INTERNATIONAL CENTRE
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EDUCATION IN
MATHEMATICS

The Improving Mathematics Education in Schools (TIMES) Project

CIRCLE GEOMETRY

A guide for teachers - Years 9–10

MEASUREMENT AND
GEOMETRY : Module 26

June 2011

YEARS
9
10

Circle Geometry

(Measurement and Geometry: Module 26)

For teachers of Primary and Secondary Mathematics

510

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CIRCLE GEOMETRY

A guide for teachers - Years 9–10

MEASUREMENT AND GEOMETRY : Module 26

June 2011

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YEARS 9 10

CIRCLE GEOMETRY

ASSUMED KNOWLEDGE

- Introductory plane geometry involving points and lines, parallel lines and transversals, angle sums of triangles and quadrilaterals, and general angle-chasing.
- Experience with a logical argument in geometry written as a sequence of steps, each justified by a reason.
- Ruler-and-compasses constructions.
- The four standard congruence tests and their application to proving properties of and tests for special triangles and quadrilaterals.
- The four standard similarity tests and their application.
- Trigonometry with triangles.

MOTIVATION

Most geometry so far has involved triangles and quadrilaterals, which are formed by intervals on lines, and we turn now to the geometry of circles. Lines and circles are the most elementary figures of geometry – a line is the locus of a point moving in a constant direction, and a circle is the locus of a point moving at a constant distance from some fixed point – and all our constructions are done by drawing lines with a straight edge and circles with compasses. Tangents are introduced in this module, and later tangents become the basis of differentiation in calculus.

The theorems of circle geometry are not intuitively obvious to the student, in fact most people are quite surprised by the results when they first see them. They clearly need to be proven carefully, and the cleverness of the methods of proof developed in earlier modules is clearly displayed in this module. The logic becomes more involved – division into cases is often required, and results from different parts of previous geometry modules are often brought together within the one proof. Students traditionally learn a greater respect and appreciation of the methods of mathematics from their study of this imaginative geometric material.

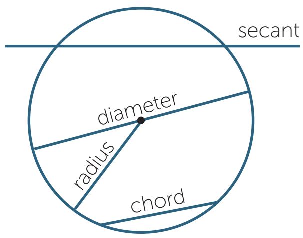
The theoretical importance of circles is reflected in the amazing number and variety of situations in science where circles are used to model physical phenomena. Circles are the first approximation to the orbits of planets and of their moons, to the movement of electrons in an atom, to the motion of a vehicle around a curve in the road, and to the shapes of cyclones and galaxies. Spheres and cylinders are the first approximation of the shape of planets and stars, of the trunks of trees, of an exploding fireball, and of a drop of water, and of manufactured objects such as wires, pipes, ball-bearings, balloons, pies and wheels.

CONTENT

RADII AND CHORDS

We begin by recapitulating the definition of a circle and the terminology used for circles. Throughout this module, all geometry is assumed to be within a fixed plane.

- A **circle** is the set of all points in the plane that are a fixed distance (**the radius**) from a fixed point (**the centre**).
- Any **interval** joining a point on the circle to the centre is called a **radius**. By the definition of a circle, any two radii have the same length. Notice that the word 'radius' is being used to refer both to these intervals and to the common length of these intervals.

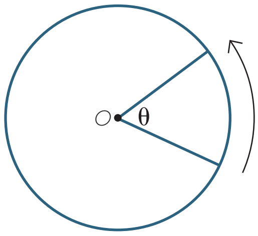


- An interval joining two points on the circle is called a **chord**.
- A chord that passes through the centre is called a **diameter**. Since a diameter consists of two radii joined at their endpoints, every diameter has length equal to twice the radius. The word 'diameter' is use to refer both to these intervals and to their common length.
- A line that cuts a circle at two distinct points is called a **secant**. Thus a chord is the interval that the circle cuts off a secant, and a diameter is the interval cut off by a secant passing through the centre of a circle centre.

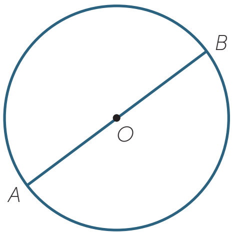
Symmetries of a circle

Circles have an abundance of symmetries:

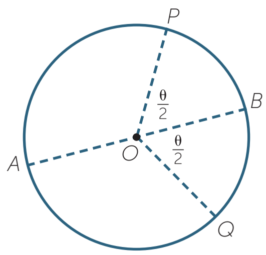
- A circle has every possible rotation symmetry about its centre, in that every rotation of the circle about its centre rotates the circle onto itself.



- If AOB is a diameter of a circle with centre O, then the reflection in the line AOB reflects the circle onto itself. Thus every diameter of the circle is an axis of symmetry.

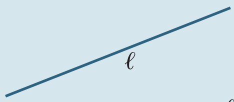


As a result of these symmetries, any point P on a circle can be moved to any other point Q on the circle. This can be done by a rotation through the angle $\theta = \angle POQ$ about the centre. It can also be done by a reflection in the diameter AOB bisecting $\angle POQ$. Thus every point on a circle is essentially the same as every other point on the circle – no other figure in the plane has this property except for lines.

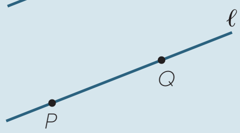


EXERCISE 1

a Identify all translations, rotations and reflections of the plane that map a line ℓ onto itself.

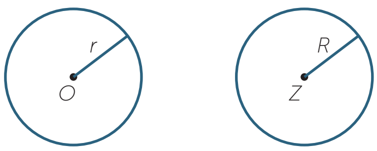


b Which of the transformations in part **a** map a particular point P on ℓ to another particular point Q on ℓ.

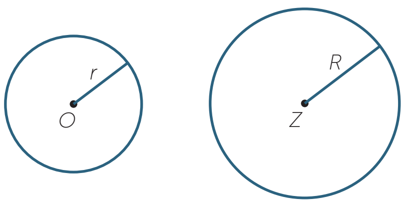


Congruence and similarity of circles

Any two circles with the same radius are congruent – if one circle is moved so that its centre coincides with the centre of the other circle, then it follows from the definition that the two circles will coincide.



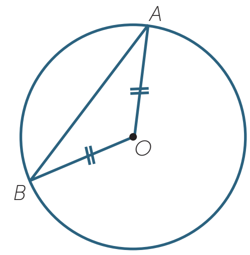
More generally, any two circles are similar – move one circle so that its centre coincides with the centre of the other circle, then apply an appropriate enlargement so that it coincides exactly with the second circle.



A circle forms a curve with a definite length, called the **circumference**, and it encloses a definite area. The similarity of any two circles is the basis of the definition of π, the ratio of the circumference and the diameter of any circle. We saw in the module, The *Circles* that if a circle has radius r, then

circumference of the circle $= 2\pi r$ and area of the circle $= \pi r^2$

Radii and chords



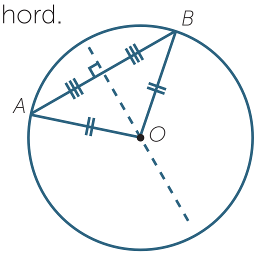
Let AB be a chord of a circle not passing through its centre O. The chord and the two equal radii OA and BO form an isosceles triangle whose base is the chord. The angle $\angle AOB$ is called the **angle at the centre subtended by the chord**.

In the module, *Rhombuses*, *Kites* and *Trapezia* we discussed the axis of symmetry of an isosceles triangle. Translating that result into the language of circles:

Theorem

Let AB be a chord of a circle with centre O. The following three lines coincide:

- The bisector of the angle $\angle AOB$ at the centre subtended by the chord.
- The interval joining O and the midpoint of the chord AB.
- The perpendicular bisector of the chord AB.



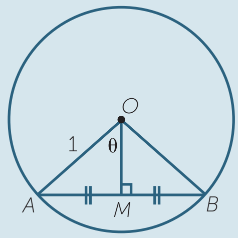
Trigonometry and chords

Constructions with radii and chords give plenty of opportunity for using trigonometry.

EXERCISE 2

In the diagram to the right, the interval AM subtends an angle θ at the centre O of a circle of radius 1. Show that:

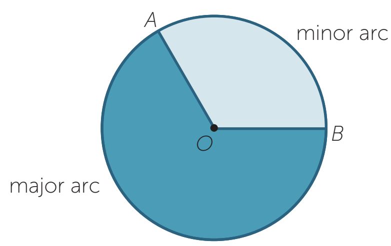
a $AM = \sin\theta$ **b** $OM = \cos\theta$



This exercise shows that sine can be regarded as the length of the semichord AM in a circle of radius 1, and cosine as the perpendicular distance of the chord from the centre. Until modern times, tables of sines were compiled as tables of chords or semichords, and the name 'sine' is conjectured to have come in a complicated and confused way from the Indian word for semichord.

Arcs and sectors

Let A and B be two different points on a circle with centre O. These two points divide the circle into two **opposite arcs**. If the chord AB is a diameter, then the two arcs are called **semicircles**. Otherwise, one arc is longer than the other – the longer arc is called the **major arc** AB and the shorter arc is called the **minor arc** AB.



Now join the radii OA and OB. The reflex angle $\angle AOB$ is called the **angle subtended by the major arc** AB. The non-reflex angle $\angle AOB$ is called the **angle subtended by the minor arc** AB, and is also the angle subtended by the chord AB.

The two radii divide the circle into two sectors, called correspondingly the **major sector** OAB and the **minor sector** OAB.

It is no surprise that equal chords and equal arcs both subtend equal angles at the centre of a fixed circle. The result for chords can be proven using congruent triangles, but congruent triangles cannot be used for arcs because they are not straight lines, so we need to identify the transformation involved.

Theorem

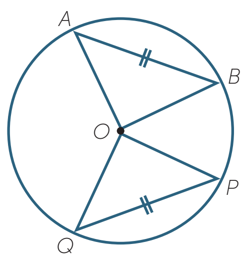
a Equal chords of a circle subtend equal angles at the centre.

b Equal arcs of a circle subtend equal angles at the centre.

Proof

a In the diagram to the right, $\triangle AOB \equiv \triangle POQ$ (SSS)

so $\angle AOB = \angle POQ$ (matching angles of congruent triangles)



b Rotate the circle so that the arc PQ coincides with the arc AB or BA. Then the angles $\angle POQ$ and $\angle AOB$ coincide, and hence are equal.

Segments

A chord AB of a circle divides the circle into two **segments**. If AB is a diameter, the two congruent segments are called **semicircles** – the word 'semicircle' is thus used both for the semicircular arc, and for the segment enclosed by the arc and the diameter. Otherwise, the two segments are called a **major segment** and a **minor segment**.

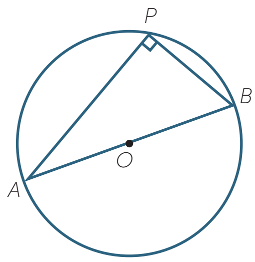
The word 'subtend'

The word 'subtend' literally means 'holds under', and is often used in geometry to describe an angle.

Suppose that we have an interval or arc AB and a point P not on AB. Join the intervals AP and BP to form the angle $\angle APB$. We say that the angle $\angle APB$ is the angle subtended by the interval or arc AB at the point P.

ANGLES IN A SEMICIRCLE

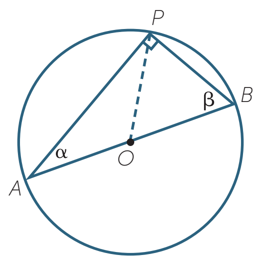
Let AOB be a diameter of a circle with centre O, and let P be any other point on the circle. The angle $\angle APB$ subtended at P by the diameter AB is called an **angle in a semicircle**. This angle is always a right angle – a fact that surprises most people when they see the result for the first time.



Theorem

An angle in a semicircle is a right angle.

Proof



Let AB be a diameter of a circle with centre O, and let P be any other point on the circle.

Join the radius PO, and let $\alpha = \angle A$ and $\beta = \angle B$.

