MATHEMATICS

MINIMUM LEVEL LEARNING MATERIAL

for

CLASS – XII

2020 – 21

Prepared by

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DEDICATED

TO

MY FATHER

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MINIMUM LEVEL DAILY REVISION SYLLABUS
FOR REMEDIAL STUDENTS
MATHEMATICS : CLASS XII

S. NO.	CHAPTER/TOPIC	MARKS COVERED AS PER LATEST CBSE SAMPLE PAPERS
1	Relations and Functions – Full Chapter	6
2	Inverse Trigonometric Functions – Full Chapter	2
3	Matrices – Full Chapter	5
4	Determinants – Full Chapter	5
5	Continuity and Differentiability – Full Chapter	8
6	Vector Algebra – Full Chapter	5
7	Linear Programming – Full Chapter	5
8	Probability – Full Chapter	8
Total Marks		44

All Remedial Students have to complete the above chapters/topics thoroughly with 100% perfection and then they can also concentrate the below topics for Board Exam:

*Application of Derivatives **– NCERT imp questions and Objective types questions**

*Integrals **– NCERT imp questions and Objective types questions**

*Differential Equations **– NCERT imp questions and Objective types questions**

*Application of the Integrals **– NCERT imp questions**

*Three Dimensional Geometry **– NCERT imp questions and Objective types questions**

NOTE: *In Probability chapter, Probability distribution based questions given only for reviewing and understanding. In Inverse Trigonometry, properties based questions also included as one question is given in CBSE Sample paper for 2020-21.*

INDEX OF MINIMUM LEVEL LEARNING STUDY MATERIAL

CLASS XII : MATHEMATICS

RELATIONS AND FUNCTIONS

CHAPTER – 1: RELATIONS AND FUNCTIONS

MARKS WEIGHTAGE – 06 marks

NCERT Important Questions & Answers

1. **Show that the relation *R* in the set *R* of real numbers, defined as $R = \{(a, b) : a \le b^2\}$ is neither reflexive nor symmetric nor transitive.**

   Ans:

   We have $R = \{(a,b) : a \le b^2\}$, where $a, b \in R$

   For reflexivity, we observe that $\frac{1}{2} \le \left(\frac{1}{2}\right)^2$ is not true.

   So, R is not reflexive as $\left(\frac{1}{2}, \frac{1}{2}\right) \notin R$

   For symmetry, we observe that $-1 \le 3^2$ but $3 > (-1)^2$

   $\therefore (-1, 3) \in R$ but $(3, -1) \notin R$.

   So, R is not symmetric.

   For transitivity, we observe that $2 \le (-3)^2$ and $-3 \le (1)^2$ but $2 > (1)^2$

   $\therefore (2, -3) \in R$ and $(-3,1) \in R$ but $(2, 1) \notin R$. So, R is not transitive.

   Hence, R is neither reflexive, nor symmetric and nor transitive.

2. **Prove that the relation *R* in *R* defined by $R = \{(a, b): a \le b^3\}$ is neither reflexive nor symmetric nor transitive.**

   Ans:

   Given that $R = \{(a, b): a \le b^3\}$

   It is observed that $\left(\frac{1}{2}, \frac{1}{2}\right) \in R$ as $\frac{1}{2} < \left(\frac{1}{2}\right)^3 = \frac{1}{8}$

   So, R is not reflexive.

   Now, (1, 2) (as $1 < 2^3 = 8$)

   But $(2, 1) \notin R$ (as $2^3 > 1$)

   So, R is not symmetric.

   We have $\left(3, \frac{3}{2}\right), \left(\frac{3}{2}, \frac{6}{5}\right) \in R$ as $3 > \left(\frac{3}{2}\right)^3$ *and* $\frac{3}{2} < \left(\frac{6}{5}\right)^3$

   But $\left(3, \frac{6}{5}\right) \notin R$ *as* $3 > \left(\frac{6}{5}\right)^3$

   Therefore, R is not transitive.

   Hence, R is neither reflexive nor symmetric nor transitive.

3. **Show that the relation R in the set *A* = {1, 2, 3, 4, 5} given by R = {(a, b) : |a – b| is even}, is an equivalence relation. Show that all the elements of {1, 3, 5} are related to each other and all the elements of {2, 4} are related to each other. But no element of {1, 3, 5} is related to any element of 2, 4}.**

   Ans:

   Given that A = {1, 2, 3, 4, 5} and R = {(a, b) : |a – b| is even}

   It is clear that for any element $a \in A$, we have (which is even).

   ∴ R is reflexive.

   Let $(a, b) \in$ R.

   ⇒ |a – b| is even

   ⇒(a – b) is even

   ⇒– (a – b) is even

⇒(b – a) is even
⇒ |b – a| is even
⇒(*b*, *a*) ∈ R
∴R is symmetric.
Now, let (*a*, *b*) ∈ R and (*b*, *c*) ∈ R.
⇒ |a – b| is even and |b – c| is even
⇒ (a – b) is even and (b – c) is even
⇒ (a – c) = (a – b) + (b – c) is even (Since, sum of two even integers is even)
⇒ |a – c| is even
⇒ (*a*, *c*) ∈ R
∴R is transitive.
Hence, R is an equivalence relation.
Now, all elements of the set {1, 2, 3} are related to each other as all the elements of this subset are odd. Thus, the modulus of the difference between any two elements will be even.
Similarly, all elements of the set {2, 4} are related to each other as all the elements of this subset are even.
Also, no element of the subset {1, 3, 5} can be related to any element of {2, 4} as all elements of {1, 3, 5} are odd and all elements of {2, 4} are even. Thus, the modulus of the difference between the two elements (from each of these two subsets) will not be even.

4. **Show that each of the relation R in the set $A = \{x \in Z : 0 \le x \le 12\}$, given by R = {(a, b): |a – b| is a multiple of 4} is an equivalence relation. Find the set of all elements related to 1.**
Ans:
$A = \{x \in Z : 0 \le x \le 12\} = \{0,1,2,3,4,5,6,7,8,9,10,11,12\}$ and
R = {(a, b): |a – b| is a multiple of 4}
For any element *a* ∈A, we have (*a*, *a*) ∈ R ⇒ |a – a| = 0 is a multiple of 4.
∴R is reflexive.
Now, let (*a*, *b*) ∈ R ⇒|a – b| is a multiple of 4.
⇒|–(a – b)| is a multiple of 4
⇒|b – a| is a multiple of 4.
⇒ (*b*, *a*) ∈ R
∴R is symmetric.
Now, let (*a*, *b*), (*b*, *c*) ∈ R.
⇒|a – b| is a multiple of 4 and |b – c| is a multiple of 4.
⇒(a – b) is a multiple of 4 and (b – c) is a multiple of 4.
⇒(a – b + b – c) is a multiple of 4
⇒(a – c) is a multiple of 4
⇒|a – c| is a multiple of 4
⇒ (*a*, *c*) ∈R
∴ R is transitive.
Hence, R is an equivalence relation.
The set of elements related to 1 is {1, 5, 9} since
|1 – 1| = 0 is a multiple of 4
|5 – 1| = 4 is a multiple of 4
|9 – 1| = 8 is a multiple of 4

5. **Let *A* = *R* − {3} and *B* = *R* − {1} . Prove that the function *f* : *A*→ *B* defined by *f(x)* = $\left(\frac{x-2}{x-3}\right)$ is *f* one-one and onto ? Justify your answer.**
Ans:

Here, $A = R - \{3\}$, $B = R - \{1\}$ and $f : A \to B$ is defined as $f(x) = \left(\frac{x-2}{x-3}\right)$

Let $x, y \in A$ such that $f(x) = f(y)$

$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3} \Rightarrow (x-2)(y-3) = (y-2)(x-3)$

$\Rightarrow xy - 3x - 2y + 6 = xy - 3y - 2x + 6$

$\Rightarrow -3x - 2y = -3y - 2x$

$\Rightarrow 3x - 2x = 3y - 2y \Rightarrow x = y$

Therefore, f is one- one. Let $y \in B = R - \{1\}$. Then, $y \neq 1$

The function f is onto if there exists $x \in A$ such that $f(x) = y$.

Now, $f(x) = y$

$\Rightarrow \frac{x-2}{x-3} = y \Rightarrow x - 2 = xy - 3y$

$\Rightarrow x(1-y) = -3y + 2$

$\Rightarrow x = \frac{2-3y}{1-y} \in A \ \ [y \neq 1]$

Thus, for any $y \in B$, there exists $\frac{2-3y}{1-y} \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\left(\frac{2-3y}{1-y}\right) - 2}{\left(\frac{2-3y}{1-y}\right) - 3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

Therefore, f is onto. Hence, function f is one-one and onto.

6. Show that $f : [-1,1] \to R$, given by $f(x) = \frac{x}{x+2}, x \neq -2$, is one-one.

Ans:

Given that $f : [-1,1] \to R$, given by $f(x) = \frac{x}{x+2}, x \neq -2$,

Let f(x) = f(y)

$\Rightarrow \frac{x}{x+2} = \frac{y}{y+2} \Rightarrow xy + 2x = xy + 2y$

$\Rightarrow 2x = 2y \Rightarrow x = y$

Therefore, f is a one-one function.

7. Show that the function $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$ is one-one and onto function.

Ans:

It is given that $f : R \to \{x \in R : -1 < x < 1\}$ defined by $f(x) = \frac{x}{1+|x|}, x \in R$

Suppose, $f(x) = f(y)$, where $x, y \in R \Rightarrow \frac{x}{1+|x|} = \frac{y}{1+|y|}$

It can be observed that if x is positive and y is negative, then we have

$$\frac{x}{1+x} = \frac{y}{1-y} \Rightarrow 2xy = x - y$$

Since, x is positive and y is negative, then $x > y \Rightarrow \ x - y > 0$

But, $2xy$ is negative. Then, $2xy \neq x - y$.
Thus, the case of x being positive and y being negative can be ruled out.
Under a similar argument, x being negative and y being positive can also be ruled out. Therefore, x and y have to be either positive or negative.

When x and y are both positive, we have $f(x) = f(y) \Rightarrow \frac{x}{1+x} = \frac{y}{1+y} \Rightarrow x + xy = y + xy \Rightarrow x = y$

When x and y are both negative, we have $f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y} \Rightarrow x - xy = y - xy \Rightarrow x = y$

Therefore, f is one-one. Now, let $y \in R$ such that $-1 < y < 1$.

If y is negative, then there exists $x = \frac{y}{1+y} \in R$ such that

$$f(x) = f\left(\frac{y}{1+y}\right) = \frac{\left(\frac{y}{1+y}\right)}{1+\left|\frac{y}{1+y}\right|} = \frac{\frac{y}{1+y}}{1+\left(\frac{-y}{1+y}\right)} = \frac{y}{1+y-y} = y$$

If y is positive, then there exists $x = \frac{y}{1-y} \in R$ such that

$$f(x) = f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{1+\left|\frac{y}{1-y}\right|} = \frac{\frac{y}{1-y}}{1+\left(\frac{y}{1-y}\right)} = \frac{y}{1-y+y} = y$$

Therefore, f is onto. Hence, f is one-one and onto.

8. Show that the function $f : R \to R$ given by $f(x) = x^3$ is injective.

Ans:

Here, $f : R \to R$ is given as $f(x) = x^3$.
Suppose, $f(x) = f(y)$, where $x, y \in R \Rightarrow x^3 = y^3$ …(i)
Now, we need to show that $x = y$
Suppose, $x \neq y$, their cubes will also not be equal.
$x^3 \neq y^3$
However, this will be a contradiction to Eq. i).
Therefore, $x = y$. Hence, f is injective.

9. Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides $a – b$} is an equivalence relation.

Ans:

R is reflexive, as 2 divides $(a – a)$ for all $a \in$ **Z**.
Further, if $(a, b) \in$ R, then 2 divides $a – b$.
Therefore, 2 divides $b – a$.
Hence, $(b, a) \in$ R, which shows that R is symmetric.
Similarly, if $(a, b) \in$ R and $(b, c) \in$ R, then $a – b$ and $b – c$ are divisible by 2.
Now, $a – c = (a – b) + (b – c)$ is even.
So, $(a – c)$ is divisible by 2. This shows that R is transitive.
Thus, R is an equivalence relation in **Z**.

CHAPTER – 1: RELATIONS AND FUNCTIONS

MARKS WEIGHTAGE – 06 marks

Previous Years Board Exam (Important Questions & Answers)

1. **Let T be the set of all triangles in a plane with R as relation in T given by $R = \{(T_1, T_2) : T_1 \cong T_2\}$. Show that R is an equivalence relation.**
Ans:
(*i*) Reflexive
R is reflexive if $T_1\ R\ T_1$
Since $T_1 \cong T_1$
$\therefore$ R is reflexive.
(*ii*) Symmetric
R is symmetric if $T_1\ R\ T_2 \Rightarrow T_2\ R\ T_1$
Since $T_1 \cong T_2 \Rightarrow T_2 \cong T_1$
$\therefore$ R is symmetric.
(*iii*) Transitive
R is transitive if $T_1\ R\ T_2$ and $T_2\ R\ T_3 \Rightarrow T_1\ R\ T_3$
Since $T_1 \cong T_2$ and $T_2 \cong T_3 \Rightarrow T_1 \cong T_3$
$\therefore$ R is transitive
From (*i*), (*ii*) and (*iii*), we get R is an equivalence relation.

2. **Let $f : N \to N$ be defined by** $f(n) = \begin{cases} \dfrac{n+1}{2}, if \ \ n \ is \ odd \\ \dfrac{n}{2}, if \ \ n \ is \ even \end{cases}$ $for \ all \ n \in N$ **. Find whether the function f is bijective.**
Ans:
Given that $f : N \to N$ be defined by $f(n) = \begin{cases} \dfrac{n+1}{2}, if \ \ n \ is \ odd \\ \dfrac{n}{2}, if \ \ n \ is \ even \end{cases}$ $for \ all \ n \in N$.
Let $x, y \in N$ and let they are odd then
$f(x) = f(y) \Rightarrow \dfrac{x+1}{2} = \dfrac{y+1}{2} \Rightarrow x = y$
If $x, y \in N$ are both even then also
$f(x) = f(y) \Rightarrow \dfrac{x}{2} = \dfrac{y}{2} \Rightarrow x = y$
If $x, y \in N$ are such that x is even and y is odd then
$f(x) = \dfrac{x+1}{2} \ \ and \ \ f(y) = \dfrac{y}{2}$
Thus, $x \neq y$ for $f(x) = f(y)$
Let $x = 6$ and $y = 5$
We get $f(6) = \dfrac{6}{2} = 3, f(5) = \dfrac{5+1}{2} = 3$
$\therefore$ $f(x) = f(y)$ but $x \neq y$ ...(*i*)
So, $f(x)$ is not one-one.
Hence, $f(x)$ is not bijective.

