Characteristics of Exponential Functions

To gain a better understanding of exponents, we will look at the graph $y = c^x$, the exponential function. The base can be any value greater than zero except ____. Therefore, we must look at $0 < c < 1$ and $c > 1$. Remember *c* cannot be less than zero. We define the exponential function and its graph as follows:

Exponential Function

The equation $f(x) = c^x$, $c > 0, c \neq 1$ is called an exponential function with a base *c*, and *x* any real value.

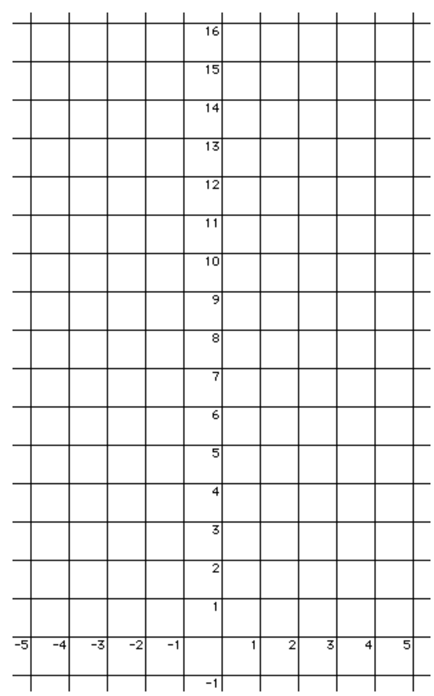
Why is $c > 0, c \neq 1$?

Exploration of $y = c^x$

Example 1: To draw a manageable graph, let $c = 2$. Let's graph $y = 2^x$ using a table of values.

x	-3	-2	-1	0	1	2	3	4
y								

a) What happens to the graph as *x* becomes more and more negative, without bound?



Note: The *x*-axis is an ___________________ of the function.

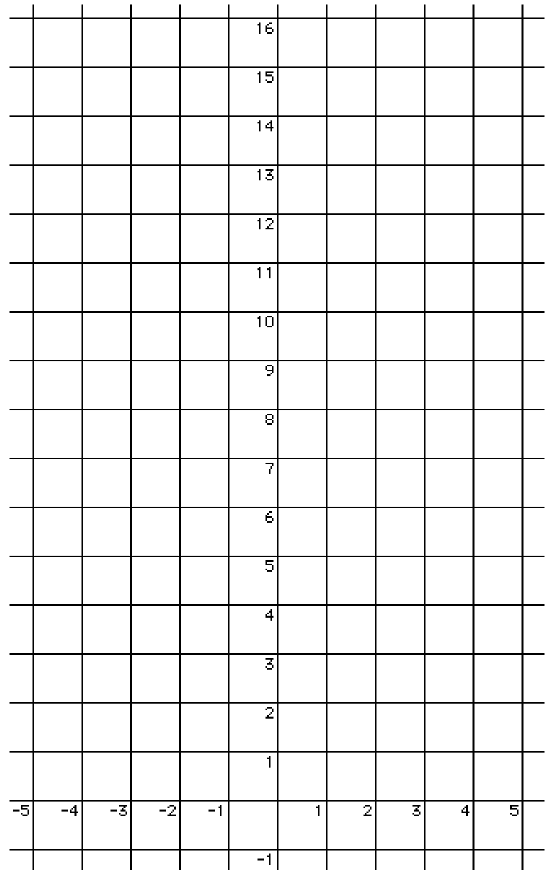
b) Determine any intercepts and the domain and range.

c) Is the graph increasing or decreasing?

Example 2: Graph the function $y = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3	4
y								

a) Determine the asymptotes, intercepts and the domain and range of $y = \left(\frac{1}{2}\right)^x$



b) Is the graph increasing or decreasing?

How does the graph of $y = \left(\frac{1}{2}\right)^x$ compare to the graph of $y = 2^x$?

1) Asymptotes: ____________________

2) y–intercept ____________________

3) Domain: ___________ Range: ____________________

What is the main difference between the two graphs?