The triangles AOP and BOP are isosceles because all radii are equal, so $\angle APO = \alpha$ and $\angle BPO = \beta$ (base angles of isosceles triangles AOP and BOP).

Hence $\alpha + \beta + (\alpha + \beta) = 180°$ (angle sum of $\triangle APB$)

$$2\alpha + 2\beta = 180°$$

$$\alpha + \beta = 90°,$$

so $\angle APB = 90°$, as required.

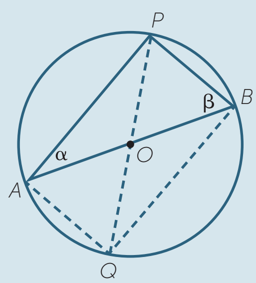
This famous theorem is traditionally ascribed to the Greek mathematician Thales, the first known Greek mathematician.

EXERCISE 3

Thales' theorem can also be proven using the following test for a rectangle developed in the module, *Parallelograms and Rectangles.*

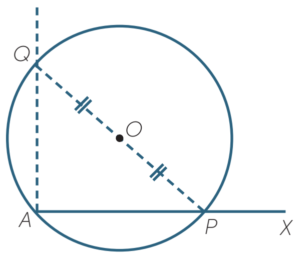
A quadrilateral whose diagonals are equal and bisect each other is a rectangle.

Join the radius PO and produce it to a diameter POQ, then join up the quadrilateral $APBQ$. Explain why $\angle APB$ is a right angle.



Constructing a right angle at the endpoint of an interval

Thales' theorem gives a quick way to construct a right angle at the endpoint of an interval AX.



1 Choose a point O above the interval AX.

2 Draw a circle with centre O through A crossing AX again at P.

3 Draw the radius PO and produce it to a diameter POQ.

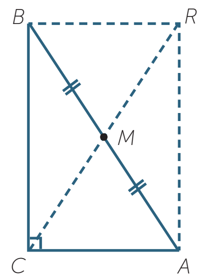
4 Join AQ, then $\angle PAQ = 90°$.

The converse theorem

The angle in a semicircle theorem has a straightforward converse that is best expressed as a property of a right-angled triangle:

Theorem

The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.



Proof

Let $\triangle ABC$ be right-angled at C, and let M be the midpoint of the hypotenuse AB.

We need to prove that $MC = MA = MB$. Complete the rectangle $ACBR$. Because $ACBR$ is a rectangle, its diagonals bisect each other and are equal. Hence M is the midpoint of the other diagonal CR, and $AM = BM = CM = RM$.

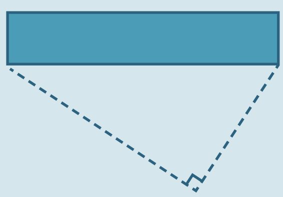
This is an excellent example of the way ideas in geometry fit together – a significant theorem about circles has been proven using a property of rectangles.

Two practical situations to illustrate the converse theorem

A set of points in the plane is often called a **locus**. The term is used particularly when the set of points is the curve traced out by a moving point. For example, a circle can be defined as the locus of a point that moves so that its distance from some fixed point is constant. The two examples below use the converse of the angle in a semicircle theorem to describe a locus.

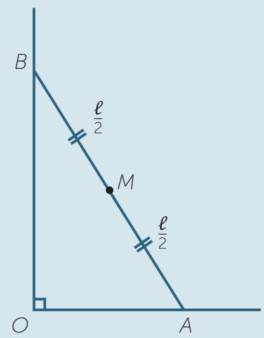
EXERCISE 4

A photographer is photographing the ornamental front of a building. He wants the two ends of the front to subtend a right angle at his camera. Describe the set of all positions where he can stand.



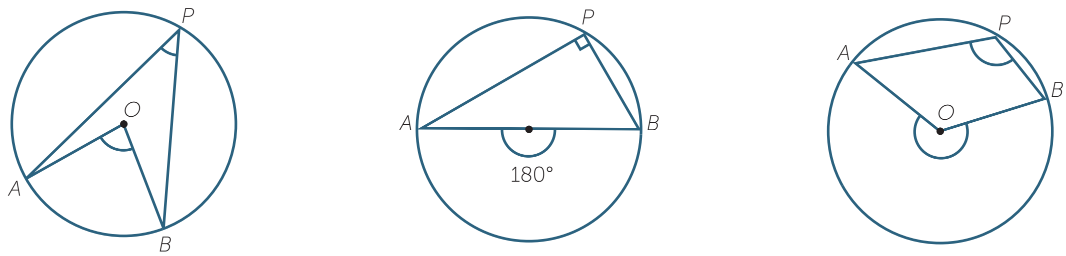
EXERCISE 5

A plank of length ℓ metres is initially resting flush against a wall, but it slips outwards, with its top sliding down the wall and its foot moving at right angles to the wall. What path does the midpoint of the plank trace out?



ANGLES AT THE CENTRE AND CIRCUMFERENCE

The angle-in-a-semicircle theorem can be generalised considerably. In each diagram below, AB is an arc of a circle with centre O, and P is a point on the opposite arc. The arc AB subtends the angle $\angle AOB$ at the centre. The arc also subtends the angle $\angle APB$, called an **angle at the circumference subtended by the arc** AB.



In the middle diagram, where the arc is a semicircle, the angle at the centre is a straight angle, and by the previous theorem, the angle at the circumference is a right angle – exactly half. We shall show that this relationship holds also for the other two cases, when the arc is a minor arc (left-hand diagram) or a major arc (right-hand diagram). The proof uses isosceles triangles in a similar way to the proof of Thales' theorem.

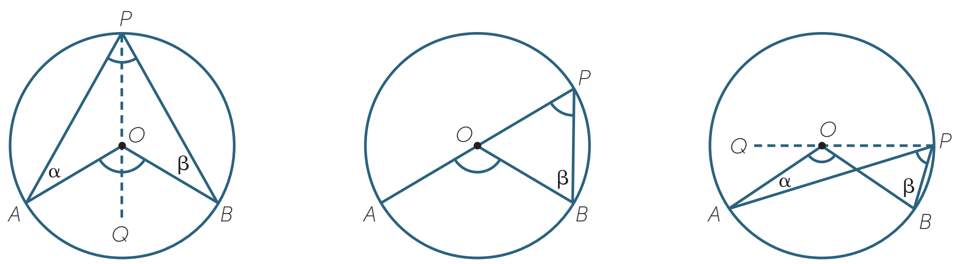
Theorem

An angle at the circumference of a circle is half the angle at the centre subtended by the same arc.

Proof

Let AB be an arc of a circle with centre O, and let P be any point on the opposite arc. We need to prove $\angle AOB = 2\angle APB$. The proof divides into three cases, depending on whether:

Case 1: O lies inside $\triangle ABP$ **Case 2:** O lies on $\triangle APB$ **Case 3:** O lies outside $\triangle APB$.



Case 1: Join PO and produce PO to Q. Then $OA = OB = OP$ (radii), so we have two isosceles triangles OAP and OAQ.

Let	$\angle OAP = \alpha$ and $\angle OBP = \beta$.	
Then	$\angle APO = \alpha$	(base angles of isosceles $\triangle OAP$)
and	$\angle BPO = \beta$	(base angles of isosceles $\triangle OBP$).
Hence	$\angle AOQ = 2\alpha$	(exterior angle of $\triangle OAP$)
and	$\angle BOQ = 2\beta$ so	(exterior angle of $\triangle OBP$),
	$\angle AOB = 2\alpha + 2\beta = 2(\alpha + \beta) = 2 \times \angle APB$.	

EXERCISE 6

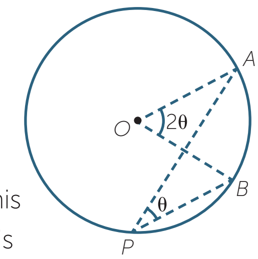
Complete the proof in the other two cases.

EXAMPLE

A punter stands on the edge of a circular racing track. With his binoculars he is following a horse that is galloping around the track at one revolution a minute. Explain why the punter's binoculars are rotating at a constant rate of half a revolution per minute.

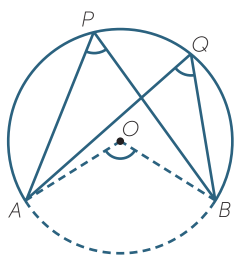
SOLUTION

As the horse moves from position A to position B, the horse moves an angle $\angle AOB = 2\theta$ about the centre O of the track. The angle at the circumference is half the angle at the centre, so $\angle APB = \theta$ which means that the punter's binoculars rotate by an angle θ the horse moves from A to B. Hence the punter is rotating his binoculars at a constant rate that is half the rate at which the horse is rotating about the centre.



Angles subtended by the same arc

In the diagram to the right, the two angles $\angle APB$ and $\angle AQB$ are subtended by the same (minor) arc AB. Each angle is half the angle $\angle AOB$ at the centre subtended by the same arc, so $\angle APB = \angle AQB$.



This corollary of the previous theorem is a particularly significant result about angles in circles:

Theorem

Two angles at the circumference subtended by the same arc are equal.

Thales' theorem is a special case of this theorem.

Some alternative terminology

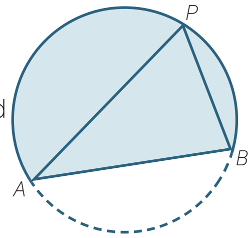
The last two theorems are often expressed in slightly different language, and some explanation is needed to avoid confusion.

1 An angle subtended by an arc is often said to be **standing on the arc**. With this terminology, the two theorems become:

- An angle at the circumference of a circle is half the angle at the centre standing on the same arc.
- Two angles at the circumference standing on the same arc are equal.

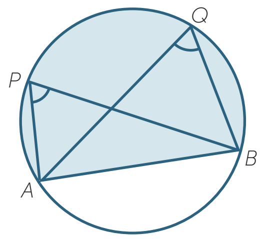
In the context of these two theorems, it is best to avoid the phrases 'standing on a chord *AB*' and 'subtended by a chord *AB*', because we need to distinguish between angles subtended by the major arc *AB* and angles subtended by the minor arc *AB*.

2 In the upper diagram to the right, the angle $\angle APB$ is called an angle in the (major) segment. Notice that it actually stands on the minor arc *AB*, which can be confusing. We have already used this terminology before in speaking about an 'angle in a semicircle'. With this terminology, the last theorem becomes:



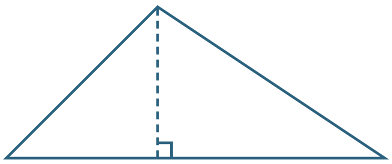
Two angles in the same segment of a circle are equal.

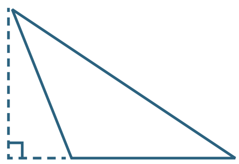
The situation is illustrated in the lower diagram to the right.



Extension – The orthocentre of a triangle

An **altitude** of a triangle is a perpendicular from any of the three vertices to the opposite side, produced if necessary. The two cases are illustrated in the diagrams below.

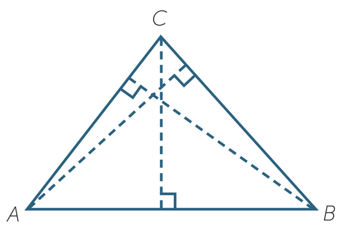




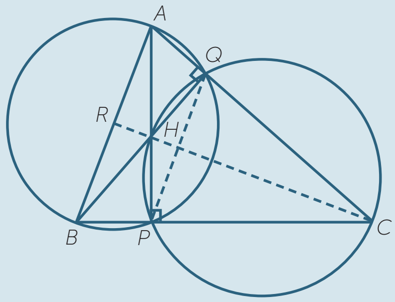
There are three altitudes in a triangle. The following theorem proves that they concurrent at a point called the **orthocentre** *H* of the triangle. It is surprising that circles can be used to prove the concurrence of the altitudes.

Theorem

The altitudes of a triangle are concurrent.



