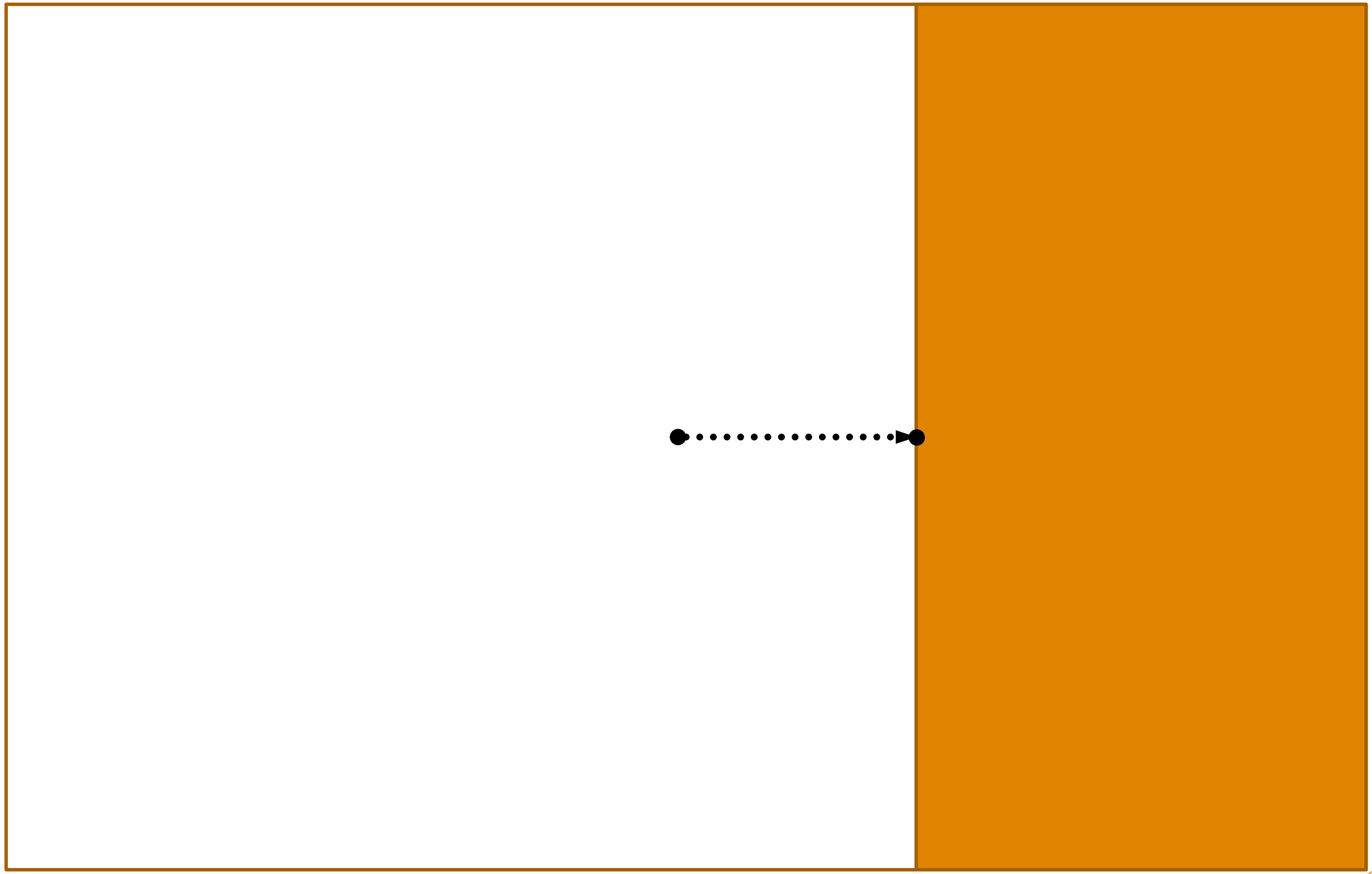
Chasing Nested Convex Bodies

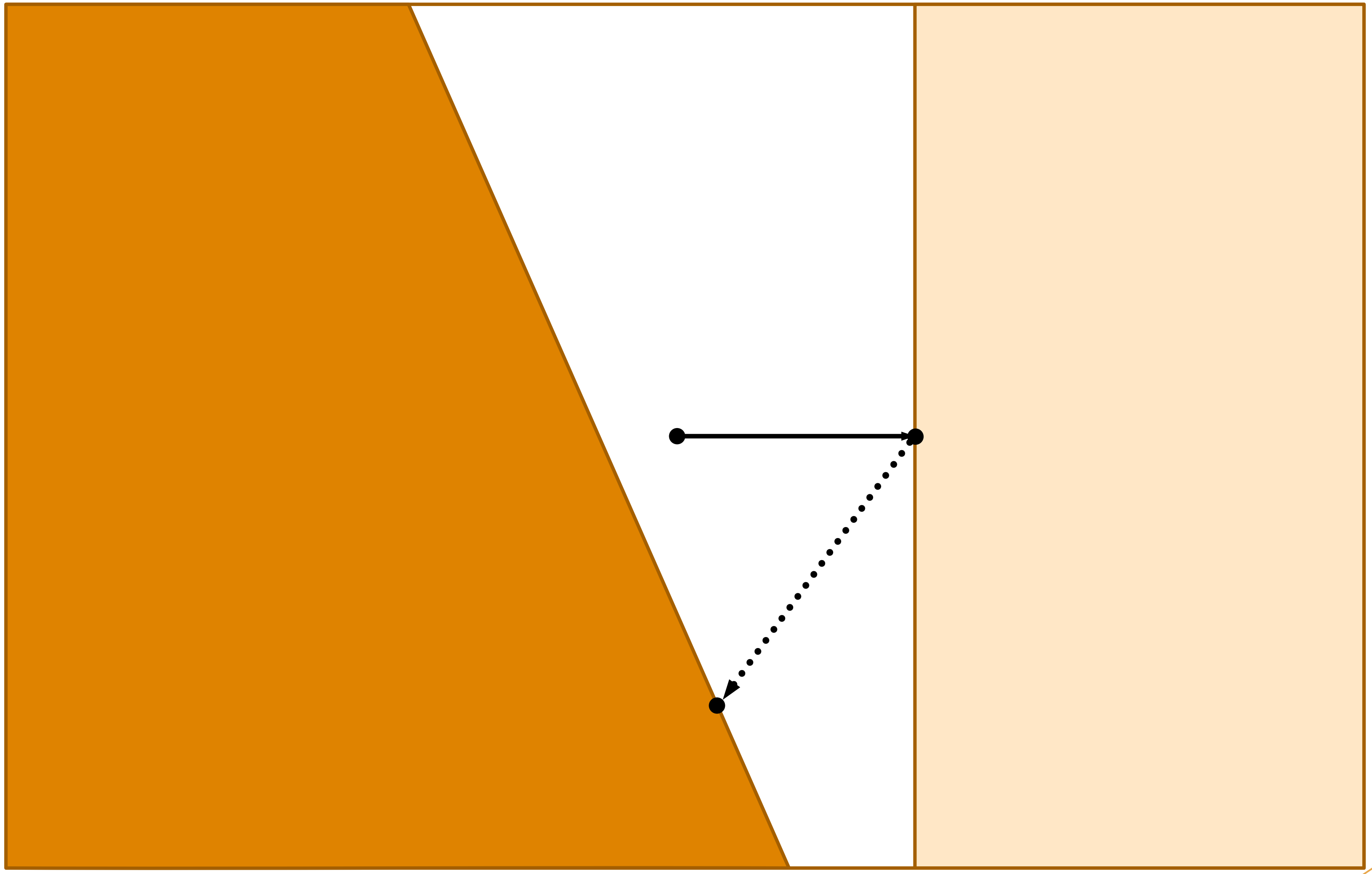
C.J. Argue

Joint with Sébastien Bubeck, Michael Cohen
Anupam Gupta, Yin Tat Lee

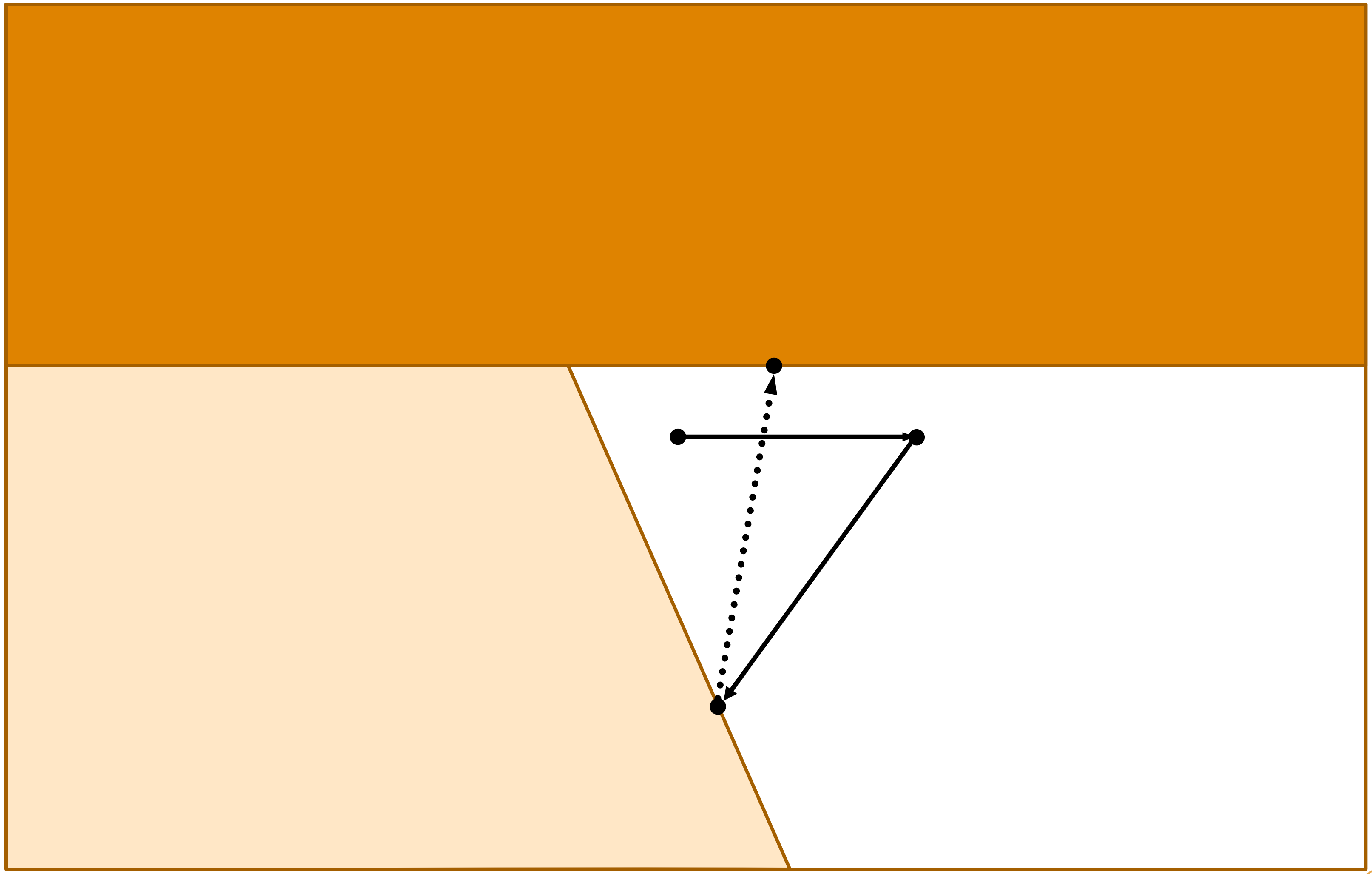
The Problem



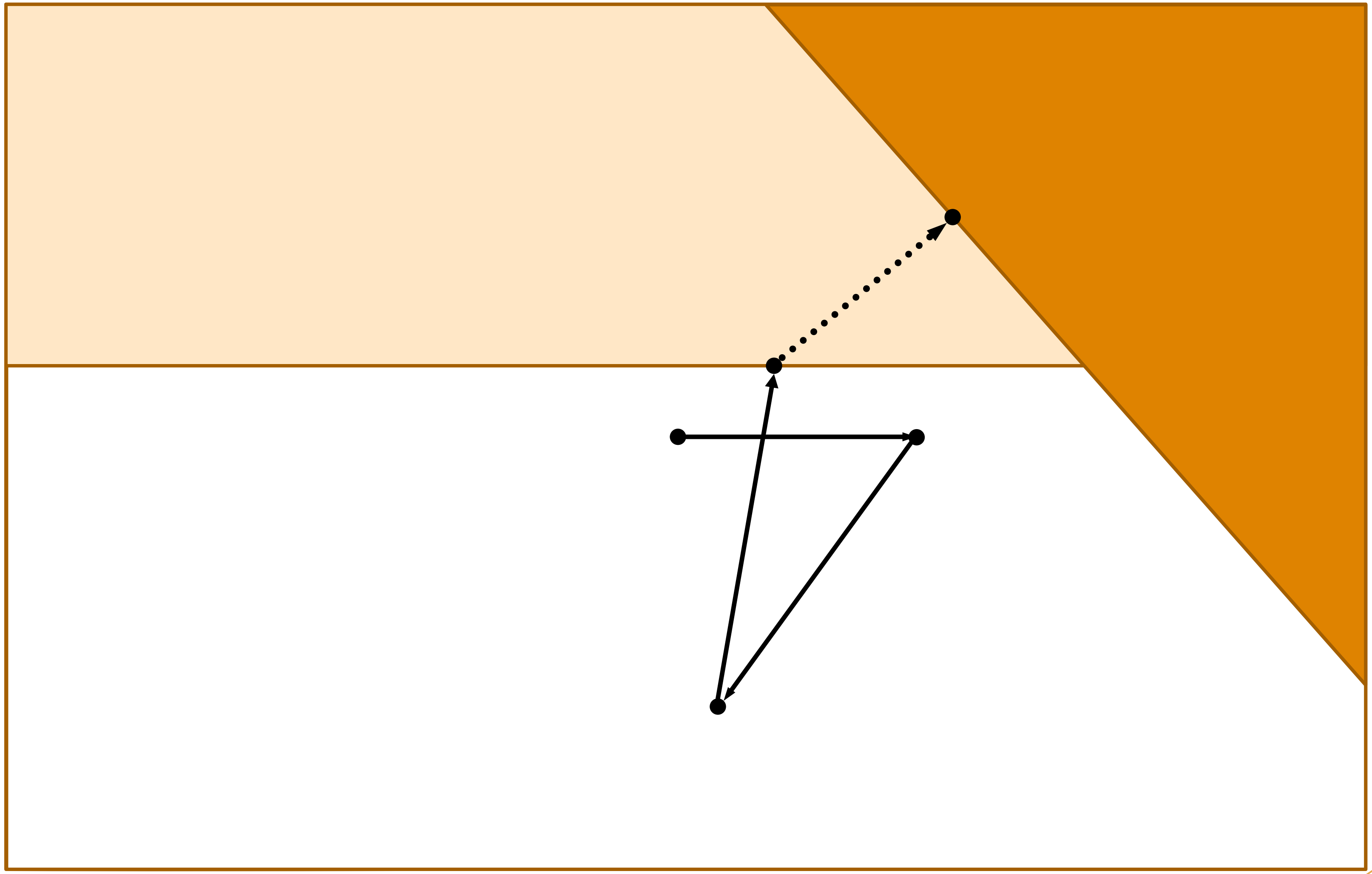
The Problem



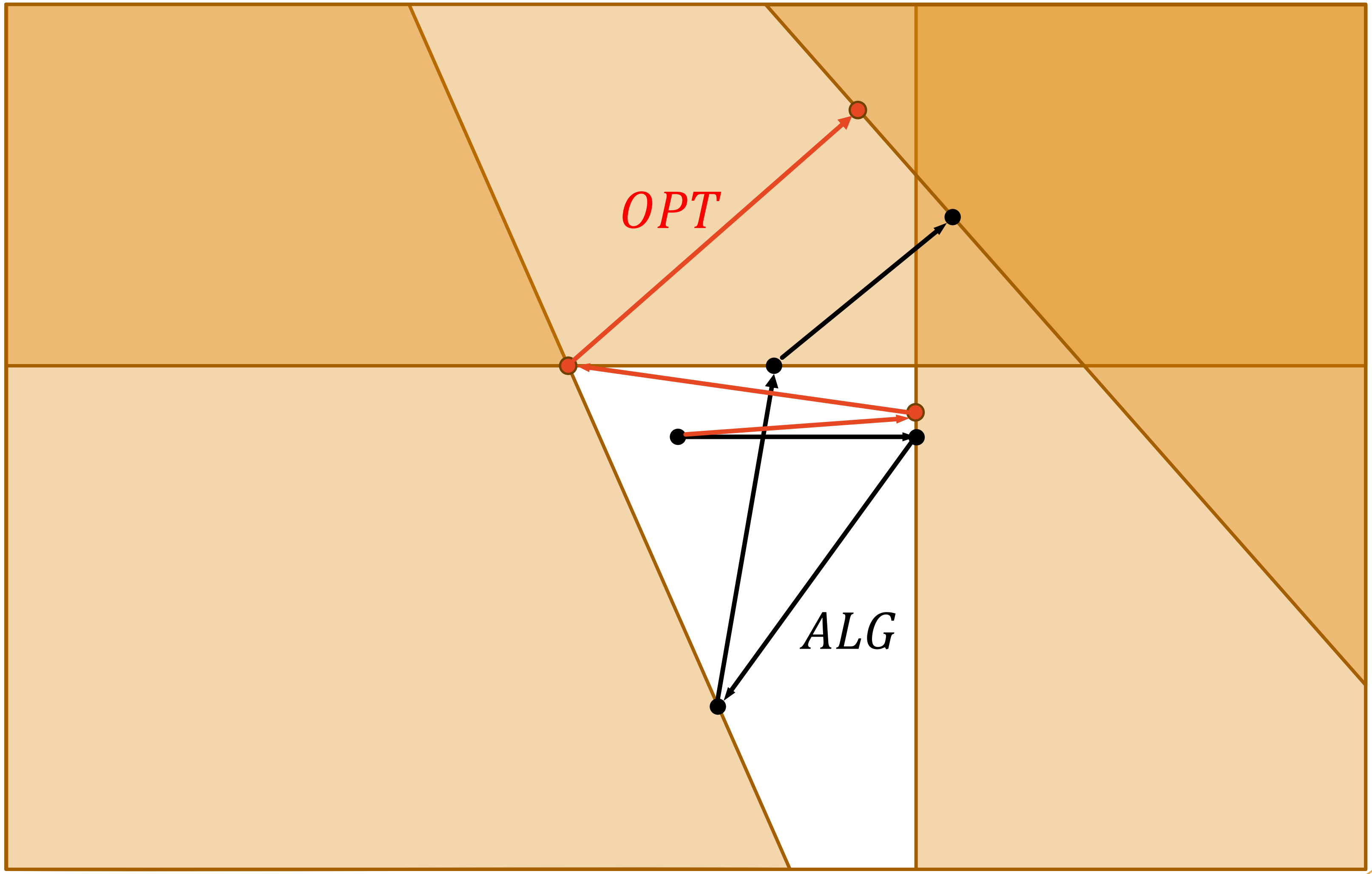
The Problem



The Problem



The Problem



