MASSACHVSETTS INSTITVTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.081—Introduction to EECS I
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Lecture 8 and 9 Notes

Circuit Equivalents and Op-Amps

Circuit Equivalents

We saw last week that pieces of circuits cannot be abstracted as functional elements; the actual voltages and currents in them will depend on how they are connected to the rest of a larger circuit. However, we can still abstract them as sets of constraints on the values involved.

In fact, when a circuit includes only resistors and voltage sources, we can derive a much simpler circuit that induces the same constraints on currents and voltages as the original one. This a kind of abstraction that's similar to the abstraction that we saw in linear systems: we can take a complex circuit and treat it as if it were a much simpler circuit.

If somebody gave you a circuit made of resistors and voltage sources, and put it in a black box with two wires coming out, labeled + and -, what could you do with it? You could try to figure out what constraints that box puts on the voltage between and current through the wires coming out of the box.

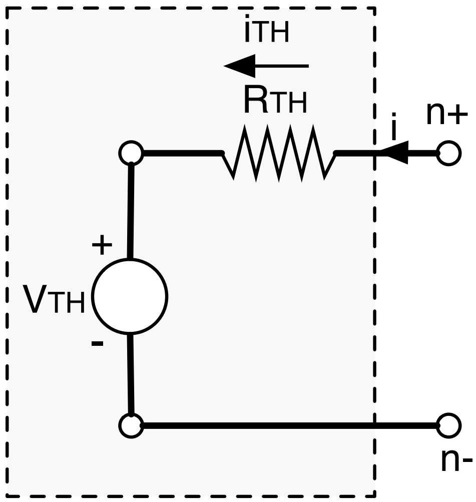
We can start by figuring out the *open-current voltage* across the two terminals. That is the voltage drop we'd see across the two wires if nothing were connected to them. We'll call that V_{oc}. Another thing we could do is connect the two wires together, and see how much current runs through them; this is called the *short-circuit current*. We'll call that i_{sc}.

It turns out that these two values are sufficient to characterize the constraint that this whole box will exert on a circuit connected to it. The constraint will be a relationship between the voltage across its terminals and the current flowing through the box. We can derive it by using Thévenin's theorem:

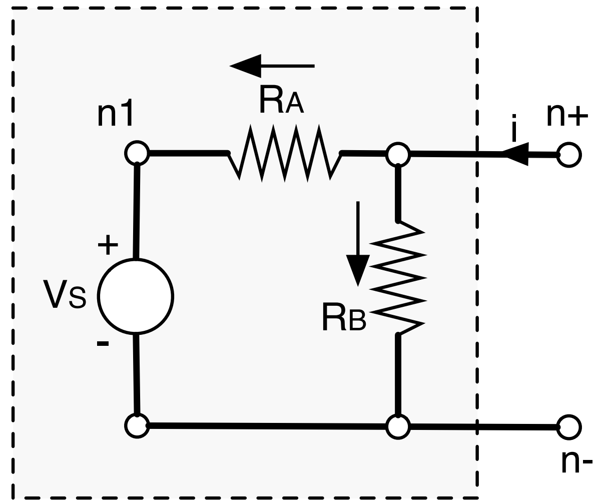
Theorem 1 *Any combination of voltage sources and resistances with two terminals can be replaced by a single voltage source* V_{th} *and a single series resistor* R_{th}*. The value of* V_{th} *is the open circuit voltage at the terminals* V_{oc}*, and the value of* R_{th} *is* V_{th} *divided by the current with the terminals short circuited* ($-i_{sc}$)*.*

Let's look at a picture, then an example. In figure 1(a) we show a picture of a black (well, gray) box, abstracted as being made up of a circuit with a single voltage source V_{th} and a single resistor R_{th} in series. The open-circuit voltage from n_+ to n_- is clearly V_{th}. The short-circuit current i_{sc} (in the direction of the arrow) is $-V_{th}/R_{th}$. So, this circuit would have the desired measured properties.[1]

[1] The minus sign here can be kind of confusing. The issue is this: when we are treating this circuit as a black box with terminals n_+ and n_-, we think of the current flowing *out* of n_+ and *in* to n_-, which is consistent with the voltage difference $V_{th} = V_+ - V_-$. But when we compute the short-circuit current by wiring n_+ and n_- together, we are continuing to think of i_{sc} as flowing out of n_+, but now it is coming *out* of n_- and *in* to n_+, which is the opposite direction. So, we have to change its sign to compute R_{th}.



