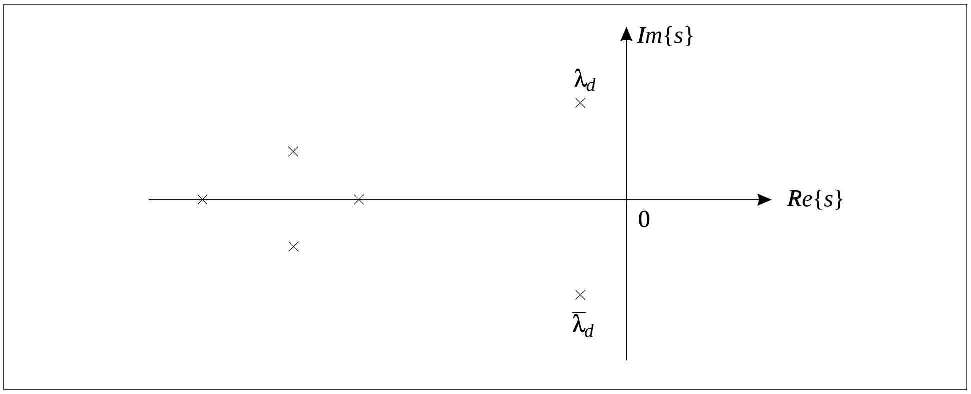

Figure 6.6: Complex conjugate dominant system poles

This analysis can be also justified by using the closed-loop system transfer function. Consider, for example, a system described by its transfer function as

$$M(s)=\frac{Y(s)}{U(s)}=\frac{12600(s+1)}{(s+3)(s+10)(s+60)(s+70)}$$

Since the poles at –60 and –70 are far to the left, their contribution to the system response is negligible (they decay very quickly to zero as e^{-60t} and e^{-70t}). The transfer function can be formally simplified as follows

$$\begin{aligned}M(s)&=\frac{12600(s+1)}{(s+3)(s+10)60\left(\frac{s}{60}+1\right)70\left(\frac{s}{70}+1\right)}\\&\approx\frac{3(s+1)}{(s+3)(s+10)}=M_r(s)\end{aligned} \tag{6.21}$$

Example 6.2: In this example we use MATLAB to compare the step responses of the original and reduced-order systems whose transfer functions are given in (6.21). The results obtained for $y(t)$ and $y_r(t)$ are given in Figure 6.7. It can be seen from this figure that step responses for the original and reduced-order (approximate) systems almost overlap.

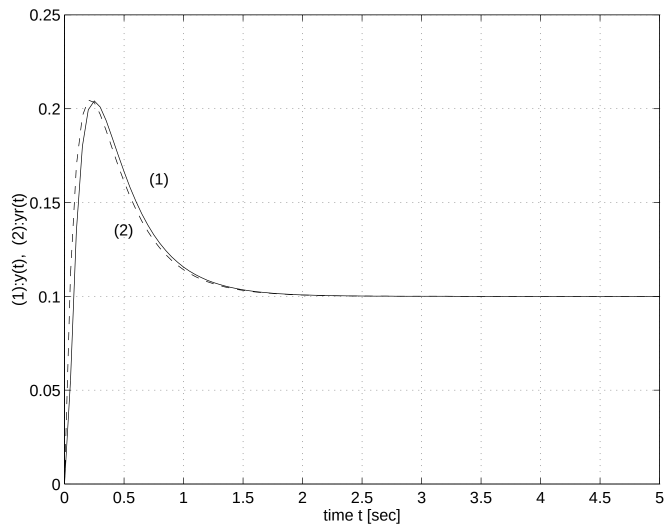

Figure 6.7: System step responses for the original (1) and reduced-order approximate (2) systems

The corresponding responses are obtained by the following sequence of MATLAB functions

```
z=-1;
p=[-3 –10 –60 –70];
k=12600;
[num,den]=zp2tf(z,p,k);
t=0:0.05:5;
[y,x]=step(num,den,t);
zr=-1;
pr=[-3 –10];
kr=3;
[numr,denr]=zp2tf(zr,pr,kr);
[yr,xr]=step(numr,denr,t);
plot(t,y,t,yr,'- –');
xlabel('time t [sec]');
ylabel('(1):y(t), (2):yr(t)');
grid;
text(0.71,0.16,'(1)');
```

```
text(0.41,0.13,'(2)');
```

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Similarly one can neglect the complex conjugate non-dominant poles, as is demonstrated in the next example.

Example 6.3: Consider the following transfer function containing two pairs of complex conjugate poles

$$M(s) = \frac{20(s+2)}{(s+1-j)(s+1+j)(s+10-j10)(s+10+j10)}$$

and the corresponding approximate reduced-order transfer function obtained by

$$\begin{aligned} M(s) &= \frac{20(s+2)}{(s^2+2s+2)(s^2+20s+200)} \\ &= \frac{20(s+2)}{(s^2+2s+2)200\left(\frac{s^2}{200}+\frac{20s}{200}+1\right)} \approx \frac{(s+2)}{10(s^2+2s+2)} = M_r(s) \end{aligned}$$

The step responses of the original and approximate reduced-order systems are presented in Figure 6.8.