3. What is the range of the function $f(x) = \frac{|x-1|}{(x-1)}$?

Ans:

We have given $f(x) = \frac{|x-1|}{(x-1)}$

$$|x-1| = \begin{cases} (x-1), & if \quad x-1>0 \quad or \quad x>1 \\ -(x-1), & if \quad x-1<0 \quad or \quad x<1 \end{cases}$$

(i) For $x > 1$, $f(x) = \frac{(x-1)}{(x-1)} = 1$

(ii) For $x < 1$, $f(x) = \frac{-(x-1)}{(x-1)} = -1$

$\therefore$ Range of $f(x) = \frac{|x-1|}{(x-1)}$ is {–1 , 1}.

4. Let Z be the set of all integers and R be the relation on Z defined as R = {(a, b) ; a, $b \in Z$, and (a – b) is divisible by 5.} Prove that R is an equivalence relation.

Ans:

We have provided R = {(a, b) : a, $b \in Z$, and(a – b) is divisible by 5}

(i) As (a – a) = 0 is divisible by 5.

$\therefore (a, a) \in R \ \forall \ a \in R$

Hence, R is reflexive.

(ii) Let $(a, b) \in R$

$\Rightarrow$ (a – b) is divisible by 5.

$\Rightarrow$ – (b – a) is divisible by 5.

$\Rightarrow$ (b – a) is divisible by 5.

$\therefore (b, a) \in R$

Hence, R is symmetric.

(iii) Let $(a, b) \in R$ and $(b, c) \in Z$

Then, (a – b) is divisible by 5 and (b – c) is divisible by 5.

(a – b) + (b – c) is divisible by 5.

(a – c) is divisible by 5.

$\therefore (a, c) \in R$

$\Rightarrow R$ is transitive.

Hence, R is an equivalence relation.

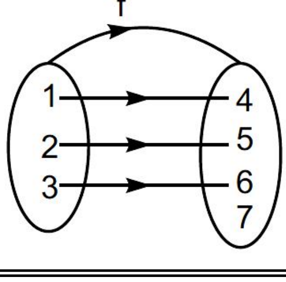
5. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2,5), (3, 6)} be a function from A to B. State whether f is one-one or not.

Ans:

f is one-one because

$f(1) = 4$; $f(2) = 5$; $f(3) = 6$

No two elements of A have same f image.



6. Let $A = R - \{3\}$ and $B = R - \{1\}$. Consider the function $f : A \to B$ defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that f is one-one and onto and hence find f^{-1}.

Ans:

Let $x_1, x_2 \in A$.

Now, $f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$\Rightarrow -3x_1 - 2x_2 = -2x_1 - 3x_2$

$\Rightarrow -x_1 = -x_2 \Rightarrow x_1 = x_2$

Hence f is one-one function.

For Onto

Let $y = \frac{x-2}{x-3} \Rightarrow xy - 3y = x - 2$

$\Rightarrow xy - x = 3y - 2 \Rightarrow x(y-1) = 3y - 2$

$\Rightarrow x = \frac{3y-2}{y-1}$ - - - - - (i)

From above it is obvious that $\forall$ y except 1, *i.e.*, $\forall y \in B = R - \{1\}$ $\exists x \in A$

Hence f is onto function.

Thus f is one-one onto function.

It f^{-1} is inverse function of f then $f^{-1}(y) = \frac{3y-2}{y-1}$ [from (i)]

7. Show that f : N → N, given by $f(x) = \begin{cases} x+1, if \quad x \text{ is odd} \\ x-1, if \quad x \text{ is even} \end{cases}$ is both one-one and onto.

Ans:

For one-one

Case I : When x_1, x_2 are odd natural number.

$\therefore f(x_1) = f(x_2) \Rightarrow x_1+1 = x_2 + 1 \ \forall \ x_1, x_2 \in N$

$\Rightarrow x_1 = x_2$

i.e., f is one-one.

Case II : When x_1, x_2 are even natural number

$\therefore f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2$

i.e., f is one-one.

Case III : When x_1 is odd and x_2 is even natural number

$f(x_1) = f(x_2) \Rightarrow x_1+1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2$ which is never possible as the difference of odd and even number is always odd number.

Hence in this case $f(x_1) \neq f(x_2)$

i.e., f is one-one.

Case IV: When x_1 is even and x_2 is odd natural number

Similar as case III, We can prove f is one-one

For onto:

$\therefore f(x) = x + 1$ if x is odd

$= x - 1$ if x is even

$\Rightarrow$ For every even number 'y' of codomain $\exists$ odd number y - 1 in domain and for every odd number y of codomain $\exists$ even number y +1 in Domain.
i.e. f is onto function.
Hence f is one-one onto function.

8. Prove that the relation R in the set A = {5, 6, 7, 8, 9} given by $R = \{(a, b) : |a - b|$, is divisible by 2}, is an equivalence relation. Find all elements related to the element 6.
Ans:
Here R is a relation defined as $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$
Reflexivity
Here $(a, a) \in R$ as $|a - a| = 0 = 0$ divisible by 2 *i.e.*, R is reflexive.
Symmetry
Let $(a, b) \in R$
$(a, b) \in R \Rightarrow |a - b|$ is divisible by 2
$\Rightarrow a - b = \pm 2m \Rightarrow b - a = \mp 2m$
$\Rightarrow |b - a|$ is divisible by 2 $\Rightarrow (b, a) \in R$
Hence R is symmetric
Transitivity Let $(a, b), (b, c) \in R$
Now, $(a, b), (b, c) \in R \Rightarrow |a - b|, |b - c|$ are divisible by 2
$\Rightarrow a - b = \pm 2m$ and $b - c = \pm 2n$
$\Rightarrow a - b + b - c = \pm 2(m + n)$
$\Rightarrow (a - c) = \pm 2k \quad [\because k = m + n]$
$\Rightarrow (a - c) = 2k$
$\Rightarrow (a - c)$ is divisible by 2 $\Rightarrow (a, c) \in R$.
Hence R is transitive.
Therefore, R is an equivalence relation.
The elements related to 6 are 6, 8.

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. Let R be a relation on the set L of lines defined by $l_1 R\, l_2$ if l_1 is perpendicular to l_2, then relation R is
(a) reflexive and symmetric (b) symmetric and transitive
(c) equivalence relation (d) symmetric

2. Given triangles with sides T_1 : 3, 4, 5; T_2 : 5, 12, 13; T_3 : 6, 8, 10; T_4 : 4, 7, 9 and a relation R in set of triangles defined as $R = \{(\Delta_1, \Delta_2) : \Delta_1 \text{ is similar to } \Delta_2\}$. Which triangles belong to the same equivalence class?
(a) T_1 and T_2 (b) T_2 and T_3 (c) T_1 and T_3 (d) T_1 and T_4.

3. Given set A = {1, 2, 3} and a relation R = {(1, 2), (2, 1)}, the relation R will be
(a) reflexive if (1, 1) is added (b) symmetric if (2, 3) is added
(c) transitive if (1, 1) is added (d) symmetric if (3, 2) is added

4. Given set A = {a, b, c}. An identity relation in set A is
(a) R = {(a, b), (a, c)} (b) R = {(a, a), (b, b), (c, c)}
(c) R = {(a, a), (b, b), (c, c), (a, c)} (d) R = {(c, a), (b, a), (a, a)}

5. A relation S in the set of real numbers is defined as xSy $\Rightarrow$ $x - y + \sqrt{3}$ is an irrational number, then relation S is
(a) reflexive (b) reflexive and symmetric (c) transitive (d) symmetric and transitive

6. Let R be a relation on the set N of natural numbers defined by nRm if n divides m. Then R is
(a) Reflexive and symmetric (b) Transitive and symmetric
(c) Equivalence (d) Reflexive, transitive but not symmetric

7. Let L denote the set of all straight lines in a plane. Let a relation R be defined by *l* R *m* if and only if *l* is perpendicular to m for all *l*, *m* $\in$ L. Then R is
(a) reflexive (b) symmetric **(c)** transitive (d) none of these

8. Let N be the set of natural numbers and the function f : N → N be defined by f (n) = 2n + 3 $\forall n \in N$. Then f is
(a) surjective (b) injective **(c)** bijective (d) none of these

9. Set A has 3 elements and the set B has 4 elements. Then the number of injective mappings that can be defined from A to B is
(a) 144 (b) 12 **(c)** 24 (d) 64

10. Let f : R → R be defined by $f(x) = x^2 + 1$. Then, pre-images of 17 and – 3, respectively, are
(a) φ, {4, – 4} (b) {3, – 3}, φ **(c)** {4, –4}, φ (d) {4, – 4, {2, – 2}

11. For real numbers x and y, define xRy if and only if x – y + 2 is an irrational number. Then the relation R is
(a) reflexive (b) symmetric **(c)** transitive (d) none of these

12. Let T be the set of all triangles in the Euclidean plane, and let a relation R on T be defined as aRb if a is congruent to b $\forall$ a, b $\in$ T. Then R is
(a) reflexive but not transitive (b) transitive but not symmetric
(c) equivalence (d) none of these

13. Consider the non-empty set consisting of children in a family and a relation R defined as aRb if a is brother of b. Then R is
(a) symmetric but not transitive (b) transitive but not symmetric
(c) neither symmetric nor transitive (d) both symmetric and transitive

14. The maximum number of equivalence relations on the set A = {1, 2, 3} are
(a) 1 (b) 2 **(c)** 3 (d) 5

15. If a relation R on the set {1, 2, 3} be defined by R = {(1, 2)}, then R is
(a) reflexive (b) transitive **(c)** symmetric (d) none of these

16. Let us define a relation R in R as aRb if a ≥ b. Then R is
(a) an equivalence relation (b) reflexive, transitive but not symmetric
(c) symmetric, transitive but not reflexive (d) neither transitive nor reflexive but symmetric.

17. Let A = {1, 2, 3} and consider the relation R = {1, 1), (2, 2), (3, 3), (1, 2), (2, 3), (1,3)}. Then R is
(a) reflexive but not symmetric (b) reflexive but not transitive
(c) symmetric and transitive (d) neither symmetric, nor transitive

18. If the set A contains 5 elements and the set B contains 6 elements, then the number of one-one and onto mappings from A to B is
(a) 720 (b) 120 **(c)** 0 (d) none of these

19. Let A = {1, 2, 3, ...n} and B = {a, b}. Then the number of surjections from A into B is
(a) ${}^nP_2$ (b) $2^n - 2$ **(c)** $2^n - 1$ (d) None of these

20. Let f : R → R be defined by f (x) = $\frac{1}{x}$. Then f is

(a) one-one (b) onto **(c)** bijective (d) f is not defined

21. Which of the following functions from Z into Z are bijections?

(a) f (x) = x^3 (b) f (x) = x + 2 **(c)** f (x) = 2x + 1 (d) f (x) = x2 + 1

22. Let f : [2, ∞) → R be the function defined by f (x) = x^2 – 4x + 5, then the range of f is

(a) R (b) [1, ∞) **(c)** [4, ∞) (d) [5, ∞)

23. Let f : N → R be the function defined by f (x) = $\frac{2x-1}{2}$ and g : Q → R be another function defined by g(x) = x + 2. Then (g o f)$\frac{3}{2}$ is

(a) 1 (b) 1 (c) $\frac{7}{2}$ (d) none of these

24. Let f : R → R be defined by $f(x)=\begin{cases} 2x : x>3 \\ x^2 : 1<x\le 3 \\ 3x : x\le 1 \end{cases}$. Then f (– 1) + f (2) + f (4) is

(a) 9 (b) 14 (c) 5 (d) none of these

25. Let the function 'f ' : N → N be defined by f(x) = 2x + 3, x ∈ N. Then 'f ' is

(a) not onto (b) bijective function (c) many-one, into function (d) none of these

26. A relation defined in a non-empty set A, having n elements, has

(a) n relations (b) 2 relations (c) n^2 relations (d) $2n^2$ relations

27. If f (x) = x^3 and g(x) = cos 3x , then fog is

(a) x^3. cos 3x (b) cos $3x^3$ (c) $\cos^3 3x$ (d) 3cos x^3.

28. A relation R in human beings defined as R = {(a, b) : a, b human beings ; a loves b} is

(a) reflexive (b) symmetric and transitive (c) equivalence (d) neither of these

29. Consider the set A = {1, 2, 3} and R be the smallest equivalence relation on A, then R = ________

30. The domain of the function f : R→R defined by f (x) = $\sqrt{x^2-3x+2}$ is ________.

31. The domain of the function f : R→R defined by f (x) = $\sqrt{4-x^2}$ is ________.

32. Consider the set A containing n elements. Then, the total number of injective functions from A onto itself is ________.

33. Let Z be the set of integers and R be the relation defined in Z such that aRb if a – b is divisible by 3. Then R partitions the set Z into ________ pairwise disjoint subsets.

34. Consider the set A = {1, 2, 3} and the relation R = {(1, 2), (1, 3)}. R is a __________relation.

35. Let the relation R be defined in N by aRb if 2a + 3b = 30. Then R = ______.

36. Let the relation R be defined on the set A = {1, 2, 3, 4, 5} by R = {(a, b) : $|a^2 - b^2| < 8$. Then R is given by _______.

37. Let R be a relation defined as R = {(x, x), (y, y), (z, z), (x, z)} in set A = {x, y, z} then R is ______________ (reflexive/symmetric) relation.

38. Let A = {1, 2, 3, 4} and B = {a, b, c}. Then number of one-one functions from A to B are ________.

39. If n(a) = p, then number of bijective functions from set A to A are _______.

40. If f (x) = $\frac{x-1}{|x-1|}$, x($\neq$ 1) $\in$ R then range of 'f ' is _____.

INVERSE TRIGONOMETRIC FUNCTIONS

CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 02 marks

QUICK REVISION (Important Concepts & Formulae)

Inverse Trigonometrical Functions

A function $f : A \to B$ is invertible if it is a bijection. The inverse of f is denoted by f^{-1} and is defined as $f^{-1}(y) = x \Leftrightarrow f(x) = y$.

☞ Clearly, domain of f^{-1} = range of f and range of f^{-1} = domain of f.

☞ The inverse of sine function is defined as $\sin^{-1}x = \theta \Leftrightarrow$ sinq = x, where $\theta \in [-\pi/2, \pi/2]$ and $x \in [-1, 1]$.

☞ Thus, sin –1 x has infinitely many values for given $x \in [-1, 1]$

☞ There is one value among these values which lies in the interval $[-\pi/2, \pi/2]$. This value is called the principal value.

Domain and Range of Inverse Trigonometrical Functions

Function	Domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty)$	$(0, \pi)$
$\sec^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$

Properties of Inverse Trigonometrical Functions

☞ $\sin^{-1}(\sin\theta) = \theta$ and $\sin(\sin^{-1}x) = x$, provided that $-1 \le x \le 1$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

☞ $\cos^{-1}(\cos\theta) = \theta$ and $\cos(\cos^{-1}x) = x$, provided that $-1 \le x \le 1$ and $0 \le \theta \le \pi$

☞ $\tan^{-1}(\tan\theta) = \theta$ and $\tan(\tan^{-1}x) = x$, provided that $-\infty \le x \le \infty$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

☞ $\cot^{-1}(\cot\theta) = \theta$ and $\cot(\cot^{-1}x) = x$, provided that $-\infty < x < \infty$ and $0 < \theta < \pi$.