(a) The abstract Thévenin equivalent circuit



(b) An example circuit

Figure 1: Thévenin equivalence examples

Figure 1(b) shows an actual circuit. We'll compute its associated open-circuit voltage and short-circuit current, construct the associated *Thévenin equivalent* circuit, and be sure it has the same properties.

The first step is to compute the open-circuit voltage. This just means figuring out the difference between the voltage at nodes n_+ and n_-, under the assumption that the current $i = 0$. An easy way to do this is to set n_- as ground and then find the node voltage at n_+. Let's write down the equations:

$$\begin{aligned} v_+ - v_1 &= i_A R_A \\ v_1 - v_- &= V_s \\ v_+ - v_- &= i_B R_B \\ -i_A - i_B &= 0 \\ i_A - i_S &= 0 \\ v_- &= 0 \end{aligned}$$

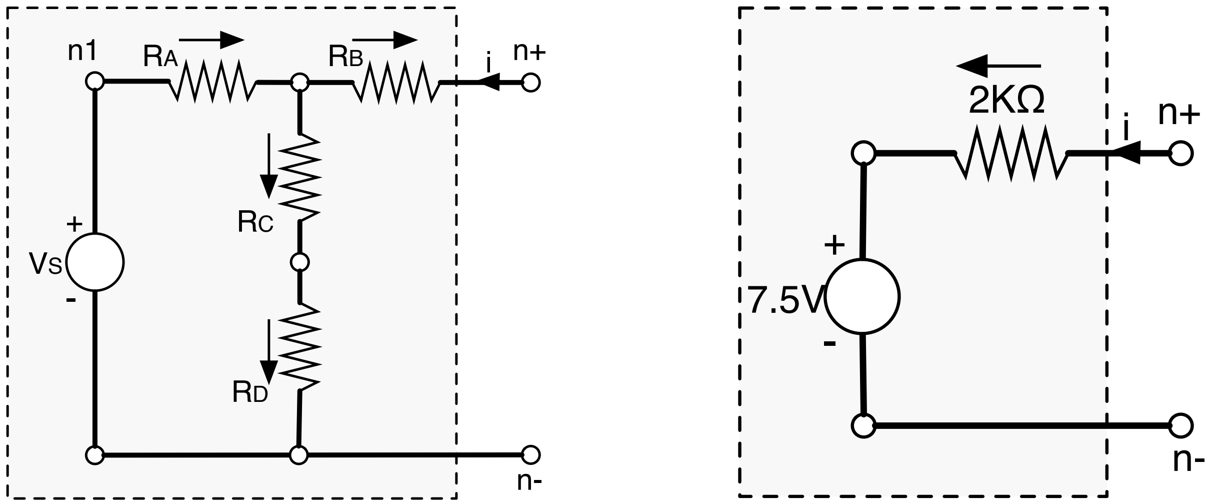
We can solve these pretty straightforwardly to find that

$$v_+ = V_s \frac{R_B}{R_A + R_B} \quad .$$

So, we know that, for this circuit, $R_{th} = V_s \frac{R_B}{R_A + R_B}$.

Now, we need the short-circuit current, i_{sc}. To find this, imagine a wire connecting n_+ to n_-; we want to solve for the current passing through this wire. We can use the equations we had before, but adding equation 4 wiring n_+ to n_-, and adding the current i_{sc} to the KCL equation 5.

$$v_+ - v_1 = i_A R_A \qquad (1)$$
$$v_1 - v_- = V_s \qquad (2)$$



(a) Another example circuit (b) Its Thévenin equivalent circuit

Figure 2: Thévenin equivalence examples

$$\begin{aligned} v_+ - v_- &= i_B R_B \quad (3) \\ v_+ &= v_- \quad (4) \\ i_{sc} - i_A - i_B &= 0 \quad (5) \\ i_A - i_S &= 0 \quad (6) \\ v_- &= 0 \quad (7) \end{aligned}$$

