Chua circuit

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Curator: Dr. Leon O. Chua, Dept. of EECS, University of California, Berkeley

The **Chua Circuit** is the simplest *electronic circuit* exhibiting *chaos*, and many well-known bifurcation phenomena, as verified from numerous laboratory experiments, computer simulations, and rigorous mathematical analysis.

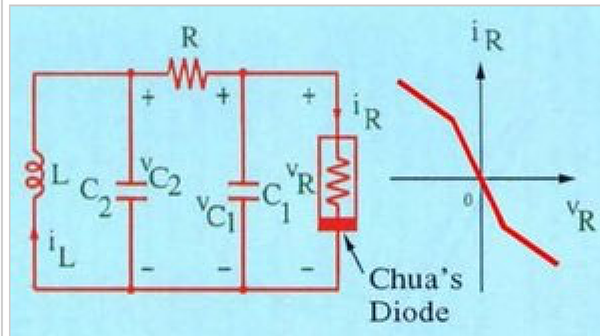


Figure 1: The Chua Circuit.

Contents

Historical Background

The Chua Circuit was invented in the fall of 1983 (Chua, 1992) in response to two unfulfilled quests among many researchers on chaos concerning two wanting aspects of the *Lorenz Equations* (Lorenz, 1963). The first quest was to devise a laboratory system which can be realistically modeled by the Lorenz Equations in order to demonstrate chaos is a robust *physical* phenomenon, and not merely an artifact of computer round-off errors. The second quest was to prove that the Lorenz attractor, which was obtained by computer simulation, is indeed chaotic in a rigorous mathematical sense. The existence of *chaotic attractors* from the Chua circuit had been confirmed *numerically* by Matsumoto (1984), observed *experimentally* by Zhong and Ayrom (1985), and *proved* rigorously in (Chua, et al, 1986). The basic approach of the proof is illustrated in a guided exercise on Chua's circuit in the well-known textbook by Hirsch, Smale and Devaney (2003).

Circuit Diagram and Realization

The circuit diagram of the Chua Circuit is shown in Figure 1. It contains 5 circuit elements. The first four elements on the left are standard off-the-shelf *linear passive* electrical components; namely, *inductance* $L > 0$, *resistance* $R > 0$, and two capacitances $C_1 > 0$ and $C_2 > 0$. They are called passive elements because they do not need a power supply (*e.g.*, battery). Interconnection of passive elements always leads to trivial dynamics, with all element voltages and currents tending to zero (Chua, 1969).

Local Activity is Necessary for Chaos

The simplest circuit that could give rise to oscillatory or chaotic waveforms must include at least one *locally active* (Chua, 1998), (Chua, 2005) *nonlinear* element, powered by a battery, such as the *Chua diode* shown in Figure 1, characterized by a current vs. voltage nonlinear function $i_R = g(v_R)$, whose slope must be negative somewhere on the curve. Such an element is called a locally active resistor. Although the function $g(\bullet)$ may assume many shapes,the original Chua circuit specifies the 3-segment piecewise-linear odd-symmetric characteristic shown in the right hand side of Figure 1, where m_0 denotes the slope of the middle segment and m_1 denotes the slope of the two outer segments ; namely,

$$g(v_R) = \begin{cases} m_1 v_R + m_1 - m_0 & ,if \quad v_R \leq -1 \\ m_0 v_R & ,if \quad -1 \leq v_R \leq 1 \\ m_1 v_R + m_0 - m_1 & ,if \quad 1 \leq v_R \end{cases}$$

where the coordinate of the two symmetric breakpoints are normalized, without loss of generality, to $v_R = \pm 1$.

The Chua Diode is Locally Active

The Chua diode is not an off-the-shelf component. However, there are many ways to synthesize such an element using off-the-shelf components and a power supply, such as batteries. The circuit for realizing the Chua diode need not concern us since the dynamical behavior of the Chua Circuit depends *only* on the 4 parameter values L, R, C_1, C_2 and the nonlinear characteristic function $g(\bullet)$.

Any locally active device requires a power supply for the same reason a mobile phone can not function without batteries (Chua, 1969). A physical circuit for realizing the Chua Circuit in Figure 1 is shown in Figure 2.