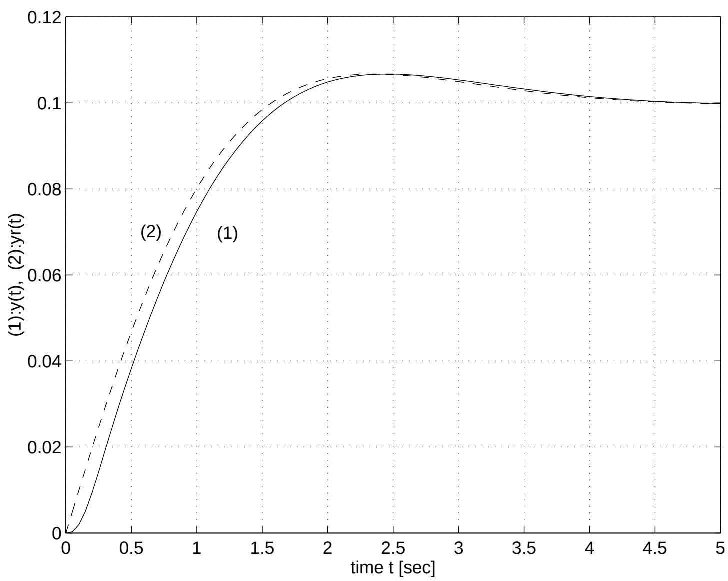

Figure 6.8: System step responses for the original (1) and approximate (2) systems with complex conjugate poles

It can be seen from this figure that a very good approximation for the step response is obtained by using the approximate reduced-order model.

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However, the above technique is rather superficial. In addition, for multi-input multi-output systems this procedure becomes computationally cumbersome. In that case we need a more systematic method. In the control literature one is able to find several techniques used for the system order reduction. One of them, the *method of singular perturbations* (Kokotović and Khalil, 1986; Kokotović *et al.*, 1986), is presented below. The method systematically generalizes the previously explained idea of dominant poles.

The eigenvalues of certain systems (having large and small time constants, or slow and fast system modes) are clustered in two or several groups (see Figure 6.9). According to the theory of singular perturbations, if it is possible to find an isolated group of poles (eigenvalues) closest to the imaginary axis, then the system response will be predominantly determined by that group of eigenvalues.

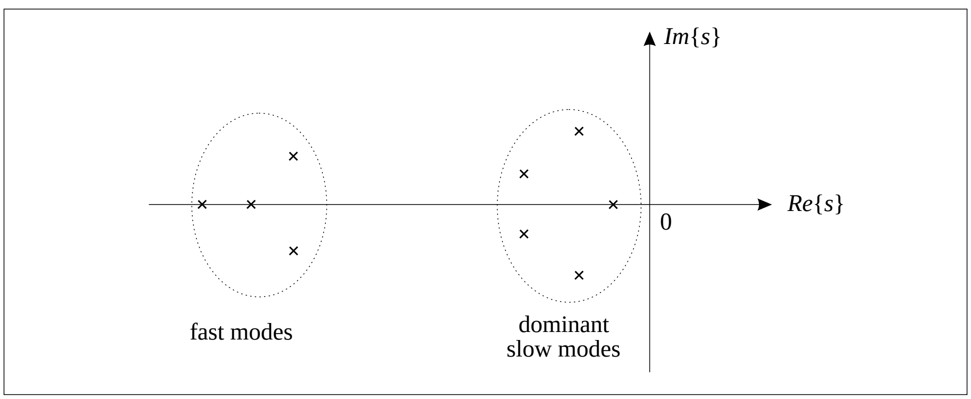

Figure 6.9: System eigenvalues clustered in two disjoint groups

The state space form of such systems is given by

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \frac{1}{\epsilon}\mathbf{A}_3 & \frac{1}{\epsilon}\mathbf{A}_4 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 \\ \frac{1}{\epsilon}\mathbf{B}_2 \end{bmatrix} u$$
$$\mathbf{y} = \mathbf{C}_1\mathbf{x}_1 + \mathbf{C}_2\mathbf{x}_2 \tag{6.22}$$

where ϵ is a small positive parameter. It indicates that the time derivatives for state variables $\mathbf{x}_2$ are large, so that variables $\mathbf{x}_2$ change quickly, in contrast to variables $\mathbf{x}_1$, which are slow. If the state variables $\mathbf{x}_2$ are asymptotically stable, then they decay very quickly, so that after the fast dynamics disappear $(\dot{\mathbf{x}}_2 = 0)$, we get an approximation for the fast subsystem as

$$0 = \mathbf{A}_3\mathbf{x}_{1app} + \mathbf{A}_4\mathbf{x}_{2app} + \mathbf{B}_2\mathbf{u} \tag{6.23}$$

From this equation we are able to find $\mathbf{x}_{2app}$ (assuming that the matrix $\mathbf{A}_4$ in nonsingular, which is the standard assumption in the theory of singular perturbations; Kokotović *et al.*, 1986) as

$$\mathbf{x}_{2app} = -\mathbf{A}_4^{-1}(\mathbf{A}_3\mathbf{x}_{1app} + \mathbf{B}_2\mathbf{u}) \tag{6.24}$$