EXERCISE 7



In the diagram to the right, the altitudes AP and BQ meet at H. The interval CH is produced to meet AB, produced if necessary, at R. We need to prove that $CR \perp AB$.

a Explain why Q, H, P and C are concyclic, and draw the circle.

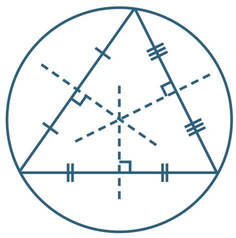
b Explain why A, B, P and Q are concyclic, and draw the circle.

c Join the common chord PQ, and explain why $\angle QCH = \angle QPH$.

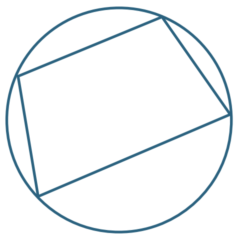
d Explain why $\angle APQ = \angle ABQ$.

e Use $\triangle ABQ$ and $\triangle ACR$ to explain why $CR \perp AB$.

CYCLIC QUADRILATERALS



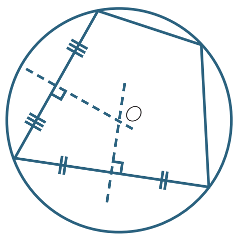
In the module, *Congruence,* we showed how to draw the **circumcircle** through the vertices of any triangle. To do this, we showed that the perpendicular bisectors of its three sides are concurrent, and that their intersection, called the **circumcentre** of the triangle, is equidistant from each vertex.



No other circle passes through these three vertices. If we tried to take as centre a point P other than the circumcentre, then P would not lie on one of the perpendicular bisectors, so it could not be equidistant from the three vertices.

When there are four points, we can always draw a circle through any three of them (provided they are not collinear), but only in special cases will that circle pass through the fourth point. A **cyclic quadrilateral** is a quadrilateral whose vertices all lie on a circle. This is the last type of special quadrilateral that we shall consider.

Constructing the circumcircle of a cyclic quadrilateral



Suppose that we are given a quadrilateral that is known to be cyclic, but whose circumcentre is not shown (perhaps it has been rubbed out). The circumcentre of the quadrilateral is the circumcentre of the triangle formed by any three of its vertices, so the construction to the right will find its circumcentre.

The opposite angles of a cyclic quadrilateral

The distinctive property of a cyclic quadrilateral is that its opposite angles are supplementary. The following proof uses the theorem that an angle at the circumference is half the angle at the centre standing on the same arc.

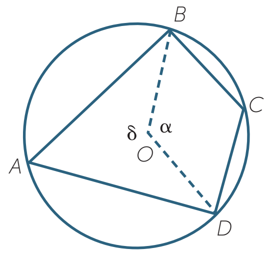
Theorem

The opposite angles of a cyclic quadrilateral are supplementary.

Proof

Let *ABCD* be a cyclic quadrilateral with *O* the centre of the circle.

Join the radii *OB* and *OD*. Let α and γ be the angles at the centre, as shown on the diagram.



Then $\alpha + \gamma = 360°$ (angles in a revolution at *O*)

Also $\angle A = \frac{1}{2}\alpha$ (angles on the same arc *BCD*)

and $\angle C = \frac{1}{2}\gamma$ (angles on the same arc *BAD*)

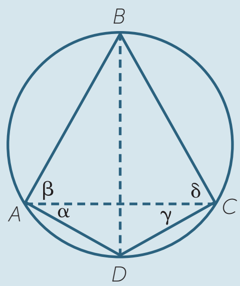
so $\angle A + \angle C = \frac{1}{2}\alpha + \frac{1}{2}\gamma = 180°$

Hence also $\angle ABC + \angle ADC = 180°$ (angle sum of quadrilateral *ABCD*)

Here is an alternative proof using the fact that two angles in the same segment are equal.

EXERCISE 8

Join the diagonals *AC* and *BD* of the cyclic quadrilateral *ABCD*. Let α, β, γ and δ be the angles shown.



a Give a reason why $\angle DBC = \alpha$

b What other angles have sizes β, γ and δ?

c Prove that $\alpha + \beta + \gamma + \delta = 180°$.

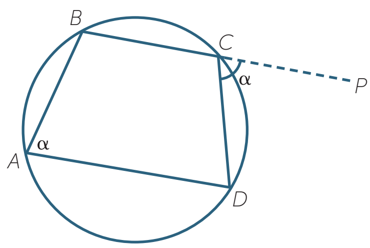
d Hence prove that the opposite angles of *ABCD* are supplementary.

Exterior angles of a cyclic quadrilateral

An exterior angle of a cyclic quadrilateral is supplementary to the adjacent interior angle, so is equal to the opposite interior angle. This gives us the corollary to the cyclic quadrilateral theorem:

Theorem

An exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

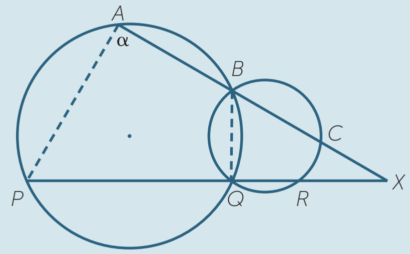


Proof

In the diagram to the right, *BC* is produced to *P* to form the exterior angle $\angle PCD$. This exterior angle and $\angle A$ are both supplementary to $\angle BCD$, so they are equal.

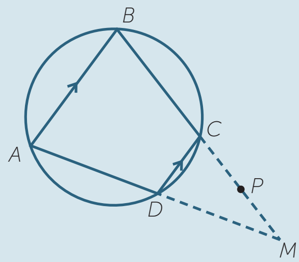
EXERCISE 9

Show that $AP \parallel CR$ in the diagram to the right.



EXERCISE 10

If a cyclic trapezium is not a rectangle, show that the other two sides are not parallel, but have equal length.



Extension – A test for a cyclic quadrilateral

The property of a cyclic quadrilateral proven earlier, that its opposite angles are supplementary, is also a test for a quadrilateral to be cyclic. That is the converse is true. This theorem completes the structure that we have been following – for each special quadrilateral, we establish its distinctive properties, and then establish tests for it.

The proof uses 'proof by contradiction', and is thus a little more difficult than other Year 10 material.

Theorem

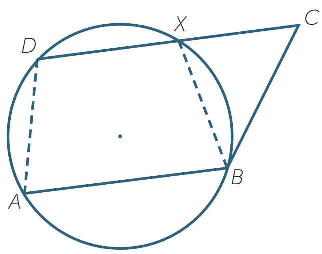
If the opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Proof

Let $ABCD$ be a quadrilateral with $\angle A + \angle C = 180°$.

Construct the circle through A, B and D, and suppose, by way of contradiction, that the circle does not pass through C.

Let DC, produced if necessary, meet the circle again at X, and join XB.



Then $\angle A$ and $\angle BXD$ are supplementary because $ABXD$ is a cyclic quadrilateral,

so the angles $\angle DXB$ and $\angle DCB$ are equal, so $XB \parallel CB$

Hence XB and CB are the same line, so C and X coincide, that is the circle does pass through C.

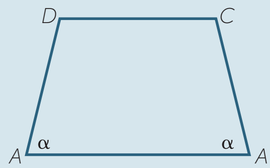
EXERCISE 11

Prove the following alternative form of the above theorem:

If an exterior angle of a quadrilateral equals the opposite interior angle, then the quadrilateral is cyclic.

EXERCISE 12

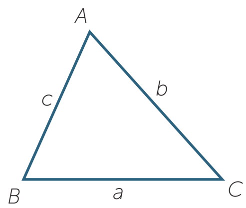
In the diagram to the right, the two adjacent acute angles of the trapezium are equal.
Prove that the trapezium is cyclic.



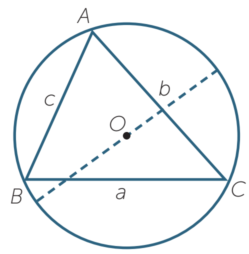
Extension – The sine rule and circle geometry

The sine rule states that for any triangle *ABC*, the ratio of any side over the sine of its opposite angle is a constant,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Each term is the ratio of a length over a pure number, so their common value seems to be a length. Thus it reasonable to ask, what is this common length? The answer is a surprise – the common length is the diameter of the circumcircle through the vertices of the triangle.



The proof of this result provides a proof of the sine rule that is independent of the proof given in the module, *Further Trigonometry*.

Theorem

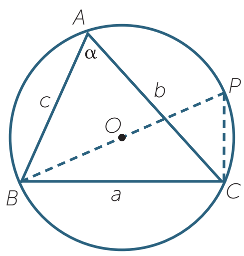
In any triangle *ABC*, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, where *R* is the radius of the circumcircle.

Proof

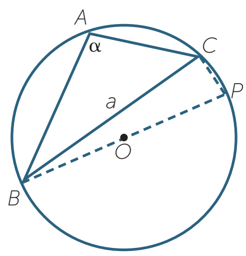
Let *O* be the centre of the circumcircle through *A*, *B* and *C*, and let $\angle A = \alpha$.

It is sufficient to prove that $\frac{a}{\sin A}$ is the diameter of the circumcircle.

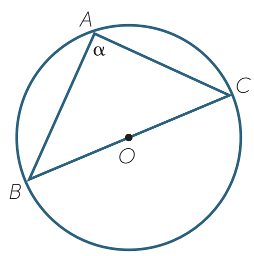
There are three cases, as shown below.



Case 1: *A* and *O* lie on the same side of *BC*.



Case 2: *A* and *O* lie on opposite sides of *BC*.



Case 3: *O* lies on *BC*

In cases 1 and 2, construct the diameter *BOP*, and join *PC* .

EXERCISE 13

a Complete the following steps of the proof in Case 1.

i Explain why $\angle BCP = 90°$ and $\angle P = \alpha$

ii Complete the proof using $\triangle BCP$.

b Complete the following steps of the proof in Case 2.

i Explain why $\angle BCP = 90°$ and $\sin P = \sin \alpha$

ii Complete the proof using $\triangle BCP$.

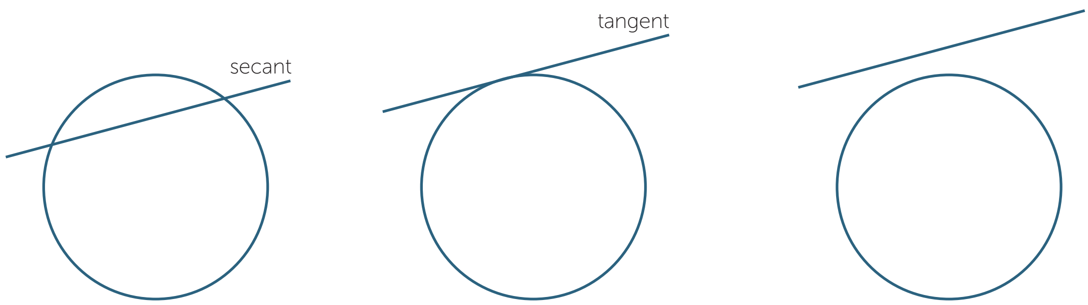
c Complete the proof in Case 3.

Tangents to circles

A **tangent** to a circle is a line that meets the circle at just one point. The diagram below shows that given a line and a circle, can arise three possibilities:

- The line may be a **secant**, cutting the circle at two points.
- The line may be a **tangent**, touching the circle at just one point.
- The line may miss the circle entirely.

The words 'secant' and 'tangent' are from Latin – 'secant' means 'cutting' (as in 'cross-section'), and 'tangent' means 'touching' (as in 'tango').



The point where a tangent touches a circle is called a **point of contact**. It is not immediately obvious how to draw a tangent at a particular point on a circle, or even whether there may be more than one tangent at that point. The following theorem makes the situation clear, and uses Pythagoras' theorem in its proof.

Theorem

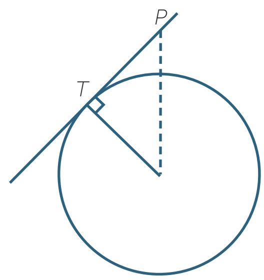
Let T be a point on a circle with centre O.

a The line through T, perpendicular to the radius OT, is a tangent to the circle.

b This line is the only tangent to the circle at T.

c Every point on the tangent, except for T itself, lies outside the circle.