We can solve this system to find that

$$i_{sc} = -\frac{V_s}{R_A} \; ,$$

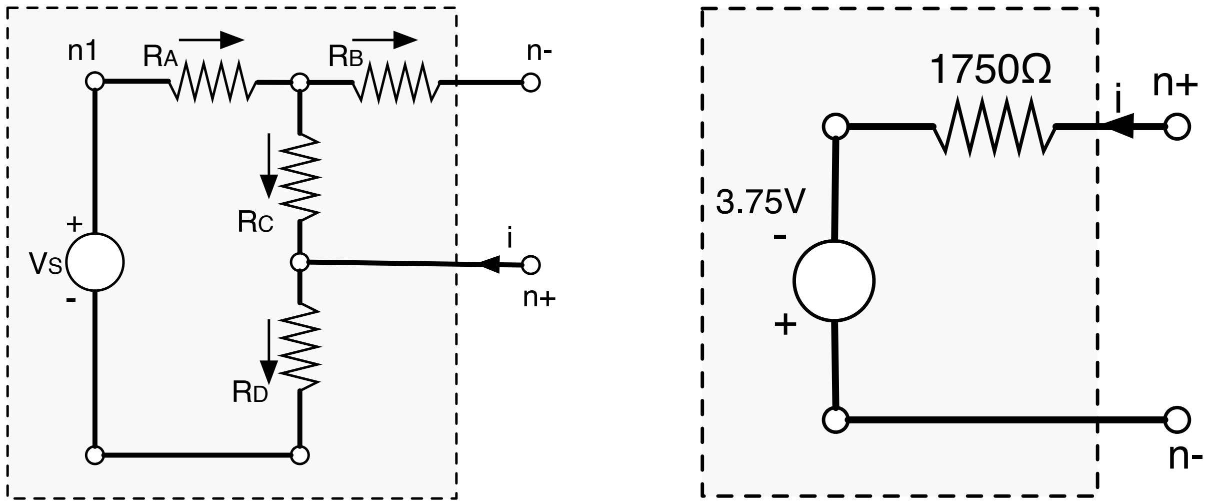
and therefore that

$$\begin{aligned} R_{th} &= -\frac{V_{th}}{i_{sc}} \\ &= V_s \frac{R_B}{R_A + R_B} \frac{V_s}{R_A} \\ &= \frac{R_A R_B}{R_A + R_B} \end{aligned}$$

What can we do with this information? We could use it during circuit analysis to simplify parts of a circuit model, individually, making it easier to solve the whole system. We could also use it in design, to construct a simpler implementation of a more complex network design. One important point is that the Thévenin equivalent circuit is not exactly the same as the original one. It will exert the same constraints on the voltages and currents of a circuit that it is connected to, but will, for example, have different heat dissipation properties.

Example

Here's another example, in figure 2(a). It's a bit more hassle than the previous one, but you can write down the equations to describe the constituents and KCL constraints, as before. If we



(a) Same circuit, but with different ports

(b) Its Thévenin equivalent circuit

Figure 3: Thévenin equivalence examples

let $R_A = 2K\Omega$, $R_B = R_C = R_D = 1K\Omega$, and $V_S = 15V$, then we can solve for $V_{th} = 7.5V$ and $R_{th} = 2K\Omega$. So, it is indistinguishable by current and voltage from the circuit shown in figure 2(b).

In figure 3(a) we show the same circuit, but with the connections that run outside the box made to different nodes in the circuit. Note also that the top lead is marked n_- and the bottom one n_+. If we solve, using the same values for the resistors and voltage source as before, we find that $V_{th} = -3.75V$ and $R_{th} = 1750\Omega$. We show the Thévenin equivalent circuit in figure 3(b). We've changed the polarity of the voltage source and made it 3.75V (instead of having the + terminal at the top and a voltage of -3.75), but that's just a matter of drawing.