Substituting this approximation in (6.22), we get an approximate reduced-order slow subsystem as

$$\begin{aligned} \dot{\mathbf{x}}_{1app} &= \mathbf{A}_s\mathbf{x}_{1app} + \mathbf{B}_s\mathbf{u} \\ \mathbf{y}_{app} &= \mathbf{C}_s\mathbf{x}_{1app} + \mathbf{D}_s\mathbf{u} \end{aligned}$$
$$\mathbf{A}_s = \mathbf{A}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{A}_3, \quad \mathbf{B}_s = \mathbf{B}_1 - \mathbf{A}_2\mathbf{A}_4^{-1}\mathbf{B}_2$$
$$\mathbf{C}_s = \mathbf{C}_1 - \mathbf{C}_2\mathbf{A}_4^{-1}\mathbf{A}_3, \quad \mathbf{D}_s = -\mathbf{C}_2\mathbf{A}_4^{-1}\mathbf{B}_2 \tag{6.25}$$

From the theory of singular perturbations it is known that $\mathbf{x}_1(t)$ is close to $\mathbf{x}_{1app}(t)$ for every $t \geq t_0$, and $\mathbf{y}_{app}(t)$ is a good approximation for $\mathbf{y}(t)$ for $t \geq t_1 > t_0$, where $t \geq t_1$ indicates the fact that this approximation becomes valid shortly after the fast transient disappears (Kokotović *et al.*, 1986).

Example 6.4: Consider a mathematical model of a singularly perturbed fluid catalytic cracker considered in Arkun and Ramakrishnan (1983). The problem matrices are given by

$$\mathbf{A} = \begin{bmatrix} -16.11 & -0.39 & 27.2 & 0 & 0 \\ 0.01 & -16.99 & 0 & 0 & 12.47 \\ 15.11 & 0 & -53.6 & -16.57 & 71.78 \\ -53.36 & 0 & 0 & -107.2 & 232.11 \\ 2.27 & 69.1 & 0 & 2.273 & -102.99 \end{bmatrix}$$

$$\mathbf{B}^T = \begin{bmatrix} 11.12 & -3.61 & -21.91 & -53.6 & 69.1 \\ -12.6 & 3.36 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of this system are

$$\lambda(\mathbf{A}) = \{-2.85, -7.78, -74.32, -82.86, -129.08\}$$

which indicates that the system has two slow (–2.85 and –7.78) and three fast modes. The small parameter ϵ represents the separation of system eigenvalues into two disjoint groups. It can be roughly estimated as $\epsilon \approx 7.78/74.32 \approx 0.1$ (the ratio of the smallest and largest eigenvalues in the given slow and fast subsets). We use MATLAB to partition matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as follows

```
eps=0.1;
A1=A(1:2,1:2);
A2=A(1:2,3:5);
A3=A(3:5,1:2)*eps;
A4=A(3:5,3:5)*eps;
B1=B(1:2,1:2);
B2=B(3:5,1:2)*eps;
C1=C(1:2,1:2);
C2=C(1:2,3:5);
```

The slow subsystem matrices, obtained from (6.25), are given by

$$\mathbf{A}_s = \begin{bmatrix} -4.0452 & 12.4474 \\ 0.1548 & -8.2035 \end{bmatrix}, \quad \mathbf{B}_s = \begin{bmatrix} 16.8321 & -12.6 \\ 5.0320 & 3.36 \end{bmatrix}$$

$$\mathbf{C}_s = \begin{bmatrix} 0.0116 & 0.7046 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D}_s = \begin{bmatrix} 0.693 & 0 \\ 0 & 0 \end{bmatrix}$$

The eigenvalues of the slow subsystem matrix are $\lambda(\mathbf{A}_s) = \{-3.6245, -8.6243\}$. This reflects the impact of the fast modes on the slow modes so that the original slow eigenvalues located at –2.85 and –7.78 are now changed to –3.6245 and –8.6243. In Figure 6.10 the outputs of the original (solid lines) and reduced (dashed lines) systems are presented in the time interval specified by MATLAB as `t=0:0.025:5`. It can be seen that the output responses of these systems are remarkably close to each other.

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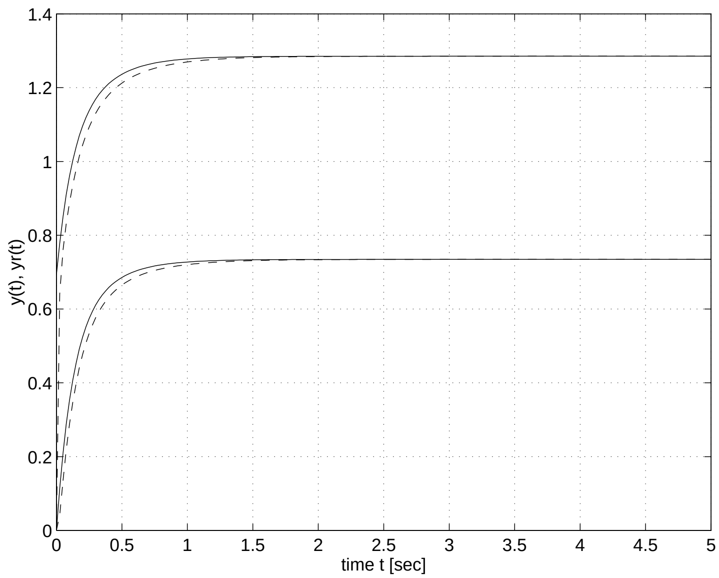