Proof

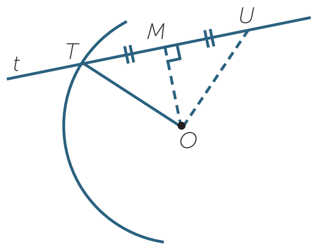


First we prove parts **a** and **c**. Let ℓ be the line through T perpendicular to the radius OT. Let P be any other point on ℓ, and join the interval OP. Then using Pythagoras' theorem in $\triangle OTP$,

$$OP^2 = OT^2 + PT^2\text{; which is greater than } OT^2,$$

so OP is greater than the radius OT. Hence P lies outside the circle, and not on it. This proves that the line ℓ is a tangent, because it meets the circle only at T. It also proves that every point on ℓ, except for T, lies outside the circle.

It remains to prove part **b**, that there is no other tangent to the circle at T.

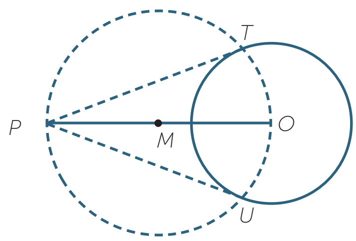


Let t be a tangent at T, and suppose, by way of contradiction, that t were not perpendicular to OT. Let M be the foot of the perpendicular from O to t, and construct the point U on t,on the other side of M, so that $UM = MT$. Then using Pythagoras' theorem in $\triangle OMT$ and $\triangle OMU$,

$$OT^2 = OM^2 + MT^2 = OM^2 + MU^2 = OU^2,$$

So $OU = OT$. Hence U also lies on the circle, contradicting the fact that t is a tangent.

Construction – Tangents from an external point



Using this radius and tangent theorem, and the angle in a semi circle theorem, we can now construct tangents to a circle with centre O from a point P outside the circle.

1 Join OP and construct the midpoint M of OP .

2 Draw the circle with centre M through O and P, and let it meet the circle at T and U.

The angles $\angle PTO$ and $\angle PUO$ are right angles, because they are angles in a semicircle. Hence the lines PT and PU are tangents, because they are perpendicular to the radii OT and OU, respectively.

Tangents from an external point have equal length

It is also a simple consequence of the radius-and-tangent theorem that the two tangents PT and PU have equal length.

Notice that here, and elsewhere, we are using the word 'tangent' in a second sense, to mean not the whole line, but just the interval from an external point to the point of contact.

Theorem

Tangents to a circle from an external point have equal length.

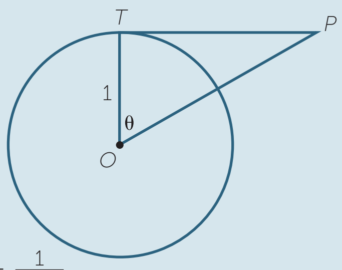
EXERCISE 14

Prove this result using either congruence or Pythagoras' theorem.

Tangents and trigonometry

The right angle formed by a radius and tangent gives further opportunities for simple trigonometry. The following exercise shows how the names 'tangent' and 'secant', and their abbreviations tan θ and sec θ, came to be used in trigonometry. In a circle of radius 1, the length of a tangent subtending an angle θ at the centre is tan θ , and the length of the secant from the external point to the centre is sec θ .

EXERCISE 15



A tangent *PT* to a circle of radius 1 touches the circle at *T* and subtends an angle θ at the centre *O*. Show that

$$PT = \tan\theta \text{ and } PO = \sec\theta$$

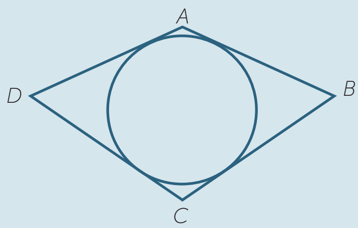
where the trigonometry function secant is defined by $\sec\theta = \frac{1}{\cos\theta}$.

Quadrilaterals with incircles

The following exercise involves quadrilaterals within which an incircle can be drawn tangent to all four sides. These quadrilaterals form yet another class of special quadrilaterals.

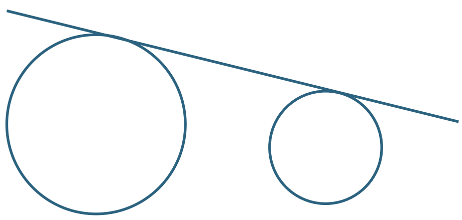
EXERCISE 16

The sides of a quadrilateral are tangent to a circle drawn inside it. Show that the sums of opposite sides of the quadrilateral are equal.

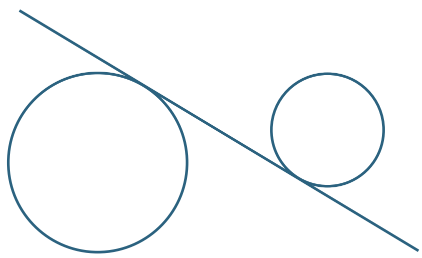


Common tangents and touching circles

A line that is tangent to two circles is called a **common tangent** to the circles. When the points of contact are distinct, there are two cases, as in the diagrams below. The two circles lie on the same side of a **direct common tangent**, and lie on opposite sides of an **indirect common tangent**.

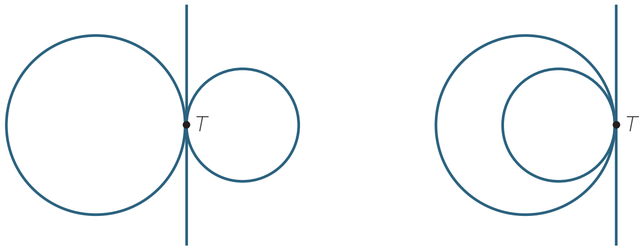


Direct common tangent



Indirect common tangent

Two circles are said to touch at a common point T if there is a common tangent to both circles at the point T. As in the diagram below, the circles **touch externally** when they are on opposite sides of the common tangent, and **touch internally** when they are on the same side of the common tangent.



Provided that they are distinct, touching circles have only the one point in common.

EXERCISE 17

a How many common tangents are there to two circles when the circles:

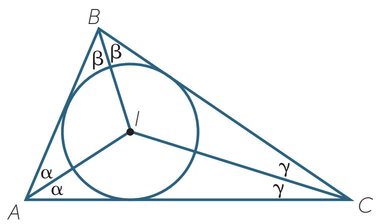
- **i** are one inside the other,
- **ii** touch internally,
- **iii** overlap,
- **iv** touch externally,
- **v** are next to each other without touching?

b Prove that the intervals between the points of contact on two indirect common tangents have equal length.

c Prove that the intervals between the points of contact on two direct common tangents have equal length.

Extension - The incentre of a triangle

We are now in a position to prove a wonderful theorem on the angle bisectors of a triangle. These three bisectors are concurrent, and their point of intersection is called the **incentre** of the triangle. The incentre is the centre of the **incircle** tangent to all three sides of the triangle, as in the diagram to the right.



Theorem

The angle bisectors of a triangle are concurrent, and the resulting incentre is the centre of the incircle, that is tangent to all three sides.

Proof

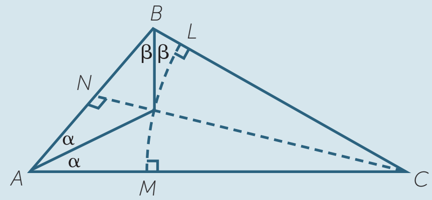
We have to prove IC bisects $\angle BCD$

Given $\triangle ABC$, construct the angle bisectors of $\angle A$ and $\angle B$, and let I be their point of intersection.

Join IC, and let $\alpha = \angle BAI = \angle CAI$ and $\beta = \angle ABI = \angle CBI$.

Drop perpendiculars IL, IM and IN to BC, CA and AB respectively.

EXERCISE 18



a Prove that $\triangle MAI \equiv \triangle NAI$ and that $\triangle NBI \equiv \triangle LBI$.

b Explain why $LI = MI = NI$.

c Prove that $\triangle LCI \equiv \triangle MCI$.

d Hence complete the proof that the angle bisectors meet at I, and that I is the incentre.

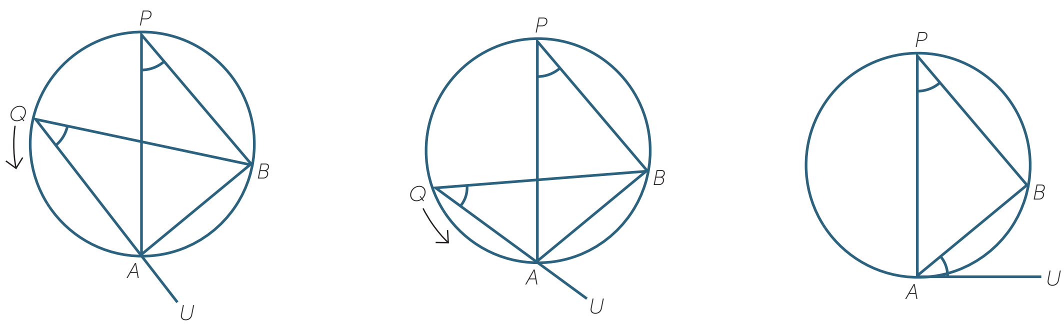
Summary - The four centres of a triangle

This completes the development of the four best-known centres of a triangle. The results, and the associated terminology and notation, are summarised here for reference.

- The perpendicular bisectors of the three sides of a triangle are concurrent at the **circumcentre** O, which is the centre of the **circumcircle** through all three vertices. This was proven in the module, *Congruence*.
- The angle bisectors of the three angles of a triangle are concurrent at the **incentre** I, which is the centre of the **incircle** that is tangent to all three sides. This was proven above.
- A median of a triangle is an interval joining a vertex to the midpoint of an opposite side. The medians of a triangle are concurrent at the **centroid** G, which divides each median in the ratio 2 : 1. This was proven in the module, Scale Drawings and *Similarity*.
- An altitude of a triangle is a perpendicular from a vertex to the opposite side, produced if necessary. The altitudes of a triangle, also produced if necessary, are concurrent at the **orthocentre** H. This was proven earlier in the present module as an extension in the section on angles at the centre and circumference.

THE ALTERNATE SEGMENT THEOREM

The left-hand diagram below shows two angles $\angle P$ and $\angle Q$ lying in the same segment of a circle – we have proven that these two angles are equal. In the next two diagrams, the angle $\angle BQU$ remains equal to $\angle P$ as the point Q moves around the arc closer and closer to A. In the last diagram, Q coincides with A, and AU is a tangent.

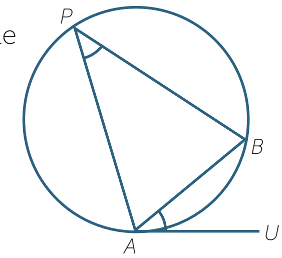


It seems reasonable from these diagrams that in the limiting case, the angle $\angle BAU$ will still equal $\angle P$ – this is what the alternate segment theorem says.

Theorem

An angle between a chord and a tangent is equal to any angle in the alternate segment.

Here $\angle BAU$ is 'an angle between a chord and a tangent'. The word 'alternate' means 'other' – the chord AB divides the circle into two segments, and the **alternate segment** is the segment on the left containing the angle $\angle P$. We have already proven that all the angles in this left-hand segment are equal.



There are two equally satisfactory proofs of this theorem. One is written out below and the other is left as an exercise.

Proof

Let AB be a chord of a circle and let BU be a tangent at B.

Let P be a point on the arc that is not within the arms of $\angle ABU$, and let $\angle P = \alpha$

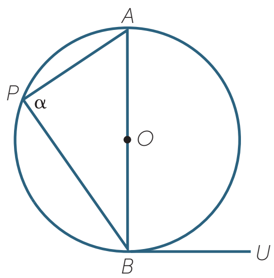
There are three cases, depending on whether:

Case 1: The centre O lies on the chord AB.