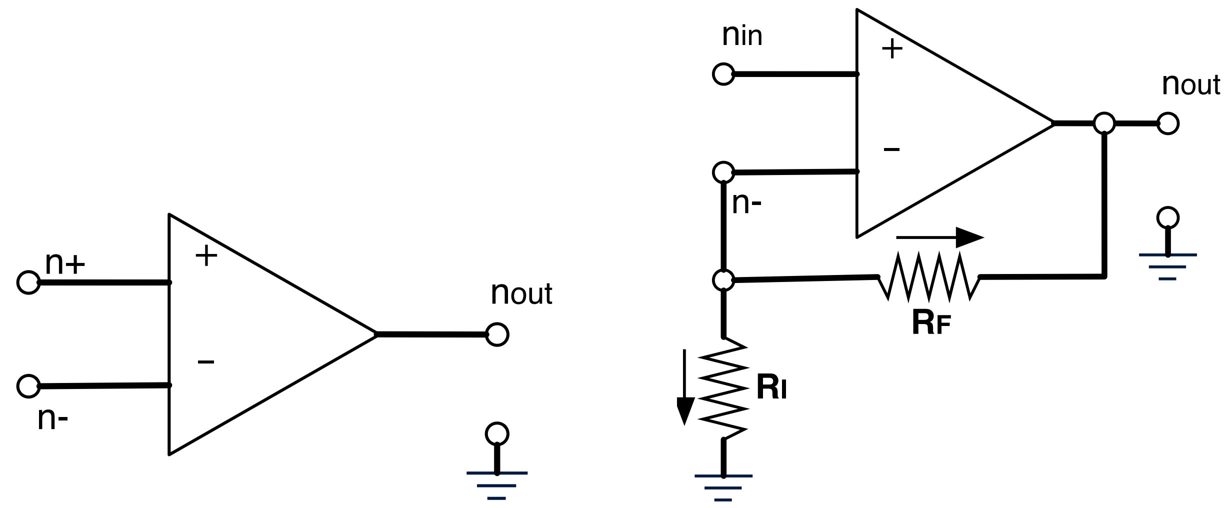
These results are quite different: so, the moral is, it matters which wires you connect up to what!

Op Amps

So far, we have considered circuits with resistors and voltage sources. Now we are going introduce a new component, called an *operational amplifier* or op-amp, for short. We are studying op-amps because they are a very important circuit element, as well as because they will allow us to explore a sequence of models of how they work. These models vary in complexity and fidelity. The simplest is the easiest to use for basic circuit designs, but doesn't capture some important behavioral properties. The more complex models give us a more complete picture, but are often unnecessarily complicated. There is no right model of an op-amp: it all depends on the question that you are trying to answer.

Basic model

Figure 4(a) shows a diagram of our simplest op-amp model. The basic behavioral model is that it adjusts v_{out} in order to try to maintain the constraint that $v_+ \approx v_-$ and that no current flows in to n_+ or n_-.



(a) Basic op-amp model (b) A non-inverting amplifier circuit

Figure 4: Basic op-amp models

The best way to understand why we might want such a device is to see how it behaves in some small circuit configurations.

Non-inverting amplifier Not surprisingly, a primary use of an op-amp is as an amplifier. Here is an amplifier configuration, shown in figure 4(b). Let's see if we can figure out the relationship between v_{in} and v_{out}. The circuit constraints tell us that

$$
\begin{aligned}
v_- &= i_I R_I && (8)\\
v_- - v_{\text{out}} &= i_F R_F && (9)\\
-i_I - i_F &= 0 && (10)\\
v_{\text{in}} &= v_- && (11)
\end{aligned}
$$

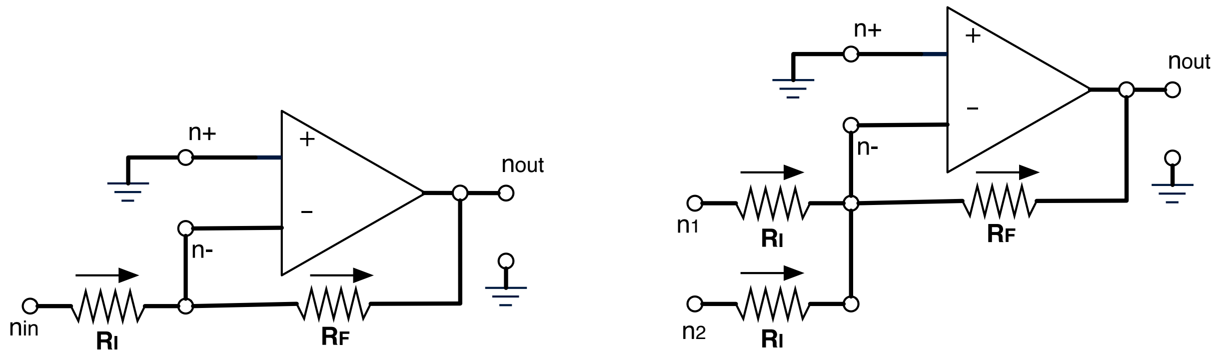
The KCL equation 10 has no term for the current into the op-amp, because we assume it is zero. Equation is the op-amp contraint. So, we find that