Figure 6.10: Outputs of the original fifth-order system and reduced second-order system obtained by the method of singular perturbations

Models of many real physical linear control systems that have the singularly perturbed structure, displaying slow and fast state variables, can be found in Gajić and Shen (1993).

A MATLAB laboratory experiment involving system order reduction and comparison of corresponding system trajectories and outputs of a real physical control system by using the method of singular perturbations is formulated in Section 6.7.

6.4 Steady State Errors

The response of an asymptotically stable linear system is in the long run determined by its steady state component. During the initial time interval the transient response decays to zero, according to the asymptotic stability requirement (6.2), so that in the remaining part of the time interval the system response is represented by its steady state component only. Control engineers are interested in having steady state responses as close as possible to the desired ones so that we define the so-called steady state errors, which represent the differences at steady state of the actual and desired system responses (outputs).

Before we proceed to steady state error analysis, we introduce a simplified version of the basic linear control system problem defined in Section 1.1.

Simplified Basic Linear Control Problem

As defined in Section 1.1 the basic linear control problem is still very difficult to solve. A simplified version of this problem can be formulated as follows. Apply to the system input a time function equal to the desired system output. This time function is known as the system's *reference input* and is denoted by $r(t)$. Note that $r(t) = u(t)$. Compare the actual and desired outputs by feeding back the actual output variable. The difference $y(t) - r(t) = e(t)$ represents the error signal. Use the error signal together with simple controllers (if necessary) to drive the system under consideration such that $e(t)$ is reduced as much as possible, at least at steady state. If a simple controller is used in the feedback loop (Figure 6.11) the error signal has to be slightly redefined, see formula (6.26).

In the following we use this simplified basic linear control problem in order to identify the structure of controllers (feedback elements) that for certain types of reference inputs (desired outputs) produce zero steady state errors.

Consider the simplest feedback configuration of a single-input single-output system given in Figure 6.11.

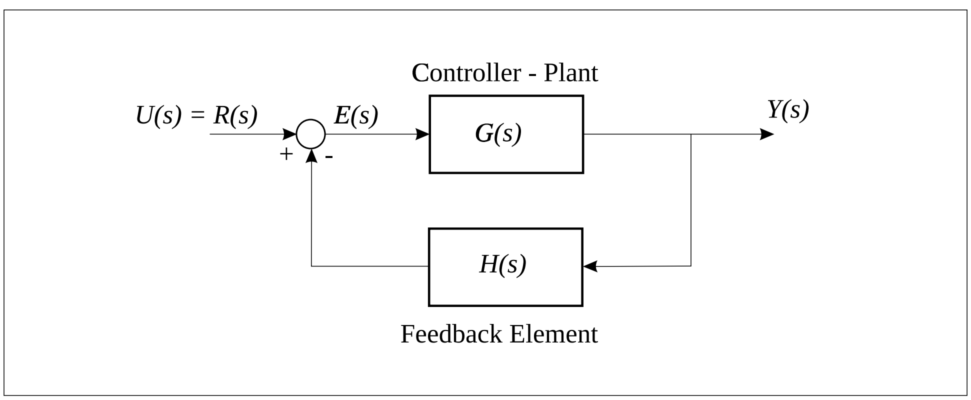

Figure 6.11: Feedback system and steady state errors

Let the input signal $U(s) = R(s)$ represent the Laplace transform of the desired output (in this feedback configuration the desired output signal is used as an input signal); then for $H(s) = 1$, we see that in Figure 6.11 the quantity $E(s)$ represents the difference between the desired output $R(s) = U(s)$ and the actual output $Y(s)$. In order to be able to reduce this error as much as possible, we allow dynamic elements in the feedback loop. Thus, $H(s)$ as a function of s has to be chosen such that for the given type of reference input, the error, now defined by

$$E(s) = R(s) - H(s)Y(s) \tag{6.26}$$

is eliminated or reduced to its minimal value at steady state.

From the block diagram given in Figure 6.11 we have

$$E(s) = R(s) - H(s)G(s)E(s)$$

so that the expression for the error is given by

$$E(s) = \frac{R(s)}{1 + H(s)G(s)} \tag{6.27}$$

The steady state error component can be obtained by using the final value theorem of the Laplace transform as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \{sE(s)\} = \lim_{s\to 0} \left\{ \frac{sR(s)}{1 + H(s)G(s)} \right\} \tag{6.28}$$

This expression will be used in order to determine the nature of the feedback element $H(s)$ such that the steady state error is reduced to zero for different types of desired outputs. We will particularly consider step, ramp, and parabolic functions as desired system outputs.