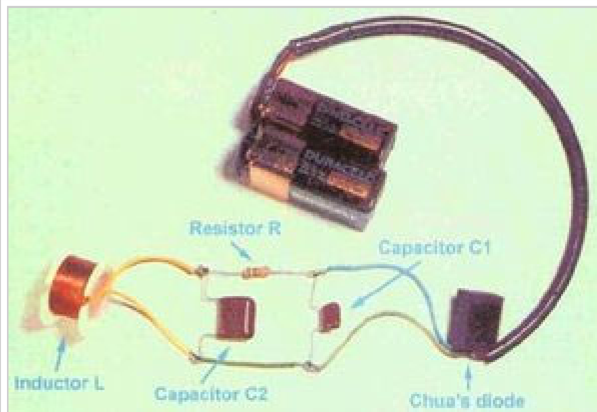
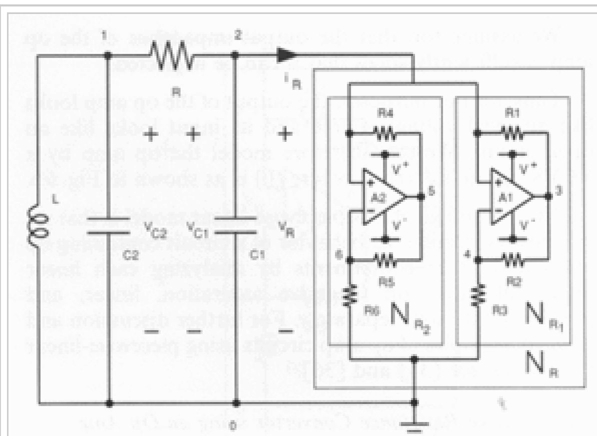


Figure 2: Physical realization of the Chua Circuit.

Observe the one-to-one correspondence between each linear circuit element in Figure 1 and its corresponding physical component in Figure 2 (Gandhi *et al*, 2007). The Chua diode in Figure 1 corresponds to the small black box with two external wires soldered across capacitance C_1. Two batteries are used to supply power for the Chua diode. The parameter values for L, R, C_1, and C_2, as well as instructions for building the Chua diode in Figure 1 are given in (Kennedy, 1992).

Figure 3 shows the complete Chua Circuit, including the circuit schematic diagram (enclosed inside the box N_R) for realizing the Chua diode, using 2 standard *Operational Amplifiers* (Op Amps) and 6 linear resistors (Gandhi *et al*. 2007).



The two vertical terminals emanating from each Op Amp (labeled V^+ and V^-, respectively) in Figure 3 must be connected to the *plus* and *minus* terminals of a 9 volt battery, respectively.

There are many other circuits for realizing the Chua diode. The most

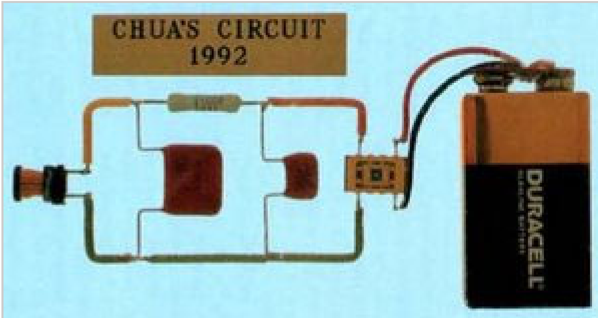


Figure 4: A Chua Circuit where the Chua diode is implemented by a specially designed IC chip.

Figure 3: Realization of Chua's Circuit using two Op Amps and six linear resistors to implement the Chua diode N_R.

compact albeit expensive way is to design an integrated electronic circuit, such as the physical circuit shown in Figure 4, where the black box in Figure 2 had been replaced by a single IC chip (Cruz and Chua, 1993), and powered by only one battery.

Oscilloscope Displays of Chaos

Using the Chua Circuit shown in Figure 4, the *voltage* waveforms $v_{C_1}(t)$ and $v_{C_2}(t)$ across capacitors C_1 and C_2, and the current waveform $i_L(t)$ through the inductor L in Figure 1, were observed using an oscilloscope and displayed in Figure 5 (a), (b), and (c) (left column), respectively.

The *Lissajous* figures associated with 3 permutated pairs of waveforms are displayed on the right column Figure 5; namely, in the $v_{C_1} - i_L$ plane in Figure 5(d), the $v_{C_1} - v_{C_2}$ plane in Figure 5(e), and the $v_{C_2} - i_L$ plane Figure 5 (f). They are 2-dimensional projections of the *chaotic* attractor, called the *double scroll*, traced out by the 3 waveforms from the left column in the 3-dimensional $v_{C_1} - v_{C_2} - i_L$ space.

