Chapter 2 Unit 2: Exponents

Chapter Outline

KEY STANDARDS

Work with radicals and integer exponents.

MCC8.EE.1 Know and apply the properties of integer exponents to generate equivalent numerical expressions. For example, $3^2 \times 3^{(\tilde{}5)} = 3^{(\tilde{}3)} = \frac{1}{3^3} = \frac{1}{27}$.

MCC8.EE.2 ~~Use square root and cube root symbols to represent solutions to equations of the form x^2 = p and x^3 =p, where p is a positive rational number.~~Evaluate square roots of small perfect squares and cube roots of small perfect cubes. Know that $\sqrt{2}$ is irrational.

MCC8.EE.3 Use numbers expressed in the form of a single digit times an integer power of 10 to estimate very large or very small quantities, and to express how many times as much one is than the other. For example, estimate the population of the United States as 3×10^8 and the population of the world as 7×10^9, and determine that the world population is more than 20 times larger.

MCC8.EE.4 Perform operations with numbers expressed in scientific notation, including problems where both decimal and scientific notation are used. Use scientific notation and choose units of appropriate size for measurements of very large or very small quantities (e.g., use millimeters per year for seafloor spreading). Interpret scientific notation that has been generated by technology.

MCC8.EE.7 Solve linear equations in one variable.

a. Give examples of linear equations in one variable with one solution, infinitely many solutions, or no solutions. Show which of these possibilities is the case by successively transforming the given equation into simpler forms, until an equivalent equation of the form $x = a$, $a = b$, or $a = a$ results (where a and b are different numbers).

b. Solve linear equations with rational number coefficients, including equations whose solutions require expanding expressions using the distributive property and collecting like terms.

MCC8.NS.1 Know that numbers that are not rational are called irrational. Understand informally that every number has a decimal expansion; for rational numbers show that the decimal

MCC8.NS.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions (e.g., π^2). *For example, by truncating the decimal expansion of $\sqrt{2}$ (square root of 2), show that $\sqrt{2}$ is between 1 and 2, then between 1.4 and 1.5, and explain how to continue on to get better approximations.*

2.1 Chapter 9 – Concept 9.3: Zero, Negative, and Fractional Exponents (Lesson)

- Simplify expressions with zero exponents.
- Simplify expressions with negative exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Learning Objectives

- Simplify expressions with zero exponents.
- Simplify expressions with negative exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Introduction

There are many interesting concepts that arise when contemplating the product and quotient rule for exponents. You may have already been wondering about different values for the exponents. For example, so far we have only considered positive, whole numbers for the exponent. So called **natural numbers** (or **counting numbers**) are easy to consider, but even with the everyday things around us we think about questions such as "is it possible to have a negative amount of money?" or "what would one and a half pairs of shoes look like?" In this lesson, we consider what happens when the exponent is not a natural number. We will start with "What happens when the exponent is zero?"

Simplify Expressions with Exponents of Zero

Let us look again at the quotient rule for exponents (that $\frac{x^n}{x^m} = x^{n-m}$) and consider what happens when $n = m$. Let's take the example of x^4 divided by x^4.

$$\frac{x^4}{x^4} = x^{(4-4)} = x^0$$

Now we arrived at the quotient rule by considering how the factors of x cancel in such a fraction. Let's do that again with our example of x^4 divided by x^4.

$$\frac{x^4}{x^4} = \frac{\not{x} \cdot \not{x} \cdot \not{x} \cdot \not{x}}{\not{x} \cdot \not{x} \cdot \not{x} \cdot \not{x}} = 1$$

So $x^0 = 1$.

This works for any value of the exponent, not just 4.

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

Since there is the same number of factors in the numerator as in the denominator, they cancel each other out and we obtain $x^0 = 1$. The zero exponent rule says that any number raised to the power zero is one.

Zero Rule for Exponents: $x^0 = 1,\ x \neq 0$

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Simplify Expressions With Negative Exponents

Again we will look at the quotient rule for exponents (that $\frac{x^n}{x^m} = x^{n-m}$) and this time consider what happens when $m > n$. Let's take the example of x^4 divided by x^6.

$\frac{x^4}{x^6} = x^{(4-6)} = x^{-2}$ for $x \neq 0$.

By the quotient rule our exponent for x is -2. But what does a negative exponent really mean? Let's do the same calculation long-hand by dividing the factors of x^4 by the factors of x^6.

$$\frac{x^4}{x^6} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot x \cdot x} = \frac{1}{x \cdot x} = \frac{1}{x^2}$$

So we see that x to the power -2 is the same as one divided by x to the power +2. Here is the negative power rule for exponents.

Negative Power Rule for Exponents $\frac{1}{x^n} = x^{-n}\ x \neq 0$

You will also see negative powers applied to products and fractions. For example, here it is applied to a product.

$$(x^3y)^{-2} = x^{-6}y^{-2} \qquad \text{using the power rule}$$

$$x^{-6}y^{-2} = \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2} \qquad \text{using the negative power rule separately on each variable}$$

Here is an example of a negative power applied to a quotient.

$$\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}} \qquad \text{using the power rule for quotients}$$

$$\frac{a^{-3}}{b^{-3}} = \frac{a^{-3}}{1} \cdot \frac{1}{b^{-3}} = \frac{1}{a^3} \cdot \frac{b^3}{1} \qquad \text{using the negative power rule on each variable separately}$$

$$\frac{1}{a^3} \cdot \frac{b^3}{1} = \frac{b^3}{a^3} \qquad \text{simplifying the division of fractions}$$

$$\frac{b^3}{a^3} = \left(\frac{b}{a}\right)^3 \qquad \text{using the power rule for quotients in reverse.}$$

The last step is not necessary but it helps define another rule that will save us time. A fraction to a negative power is "flipped".

Negative Power Rule for Fractions $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n,\ x \neq 0, y \neq 0$

In some instances, it is more useful to write expressions without fractions and that makes use of negative powers.

Example 1

Write the following expressions without fractions.

(a) $\frac{1}{x}$

(b) $\frac{2}{x^2}$

(c) $\frac{x^2}{y^3}$

(d) $\frac{3}{xy}$

Solution

We apply the negative rule for exponents $\frac{1}{x^n} = x^{-n}$ on all the terms in the denominator of the fractions.

(a) $\frac{1}{x} = x^{-1}$

(b) $\frac{2}{x^2} = 2x^{-2}$

(c) $\frac{x^2}{y^3} = x^2y^{-3}$

(d) $\frac{3}{xy} = 3x^{-1}y^{-1}$

Sometimes, it is more useful to write expressions without negative exponents.

Example 2

Write the following expressions without negative exponents.

(a) $3x^{-3}$

(b) $a^2b^{-3}c^{-1}$

(c) $4x^{-1}y^3$

(d) $\frac{2x^{-2}}{y^{-3}}$

Solution

We apply the negative rule for exponents $\frac{1}{x^n} = x^{-n}$ on all the terms that have negative exponents.

(a) $3x^{-3} = \frac{3}{x^3}$

(b) $a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$

(c) $4x^{-1}y^3 = \frac{4y^3}{x}$

(d) $\frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$

Example 3

Simplify the following expressions and write them without fractions.

(a) $\frac{4a^2b^3}{2a^5b}$

(b) $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$

Solution

(a) Reduce the numbers and apply quotient rule on each variable separately.

$$\frac{4a^2b^3}{6a^5b} = 2 \cdot a^{2-5} \cdot b^{3-1} = 2a^{-3}b^2$$

(b) Apply the power rule for quotients first.

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^2}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, use product rule on the x's and the quotient rule on the y's.

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

Example 4

Simplify the following expressions and write the answers without negative powers.

(a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

(b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

(a) Apply the quotient rule inside the parenthesis.

$$\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$$

Apply the power rule.

$$(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$$

(b) Apply the quotient rule on each variable separately.

$$\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$$

Simplify Expressions With Fractional Exponents

The exponent rules you learned in the last three sections apply to all powers. So far we have only looked at positive and negative integers. The rules work exactly the same if the powers are fractions or irrational numbers. Fractional exponents are used to express the taking of roots and radicals of something (square roots, cube roots, etc.). Here is an exmaple.

$\sqrt{a} = a^{\frac{1}{2}}$ and $\sqrt[3]{a} = a^{\frac{1}{3}}$ and $\sqrt[5]{a^2} = \left(a^2\right)^{\frac{1}{5}} = a^{\frac{2}{5}} = a^{\frac{2}{5}}$

Roots as Fractional Exponents $\sqrt[m]{a^n} = a^{\frac{n}{m}}$

We will examine roots and radicals in detail in a later chapter. In this section, we will examine how exponent rules apply to fractional exponents.

Example 5

Simplify the following expressions.

(a) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$

(b) $\left(a^{\frac{1}{3}}\right)^2$

(c) $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}}$

(d) $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}}$

Solution

(a) Apply the product rule.