Before we proceed to the actual steady state error analysis, we introduce one additional definition.

Definition 6.1 *The type of feedback control system* is determined by the number of poles of the open-loop feedback system transfer function located at the origin, i.e. it is equal to j, where j is obtained from

$$G(s)H(s) = \frac{K(s + z_1)\cdots(s + z_m)}{s^j(s + p_1)(s + p_2)\cdots(s + p_{n-j})} \tag{6.29}$$

Now we consider the steady state errors for different desired outputs, namely unit step, unit ramp, and unit parabolic outputs.

Unit Step Function as Desired Output

Assuming that our goal is that the system output follows as close as possible the unit step function, i.e. $U(s) = R(s) = 1/s$, we get from (6.28)

$$e_{ss} = \lim_{s\to 0}\left\{\frac{s}{1+H(s)G(s)}\frac{1}{s}\right\} = \frac{1}{1+\lim_{s\to 0}\{H(s)G(s)\}} = \frac{1}{1+K_p} \tag{6.30}$$

where K_p is known as the *position constant* and from (6.30) is given by

$$K_p = \lim_{s\to 0}\{H(s)G(s)\} \tag{6.31}$$

It can be seen from (6.30) that the steady state error for the unit step reference is reduced to zero for $K_p = \infty$. Examining closely (6.31), taking into account (6.29), we see that this condition is satisfied for $j \geq 1$.

Thus, we can conclude that the feedback type system of order at least one allows the system output at steady state to track the unit step function perfectly.

Unit Ramp Function as Desired Output

In this case the steady state error is obtained as

$$e_{ss} = \lim_{s\to 0}\{sE(s)\} = \lim_{s\to 0}\left\{\frac{s}{1+H(s)G(s)}\frac{1}{s^2}\right\} = \frac{1}{\lim_{s\to 0}\{sH(s)G(s)\}} = \frac{1}{K_v} \tag{6.32}$$

where

$$K_v = \lim_{s\to 0}\{sH(s)G(s)\} \tag{6.33}$$

is known as the *velocity constant*. It can be easily concluded from (6.29) and (6.33) that $K_v = \infty$, i.e. $e_{ss} = 0$ for $j \geq 2$. Thus, systems having two and more pure integrators ($1/s$ terms) in the feedback loop will be able to perfectly track the unit ramp function as a desired system output.

Unit Parabolic Function as Desired Output

For a unit parabolic function we have $R(s) = 2/s^3$ so that from (6.28)

$$e_{ss} = \lim_{s\to 0}\left\{\frac{s}{1+H(s)G(s)}\frac{2}{s^3}\right\} = \frac{2}{\lim_{s\to 0}\{s^2H(s)G(s)\}} = \frac{2}{K_a} \tag{6.34}$$

where the so-called *acceleration constant*, K_a, is defined by

$$K_a = \lim_{s\to 0}\left\{s^2H(s)G(s)\right\} \tag{6.35}$$

From (6.29) and (6.35), we conclude that $K_a = \infty$ for $j \geq 3$, i.e. the feedback loop must have three pure integrators in order to reduce the corresponding steady state error to zero.

Example 6.5: The steady state errors for a system that has the open-loop transfer function as

$$H(s)G(s) = \frac{20(s+1)}{s(s+2)(s+5)}$$

are

$$\begin{aligned} K_p &= \infty \;\; \Rightarrow \;\; e_{ss} = 0 \quad (\text{step}) \\ K_v &= 2 \;\; \Rightarrow \;\; e_{ss} = 0.5 \;\; (\text{ramp}) \\ K_a &= 0 \;\; \Rightarrow \;\; e_{ss} = \infty \quad (\text{parabolic}) \end{aligned}$$

Since the open-loop transfer function of this system has one integrator the output of the closed-loop system can perfectly track only the unit step.

◇

Example 6.6: Consider the second-order system whose open-loop transfer function is given by

$$H(s)G(s) = \frac{(s+3)}{(s+1)(s+2)}$$

The position constant for this system is $K_p = 1.5$ so that the corresponding steady state error is

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+1.5} = 0.4$$

The unit step response of this system is presented in Figure 6.12, from which it can be clearly seen that the steady state output is equal to 0.6; hence the steady state error is equal to $1 - 0.6 = 0.4$.

◇

Note that the transient analysis and the study of steady state errors can be performed for discrete-time linear systems in exactly the same way as was used for continuous-time systems. The steady state errors for discrete-time systems are obtained by using the final value theorem of the $\mathcal{Z}$-transform and following the same procedure as in Section 6.4.