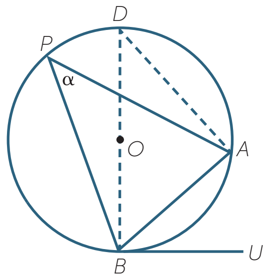
Case 2: The centre O lies outside the arms of $\angle ABU$.

Case 3: The centre O lies within the arms of $\angle ABU$.

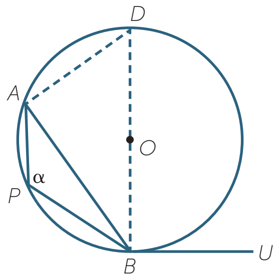
In each case, we need to prove that $\angle ABU = \alpha$



Case 1



Case 2



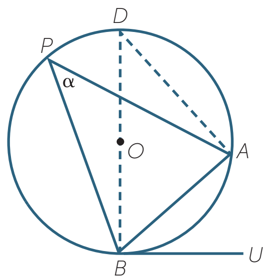
Case 3

Case 1 is easily settled:

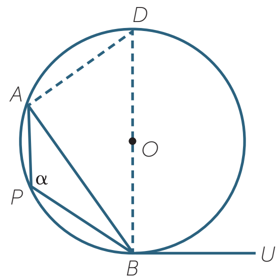
$\alpha = 90°$ (angle in a semicircle) and $\angle ABU = 90°$ (radius and tangent)

For the other two cases, construct the diameter BOD, and join DA.

Case 2:	$\angle ADB = \alpha$	(angles on the same arc AB)
and	$\angle BAD = 90°$	(angle in a semicircle)
so	$\angle DBA = 90° - \alpha$	(angle sum of ΔABD)
But	$\angle DBU = 90°$	(radius and tangent)
so	$\angle ABU = \alpha$	(adjacent angles at B)

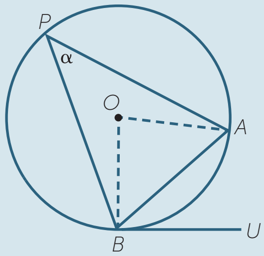


Case 3:	$\angle ADB = 180° - \alpha$	(cyclic quadrilateral *APBD*)
and	$\angle BAD = 90°$	(angle in a semicircle)
so	$\angle ABD = \alpha - 90°$	(angle sum of $\triangle ABD$)
But	$\angle DBU = 90°$	(radius and tangent)
so	$\angle ABU = \alpha$	(adjacent angles at *B*)

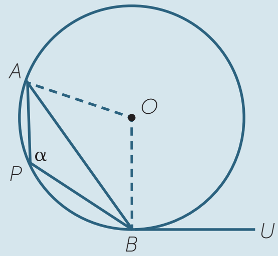


EXERCISE 19

Find an alternative proof in cases 2 and 3 by constructing the radii *AO* and *BO* and using angles at the centre.



Case 2

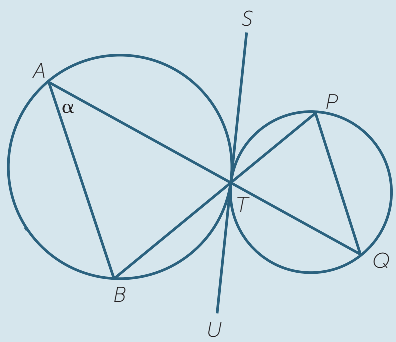


Case 3

The result in the following exercise is surprising. One would not expect parallel lines to emerge so easily in a diagram with two touching circles.

EXERCISE 20

Prove that $AB \parallel PQ$ in the diagram to the right, where *STU* is a common tangent to both circles. *AQ* and *BP* are straight lines.

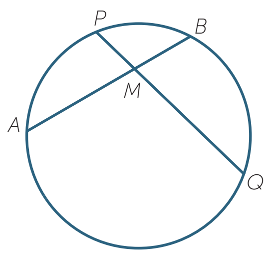


SIMILARITY AND CIRCLES

The final theorems in this module combine similarity with circle geometry to produce three theorems about intersecting chords, intersecting secants, and the square on a tangent.

Intersecting chords

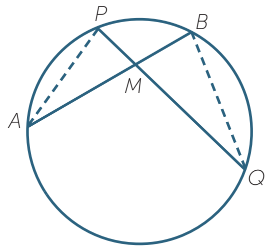
The first theorem deals with chords that intersect within the circle. In this situation, each chord cuts the other into two sub-intervals called **intercepts**. It is an amazing consequence of similar triangles that, in this situation, the products of the intercepts on each chord are equal. That is, in the diagram to the right, $AM \times MB = PM \times MQ$.



Theorem

When two chords intersect within a circle, the products of the intercepts are equal.

Proof



Let AB and PQ be chords intersecting at M. Join AP and BQ.

In the triangles APM and QBM:

$\angle PAM = \angle BQM$ (angles on the same arc PB)

$\angle APM = \angle QBM$ (angles on the same arc AQ)

so $\triangle APM$ is similar to $\triangle QBM$ (AA):

Hence $\frac{AM}{QM} = \frac{PM}{BM}$ (matching sides of similar triangles)

so $AM \times BM = PM \times QM$.

The very last step is particularly interesting. It converts the equality of two ratios of lengths to the equality of two products of lengths. This is a common procedure when working with similarity.

The sine rule and similarity

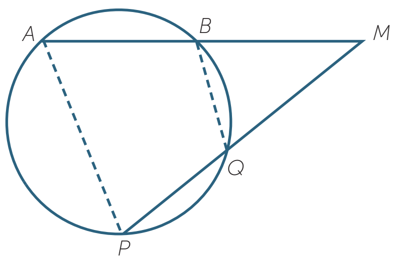
Many problems involving similarity can be handled using the sine rule. The exercise below gives an alternative proof of the intersecting chord theorem using the sine rule to deal directly with the ratio of two sides of the triangles.

The remaining two theorems of this section also have alternative proofs using the sine rule.

EXERCISE 21

Use the sine rule in the diagram in the above proof to prove that $\frac{AM}{PM} = \frac{QM}{BM}$.

Secants from an external point



Now suppose that the chords do not meet within the circle. If they meet on the circle, the identity above holds trivially, and if they are parallel, there is nothing to say. Otherwise, we can produce the chords until they intersect outside the circle, and an analogous theorem applies.

We are now dealing with secants from an external point. When a secant through an external point M meets a circle at two points A and B, the lengths AM and BM are called the **intercepts** of the secant from the external point, and as before $AM \times MB = PM \times MQ$.

Be careful here, the chord AB is not one of the two intercepts. With this definition of intercept, the previous theorem can now be stated as follows. Its proof is word for word almost the same, apart from the reasons:

Theorem

The product of the intercepts on two secants from an external point are equal.

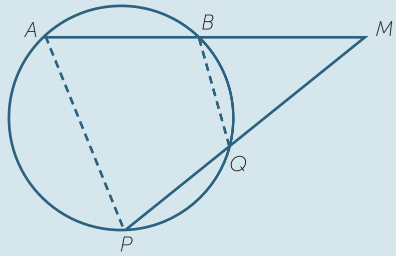
Proof

Let *ABM* and *PQM* be two secants from an external point *M*, as shown.

Join *AP* and *BQ*.

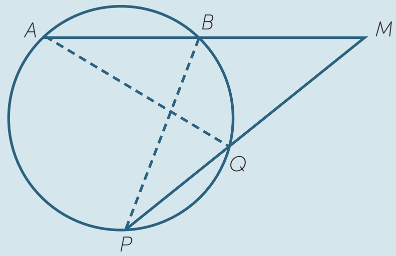
EXERCISE 22

Complete the proof.



EXERCISE 23

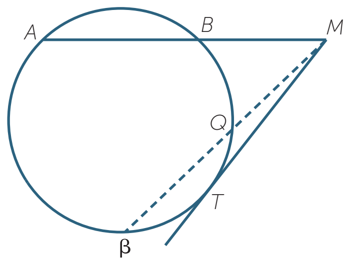
If instead we had joined the intervals *AQ* and *BP*, what corresponding changes should be made to the proof?



Tangent and secant from an external point

Now imagine the secant *MPQ* in the previous diagrams rotating until it becomes a tangent at a point T on the circle. As the secant rotates, the length of each intercept *PM* and *QM* gets closer to the length of the tangent *TM* to the circle from *M*, so the product $PM \times QM$ gets closer to the square TM^2 of the tangent from *M*. This is a proof using limits.

When we draw a secant and a tangent from *M*, we have seen that the product $AM \times BM$ equals the square TM^2 of the tangent. The geometric proof is similar to the previous two proofs, but it does require the alternate segment theorem to establish the similarity.



Theorem

The product of the intercepts on a secant from an external point equals the square of the tangent from that point.

Proof

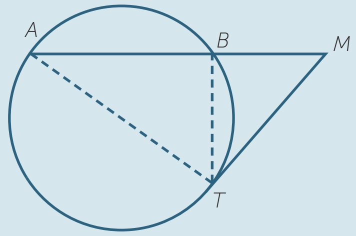
Let ABM be a secant, and TM a tangent, from an external point M, as shown. We need to prove that $AM \times BM = TM^2$. Join AT and BT.

EXERCISE 24

a Explain why $\angle A = \angle MTB$.

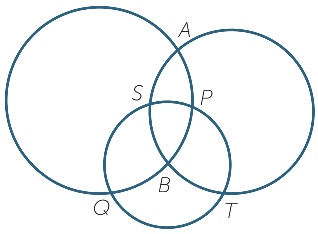
b Prove that $\triangle ATM$ is similar to $\triangle TBM$.

c Complete the proof.



The concurrence of three common chords

Proving concurrence usually involves logic that is a little more sophisticated than required in most proofs. The following theorem is a difficult application of the intersecting chord theorem.



EXERCISE 25

Show that the three common chords AB, PQ and ST to the three circles in the diagram above are concurrent.

LINKS FORWARD

Circle geometry is often used as part of the solution to problems in trigonometry and calculus.

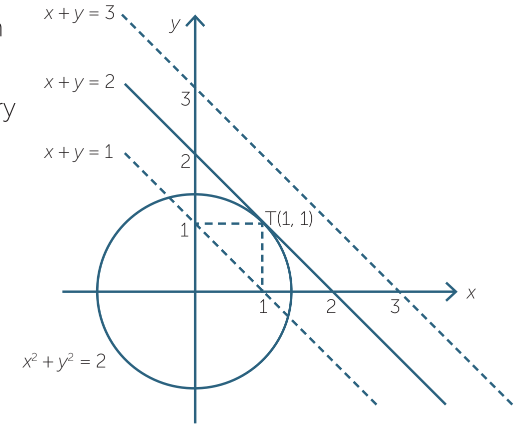
CONVERSE OF THE CIRCLE THEOREMS

The circle theorems proven in this module all have dramatic and important converse theorems, which are tests for points to lie on a circle. The proofs of these converses, and their applications, are usually regarded as inappropriate for Years 9–10, apart from the converse of the angle in a semicircle theorem, which was developed within the module. They are so closely related to the material in this module, however, that they have been fully developed in an Appendix.

COORDINATE GEOMETRY

Tangents to circles

We defined a tangent to a circle as a line with one point in common with the circle. This definition can be used in coordinate geometry using simultaneous equations. For example, the diagram to the right shows the line $x + y = 2$ and the circle $x^2 + y^2 = 2$.



Substituting the equation of the line into the equation of the circle gives

$$x^2 + (2 - x)^2 = 2$$

$$2x^2 - 4x + 4 = 2$$

$$x^2 - 2x + 1 = 0$$

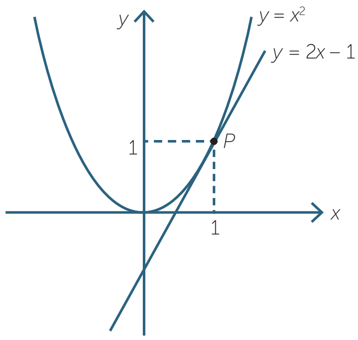
$$(x - 1)^2 = 0$$

and the only solution is thus $x = 1$. Hence the line and the circle have only the single point of intersection $T(1, 1)$, proving that the line is a tangent to the circle.