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}$$

(b) Apply the power rule.

$$\left(a^{\frac{1}{3}}\right)^2 = a^{\frac{2}{3}}$$

(c) Apply the quotient rule.

$$\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}} = a^{\frac{5}{2}-\frac{1}{2}} = a^{\frac{4}{2}} = a^2$$

(d) Apply the power rule for quotients.

$$\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}} = \frac{x^{\frac{2}{3}}}{y}$$

Evaluate Exponential Expressions

When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS.**

Evaluate inside the **P**arenthesis.

Evaluate **E**xponents.

Perform **M**ultiplication and **D**ivision operations from left to right.

Perform **A**ddition and **S**ubtraction operations from left to right.

Example 6

Evaluate the following expressions to a single number.

(a) 5^0

(b) 7^2

(c) $\left(\frac{2}{3}\right)^3$

(d) 3^{-3}

(e) $16^{\frac{1}{2}}$

(f) $8^{\frac{-1}{3}}$

Solution

(a) $5^0 = 1$ Remember that a number raised to the power 0 is always 1.

(b) $7^2 = 7 \cdot 7 = 49$

(c) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

(d) $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$

(e) $16^{\frac{1}{2}} = \sqrt{16} = 4$ Remember that an exponent of $\frac{1}{2}$ means taking the square root.

(f) $8^{\frac{-1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ Remember that an exponent of $\frac{1}{3}$ means taking the cube root.

Example 7

Evaluate the following expressions to a single number.

(a) $3 \cdot 5^5 - 10 \cdot 5 + 1$

(b) $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2}$

(c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4}$

Solution

(a) Evaluate the exponent.

$$3 \cdot 5^2 - 10 \cdot 6 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1$$

Perform multiplications from left to right.

$$3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1$$

Perform additions and subtractions from left to right.

$$75 - 50 + 1 = 26$$

(b) Treat the expressions in the numerator and denominator of the fraction like they are in parenthesis.

$$\frac{(2 \cdot 4^2 - 3 \cdot 5^2)}{(3^2 - 2^2)} = \frac{(2 \cdot 16 - 3 \cdot 25)}{(9 - 4)} = \frac{(32 - 75)}{5} = \frac{-43}{5}$$

(c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4} = \left(\frac{2^2}{3^3}\right)^2 \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{2^2} = \frac{2^2}{3^5} = \frac{4}{243}$

Example 8

Evaluate the following expressions for $x = 2, y = -1, z = 3$.

(a) $2x^2 - 3y^3 + 4z$

(b) $(x^2 - y^2)^2$

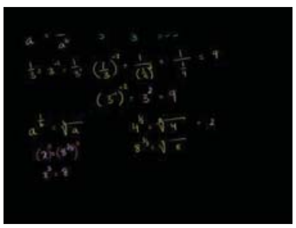
(c) $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

Solution

(a) $2x^2 - 3y^3 + 4z = 2 \cdot 2^2 - 3 \cdot (-1)^3 + 4 \cdot 3 = 2 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 8 + 3 + 12 = 23$

(b) $(x^2 - y^2)^2 = (2^2 - (-1)^2)^2 = (4 - 1)^2 = 3^2 = 9$

(c) $\left(\frac{3x^2 - y^5}{4z}\right)^{-2} = \left(\frac{3 \cdot 2^2 \cdot (-1)^5}{4 \cdot 3}\right)^{-2} = \left(\frac{3 \cdot 4 \cdot (-1)}{12}\right)^{-2} = \left(\frac{-12}{12}\right)^{-2} = \left(\frac{-1}{1}\right)^{-2} = \left(\frac{1}{-1}\right)^2 = (-1)^2 = 1$



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Review Questions

Simplify the following expressions, be sure that there aren't any negative exponents in the answer.

1. $x^{-1} \cdot y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{x^{-3}y^{-5}}{z^{-7}}$
5. $\left(x^{\frac{1}{2}}y^{-\frac{2}{3}}\right)\left(x^2y^{\frac{1}{3}}\right)$
6. $\left(\frac{a}{b}\right)^{-2}$
7. $(3a^{-2}b^2c^3)^3$
8. $x^{-3} \cdot x^3$

Simplify the following expressions so that there aren't any fractions in the answer.

1. $\frac{a^{-3}(a^5)}{a^{-6}}$
2. $\frac{5x^6y^2}{x^8y}$
3. $\frac{(4ab^6)^3}{(ab)^5}$
4. $\left(\frac{3x}{y^{\frac{1}{3}}}\right)^3$
5. $\frac{3x^2y^{\frac{3}{2}}}{xy^{\frac{1}{2}}}$
6. $\frac{(3x^3)(4x^4)}{(2y)^2}$
7. $\frac{a^{-2}b^{-3}}{c^{-1}}$

8. $\frac{x^{\frac{1}{2}}y^{\frac{5}{2}}}{x^{\frac{3}{2}}y^{\frac{3}{2}}}$

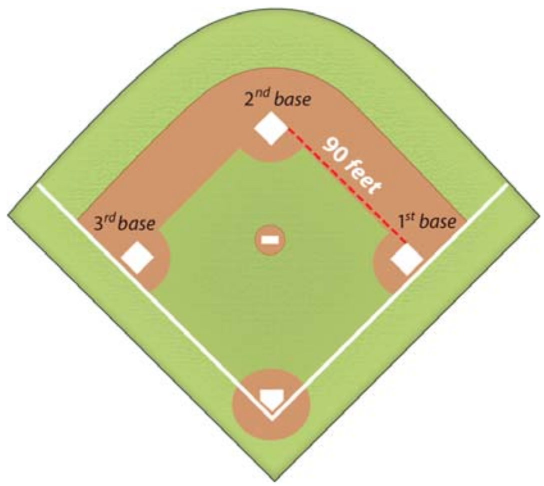
Evaluate the following expressions to a single number.

1. 3^{-2}
2. $(6.2)^0$
3. $8^{-4} \cdot 8^6$
4. $\left(16^{\frac{1}{2}}\right)^3$
5. $x^2 4x^3 y^4 4y^2$ if $x = 2$ and $y = -1$
6. $a^4(b^2)^3 + 2ab$ if $a = -2$ and $b = 1$
7. $5x^2 - 2y^3 + 3z$ if $x = 3$, $y = 2$, and $z = 4$
8. $\left(\frac{a^2}{b^3}\right)^{-2}$ if $a = 5$ and $b = 3$

2.2 Evaluation of Square Roots

Here you'll learn to evaluate square roots.

Have you ever loved a sport?



Miguel loves baseball. He is such a fan that he is volunteering all summer for the University. The University team, The "Wildcats" is an excellent team and Miguel is very excited to be helping out. He doesn't even mind not being paid because he will get to see all of the games for free while he has the opportunity to learn more about baseball.

On the day of the first game, Miguel notices some big dark clouds as he rides his bike to the ball park. Sure enough as soon as the game is about to start, the rain begins. Like magic, a bunch of different people drag a huge tarp over the entire baseball infield. Miguel has never seen a tarp so big in his whole life.

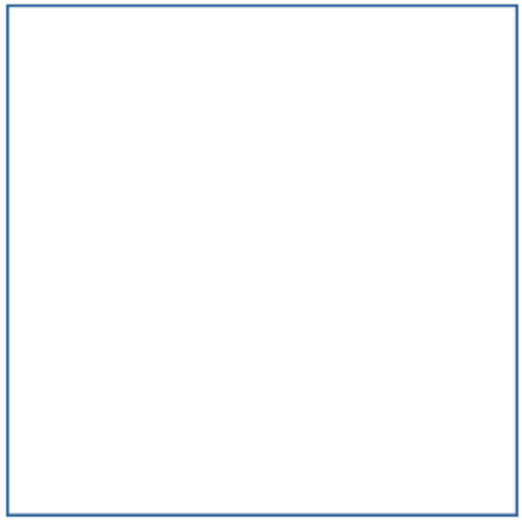
He wonders how big the tarp actually is if it covers the entire infield. Miguel, being the fan that he is knows that the distance from one base to another, say 1^{st} to 2^{nd} is 90 feet. If the infield is in the shape of a square, then how many square feet is the infield? How can he be sure that his answer is correct?

Miguel begins to figure this out in his head.

Can you figure this out? Squaring numbers and finding their square roots is just one way to solve this problem. This Concept will teach you all about square roots and squaring. Pay close attention and at the end of the Concept you will be able to figure out the size of the tarp.

Guidance

Think about a square for a minute. We can look at a ***square*** in a couple of different ways. First, we can look at just the outline of the square.

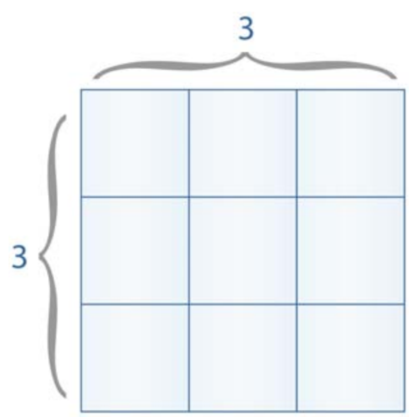


When you look at this square, you can see only the outside, but we all know that the side of a square can be measured and for a square to be a square it has to have four ***congruent*** sides.