What conclusion can be drawn by the above statement?

Features of the graph $y = c^x$

1) y–intercept

2) x-intercept

2) The function has an asymptote whose equation is ________________

3) Range:__________ Domain: ___________

4) If $c > 1$, ____.

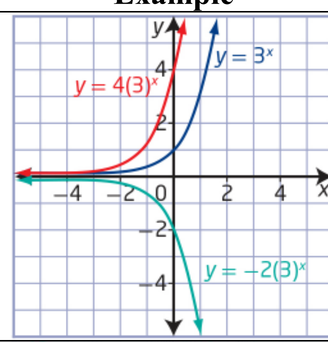
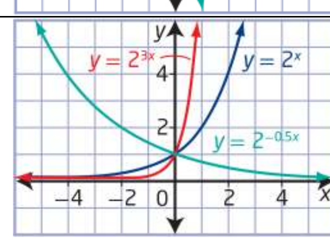
If $0 < c < 1$, ____.

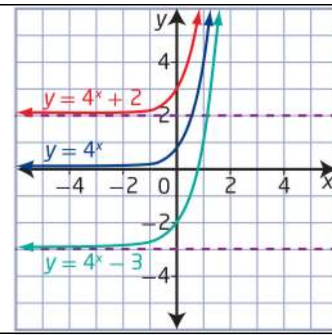
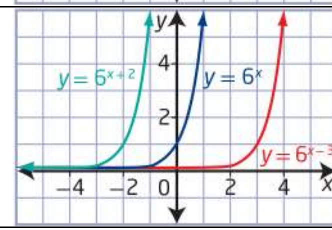
Assignment: page 342-343 #1-5

Transformations of Exponential Functions

Like many functions, you can apply a transformation on an exponential graph $y = c^x$ to obtain

$$y = a(c)^{b(x-h)} + k$$

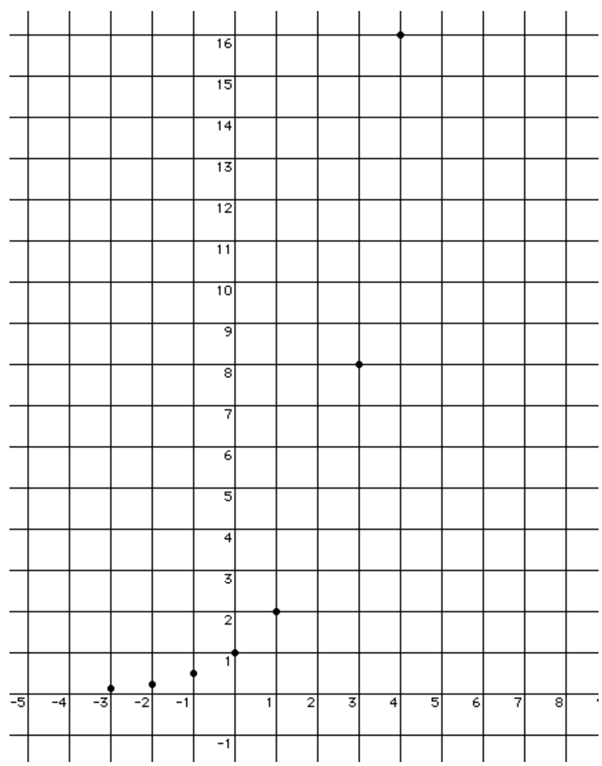
Parameter	Transformation	Example
a		
b		

h		
k		

The general transformation is (x, y) corresponds to $\left(\frac{x}{b} + h, ay + k\right)$

Example 1: Graph the function $y = 3(2)^{(x-3)}$.

Let's compare the graphs of $y = 2^x$ and $y = 3(2)^{(x-3)}$. Identify the domain, range, equation of the horizontal asymptote and any intercepts.