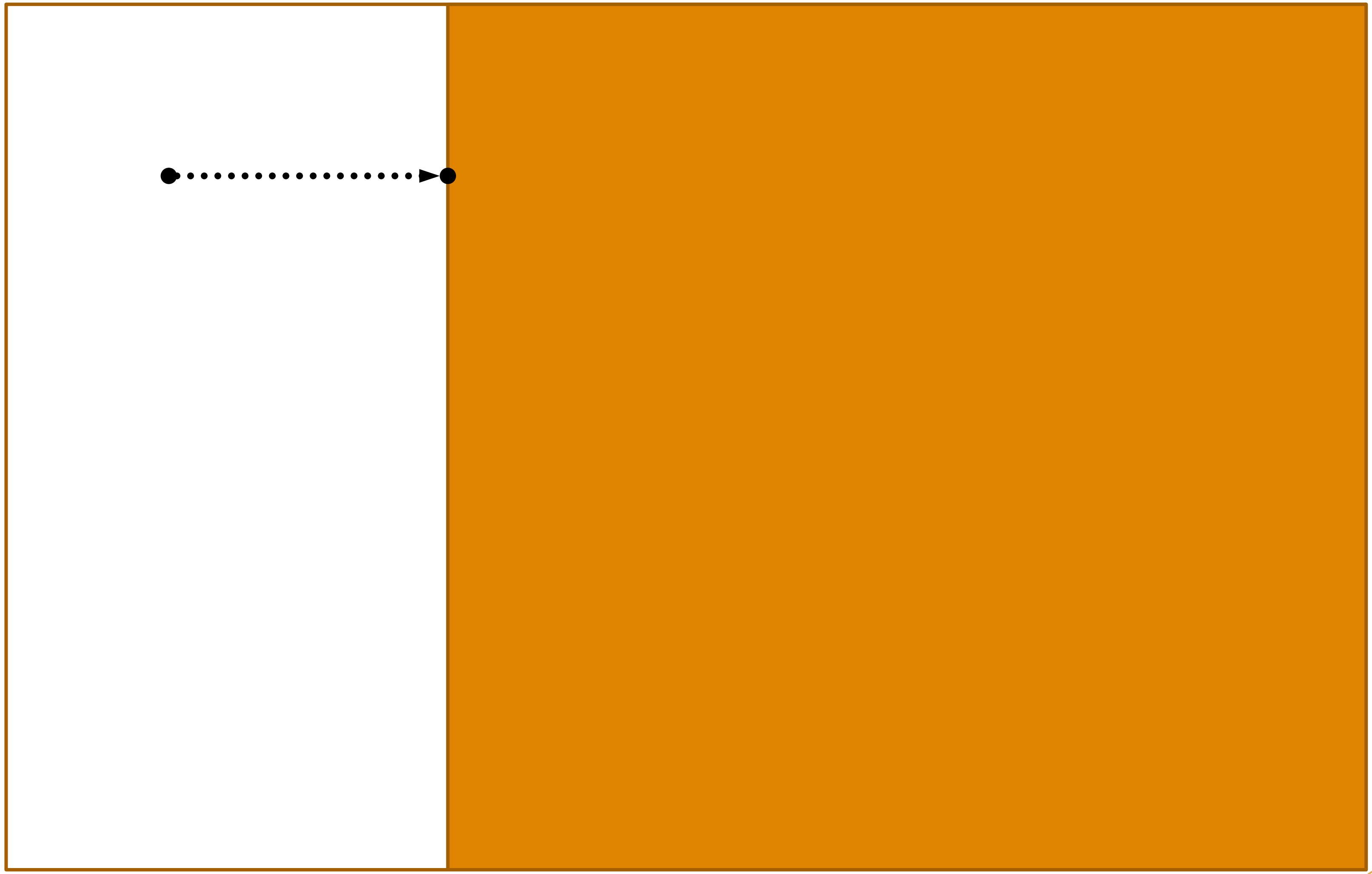
The Problem – Formal Definition

- Given convex sets $K^1, K^2, \ldots, K^t$ in $\mathbb{R}^d$
- Choose $x^i \in K^i$ online ($x^0 = 0$)
- Cost $ALG = \sum_{i=1}^{t} ||x^i - x^{i-1}||$
- Goal – minimize competitive ratio

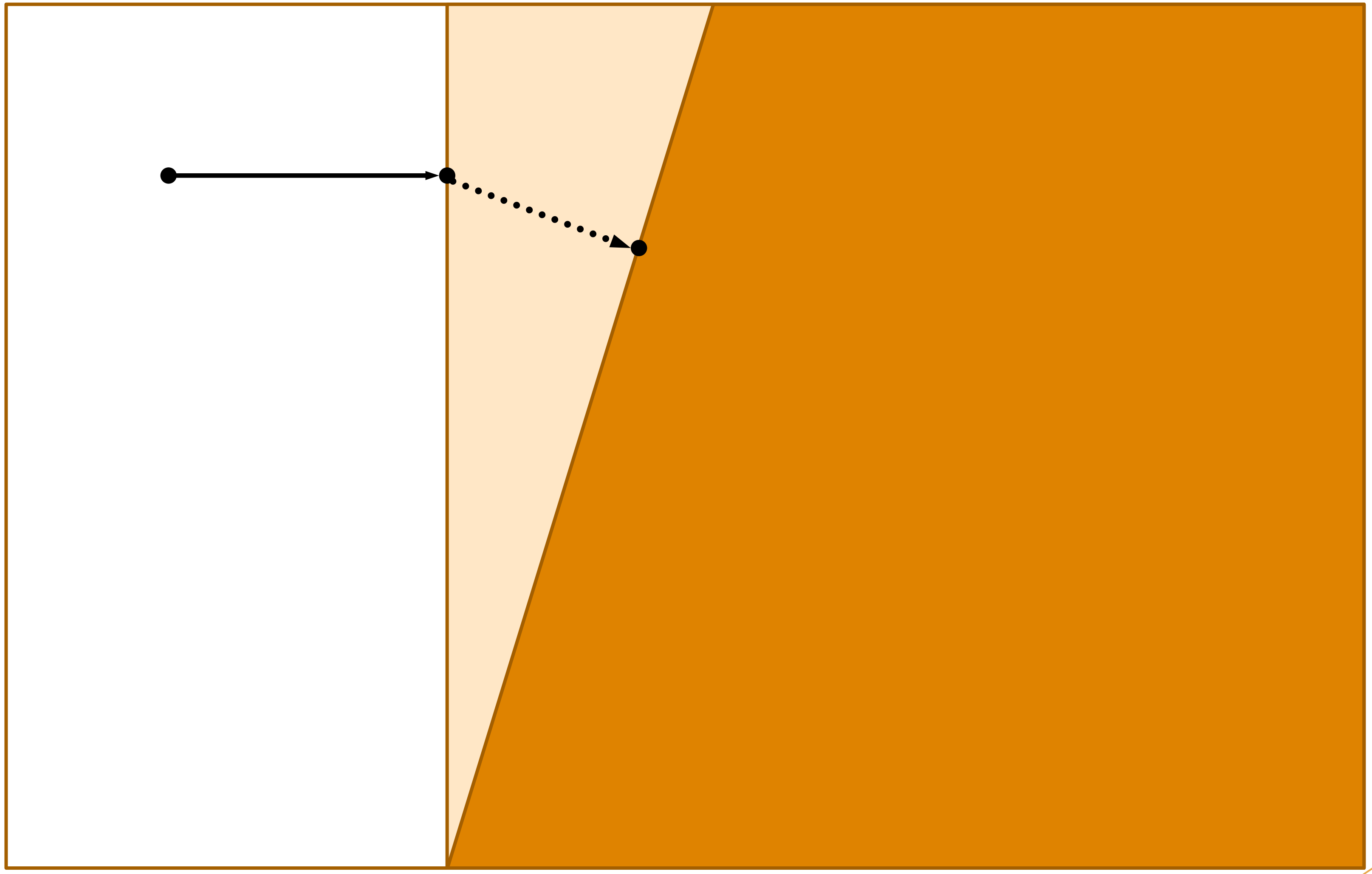
$$\mathrm{cr}(ALG) \coloneqq \max_{\sigma} \frac{ALG(\sigma)}{OPT(\sigma)}$$