Do you remember what congruent means?

It means exactly the same. So if a square has congruent sides, then they are the same length.

Now let's say the side of a square is 3 units long. That means that each side of the square is 3 units long. Look at this picture of a square.



We call a number like this one a *square number* because it makes up a square. 3^2 **is represented in this square.**

How many units make up the entire square?

If we count, we can see that this square is made up of 9 units. It is the same answer as 3^2**, because** 3^2 **is equal to 9.**

Do you see a connection?

Think back to exponents, when we square a number, we multiply the number by itself. All squares have congruent side lengths, so the side length of a square multiplied by itself will tell you the number of units in the square.

We square the side length to find the number of units in the square.

This Concept is all about *square roots*. A square root is the number that we multiply by itself, or square, to get a certain result. In fact, if you square a number, when you take the square root of that number you will be back to the original number again.

Let's think about the square that we just looked at. The dimensions of the square is 3×3. We square the three to find the units in the square. **The square root of the 3×3 square is 3. This is the value that we would multiply by itself.**

We can find the square root of a number. How do we do this?

Finding the square root is the inverse operation of squaring a number. Inverse operations are simply the opposite of each other. Subtraction and addition are inverse operations, because one "undoes" the other. Similarly, squaring and finding the square root are inverse operations. **When we find the square root, we look for the number that, times itself, will produce a given number.**

We also use a symbol to show that we are looking for the square root of a number. Here is the symbol for square root.

$\sqrt{9}$

If this were the problem, we would be looking for the square root of 9.

You could think of this visually as a square that has nine units in it. What would be the length of the side? It would be three.

You could also think of it using mental math to solve it. What number times itself is equal to nine. The answer is three.

When we find the square root of a number, we evaluate that square root.

$\sqrt{25}$

This problem is asking us for the square root of 25. What number times itself is equal to 25? If you don't know right away, you can think about this with smaller numbers.

$$3 \times 3 = 9$$
$$4 \times 4 = 16$$
$$5 \times 5 = 25$$

That's it! The square root of 25 is 5.

We can also evaluate numbers where the square root is not a whole number.

$\sqrt{7}$

To find the square root of seven, we can think about which two squares it is closest to.

$$2 \times 2 = 4$$
$$3 \times 3 = 9$$

Seven is between four and nine, so we can say that the square root of seven is between 2 and 3.

Our answer would be that the $\sqrt{7}$ is between 2 and 3.

We can get a more exact number, but we aren't going to worry about that for right now. Here is another one.

$\sqrt{10}$

The square root of ten is between which two numbers?

$$3 \times 3 = 9$$
$$4 \times 4 = 16$$

Our answer is that the $\sqrt{10}$ is between 3 and 4.

Now its time for you to try a few on your own. Evaluate each square root.

Example A

$\sqrt{36}$

Solution:6

Example B

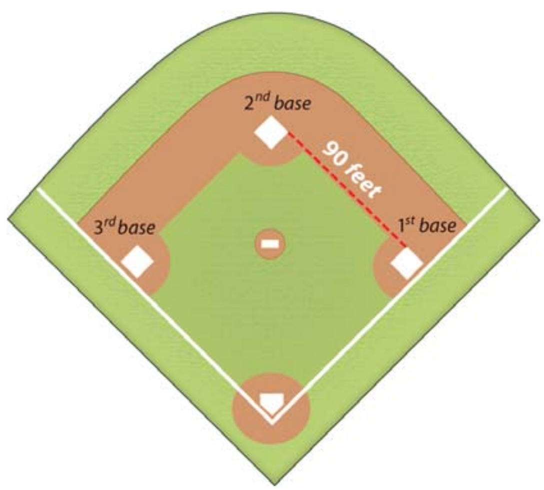
$\sqrt{49}$

Solution:7

Example C

$\sqrt{12}$

Solution: Between 3 and 4.



Here is the original problem once again.

Miguel loves baseball. He is such a fan that he is volunteering all summer for the University. The University team, The "Wildcats" is an excellent team and Miguel is very excited to be helping out. He doesn't even mind not being paid because he will get to see all of the games for free while he has the opportunity to learn more about baseball.

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He wonders how big the tarp actually is if it covers the entire infield. Miguel, being the fan that he is knows that the distance from one base to another, say 1^{st} to 2^{nd} is 90 feet. If the infield is in the shape of a square, then how many square feet is the infield? How can he be sure that his answer is correct?

Miguel begins to figure this out in his head.

We can use what we know about squares to help us with this problem. We know that a square has four equal sides. This makes sense with baseball too. You want the distance from 1^{st} to 2^{nd} base to be the same as from

3^{rd} **to Home. Therefore, if you know the distance from one base to another is 90 feet, then you know each distance from base to base.**

However, Miguel wants to figure out the size of the tarp. He can do this by squaring the distance from 1^{st} **to** 2^{nd} **base. This will give him the area of the square.**

$90^2 = 90 \times 90 = 8100\ square feet$

This is the size of the tarp.

How can Miguel check the accuracy of his answer? He can do this by finding the square root of the area of the tarp. Remember that finding a square root is the inverse operation for squaring a number.

$\sqrt{8100}$

To complete this, worry about the 81 and not the 8100. 81 is a perfect square. $9 \times 9 = 81$ **so** $90 \times 90 = 8100$

$\sqrt{8100} = 90\ ft$

Miguel's answer checks out.

Vocabulary

Here are the vocabulary words in this Concept.

Square
: a four sided figure with congruent sides.

Congruent
: exactly the same

Square Number
: a number of units which makes a perfect square.

Square root
: a number that when multiplied by itself equals the square of the number.

Guided Practice

Here is one for you to try on your own.

$\sqrt{64}$

Answer

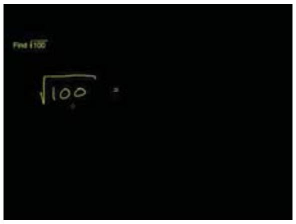
What is the square root of 64? What number times itself is 49? Let's start where we left off with five.

$$6 \times 6 = 36$$
$$7 \times 7 = 49$$
$$8 \times 8 = 64$$

That's it! The square root of 64 is 8.

Video Review

Here is a video for review.



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- This is a KhanAcademyvideo on understanding square roots.

Practice

Directions: Evaluate each square root.

1. $\sqrt{16}$
2. $\sqrt{25}$
3. $\sqrt{1}$
4. $\sqrt{49}$
5. $\sqrt{144}$
6. $\sqrt{81}$
7. $\sqrt{169}$
8. $\sqrt{121}$
9. $\sqrt{100}$
10. $\sqrt{256}$

Directions: Name the two values each square root is in between.

11. $\sqrt{12}$
12. $\sqrt{14}$
13. $\sqrt{30}$
14. $\sqrt{40}$
15. $\sqrt{50}$
16. $\sqrt{62}$
17. $\sqrt{70}$
18. $\sqrt{101}$
19. $\sqrt{5}$
20. $\sqrt{15}$

2.3 Square Roots and Irrational Numbers

Here you'll learn how to decide whether a number is rational or irrational and how to take the square root of a number.

Suppose an elementary school has a square playground with an area of 3000 square feet. Could you find the width of the playground? Would the width be a rational or irrational number? In this Concept, you'll learn how to take the square root of a number and decide whether the result is rational or irrational so that you can answer questions such as these.

Guidance

Human chess is a variation of chess, often played at Renaissance fairs, in which people take on the roles of the various pieces on a chessboard. The chessboard is played on a square plot of land that measures 324 square meters with the chess squares marked on the grass. How long is each side of the chessboard?

To answer this question, you will need to know how to find the square root of a number.

The **square root** of a number n is any number s such that $s^2 = n$.

Every positive number has two square roots, the positive and the negative. The symbol used to represent the square root is $\sqrt{x}$. It is assumed that this is the positive square root of x. To show both the positive and negative values, you can use the symbol $\pm$, read "plus or minus."

For example:

$\sqrt{81} = 9$ means the positive square root of 81.

$-\sqrt{81} = -9$ means the negative square root of 81.

$\pm\sqrt{81} = \pm 9$ means the positive or negative square root of 81.

Example A

The human chessboard measures 324 square meters. How long is one side of the square?

Solution: The area of a square is $s^2 = Area$. The value of $Area$ can be replaced with 324.

$$s^2 = 324$$

The value of s represents the square root of 324.

$$s = \sqrt{324} = 18$$

The chessboard is 18 meters long by 18 meters wide.

Approximating Square Roots

When the square root of a number is a whole number, this number is called a **perfect square.** 9 is a perfect square because $\sqrt{9} = 3$.

Not all square roots are whole numbers. Many square roots are irrational numbers, meaning there is no rational number equivalent. For example, 2 is the square root of 4 because $2 \times 2 = 4$. The number 7 is the square root of 49 because $7 \times 7 = 49$. What is the square root of 5?