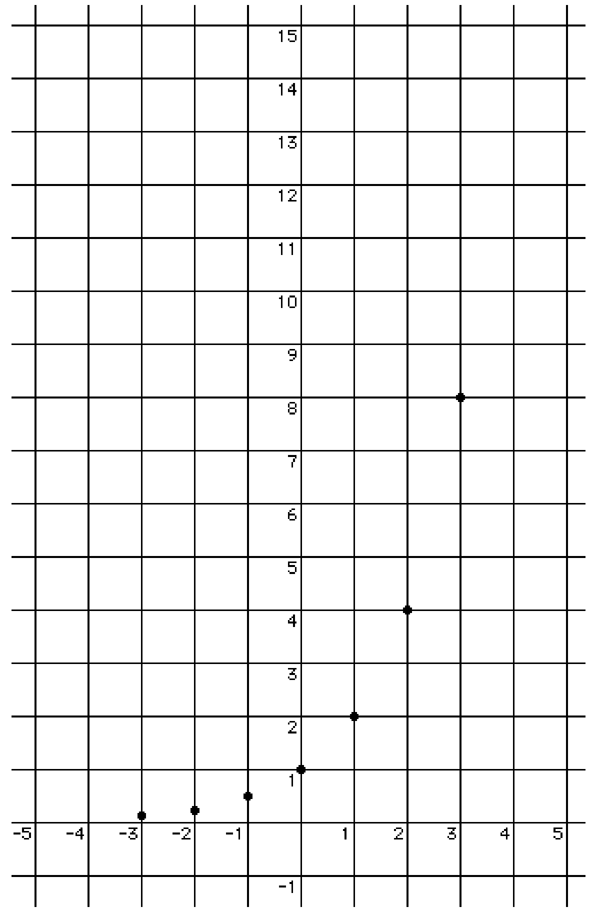


To graph the function, you must determine the points of the base function and the transformation. It would help if you made a table of values.

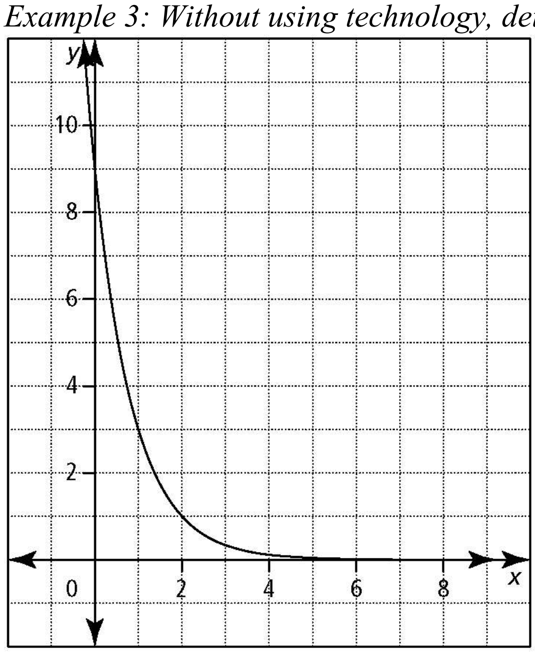
Example 2: Draw the graph of $y = 3(2^x - 1)$.

How does the graph of $y = 2^{-2x} + 1$ compare to the graph of $y = 2^x$?

Identify the domain, range, equation of the horizontal asymptote and any intercepts.



Example 3: Without using technology, determine the equation of the graph.



Example 4: The population, P thousands, of bacteria culture can be modelled with the equation $P = 1.4(1.38)^n$, where n is the number of hours elapsed.

a) Use a table of values to graph the function for $0 \leq n \leq 10$

b) Explain the roles of the numbers 1.4 and 1.38.

c) What are the domain and range of this function?

Assignment: page 354-355 #1, 2, 3(aceg), 4, 6, 9, 11

Solving Exponential Equations

Definition: exponential equation is an equation that has ___________________________ in the exponent.