Similarly, the dotted line $x + y = 1$ is a secant, intersecting the circle in two points, and the dotted line $x + y = 3$ does not intersect the circle at all.

Tangents to parabolas

This simultaneous equations approach to tangents can be generalised to other curves defined by algebraic equations.



For example, the line $y = 2x - 1$ is a tangent to the graph of the parabola of $y = x^2$ at the point $P(1, 1)$, because solving the equations simultaneously gives

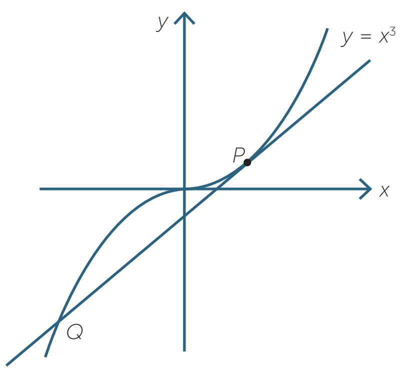
$$x^2 = 2x - 1$$

$$x^2 - 2x + 1 = 0$$

Hence $P(1, 1)$ is the only point of intersection.

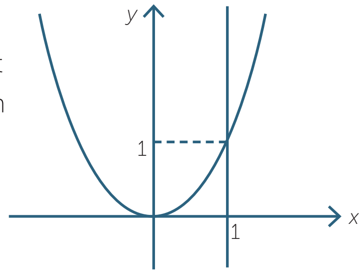
CALCULUS

For graphs defined by cubics and higher powers, however, the definition of 'tangent' has to be adapted. The first diagram to the right shows that we would think of as a 'tangent' at a point P can cross the curve again at some other point, Q.

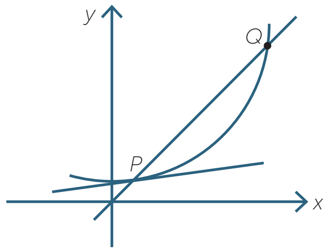


The following diagram shows that even with a quadratic graph, our current definition of 'tangent' would mean that every vertical line would be a tangent to the parabola!

Clearly we need to change the requirement of a single point of intersection, and instead develop some idea about a tangent being a straight line that 'approximates a curve' in the neighbourhood of the point of contact.



These considerations lead naturally to the well-known limiting process. To construct a tangent at a point P on a general curve, we construct the secant through P and another point Q on the curve, and then move the point Q closer and closer to P. This is the traditional beginning of calculus at school.

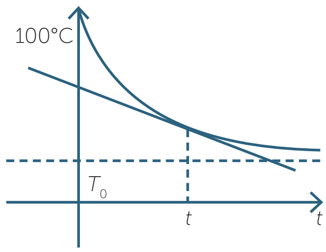


Applications in motion and rates of change

When a stone on a string is whirled around in a circle, then suddenly let go, it flies off at a tangent to the circle (ignoring the subsequent fall to the ground). We can interpret this situation by saying that the tangent describes the direction in which the stone was travelling at the instant when it was released. This leads to the concept of a vector which has both magnitude and direction representing the velocity particle.

The study of motion begins with motion in a straight line, that is, in one dimension. When motion in two dimensions is first considered, circular paths and parabolic paths are the first paths to be considered, because they are reasonably straightforward to describe mathematically, and they arise in so many practical situations.

A rate of change is also described by a tangent. This time the tangent does not have any physical significance, but it is the gradient of the tangent that gives the instantaneous rate of change. For example, when boiling water is removed from the stove and cools, the temperature–time graph looks something like the graph to the right. At any time t, the rate at which the water is cooling is given by the gradient of the tangent at the corresponding point on the curve.



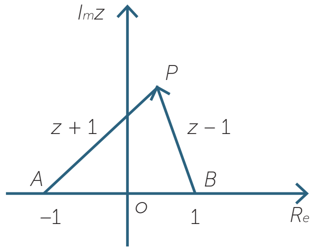
Vectors and complex numbers

Problems in complex numbers often require locating a set of complex numbers on the complex plane. The following example requires some knowledge of vectors and circle geometry.

EXAMPLE

Sketch the set of complex numbers such that the ratio $\frac{z-1}{z+1}$ is an imaginary number, that is a real multiple of i.

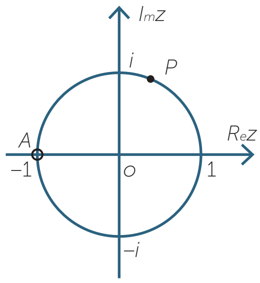
SOLUTION



If $z = 1$, then $\frac{z-1}{z+1} = 0$, which is imaginary.

If $z = -1$, then $\frac{z-1}{z+1}$ is not defined.

So suppose that z is neither 1 nor −1.

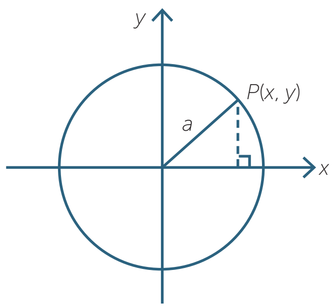


The condition on z is that $z - 1$ is ki times $z + 1$, where k is real. We know that multiplying by ki rotates the direction of a complex number by 90° or by −90°. That is, the vector $z - 1$ is perpendicular to the vector $z + 1$. We know that $z - 1$ is a vector with tail at 1 and head at z, and that $z + 1$ is a vector with tail at −1 and head at z. Let A, B and P represent the points −1, 1 and z. Then the condition becomes $\angle APB$ is a right angle. Thus the locus of z is a circle with diameter AB, that is a circle of radius 1 and centre 0, excluding −1.

Many problems similar to this involve not just the theorems developed in the module, but their converses as well, as developed in the Appendix to this module.

HISTORY AND APPLICATIONS

Greek geometry was based on the constructions of straight lines and circles, using a straight edge and compasses, which naturally gave circles a central place in their geometry. All the theorems developed in the Content and Appendix of this module were developed by the Greeks, and appear in Euclid's Elements.



In coordinate geometry, developed later by Descartes in the 17th century, horizontal and vertical lengths are measured against the two axes, and diagonal lengths are related to them using Pythagoras' theorem. The resulting connection between circles and Pythagoras' theorem is seen in the equation of a circle.

The square of the distance of a point $P(x, y)$ from the origin is $x^2 + y^2$ by Pythagoras' theorem, which means that the equation of the circle with radius a and centre the origin is $x^2 + y^2 = a^2$.

A circle is a simple closed curve with an inside and an outside, a property that it shares with triangles and quadrilaterals. In three dimensions, spheres, cubes and toruses (doughnuts) have an inside and an outside, but a torus is clearly connected in a different way from a sphere. The subject called **topology**, begun by Euler and developed extensively in the 20th century, begins with such observations.

Greek astronomy made great use of circles and spheres. They knew that the Earth was round, and were able to calculate its circumference with reasonable accuracy. Ptolemy described the heavenly bodies in terms of concentric spheres on which the Moon, the planets, the Sun and the stars were embedded. When astronomy was reconstructed in the 16th century by Copernicus, the heavenly spheres disappeared, but he still used circles for orbits, with the centre moved from the Earth to the Sun. Ellipses and other refinements of the orbits were soon introduced into this basic model of circles by Kepler, who empirically found three laws of motion for planets. At the end of the 17th century Newton used calculus, his laws of motion and the universal law of gravitation to derive Kepler's laws.

John Dalton reconstructed chemistry at the start of the 19th century on the basis of atoms, which he regarded as tiny spheres, and in the 20th century, models of circular orbits and spherical shells were originally used to describe the motion of electrons around the spherical nucleus. Thus circles and their geometry have always remained at the heart of theories about the microscopic world of atoms and theories about the cosmos and the universe.

Geometry continues to play a central role in modern mathematics, but its concepts, including many generalisations of circles, have become increasingly abstract. For example, spheres in higher dimensional space came to notice in 1965, when John Leech and John Conway made a spectacular contribution to modern algebra by studying an extremely close packing of spheres in 24-dimensional space.

On the other hand, classical Euclidean geometry in the form presented in this module has nevertheless advanced in modern times – here are three results obtained in recent centuries.

THE EULER LINE

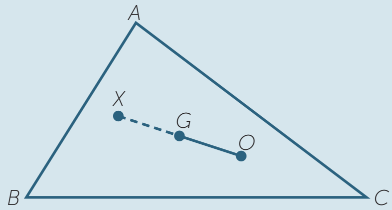
In 1765, Euler discovered that:

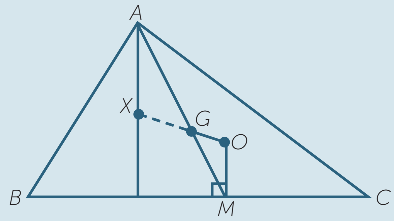
> *The orthocentre, the circumcentre, and the centroid of any triangle are collinear. The centroid divides the interval joining the circumcentre and the orthocentre in the ratio* 2:1.

The line joining these three centres is now called the Euler line. The proof is reasonably straightforward, and is presented in the following exercise.

EXERCISE 26

In the diagram to the right, G and O are respectively the centroid and the circumcentre of $\triangle ABC$. The interval OG is produced to X so that $OG : GX = 1 : 2$. We shall prove that X is the orthocentre of $\triangle ABC$ by proving that it lies on each of the three altitudes of the triangle. Join AX. Let M be the midpoint of BC. Join the median AGM, and the perpendicular OM.





a Prove that $\triangle AGX$ is similar to $\triangle MGO$, with similarity ratio 2:1.

b Hence prove that $AX \parallel OM$.

c Hence explain why AX produced is an altitude of $\triangle ABC$.

d Complete the proof is the orthocentre of the triangle.

THE NINE-POINT CIRCLE

In the early 19th century, Poncelet, Feuerbach and others showed that in any triangle, the following nine points are cyclic:

- the midpoint of each side of the triangle,
- the foot of each altitude,
- the midpoint of the interval joining each vertex of the triangle to the orthocentre.

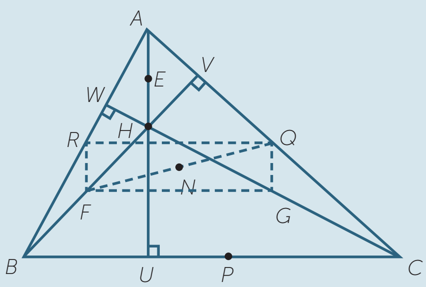
That is, these nine points lie on a circle.

He also showed that the centre of this nine-point circle lies on the Euler line, and is the midpoint of the interval joining the circumcentre to the orthocentre.

Again, the proof is straightforward enough to present here as a structured exercise, although proving that the centre of the nine-point circle is the midpoint of OH is rather fiddly.

EXERCISE 27

The diagram to the right shows $\triangle ABC$ with the altitudes AU, BV and CW meeting at the orthocenter H. The points E, F and G are the respective midpoints of AH, BH and CH. The points P, Q and R are the midpoints of the sides BC, CA and AB.



a Prove that RQ and FG are each parallel to BC and half its length.

b Prove that RF and QG are each parallel to AH and half its length.

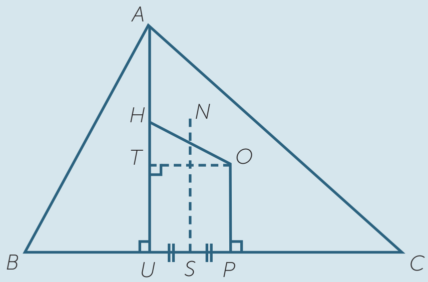
c Hence prove that $FGQR$ is a rectangle.

d Prove similarly that $FPQE$ is a rectangle.

e Let N be the midpoint of FQ, and use the properties of a rectangle to prove that $RN = GN = FN = QN = PN = EN$; and hence that R, G, F, Q, P and E are concyclic, lying on a circle with centre N .

f By considering $\triangle FQV$, prove that N is equidistant from Q and V, and hence that the circle also passes through U, V and W.

g The diagram to the right shows the altitude AU, the midpoint P of BC, the midpoint S of UP, and the circumcentre O, orthocentre H, and nine-point centre N.



i Explain why SN is the perpendicular bisector of UP.

ii Explain why SN, produced if necessary, bisects OH.

iii Explain why N lies on OH, and is thus its midpoint.