  - σ arbitrary instance
  - $OPT(\sigma)$ optimal *offline* cost

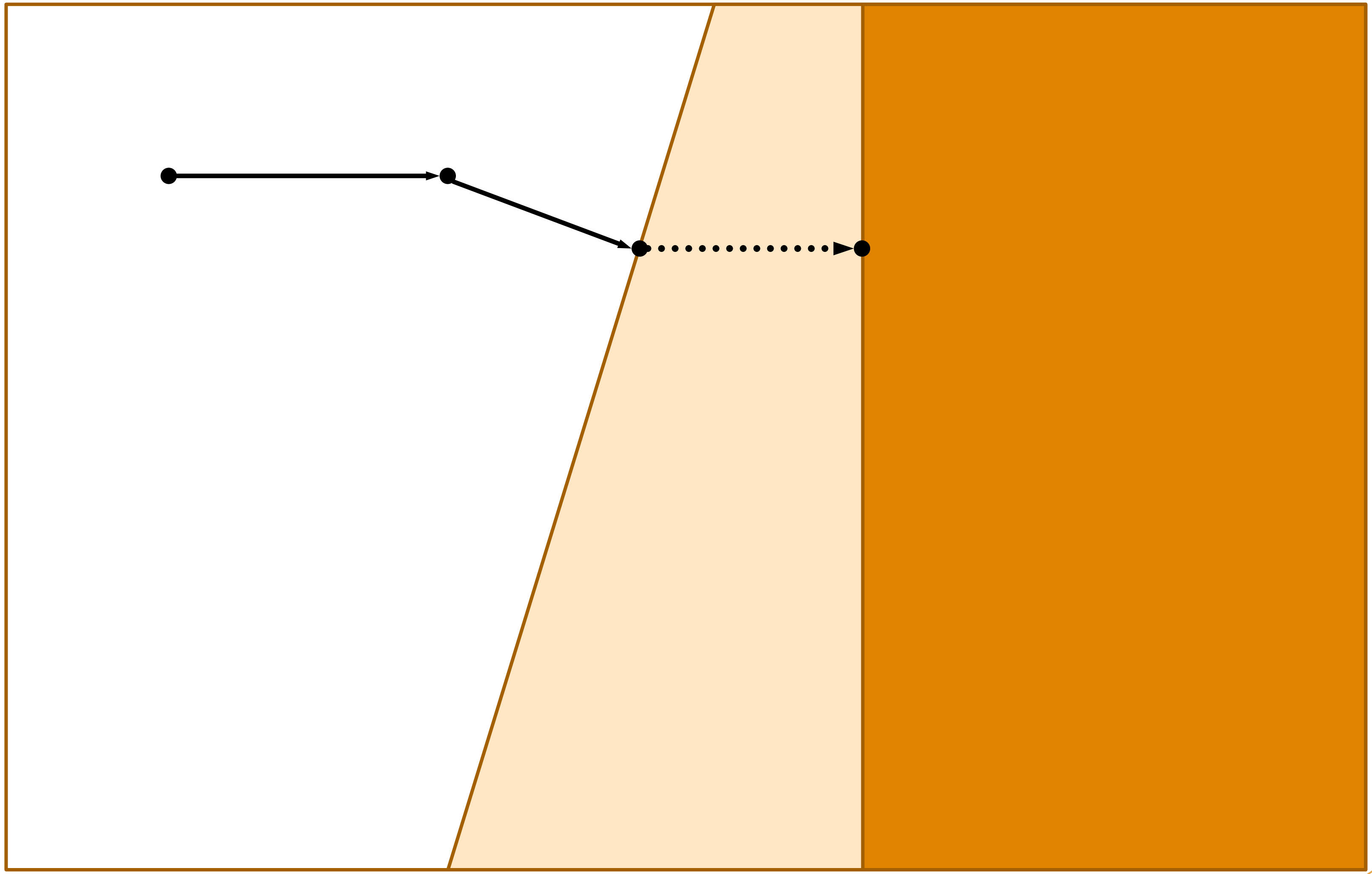
Nested Version



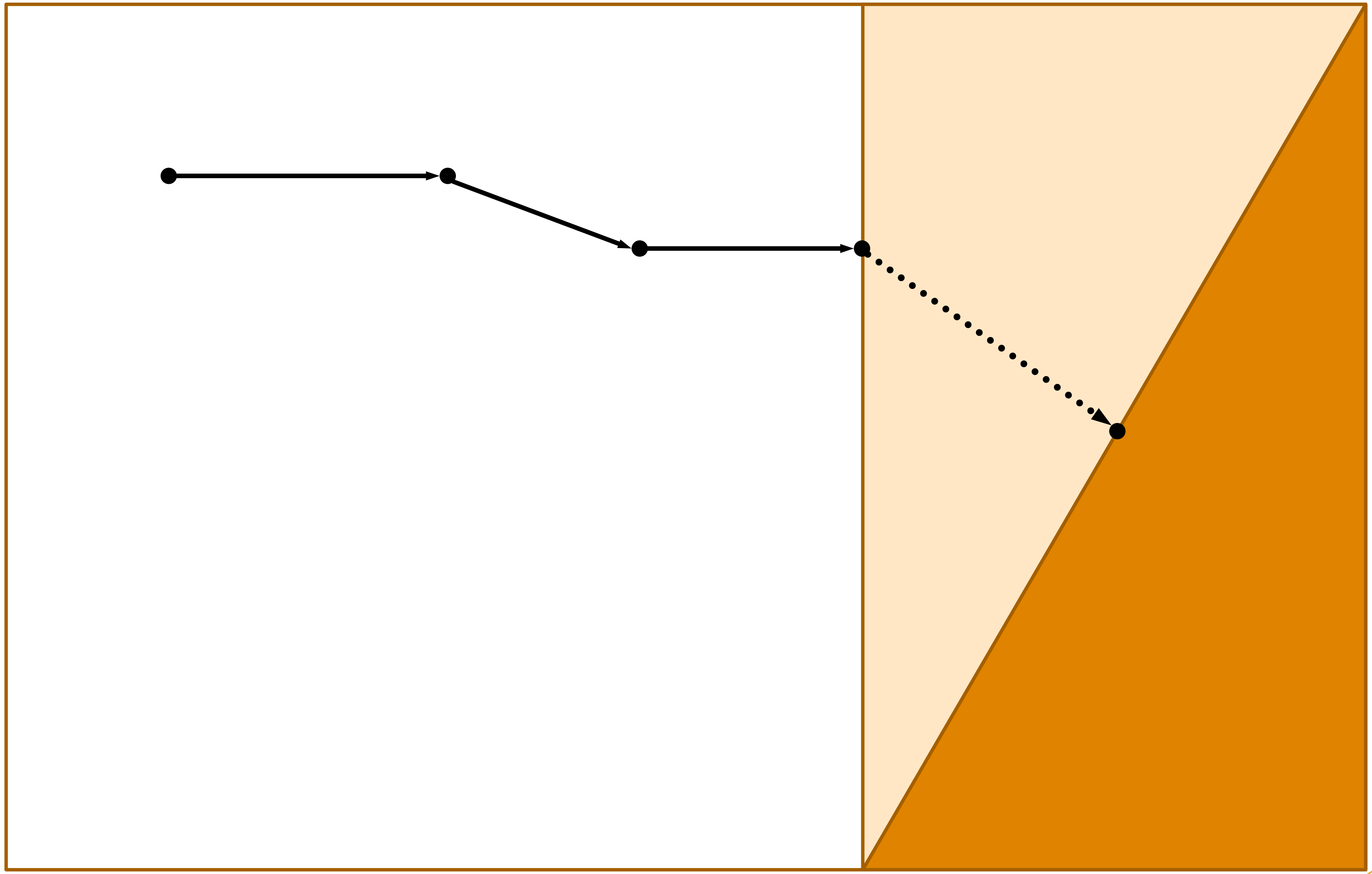
Nested Version



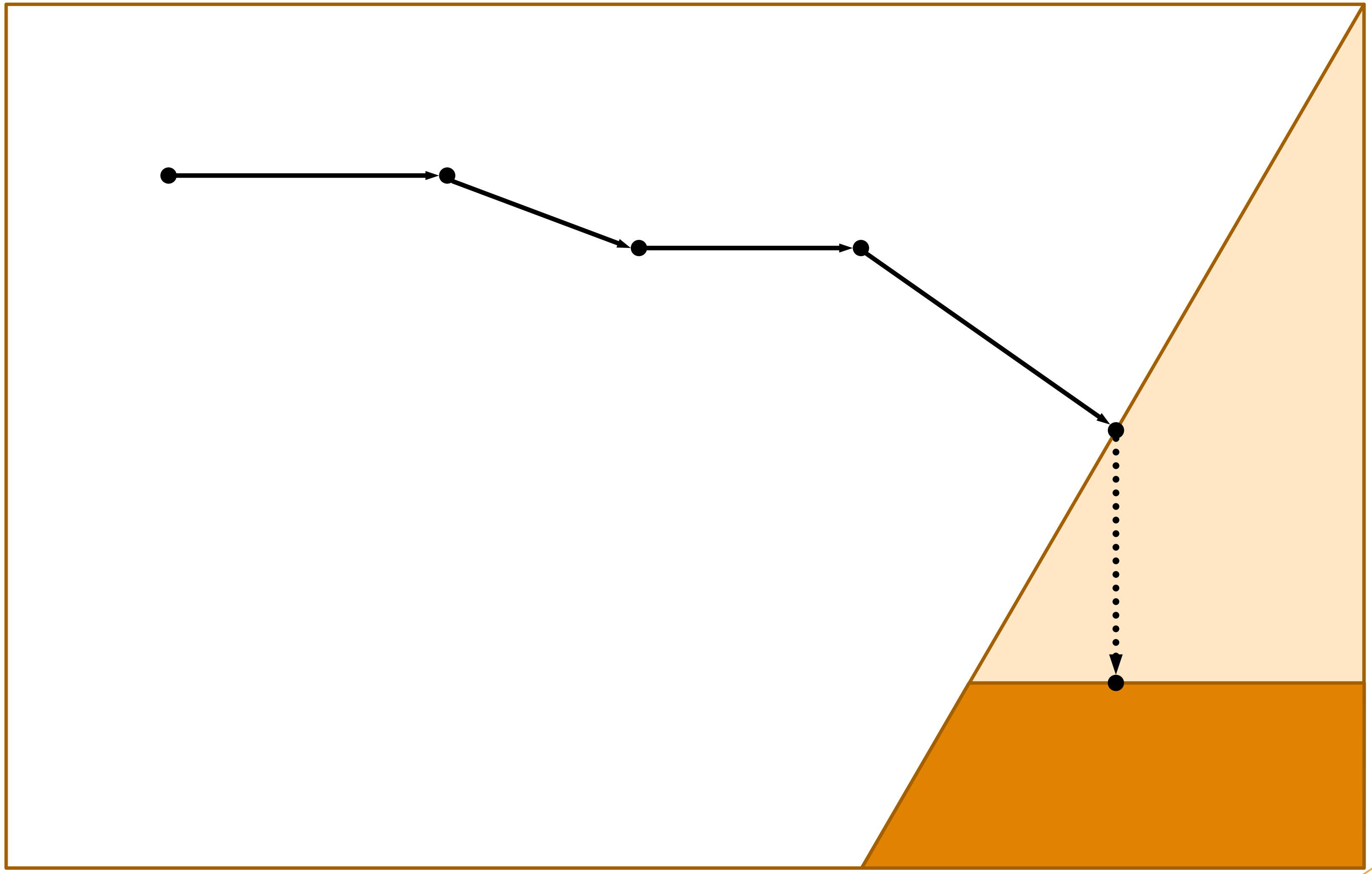
Nested Version



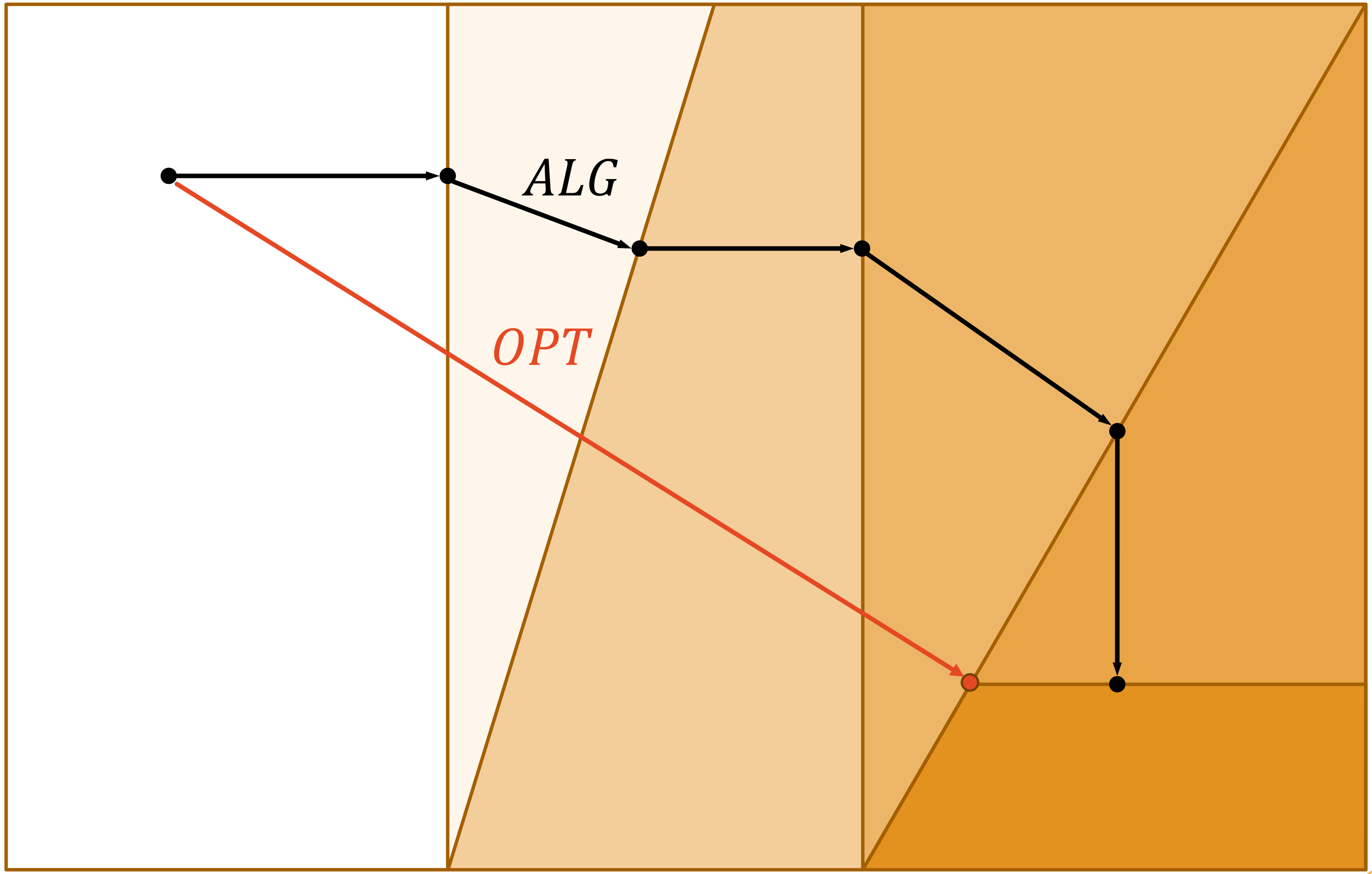
Nested Version



Nested Version



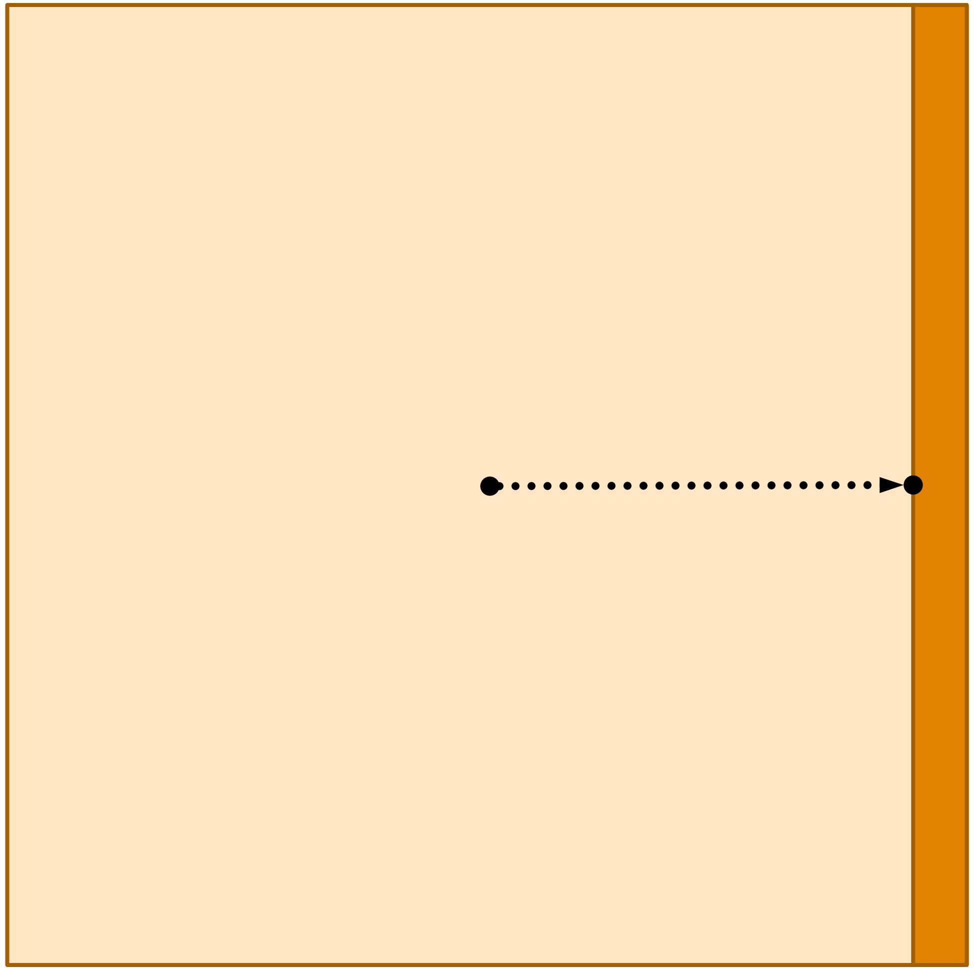
Nested Version



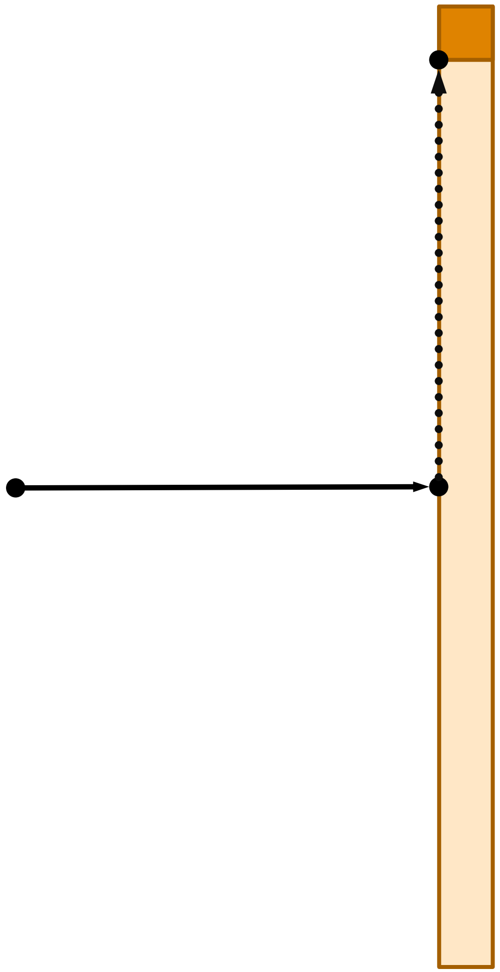
Motivation

- Metrical task systems (MTS)
  - Given convex functions $f_1, f_2, \dots, f_t$
  - Choose x^i online ($x^0 = 0$)
  - Cost $ALG = \sum_{i=1}^{t} ||x^i - x^{i-1}|| + f_i(x^i)$
- Convex body chasing – role of geometry in MTS
- Nested – manageable, gives insight into general problem

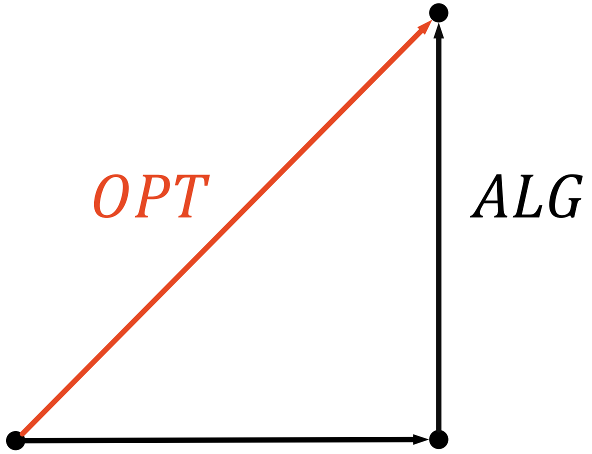
Lower Bound



Lower Bound



Lower Bound



$$ALG \geq \sqrt{2} \cdot OPT$$

$$\boxed{ALG \geq \sqrt{d} \cdot OPT}$$

Results

- [FL 93] $\sqrt{d}$ lower bound,
  Competitive general chasing for $d = 2$ case
- [BB+ 17] $d^{O(d)}$-competitive nested chasing
- **[AB+ 18] $\boldsymbol{O(d \log d)}$-competitive nested chasing**