Review: Recall Exponent Laws

1. Product Law: $a^m \times a^n =$
2. Quotient Law: $\dfrac{a^m}{a^n} =$
3. Power Law: $\left(a^m\right)^n =$
4. Zero Exponent: $a^0 =$
5. Negative Exponent: $a^{-m} =$
6. $\left(ab\right)^n =$
7. $\left(\dfrac{a}{b}\right)^n =$
8. Fraction Exponent: $a^{\frac{m}{n}} =$
9. $a^m = a^n$ *ONLY IF* $m = n$

Example 1: Rewrite each expression as a power with a base 2.

a) 32

b) $\dfrac{1}{8}$

c) $8^{\frac{2}{3}}\left(\sqrt{16}\right)^3$

Solving equations where the exponent involves a variable:
Equations which can be converted to the same integral base

Example 2: Solve for *x*. $8^{2x+1} = 16^{3x-5}$

Try this… $9^{x-2} = \left(\frac{1}{27}\right)^{2x+1}$

Example 2: Solve for *x*. $\left(\sqrt{125}\right)^{2x+1} = \sqrt[3]{625}$

Note: Not all exponential equations can be solved using the above method. For example $9^x = 5$ ***does not have the same base. Another method will be taught at a later class.***

Assignment: page 364 #1-5

Understanding Logarithms

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a logarithmic function. It is written as:

Definition:

It is read as ____

It is extremely important to be aware of the restrictions on the above statement.

Changing between Logarithmic form and Exponential form

Note: If the log has a base of 10, $f(x) = \log_{10}(x)$*, it can be written as just* $f(x) = \log(x)$*; 10 is assumed.*

Example 1:

a) Express $5^4 = 625$ in logarithmic form

b) Express $\log 1000 = 3$ in exponential form.

c) Express $4^{-3} = \frac{1}{64}$ in logarithmic form.

d) Rewrite $y = 2^x$ as a logarithmic function

Example 2: Without a calculator, evaluate

a) $\log_4 64 =$

b) $\log_8(1) =$

c) $\log 0.001 =$

d) $\log_2 \sqrt{8} =$

Using Benchmarks to Estimate the values of Logarithm

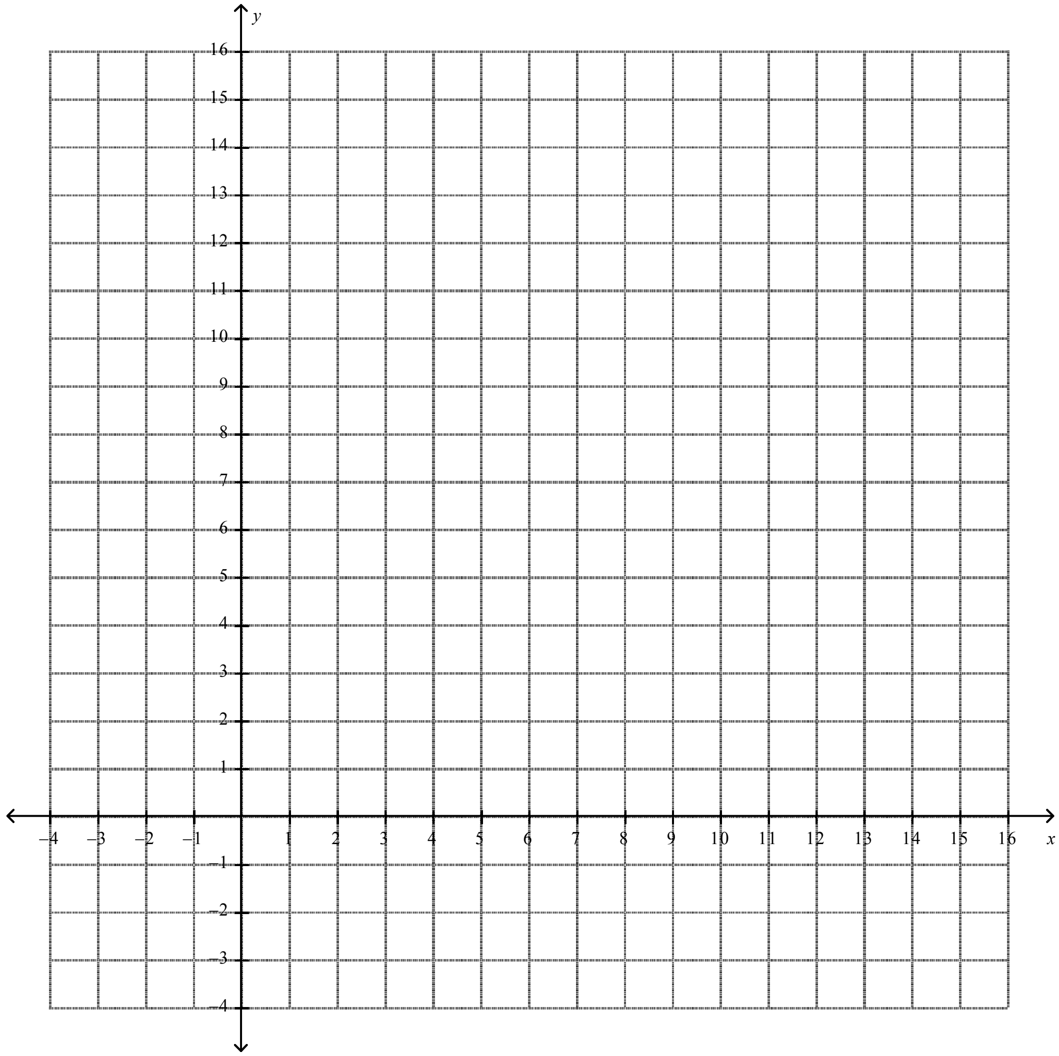
Example 3: To the nearest tenth, estimate $\log_2 5$.