There is no whole number multiplied by itself that equals five, so $\sqrt{5}$ is not a whole number. To find the value of $\sqrt{5}$, we can use estimation.

To estimate the square root of a number, look for the perfect integers **less than** and **greater than** the value, and then estimate the decimal.

Example B

Estimate $\sqrt{5}$.

Solution: The perfect square below 5 is 4 and the perfect square above 5 is 9. Therefore, $4 < 5 < 9$. Therefore, $\sqrt{5}$ is between $\sqrt{4}$ and $\sqrt{9}$, or $2 < \sqrt{5} < 3$. Because 5 is closer to 4 than 9, the decimal is a low value: $\sqrt{5} \approx 2.2$.

Identifying Irrational Numbers

Recall the number hierarchy from a previous Concept. Real numbers have two categories: rational and irrational. If a value is not a perfect square, then it is considered an **irrational number.** These numbers cannot be written as a fraction because the decimal does not end **(non-terminating)** and does not repeat a pattern **(non-repeating)**. Although irrational square roots cannot be written as fractions, we can still write them **exactly**, without typing the value into a calculator.

For example, suppose you do not have a calculator and you need to find $\sqrt{18}$. You know there is no whole number squared that equals 18, so $\sqrt{18}$ is an irrational number. The value is between $\sqrt{16} = 4$ and $\sqrt{25} = 5$. However, we need to find the exact value of $\sqrt{18}$.

Begin by writing the **prime factorization** of $\sqrt{18}$. $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2}$. $\sqrt{9} = 3$ but $\sqrt{2}$ does not have a whole number value. Therefore, the exact value of $\sqrt{18}$ is $3\sqrt{2}$.

You can check your answer on a calculator by finding the decimal approximation for each square root.

Example C

Find the exact value of $\sqrt{75}$.

Solution:

$$\sqrt{75} = \sqrt{25 \times 3} = \sqrt{25} \times \sqrt{3} = 5\sqrt{3}$$

Guided Practice

The area of a square is 50 square feet. What are the lengths of its sides?

Solution:

We know that the formula for the area of a square is $a = s^2$. Using this formula:

$$amp; a = s^2$$
$$50 = s^2$$
$$\sqrt{50} = \sqrt{s^2}$$
$$\sqrt{50} = s$$

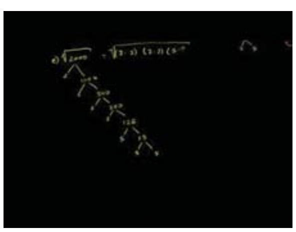
Now we will simplify:

$\sqrt{50} = \sqrt{25 \cdot 2} = 5\sqrt{2}$.

The length of each side of the square is $5\sqrt{2}$ feet.

Practice

Sample explanations for some of the practice exercises below are available by viewing the following video. Note that there is not always a match between the number of the practice exercise in the video and the number of the practice exercise listed in the following exercise set. However, the practice exercise is the same in both. CK-12 Basic Algebra:Square Roots and Real Numbers (10:18)



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Find the following square roots **exactly without using a calculator**. Give your answer in the simplest form.

1. $\sqrt{25}$
2. $\sqrt{24}$
3. $\sqrt{20}$
4. $\sqrt{200}$
5. $\sqrt{2000}$
6. $\sqrt{\frac{1}{4}}$
7. $\sqrt{\frac{9}{4}}$
8. $\sqrt{0.16}$
9. $\sqrt{0.1}$
10. $\sqrt{0.01}$

Use a calculator to find the following square roots. Round to two decimal places.

11. $\sqrt{13}$
12. $\sqrt{99}$
13. $\sqrt{123}$
14. $\sqrt{2}$
15. $\sqrt{2000}$
16. $\sqrt{0.25}$
17. $\sqrt{1.35}$
18. $\sqrt{0.37}$
19. $\sqrt{0.7}$
20. $\sqrt{0.01}$

2.4 Zero, Negative, and Fractional Exponents

Learning Objectives

- Simplify expressions with negative exponents.
- Simplify expressions with zero exponents.
- Simplify expression with fractional exponents.
- Evaluate exponential expressions.

Introduction

The product and quotient rules for exponents lead to many interesting concepts. For example, so far we've mostly just considered positive, whole numbers as exponents, but you might be wondering what happens when the exponent isn't a positive whole number. What does it mean to raise something to the power of zero, or -1, or $\frac{1}{2}$? In this lesson, we'll find out.

Simplify Expressions With Negative Exponents

When we learned the quotient rule for exponents $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$, we saw that it applies even when the exponent in the denominator is bigger than the one in the numerator. Canceling out the factors in the numerator and denominator leaves the leftover factors in the denominator, and subtracting the exponents leaves a negative number. So negative exponents simply represent fractions with exponents in the denominator. This can be summarized in a rule:

Negative Power Rule for Exponents: $x^{-n} = \frac{1}{x^n}$, where $x \neq 0$

Negative exponents can be applied to products and quotients also. Here's an example of a negative exponent being applied to a product:

$$(x^3y)^{-2} = x^{-6}y^{-2} \qquad \text{using the power rule}$$

$$x^{-6}y^{-2} = \frac{1}{x^6} \cdot \frac{1}{y^2} = \frac{1}{x^6y^2} \qquad \text{using the negative power rule separately on each variable}$$

And here's one applied to a quotient:

$\left(\frac{a}{b}\right)^{-3} = \frac{a^{-3}}{b^{-3}}$	using the power rule for quotients
$\frac{a^{-3}}{b^{-3}} = \frac{a^{-3}}{1} \cdot \frac{1}{b^{-3}} = \frac{1}{a^3} \cdot \frac{b^3}{1}$	using the negative power rule on each variable separately
$\frac{1}{a^3} \cdot \frac{b^3}{1} = \frac{b^3}{a^3}$	simplifying the division of fractions
$\frac{b^3}{a^3} = \left(\frac{b}{a}\right)^3$	using the power rule for quotients in reverse.

That last step wasn't really necessary, but putting the answer in that form shows us something useful: $\left(\frac{a}{b}\right)^{-3}$ is equal to $\left(\frac{b}{a}\right)^3$. This is an example of a rule we can apply more generally:

Negative Power Rule for Fractions: $\left(\frac{x}{y}\right)^{-n} = \left(\frac{y}{x}\right)^n$, where $x \neq 0, y \neq 0$

This rule can be useful when you want to write out an expression without using fractions.

Example 1

Write the following expressions without fractions.

a) $\frac{1}{x}$

b) $\frac{2}{x^2}$

c) $\frac{x^2}{y^3}$

d) $\frac{3}{xy}$

Solution

a) $\frac{1}{x} = x^{-1}$

b) $\frac{2}{x^2} = 2x^{-2}$

c) $\frac{x^2}{y^3} = x^2y^{-3}$

d) $\frac{3}{xy} = 3x^{-1}y^{-1}$

Example 2

Simplify the following expressions and write them without fractions.

a) $\frac{4a^2b^3}{2a^5b}$

b) $\left(\frac{x}{3y^2}\right)^3 \cdot \frac{x^2y}{4}$

Solution

a) Reduce the numbers and apply the quotient rule to each variable separately:

$$\frac{4a^2b^3}{2a^5b} = 2 \cdot a^{2-5} \cdot b^{3-1} = 2a^{-3}b^2$$

b) Apply the power rule for quotients first:

$$\left(\frac{2x}{y^2}\right)^3 \cdot \frac{x^2y}{4} = \frac{8x^3}{y^6} \cdot \frac{x^2y}{4}$$

Then simplify the numbers, and use the product rule on the x's and the quotient rule on the y's:

$$\frac{8x^3}{y^6} \cdot \frac{x^2y}{4} = 2 \cdot x^{3+2} \cdot y^{1-6} = 2x^5y^{-5}$$

You can also use the negative power rule the other way around if you want to write an expression without negative exponents.

Example 3

Write the following expressions without negative exponents.

a) $3x^{-3}$

b) $a^2b^{-3}c^{-1}$

c) $4x^{-1}y^3$

d) $\frac{2x^{-2}}{y^{-3}}$

Solution

a) $3x^{-3} = \frac{3}{x^3}$

b) $a^2b^{-3}c^{-1} = \frac{a^2}{b^3c}$

c) $4x^{-1}y^3 = \frac{4y^3}{x}$

d) $\frac{2x^{-2}}{y^{-3}} = \frac{2y^3}{x^2}$

Example 4

Simplify the following expressions and write the answers without negative powers.

a) $\left(\frac{ab^{-2}}{b^3}\right)^2$

b) $\frac{x^{-3}y^2}{x^2y^{-2}}$

Solution

a) Apply the quotient rule inside the parentheses: $\left(\frac{ab^{-2}}{b^3}\right)^2 = (ab^{-5})^2$

Then apply the power rule: $(ab^{-5})^2 = a^2b^{-10} = \frac{a^2}{b^{10}}$

b) Apply the quotient rule to each variable separately: $\frac{x^{-3}y^2}{x^2y^{-2}} = x^{-3-2}y^{2-(-2)} = x^{-5}y^4 = \frac{y^4}{x^5}$

Simplify Expressions with Exponents of Zero

Let's look again at the quotient rule for exponents $\left(\frac{x^n}{x^m} = x^{(n-m)}\right)$ and consider what happens when $n = m$. For example, what happens when we divide x^4 by x^4? Applying the quotient rule tells us that $\frac{x^4}{x^4} = x^{(4-4)} = x^0$—so what does that zero mean?