- [BL+ 18] $O\left(\sqrt{d \log d}\right)$-competitive nested chasing,
  $exp(d)$-competitive general chasing

Talk outline

1. Motivating ideas
   - Reduction to *"Tighten"* problem
   - *Centroid* and *Recursive Greedy*
2. *Recursive Centroid*
   - $O(d \log d)$-competitive algorithm
   - Analysis (sketch)

Part 1 – Motivating ideas

Centroid, *Recursive Greedy*, and why neither is good enough

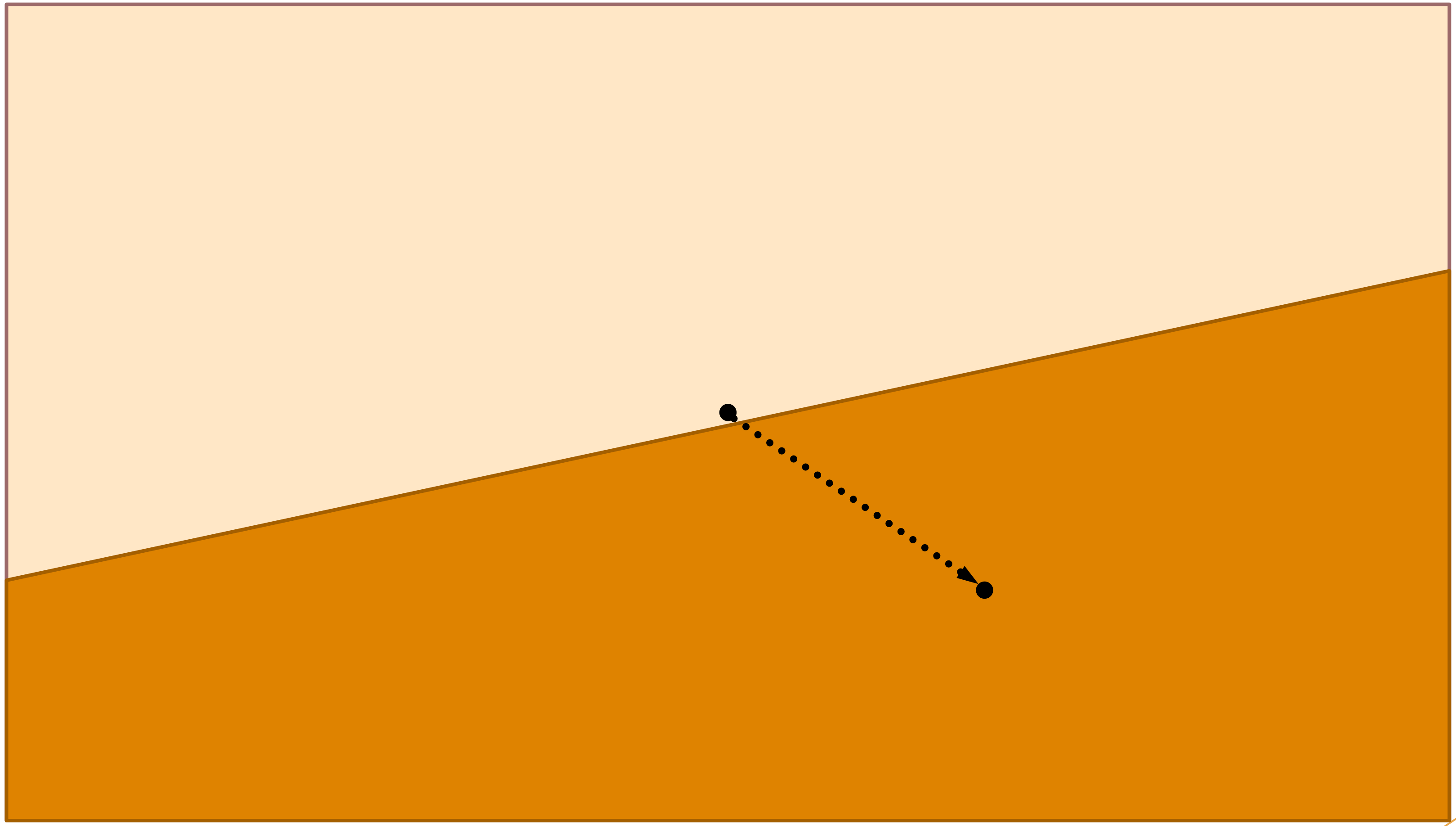
Reduction to Tighten

- Bounded – $diam(K^1) = O(1),\ \ OPT = \Omega(1)$
  - $f(d) \cdot diam(K^1)$ total cost $\Rightarrow f(d)$-competitive
  - Guess-and-double
- Tighten – end when $diam(K^t) \leq \frac{1}{2} diam(K^1)$
  - Apply repeatedly
  - Cost decreases geometrically

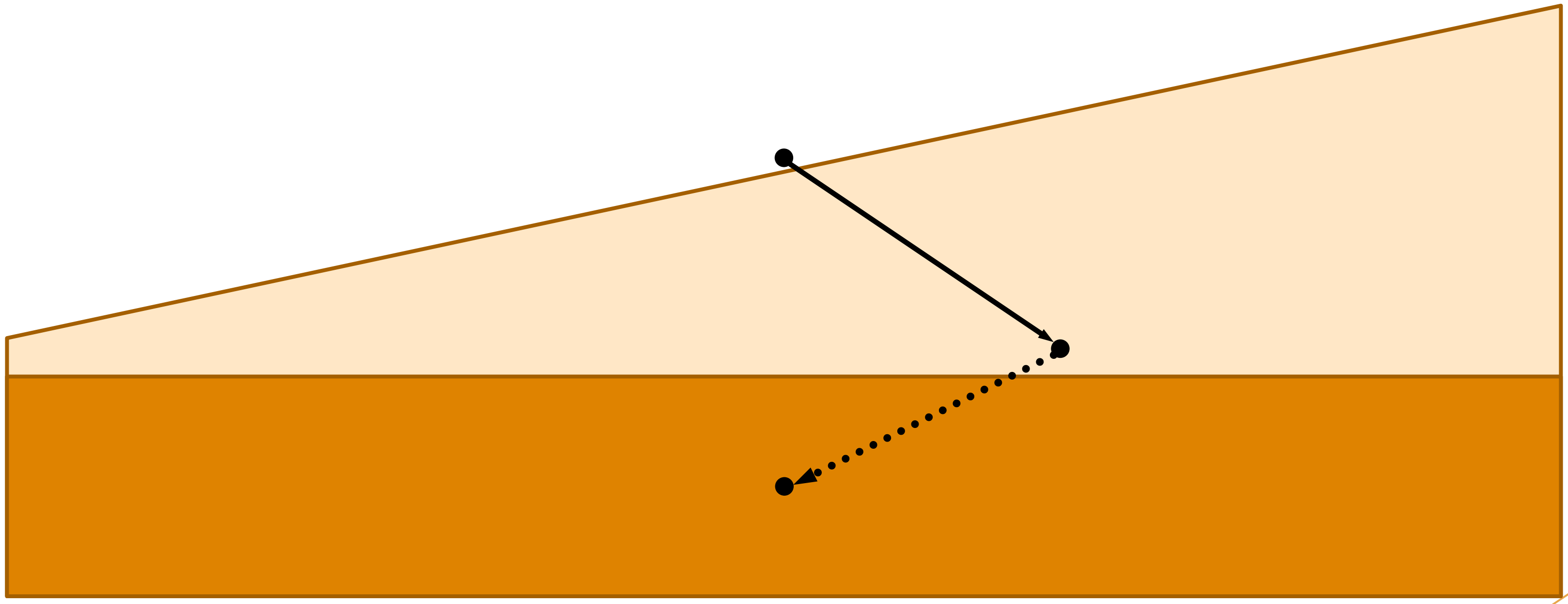
Idea 1 – *Centroid*

- "Move to center so any cut is good"
- *Centroid algorithm:* $x^t = \mu(K^t) \coloneqq \int_{K^t} x \, dx$
  - (K^t bounded)
- Grünbaum ['60] $\Rightarrow Vol(K^t) \leq (1-c) \cdot Vol(K^{t-1})$
  $\leq (1-c)^t \cdot Vol(\mathrm{K}^0)$
- Volume drops $O(2^d)$ in $O(d)$ steps