It is important to point out that the Chua Circuit is *not* an *analog computer*. Rather it is a *physical system* where the *voltage*, *current*, and *power* associated with *each* of the 5 circuit elements in Figure 1 can be measured and observed on an oscilloscope, and where the *power flow* among the elements makes physical sense. In an *analog computer* (usually using Op Amps interconnected with other electronic components to mimic some prescribed set of differential equations), the measured voltages have no physical meanings because the corresponding currents and powers can not be identified, let alone measured, from the analog computer.

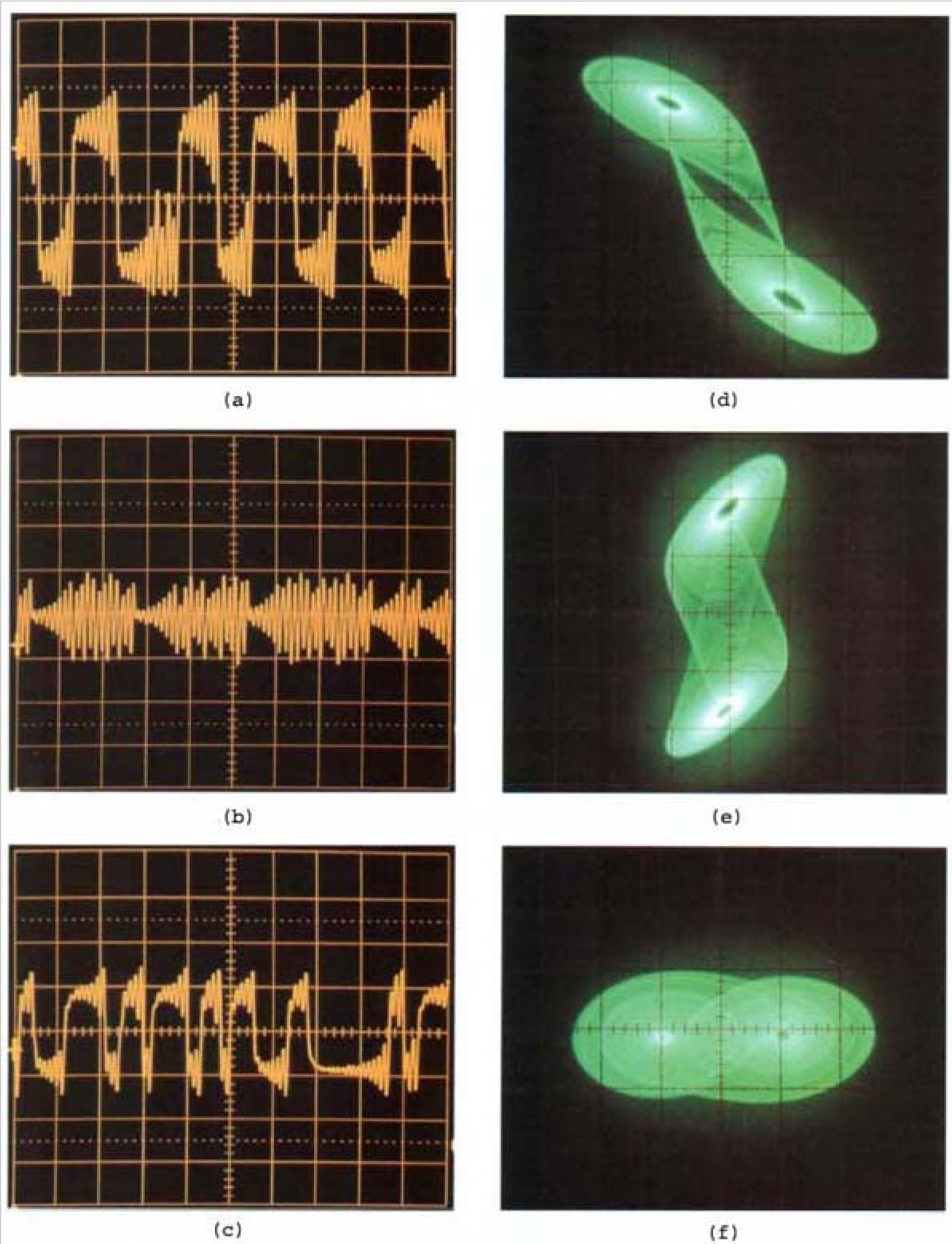


Figure 5: .Waveforms and Lissajous figures recorded from experimental measurements on the Chua Circuit shown in Figure 4. The three waveforms displayed in (a), (b), and (c) (left column) correspond to to $v_{C_1}(t)$, $v_{C_2}(t)$ and $i_L(t)$, respectively. The three Lissajous figures displayed in (d), (e), and (f) (right column) correspond to the pair of variables (v_{C_1}, i_L), (v_{C_1}, v_{C_2}), and (v_{C_2}, i_L), respectively

Chua Equations

By rescaling the circuit variables v_{C_1}, v_{C_2}, and i_L from Figure 1, we obtain the following dimensionless Chua Equations involving 3 dimensionless state variables *x*, *y*, *z*, and only 2 dimensionless parameters α and β :

Chua Equations

$$\begin{aligned} \dot{x} &= \alpha(y - \phi(x)) \\ \dot{y} &= x - y + z \\ \dot{z} &= -\beta y \end{aligned}$$

where α and β are real numbers, and $\phi(x)$ is a scalar function of the single variable x . The Chua Equations are simpler than the Lorenz Equations in the sense that it contains only one scalar nonlinearity, whereas the Lorenz Equations contains 3 nonlinear terms, each consisting of a product of two variables (Pivka *et al*, 1996). In the original version studied in-depth in (Chua *et al*, 1986), $\phi(x)$ is defined as a *piecewise-linear* function