☞ $\sec^{-1}(\sec\theta) = \theta$ and $\sec(\sec^{-1}x) = x$

☞ $\text{cosec}^{-1}(\text{cosec}\theta) = \theta$ and $\text{cosec}(\text{cosec}^{-1}x) = x$,

☞ $\sin^{-1}x = \text{cosec}^{-1}\frac{1}{x}$ *or* $\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}$

☞ $\cos^{-1} x = \sec^{-1}\frac{1}{x}$ *or* $\sec^{-1} x = \cos^{-1}\frac{1}{x}$

☞ $\tan^{-1} x = \cot^{-1}\frac{1}{x}$ *or* $\cot^{-1} x = \tan^{-1}\frac{1}{x}$

☞ $\sin^{-1} x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x} = \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{x}$

☞ $\cos^{-1} x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}} = \operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}} = \sec^{-1}\frac{1}{x}$

☞ $\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x} = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x}$

☞ $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, where -1 \le x \le 1$

☞ $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, where -\infty \le x \le \infty$

☞ $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}, where \quad x \le -1 \quad or \quad x \ge 1$

☞ $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), if \quad xy < 1$

☞ $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right), if \quad xy > 1$

☞ $\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$

☞ $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), if \quad x, y \ge 0, x^2 + y^2 \le 1$

☞ $\sin^{-1} x - \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), if \quad x, y \ge 0, x^2 + y^2 \le 1$

☞ $\sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), if \quad x, y \ge 0, x^2 + y^2 > 1$

☞ $\sin^{-1} x - \sin^{-1} y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right), if \quad x, y \ge 0, x^2 + y^2 > 1$

☞ $\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \ge 0, x^2 + y^2 \le 1$

☞ $\cos^{-1} x - \cos^{-1} y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \geq 0, x^2 + y^2 \leq 1$

☞ $\cos^{-1} x + \cos^{-1} y = \pi - \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \geq 0, x^2 + y^2 > 1$

☞ $\cos^{-1} x - \cos^{-1} y = \pi - \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right), if \quad x, y \geq 0, x^2 + y^2 > 1$

☞ $\sin^{-1}(-x) = -\sin^{-1} x, \quad \cos^{-1}(-x) = \pi - \cos^{-1} x$

☞ $\tan^{-1}(-x) = -\tan^{-1} x, \quad \cot^{-1}(-x) = \pi - \cot^{-1} x$

☞ $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right), \qquad 2\cos^{-1} x = \cos^{-1}\left(2x^2 - 1\right)$

☞ $2\tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

☞ $3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right), \quad 3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right)$

☞ $3\tan^{-1} x = \tan^{-1}\left(\frac{3x - x^3}{1-3x^2}\right)$

CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 02 marks

NCERT Important Questions & Answers

1. Find the values of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Ans:

Let $\tan^{-1}(1)=x \Rightarrow \tan x = 1 = \tan\frac{\pi}{4} \Rightarrow x=\frac{\pi}{4}\ where\ \ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \tan^{-1}(1)=\frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right)=y \Rightarrow \cos y=-\frac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$ $\quad (\because \cos(\pi-\theta)=-\cos\theta)$

$\Rightarrow y=\frac{2\pi}{3}\ where\ \ y\in[0,\pi]$

Let $\sin^{-1}\left(-\frac{1}{2}\right)=z \Rightarrow \sin z=-\frac{1}{2}=-\sin\frac{\pi}{6}=\sin\left(-\frac{\pi}{6}\right)\Rightarrow z=-\frac{\pi}{6}\ where\ \ z\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)=x+y+z=\frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$=\frac{3\pi+8\pi-2\pi}{12}=\frac{9\pi}{12}=\frac{3\pi}{4}$

2. Prove that $3\sin^{-1}x=\sin^{-1}(3x-4x^3), x\in\left(-\frac{1}{2},\frac{1}{2}\right)$

Ans:

Let $\sin^{-1}x=\theta \Rightarrow x=\sin\theta$, *then*

We know that $\sin 3\theta = 3\sin\theta-4\sin^3\theta$

$\therefore 3\theta=\sin^{-1}(3\sin\theta-4\sin^3\theta)=\sin^{-1}(3x-4x^3)$

$\Rightarrow 3\sin^{-1}x=\sin^{-1}(3x-4x^3)$

3. Prove that $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

Ans:

Given $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\frac{1}{2}$

$LHS=\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}=\tan^{-1}\left(\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}.\frac{7}{24}}\right)$ $\quad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$

$=\tan^{-1}\left(\frac{\frac{48+77}{264}}{1-\frac{14}{264}}\right)=\tan^{-1}\left(\frac{\frac{125}{264}}{\frac{264-14}{264}}\right)=\tan^{-1}\left(\frac{\frac{125}{264}}{\frac{250}{264}}\right)=\tan^{-1}\frac{125}{250}=\tan^{-1}\frac{1}{2}=RHS$

4. Prove that $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Ans:

$$Given\ \ 2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$$

$$LHS=2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right)+\tan^{-1}\frac{1}{7}\qquad\left(\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)$$

$$=\tan^{-1}\frac{1}{1-\frac{1}{4}}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}.\frac{1}{7}}\right)\qquad\left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$=\tan^{-1}\left(\frac{\frac{28+3}{21}}{1-\frac{4}{21}}\right)=\tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right)=\tan^{-1}\frac{31}{17}=RHS$$

5. Simplify : $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x},x\neq 0$

Ans:

Let $x=\tan\theta$, then $\theta=\tan^{-1}x$ ………………… (i)

$$\tan^{-1}\frac{\sqrt{1+x^2}-1}{2}=\tan^{-1}\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}=\tan^{-1}\frac{\sqrt{\sec^2\theta}-1}{\tan\theta}$$

$$=\tan^{-1}\frac{\sec\theta-1}{\tan\theta}=\tan^{-1}\left(\frac{\frac{1}{\cos\theta}-1}{\frac{\sin\theta}{\cos\theta}}\right)=\tan^{-1}\left(\frac{\frac{1-\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)=\tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\qquad\left[\begin{array}{l}\because 1-\cos\theta=2\sin^2\frac{\theta}{2}\\ and\ \ \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\end{array}\right]$$

$$=\tan^{-1}\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)=\tan^{-1}\tan\frac{\theta}{2}=\frac{\theta}{2}=\frac{1}{2}\tan^{-1}x\quad\text{[using (i)]}$$

6. Simplify : $\tan^{-1}\frac{1}{\sqrt{x^2-1}},|x|>1$

Ans:

Let $x=\sec\theta$, then $\theta=\sec^{-1}x$ ………………… (i)

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}}=\tan^{-1}\frac{1}{\sqrt{\sec^2\theta-1}}=\tan^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$=\tan^{-1}\frac{1}{\tan\theta}=\tan^{-1}(\cot\theta)=\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\theta\right)\right)\left(\because \tan\left(\frac{\pi}{2}-\theta\right)=\cot\theta\right)$$

$= \frac{\pi}{2} - \theta = \frac{\pi}{2} - \sec^{-1} x$ [using (i)]

7. Simplify : $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), 0 < x < \pi$

Ans:

$$\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right) = \tan^{-1}\left(\frac{\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x} + \frac{\sin x}{\cos x}}\right)$$

(inside the bracket divide numerator and denominator by cos *x*)

$$= \tan^{-1}\left(\frac{1 - \tan x}{1 + \tan x}\right) = \tan^{-1}\left(\tan\left(\frac{\pi}{4} - x\right)\right) \quad \left(\because \tan\left(\frac{\pi}{4} - x\right) = \frac{1 - \tan x}{1 + \tan x}\right)$$

$$= \frac{\pi}{4} - x$$

8. Simplify : $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \ \ and \ \ xy < 1$

Ans:

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right], |x| < 1, y > 0 \ \ and \ \ xy < 1$$

$$\left[\because 2\tan^{-1} x = \sin^{-1}\frac{2x}{1+x^2} \ \ and \ \ 2\tan^{-1} y = \cos^{-1}\frac{1-y^2}{1+y^2}\right]$$

$$= \tan\frac{1}{2}\left[(2\tan^{-1} x + 2\tan^{-1} y)\right] = \tan\left[\frac{1}{2}.2(\tan^{-1} x + \tan^{-1} y)\right] = \tan(\tan^{-1} x + \tan^{-1} y)$$

$$= \tan\left(\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right) \quad \left(\because \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$= \frac{x+y}{1-x.y}$$

9. Find the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$**.**

Ans:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right) where, \frac{5\pi}{6} \in [0, \pi]$$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\left(\frac{5\pi}{6}\right)\right) = \frac{5\pi}{6} \quad (\because \cos(2\pi - \theta) = \cos\theta)$$

10. Prove that $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Ans:

Given $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

Let $\cos^{-1}\frac{12}{13} = x \Rightarrow \cos x = \frac{12}{13}$

$$\therefore \sin x = \sqrt{1 - \cos^2 x} = \sqrt{1 - \left(\frac{12}{13}\right)^2} = \sqrt{\frac{25}{169}} = \frac{5}{13}$$

$\Rightarrow x = \sin^{-1}\frac{5}{13}$

$LHS = \cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{3}{5}$

$= \sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right) \quad \left[\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)\right]$

$= \sin^{-1}\left(\frac{5}{13}\sqrt{\frac{16}{25}} + \frac{3}{5}\sqrt{\frac{144}{169}}\right) = \sin^{-1}\left(\frac{5}{13}\times\frac{4}{5} + \frac{3}{5}\times\frac{12}{13}\right)$

$= \sin^{-1}\left(\frac{20}{65} + \frac{36}{65}\right) = \sin^{-1}\frac{56}{65} = RHS$

11. Prove that $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Ans:

$RHS = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

$Let\ \sin^{-1}\frac{5}{13} = x \Rightarrow \sin x = \frac{5}{13}$

$\therefore \cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{5}{13}\right)^2} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\Rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{\frac{5}{13}}{\frac{12}{13}} = \frac{5}{12} \Rightarrow x = \tan^{-1}\frac{5}{12}$

$Let\ \cos^{-1}\frac{3}{5} = y \Rightarrow \cos y = \frac{3}{5}$

$\therefore \sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\Rightarrow \tan y = \frac{\sin x}{\cos x} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3} \Rightarrow y = \tan^{-1}\frac{4}{3}$

then the equation becomes $\tan^{-1}\frac{63}{16} = x + y$

$\Rightarrow \tan^{-1}\frac{63}{16} = \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3}$

$RHS = \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{4}{3} = \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}.\frac{4}{3}}\right) \quad \left(\because \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$

$= \tan^{-1}\left(\frac{\frac{15+48}{36}}{1-\frac{20}{36}}\right) = \tan^{-1}\left(\frac{\frac{63}{36}}{\frac{16}{36}}\right) = \tan^{-1}\left(\frac{63}{16}\right) = LHS$

12. Prove that $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8} = \frac{\pi}{4}$

Ans:

$LHS = \left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}\right)+\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{8}\right)$

$= \tan^{-1}\left(\dfrac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{5}.\frac{1}{7}}\right) + \tan^{-1}\left(\dfrac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{3}.\frac{1}{8}}\right) \quad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$

$= \tan^{-1}\left(\dfrac{\frac{7+5}{35}}{1-\frac{1}{35}}\right)+\tan^{-1}\left(\dfrac{\frac{8+3}{24}}{1-\frac{1}{24}}\right)=\tan^{-1}\left(\dfrac{\frac{12}{35}}{\frac{30}{35}}\right)+\tan^{-1}\left(\dfrac{\frac{11}{24}}{\frac{23}{24}}\right)$

$= \tan^{-1}\left(\frac{12}{34}\right)+\tan^{-1}\left(\frac{11}{23}\right)=\tan^{-1}\left(\frac{6}{17}\right)+\tan^{-1}\left(\frac{11}{23}\right)$

$= \tan^{-1}\left(\dfrac{\frac{6}{17}+\frac{11}{23}}{1-\frac{6}{17}.\frac{11}{23}}\right) = \tan^{-1}\left(\dfrac{\frac{138+187}{391}}{1-\frac{66}{391}}\right)=\tan^{-1}\left(\dfrac{\frac{325}{391}}{\frac{325}{391}}\right)=\tan^{-1}(1)=\frac{\pi}{4}=RHS$

13. Prove that $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right)$

Ans:

Given $\cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)=\frac{x}{2}, x\in\left(0,\frac{\pi}{4}\right)$

$LHS = \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$

$= \cot^{-1}\left(\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\times\dfrac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}+\sqrt{1-\sin x}}\right)$ (by rationalizing the denominator)

$= \cot^{-1}\left(\dfrac{\left(\sqrt{1+\sin x}+\sqrt{1-\sin x}\right)^2}{\left(\sqrt{1+\sin x}\right)^2-\left(\sqrt{1-\sin x}\right)^2}\right)=\cot^{-1}\left(\dfrac{1+\sin x+1-\sin x+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$

$= \cot^{-1}\left(\dfrac{2+2\cos x}{\sin x}\right)=\cot^{-1}\left(\dfrac{2(1+\cos x)}{2\sin x}\right)=\cot^{-1}\left(\dfrac{1+\cos x}{\sin x}\right)$

$= \cot^{-1}\left(\dfrac{2\cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) \quad \left(\because 1+\cos x=2\cos^2\frac{x}{2} \textit{ and } \sin x=2\sin\frac{x}{2}\cos\frac{x}{2}\right)$

$= \cot^{-1}\left(\dfrac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right) == \cot^{-1}\left(\cot\frac{x}{2}\right)=\frac{x}{2}=RHS$

14. Prove that $\tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)=\frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$

Ans:

Let $x = \cos y \Rightarrow y=\cos^{-1}x$

$$LHS = \tan^{-1}\left(\frac{\sqrt{1+\cos y}-\sqrt{1-\cos y}}{\sqrt{1+\cos y}+\sqrt{1-\cos y}}\right) = \tan^{-1}\left(\frac{2\cos\frac{y}{2}-2\sin\frac{y}{2}}{2\cos\frac{y}{2}+2\sin\frac{y}{2}}\right)$$

$$\left(\because 1+\cos y = 2\cos^2\frac{y}{2} \ \ and \ \ 1-\cos y = 2\sin^2\frac{y}{2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{y}{2}-\sin\frac{y}{2}}{\cos\frac{y}{2}+\sin\frac{y}{2}}\right) = \tan^{-1}\left(\frac{1-\tan\frac{y}{2}}{1+\tan\frac{y}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{y}{2}\right) = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

$$\left(\because \tan\left(\frac{\pi}{4}-x\right) = \frac{1-\tan x}{1+\tan x}\right)$$

15. Solve for x: $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

Ans:

Given $\tan^{-1}\frac{1-x}{1+x} = \frac{1}{2}\tan^{-1}x, (x>0)$

$$\Rightarrow 2\tan^{-1}\frac{1-x}{1+x} = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2}\right) = \tan^{-1}x \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2\left(\frac{1-x}{1+x}\right)}{\frac{(1+x)^2-(1-x)^2}{(1+x)^2}}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x)(1+x)}{(1+x)^2-(1-x)^2}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x^2)}{1+2x+x^2-1+2x-x^2}\right) = \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{2(1-x^2)}{4x}\right) = \tan^{-1}x \ \Rightarrow \tan^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}x$$

$$\Rightarrow \frac{1-x^2}{2x} = x \Rightarrow 1-x^2 = 2x^2 \Rightarrow 1 = 3x^2 \Rightarrow x^2 = \frac{1}{3} \Rightarrow x = \pm\frac{1}{\sqrt{3}}$$

$$\left[\because x>0 \text{ given, so we do not take } x = -\frac{1}{\sqrt{3}}\right]$$

$$\Rightarrow x = \frac{1}{\sqrt{3}}$$

16. Solve for x: $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

Ans:

Given $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{1-\cos^2 x}\right) = \tan^{-1}(2\operatorname{cosec} x) \qquad \left[\because 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{2\cos x}{\sin^2 x}\right) = \tan^{-1}\left(\frac{2}{\sin x}\right)$$

$$\Rightarrow \frac{2\cos x}{\sin^2 x} = \frac{2}{\sin x} \Rightarrow \frac{\cos x}{\sin x} = 1$$

$$\Rightarrow \cot x = 1 \Rightarrow \cot x = \cot\frac{\pi}{4} \Rightarrow x = \frac{\pi}{4}$$

17. Solve for x: $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

Ans:

Given $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

$$\Rightarrow -2\sin^{-1} x = \frac{\pi}{2} - \sin^{-1}(1-x) \Rightarrow -2\sin^{-1} x = \cos^{-1}(1-x)$$

$$\left[\because \sin^{-1}(1-x) + \cos^{-1}(1-x) = \frac{\pi}{2}\right]$$

$$\Rightarrow \cos\left(-2\sin^{-1} x\right) = 1 - x$$

$$\Rightarrow \cos\left(2\sin^{-1} x\right) = 1 - x \qquad \left[\because \cos(-x) = \cos x\right]$$

$$\Rightarrow 1 - 2\sin^2\left(\sin^{-1} x\right) = 1 - x \qquad \left[\because \cos 2x = 1 - 2\sin^2 x\right]$$

$$\Rightarrow 1 - 2\left[\sin\left(\sin^{-1} x\right)\right]^2 = 1 - x$$

$$\Rightarrow 1 - 2x^2 = 1 - x \Rightarrow 2x^2 - x = 0$$

$$\Rightarrow x(2x-1) = 0 \Rightarrow x = 0 \ \ or \ \ 2x - 1 = 0$$

$$\Rightarrow x = 0 \ \ or \ \ x = \frac{1}{2}$$

But $x = \frac{1}{2}$ does not satisfy the given equation, so $x = 0$.

18. Simplify: $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right)$

Ans:

Given $\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{x-y}{x+y}\right) = \tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1}\left(\frac{\frac{x}{y}-1}{\frac{x}{y}+1}\right)$

$$= \tan^{-1}\left(\frac{x}{y}\right) - \left(\tan^{-1}\left(\frac{x}{y}\right) - \tan^{-1} 1\right) \qquad \left(\because \tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+x.y}\right)\right)$$

$$\Rightarrow \tan^{-1} 1 = \frac{\pi}{4}$$

19. Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$ **in the simplest form.**

Ans:

Given $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$= \tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\cos\frac{x}{2}\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right)$$

$$\left(\because 1-\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2}, \sin^2\frac{x}{2}+\cos^2\frac{x}{2}=1 \;\; and \;\; \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) = \frac{\pi}{4}+\frac{x}{2}$$

20. Simplify : $\cot^{-1}\frac{1}{\sqrt{x^2-1}}, |x|>1$

Ans:

Let $x = \sec\theta$, then $\theta = \sec^{-1}x$ ………………… (i)

$$\cot^{-1}\frac{1}{\sqrt{x^2-1}} = \cot^{-1}\frac{1}{\sqrt{\sec^2\theta-1}} = \cot^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$= \cot^{-1}\frac{1}{\tan\theta} = \cot^{-1}(\cot\theta) = \theta = \sec^{-1}x$$

21. Prove that $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

Ans:

Let $\sin^{-1}\frac{3}{5} = x$ and $\sin^{-1}\frac{8}{17} = y$

Therefore $\sin x = \frac{3}{5}$ and $\sin y = \frac{8}{17}$

Now, $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

and $\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\left(\frac{8}{17}\right)^2} = \sqrt{1-\frac{64}{289}} = \frac{15}{17}$

We have $\cos(x-y) = \cos x\cos y + \sin x\sin y = \frac{4}{5}\times\frac{15}{17} + \frac{3}{5}\times\frac{8}{17} = \frac{60}{85}+\frac{24}{85} = \frac{84}{85}$

$\Rightarrow x - y = \cos^{-1}\frac{84}{85}$

$\Rightarrow \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

22. Prove that $\sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

Ans:

Let $\sin^{-1}\frac{12}{13} = x$, $\cos^{-1}\frac{4}{5} = y$ and $\tan^{-1}\frac{63}{16} = z$

Then $\sin x = \frac{12}{13}$, $\cos y = \frac{4}{5}$ and $\tan z = \frac{63}{16}$

Now, $\cos x = \sqrt{1-\sin^2 x} = \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \frac{5}{13}$

and $\sin y = \sqrt{1-\cos^2 y} = \sqrt{1-\left(\frac{4}{5}\right)^2} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$

$$\Rightarrow \tan x = \frac{\sin x}{\cos x} = \frac{\frac{12}{13}}{\frac{5}{13}} = \frac{12}{5} \quad and \quad \tan y = \frac{\sin y}{\cos y} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1-\tan x.\tan y} = \frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}.\frac{3}{4}} = \frac{\frac{48+15}{20}}{1-\frac{36}{20}} = \frac{\frac{63}{20}}{-\frac{16}{20}} = -\frac{63}{16} = -\tan z$$

$\Rightarrow \tan(x+y) = -\tan z = \tan(-z) = \tan(\pi - z)$

$\Rightarrow x+y = \pi - z$

$\Rightarrow x+y+z = \pi$

$\Rightarrow \sin^{-1}\frac{12}{13} + \cos^{-1}\frac{4}{5} + \tan^{-1}\frac{63}{16} = \pi$

23. Simplify: $\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right), if\ \frac{a}{b}\tan x > -1$

Ans:

$$\tan^{-1}\left(\frac{a\cos x - b\sin x}{b\cos x + a\sin x}\right) = \tan^{-1}\left(\frac{\frac{a\cos x - b\sin x}{b\cos x}}{\frac{b\cos x + a\sin x}{b\cos x}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{a}{b} - \tan x}{1+\frac{a}{b}\tan x}\right) = \tan^{-1}\frac{a}{b} - \tan^{-1}(\tan x) = \tan^{-1}\frac{a}{b} - x$$

CHAPTER – 2: INVERSE TRIGONOMETRIC FUNCTIONS

MARKS WEIGHTAGE – 02 marks

Previous Years Board Exam (Important Questions & Answers)

1. Evaluate : $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$

Ans:

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]=\sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]=\sin\frac{\pi}{2}=1$$

2. Write the value of cot (tan^{-1}a + cot^{-1}a).

Ans:

$$\cot\left(\tan^{-1}a+\cot^{-1}a\right)=\cot\left(\frac{\pi}{2}-\cot^{-1}a+\cot^{-1}a\right)\;=\cot\frac{\pi}{2}=0$$

3. Find the principal values of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$.

Ans:

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right)=\cos^{-1}\left(\cos\left(\pi+\frac{\pi}{6}\right)\right)$$

$$=\cos^{-1}\left(-\cos\frac{\pi}{6}\right)=\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$$

4. Find the principal values of $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$

Ans:

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right)=\tan^{-1}\left(\tan\left(\pi-\frac{\pi}{4}\right)\right)$$

$$=\tan^{-1}\left(-\tan\frac{\pi}{4}\right)=\tan^{-1}(-1)=-\frac{\pi}{4}$$

5. Prove that: $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$

Ans:

$$LHS=\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)$$

$$=\frac{\tan\frac{\pi}{4}+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}+\frac{\tan\frac{\pi}{4}-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1+\tan\frac{\pi}{4}\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$=\frac{1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}+\frac{1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{\left[1+\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]^2 + \left[1-\tan\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]^2}{\left[1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right]}$$

$$= \frac{2+2\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2\left(1+\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)\right)}{1-\tan^2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= \frac{2}{\cos 2\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\cos\left(\cos^{-1}\frac{a}{b}\right)} = \frac{2}{\frac{a}{b}} = \frac{2b}{a} = RHS$$

6. Prove that $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

Ans:

$\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\frac{\pi}{2}-\sin^{-1}\frac{16}{65}=\cos^{-1}\frac{16}{65}$

Let $\sin^{-1}\frac{4}{5}=x$ and $\sin^{-1}\frac{5}{13}=y$

Therefore $\sin x=\frac{4}{5}$ and $\sin y=\frac{5}{13}$

Now, $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{4}{5}\right)^2}=\sqrt{1-\frac{16}{25}}=\frac{3}{5}$

and $\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-\left(\frac{5}{13}\right)^2}=\sqrt{1-\frac{25}{144}}=\frac{12}{13}$

We have $\cos(x+y)=\cos x\cos y-\sin x\sin y=\frac{3}{5}\times\frac{12}{13}-\frac{4}{5}\times\frac{5}{13}=\frac{36}{65}-\frac{20}{65}=\frac{16}{65}$

$\Rightarrow x+y=\cos^{-1}\frac{16}{65}$

$\Rightarrow \sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}=\cos^{-1}\frac{16}{65}$

7. Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$

Ans:

$LHS=\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}$

$$=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}.\frac{2}{9}}\right)=\tan^{-1}\left(\frac{\frac{9+8}{36}}{1-\frac{2}{36}}\right)=\tan^{-1}\left(\frac{\frac{17}{36}}{\frac{34}{36}}\right) \qquad \left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$=\tan^{-1}\frac{17}{34}=\tan^{-1}\frac{1}{2}=\frac{1}{2}\left(2\tan^{-1}\frac{1}{2}\right)$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{1-\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2}\right)=\frac{1}{2}\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right) \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{\frac{3}{4}}{\frac{5}{4}}\right)=\frac{1}{2}\cos^{-1}\frac{3}{5}=RHS$$

8. Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x\in(0,1)$

Ans:

$$LHS=\tan^{-1}\sqrt{x}=\frac{1}{2}\left(2\tan^{-1}\sqrt{x}\right)$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{1-\left(\sqrt{x}\right)^2}{1+\left(\sqrt{x}\right)^2}\right) \qquad \left[\because 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)=RHS$$

9. Prove that $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\left(\frac{1}{3}\right)=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$

Ans:

$$LHS=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)=\frac{9}{4}\cos^{-1}\frac{1}{3}$$

$$Let\ \ \cos^{-1}\frac{1}{3}=x\Rightarrow\cos x=\frac{1}{3}\Rightarrow\sin x=\sqrt{1-\cos^2 x}$$

$$\Rightarrow\sin x=\sqrt{1-\left(\frac{1}{3}\right)^2}=\sqrt{1-\frac{1}{9}}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3}$$

$$\Rightarrow x=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)\Rightarrow\cos^{-1}\frac{1}{3}=\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\therefore\frac{9}{4}\cos^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)=RHS$$

10. Find the principal value of $\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$

Ans:

$$\tan^{-1}\sqrt{3}-\sec^{-1}(-2)$$

$$=\tan^{-1}\left(\tan\frac{\pi}{3}\right)-\sec^{-1}\left(-\sec\frac{\pi}{3}\right)$$

$$=\frac{\pi}{3}-\sec^{-1}\left(\sec\left(\pi-\frac{\pi}{3}\right)\right)=\frac{\pi}{3}-\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

$$=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$$

11. Prove that : $\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}$

Ans:

Let $\sin^{-1}\frac{3}{5}=x$ and $\cot^{-1}\frac{3}{2}=y$

Then $\sin x=\frac{3}{5}$ and $\cot y=\frac{3}{2}$

Now $\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$

and $\sin y=\frac{1}{\sqrt{1+\cot^2 y}}=\frac{1}{\sqrt{1+\left(\frac{3}{2}\right)^2}}=\frac{1}{\sqrt{1+\frac{9}{4}}}=\frac{1}{\sqrt{\frac{13}{4}}}=\frac{2}{\sqrt{13}}$

$\Rightarrow \cos y=\frac{3}{\sqrt{13}}$

$LHS=\cos(x+y)=\cos x\cos y-\sin x\sin y=\frac{4}{5}\times\frac{3}{\sqrt{13}}-\frac{3}{5}\times\frac{2}{\sqrt{13}}$

$=\frac{12}{5\sqrt{13}}-\frac{6}{5\sqrt{13}}=\frac{6}{5\sqrt{13}}=RHS$

Write the value of $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

Ans:

Let $2\tan^{-1}\frac{1}{5}=x\Rightarrow\tan^{-1}\frac{1}{5}=\frac{x}{2}\Rightarrow\tan\frac{x}{2}=\frac{1}{5}$

$\tan\left(2\tan^{-1}\frac{1}{5}\right)=\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}=\frac{2}{5}\times\frac{25}{24}=\frac{5}{12}$

12. Find the value of the following: $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right], |x|<1, y>0 \;\; and \;\; xy<1$

Ans:

$Let \;\; x=\tan\alpha \;\; and \;\; y=\tan\beta\Rightarrow\alpha=\tan^{-1}x, \beta=\tan^{-1}y$

$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right]$

$=\tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\alpha}{1+\tan^2\alpha}+\cos^{-1}\frac{1-\tan^2\beta}{1+\tan^2\beta}\right]$

$=\tan\frac{1}{2}\left[\sin^{-1}(\sin 2\alpha)+\cos^{-1}(\cos 2\beta)\right] \quad \left[\because \sin 2\alpha=\frac{2\tan\alpha}{1+\tan^2\alpha} \;\; and \;\; \cos 2\beta=\frac{1-\tan^2\beta}{1+\tan^2\beta}\right]$

$=\tan\frac{1}{2}\left[2\alpha+2\beta\right]=\tan\left[\alpha+\beta\right]=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha.\tan\beta}=\frac{x+y}{1-xy}$

13. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$

Ans:

$$\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$$

$$= \tan^{-1}\left[2\sin\left(2\frac{\pi}{6}\right)\right] = \tan^{-1}\left[2\sin\left(\frac{\pi}{3}\right)\right] = \tan^{-1}\left[2\times\frac{\sqrt{3}}{2}\right]$$

$$= \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

14. Prove that $\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$

Ans:

$Let \ \sin^{-1}\frac{3}{4} = \alpha \Rightarrow \sin\alpha = \frac{3}{4}$

$$\Rightarrow \frac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \frac{3}{4} \qquad \left[\because \sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}\right]$$

$$\Rightarrow 3+3\tan^2\frac{\alpha}{2} = 8\tan\frac{\alpha}{2} \Rightarrow 3\tan^2\frac{\alpha}{2} - 8\tan\frac{\alpha}{2} + 3 = 0$$

$$\Rightarrow \tan\frac{\alpha}{2} = \frac{8\pm\sqrt{64-36}}{6} = \frac{8\pm\sqrt{28}}{6}$$

$$\Rightarrow \tan\frac{\alpha}{2} = \frac{8\pm 2\sqrt{7}}{6} \Rightarrow \tan\frac{\alpha}{2} = \frac{4\pm\sqrt{7}}{3}$$

$$\Rightarrow \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$

15. If $y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$**, then prove that** $\sin y = \tan^2\frac{x}{2}$

Ans:

$$y = \cot^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right)$$

$$\Rightarrow y = \frac{\pi}{2} - \tan^{-1}\left(\sqrt{\cos x}\right) - \tan^{-1}\left(\sqrt{\cos x}\right) = \frac{\pi}{2} - 2\tan^{-1}\left(\sqrt{\cos x}\right)$$

$$\Rightarrow y = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-\cos x}{1+\cos x}\right) \qquad \left[\because 2\tan^{-1}x = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$$

$$\Rightarrow y = \sin^{-1}\left(\frac{1-\cos x}{1+\cos x}\right)$$

$$\Rightarrow \sin y = \frac{1-\cos x}{1+\cos x} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \tan^2\frac{x}{2}$$

16. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$**, then find the value of x.**

Ans:

$$\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1 \Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \sin^{-1}1$$

$$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1}x = \frac{\pi}{2} \Rightarrow \sin^{-1}\frac{1}{5} = \frac{\pi}{2} - \cos^{-1}x = \sin^{-1}x \qquad \Rightarrow x = \frac{1}{5}$$

17. Prove that $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

Ans:

$$LHS=2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}$$

$$=2\tan^{-1}\frac{1}{5}+2\tan^{-1}\frac{1}{8}+\sec^{-1}\frac{5\sqrt{2}}{7}=2\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right)+\sec^{-1}\frac{5\sqrt{2}}{7}$$

$$=2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}.\frac{1}{8}}\right)+\tan^{-1}\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1}$$

$$=2\tan^{-1}\left(\frac{\frac{8+5}{40}}{1-\frac{1}{40}}\right)+\tan^{-1}\sqrt{\frac{50}{49}-1}$$

$$=2\tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right)+\tan^{-1}\sqrt{\frac{1}{49}}=2\tan^{-1}\left(\frac{13}{39}\right)+\tan^{-1}\frac{1}{7}$$

$$=2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right)+\tan^{-1}\frac{1}{7}\qquad\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$=\tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)+\tan^{-1}\frac{1}{7}$$

$$=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{1}{7}=\tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}.\frac{1}{7}}\right)\qquad\left(\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-x.y}\right)\right)$$

$$=\tan^{-1}\left(\frac{\frac{21+4}{28}}{1-\frac{3}{28}}\right)=\tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)=\tan^{-1}1=\frac{\pi}{4}$$

18. If $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$ **then write the value of** $x+y+xy$.

Ans:

$$\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$$

$$\Rightarrow\tan^{-1}\left(\frac{x+y}{1-x.y}\right)=\frac{\pi}{4}\Rightarrow\frac{x+y}{1-x.y}=\tan\frac{\pi}{4}=1$$

$$\Rightarrow x+y=1-xy\Rightarrow x+y+xy=1$$

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. Which of the following corresponds to the principal value branch of $\tan^{-1}x$?

(a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (c) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ (d) $(0, \pi)$

2. Which of the following is the principal value branch of $\cos^{-1}x$?

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $(0, \pi) - \left\{\frac{\pi}{2}\right\}$ (c) $(0, \pi)$ (d) $[0, \pi]$

3. Which of the following is the principal value branch of $\text{cosec}^{-1}x$?

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ (b) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4. The principal value branch of $\sec^{-1}x$ is

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ (b) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (c) $(0, \pi)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

5. One branch of $\cos^{-1}$ other than the principal value branch corresponds to

(a) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ (b) $[\pi, 2\pi] - \left\{\frac{3\pi}{2}\right\}$ (c) $(0, \pi)$ (d) $[2\pi, 3\pi]$

6. The value of $\sin^{-1}\left(\cos\left(\frac{43\pi}{5}\right)\right)$

(a) $\frac{3\pi}{5}$ (b) $\frac{-7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $-\frac{\pi}{10}$

7. The principal value of the expression $\cos^{-1}[\cos(-680°)]$ is

(a) $\frac{2\pi}{9}$ (b) $\frac{-2\pi}{9}$ (c) $\frac{34\pi}{9}$ (d) $\frac{\pi}{9}$

8. The value of $\cot(\sin^{-1}x)$ is

(a) $\frac{\sqrt{1+x^2}}{x}$ (b) $\frac{x}{\sqrt{1+x^2}}$ (c) $\frac{1}{x}$ (d) $\frac{\sqrt{1-x^2}}{x}$

9. If $\tan^{-1}x = \frac{\pi}{10}$ for some $x \in R$, then the value of $\cot^{-1}x$ is

(a) $\frac{\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{3\pi}{5}$ (d) $\frac{4\pi}{5}$

10. The domain of $\sin^{-1} 2x$ is

(a) $[0, 1]$ (b) $[-1, 1]$ (c) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (d) $[-2, 2]$

11. The principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $-\frac{\pi}{3}$ (c) $-\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

12. If $3\tan^{-1} x + \cot^{-1} x = \pi$, then x equals

(a) 0 (b) 1 (c) –1 (d) $\frac{1}{2}$

13. The domain of the function $\cos^{-1}(2x - 1)$ is

(a) [0, 1] (b) [–1, 1] (c) (–1, 1) (d) $[0, \pi]$

14. The domain of the function defined by $f(x) = \sin^{-1}\sqrt{x-1}$ is

(a) [1, 2] (b) [–1, 1] (c) [0, 1] (d) none of these

15. If $\cos\left(\sin^{-1}\frac{2}{5}+\cos^{-1}x\right) = 0$, then x is equal to

(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) 0 (d) 1

16. The value of $\sin(2\tan^{-1}(0.75))$ is equal to

(a) 0.75 (b) 1.5 (c) 0.96 (d) sin 1.5

17. The value of $\sin^{-1}\left(\cos\left(\frac{33\pi}{5}\right)\right)$ is

(a) $\frac{3\pi}{5}$ (b) $\frac{-7\pi}{5}$ (c) $\frac{\pi}{10}$ (d) $\frac{-\pi}{10}$

18. The value of $\cos^{-1}\left(\cos\frac{3\pi}{2}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

19. The value of the expression $2\sec^{-1} 2 + \sin^{-1}\frac{1}{2}$ is

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$ (c) $\frac{7\pi}{6}$ (d) 1

20. If $\tan^{-1} x + \tan^{-1} y = \frac{4\pi}{5}$, then $\cot^{-1} x + \cot^{-1} y$ equals

(a) $\frac{\pi}{5}$ (b) $\frac{2\pi}{5}$ (c) $\frac{3\pi}{5}$ (d) π

21. If $\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where a, x $\in$]0, 1, then the value of x is

(a) 0 (b) $\frac{a}{2}$ (c) a (d) $\frac{2a}{1-a^2}$

22. The greatest and least values of $(\sin^{-1}x)^2 + (\cos^{-1}x)^2$ are respectively

(a) $\frac{5\pi^2}{4}$ and $\frac{\pi^2}{8}$ (b) $\frac{\pi}{2}$ and $\frac{-\pi}{2}$ (c) $\frac{\pi^2}{4}$ and $\frac{-\pi^2}{4}$ (d) $\frac{\pi^2}{4}$ and 0

23. Let θ = $\sin^{-1}$ (sin (– 600°), then value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{-2\pi}{3}$

24. The domain of the function y = $\sin^{-1}(-x^2)$ is

(a) [0, 1] (b) (0, 1) (c) [–1, 1] (d) φ

25. The domain of y = $\cos^{-1}(x^2 - 4)$ is

(a) [3, 5] (b) [0, π] (c) $\left[-\sqrt{5},-\sqrt{3}\right]\cap\left[-\sqrt{5},\sqrt{3}\right]$ (d) $\left[-\sqrt{5},-\sqrt{3}\right]\cup\left[\sqrt{3},\sqrt{5}\right]$

26. The domain of the function defined by f (x) = $\sin^{-1}$x + cosx is

(a) [–1, 1] (b) [–1, π + 1] (c) (–∞,∞) (d) φ

27. The value of sin (2 $\sin^{-1}$ (0.6)) is

(a) 0.48 (b) 0.96 (c) 1.2 (d) sin 1.2

28. If $\sin^{-1}$ x + $\sin^{-1}$ y = $\frac{\pi}{2}$, then value of $\cos^{-1}$ x + $\cos^{-1}$ y is

(a) $\frac{\pi}{2}$ (b) π (c) 0 (d) $\frac{2\pi}{3}$

29. The value of tan ($\cos^{-1}\frac{3}{5}$ + $\tan^{-1}\frac{1}{4}$) is

(a) $\frac{19}{8}$ (b) $\frac{8}{19}$ (c) $\frac{19}{12}$ (d) $\frac{3}{4}$

30. The value of the expression sin [$\cot^{-1}$ (cos ($\tan^{-1}$ 1))] is

(a) 0 (b) 1 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{\frac{2}{3}}$

31. The equation $\tan^{-1}$x – $\cot^{-1}$x = $\tan^{-1}\frac{1}{\sqrt{3}}$ has

(a) no solution (b) unique solution (c) infinite number of solutions (d) two solutions

32. If α ≤ 2 $\sin^{-1}$x + $\cos^{-1}$x ≤ β , then

(a) $\alpha = \frac{-\pi}{2}, \beta = \frac{\pi}{2}$ (b) α = 0, β = π (c) $\alpha = \frac{-\pi}{2}, \beta = \frac{3\pi}{2}$ (d) α = 0, β = 2π

33. If $\cos^{-1}$ α + $\cos^{-1}$ β + $\cos^{-1}$ γ = 3π, then α (β + γ) + β (γ + α) + γ (α + β) equals

(a) 0 (b) 1 (c) 6 (d) 12

34. The value of $\tan^2(\sec^{-1}2) + \cot^2(\mathrm{cosec}^{-1}3)$ is

(a) 5 (b) 11 (c) 13 (d) 15

35. The number of real solutions of the equation $\sqrt{1+\cos 2x} = \sqrt{2}\cos^{-1}(\cos x)$ in $\left[\frac{\pi}{2}, \pi\right]$ is

(a) 0 (b) 1 (c) 2 (d) Infinite

36. The principal value of $\cos^{-1}$x, for x = $\frac{\sqrt{3}}{2}$ is ____________

37. The value of $\tan^{-1}\sin\frac{-\pi}{2}$ is ____________

38. The value of $\cos^{-1}\cos\frac{13\pi}{6}$ is ____________

39. The value of $\tan^{-1}\tan\frac{9\pi}{8}$ is ____________

40. The value of tan ($\tan^{-1}(-4)$) is ____________

41. The value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ is ____________

42. The value of $\sin^{-1}(\cos(\sin^{-1}\frac{\sqrt{3}}{2}))$ is ____________

43. The value of $\sec\left(\tan^{-1}\frac{y}{2}\right)$ is ____________

44. The value of $\sin 2\cot^{-1}\frac{-5}{12}$ is ____________

45. The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is___________.

46. The value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$ is___________.

47. If $\cos(\tan^{-1}x + \cot^{-1}\sqrt{3}) = 0$, then value of x is___________.

48. The set of values of $\sec^{-1}\left(\frac{1}{2}\right)$ is___________.

49. The principal value of $\tan^{-1}\sqrt{3}$ is___________.

50. The value of $\cos^{-1}\left(\cos\frac{14\pi}{3}\right)$ is___________.

51. The value of $\cos(\sin^{-1}x + \cos^{-1}x)$, $|x| \le 1$ is_______ .

52. The value of expression $\tan\left(\frac{\sin^{-1}x + \cos^{-1}x}{2}\right)$,when $x = \frac{\sqrt{3}}{2}$ is__________.

53. If $y = 2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ for all x, then____< y <____.

54. The result $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$ is true when value of xy is _____

55. The value of $\cot^{-1}(-x)$ for all $x \in R$ in terms of $\cot^{-1}x$ is _______.

MATRICES

CHAPTER – 3: MATRICES

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. If a matrix has 18 elements, what are the possible orders it can have? What, if it has 5 elements?

Ans:

Since, a matrix containing 18 elements can have any one of the following orders :

$$1 \times 18, 18 \times 1, 2 \times 9, 9 \times 2, 3 \times 6, 6 \times 3$$

Similarly, a matrix containing 5 elements can have order 1×5 or 5×1.

2. Construct a 3 × 4 matrix, whose elements are given by:

(i) $a_{ij} = \frac{1}{2}|-3i + j|$ (ii) $a_{ij} = 2i - j$

Ans:

(i) The order of given matrix is 3 × 4, so the required matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}, \text{ where } a_{ij} = \frac{1}{2}|-3i + j|$$

Putting the values in place of i and j, we will find all the elements of matrix A.

$$a_{11} = \frac{1}{2}|-3+1| = 1,\ a_{12} = \frac{1}{2}|-3+2| = \frac{1}{2},\ a_{13} = \frac{1}{2}|-3+3| = 0$$

$$a_{14} = \frac{1}{2}|-3+4| = \frac{1}{2},\ a_{21} = \frac{1}{2}|-6+1| = \frac{5}{2},\ a_{22} = \frac{1}{2}|-6+2| = 2$$

$$a_{23} = \frac{1}{2}|-6+3| = \frac{3}{2},\ a_{24} = \frac{1}{2}|-6+4| = 1,\ a_{31} = \frac{1}{2}|-9+1| = 4$$

$$a_{32} = \frac{1}{2}|-9+2| = \frac{7}{2},\ a_{33} = \frac{1}{2}|-9+3| = 3,\ a_{34} = \frac{1}{2}|-9+4| = \frac{5}{2}$$

Hence, the required matrix is $A = \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{5}{2} & 2 & \frac{3}{2} & 1 \\ 4 & \frac{7}{2} & 3 & \frac{5}{2} \end{bmatrix}_{3\times 4}$

(ii) Here, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}_{3\times 4}$,where $a_{ij} = 2i - j$

$a_{11} = 2 - 1 = 1,$ $a_{12} = 2 - 2 = 0,$
$a_{13} = 2 - 3 = -1,$ $a_{14} = 2 - 4 = -2,$
$a_{21} = 4 - 1 = 3,$ $a_{22} = 4 - 2 = 2,$
$a_{23} = 4 - 3 = 1,$ $a_{24} = 4 - 4 = 0$
$a_{31} = 6 - 1 = 5,$ $a_{32} = 6 - 2 = 4,$
$a_{33} = 6 - 3 = 3$ and $a_{34} = 6 - 4 = 2$

Hence, the required matrix is $A=\begin{bmatrix}1 & 0 & -1 & -2\\ 3 & 2 & 1 & 0\\ 5 & 4 & 3 & 2\end{bmatrix}_{3\times 4}$

3. Find the value of *a*, *b*, *c* and *d* from the equation: $\begin{bmatrix}a-b & 2a+c\\ 2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$

Ans:

Given that $\begin{bmatrix}a-b & 2a+c\\ 2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\ 0 & 13\end{bmatrix}$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$a - b = -1$ …(i)

$2a - b = 0$ …(ii)

$2a + c = 5$ …(iii)

and $3c + d = 13$ …(iv)

Subtracting Eq.(i) from Eq.(ii), we get $a = 1$

Putting $a = 1$ in Eq. (i) and Eq. (iii), we get

$1 - b = -1$ and $2 + c = 5$

$\Rightarrow b = 2$ and $c = 3$

Substituting $c = 3$ in Eq. (iv), we obtain

$3 \times 3 + d = 13 \Rightarrow d = 13 - 9 = 4$

Hence, $a = 1, b = 2, c = 3$ and $d = 4$.