Well, we first discovered the quotient rule by considering how the factors of x cancel in such a fraction. Let's do that again with our example of x^4 divided by x^4:

$$\frac{x^4}{x^4} = \frac{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}}{\cancel{x} \cdot \cancel{x} \cdot \cancel{x} \cdot \cancel{x}} = 1$$

So $x^0 = 1$! You can see that this works for any value of the exponent, not just 4:

$$\frac{x^n}{x^n} = x^{(n-n)} = x^0$$

Since there is the same number of x's in the numerator as in the denominator, they cancel each other out and we get $x^0 = 1$. This rule applies for all expressions:

Zero Rule for Exponents: $x^0 = 1$, where $x \neq 0$

For more on zero and negative exponents, watch the following video at squidoo.com: http://www.google.com/url?sa=t&source=video&cd=4&ved=0CFMQtwIwAw&url=http%3A%2F%2Fwww.youtube.com%2Fwatch%3Fv%3D9svqGWwyN8Q&rct=j&q=negative%20exponents%20applet&ei=1fH6TP2IGoX4sAOnlbT3DQ&usg=AFQjCNHzLF4_-2aeo0dMWsa2wJ_CwzckXNA&cad=rja.

Simplify Expressions With Fractional Exponents

So far we've only looked at expressions where the exponents are positive and negative integers. The rules we've learned work exactly the same if the powers are fractions or irrational numbers—but what does a fractional exponent even mean? Let's see if we can figure that out by using the rules we already know.

Suppose we have an expression like $9^{\frac{1}{2}}$—how can we relate this expression to one that we already know how to work with? For example, how could we turn it into an expression that doesn't have any fractional exponents?

Well, the power rule tells us that if we raise an exponential expression to a power, we can multiply the exponents. For example, if we raise $9^{\frac{1}{2}}$ to the power of 2, we get $\left(9^{\frac{1}{2}}\right)^2 = 9^{2\cdot\frac{1}{2}} = 9^1 = 9$.

So if $9^{\frac{1}{2}}$ *squared* equals 9, what does $9^{\frac{1}{2}}$ itself equal? Well, 3 is the number whose square is 9 (that is, it's the square root of 9), so $9^{\frac{1}{2}}$ must equal 3. And that's true for all numbers and variables: a number raised to the power of $\frac{1}{2}$ is just the square root of the number. We can write that as $\sqrt{x} = x^{\frac{1}{2}}$, and then we can see that's true because $\left(\sqrt{x}\right)^2 = x$ just as $\left(x^{\frac{1}{2}}\right)^2 = x$.

Similarly, a number to the power of $\frac{1}{3}$ is just the cube root of the number, and so on. In general, $x^{\frac{1}{n}} = \sqrt[n]{x}$. And when we raise a number to a power and then take the root of it, we still get a fractional exponent; for example, $\sqrt[3]{x^4} = \left(x^4\right)^{\frac{1}{3}} = x^{\frac{4}{3}}$. In general, the rule is as follows:

Rule for Fractional Exponents: $\sqrt[m]{a^n} = a^{\frac{n}{m}}$ and $\left(\sqrt[m]{a}\right)^n = a^{\frac{n}{m}}$

We'll examine roots and radicals in detail in a later chapter. In this section, we'll focus on how exponent rules apply to fractional exponents.

Example 5

Simplify the following expressions.

a) $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}}$

b) $\left(a^{\frac{1}{3}}\right)^2$

c) $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}}$

d) $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}}$

Solution

a) Apply the product rule: $a^{\frac{1}{2}} \cdot a^{\frac{1}{3}} = a^{\frac{1}{2}+\frac{1}{3}} = a^{\frac{5}{6}}$

b) Apply the power rule: $\left(a^{\frac{1}{3}}\right)^2 = a^{\frac{2}{3}}$

c) Apply the quotient rule: $\frac{a^{\frac{5}{2}}}{a^{\frac{1}{2}}} = a^{\frac{5}{2}-\frac{1}{2}} = a^{\frac{4}{2}} = a^2$

d) Apply the power rule for quotients: $\left(\frac{x^2}{y^3}\right)^{\frac{1}{3}} = \frac{x^{\frac{2}{3}}}{y}$

Evaluate Exponential Expressions

When evaluating expressions we must keep in mind the order of operations. You must remember **PEMDAS:**

1. Evaluate inside the **P**arentheses.
2. Evaluate **E**xponents.
3. Perform **M**ultiplication and **D**ivision operations from left to right.
4. Perform **A**ddition and **S**ubtraction operations from left to right.

Example 6

Evaluate the following expressions.

a) 5^0

b) $\left(\frac{2}{3}\right)^3$

c) $16^{\frac{1}{2}}$

d) $8^{-\frac{1}{3}}$

Solution

a) $5^0 = 1$ A number raised to the power 0 is always 1.

b) $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$

c) $16^{\frac{1}{2}} = \sqrt{16} = 4$ Remember that an exponent of $\frac{1}{2}$ means taking the square root.

d) $8^{-\frac{1}{3}} = \frac{1}{8^{\frac{1}{3}}} = \frac{1}{\sqrt[3]{8}} = \frac{1}{2}$ Remember that an exponent of $\frac{1}{3}$ means taking the cube root.

Example 7

Evaluate the following expressions.

a) $3 \cdot 5^2 - 10 \cdot 5 + 1$

b) $\frac{2 \cdot 4^2 - 3 \cdot 5^2}{3^2 - 2^2}$

c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4}$

Solution

a) Evaluate the exponent: $3 \cdot 5^2 - 10 \cdot 5 + 1 = 3 \cdot 25 - 10 \cdot 5 + 1$

Perform multiplications from left to right: $3 \cdot 25 - 10 \cdot 5 + 1 = 75 - 50 + 1$

Perform additions and subtractions from left to right: $75 - 50 + 1 = 26$

b) Treat the expressions in the numerator and denominator of the fraction like they are in parentheses: $\frac{(2 \cdot 4^2 - 3 \cdot 5^2)}{(3^2 - 2^2)} = \frac{(2 \cdot 16 - 3 \cdot 25)}{(9-4)} = \frac{(32-75)}{5} = \frac{-43}{5}$

c) $\left(\frac{3^3}{2^2}\right)^{-2} \cdot \frac{3}{4} = \left(\frac{2^2}{3^3}\right)^2 \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{4} = \frac{2^4}{3^6} \cdot \frac{3}{2^2} = \frac{2^2}{3^5} = \frac{4}{243}$

Example 8

Evaluate the following expressions for $x = 2, y = -1, z = 3$.

a) $2x^2 - 3y^3 + 4z$

b) $(x^2 - y^2)^2$

c) $\left(\frac{3x^2y^5}{4z}\right)^{-2}$

Solution

a) $2x^2 - 3y^3 + 4z = 2 \cdot 2^2 - 3 \cdot (-1)^3 + 4 \cdot 3 = 2 \cdot 4 - 3 \cdot (-1) + 4 \cdot 3 = 8 + 3 + 12 = 23$

b) $(x^2 - y^2)^2 = (2^2 - (-1)^2)^2 = (4 - 1)^2 = 3^2 = 9$

c) $\left(\frac{3x^2y^5}{4z}\right)^{-2} = \left(\frac{3 \cdot 2^2 \cdot (-1)^5}{4 \cdot 3}\right)^{-2} = \left(\frac{3 \cdot 4 \cdot (-1)}{12}\right)^{-2} = \left(\frac{-12}{12}\right)^{-2} = \left(\frac{-1}{1}\right)^{-2} = \left(\frac{1}{-1}\right)^2 = (-1)^2 = 1$

Review Questions

Simplify the following expressions in such a way that there aren't any negative exponents in the answer.

1. $x^{-1}y^2$
2. x^{-4}
3. $\frac{x^{-3}}{x^{-7}}$
4. $\frac{x^{-3}y^{-5}}{z^{-7}}$
5. $(x^{\frac{1}{2}}y^{\frac{-2}{3}})(x^2y^{\frac{1}{3}})$
6. $\left(\frac{a}{b}\right)^{-2}$
7. $(3a^{-2}b^2c^3)^3$
8. $x^{-3} \cdot x^3$

Simplify the following expressions in such a way that there aren't any fractions in the answer.