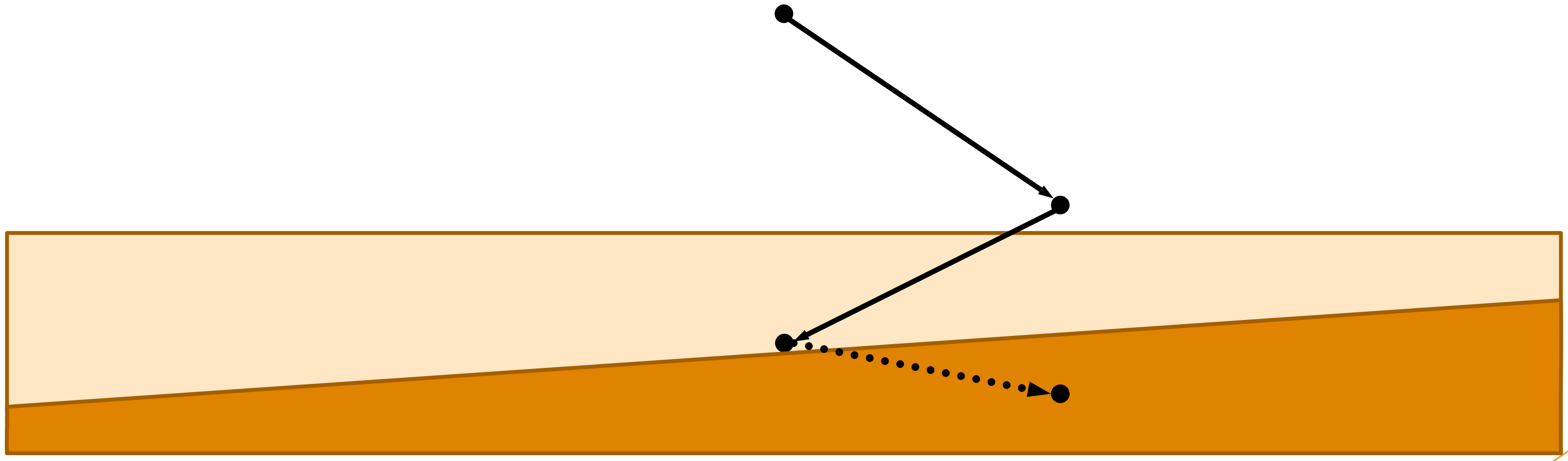
Problem with *Centroid*



Problem with *Centroid*



Problem with *Centroid*

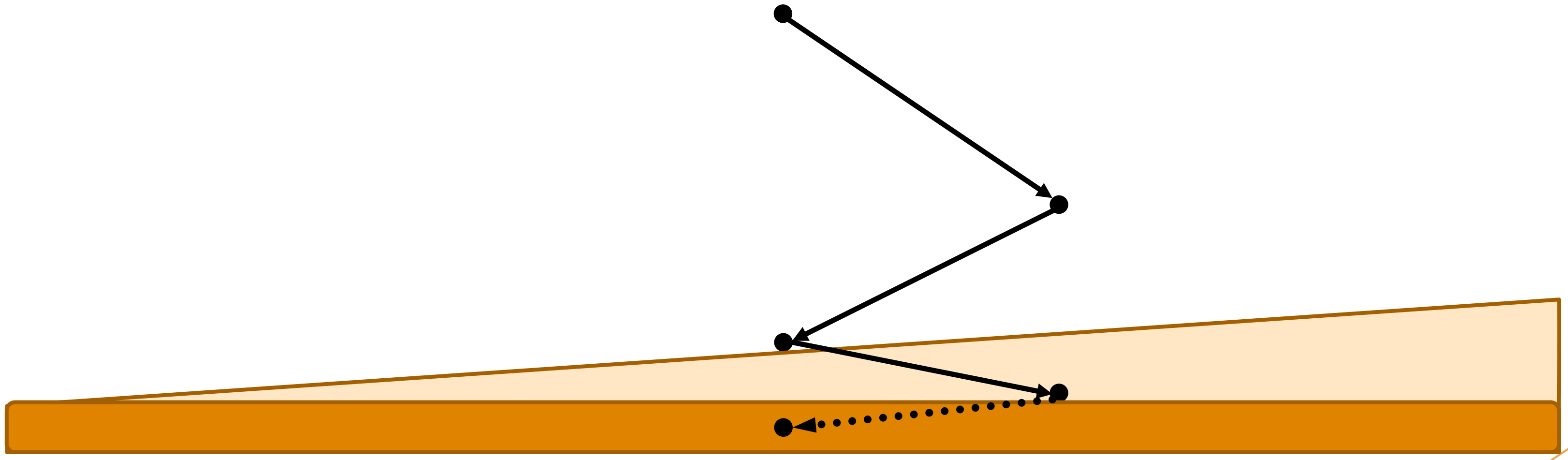


Problem with *Centroid*

Not competitive

☹

Diameter constant

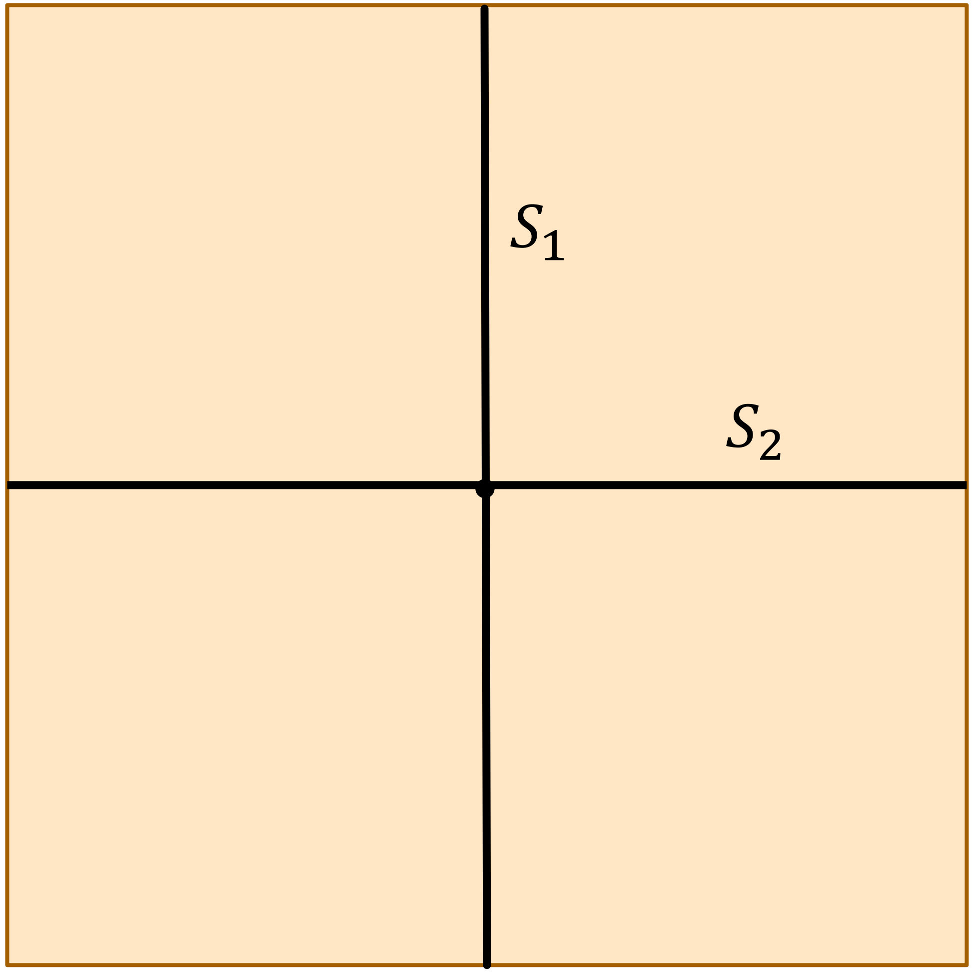


Idea 2 – *Recursive Greedy*

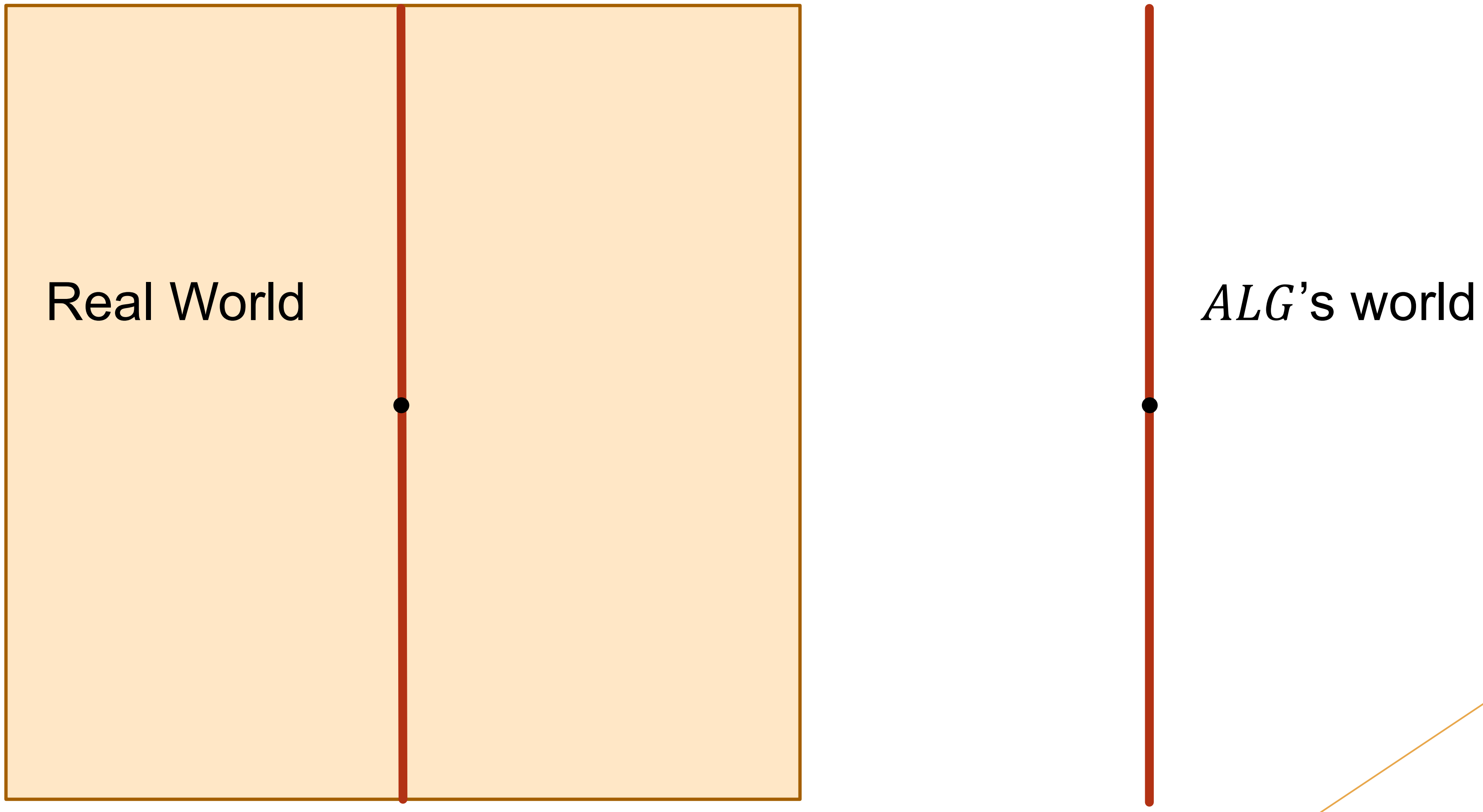
- “Refuse to move back and forth”
- In $\mathbb{R}^1$, run *Greedy*
- In $\mathbb{R}^d$
  - Fix orthogonal hyperplanes $S_1, \dots, S_d$
  - For $i = 1, \dots, d$
    - Run RG^{d-1} on sets $K^t \cap S_i$