$$\phi(x) \triangleq x + g(x) = m_1 x + \frac{1}{2}(m_0 - m_1)[|x+1| - |x-1|]$$

where m_0 and m_1 denote the slope of the inner and outer segments of the piecewise-linear function in Figure 1, respectively. Although simpler smooth scalar functions, such as polynomials, could be chosen for $\phi(x)$ without affecting the *qualitative* behaviors of the Chua Equations, a continuous (but not differentiable) piecewise-linear function was chosen strategically from the outset in (Chua *et al*, 1986) in order to devise a *rigorous* proof showing the experimentally and numerically derived *double scroll attractor is indeed chaotic*. Unlike the Lorenz attractor (Lorenz, 1963), which had not been proven to be chaotic until 36 years later (Stewart, 2000) by Tucker (1999), it was possible to prove the double scroll attractor from the Chua Circuit is chaotic by virtue of the fact that certain Poincare return maps associated with the attractor can be derived *explicitly* in analytical form via *compositions* of eigen vectors within each linear region of the 3-dimensional state space (Chua *et al*, 1986), (Shilnikov, 1994).

Fractal Geometry of the Double Scroll Attractor

Based on an in-depth analysis of the phase portrait located in each of the 3 linear regions of the *x-y-z* state space, as well as from a detailed numerical analysis of the double scroll attractor shown in Figure 6, the geometrical structure of the double scroll attractor is found to consist of a juxtaposition of infinitely many thin, concentric, oppositely-directed fractal-like layers. The local geometry of each cross section appears to be a *fractal* at all cross sections and scales. This fractal geometry is depicted in the caricature shown in Figure 7. A 3-dimensional model of the double scroll attractor, accurate to millimeter scales, has been carefully sculpted using red and blue fiber glass, and displayed in Figure 8.

Figure 6: The double scroll attractor derived by computer simulations of the Chua Equations.