An enlargement transformation associated with Euler's line and the nine-point circle

Here is a rather dramatic proof using a single enlargement that establishes the existence of the Euler line and the positions on it of the centroid G, the circumcentre O, the orthocentre H, and the nine-point centre N. The notation is the same as that used in exercises 25 and 26.

In this construction, all that is used about the nine-point circle point is that it is the circle through P, Q and R. The fact that the other six points lie on it would be proven afterwards.

The medians trisect each other, so

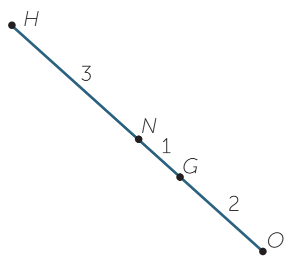
$$AG : GP = BG : GQ = CG : GR = 2 : 1.$$

Consider the enlargement with centre G and enlargement factor $-\frac{1}{2}$.

This enlargement fixes G, and maps A to P, B to Q and C to R, so the circumcircle of $\triangle ABC$ is mapped to the circumcircle of $\triangle PQR$. Thus the circumcentre O is mapped to the nine-point centre N, and G thus divides NO in the ratio 1 : 2. The orthocentre H of $\triangle ABC$ is mapped to the orthocentre of $\triangle PQR$, and the orthocentre of $\triangle PQR$ is the circumcentre O of $\triangle ABC$, because the perpendicular bisectors of the sides of $\triangle ABC$ are the altitudes of $\triangle PQR$, so G divides HO in the ratio 2 : 1. Thus the four points H, N, G and O are collinear, with

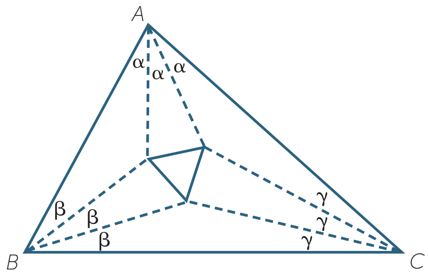
$$HN : NG : GO = 3 : 1 : 2.$$

It also follows that the nine-point circle has half the radius of the circumcircle.



MORLEY'S TRISECTOR THEOREM

In 1899, the American mathematician Frank Morley discovered an amazing equilateral triangle that is formed inside every triangle. When each angle of a triangle is trisected, the points of intersection of trisectors of adjacent vertices form an equilateral triangle.



The result has many proofs by similar triangles, and we refer the reader particularly to John Conway's proof at

http://www.cut-the-knot.org/triangle/Morley/conway.shtml

and Bollobas' version of it at http://www.cut-the-knot.org/triangle/Morley/Bollobas.shtml.

The result can also be proven using the compound angle formulae of trigonometry, and is thus reasonably accessible to students in senior calculus courses. See

http://en.wikipedia.org/wiki/Morley%27s_triangle

http://www.cut-the-knot.org/triangle/Morley/

APPENDIX – CONVERSES TO THE CIRCLE THEOREMS

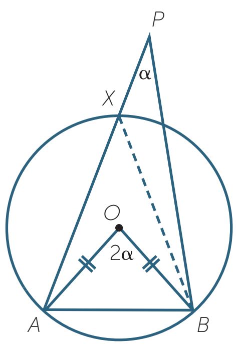
These proofs are best written as proofs by contradiction. The original theorem is used in the proof of each converse theorem. We have seen this approach when Pythagoras' theorem was used to prove the converse of Pythagoras' theorem.

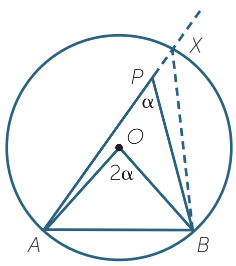
The converse of the angle at the centre theorem

The key theorem of the present module is, 'An angle at the circumference of a circle is half the angle at the centre subtended by the same arc.' The most straightforward converse of this is:

Theorem

Let $\triangle OAB$ be isosceles with $OA = OB$, and let P be a point on the same side of the interval AB as O. Suppose that AB subtends an angle at P half the size of the angle $\angle AOB$. Then the circle with centre O through A and B passes through P .





Proof

Let $\angle APB = \alpha$ and $\angle AOB = 2\alpha$. Construct the circle through A and B. Suppose, by way of contradiction, that the circle does not pass through P, and let the circle cross AP, produced if necessary, at X. Join BX.

Then $\angle AXB = \frac{1}{2}\angle AOB$ (angles at centre and circumference on the same arc AB)

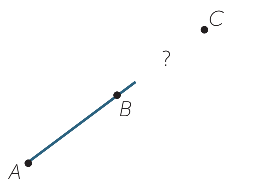
$= \alpha$

so $BX \parallel BP$ (corresponding angles are equal);

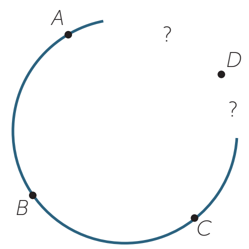
so the lines BX and PX coincide, and hence the points X and P coincide. Thus P lies on the circle, contradicting the assumption that the circle does not pass through P.

Testing whether four given points are concyclic

One of the basic axioms of geometry is that a line can be drawn through any two distinct points A and B. When there are three distinct points A, B and C, we can ask whether the three points are collinear, or form a triangle.



Similarly, any three non-collinear points A, B and C are concyclic. When there are four points A, B, C and D, no three collinear, we can ask whether these four points are concyclic, that is, do they lie on a circle.



The remaining converse theorems all provide tests as to whether four given points are concyclic. In each case, we draw the unique circle through three of them and prove that the fourth point lies on this circle.

We shall assume that the fourth point does not lie on the circle and produce a contradiction.

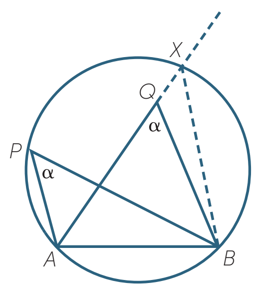
The converse of the angles on the same arc theorem

Theorem

If an interval subtends equal angles at two points on the same side of the interval, then the two points and the endpoints of the interval are concyclic.

Proof

Let the interval AB subtend equal angles a at points P and Q on the same side of BC. Construct the circle through A, B and P, and suppose, by way of contradiction, that the circle does not pass through Q.



Let AQ, produced if necessary, meet the circle again at X.

Then	$\angle AXB = \alpha$	(angles on the same arc AB),
so	$BX \parallel BQ$	(corresponding angles are equal)

Hence BQ and BX are the same line, so Q coincides with X, which is a contradiction.

A test for a cyclic quadrilateral

We proved earlier, as extension content, two tests for a cyclic quadrilateral:

- If the opposite angles of a cyclic quadrilateral are supplementary, then the quadrilateral is cyclic.
- If an exterior angle of a quadrilateral equals the opposite interior angle, then the quadrilateral is cyclic.

The proof by contradiction of the first test is almost identical to the proof of the previous converse theorem. The second test is a simple corollary of the first test.

The intersecting chord test

We proved that the product of the intercepts of intersecting chords are equal. The converse of this gives yet another test for four points to be concyclic. The proof proceeds along exactly the same lines.

Theorem

Let AB and PQ be intervals intersecting at M, with $AM \times BM = PM \times QM$. Then the four endpoints A, B, P and Q are concyclic.

EXERCISE 28

Complete the proof of this theorem.

We leave it the reader to formulate and prove the (true) converses to the remaining two theorems about secants from an external point, and a tangent and a secant from an external point. By similar methods, one can also prove the converse of the theorem that if a quadrilateral has an incircle, then the sums of its opposite sides are equal.

ANSWERS TO EXERCISES

EXERCISE 1

a Every translation of any distance along ℓ 'maps' ℓ onto itself.

Let O be any point on ℓ. Then reflection in the line m through O perpendicular to ℓ, and rotation of 180° about O, both map ℓ onto itself (and are identical in their action on the points of ℓ).

Reflection of the plane in the line ℓ fixes the line ℓ, because it fixes every point on ℓ.

b The translation of the plane moving P to Q maps ℓ onto itself, and maps P to Q.

Reflection in the perpendicular bisector of the interval PQ maps ℓ onto itself, and exchanges the points P and Q. (Rotation of 180° about the midpoint of PQ does the same.)

EXERCISE 2

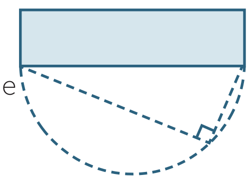
Use simple trigonometry in $\triangle AMO$.

EXERCISE 3

All radii of a circle are equal, so $OA = OB = OP = OQ$, so the quadrilateral $APBQ$ is a rectangle because its diagonals are equal and bisect each other. Hence $\angle APB$ is a right angle.

EXERCISE 4

At all times, the front of the building is the hypotenuse of a right-angled triangle whose third vertex is the photographer. Hence the circle with diameter, the front of the building, always pass through the photographer, and his possible positions are the points on the semicircle in front of the building.



EXERCISE 5

At all times, the triangle formed by the plank, the wall and the floor is right-angled with hypotenuse of length ℓ metres. By the converse theorem above, the midpoint of the plank is therefore always $\frac{l}{2}$ metres from the corner. Hence the midpoint traces out a quadrant of the circle with centre at the corner and radius $\frac{l}{2}$ metres.

EXERCISE 6

Case 2: It will be sufficient to prove the result when O lies on AP.

Let $\angle OBP = \beta$.

Then $\angle BPO = \beta$ (base angles of isosceles $\triangle OBP$),

so $\angle AOB = 2\beta = 2 \times \angle APB$ (exterior angle of $\triangle OBP$).

Case 3: Join PO and produce PO to Q.

Let $\angle OAP = \alpha$

and $\angle OBP = \beta$

Then $\angle APO = \alpha$ (base angle of isosceles $\triangle AOP$)

$\angle OPB = \beta$ (base angle of $\triangle OBP$)

Hence $\angle APB = \beta - \alpha$

Also $\angle AOQ = 2\alpha$ (exterior angle of $\triangle OAP$)

and $\angle BOQ = 2\beta$ (exterior angle of $\triangle OBP$)

Hence $\angle AOB = 2\beta - 2\alpha = 2(\beta - \alpha) = 2 \times \angle APB$

EXERCISE 7

a The interval HC subtends right angles at P and Q, so the circle with diameter HC passes through them.

b The interval AB subtends right angles at P and Q, so the circle with diameter AB passes through them.

c They stand on the same arc HQ of the first circle.

d They stand on the same arc AQ of the second circle.

e In the triangles ABQ and ACR, $\angle ABQ = \angle RCA$ by parts **c** and **d**, and $\angle A$ is common, so $\angle ARC = \angle AQB = 90°$.

EXERCISE 8

a The angles $\angle DAC$ and $\angle DBC$ stand on the same arc DC, so $\angle DBC = \alpha$.

b $\angle BDC = \beta$ (angles on the same arc BC),

$\angle ABD = \gamma$ (angles on the same arc AD),

$\angle ADB = \delta$ (angles on the same arc AB).

c $2\alpha + 2\beta + 2\gamma + 2\delta = 360°$ (angle sum of quadrilateral $ABCD$)

The rest is clear.

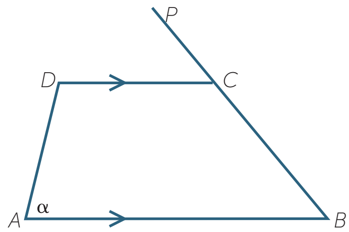
EXERCISE 9

Join the common chord BQ, and produce ABC to X, and let $\alpha = \angle A$.