9. $\frac{a^{-3}(a^5)}{a^{-6}}$
10. $\frac{5x^6y^2}{x^8y}$
11. $\frac{(4ab^6)^3}{(ab)^5}$
12. $\left(\frac{3x}{y^{\frac{1}{3}}}\right)^3$
13. $\frac{3x^2y^{\frac{3}{2}}}{xy^{\frac{1}{2}}}$
14. $\frac{(3x^3)(4x^4)}{(2y)^2}$
15. $\frac{a^{-2}b^{-3}}{c^{-1}}$
16. $\frac{x^{\frac{1}{2}}y^{\frac{5}{2}}}{x^{\frac{3}{2}}y^{\frac{3}{2}}}$

Evaluate the following expressions to a single number.

17. 3^{-2}
18. $(6.2)^0$

19. $8^{-4} \cdot 8^{6}$
20. $\left(16^{\frac{1}{2}}\right)^{3}$
21. $x^{2} \cdot 4x^{3} \cdot y^{4} \cdot 4y^{2}$, if $x = 2$ and $y = -1$
22. $a^{4}(b^{2})^{3} + 2ab$, if $a = -2$ and $b = 1$
23. $5x^{2} - 2y^{3} + 3z$, if $x = 3, y = 2$, and $z = 4$
24. $\left(\frac{a^{2}}{b^{3}}\right)^{-2}$, if $a = 5$ and $b = 3$
25. $\left(\frac{x^{-2}}{y^{4}}\right)^{\frac{1}{2}}$, if $x = -3$ and $y = 2$

2.5 Evaluate Numerical and Variable Expressions Involving Powers

Here you'll evaluate numerical and variable expressions involving powers.

Do you know how to evaluate a numerical expression when it has powers in it? Casey is having a difficult time doing exactly that. When Casey arrived home from school he looked at his homework. He immediately noticed this problem.

$5^4 + -2^4 + 12$

Casey isn't sure how to evaluate this expression. Do you know how to do it? This Concept will show you how to evaluate numerical expressions involving powers. Then you will be able to help Casey at the end of the Concept.

Guidance

Did you know that you can evaluate numerical and variable expressions involving powers? First, let's identify a numerical and a variable expression.

A *numerical expression* is a group of numbers and operations that represent a quantity, there isn't an equal sign.

A *variable expression* is a group of numbers, operations and variables that represents a quantity, there isn't an equal sign.

We can combine the order of operations, numerical expressions and variable expressions together with powers.

Let's talk about powers.

A *power* is a number with an *exponent* and a *base*. An *exponent* is a little number that shows the number of times a base is multiplied by itself. The *base* is the regular sized number that is being worked with.

Take a minute to write these definitions in your notebook.

Now let's apply this information.

$4^2 = 16$

What happened here?

We can break down this problem to better understand powers and exponents. In the power 4^2 or four-squared, four is the base and two is the exponent. 4^2 means four multiplied two times or 4×4. Therefore, 4^2 is sixteen.

Let's go back a step and evaluate an expression with a power in it. Take a look.

Evaluate 6^3

First, we have to think about what this means. It means that we take the base, 6 and multiply it by itself three times.

$$\begin{aligned}&6 \times 6 \times 6\\&36 \times 6\\&216\end{aligned}$$

The answer is 216.

Here is another one with a negative number in it.

Evaluate -8^2

To work on this one, we have to work on remembering integer rules. Think back remember that we multiply a negative times a negative to get a positive.

$-8 \cdot -8 = 64$

The answer is 64.

This is called evaluating a power.

Let's look at evaluating powers within expressions.

Simplify the expression $6^4 + 2^5 + 12$.

Step 1: Simplify 6^4.

$6^4 = 6 \times 6 \times 6 \times 6 = 1,296$

Step 2: Simplify 2^5.

$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$

Step 3: Add to solve.

$1,296 + 32 + 12 = 1,340$

The answer is 1,340.

We can also evaluate variable expressions by substituting given values into the expressions.

Evaluate the expression $4a^2$ when $a = 3$.

Step 1: Substitute 3 for the variable "a."

$$4(3)^2$$

Step 2: Simplify the powers.

$$\begin{aligned}&4(3)^2\\&4(3 \cdot 3)\\&4(9)\end{aligned}$$

Step 3: Multiply to solve.

$$4(9)$$
$$36$$

The answer is 36.

Evaluate each numerical expression.

Example A

6^3+5^2+25

Solution: 266

Example B

$16(12^3)$

Solution: $27,648$

Example C

$6^2+5^3+15-11$

Solution: 165

Now let's go back to the dilemma at the beginning of the Concept. Here is the problem that was puzzling to Casey.

5^4+-2^4+12

First, Casey will need to evaluate the powers.

$5^4=(5)(5)(5)(5)=625$

$-2^4=(-2)(-2)(-2)(-2)=16$

Now we can substitute these values back into the expression.

$625+16+12=653$

This is the answer to Casey's problem.

Vocabulary

Numerical Expression
: a group of numbers and operations used to represent a quantity without an equals sign.

Variable Expression
: a group of numbers, operations and variables used to represent a quantity without an equals sign.

Powers
: the value of a base and an exponent.

Base

the regular sized number that the exponent works upon.

Exponent

the little number that tells you how many times to multiply the base by itself.

Guided Practice

Here is one for you to try on your own.

Evaluate the expression $5b^4 + 17$. Let $b = 5$.

Solution

Step 1: Substitute 5 for "*b*."

$$5(5)^4 + 17$$

Step 2: Simplify the powers.

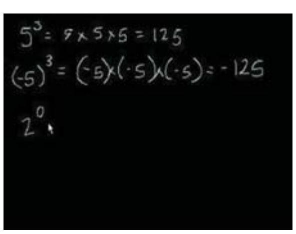
$$\begin{aligned}&5(5 \cdot 5 \cdot 5 \cdot 5) + 17\\&5(625) + 17\end{aligned}$$

Step 3: Multiply then add to solve.

$$\begin{aligned}&5(625) + 17\\&3,125 + 17 = 3,142\end{aligned}$$

The answer is 3,142.

Video Review



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Click image to the left for more content.

[http://www.khanacademy.org/video?v=8htcZca0JIA Khan Academy Level 1 Exponents]

Practice

Directions: Evaluate each power.

1. 3^3

2. 4^2
3. -2^4
4. -8^2
5. 5^3
6. 2^6
7. -9^2
8. -2^6

Directions: Evaluate each numerical expression.

9. 6^2+22
10. -3^3+18
11. 2^3+16-4
12. -5^2-19
13. -7^2+52-2
14. $18+9^2-3$
15. $22-3^3+7$

Directions: Evaluate each variable expression using the given values.

16. $6a+4^2-2$, when $a=3$
17. a^3+14, when $a=6$
18. $2a^2-16$, when $a=4$
19. $5b^3+12$, when $b=-2$
20. $2x^2+52$, when $x=4$

2.6 Algebra Expressions with Exponents

Here you'll learn to evaluate powers with variable bases.

Remember the tent dilemma from the last Concept?

The hikers were given a specific tent with specific dimensions. Remember, they were given a Kelty Trail Dome 6.

What if a different tent was used? What if many different tents were used?

The square footage of the floor would always have an exponent of 2, but a variable would be needed for the base because different size tents would be being used.

Here is how we could write this.

a^2

In this case, a is the length of one side of a square tent.

What if a tent with 8 feet on one side was being used?

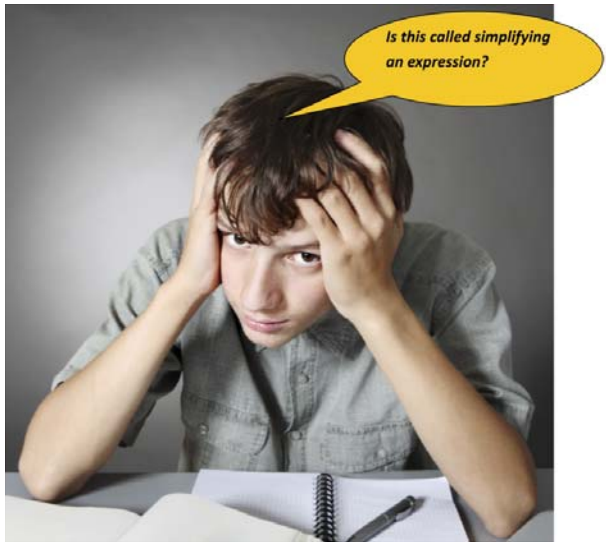
What if a tent with 15 feet on one side was being used?

What would the square footage of each tent be?

This Concept will teach you how to evaluate powers with variable bases. Pay attention and you will know how to work through this at the end of the Concept.

Guidance

When we are dealing with numbers, it is often easier to just simplify. It makes more sense to deal with 16 than with 4^2. Exponential notation really comes in handy when we're dealing with variables. It is easier to write y^{12} than it is to write *yyyyyyyyyyyy*.



Yes, and we can simplify by using exponential form and we can also write out the variable expression by using expanded form.

Write the following in expanded form: x^5

To write this out, we simply write out each x five times.

$x^5 = xxxxx$

We can work the other way to by taking an variable expression in expanded form and write it in exponential form.

$aaaa$

Our answer is a^4.

What about when we multiply two variable terms with exponents?