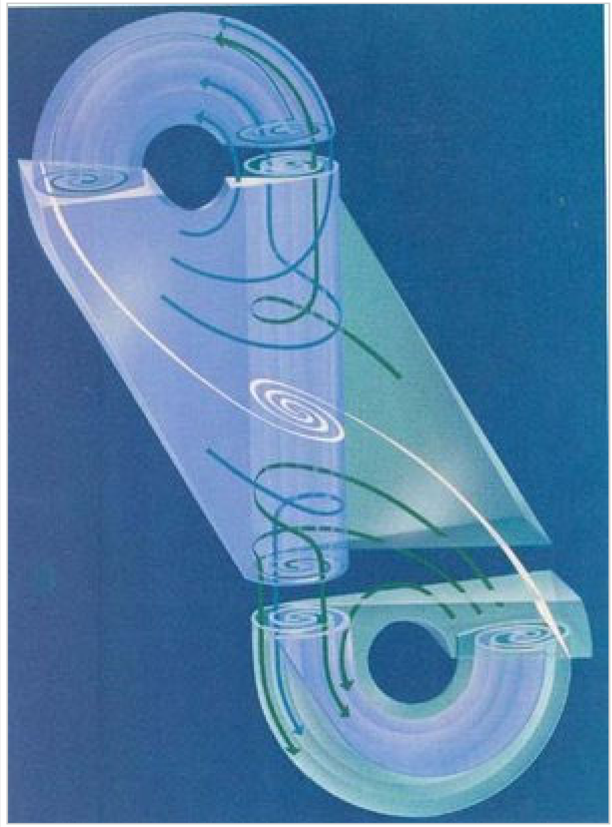


Figure 7: A caricature of the double spiral fractal geometry of the double scroll attractor.

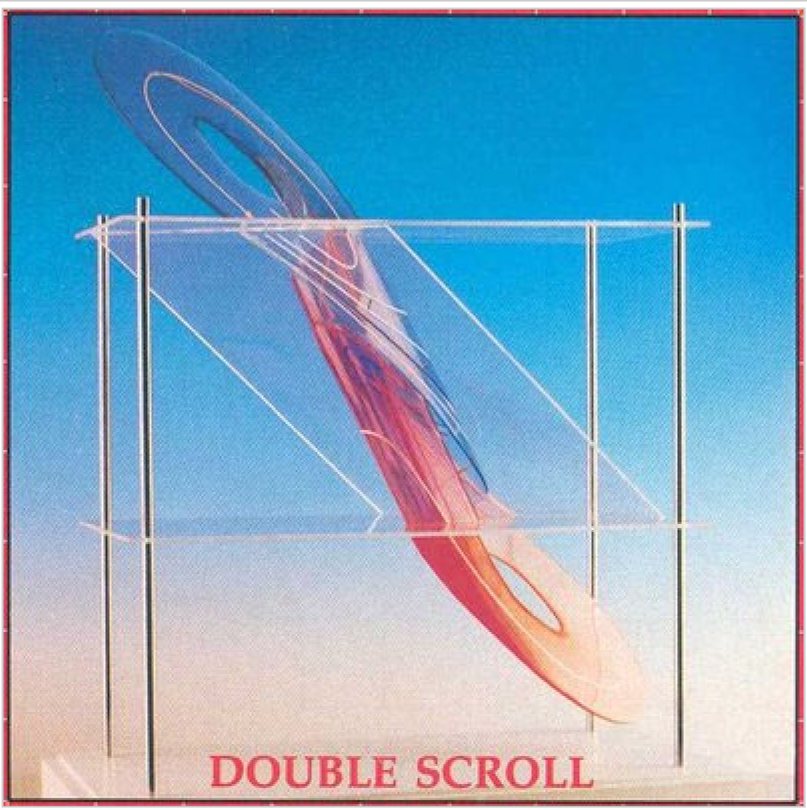


Figure 8: Three-dimensional fiber glass model of the double scroll attractor.

Period-Doubling Route to Chaos

By fixing the parameters of the Chua Equations at α = 15.6, m_0 = -8/7 and m_1 = -5/7, and varying the parameter β from β = 25 to β = 51, one observes a classic *period-doubling* bifurcation route to chaos (Kennedy, 2005). This is depicted in Figure 9, reproduced from page 377 of (Alligood *et al*, 1997).