Then	$\angle BQR = \alpha$	(opposite exterior angle of cyclic quadrilateral)
and	$\angle RCX = \alpha$	(opposite exterior angle of cyclic quadrilateral),
so	$AP \parallel CR$	(corresponding angles are equal)

EXERCISE 10

In this proof, we construct two isosceles triangles. Let $ABCD$ be a cyclic trapezium with $AB \parallel DC$. Since $ABCD$ is not a rectangle, and its angles add to 360°, one of its angles is acute.



Let $\angle A = \alpha$ be acute, and produce BC to P.

Then	$\angle DCP = \alpha$	(opposite exterior angle of cyclic quadrilateral)
so	$\angle B = \alpha$	(corresponding angles, $AB \parallel DC$).

Hence $\angle B$ is also acute, so AD and BC produced meet at a point M,

and	$\angle CDM = \alpha$	(corresponding angles, $AB \parallel DC$).

Hence $\angle DMC$ and $\angle AMB$ are both isosceles, with $DM = CM$ and $AM = BM$, so $AD = BC$.

EXERCISE 11

The adjacent interior angle is supplementary to the exterior angle, and therefore equal to the opposite interior angle. Hence the quadrilateral is cyclic by the theorem.

EXERCISE 12

$\angle D = 180° - \alpha$ (co-interior angles, $AB \parallel DC$)

so $\angle D + \angle B = 180°$

Hence the quadrilateral $ABCD$ is cyclic (opposite angles are supplementary).

EXERCISE 13

a In **Case 1**, $\angle BCP = 90°$ (angle in a semicircle)

and	$\angle P = \alpha$	(angles on the same arc BC)
Hence	$\frac{a}{2R} = \sin \alpha$	(simple trigonometry in $\triangle BCP$)
so	$\frac{a}{\sin A} = 2R$.	

b In **Case 2**, $\angle BCP = 90°$ (angle in a semicircle)

and $\angle P = 180° - \alpha$ (opposite angles of cyclic quadrilateral *BACP*)

so $\sin P = \sin(180° - \alpha) = \sin\alpha$

Hence $\frac{a}{2R} = \sin\alpha$ (simple trigonometry in $\triangle BCP$)

$\frac{a}{\sin A} = 2R.$

c In **Case 3**, $\alpha = 90°$ (angle in a semicircle),

so $\sin\alpha = 1.$

The diameter of the circumcircle is *a*,

so $\frac{a}{\sin A} = \frac{2R}{1} = 2R.$

EXERCISE 14

Let tangents from an external point *P* touch the circle at *T* and *U*.

Join the radii *OT* and *OU* and the interval *OP*.

Since *OP* is common, the radii are equal, and the radii are perpendicular to the tangents,

$\triangle OPT \equiv \triangle OPU$ (RHS)

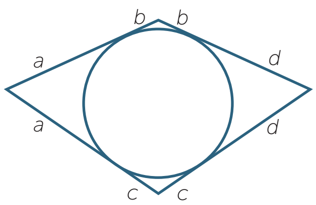
so $PT = PU$ (matching sides of congruent triangles).

EXERCISE 15

The triangle *OPT* is right-angled at *T*. The rest is simple trigonometry.

EXERCISE 16

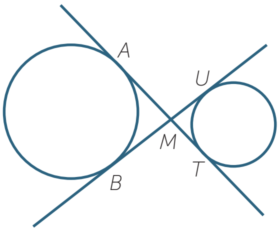
Tangents from an external point are equal, so we can label the lengths in the figure as shown. Then both sums of opposite sides of the quadrilateral are $a + b + c + d$.

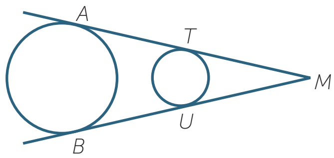


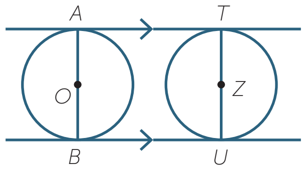
EXERCISE 17

a **i** 0 **ii** 1 **iii** 2 **iv** 3 **v** 4.

b The first diagram below illustrates the situation with two indirect common tangents.





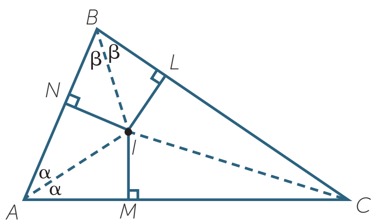


The indirect tangents *AT* and B*U* intersect at *M*. Then *AM* = *BM* and *TM* = *UM* because tangents from an external point are equal, and adding the lengths gives *AT* = *BU*.

c With direct common tangents, there are two cases to consider. Suppose first that the direct tangents *AT* and *BU* intersect at *M* when produced, as in the second diagram above. Then again *AM* = *BM* and *TM* = *UM*, and subtracting gives *AT* = *BU*.

Now suppose that the direct common tangents *AT* and *BU* are parallel, as in the third diagram. The radii *AO* and *BO* are perpendicular to the parallel tangents, so they lie on the one line and form a diameter, and the radii *TZ* and *UZ* also form a diameter. Hence *ABUT* is a rectangle, so its opposite sides *AT* and *BU* are equal.

EXERCISE 18



a Use the AAS test.

b They are matching sides of congruent triangles.

c Use the RHS test.

d Hence $\angle LCI = \angle MCI$, so *IC* is also an angle bisector, thus proving the concurrence of the angle bisectors. Also the circle with centre *I* and radius $LI = MI = NI$ is tangent to all three sides by the radius-and-tangent theorem.

EXERCISE 19

In **Case 2**,	$\angle AOB = 2\alpha$	(angles at centre and circumference on same arc *AB*)
so	$\angle OBA = 90° - \alpha$	(angle sum of isosceles ΔOAB)
But	$\angle OBU = 90°$	(radius and tangent)
so	$\angle ABU = \alpha$	(adjacent angles at *B*).
In **Case 3**,	reflex $\angle AOB = 2\alpha$	(angles at centre and circumference on same arc *AB*)
so	non-reflex $\angle AOB = 360° - 2\alpha$	(angles in a revolution)
	$\angle OBA = \alpha - 90°$	(angle sum of isosceles ΔOAB)
	$\angle OBU = 90°$	(radius and tangent)
	$\angle ABU = \alpha$	(adjacent angles at *B*).

EXERCISE 20

Let $\alpha = \angle A$.

Then $\angle BTU = \alpha$ (alternate segment theorem)

so $\angle STP = \alpha$ (vertically opposite angles)

so $\angle Q = \alpha$ (alternate segment theorem)

Hence $AB \parallel PQ$ (alternate angles are equal).

EXERCISE 21

We have already proven that $\angle A = \angle Q$ and that $\angle P = \angle B$.

The sine rule $\frac{x}{\sin X} = \frac{y}{\sin Y}$ in a triangle XYZ can also be written in the form

$$\frac{x}{y} = \frac{\sin X}{\sin Y}$$

Using this form of the sine rule in the two triangles,

$$\frac{AM}{PM} = \frac{\sin P}{\sin A} = \frac{\sin B}{\sin Q} = \frac{QM}{BM}.$$

EXERCISE 22

Proof: In the triangles APM and QBM :

$\angle PAM = \angle BQM$ (exterior angle of cyclic quadrilateral $ABQP$)

$\angle AMP = \angle BMQ$ (common angle)

so $\triangle APM$ is similar to $\triangle QBM$ (AA).

Hence $\frac{AM}{QM} = \frac{PM}{BM}$. (matching sides of similar triangles)

so $AM \times BM = PM \times QM$.

EXERCISE 23

Proof: In the triangles AQM and PBM :

$\angle A = \angle P$ (angles on the same arc BQ)

$\angle M$ is common to both triangles

so $\triangle AQM$ is similar to $\triangle PBM$

Hence $\frac{AM}{QM} = \frac{PM}{BM}$ (matching sides of similar triangles)

So $AM \times BM = PM \times QM$

EXERCISE 24

Proof: In the triangles ATM and TBM:

$\angle TAM = \angle BTM$ (alternate segment theorem)

$\angle AMT = \angle TMB$ (common)

so $\triangle ATM$ is similar to $\triangle TBM$ (AA)

$\frac{AM}{TM} = \frac{TM}{BM}$ (matching sides of similar triangles)

$AM \times \text{BM} = TM^2$

EXERCISE 25

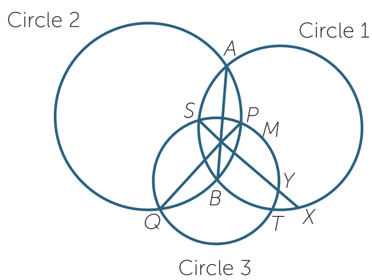
Let the chords AB and PQ meet at M. Join SM and produce it to meet the circles at X and Y. We need to prove that the points X and Y coincide.

Using the intersecting chord theorem in each circle in turn,

$SM \times MX = AM \times MB$ (from Circle 1)

$= PM \times MQ$ (from Circle 2)

$= SM \times MY$ (from Circle 3).



EXERCISE 26

a First, $AG : GM = 2 : 1$ because the centroid divides each median in the ratio 2 : 1.

Secondly, $XG : GO = 2 : 1$ by the construction of the point X.

Thirdly, $\angle AGX = \angle MGO$ because they are vertically opposite angles.

b Hence $\triangle AGX$ is similar to $\triangle MGO$ by the SAS similarity test.

Hence the matching angles $\angle XAG$ and $\angle OMG$ are equal,

and since these are alternate angles, $AX \parallel OM$.

c Thus AX meets BC at right angles so X lies on the altitude from A. AX produced is an altitude of $\triangle ABC$.

d We have shown X is on one altitude and the same argument shows X is on all three altitudes. Thus X is H orthocentre of the triangle.

EXERCISE 27

a & b These follow because the interval joining the midpoints of two sides of a triangle is parallel to the third side and half its length.

c Hence *FGQR* is a parallelogram, and is a rectangle because $AU \perp BC$

d This follows by a similar argument to parts **a** – **c**.

e This follows because the diagonals of each rectangles are equal and bisect each other. Note that *FQ* is a diagonal of both rectangles.

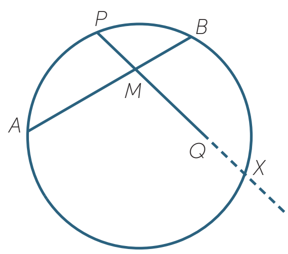
f Since *N* is the midpoint of the hypotenuse of the right-angled triangle *FQV*, it follows by the converse of the angle-in-a-semicircle theorem, proven earlier in this module, that *N* is equidistant from *F*, *Q* and *V*. Similarly, *N* is equidistant from *E*, *P* and *U*, and from *G*, *R* and *W*. Hence the circle passes through all nine points.

g **i** The perpendicular bisector of *UP* is the locus of all points equidistant from *U* and *P*, and this includes the nine-point centre *N*.

ii Complete the rectangle *UPOT*. Then $NS \parallel HTU$, so *NS* bisects *OT* using opposite sides of rectangles, and so also bisects *OH* using a similarity theorem on triangles. Hence *NS* meets *OH* at *Y* .

iii By similar arguments to parts **i** and **ii**, the perpendicular bisectors of *UP*, *VQ* and *WR* all pass through *N*, and all bisect *OH*. Hence the three perpendicular bisectors are concurrent, and the midpoint of *OH* and the point *N* coincide at the intersection of these three lines.

EXERCISE 28



Construct the circle through *A*, *B* and *P*, and suppose by way of contradiction that the circle does not pass through *Q*.

Let *PQ*, produced if necessary, meet the circle again at *X* .

Then $AM \times BM = PM \times XM$ (intersecting chords)

so $XM = QM$, and *Q* and *X* both lie on the same side of *M*.

Hence *X* coincides with *Q*, which is a contradiction.

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