RG^{d-1} – *Recursive Greedy* in $(d-1)$ dimensions

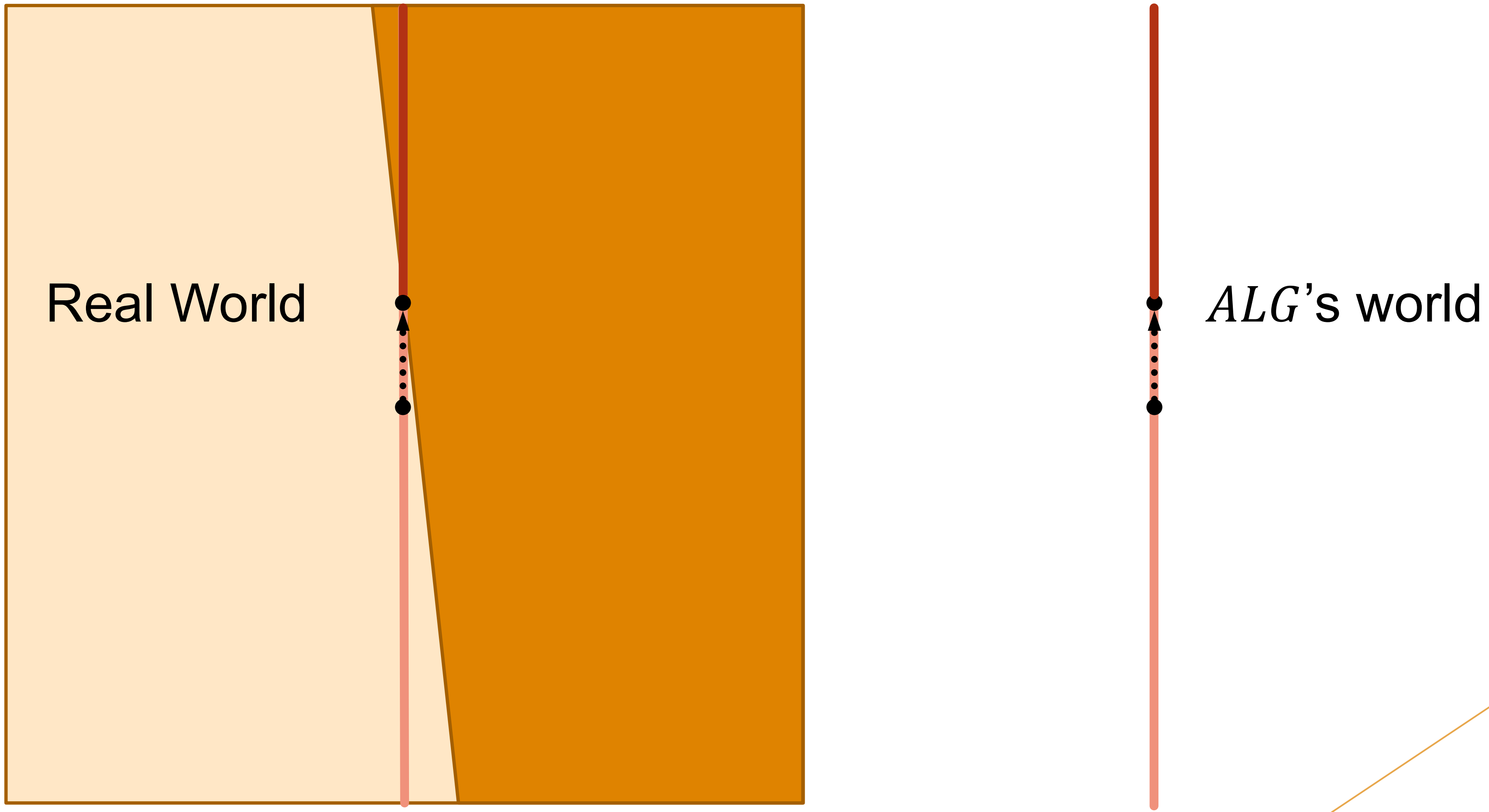
Idea 2 – *Recursive Greedy*



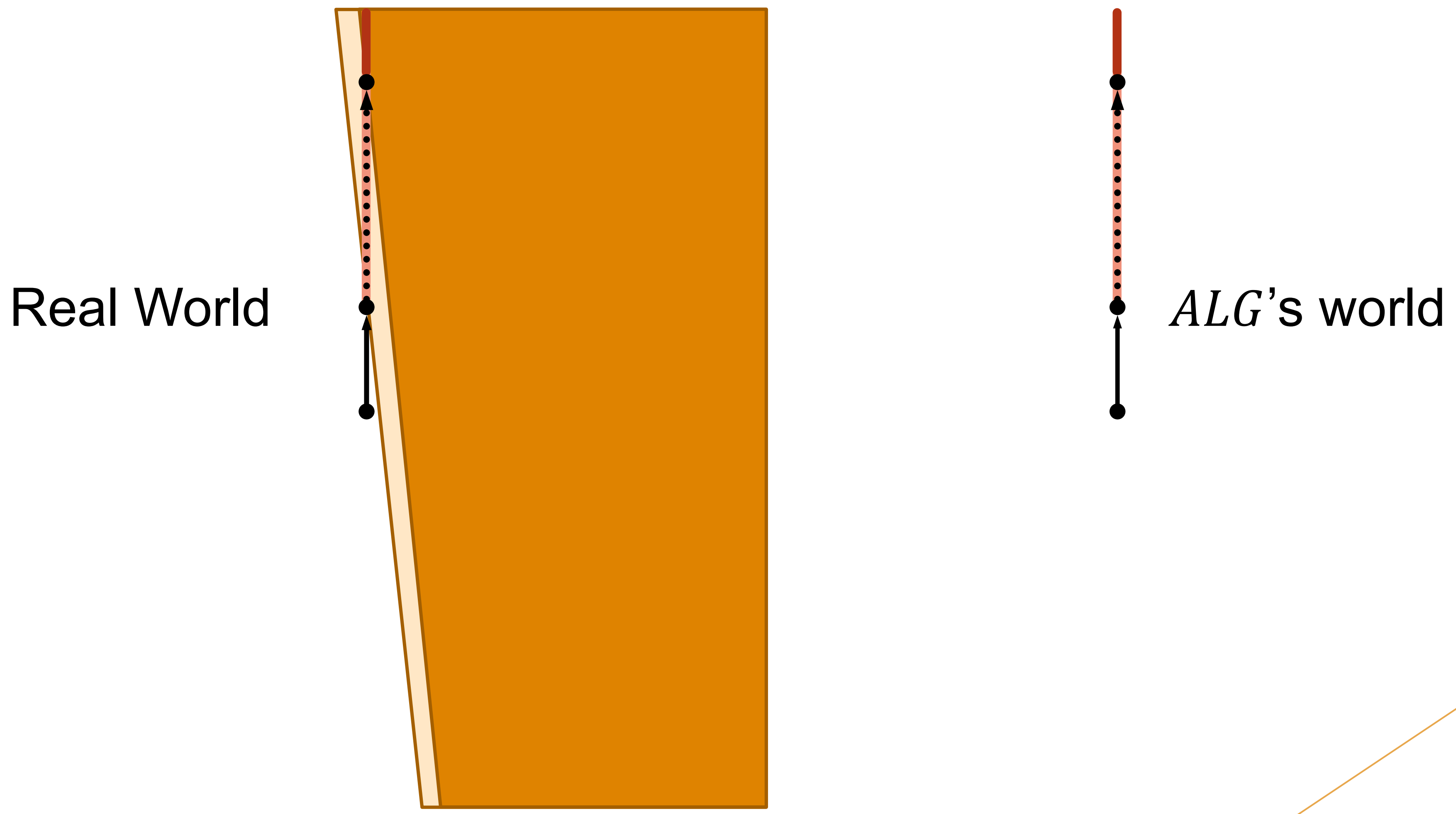
Idea 2 – *Recursive Greedy*



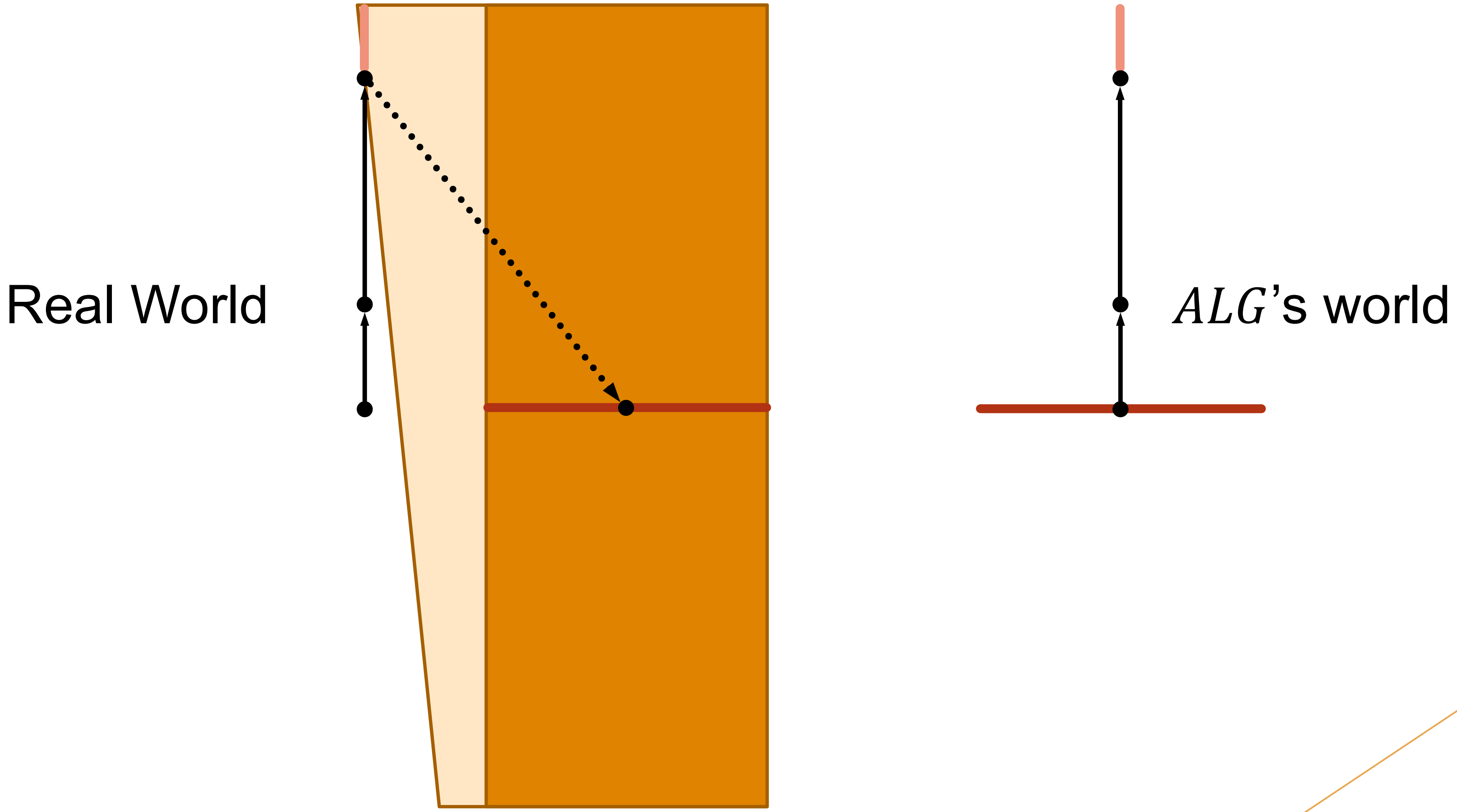
Idea 2 – *Recursive Greedy*



Idea 2 – *Recursive Greedy*

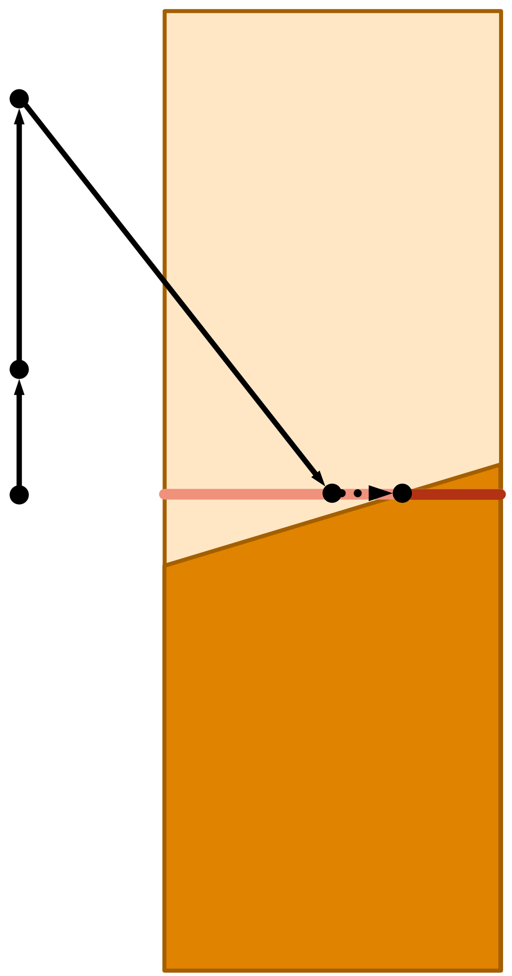


Idea 2 – *Recursive Greedy*



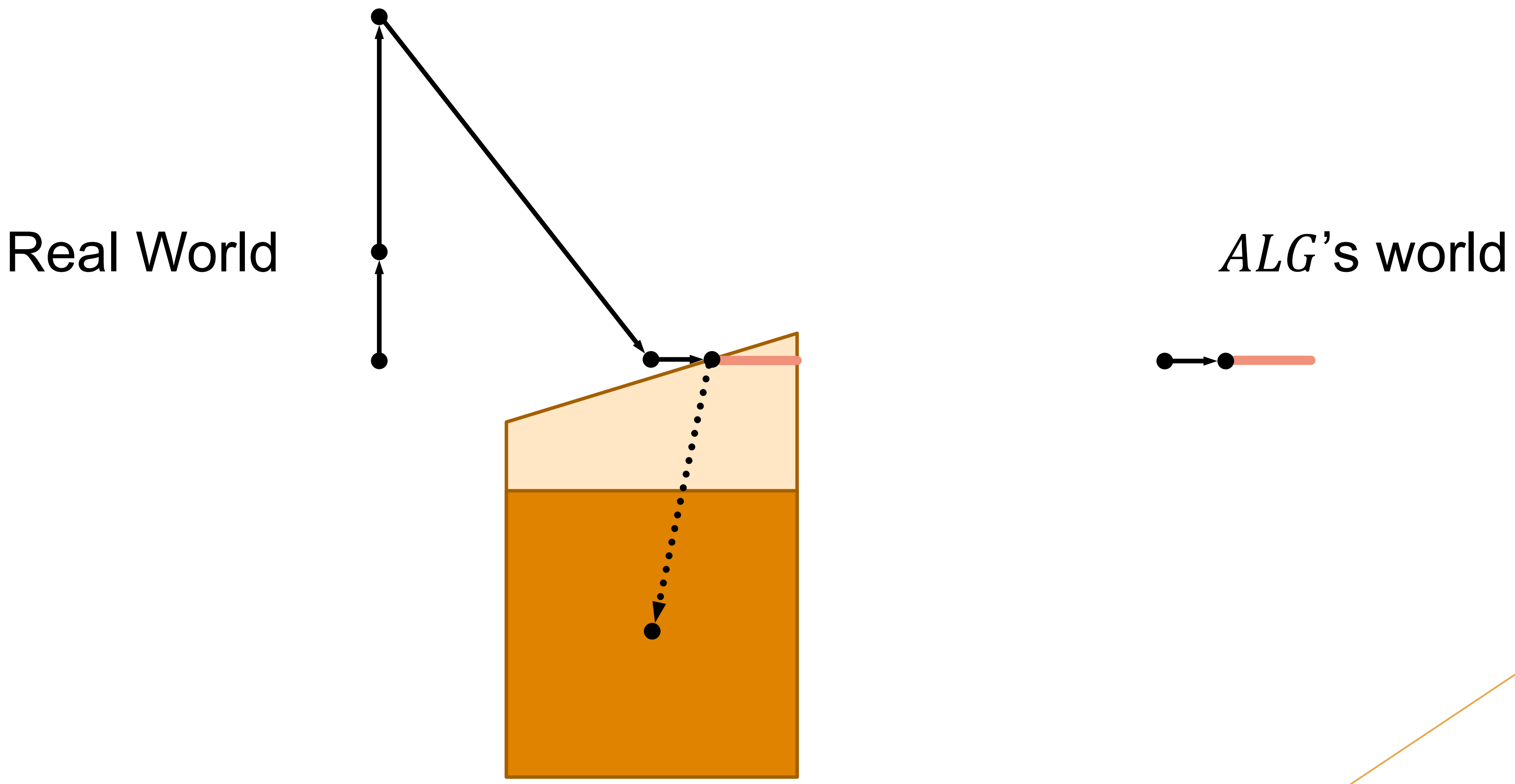
Idea 2 – *Recursive Greedy*

Real World

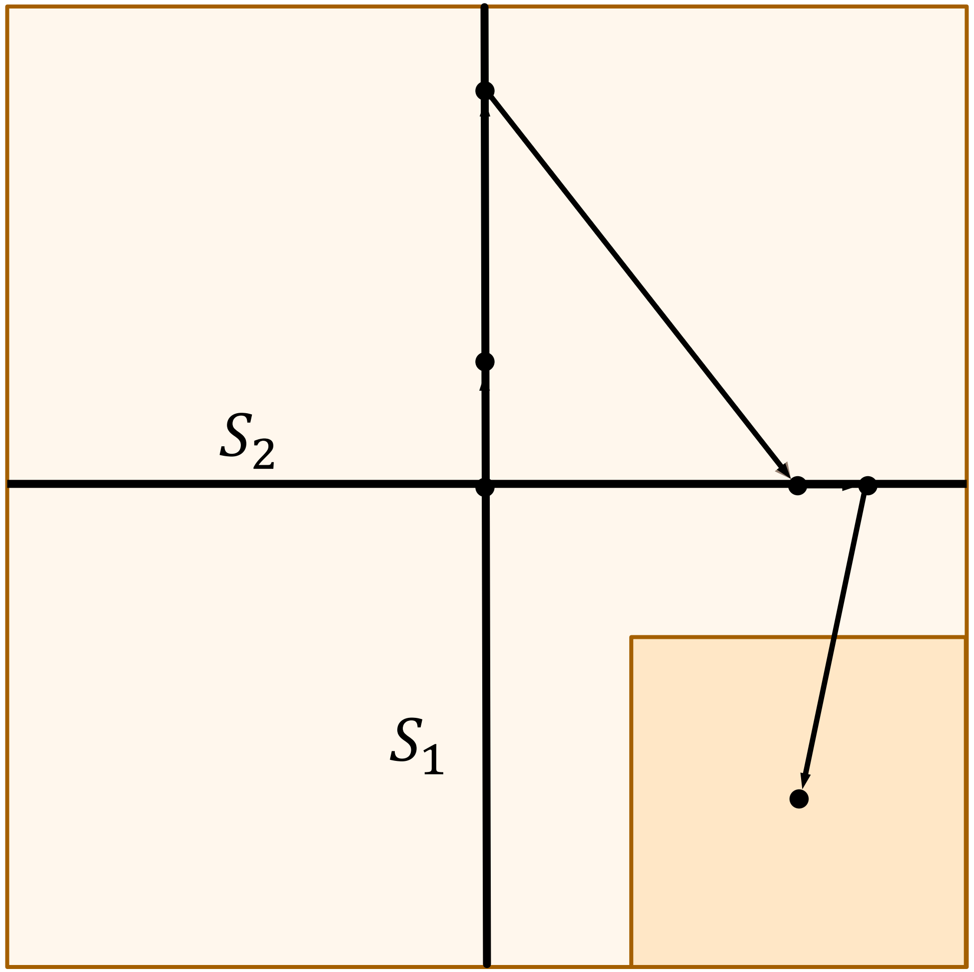


ALG's world

Idea 2 – *Recursive Greedy*



Idea 2 – *Recursive Greedy*



Diameter ↓↓↓

☺

Competitive algorithm
[BB+ '17]

Problem with *Recursive Greedy*

- $d^{O(d)}$-competitive
  - Expensive recursive calls
  - Diameter ↓ only $O\left(\sqrt{1-1/d}\right)$ after d recursive calls

Recap of Part 1

- *Centroid*
  - Volume drops quickly
  - Diameter stays constant
- *Recursive Greedy*
  - Controls individual dimensions