4. Find X and Y, if X + Y = $\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}$ **and X – Y =** $\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$**.**

Ans:

$(X+Y)+(X-Y)=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}+\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$

$\Rightarrow 2X=\begin{bmatrix}8 & 8\\ 0 & 8\end{bmatrix}\Rightarrow X=\frac{1}{2}\begin{bmatrix}8 & 8\\ 0 & 8\end{bmatrix}$

$\Rightarrow X=\begin{bmatrix}4 & 4\\ 0 & 4\end{bmatrix}$

$Now,(X+Y)-(X-Y)=\begin{bmatrix}5 & 2\\ 0 & 9\end{bmatrix}-\begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}$

$\Rightarrow 2Y=\begin{bmatrix}2 & -4\\ 0 & 10\end{bmatrix}\Rightarrow X=\frac{1}{2}\begin{bmatrix}2 & -4\\ 0 & 10\end{bmatrix}$

$\Rightarrow Y=\begin{bmatrix}1 & -2\\ 0 & 5\end{bmatrix}$

5. Find the values of *x* and *y* from the following equation:

$2\begin{bmatrix}x & 5\\ 7 & y-3\end{bmatrix}+\begin{bmatrix}3 & -4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 6\\ 15 & 14\end{bmatrix}$

Ans:

$2\begin{bmatrix}x & 5\\ 7 & y-3\end{bmatrix}+\begin{bmatrix}3 & -4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 6\\ 15 & 14\end{bmatrix}$

$\Rightarrow\begin{bmatrix}2x & 10\\ 14 & 2y-6\end{bmatrix}+\begin{bmatrix}3 & -4\\ 1 & 2\end{bmatrix}=\begin{bmatrix}7 & 6\\ 15 & 14\end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

or $2x + 3 = 7$ and $2y - 4 = 14$

or $2x = 7 - 3$ and $2y = 18$

or $x = \frac{4}{2}$ and $y = \frac{18}{2}$

i.e. $x = 2$ and $y = 9$.

6. Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ **and** $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$**.**

Ans:

We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Thus, if the product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

7. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$**, then show that** $A^3 - 23A - 40I = O$

Ans:

$$A^2 = A.A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$$So, A^3 = A.A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix}$$

$$Now, A^3 - 23A - 40I = \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} + \begin{bmatrix} -23 & -46 & -69 \\ -69 & 46 & -23 \\ -92 & -46 & -23 \end{bmatrix} + \begin{bmatrix} -40 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-46+0 & 69-69+0 \\ 69-69+0 & -6+46-40 & 23-23+0 \\ 92-92+0 & 46-46+0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

8. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$**, find the values of *x* and *y*.**

Ans:

$$x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2x-y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$2x - y = 10$ …(i)

and $3x + y = 5$ …(ii)

Adding Eqs. (i) and (ii), we get

$5x = 15 \Rightarrow x = 3$

Substituting $x = 3$ in Eq. (i), we get

$2 \times 3 - y = 10 \Rightarrow y = 6 - 10 = -4$

9. Given $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, find the values of *x*, *y*, *z* and *w*.

Ans:

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$3x = x + 4 \Rightarrow 2x = 4 \Rightarrow x = 2$

and $3y = 6 + x + y \Rightarrow 2y = 6 + x \Rightarrow y = \frac{6+x}{2}$

Putting the value of *x*, we get

$$y = \frac{6+2}{2} = \frac{8}{2} = 4$$

Now, $3z = -1 + z + w$, $2z = -1 + w$

$$z = \frac{-1+w}{2} \quad \ldots(i)$$

Now, $3w = 2w + 3 \Rightarrow w = 3$

Putting the value of *w* in Eq. (i), we get

$$z = \frac{-1+3}{2} = \frac{2}{2} = 1$$

Hence, the values of *x*, *y*, *z* and *w* are 2, 4, 1 and 3.

10. If $F(x) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}$, show that F(*x*) F(*y*) = F(*x* + *y*).

Ans:

$$LHS = F(x)F(y) = \begin{bmatrix} \cos x & -\sin x & 0 \\ \sin x & \cos x & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos y & -\sin y & 0 \\ \sin y & \cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos x\cos y - \sin x\sin y & -\sin y\cos x - \sin x\cos y & 0 \\ \sin x\cos y + \cos x\sin y & -\sin x\sin y + \cos x\cos y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(x+y) & -\sin(x+y) & 0 \\ \sin(x+y) & \cos(x+y) & 0 \\ 0 & 0 & 1 \end{bmatrix} = F(x+y) = RHS$$

11. Find $A^2 - 5A + 6I$, if $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$

Ans:

$$A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$$

$$\therefore A^2 - 5A + 6I = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 5-10+6 & -1-0+0 & 2-5+0 \\ 9-10+0 & -2-5+6 & 5-15+0 \\ 0-5+0 & -1+5+0 & -2+0+6 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

12. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, prove that $A^3 - 6A^2 + 7A + 2I = 0$

Ans:

$$A^2 = A.A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}$$

$$A^3 = A^2.A = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix}$$

$$\therefore A^3 - 6A^2 + 7A + 2I = \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - 6\begin{bmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{bmatrix} - \begin{bmatrix} 30 & 0 & 48 \\ 12 & 24 & 30 \\ 48 & 0 & 78 \end{bmatrix} + \begin{bmatrix} 7 & 0 & 14 \\ 0 & 14 & 7 \\ 14 & 0 & 21 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 21-30+7+2 & 0-0+0+0 & 34-48+14+0 \\ 12-12+0+0 & 8-24+14+2 & 23-30+7+0 \\ 34-48+14+0 & 0-0+0+0 & 55-78+21+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

13. If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find *k* so that $A^2 = kA - 2I$

Ans:

Given than $A^2 = kA - 2I$

$$\Rightarrow \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix} = \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} = \begin{bmatrix} 3k-2 & -2k \\ 4k & -2k-2 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. Comparing the corresponding elements, we get

$3k - 2 = 1 \Rightarrow k = 1$
$-2k = -2 \Rightarrow k = 1$
$4k = 4 \Rightarrow k = 1$
$-4 = -2k - 2 \Rightarrow k = 1$

Hence, $k = 1$

14. If $A = \begin{bmatrix} 0 & -\tan\frac{\alpha}{2} \\ \tan\frac{\alpha}{2} & 0 \end{bmatrix}$ **and I is the identity matrix of order 2, show that**

I + A = (I – A)$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Ans:

Let $A = \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}$ *where* $x = \tan\frac{\alpha}{2}$

Now, $\cos\alpha = \dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \dfrac{1-x^2}{1+x^2}$ *and* $\cos\alpha = \dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}} = \dfrac{2x}{1+x^2}$

$$RHS = (I-A)\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix}\right)\begin{bmatrix} \frac{1-x^2}{1+x^2} & -\frac{2x}{1+x^2} \\ \frac{2x}{1+x^2} & \frac{1-x^2}{1+x^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}\begin{bmatrix} \frac{1-x^2}{1+x^2} & -\frac{2x}{1+x^2} \\ \frac{2x}{1+x^2} & \frac{1-x^2}{1+x^2} \end{bmatrix} = \begin{bmatrix} \frac{1-x^2+2x^2}{1+x^2} & \frac{-2x+x(1-x^2)}{1+x^2} \\ \frac{-x(1-x^2)+2x}{1+x^2} & \frac{2x^2+1-x^2}{1+x^2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+x^2}{1+x^2} & \frac{-2x+x-x^3}{1+x^2} \\ \frac{-x+x^3+2x}{1+x^2} & \frac{1+x^2}{1+x^2} \end{bmatrix} = \begin{bmatrix} 1 & \frac{-x-x^3}{1+x^2} \\ \frac{x^3+x}{1+x^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{-x(1+x^2)}{1+x^2} \\ \frac{x(x^2+1)}{1+x^2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix}$$

$$LHS = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -x \\ x & 0 \end{bmatrix} = \begin{bmatrix} 1 & x \\ -x & 1 \end{bmatrix} = RHS$$

15. Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ **as the sum of a symmetric and a skew symmetric matrix.**

Ans:

$$B=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}\Rightarrow B'=\begin{bmatrix}2 & -1 & 1\\ -2 & 3 & -2\\ -4 & 4 & -3\end{bmatrix}$$

$$Let\ \ P=\frac{1}{2}(B+B')=\frac{1}{2}\begin{bmatrix}4 & -3 & -3\\ -3 & 6 & 2\\ -3 & 2 & -6\end{bmatrix}=\begin{bmatrix}2 & -3/2 & -3/2\\ -3/2 & 3 & 1\\ -3/2 & 1 & -3\end{bmatrix}$$

$$Now\ \ P'=\begin{bmatrix}2 & -3/2 & -3/2\\ -3/2 & 3 & 1\\ -3/2 & 1 & -3\end{bmatrix}=P$$

$Thus\ \ P=\frac{1}{2}(B+B')$ is a symmetric matrix.

$$Also, let\ \ Q=\frac{1}{2}(B-B')=\frac{1}{2}\begin{bmatrix}0 & -1 & -5\\ 1 & 0 & 6\\ 5 & -6 & 0\end{bmatrix}=\begin{bmatrix}0 & -1/2 & -5/2\\ 1/2 & 0 & 3\\ 5/2 & -3 & 0\end{bmatrix}$$

$$Now\ \ Q'=\begin{bmatrix}0 & 1/2 & 5/2\\ -1/2 & 0 & -3\\ -5/2 & 3 & 0\end{bmatrix}=-Q$$

$Thus\ \ Q=\frac{1}{2}(B-B')$ is a skew symmetric matrix.

$$Now, P+Q=\begin{bmatrix}2 & -3/2 & -3/2\\ -3/2 & 3 & 1\\ -3/2 & 1 & -3\end{bmatrix}+\begin{bmatrix}0 & -1/2 & -5/2\\ 1/2 & 0 & 3\\ 5/2 & -3 & 0\end{bmatrix}=\begin{bmatrix}2 & -2 & -4\\ -1 & 3 & 4\\ 1 & -2 & -3\end{bmatrix}=B$$

Thus, B is represented as the sum of a symmetric and a skew symmetric matrix.

16. Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

$$(i)\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}\qquad (ii)\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}\qquad (iii)\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}\qquad (iv)\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}$$

Ans:

(i)

Let $A=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}, then\ \ A=P+Q$

$$where,\ \ P=\frac{1}{2}(A+A')\ \ and\ \ Q=\frac{1}{2}(A-A')$$

$$Now, P=\frac{1}{2}(A+A')=\frac{1}{2}\left(\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}+\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}6 & 6\\ 6 & -2\end{bmatrix}=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}$$

$$\therefore P'=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}=P$$

$Thus\ \ P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$Now, Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = -Q$$

$Thus\ \ Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q,

$$P + Q = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} = A$$

(ii)

Let $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}, then\ \ A = P + Q$

$where,\ \ P = \frac{1}{2}(A + A')\ \ and\ \ Q = \frac{1}{2}(A - A')$

$$Now, P = \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$\therefore P' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = P$$

$Thus\ \ P = \frac{1}{2}(A + A')$ is a symmetric matrix.

$$Now, Q = \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore Q' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = -Q$$

$Thus\ \ Q = \frac{1}{2}(A - A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q,

$$P + Q = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = A$$

(iii)

Let $A = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}, then\ \ A = P + Q$

$where,\ \ P = \frac{1}{2}(A + A')\ \ and\ \ Q = \frac{1}{2}(A - A')$

$$Now, P=\frac{1}{2}(A+A')=\frac{1}{2}\left(\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}+\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}6 & 1 & -5\\ 1 & -4 & -4\\ -5 & -4 & 4\end{bmatrix}=\begin{bmatrix}3 & 1/2 & -5/2\\ 1/2 & -2 & -2\\ -5/2 & -2 & 2\end{bmatrix}$$

$$\therefore P'=\begin{bmatrix}3 & 1/2 & -5/2\\ 1/2 & -2 & -2\\ -5/2 & -2 & 2\end{bmatrix}=\begin{bmatrix}3 & 1/2 & -5/2\\ 1/2 & -2 & -2\\ -5/2 & -2 & 2\end{bmatrix}=P$$

Thus $P=\frac{1}{2}(A+A')$ is a symmetric matrix.

$$Now, Q=\frac{1}{2}(A-A')=\frac{1}{2}\left(\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}-\begin{bmatrix}3 & -2 & -4\\ 3 & -2 & -5\\ -1 & 1 & 2\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}0 & 5 & 3\\ -5 & 0 & 6\\ -3 & -6 & 0\end{bmatrix}=\begin{bmatrix}0 & 5/2 & 3/2\\ -5/2 & 0 & 2\\ -3/2 & -2 & 0\end{bmatrix}$$

$$\therefore Q'=\begin{bmatrix}0 & -5/2 & -3/2\\ 5/2 & 0 & 2\\ 3/2 & -2 & 0\end{bmatrix}=-Q$$

Thus $Q=\frac{1}{2}(A-A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q,

$$P+Q=\begin{bmatrix}3 & 1/2 & -5/2\\ 1/2 & -2 & -2\\ -5/2 & -2 & 2\end{bmatrix}+\begin{bmatrix}0 & 5/2 & 3/2\\ -5/2 & 0 & 2\\ -3/2 & -2 & 0\end{bmatrix}=\begin{bmatrix}3 & 3 & -1\\ -2 & -2 & 1\\ -4 & -5 & 2\end{bmatrix}=A$$

(iv)

Let $A=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}, then \ \ A=P+Q$

where, $P=\frac{1}{2}(A+A')$ *and* $Q=\frac{1}{2}(A-A')$

$$Now, P=\frac{1}{2}(A+A')=\frac{1}{2}\left(\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}+\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}2 & 4\\ 4 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$$

$$\therefore P'=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}=P$$

Thus $P=\frac{1}{2}(A+A')$ is a symmetric matrix.

$$Now, Q=\frac{1}{2}(A-A')=\frac{1}{2}\left(\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}-\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}\right)=\frac{1}{2}\begin{bmatrix}0 & 6\\ -6 & 0\end{bmatrix}=\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}$$

$$\therefore Q'=\begin{bmatrix}0 & -3\\ 3 & 0\end{bmatrix}=-Q$$

Thus $Q=\frac{1}{2}(A-A')$ is a skew symmetric matrix.