Note: unlike the previous example, 5 cannot be easily written as base of 2, and more difficult to evaluate. Therefore you can use a process called benchmark.

Graph $y = \log_c x$

Since this is an inverse of an exponential function, the graph is a reflection of the exponential function *y* = _______ in the line *y* = *x*.

Example 4: Recall the graph of $y = 2^x$, draw the graph for $y = \log_2 x$. Determine the domain and range, equations of any asymptotes, and intercepts.



Summary:

Since $y = \log_b x$ is the inverse of $y = b^x$, then the following can be stated:

	$y = b^x$	$y = \log_b x$
Domain		
Range		
x-intercept		
y-intercept		
Equation of asymptote		
Restrictions		

Assignment: page 380 #1-5, 8, 9

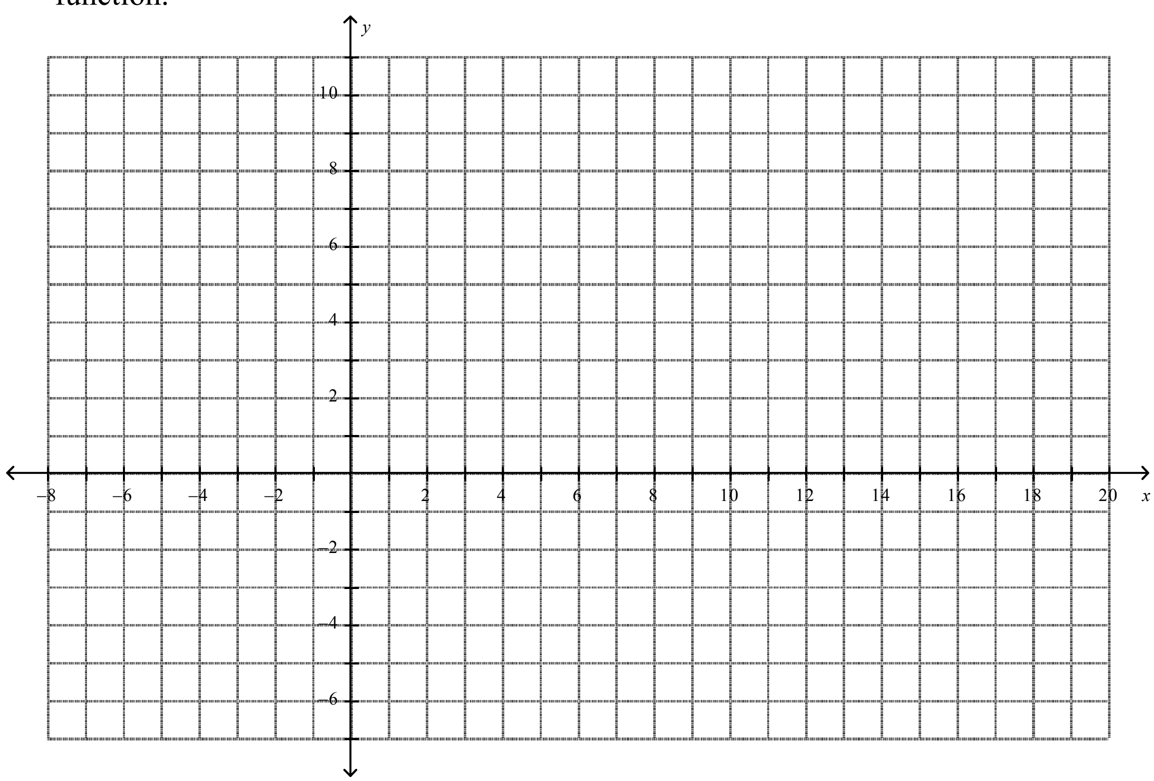
Transformations of Logarithmic Functions