$$v_{\text{out}} = v_{\text{in}} \frac{R_F + R_I}{R_I} \quad .$$

This is cool. We've arranged for the output voltage to be greater than the input voltage, and we can arrange just about any relationship we want, by choosing values of R_F and R_I.

We can think intuitively about how it works by examining some cases. First, if $R_F = 0$, then we'll have $v_{\text{out}} = v_{\text{in}}$, so there's not a particularly interesting change in the voltages. This is still a useful device, called a *voltage follower*, which we'll study a bit later.

Now let's think about a more interesting case, but simplify matters by setting $R_F = R_I$. We can look at the part of the circuit running from V_{out} through R_F and R_I to ground. This looks a lot like a voltage divider, with v_- coming out of the middle of it. Because v_- needs to be the same as v_{in}, and it is v_{out} being divided in half, then v_{out} clearly has to be $2v_{\text{in}}$.



(a) An inverting amplifier

(b) A voltage summer

Figure 5: Basic op-amp models

Inverting amplifier Figure 5(a) shows a very similar configuration, called an *inverting amplifier*. The difference is that the + terminal of the op-amp is connected to ground, and the we're thinking of the path through the resistors as the terminal of the resulting circuit. Let's figure out the relationship between v_{in} and v_{out} for this one. The circuit constraints tell us that

$$\begin{aligned} v_{\text{in}} - v_- &= i_I R_I \\ v_- - v_{\text{out}} &= i_F R_F \\ i_I - iF &= 0 \\ v_+ &= v_- \\ v_+ &= 0 \end{aligned}$$

Solving, we discover that

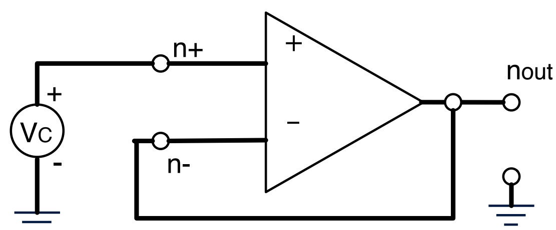
$$v_{\text{out}} = -v_{\text{in}} \frac{R_F}{R_I} \quad .$$

If $R_F = R_I$, then this circuit simply inverts the incoming voltage. So, for example, if v_{in} is $+10V$ with respect to ground, then v_{out} will be $-10V$. Again, we can see the path from n_{in} through the resistors, to n_{out}, as a voltage divider. Knowing that v_- has to be 0, we can see that v_{out} has to be equal to $-v_{\text{in}}$. If we want to scale the voltage, as well as invert it, we can do that by selecting appropriate values of R_F and R_I.

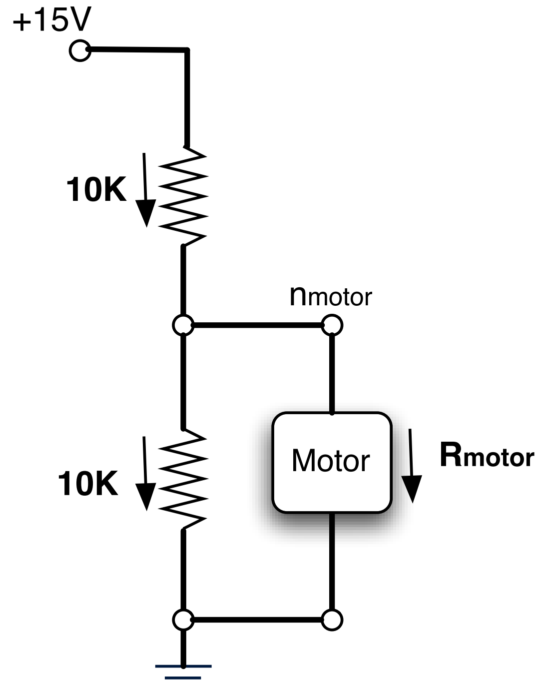
Voltage summer A *voltage summer*[2] circuit, as shown in figure 5(b), can be thought of as having three terminals, with the voltage at n_{out} constrained to be a scaled, inverted, sum of the voltages at n_1 and n_2. You should be able to write down the equations for this circuit, which is very similar to the inverting amplifier, and derive the relationship:

$$v_{\text{out}} = -\frac{R_F}{R_I}(v_1 + v_2) \quad .$$

[2]As in thing that sums, not as in endless summer.



(a) A voltage follower



(b) A motor connected to a voltage divider

Figure 6:

Voltage follower Figure 6(a) shows a basic *voltage follower* circuit. What will it do? We can see from basic wiring constraints that:

$$\begin{aligned} v_+ &= V_c \\ v_{out} &= v_- \end{aligned}$$

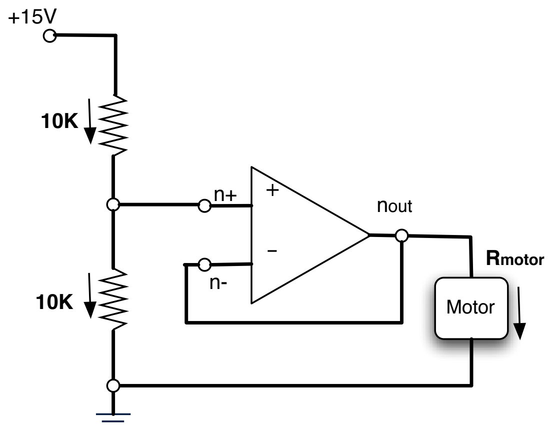
Adding in the op-amp constraint that $v_+ = v_-$, then we can conclude that $v_{out} = V_c$. So, we've managed to make a circuit with the same voltage at n_{out} as at the positive terminal of the voltage source. What good is that? We'll see in the next section.

Voltage-controlled voltage-source model

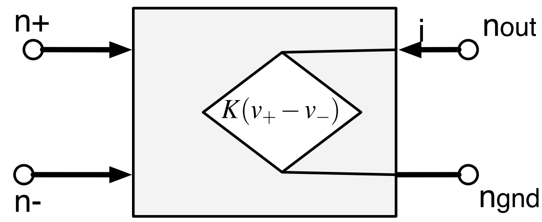
Let's start by thinking about using a variable voltage to control a motor. If we have a 15V supply, but only want to put 7.5V across the motor terminals, what should we do? A voltage divider seems like a good strategy: we can use one with two equal resistances, to make 7.5V, and then connect it to the motor as shown in figure 6(b). But what will the voltage v_{motor} end up being? It all depends on the resistance of the motor. If the motor is offering little resistance, say 100Ω, then the voltage v_{motor} will be very close to 0.[3] So, this is not an effective solution to the problem of supplying 7.5V to the motor.

In figure 7(a), we have used a voltage follower to connect the voltage divider to the motor. Based on our previous analysis of the follower, we expect the voltage at n_{out} to be 7.5V, at least before

[3] Go back and review the discussion of adding a load to a voltage divider, if this doesn't seem clear.