  - Expensive recursive calls
  - Diameter shrinks slowly

Part 2 – *Recursive Centroid*

Fusion of *Centroid* and *Recursive Greedy*

New Ideas

- Recursion on skinny subspace
  - Cheap
  - Hyperplane separation ⇒ cut parallel to skinny subspace
    - Progress on fat subspace
- Play centroid in recursion

Skinny Subspace

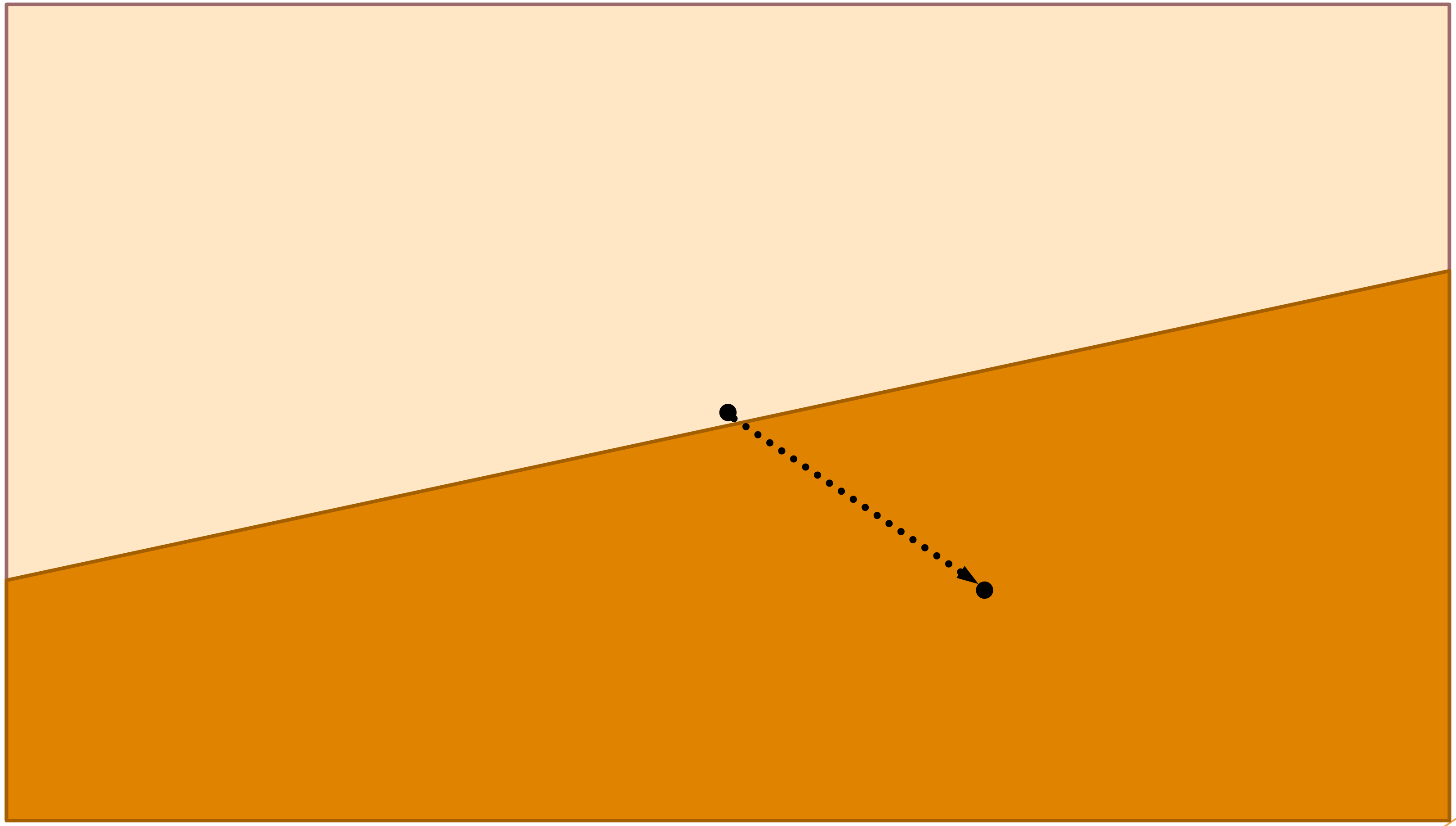
- Directional width – $w(K, v) \coloneqq \max_{x,y \in K} \langle x - y, v \rangle$
- Skinny direction – v such that $w(K^t, v) \lesssim 1/d^2$
- $S \coloneqq$ span of k skinny directions
  - Add directions over time
- $F \coloneqq S^\perp$ (fat subspace)

Recursive Centroid

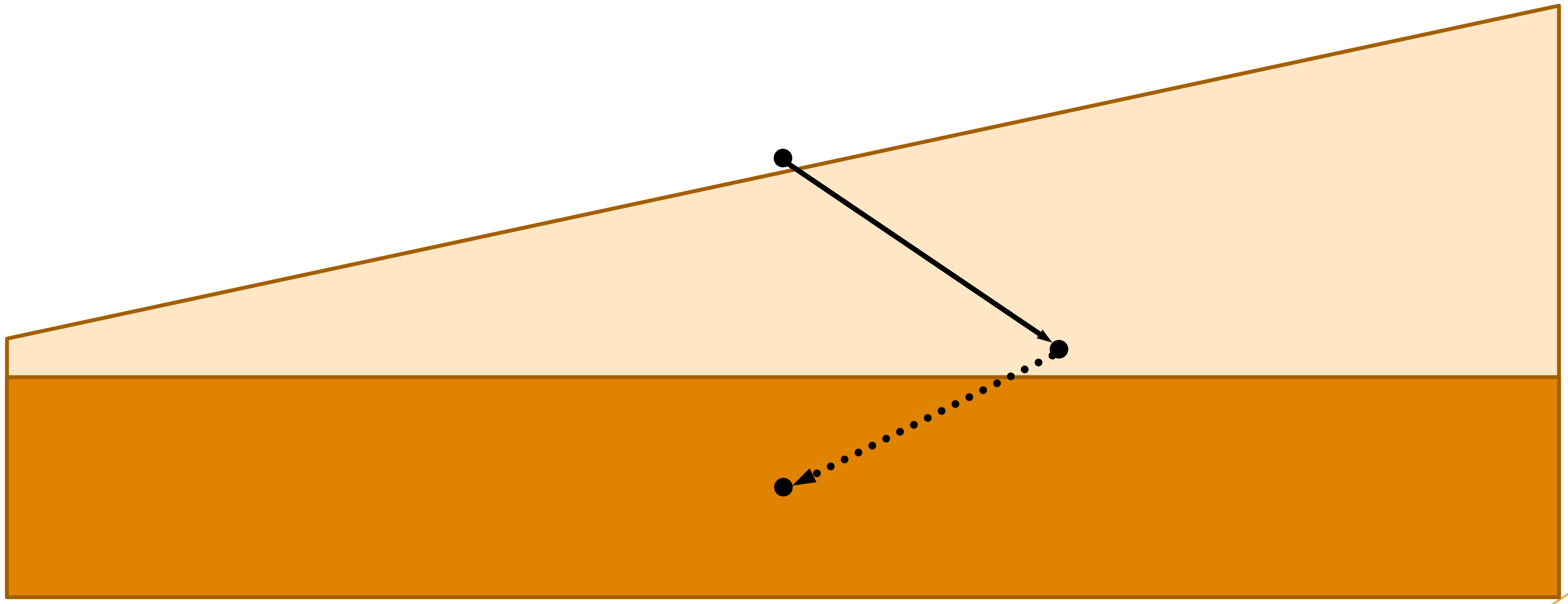
- If $S \neq \{0\}$
  - $S' \leftarrow x_t + S$
  - Run $RC^{\dim(S)}$ on $K^t \cap S'$ until empty
- $x_t \leftarrow \mu(K^t)$
- While $\exists$ skinny direction $v \in F$
  - $S \leftarrow span(S, v)$
- Repeat until $diam(K^t) \leq 1/2 \cdot diam(K^1)$