The graph of the logarithmic function $y = a\log_c\left(b(x-h)\right) + k$ can be obtained by transforming the graph of $y = \log_c x$. The table below uses mapping notation to show how each parameter affects the point (x, y) on the graph $y = \log_c x$.

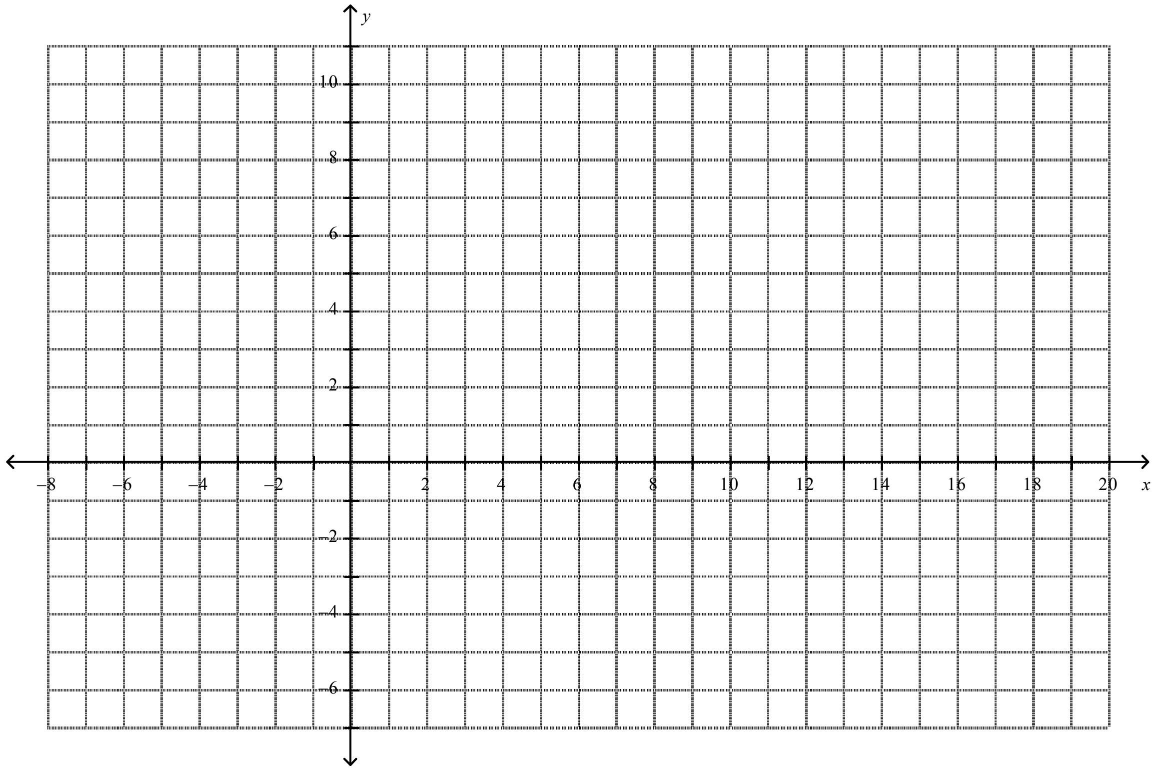
Parameter	Translation
a	$(x, y) \to (x, ay)$
b	$(x, y) \to \left(\frac{x}{b}, y\right)$
h	$(x, y) \to (x + h, y)$
k	$(x, y) \to (x, y + h)$