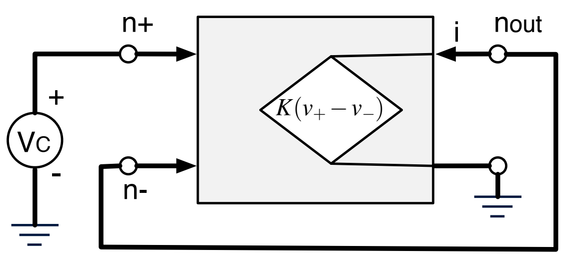


(a) Motor connected to a voltage follower

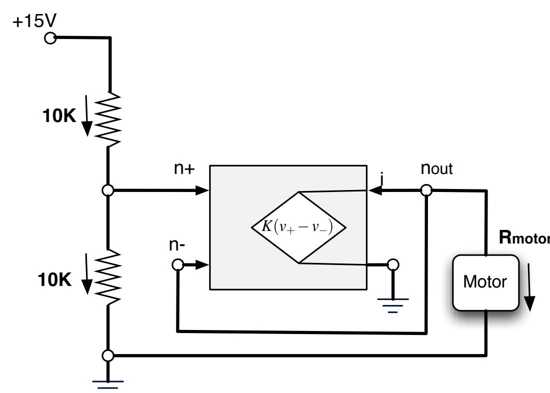


(b) Op-amp as a voltage-controlled voltage source

Figure 7:



(a) Voltage follower



(b) Motor controller with buffer

Figure 8:

we connect it up to the motor. But our simple model of the op-amp doesn't let us understand how connecting the motor to the output of the follower will affect the behavior of the voltage divider or what exactly will happen to the motor.

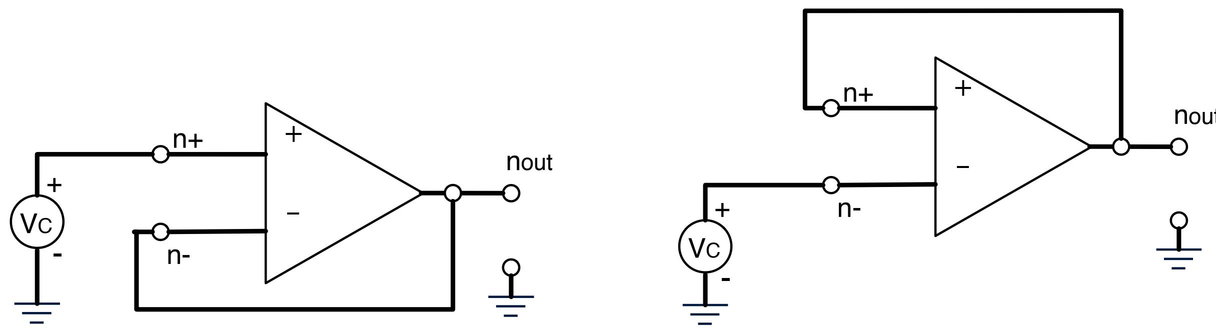
So, now, we need a somehwat more sophisticated model of the op-amp, which is shown schematically in figure 7(b). The constraint model relates the voltages at all four terminals:

$$v_{out} - v_{gnd} = K(v_+ - v_-) \ ,$$

where K is a very large gain, on the order of 10,000, and asserts that

$$i_+ = i_- = 0 \ .$$

We can think of n_{out} and n_{gnd} as constituting a voltage source, whose voltage is defined to be $K(v_+ - v_-)$. We can see it as *amplifying* the voltage difference $v_+ - v_-$. It's hard to really understand how this model works without seeing it in context. So, let's go back to the voltage follower, but



(a) "Good" voltage follower (b) "Bad" voltage follower

Figure 9: A tale of two voltage followers

think about how it works using this model. Figure 8(a) shows a voltage follower, with the VCVS model of an op-amp. We can write down the equations:

$$\begin{aligned} v_+ &= V_c \\ v_{out} - v_{gnd} &= K(v_+ - v_=) \\ v_{gnd} &= 0 \end{aligned}$$

Solving this system, we find that

$$v_{out} = V_c \frac{K}{K+1} \quad .$$

So, for large K, it has very nearly the same prediction about the output voltage as our simple model.

What did we gain by moving to this more complex model? Now we have a model of what will happen when we connect a load to the output of the op-amp. There is still no current flowing into the op-amp, and therefore no influence of the current in the part of the network connected to n_{in} on the current in the part of the network connected to n_{out}. An op-amp in this configuration is sometimes called a *buffer*, because it provides a buffer, or disconnect, between the currents on either side of it. This is a big deal; it gives us a kind of modularity in our circuits that we haven't had before, by limiting the kinds of influence of one part of the circuit on the other. The ability to partially disconnect subparts of our circuits will make it easier to do complex designs.