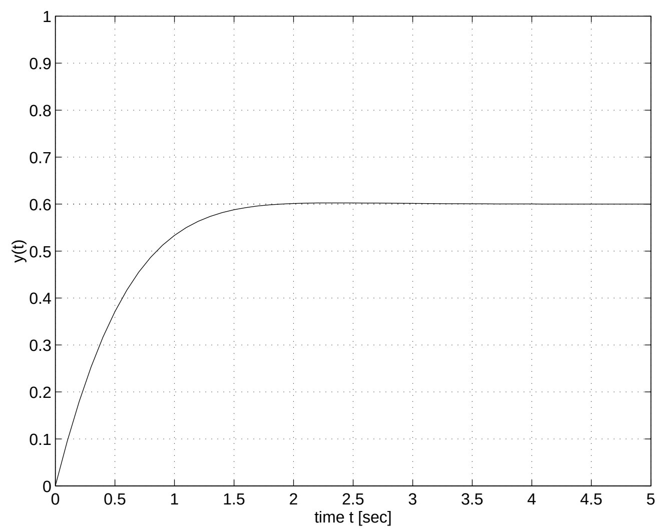

Figure 6.12: System step response for Example 6.6

6.5 Response of High-Order Systems by MATLAB

For high-order systems, analytical expressions for system step responses are quite complex. However, we are still able to determine approximately the response parameters in many cases. In this section, we plot the unit step response of a high-order control system by using MATLAB and determine approximately from the graph obtained some of transient response parameters and the corresponding steady state error.

Consider the mathematical model of a synchronous machine connected to an infinite bus. The matrix $\mathbf{A}$ of this seventh-order system is given in Problem 3.28. The remaining matrices are chosen as

$$\mathbf{B} = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 20]^T$$
$$\mathbf{C} = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1], \quad \mathbf{D} = 0$$

For a system represented in the state space form, the step response is obtained by using the MATLAB function `[y,x]=step(A,B,C,D,1,t)`, where `1` indicates that the step signal is applied to the first system input and `t` represents time. The step response of this system is given in Figures 6.13 and 6.14.

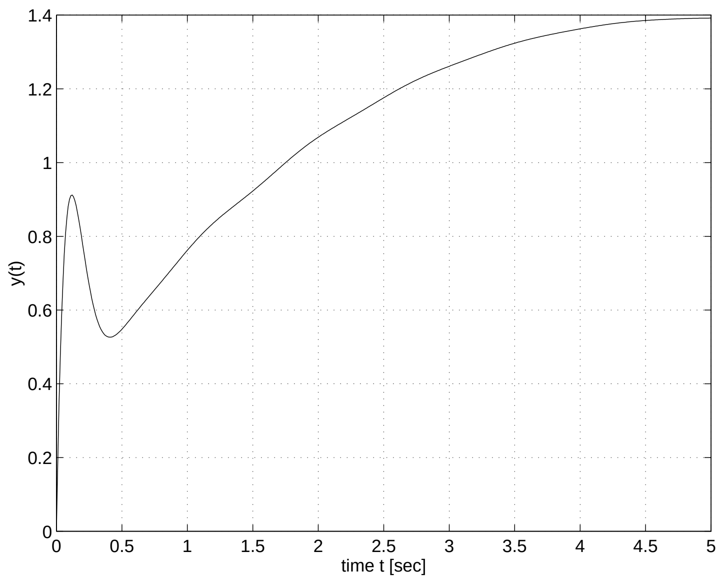

Figure 6.13: Step response of a synchronous machine for $t \in [0, 5]$

Figure 6.13 is obtained for the initial time interval of $t \in [0, 5]$. It shows the actual response shape, but it is hard to draw conclusions about the transient response parameters from this figure. However, if we plot the system step response for time interval $t \in [0, 40]$, then a response shape very similar to that in Figure 6.4 is obtained. It is pretty straightforward to read from Figure 6.14 that the peak time is $t_p \approx 5\,\mathrm{s}$, the overshot is approximately equal to 0.4, the rise time is $t_r \approx 2\,\mathrm{s}$, and the settling time is roughly equal to $12\,\mathrm{s}$. By using MATLAB, it is obtained that $y_{ss} \to 1.0226$ so that the steady state error is $e_{ss} \to 0.0226$. This can be obtained either by finding $y(t)$ for some t long enough or by using the final value theorem of the Laplace transform as

$$y_{ss} = \lim_{s \to 0} \{sY(s)\} = \lim_{s \to 0} \{sM(s)U(s)\} = \lim_{s \to 0} \left\{ sM(s)\frac{1}{s} \right\} = M(0) \quad (6.36)$$

where $M(s)$ is the system closed-loop transfer function, which can be obtained by using MATLAB as `[num,den]=ss2tf(A,B,C,D,1)`. Then, for this particular example of order seven, we have `yss=num(1,8)/den(1,8)`. Note that `num(1,8)=5048.8` and `den(1,8)=4937.2`.