The general transformation is (x, y) corresponds to $\left(\frac{x}{b} + h, ay + k\right)$

Example 1: Describe how the graph of $y = \log_2(x+3) + 4$ can be obtained from the graph of $y = \log_2 x$ then graph the function.
Identify the intercepts and the equation of the asymptotes of the graph, and the domain and range of the function.



Example 2: Draw the function $y = -3\log_4(2x) + 1$.



Assignment: page 389-390 #1-7

Laws of Logarithms

Since Logarithms are exponents, there are laws of logarithms

The Product Law

The Quotient Law

The Power Law

Example 1: Write each expression in terms of individual *x*, *y*, and *z*.

a) $\log_7\left(xy^3\sqrt{z}\right)$

b) $\log\left(\frac{x^2}{y^3\sqrt{z}}\right)$

Example 2: Express each expression as a single logarithm. State any restriction on the variable.

a) $\log x + 3\log y - \frac{1}{2}\log w$

b) $3\log x - 6\log y - \frac{1}{2}\log w$

Example 3: Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_3 9\sqrt{3}$

b) $2\log_3 6 - \frac{1}{2}\log_3 64 + \log_3 2$

Example 4: If $\log 3 = P$ and $\log 5 = Q$, write an algebraic expression in terms of P and Q for each of the following.

a) $\log 25\sqrt{3}$

b) $\log 1500$

Assignment: page 400-401 #1-3(acd), 7-11

Properties of logarithmic functions *(must know rules)*

1. $\log_b 1 = 0$
2. $\log_b b = 1$
3. Product Law
   $\log_b xy = \log_b x + \log_b y$
4. Quotient Law
   $\log_b \frac{x}{y} = \log_b x - \log_b y$
5. Power Law
   $\log_b a^n = n\log_b a$
6. Change of Base Law
   $\log_b a = \frac{\log_x a}{\log_x b}$

Additional of logarithmic functions *(helpful rules)*

7. $b^{\log_b a} = a$, $a > 0$

8. $\log_b a = \dfrac{1}{\log_a b}$

9. $\log_b a = -\log_{\frac{1}{b}} a$

10. $\log_b \dfrac{1}{x} = -\log_b x$

11. $\dfrac{\log_a x}{\log_a y} = \dfrac{\log_b x}{\log_b y}$

12. $\log_b x = \log_b y$, if and only if $x = y$

13. $\log_{B^y} B^x = \dfrac{x}{y}$

14. $\log_B B^x = x$

Logarithmic and Exponential Equations

Logs on both sides

Example 1: Solve for x: $\log_2(x-2) + \log_2 x = \log_2 3$

Check the answers in the original equation.

LHS	*RHS*
$\log_2(x-2) + \log_2 x$	$\log_2 3$

Once you have solved a logarithmic equation, you must **check each value of your solution**. Substitute each value into the original equation, and **make sure** that the equation is defined for this value of the variable. In other words, if you find a value for which the original statement of the equation is undefined – forcing you to take the log of a negative number, or zero – then you must **reject** that value as part of the solution.

You try…Solve for x: $\log(x-1)+\log(2x-3)=\log(2x^2-5)$

Logs on one side

Example 2: Solve for x, checking for any extraneous solutions.

$$\log_5(3x+1)+\log_5(x-3)=3$$

Recall the problem on page 8 where the bases were not similar, $9^x = 5$. These exponent equations which cannot be easily converted to the same base, then you have to use logs.