So, now, back to the motor. If we put our new op-amp model into the motor-control circuit, as shown in figure 8(b), things are much better. We find that v_{out}, the current into the motor is $7.5V$, and because of the isolation provided by the op-amp, it will remain that way, no matter what the resistance of the motor. Further, if in fact the motor has resistance of 100Ω, then we can find that the current through the motor i_{motor} is $.075A$.

Dynamic model

Figure 9 shows two voltage followers, a "good" one and a "bad" one. If we use our simplest model to predict their behavior, in both cases, we'll predict that $v_{out} = V_c$. If we use the more sophisticated model we developed in the previous section, we'll predict that, for the good follower $v_{out} = V_c \frac{K}{K+1}$. For the bad follower, things are connected a little bit differently:

$$\begin{aligned} v_- &= V_c \\ v_{out} - v_{gnd} &= K(v_+ - v_=) \\ v_+ &= v_{out} \\ v_{gnd} &= 0 \end{aligned}$$

Solving this system, we find that

$$v_{out} = V_c \frac{K}{K-1} \quad .$$

Those two predictions are basically the same for large K. But, in fact, the prediction about the behavior of the bad follower is **completely bogus**! To see why, we'll actually need to make a more detailed model.

Our new model is going to take the actual dynamics of the circuit into account. We can model what is going on in an op-amp by using a difference equation to describe the value of the output at a time n as a linear function of its values at previous times and of the values of v_+ and v_-.

Let's start by modeling a simple linear system that is trying to approach a target value. We saw in the lecture 6 notes that a system of the form

$$y[n] = \alpha y[n-1] + c$$

in the case where $|\alpha| < 1$ will converge to $c/(1-\alpha)$. If we let c be $v_+ - v_-$ (assuming for a minute that they are unchanging), and let $\alpha = (K-1)/K$, and define

$$v_{out}[n] = \frac{K-1}{K} v_{out}[n-1] + (v_+ - v_-) \quad ,$$

then we have a system that will, in the limit, converge to $K(v_+ - v_-)$.[4] To allow for time-varying inputs, we can generalize this to

$$v_{out}[n] = \frac{K-1}{K} v_{out}[n-1] + (v_+[n] - v_-[n]) \quad . \tag{12}$$

Now, we can use this model to predict the temporal behavior of the voltage followers. Let's start with the good one.

Good follower In this case, we have $v_+[n] = V_c$, for all n, and $v_-[n] = v_{out}[n]$ for all n. So, equation 12 becomes

$$\begin{aligned} v_{out}[n] &= \frac{K-1}{K} v_{out}[n-1] + (V_c - v_{out}[n]) \\ &= -\frac{1}{K} v_{out}[n-1] + V_c \end{aligned}$$

This system has natural frequency $-1/K$, which has magnitude less than 1, so it will converge to $\frac{K}{K+1} V_c$, which agrees with our previous model.

[4] You should be able to show this, as an exercise.

Bad follower In this case, we have $v_-[n] = V_c$, for all n, and $v_+[n] = v_{out}[n]$ for all n. So, equation 12 becomes

$$\begin{aligned} v_{out}[n] &= \frac{K-1}{K} v_{out}[n-1] + (v_{out}[n] - V_c) \\ &= \frac{2K-1}{K} v_{out}[n-1] - V_c \end{aligned}$$

This system has natural frequency $(2K-1)/K$, which has magnitude greater than 1, so it will diverge! What does divergence mean in practice? Sometimes, a bad smell and then a loud pop and then smoke.

Models

We have seen different kinds of models here. Thévenin-equivalent models have the same voltage and resistance characteristics as the circuits they model, but may have different heat-dissipation properties (consider circuit with four parallel $10K\Omega$ resistors: it looks the same as a single $2.5K\Omega$ resistor from the Thévenin-equivalent perspective, but the resistors are a lot less likely to melt).

In the case of the op-amps, we saw three different models, each of which was appropriate for different levels of modeling.

Sometimes, we do design using the most abstract model, and then check our designs using a more detailed one, just in case we have forgotten something. In the case of op-amps, people have used the detailed model to understand certain standard feedback configurations that are stable, and a designer is well-advised to stick to those configurations, unless they are prepared to analyze their results using a more detailed model.