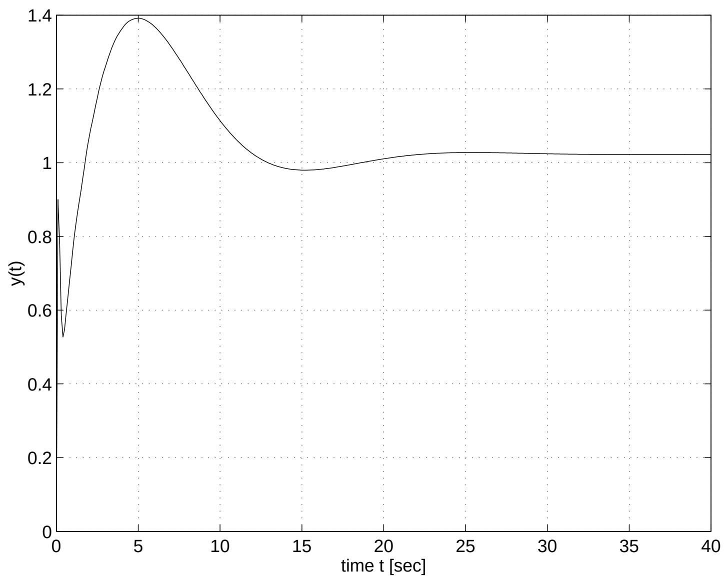

Figure 6.14: Step response of a synchronous machine for $t \in [0, 40]$

6.6 Control System Performance Specifications

Control systems should satisfy certain specifications such that systems under consideration have the desired behavior for both transient and steady state responses. If the desired specifications are not met, controllers should be designed and placed either in the forward path or in the feedback loop such that the desired specifications are obtained. The desired specifications include the required values (or upper and/or lower limits) of already defined quantities such as phase and gain margins, settling time, rise time, peak time, maximum percent overshoot, and steady state errors. Additional specifications can be defined in the frequency domain like control system frequency bandwidth, resonant frequency, and resonance peak, which will be presented in Chapter 9.

Of course, it is impossible to meet all the specifications mentioned above. Sometimes some requirements are contradictory and sometimes some of them are not affordable. Thus, control engineers have to compromise while trying to satisfy all of imposed control system requirements. Fortunately, we are able to identify the most important ones. First of all, *systems must be stable*; hence the main goal of controller design is to stabilize the system under consideration, in other words, the system phase and gain stability margins should be handled with increased care. Secondly, *systems should have limited overshoot and settling time* and *the steady state errors should be kept within admissible bounds*. In the most of cases only these specifications will be taken into account while designing controllers in Chapters 8 and 9.

In addition to the above specifications control systems should be insensitive to variation of system parameters and components. Linear models are very often obtained by performing linearization of nonlinear models, i.e. the linear models are in many cases just approximations of nonlinear systems at given operating points. That is why it is required that controllers used for control of such systems be robust, i.e. they should produce satisfactory results for broad families of linear systems that are close to linearized systems at given operating points. The importance of control system sensitivity to parameter changes has been recognized since the beginning of modern control theory (Tomović, 1963; Kokotović and Rutman, 1965; Tomović and Vukobratović, 1972). Control system robustness has been the trend of the eighties and nineties (Morari and Zafiriou, 1989; Chiang and Safonov, 1992; Grimble, 1994; Green and Limebeer, 1995). Studying these control system specifications (reduced sensitivity and increased robustness) in detail is beyond the scope of this book. Here, we just introduce the basic system sensitivity result and define the control system sensitivity function.

Consider the feedback control system given in Figure 6.11. The plant transfer function $G(s)$ is obtained through mathematical modeling either analytically or experimentally and is assumed to be known. However, due to plant parameter changes, e.g. due to components aging or parameter uncertainties, the actual plant transfer function is in fact $G_a(s) = G(s) + \Delta G(s)$, where $\Delta G(s)$ represents the absolute error of the plant transfer function, so that the corresponding relative error is $\Delta G(s)/G(s)$.

The closed-loop transfer function for the system in Figure 6.11 is given by

$$M(s) = \frac{G(s)}{1 + G(s)H(s)} \tag{6.37}$$

and the actual closed-loop system transfer function is

$$M_a(s) = \frac{G_a(s)}{1 + G_a(s)H(s)} \tag{6.38}$$

The corresponding absolute transfer function error is obtained as

$$\begin{aligned} \Delta M(s) &= M_a(s) - M(s) = M(s)\left(\frac{M_a(s)}{M(s)} - 1\right) \\ &= M(s)\frac{1}{1 + G_a(s)H(s)}\frac{G_a(s) - G(s)}{G(s)} \end{aligned} \tag{6.39}$$

This leads to

$$\frac{\Delta M(s)}{M(s)} = \frac{1}{1 + G_a(s)H(s)}\frac{\Delta G(s)}{G(s)} = S_a(s)\frac{\Delta G(s)}{G(s)} \tag{6.40}$$

where

$$S_a(s) = \frac{1}{1 + G_a(s)H(s)} \tag{6.41}$$

represents the so-called control *system sensitivity function*. Note that the sensitivity function depends on the complex frequency s. It follows from formula (6.40) that the magnitude of the sensitivity function should be chosen to be as small as possible over the frequency range of interest. Since from (6.41) $|S_a(s)| < 1$, it follows that the closed-loop relative transfer function error is reduced compared to the open-loop relative plant transfer function error. In conclusion, *feedback alone reduces system sensitivity to system parameter variations.*

Finally, let us point out that feedback also decreases system sensitivity to external disturbances. This problem has been already tacitly studied in Section 2.2—see the block diagram presented in Figure 2.3 and formula (2.18).

6.7 MATLAB Laboratory Experiment

Part 1. Consider a general second-order system given in (6.3). Choose values for parameters K and T such that all three cases appear (over-damped, under-damped, and critically damped). Using MATLAB, plot the unit step responses for all cases. Find the transient response parameters for the under-damped case.