Example 3: Solve $36 = 3\left(2^{x+1}\right)$

Example 4: Solve $5^{x+1} = 2^{x-3}$

Assignment: page 412-413 #1-2(acd), 4(acd), 5-8

Solving Problems w/ Exponents and Logarithms

Solving Problem with One Initial Principle Amount

The compound interest formula is $A = P(1+i)^n$, where A is future amount, P is the present amount principle, i is the interest rate per compounding period expressed as a decimal, and n is the number compounding periods. All interest rates are annual percentage rates (APR).

Example 1: David inherits \$10 000 and invests in a guaranteed investment certificate (GIC) that earns 6%, compounded semi-annually. How long will it take for the GIC to be worth \$11 000?

Solving Problem Involving Future Value

When a series of equal investments is made at equal time intervals, and the compounding period for the interest is equal to the time interval for the investments, the amount in dollars, or future value FV, of these investments can be determined using this formula:

$$FV = \frac{R\left[(1+i)^n - 1\right]}{i},$$

where R dollars is the regular investment, i is the investment rate per compounding period, and n is the number of investments.

Example 2: Determine how many monthly investments of \$200 would have to be made into an account that pays 6% annual interest, compounded monthly, for the future value to be \$100 000.

Solving Problems Involving Loans

Many people borrow money to finance a purchase. A loan usually repaid by making regular equal payments for a fixed period of time. The amount borrowed is called the present value, *PV*, of the loan. The following formula relates the present value to *n* equal payments of *R* dollars each, when the interest rate per compounding period is *i*. The compounding period is equal to the time between payments. The first payment is made after a time equal to the time between payments. The first payment is made after a time equal to the compounding period.

$$PV = \frac{R\left[1-(1+i)^{-n}\right]}{i}$$

Example 3: A person borrows $15 000 to buy a car. The person can afford to pay $300 a month. The loan will be repaid with equal monthly payments at 6% annual interest, compounded monthly. Hoy many payments will the person make?

Solving Problems Involving growth/decay of a population

Growth and decay formula follows the similar pattern as compound interest.

$$A = A_0(r)^{\frac{t}{T}}$$

Where A is the final amount, A_0 is the initial amount, r is growth or decay percentage as a decimal (note if it is a growth, the value is greater than 100% and if it is a decay the value is less than 100%), t is the total time that item is left, T is the total time of growth or decay per period.

Example 4: Population of a country is 8 million and growing at a rate of 2.13% per year. Determine the number of years for the population to double.

Solve A Problem Involving the Richter Scale/ Decibel Scale/pH Scale

When physical quantities have a large range of values, they are measured using a logarithmic scale. Some examples include the Richter scale, the decibel scale and the pH scale.

The Richter and pH scales can be defined as: $M = \log\frac{I}{I_0}$, where M represent magnitude, I represent intensity and I_0 is the intensity of the standard. For the measure of intensity of sound use: $L = 10\log\left(\frac{I}{I_0}\right)$, where L represent the loudness, I is the intensity and I_0 is the intensity of the quietest sound.

Example 5: The Olympia, Washington earthquake on February 28, 2001 had a magnitude of 6.8 on the Richter Scale. A more recent earthquake off the coast of Vancouver Island had a magnitude of 4.

a) Calculate the intensity of the Olympia, Washington earthquake in terms of a standard earthquake.

b) Calculate the intensity of the Vancouver Island earthquake in terms of a standard earthquake.

c) How many times as intense as the Vancouver Island earthquake was the Olympia earthquake?

Example 6: A normal conversation registers 60 dB, while a rating of over 90 dB could cause temporary hearing loss.

a) Calculate the intensity of a normal conversation in terms of I_0.

b) Calculate the intensity of over 90 dB in terms of I_0.

c) How many times as intense as the sound of a normal conversation to sounds with a rating of 90dB?

Assignment: worksheet