To do this, we are going to need to follow a few rules.

$(m^3)(m^2)$

The first thing to notice is that these terms have the same base. Both bases are m's. Because of this, we can simplify the expression quite easily.

Let's write it out in expanded form.

$mmm(mm)$

Here we have five m's being multiplied our answer is m^5.

Here is the rule.

Rule 1: When multiplying variable bases with exponents, if the bases are the same, we simply add the exponents.

Let's apply this rule to the next one.

$(x^6)(x^3)$

The bases are the same, so we add the exponents.

$x^{6+3} = x^9$

This is the answer.

We can also have an exponential term raised to a power. When this happens, one exponent is outside the parentheses. This means something different.

$(x^2)^3$

Let's think about what this means. It means that we are multiplying x squared by itself three times. We can write this out in expanded form.

$(x^2)(x^2)(x^2)$

Now we are multiplying three bases that are the same so we use Rule 1 and add the exponents.

Our answer is x^6.

We could have multiplied the two exponents in the beginning.

$(x^2)^3 = x^{2(3)} = x^6$

Here is Rule 2.

Rule 2: When raising a variable expression with an exponent to a power, you multiply the two exponents together.

Simplify x^0

Our answer is $x^0 = 1$

Anything to the power of 0 equals 1.

Rule 3: Anything to the power of 0 equals 1.

Now it's time for you to try a few on your own.

Example A

Write the following in exponential form: $aaaaaaa$

Solution:a^7

Example B

Simplify:$(a^3)(a^8)$

Solution:a^{11}

Example C

Simplify:$(x^4)^2$

Solution:x^8

Remember the tent dilemma from the beginning of the Concept? Well let's take a look at it again.

The hikers were given a specific tent with specific dimensions. Remember, they were given a Kelty Trail Dome 6.

What if a different tent was used? What if many different tents were used?

The square footage of the floor would always have an exponent of 2, but a variable would be needed for the base because different size tents would be being used.

Here is how we could write this.

a^2

In this case, a is the length of one side of a square tent.

What if a tent with 8 feet on one side was being used?

What if a tent with 15 feet on one side was being used?

What would the square footage of each tent be?

Here is our solution.

$8^2 = 64$ square feet is the first tent.

$15^2 = 225$ square feet is the second tent.

Vocabulary

Here are the vocabulary words in this Concept.

Exponent
: a little number that tells you how many times to multiply the base by itself.

Base
the big number in a variable expression with an exponent.

Exponential Notation
writing long multiplication using a base and an exponent

Expanded Form
taking a base and an exponent and writing it out as a long multiplication problem.

Guided Practice

Here is one for you to try on your own.

Simplify:$(x^6)(x^2)$

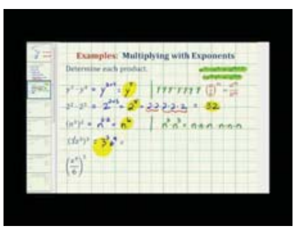
Answer

When we multiply variables with exponents, we add the exponents.

Our answer is x^8.

Video Review

Here is a video for review.



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- This is a James Sousa video onevaluating powers with variablebases.

Practice

Directions: Evaluate each expression.

1. 2^3
2. 4^2
3. 5^2
4. 9^0
5. 5^3
6. 2^6
7. 3^3
8. 3^2+4^2
9. 5^3+2^2
10. 6^2+2^3

11. $6^2 - 5^2$

12. $2^4 - 2^2$

13. $7^2 + 3^3 + 2^2$

Directions: Simplify the following variable expressions.

14. $(m^2)(m^5)$

15. $(x^3)(x^4)$

16. $(y^5)(y^3)$

17. $(b^7)(b^2)$

18. $(a^5)(a^2)$

19. $(x^9)(x^3)$

20. $(y^4)(y^5)$

Directions: Simplify.

21. $(x^2)^4$

22. $(y^5)^3$

23. $(a^5)^4$

24. $(x^2)^8$

25. $(b^3)^4$

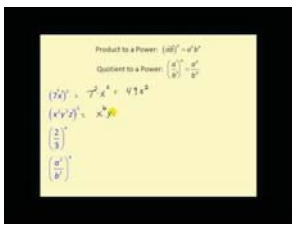
2.7 Power Properties of Exponents

Here you'll discover and use the power properties of exponents.

There are 1,000 bacteria present in a culture. When the culture is treated with an antibiotic, the bacteria count is halved every 4 hours. How many bacteria remain 24 hours later?

Watch This

Watch the second part of this video, starting around 3:30.



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James Sousa:Properties of Exponents

Guidance

The last set of properties to explore are the power properties. Let's investigate what happens when a power is raised to another power.

Investigation: Power of a Power Property

1. Rewrite $(2^3)^5$ as 2^3 five times.

$(2^3)^5 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3$

2. Expand each 2^3. How many 2's are there?

$$(2^3)^5 = \underbrace{2\cdot2\cdot2}_{2^3}\cdot\underbrace{2\cdot2\cdot2}_{2^3}\cdot\underbrace{2\cdot2\cdot2}_{2^3}\cdot\underbrace{2\cdot2\cdot2}_{2^3}\cdot\underbrace{2\cdot2\cdot2}_{2^3} = 2^{15}$$

3. What is the *product* of the powers?

$3 \cdot 5 = 15$

4. Fill in the blank. $(a^m)^n = a^{\_\cdot\_}$

$(a^m)^n = a^{mn}$

The other two exponent properties are a form of the distributive property.

Power of a Product Property: $(ab)^m = a^m b^m$

Power of a Quotient Property: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Example A

Simplify the following.

(a) $(3^4)^2$

(b) $(x^2y)^5$

Solution: Use the new properties from above.

(a) $(3^4)^2 = 3^{4 \cdot 2} = 3^8 = 6561$

(b) $(x^2y)^5 = x^{2 \cdot 5}y^5 = x^{10}y^5$

Example B

Simplify $\left(\frac{3a^{-6}}{2^2a^2}\right)^4$ without negative exponents.

Solution: This example uses the Negative Exponent Property from the previous concept. Distribute the 4^{th} power first and then move the negative power of a from the numerator to the denominator.

$$\left(\frac{3a^{-6}}{2^2a^2}\right)^4 = \frac{3^4a^{-6 \cdot 4}}{2^{2 \cdot 4}a^{2 \cdot 4}} = \frac{81a^{-24}}{2^8a^8} = \frac{81}{256a^{8+24}} = \frac{81}{256a^{32}}$$

Example C

Simplify $\frac{4x^{-3}y^4z^6}{12x^2y} \div \left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2$ without negative exponents.

Solution: This example is definitely as complicated as these types of problems get. Here, all the properties of exponents will be used. Remember that dividing by a fraction is the same as multiplying by its reciprocal.

$$\begin{aligned}\frac{4x^{-3}y^4z^6}{12x^2y} \div \left(\frac{5xy^{-1}}{15x^3z^{-2}}\right)^2 &= \frac{4x^{-3}y^4z^6}{12x^2y} \cdot \frac{225x^6z^{-4}}{25x^2y^{-2}} \\ &= \frac{y^3z^6}{3x^5} \cdot \frac{9x^4y^2}{z^4} \\ &= \frac{3x^4y^5z^6}{x^5z^4} \\ &= \frac{3y^5z^2}{x}\end{aligned}$$

Intro Problem Revisit To find the number of bacteria remaining, we use the exponential expression $1000(\frac{1}{2})^n$ where n is the number of four-hour periods.

There are 6 four-hour periods in 24 hours, so we set n equal to 6 and solve.

$1000(\frac{1}{2})^6$

Applying the Power of a Quotient Property, we get:

$1000(\frac{1^6}{2^6}) = \frac{1000 \cdot 1}{2^6} = \frac{1000}{64} = 15.625$

Therefore, there are 15.625 bacteria remaining after 24 hours.

Guided Practice

Simplify the following expressions without negative exponents.

1. $\left(\frac{5a^3}{b^4}\right)^7$

2. $(2x^5)^{-3}(3x^9)^2$

3. $\frac{(5x^2y^{-1})^3}{10y^6} \cdot \left(\frac{16x^8y^5}{4x^7}\right)^{-1}$

Answers

1. Distribute the 7 to every power within the parenthesis.

$$\left(\frac{5a^3}{b^4}\right)^7 = \frac{5^7a^{21}}{b^{28}} = \frac{78,125a^{21}}{b^{28}}$$

2. Distribute the -3 and 2 to their respective parenthesis and then use the properties of negative exponents, quotient and product properties to simplify.

$$(2x^5)^{-3}(3x^9)^2 = 2^{-3}x^{-15}3^2x^{18} = \frac{9x^3}{8}$$

3. Distribute the exponents that are outside the parenthesis and use the other properties of exponents to simplify. Anytime a fraction is raised to the -1 power, it is equal to the reciprocal of that fraction to the first power.

$$\begin{aligned}\frac{(5x^2y^{-1})^3}{10y^6} \cdot \left(\frac{16x^8y^5}{4x^7}\right)^{-1} &= \frac{5^3x^{-6}y^{-3}}{10y^6} \cdot \frac{4x^7}{16x^8y^5} \\ &= \frac{500xy^{-3}}{160x^8y^{11}} \\ &= \frac{25}{8x^7y^{14}}\end{aligned}$$

Vocabulary

Power of Power Property
: $(a^m)^n = a^{mn}$

Power of a Product Property
: $(ab)^m = a^mb^m$

Power of a Quotient Property
: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Practice

Simplify the following expressions without negative exponents.