$RC^{\dim(S)}$ – *Recursive Centroid* in $\dim(S)$ dimensions

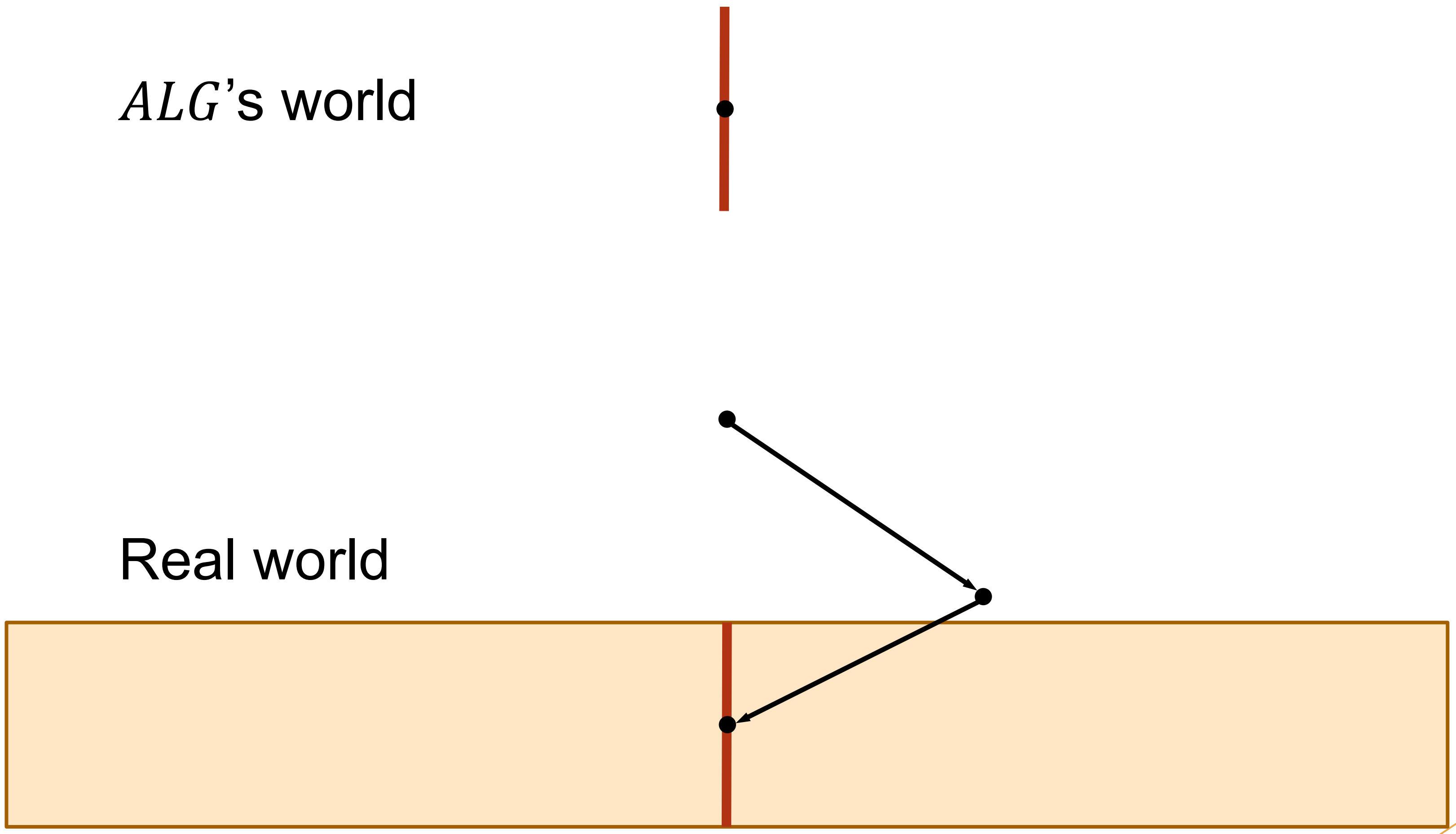
Recursive Centroid



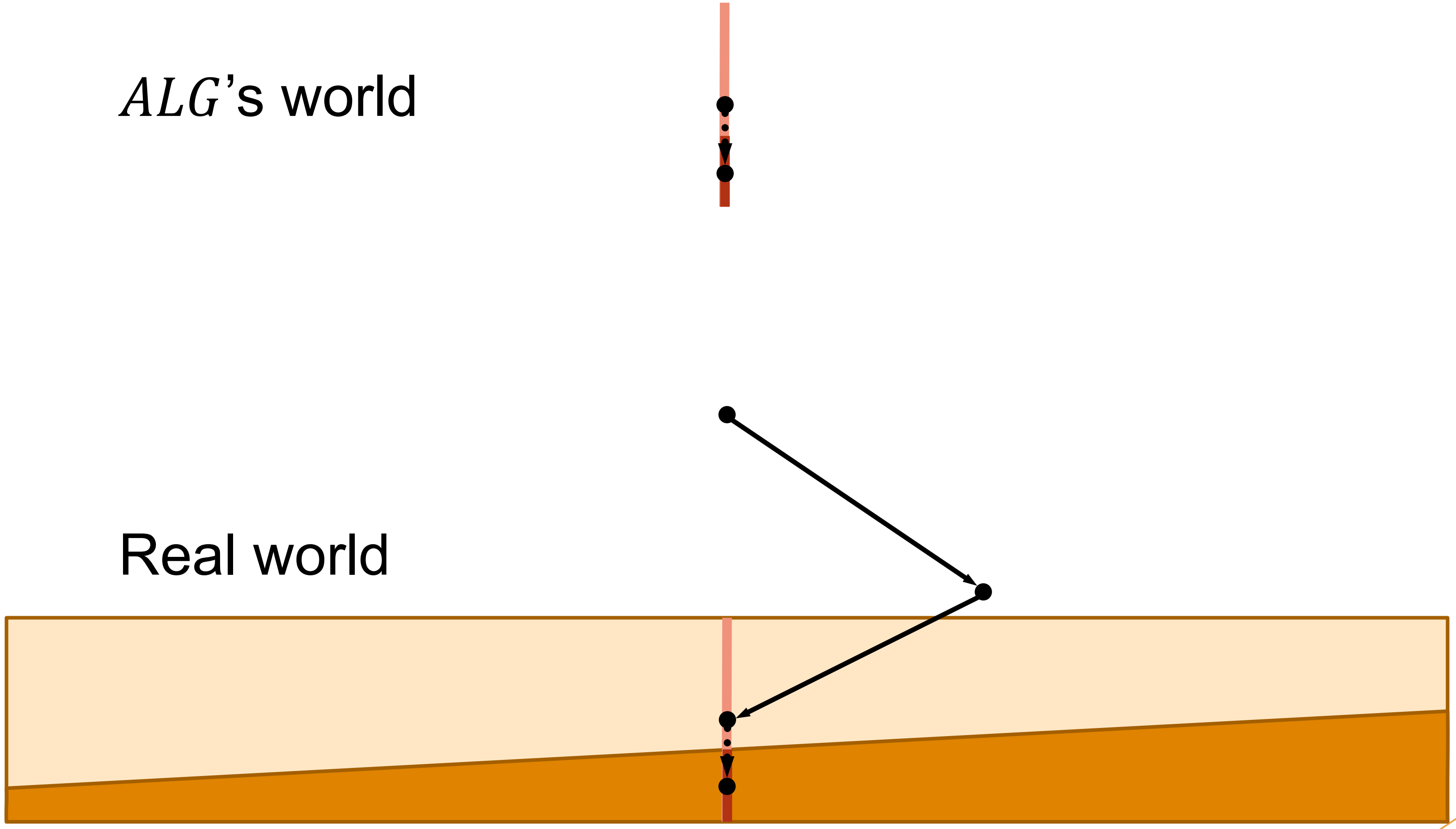
Recursive Centroid



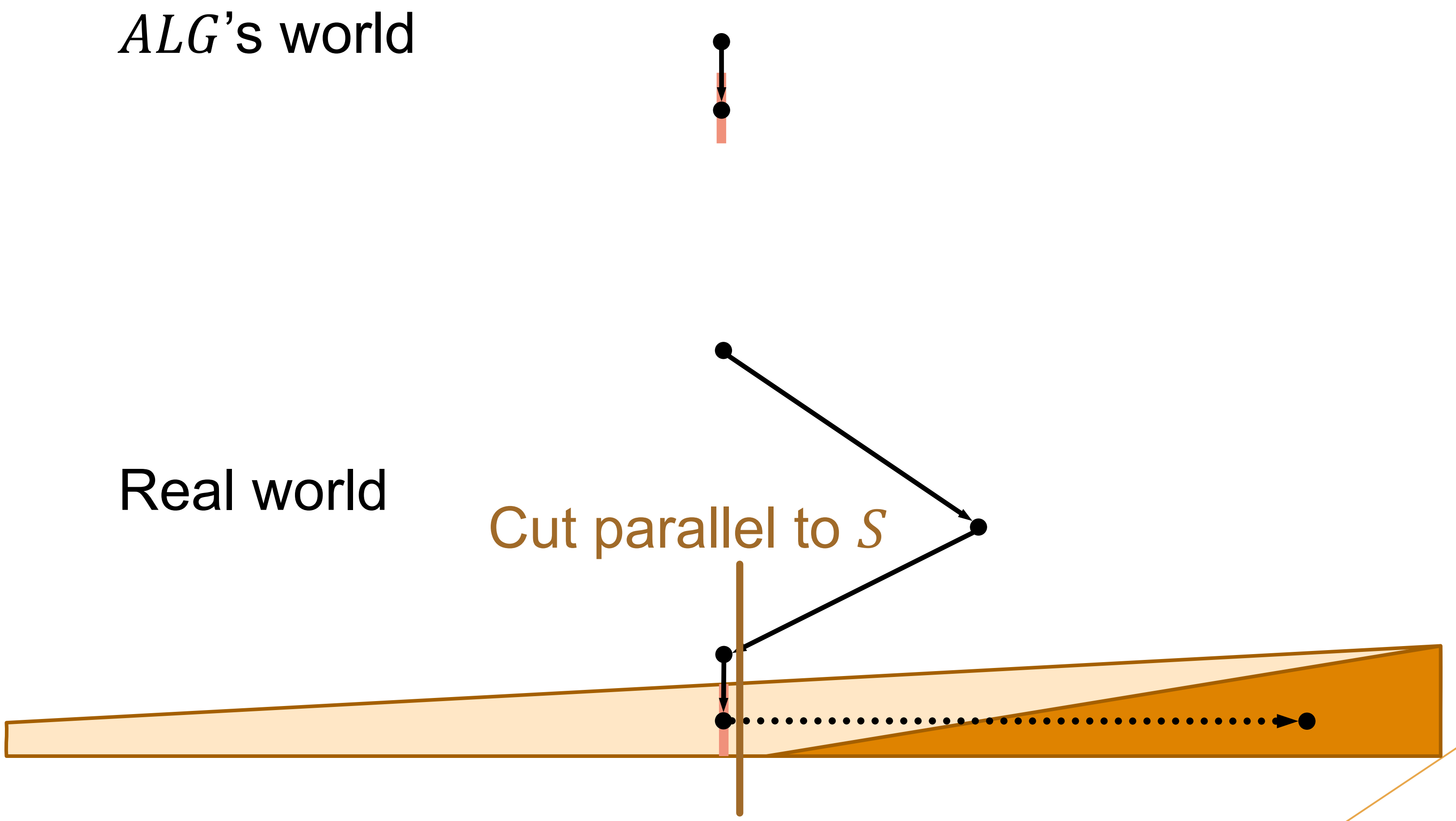
Recursive Centroid



Recursive Centroid



Recursive Centroid



Main theorem

Recursive Centroid is $O(d \log d)$-competitive [ABCGL '18]

- Recall $\sqrt{d}$ lower bound

Proof outline

- Potential $\Phi^t := Vol\left(Proj_F(K^t)\right)$

- 'Step' = Recursive call + move to centroid of K^t
  1. Cost of 1 step = $O(1)$
  2. $O(d \log d)$ steps
  - $O(d \log d)$ total cost

Proof part I – A single step

$$\Phi^t = Vol\left(Proj_F(K^t)\right)$$

- Cost $O(1)$
  - Recursion: $O(d \log d) \cdot 1/d^2 = o(1)$
  - Move to centroid: $O(1)$
- Φ^t drops $(1 - c)$
  - K^t cut by halfspace *parallel to* S

Proof part II – $O(d \log d)$ steps

$$\Phi^t = Vol\left(Proj_F(K^t)\right)$$

- Φ^t drops $\geq (1-c)^m$
  - m = # of steps
- Φ^t increases $\leq d^{O(d)}$
- $\Phi^{T-1} \geq d^{-O(d)}$

$$d^{O(d)}(1-c)^{m-1} \geq \Phi^{T-1}/\Phi^0 \geq d^{-O(d)}$$

$$m \leq O(d \log d)$$

Recap of Part 2

- Recursion on skinny subspaces
  - Cheap, good cuts
- Play centroid
  - Volume drop
- $\Phi^t = Vol\left(Proj_F(K^t)\right)$

Open questions

- $poly(d)$-competitive general chasing
- $exp(d)$ lower bound for general chasing
- Efficient algorithms

Thank you!

Questions?

In memory of Michael Cohen

References

- “A Nearly-Linear Bound for Chasing Nested Convex Bodies”
  Argue Bubeck Cohen Gupta Lee, *SODA ‘19*
- “Nested Convex Bodies are Chasable”
  Bansal Bohm Elias Koumoutsos Umboh, *SODA ‘18*
- “Chasing Nested Convex Bodies Nearly Optimally,”
  “Competitively Chasing Convex Bodies”
  Bubeck Lee Li Selke, *Preprints ‘18*
- “Chasing Convex Bodies and Functions”
  Friedman Linial, *Discrete and Computational Geometry ‘93*