Part 2. Consider the second-order system as given by (6.3) with $T = 1$. Take several values for the static gain K such that $1 = K_1 < K_2 < K_3 < K_4 = 50$ and plot the corresponding unit step responses. Draw conclusions about the impact of K on the maximum percent overshoot and the steady state errors.

Part 3. Use the method of dominant complex conjugate poles in order to approximate the step response for the second output of the F-8 aircraft, given in Section 3.5.2, by an equivalent second-order system. Hint: Find the fourth-order transfer function for the second output and reduce it to the second-order transfer function by following the procedure of Example 6.3. Note that the same reduction technique has to be applied to the transfer function zeros. In that respect eliminate the pair of complex conjugate zeros.

Part 4. Use MATLAB in order to find approximately the transient response parameters and the steady state error for the synchronous machine considered in Section 6.5, this time with the matrix $\mathbf{C}$ equal to

$$\mathbf{C} = [0 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \quad 1]$$

Hint: Use `t=0:0.5;30` while plotting the step response. Find the exact value for y_{ss} by using formula (6.36).

Part 5.[1] Use the method of singular perturbations in order to reduce the fifth-order model of a voltage regulator considered in Kokotović (1972) to an equivalent second-order slow model. The voltage regulator matrices are given by

$$\mathbf{A} = \begin{bmatrix} -0.2 & 0.5 & 0 & 0 & 0 \\ 0 & -0.5 & 1.6 & 0 & 0 \\ 0 & 0 & -14.28 & 85.71 & 0 \\ 0 & 0 & 0 & -25 & 75 \\ 0 & 0 & 0 & 0 & -10 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 30 \end{bmatrix}$$

$$\mathbf{C} = [1 \quad 0 \quad 0 \quad 0 \quad 0], \qquad \mathbf{D} = 0$$

Use MATLAB to partition this system as a singularly perturbed system having two slow and three fast modes with `A1=A(1:2,1:2)` and so on. Take $\epsilon = 0.05$. Show that the step responses of the original and reduced-order systems are very close to each other by plotting them on the same graph.

[1] This part is optional.

6.8 References

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6.9 Problems

6.1 Find expressions for constants k_1 and k_2 in (6.10) and derive formula (6.11).

6.2 Derive formula (6.13).

6.3 Find the transient response parameters for the following second-order systems

$$\text{(a)} \quad \frac{Y(s)}{U(s)} = \frac{5}{s^2 + 2s + 2}$$

$$\text{(b)} \quad \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 2s + 1}$$

$$\text{(c)} \quad \frac{Y(s)}{U(s)} = \frac{10}{s^2 + 6s + 8}$$

6.4 Consider the second-order system that has a zero in its transfer function, that is

$$\frac{Y(s)}{U(s)} = \frac{5(s+1)}{s^2 + 2s + 2}$$

Use the Laplace transform to obtain its step response. Find the transient response parameters and the steady state error for a unit step. Compare the step responses of this system and the system considered in Problem 6.3a. Plot the corresponding responses by using MATLAB.

6.5 Determine the steady state errors for unit step, unit ramp, and unit parabolic inputs of a unity feedback control system having the plant transfer function

$$G(s) = \frac{50(s+1)}{s^2(s+3)(s+5)(s+10)}$$

6.6 Compare the steady state errors for unit feedback control systems represented by

$$G_1(s) = \frac{10}{s(s+2)}, \quad G_2(s) = \frac{5}{s^2(s+1)(2s+1)}$$

assuming that the input signal (desired output) is given by $2t^2 - 3t - 2,\ t > 0$.

6.7 For a linear system with a unit feedback represented by

$$G(s) = \frac{10}{(s+1)(s+5)}$$

calculate steady state errors, pick time, 5 percent settling time, and maximum percent overshoot.

6.8 Find the values for the static gain K and the time constant T such that the second-order system represented by

$$H(s)G(s) = \frac{K}{s(Ts+5)}, \qquad H(s) = 1$$

has prespecified values for the maximum percent overshoot and the peak time.

6.9 Solve Problem 6.8 by requiring that the peak time and settling time be prespecified.

6.10 Find the closed-loop system transfer function(s) for the F-8 aircraft from Section 3.5.2 by using MATLAB, and calculate the steady state error(s) due to a unit step input by using formula (6.36). Note that there are two outputs in this problem.

6.11 Repeat Problem 6.10 for the ninth-order model of a power system having two inputs and four outputs. This model is given in Section 5.8, Part 3.

6.12 Repeat Problem 6.10 for the fifth-order distillation column considered in Problem 5.19 with the following output matrices

$$\mathrm{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

6.13 Generalize the order-reduction procedure by the method of singular perturbations, presented in Section 6.3, to the case when the output equation has the matrix **D** different from zero.

6.14 Write a MATLAB program for finding the transient response settling time. Hint: See the MATLAB program presented in Example 8.8.

6.15 Write a MATLAB program for finding the transient response rise time. Hint: First solve Problem 6.14.