1. $(2^5)^3$
2. $(3x)^4$
3. $\left(\frac{4}{5}\right)^2$
4. $(6x^3)^3$
5. $\left(\frac{2a^3}{b^5}\right)^7$
6. $(4x^8)^{-2}$
7. $\left(\frac{1}{7^2h^9}\right)^{-1}$
8. $\left(\frac{2x^4y^2}{5x^{-3}y^5}\right)^3$
9. $\left(\frac{9m^5n^{-7}}{27m^6n^5}\right)^{-4}$
10. $\frac{(4x)^2(5y)^{-3}}{(2x^3y^5)^2}$
11. $(5r^6)^4\left(\frac{1}{3}r^{-2}\right)^5$
12. $(4t^{-1}s)^3(2^{-1}ts^{-2})^{-3}$
13. $\frac{6a^2b^4}{18a^{-3}b^4}\cdot\left(\frac{8b^{12}}{40a^{-8}b^5}\right)^2$
14. $\frac{2(x^4y^4)^0}{2^4x^3y^5z}\div\frac{8z^{10}}{32x^{-2}y^5}$
15. $\frac{5g^6}{15g^0h^{-1}}\cdot\left(\frac{h}{9g^{15}j^7}\right)^{-3}$
16. **Challenge** $\frac{a^7b^{10}}{4a^{-5}b^{-2}}\cdot\left[\frac{(6ab^{12})^2}{12a^9b^{-3}}\right]^2\div(3a^5b^{-4})^3$
17. Rewrite 4^3 as a power of 2.
18. Rewrite 9^2 as a power of 3.
19. Solve the equation for x. $3^2\cdot 3^x=3^8$
20. Solve the equation for x. $(2^x)^4=4^8$

2.8 Exponent of a Quotient

Here you'll learn how to simplify a fraction with exponential expressions in both its numerator and denominator that is raised to another secondary power.

What if you had a fractional expression containing exponents that was raised to a secondary power, like $\left(\frac{x^8}{x^4}\right)^5$? How could you simplify it? After completing this Concept, you'll be able to use the power of a quotient property to simplify exponential expressions like this one.

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CK-12 Foundation: 0804S Power ofa Quotient

Guidance

When we raise a whole quotient to a power, another special rule applies. Here is an example:

$$\begin{aligned}\left(\frac{x^3}{y^2}\right)^4 &= \left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\cdot\left(\frac{x^3}{y^2}\right)\\ &= \frac{(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)\cdot(x\cdot x\cdot x)}{(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)\cdot(y\cdot y)}\\ &= \frac{x^{12}}{y^8}\end{aligned}$$

Notice that the exponent outside the parentheses is multiplied by the exponent in the numerator and the exponent in the denominator, separately. This is called the power of a quotient rule:

Power Rule for Quotients: $\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n\cdot p}}{y^{m\cdot p}}$

Let's apply these new rules to a few examples.

Example A

Simplify the following expressions.

a) $\frac{4^5}{4^2}$

b) $\frac{5^3}{5^7}$

c) $\left(\frac{3^4}{5^2}\right)^2$

Solution

Since there are just numbers and no variables, we can evaluate the expressions and get rid of the exponents completely.

a) We can use the quotient rule first and then evaluate the result: $\frac{4^5}{4^2} = 4^{5-2} = 4^3 = 64$

OR we can evaluate each part separately and then divide: $\frac{4^5}{4^2} = \frac{1024}{16} = 64$

b) Use the quotient rule first and hen evaluate the result: $\frac{5^3}{5^7} = \frac{1}{5^4} = \frac{1}{625}$

OR evaluate each part separately and then reduce: $\frac{5^3}{5^7} = \frac{125}{78125} = \frac{1}{625}$

Notice that it makes more sense to apply the quotient rule first for examples (a) and (b). Applying the exponent rules to simplify the expression *before* plugging in actual numbers means that we end up with smaller, easier numbers to work with.

c) Use the power rule for quotients first and then evaluate the result: $\left(\frac{3^4}{5^2}\right)^2 = \frac{3^8}{5^4} = \frac{6561}{625}$

OR evaluate inside the parentheses first and then apply the exponent: $\left(\frac{3^4}{5^2}\right)^2 = \left(\frac{81}{25}\right)^2 = \frac{6561}{625}$

Example B

Simplify the following expressions:

a) $\frac{x^{12}}{x^5}$

b) $\left(\frac{x^4}{x}\right)^5$

Solution

a) Use the quotient rule: $\frac{x^{12}}{x^5} = x^{12-5} = x^7$

b) Use the power rule for quotients and then the quotient rule: $\left(\frac{x^4}{x}\right)^5 = \frac{x^{20}}{x^5} = x^{15}$

OR use the quotient rule inside the parentheses first, then apply the power rule: $\left(\frac{x^4}{x}\right)^5 = (x^3)^5 = x^{15}$

Example C

Simplify the following expressions.

a) $\frac{6x^2y^3}{2xy^2}$

b) $\left(\frac{2a^3b^3}{8a^7b}\right)^2$

Solution

When we have a mix of numbers and variables, we apply the rules to each number or each variable separately.

a) Group like terms together: $\frac{6x^2y^3}{2xy^2} = \frac{6}{2} \cdot \frac{x^2}{x} \cdot \frac{y^3}{y^2}$

Then reduce the numbers and apply the quotient rule on each fraction to get $3xy$.

b) Apply the quotient rule inside the parentheses first: $\left(\frac{2a^3b^3}{8a^7b}\right)^2 = \left(\frac{b^2}{4a^4}\right)^2$

Then apply the power rule for quotients: $\left(\frac{b^2}{4a^4}\right)^2 = \frac{b^4}{16a^8}$

Watch this video for help with the Examples above.

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CK-12 Foundation: Power of aQuotient

Vocabulary

- **Quotient of Powers Property:** For all real numbers x,

$\frac{x^n}{x^m} = x^{n-m}$.

- **Power of a Quotient Property:**

$\left(\frac{x^n}{y^m}\right)^p = \frac{x^{n \cdot p}}{y^{m \cdot p}}$

Guided Practice

Simplify the following expressions.

a) $(x^2)^2 \cdot \frac{x^6}{x^4}$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Solution

In problems where we need to apply several rules together, we must keep the order of operations in mind.

a) We apply the power rule first on the first term:

$$(x^2)^2 \cdot \frac{x^6}{x^4} = x^4 \cdot \frac{x^6}{x^4}$$

Then apply the quotient rule to simplify the fraction:

$$x^4 \cdot \frac{x^6}{x^4} = x^4 \cdot x^2$$

And finally simplify with the product rule:

$$x^4 \cdot x^2 = x^6$$

b) $\left(\frac{16a^2}{4b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$

Simplify inside the parentheses by reducing the numbers:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}}$$

Then apply the power rule to the first fraction:

$$\left(\frac{4a^2}{b^5}\right)^3 \cdot \frac{b^2}{a^{16}} = \frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}}$$

Group like terms together:

$$\frac{64a^6}{b^{15}} \cdot \frac{b^2}{a^{16}} = 64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}}$$

And apply the quotient rule to each fraction:

$$64 \cdot \frac{a^6}{a^{16}} \cdot \frac{b^2}{b^{15}} = \frac{64}{a^{10}b^{13}}$$

Practice

Evaluate the following expressions.

1. $\left(\frac{3}{8}\right)^2$
2. $\left(\frac{2^2}{3^3}\right)^3$
3. $\left(\frac{2^3 \cdot 4^2}{2^4}\right)^2$

Simplify the following expressions.

4. $\left(\frac{a^3b^4}{a^2b}\right)^3$
5. $\left(\frac{18a^4}{15a^{10}}\right)^4$
6. $\left(\frac{x^6y^2}{x^4y^4}\right)^3$
7. $\left(\frac{6a^2}{4b^4}\right)^2 \cdot \frac{5b}{3a}$
8. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$
9. $\frac{(2a^2bc^2)(6abc^3)}{4ab^2c}$ for $a = 2, b = 1$, and $c = 3$
10. $\left(\frac{3x^2y}{2z}\right)^3 \cdot \frac{z^2}{x}$ for $x = 1, y = 2$, and $z = -1$
11. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 2, y = -3$
12. $\frac{2x^3}{xy^2} \cdot \left(\frac{x}{2y}\right)^2$ for $x = 0, y = 6$
13. If $a = 2$ and $b = 3$, simplify $\frac{(a^2b)(bc)^3}{a^3c^2}$ as much as possible.