Representing A as the sum of P and Q,

$$P+Q=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}+\begin{bmatrix}0 & 3\\ -3 & 0\end{bmatrix}=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}=A$$

17. If A = $\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$ and A + A′ = I, find the value of α .

Ans:

$$A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\Rightarrow A=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$$

$$Now, A+A'=I$$

$$\Rightarrow\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}+\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}2\cos\alpha & 0\\ 0 & 2\cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Comparing the corresponding elements of the above matrices, we have

$$2\cos\alpha=1\Rightarrow\cos\alpha=\frac{1}{2}=\cos\frac{\pi}{3}\Rightarrow\alpha=\frac{\pi}{3}$$

18. Show that the matrix *B′ AB* is symmetric or skew-symmetric according as *A* is symmetric or skew-symmetric.

Ans:

We suppose that *A* is a symmetric matrix, then $A' = A$

Consider $(B'AB)' = \{B'(AB)\}' = (AB)'(B')'$ $[\because (AB)' = B'A']$

$= B'A'$ (b) $[\because (B')' = B]$

$= B'(A'B) = B'(AB)$ $[\because A' = A]$

$\Rightarrow (B'AB)' = B'AB$

which shows that *B′ AB* is a symmetric matrix.

Now, we suppose that *A* is a skew-symmetric matrix.

Then, $A' = -A$

Consider $(B'AB)' = [B'(AB)]' = (AB)'(B')'$ $[\because (AB)' = B'A'$ and $(A')' = A]$

$= (B'A')B = B'(-A)B = -B'AB$ $[\because A' = -A]$

$\Rightarrow (B'AB)' = -B'AB$

which shows that *B′ AB* is a skew-symmetric matrix.

19. If *A* and *B* are symmetric matrices, prove that *AB* − *BA* is a skew-symmetric matrix.

Ans:

Here, *A* and *B*are symmetric matrices, then $A' = A$ and $B' = B$

Now, $(AB - BA)' = (AB)' - (BA)'$ $(\because (A - B)' = A' - B'$ and $(AB)' = B'A')$

$= B'A' - A'B' = BA - AB$ $(\because B' = B$ and $A' = A)$

$= -(AB - BA)$

$\Rightarrow (AB - BA)' = -(AB - BA)$

Thus, $(AB - BA)$ is a skew-symmetric matrix.

20. If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, then prove that $A^n=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$, where *n* is any positive integer.

Ans:

We are required to prove that for all $n\in N$

$$P(n)=\begin{bmatrix}1+2n & -4n\\ n & 1-2n\end{bmatrix}$$

Let n = 1, then $P(1)=\begin{bmatrix}1+2(1) & -4(1)\\ 1 & 1-2(1)\end{bmatrix}=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$ ....(*i*)

which is true for *n* = 1.

Let the result be true for $n = k$.

$$P(k) = A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \text{........}(ii)$$

Let $n = k + 1$

$$P(k+1) = A^{k+1} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix} = \begin{bmatrix} 2k+3 & -4k-4 \\ k+1 & -2k-1 \end{bmatrix}$$

$$Now, LHS = A^{k+1} = A^k A^1 = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}\begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1+2k).3+(-4k).1 & (1+2k).(-4)+(-4k)(-1) \\ k.3+(1-2k).1 & k.(-4)+(1-2k)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 3+2k & -4-4k \\ k+1 & -1-2k \end{bmatrix} = \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & -1-2(k+1) \end{bmatrix}$$

Therefore, the result is true for $n = k + 1$whenever it is true for $n = k$. So, by principle of mathematical induction, it is true for all $n \in N$.

21. For what values of x : $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$?

Ans:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ x \end{bmatrix} = O$$

Since Matrix multiplication is associative, therefore $\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0+4+0 \\ 0+0+x \\ 0+0+2x \end{bmatrix} = O$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ x \\ 2x \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 4+2x+2x \end{bmatrix} = O \Rightarrow \begin{bmatrix} 4+4x \end{bmatrix} = O$$

$$\Rightarrow 4+4x = 0 \Rightarrow 4x = -4 \Rightarrow x = -1$$

22. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$**, show that $A^2 - 5A + 7I = 0$.**

Ans:

Given that $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$\therefore A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$=\begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+7 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=O$$

23. Find *x*, if $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}=O$?

Ans:

Given that $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ 4 \\ 1 \end{bmatrix}=O$

Since Matrix multiplication is associative, therefore $\begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} x+0+2 \\ 0+8+1 \\ 2x+0+3 \end{bmatrix}=O$

$$\Rightarrow \begin{bmatrix} x & -5 & -1 \end{bmatrix}\begin{bmatrix} x+2 \\ 9 \\ 2x+3 \end{bmatrix}=O$$

$$\Rightarrow \begin{bmatrix} x(x+2)+(-5).9+(-1)(2x+3) \end{bmatrix}=O$$

$$\Rightarrow \begin{bmatrix} x^2-48 \end{bmatrix}=O \Rightarrow x^2-48=0 \Rightarrow x^2=48$$

$$\Rightarrow x=\pm\sqrt{48}=\pm4\sqrt{3}$$

24. Find the matrix X so that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}=\begin{bmatrix} -7 & -8 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

Ans:

Given that $X\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}=\begin{bmatrix} -7 & -8 & 3 \\ 2 & 4 & 6 \end{bmatrix}$

The matrix given on the RHS of the equation is a 2 × 3matrix and the one given on the LHS of the equation is as a 2 × 3 matrix. Therefore, *X* has to be a 2 × 2 matrix.

Now, let $X=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$\begin{bmatrix} a+4c & 2a+5c & 3a+6c \\ b+4d & 2b+5d & 3b+6d \end{bmatrix}=\begin{bmatrix} -7 & -8 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

Equating the corresponding elements of the two matrices, we have

$a + 4c = -7$, $2a + 5c = -8$, $3a + 6c = -9$

$b + 4d = 2$, $2b + 5d = 4$, $3b + 6d = 6$

Now, $a + 4c = -7 \Rightarrow a = -7 - 4c$

$2a + 5c = -8 \Rightarrow -14 - 8c + 5c = -8$

$\Rightarrow -3c = 6 \Rightarrow c = -2$

$\therefore\ a = -7 - 4(-2) = -7 + 8 = 1$

Now, $b + 4d = 2 \Rightarrow b = 2 - 4d$ and $2b + 5d = 4 \Rightarrow 4 - 8d + 5d = 4$

$\therefore\ -3d = 0 \Rightarrow d = 0$

$\therefore\ b = 2 - 4(0) = 2$

Thus, $a = 1$, $b = 2$, $c = -2$, $d = 0$

Hence, the required matrix *X* is $\begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix}$

25. If $A=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$**, then prove that** $A^n=\begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}, n\in N$

Ans:

We shall prove the result by using principle of mathematical induction.

We have P(n) : If $A=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$, then $A^n=\begin{bmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{bmatrix}, n\in N$

Let n = 1, then $P(1)=A^1=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

Therefore, the result is true for n = 1.

Let the result be true for n = k. So

$P(k)=A^k=\begin{bmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{bmatrix}$

Now, we prove that the result holds for n = k +1

i.e. $P(k+1)=A^{k+1}=\begin{bmatrix}\cos(k+1)\theta & \sin(k+1)\theta\\ -\sin(k+1)\theta & \cos(k+1)\theta\end{bmatrix}$

Now, $P(k+1)=A^{k+1}=A^k.A=\begin{bmatrix}\cos k\theta & \sin k\theta\\ -\sin k\theta & \cos k\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$

$=\begin{bmatrix}\cos\theta\cos k\theta-\sin\theta\sin k\theta & \cos\theta\sin k\theta+\sin\theta\cos k\theta\\ -\sin\theta\cos k\theta+\cos\theta\sin k\theta & -\sin\theta\sin k\theta+\cos\theta\cos k\theta\end{bmatrix}$

$=\begin{bmatrix}\cos(k\theta+\theta) & \sin(k\theta+\theta)\\ -\sin(k\theta+\theta) & \cos(k\theta+\theta)\end{bmatrix}=\begin{bmatrix}\cos(k+1)\theta & \sin(k+1)\theta\\ -\sin(k+1)\theta & \cos(k+1)\theta\end{bmatrix}$

Therefore, the result is true for n = k + 1. Thus by principle of mathematical induction, we have

$A^n=\begin{bmatrix}\cos n\theta & \sin n\theta\\ -\sin n\theta & \cos n\theta\end{bmatrix}, n\in N$

26. Let $A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}, B=\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}, C=\begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}$. Find a matrix D such that CD – AB = O.

Ans:

Since A, B, C are all square matrices of order 2, and CD – AB is well defined, D must be a square matrix of order 2.

Let $D=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, *then* $CD-AB=0\Rightarrow\begin{bmatrix}2 & 5\\ 3 & 8\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}-\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 2\\ 7 & 4\end{bmatrix}=O$

$\Rightarrow\begin{bmatrix}2a+5c & 2b+5d\\ 3a+8c & 3b+8d\end{bmatrix}-\begin{bmatrix}3 & 0\\ 43 & 22\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

$\Rightarrow\begin{bmatrix}2a+5c-3 & 2b+5d\\ 3a+8c-43 & 3b+8d-22\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

By equality of matrices, we get

$2a + 5c – 3 = 0$ ... (1)

$3a + 8c – 43 = 0$ ... (2)

$2b + 5d = 0$ ... (3)

and $3b + 8d – 22 = 0$ ... (4)

Solving (1) and (2), we get a = –191, c = 77. Solving (3) and (4), we get b = – 110, d = 44.

Therefore $D=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\begin{bmatrix}-191 & -110\\ 77 & 44\end{bmatrix}$

CHAPTER – 3: MATRICES

MARKS WEIGHTAGE – 05 marks

Previous Years Board Exam (Important Questions & Answers)

1. If $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$ **write the value of** $a - 2b$**.**

Ans:

Give that $\begin{bmatrix} a+4 & 3b \\ 8 & -6 \end{bmatrix} = \begin{bmatrix} 2a+2 & b+2 \\ 8 & a-8b \end{bmatrix}$

On equating, we get

$a + 4 = 2a + 2$, $3b = b + 2$, $a - 8b = -6$

$\Rightarrow a = 2, b = 1$

Now the value of $a - 2b = 2 - (2 \times 1) = 2 - 2 = 0$

2. If A **is a square matrix such that** $A^2 = A$**, then write the value of** $7A - (I + A)^3$ **, where** I **is an identity matrix.**

Ans:

$7A - (I + A)^3 = 7A - \{I^3 + 3I^2A + 3I.A^2 + A^3\}$

$= 7A - \{I + 3A + 3A + A^2A\}$ $\qquad [\because I^3 = I^2 = I, A^2 = A]$

$= 7A - \{I + 6A + A^2\} = 7A - \{I + 6A + A\}$

$= 7A - \{I + 7A\} = 7A - I - 7A = -I$

3. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$**, find the value of** $x + y$**.**

Ans:

Given that $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$

Equating, we get

$x - y = -1 \ldots (i)$

$2x - y = 0 \ldots (ii)$

$z = 4, w = 5$

$(ii) - (i) \Rightarrow 2x - y - x + y = 0 + 1$

$\Rightarrow x = 1$ and $y = 2$

$\therefore x + y = 2 + 1 = 3.$

4. Solve the following matrix equation for x **:** $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$

Ans:

Given that $\begin{bmatrix} x & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$

$\Rightarrow \begin{bmatrix} x-2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$\Rightarrow x - 2 = 0$

$\Rightarrow x = 2$

5. **If** $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$**, find (*x* – *y*).**

Ans:

Given that $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$

$\Rightarrow\begin{bmatrix}6 & 8\\10 & 2x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$

$\Rightarrow\begin{bmatrix}7 & 8+y\\10 & 2x+1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$

Equating we get $8 + y = 0$ and $2x + 1 = 5$

$\Rightarrow y = -8$ and $x = 2$

$\Rightarrow x - y = 2 + 8 = 10$

6. **For what value of *x*, is the matrix** $A=\begin{bmatrix}0 & 1 & -2\\-1 & 0 & 3\\x & -3 & 0\end{bmatrix}$ **a skew-symmetric matrix?**

Ans:

A will be skew symmetric matrix if

$A = -A'$

$\Rightarrow\begin{bmatrix}0 & 1 & -2\\-1 & 0 & 3\\x & -3 & 0\end{bmatrix}=-\begin{bmatrix}0 & -1 & x\\1 & 0 & -3\\-2 & 3 & 0\end{bmatrix}=\begin{bmatrix}0 & 1 & -x\\1 & 0 & 3\\2 & -3 & 0\end{bmatrix}$

Equating, we get $x = 2$

7. **If matrix** $A=\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ **and $A^2 = kA$, then write the value of k.**

Ans:

Given $A^2 = kA$

$\Rightarrow\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}=k\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

$\Rightarrow\begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}=k\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

$\Rightarrow 2\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}=k\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

$\Rightarrow k = 2$

8. **Find the value of *a* if** $\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$

Ans:

Given that $\begin{bmatrix}a-b & 2a+c\\2a-b & 3c+d\end{bmatrix}=\begin{bmatrix}-1 & 5\\0 & 13\end{bmatrix}$

Equating the corresponding elements we get.

$a - b = -1$ …(*i*)

$2a + c = 5$ …(*ii*)

$2a - b = 0$ …(*iii*)

$3c + d = 13$ …(*iv*)

From (*iii*) $2a = b$

$\Rightarrow a = \frac{b}{2}$

Putting in (*i*) we get $\frac{b}{2} - b = -1$

$\Rightarrow -\frac{b}{2} = -1 \Rightarrow b = 2$

$\therefore\ a = 1$

(*ii*) $c = 5 - 2 \times 1 = 5 - 2 = 3$

(*iv*) $d = 13 - 3 \times (3) = 13 - 9 = 4$

i.e. $a = 1$, $b = 2$, $c = 3$, $d = 4$

9. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$, then find the matrix *A*.

Ans:

Given that $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 8 & -3 & 5 \\ -2 & -3 & -6 \end{bmatrix}$

10. If *A* is a square matrix such that $A^2 = A$, then write the value of $(I + A)^2 - 3A$.

Ans:

$(I + A)^2 - 3A = I^2 + A^2 + 2A - 3A$

$= I^2 + A^2 - A$

$= I^2 + A - A\ [\because A^2 = A]$

$= I^2 = I \,.\, I = I$

11. If $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, write the value of *x*.

Ans:

Given that $x\begin{bmatrix} 2 \\ 3 \end{bmatrix} + y\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x \\ 3x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x - y \\ 3x + y \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$

Equating the corresponding elements we get.

$2x - y = 10$ ...(*i*)

$3x + y = 5$ ...(*ii*)

Adding (*i*) and (*ii*), we get $2x - y + 3x + y = 10 + 5$

$\Rightarrow 5x = 15 \Rightarrow x = 3$.