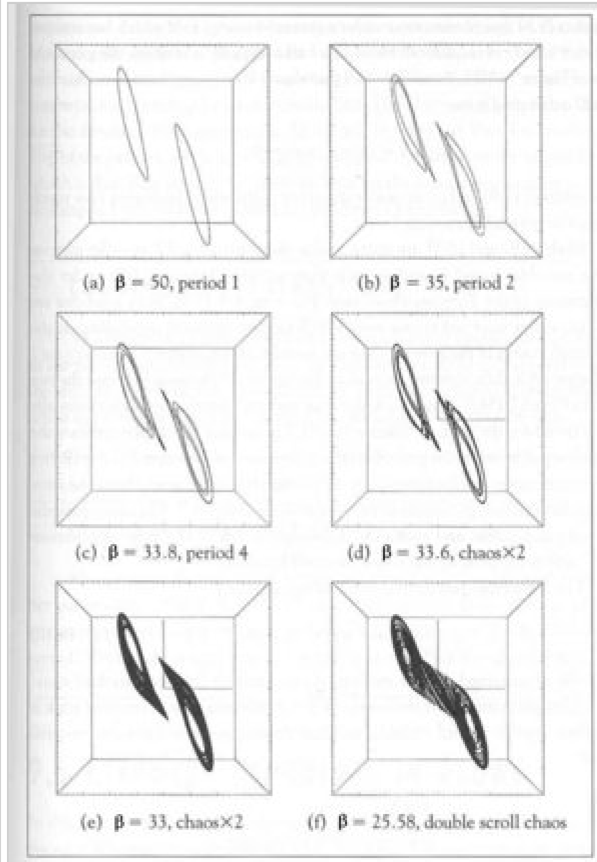


Figure 9: The Chua Circuit exhibits a period-doubling route to chaos.

Interior Crisis and Boundary Crisis

By fixing the parameters of the Chua Equations at α = 15.6, m_0 = -8/7 and m_1 = -5/7, and varying the parameter β from β = 32 to β = 30, Figure 10 (reproduced from page 421 of Alligood *et al* (1997)) shows the bifurcation of a pair of co-existing Rössler-like attractors with separate basins of attraction moving toward one another until they touch at β = 31, whereupon the two twin attractors merge into a single double scroll attractor. A further reduction to β = 30 triggers a boundary crisis, resulting in a *periodic* orbit.

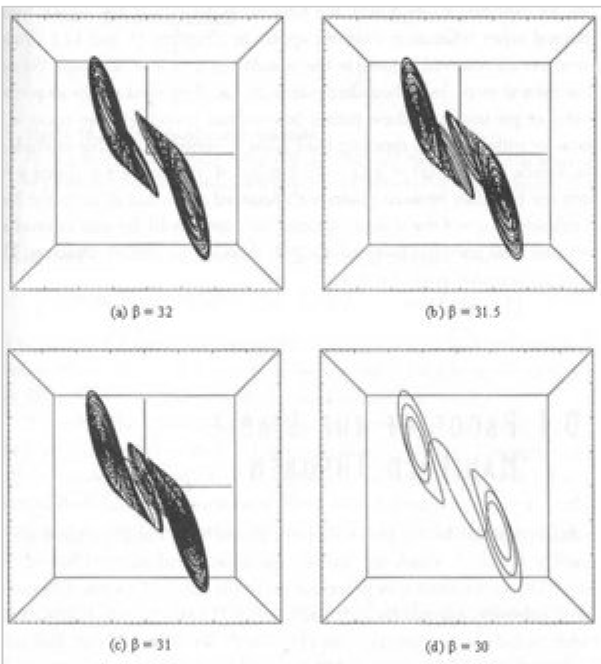


Figure 10: The Chua Circuit exhibits an interior crisis and a boundary crisis bifurcation.

Generalizations

There exists several generalized versions of the Chua Circuit. One generalization substitutes the continuous piecewise-linear function $\phi(x)$ by a smooth function, such as a *cubic* polynomial (Khibnik et al, 1993), (Shilnikov, 1994), (Huang *et al*, 1996), (Hirsch *et al*, 2003), (Tsuneda, 2005), (O'Donoghue *et al*, 2005). For example, Hirsch, Smale and Devaney chose

$$\phi(x) \triangleq = \frac{1}{16}x^3 - \frac{1}{6}x$$

with α = 10.91865 and β = 14 to obtain a pair of *homoclinic* orbits, a much coveted precursor of chaos (Shilnikov, 1994).

Another generalization replaces the third equation in the Chua Equations by

$$\dot{z} = -\beta y - \gamma z$$

thereby introducing a third parameter γ (Chua, 1993). This *unfolding* of the originl vector field gives rise to a surprisingly large number of topologically distinct chaotic attractors. For example, Bilotta had reported almost a *thousand* attractors (which appears to exhibit *different* geometrical structures) from the generalized Chua Equations (Bilotta *et al*, 2007).

Various forms of the Chua Equations can be found in textbooks on *nonlinear dynamics* (Hirsch *et al*, 2003), (Alligood *et al*, 1997) and *chaos* (van Wyk and Steeb, 1997), (Sprott, 2003), where a more detailed mathematical analysis can be found.

Applications

The Chua Circuit has been built and used in many laboratories as a physical source of pseudo random signals, and in numerous experiments on synchronization studies, such as secure communication systems and simulations of brain dynamics. It has also been used extensively in many numerical simulations, and exploited in *avant-garde* music compositions (Bilotta *et al*, 2005), and in the evolution of natural languages (Bilotta and Pantano, 2006).

Arrays of Chua Circuits have been used to generate 2-dimensional *spiral waves*, 3-dimensional *scroll waves*, (Munuzuri *et al*, 1993) and stationary patterns, such as Turing and other exotic patterns, (Munuzuri and Chua, 1997), (Madan, 1993), as illustrated in Figures 11(a), (b), and (c), respectively. Such high-dimensional *attractors* have been exploited for applications in image processing, neural networks, dynamic associative memories (Itoh and Chua, 2004), complexity (Chua, 1998), emergence (Arena *et al*, 2005), etc.

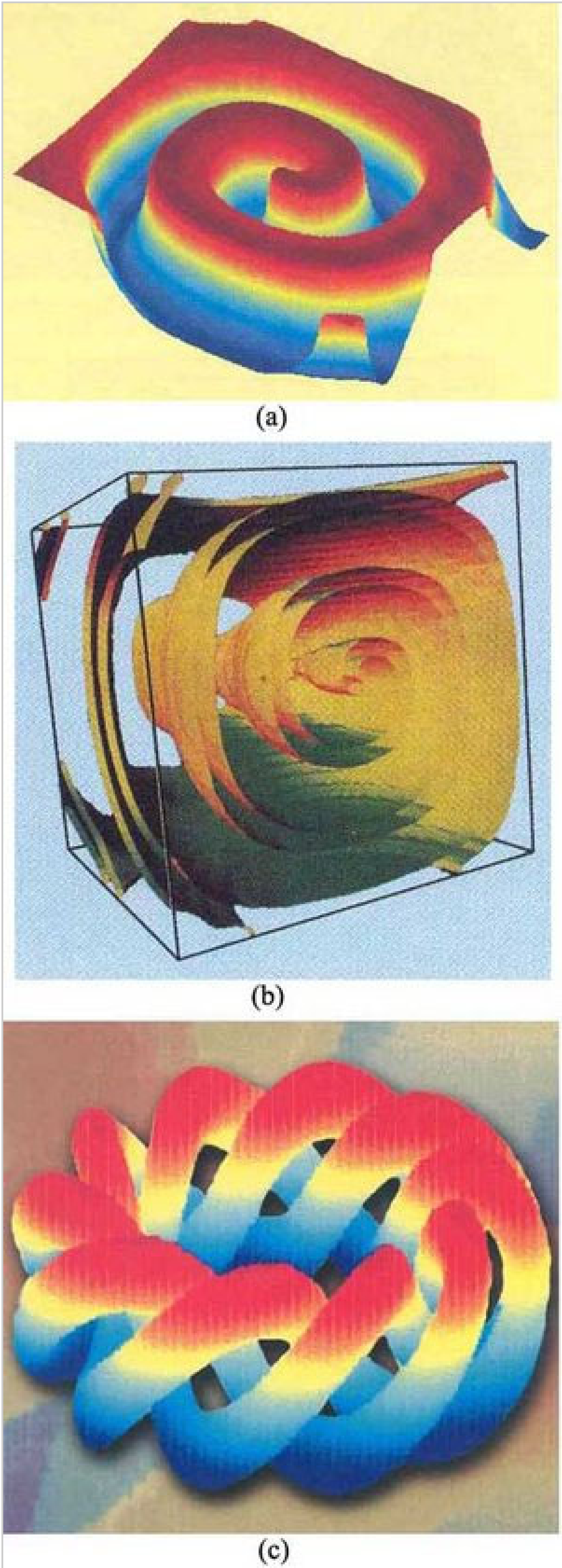


Figure 11: Spiral wave, scroll wave, and stationary pattern generated from 2- and 3-dimenstional arrays of Chua Circuits.

External Links

http://sprott.physics.wisc.edu/chaostsa/
http://www.chuacircuit.com

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See Also

Chaos, Chaotic Oscillators, FitzHugh-Nagumo Model

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