12. Find the value of $x + y$ from the following equation: $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

Ans:

Given that $2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$

Equating the corresponding element we get

$2x + 3 = 7$ and $2y - 4 = 14$

$\Rightarrow x = \frac{7-3}{2}$ *and* $y = \frac{14+4}{2}$

$\Rightarrow x = 2$ and $y = 9$

$\therefore x + y = 2 + 9 = 11$

13. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ ***and*** $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$**, then find $A^T - B^T$.**

Ans:

Given that $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$

Now, $A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$

14. If $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$**, write the value of x**

Ans:

Given that $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2-6 & -6+12 \\ 5-14 & -15+28 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -4 & 6 \\ -9 & 13 \end{bmatrix} = \begin{bmatrix} -4 & 6 \\ -9 & x \end{bmatrix}$

Equating the corresponding elements, we get

$x = 13$

15. Simplify: $\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

Ans:

$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

16. Write the values of *x – y + z* from the following equation: $\begin{bmatrix} x+y+z \\ x+z \\ y+z \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ 7 \end{bmatrix}$

Ans:

By definition of equality of matrices, we have

$x + y + z = 9$ ... (*i*)

$x + z = 5$ .... (*ii*)

$y + z = 7$ ... (*iii*)

(*i*) – (*ii*) $\Rightarrow x + y + z - x - z = 9 - 5$

$\Rightarrow y = 4$ ... (*iv*)

(*ii*) – (*iv*) $\Rightarrow x - y + z = 5 - 4$

$\Rightarrow x - y + z = 1$

17. If $\begin{bmatrix} y+2x & 5 \\ -x & 3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -2 & 3 \end{bmatrix}$**, then find the value of y.**

Ans:

Given that $\begin{bmatrix} y+2x & 5 \\ -x & 3 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ -2 & 3 \end{bmatrix}$

By definition of equality of matrices, we have

$y + 2x = 7$

$-x = -2 \Rightarrow x = 2$

$\therefore y + 2(2) = 7 \Rightarrow y = 3$

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. If a matrix has 6 elements, then number of possible orders of the matrix can be
(a) 2 (b) 4 (c) 3 (d) 6

2. If A and B are square matrices of the same order, then (A + B) (A – B) is equal to
(a) $A^2 - B^2$ (b) $A^2 - BA - AB - B^2$ (c) $A^2 - B^2 + BA - AB$ (d) $A^2 - BA + B^2 + AB$

3. If A = $\begin{bmatrix} 2 & -1 & 3 \\ -4 & 5 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & 3 \\ 4 & -2 \\ 1 & 5 \end{bmatrix}$, then
(a) only AB is defined (b) only BA is defined
(c) AB and BA both are defined (d) AB and BA both are not defined.

4. The matrix A = $\begin{bmatrix} 0 & 0 & 5 \\ 0 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ is a
(a) scalar matrix (b) diagonal matrix (c) unit matrix (d) square matrix

5. If A and B are symmetric matrices of the same order, then (AB′ –BA′) is a
(a) Skew symmetric matrix (b) Null matrix (c) Symmetric matrix (d) None of these

6. The matrix P = $\begin{bmatrix} 0 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 0 \end{bmatrix}$ is a
(a) square matrix (b) diagonal matrix (c) unit matrix (d) none

7. If A = $[a_{ij}]$ is a 2 × 3 matrix, such that aij = $\frac{(-i+2j)^2}{5}$, then a_{23} is
(a) $\frac{1}{5}$ (b) $\frac{2}{5}$ (c) $\frac{9}{5}$ (d) $\frac{16}{5}$

8. If A = diag(3, –1), then matrix A is
(a) $\begin{bmatrix} 0 & 3 \\ 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 3 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$

9. Total number of possible matrices of order 2 × 3 with each entry 1 or 0 is
(a) 6 (b) 36 (c) 32 (d) 64

10. If A is a square matrix such that $A^2 = A$, then $(I + A)^2 - 3A$ is
(a) I (b) 2A (c) 3I (d) A

11. If matrices A and B are inverse of each other then
(a) AB = BA (b) AB = BA = I (c) AB = BA = O (d) AB = O, BA = I

12. The diagonal elements of a skew symmetric matrix are
(a) all zeroes (b) are all equal to some scalar k(≠ 0)
(c) can be any number (d) none of these

13. If A = $\begin{bmatrix} 5 & x \\ y & 0 \end{bmatrix}$ and A = A′ then

(a) x = 0, y = 5 (b) x = y (c) x + y = 5 (d) x – y = 5

14. If a matrix A is both symmetric and skew symmetric then matrix A is

(a) a scalar matrix (b) a diagonal matrix

(c) a zero matrix of order n × n (d) a rectangular matrix.

15. If F(x) = $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ then F(x) F(y) is equal to

(a) F(x) (b) F(xy) (c) F(x + y) (d) F(x – y)

16. If A = $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then A^6 is equal to

(a) zero matrix (b) A (c) I (d) none of these

17. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $A^2 - 5A - 7I$ is

(a) a zero matrix (b) an identity matrix

(c) diagonal matrix (d) none of these

18. The matrix A satisfies the equation $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then matrix A is

(a) $\begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} \frac{1}{2} & -1 \\ \frac{1}{2} & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

19. If A = $\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, then A^2 is

(a) $\begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 \\ 4 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

20. Total number of possible matrices of order 3 × 3 with each entry 2 or 0 is

(a) 9 (b) 27 (c) 81 (d) 512

21. If $\begin{bmatrix} 2x+y & 4x \\ 5x-7 & 4x \end{bmatrix} = \begin{bmatrix} 7 & 7y-13 \\ y & x+6 \end{bmatrix}$, then the value of x + y is

(a) x = 3, y = 1 (b) x = 2, y = 3 (c) x = 2, y = 4 (d) x = 3, y = 3

22. If A = $\frac{1}{\pi}\begin{bmatrix} \sin^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{bmatrix}$, B = $-\frac{1}{\pi}\begin{bmatrix} -\cos^{-1}(x\pi) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{bmatrix}$, then A – B is equal to

(a) I (b) O (c) 2I (d) $\frac{1}{2}I$

23. If A and B are two matrices of the order 3 × m and 3 × n, respectively, and m = n, then the order of matrix (5A – 2B) is
(a) m × 3 (b) 3 × 3 (c) m × n (d) 3 × n

24. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then A^2 is equal to

(a) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

25. If matrix $A = [a_{ij}]_{2 \times 2}$, where $a_{ij} = 1$ if $i \neq j$ and 0 if $i = j$ then A^2 is equal to
(a) I (b) A (c) 0 (d) None of these

26. The matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a

(a) identity matrix (b) symmetric matrix (c) skew symmetric matrix (d) none of these

27. The matrix $\begin{bmatrix} 0 & -5 & 8 \\ 5 & 0 & 12 \\ -8 & -12 & 0 \end{bmatrix}$ is a

(a) diagonal matrix (b) symmetric matrix (c) skew symmetric matrix (d) scalar matrix

28. If A is matrix of order m × n and B is a matrix such that AB′ and B′A are both defined, then order of matrix B is
(a) m × m (b) n × n (c) n × m (d) m × n

29. If A and B are matrices of same order, then (AB′–BA′) is a
(a) skew symmetric matrix (b) null matrix (c) symmetric matrix (d) unit matrix

30. On using elementary column operations $C_2 \to C_2 - 2C_1$ in the following matrix equation
$\begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, we have :

(a) $\begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -5 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -5 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ 2 & 0 \end{bmatrix}$

31. If A is a square matrix such that $A^2 = I$, then $(A - I)^3 + (A + I)^3 - 7A$ is equal to
(a) A (b) I – A (c) I + A (d) 3A

32. For any two matrices A and B, we have
(a) AB = BA (b) AB ≠ BA (c) AB = O (d) None of the above

33. On using elementary row operation $R_1 \to R_1 - 3R_2$ in the following matrix equation:
$\begin{bmatrix} 4 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$, we have :

(a) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -7 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -5 & -7 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & -7 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 2 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

34. If A and B are two skew symmetric matrices of same order, then AB is symmetric matrix if ________.

35. If A and B are matrices of same order, then $(3A - 2B)'$ is equal to ________.

36. Addition of matrices is defined if order of the matrices is ________

37. _________ matrix is both symmetric and skew symmetric matrix.

38. Sum of two skew symmetric matrices is always _________ matrix.

39. The negative of a matrix is obtained by multiplying it by _________.

40. The product of any matrix by the scalar _________ is the null matrix.

41. A matrix which is not a square matrix is called a _________ matrix.

42. Matrix multiplication is _________ over addition.

43. If A is a symmetric matrix, then A^3 is a _________ matrix.

44. If A is a skew symmetric matrix, then A^2 is a _________.

45. If A and B are square matrices of the same order, then
(i) $(AB)'$ = _________.
(ii) $(kA)'$ = _________. (k is any scalar)
(iii) $[k(A - B)]'$ = _________.

46. If A is skew symmetric, then kA is a _________. (k is any scalar)

47. If A and B are symmetric matrices, then
(i) AB – BA is a _________.
(ii) BA – 2AB is a _________.

48. If A is symmetric matrix, then $B'AB$ is _________.

49. If A and B are symmetric matrices of same order, then AB is symmetric if and only if _________.

50. In applying one or more row operations while finding A^{-1} by elementary row operations, we obtain all zeros in one or more, then A^{-1} _________.

DETERMINANTS

CHAPTER – 4: DETERMINANTS

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then find the value of *x*.

Ans:

Given that $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$

On expanding both determinants, we get

$x \times x - 18 \times 2 = 6 \times 6 - 18 \times 2 \Rightarrow x^2 - 36 = 36 - 36$

$\Rightarrow x^2 - 36 = 0 \Rightarrow x^2 = 36$

$\Rightarrow x = \pm 6$

2. Find values of *k* if area of triangle is 4 sq. units and vertices are (i) (*k*, 0), (4, 0), (0, 2) (ii) (–2, 0), (0, 4), (0, *k*)

Ans:

(i) We have Area of triangle = $\frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix} = 4$

$\Rightarrow | k(0 - 2) + 1(8 - 0)| = 8$

$\Rightarrow k(0 - 2) + 1(8 - 0) = \pm 8$

On taking positive sign $- 2k + 8 = 8$

$\Rightarrow - 2k = 0$

$\Rightarrow k = 0$

On taking negative sign $-2k + 8 = - 8$

$\Rightarrow - 2k = - 16$

$\Rightarrow k = 8$

$\Rightarrow k = 0, 8$

(ii) We have Area of triangle = $\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix} = 4$

$\Rightarrow | - 2(4 - k) + 1(0 - 0)| = 8$

$\Rightarrow - 2(4 - k) + 1(0 - 0) = \pm 8$

$\Rightarrow [- 8 + 2k] = \pm 8$

On taking positive sign, $2k - 8 = 8 \Rightarrow 2k = 16 \Rightarrow k = 8$

On taking negative sign, $2k - 8 = - 8 \Rightarrow 2k = 0 \Rightarrow k = 0$

$\Rightarrow k = 0, 8$

3. If area of triangle is 35 sq units with vertices (2, – 6), (5, 4) and (*k*, 4). Then find the value of *k*.

Ans:

We have Area of triangle = $\frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix} = 35$

$\Rightarrow |2 (4 - 4) + 6(5 - k) + 1(20 - 4k)| = 70$

$\Rightarrow 2 (4 - 4) + 6 (5 - k) + 1(20 - 4k) = \pm 70$

$\Rightarrow 30 - 6k + 20 - 4k = \pm 70$

On taking positive sign, $- 10k + 50 = 70$

$\Rightarrow -10k = 20 \Rightarrow k = -2$
On taking negative sign, $-10k + 50 = -70$
$\Rightarrow -10k = -120 \Rightarrow k = 12$
$\therefore k = 12, -2$

4. Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Ans:

Given that $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Cofactors of the elements of second row

$A_{21} = (-1)^{2+1}\begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = -(9-16) = 7$

$A_{22} = (-1)^{2+2}\begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = (15-8) = 7$

and $A_{23} = (-1)^{2+3}\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = -(10-3) = -7$

Now, expansion of Δ using cofactors of elements of second row is given by
$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$
$= 2 \times 7 + 0 \times 7 + 1(-7) = 14 - 7 = 7$

5. If A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$**, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.**

Ans:

Given that A = $\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

Now, $A^2 - 5A + 7I = O$

$A^2 = A.A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 9-2 & 3+2 \\ -3-2 & -1+4 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$

$= \begin{bmatrix} 8-15+7 & 5-5+0 \\ -5+5+0 & 3-10+3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

$\therefore A^2 - 5A + 7I = O$

$\because |A| = \begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix} = 6 + 1 = 7 \neq 0$

$\therefore A^{-1}$ exists.
Now, $A.A - 5A = -7I$
Multiplying by A^{-1} on both sides, we get
$A.A\,(A^{-1}) - 5A(A^{-1}) = -7I(A^{-1})$
$\Rightarrow AI - 5I = -7A^{-1}$ (using $AA^{-1} = I$ and $IA^{-1} = A^{-1}$)

$$A^{-1} = -\frac{1}{7}(A-5I) = \frac{1}{7}(5I-A) = \frac{1}{7}\left(\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$$

6. For the matrix A = $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, find the numbers *a* and *b* such that $A^2 + aA + bI = O$.

Ans:

Given that A = $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 9+2 & 6+2 \\ 3+1 & 2+1 \end{bmatrix} = \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix}$$

Now, $A^2 + aA + bI = O$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + a\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 11 & 8 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3a & 2a \\ a & a \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix} = O$$

$$\Rightarrow \begin{bmatrix} 11+3a+b & 8+2a \\ 4+a & 3+a+b \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

If two matrices are equal, then their corresponding elements are equal.

$\Rightarrow$ $11 + 3a + b = 0$ …(i)

$8 + 2a = 0$ …(ii)

$4 + a = 0$ …(iii)

and $3 + a + b = 0$ …(iv)

Solving Eqs. (iii) and (iv), we get $4 + a = 0$

$\Rightarrow a = -4$

and $3 + a + b = 0$

$\Rightarrow 3 - 4 + b = 0 \Rightarrow b = 1$

Thus, $a = -4$ and $b = 1$

7. For the matrix A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$, Show that $A^3 - 6A^2 + 5A + 11\,I = O$. Hence, find A^{-1}.

Ans:

Given that A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1+1+2 & 1+2-1 & 1-3+3 \\ 1+2-6 & 1+4+3 & 1-6-9 \\ 2-1+6 & 2-2-3 & 2+3+9 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}$$

$$and \quad A^3 = A^2.A = \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4+2+2 & 4+4-1 & 4-6+3 \\ -3+8-28 & -3+16+14 & -3-24-42 \\ 7-3+28 & 7-6-14 & 7+9+42 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix}$$

$\therefore A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 8-24+5+11 & 7-12+5+0 & 1-6+5+0 \\ -23+18+5+0 & 27-48+10+11 & -69+84-15+0 \\ 32-42+10+0 & -13+18-5+0 & 58-84+15+11 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$$\because |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix} = 1(6-3) - 1(3+6) + 1(-1-4) = 3 - 9 - 5 = -11 \neq 0$$

$\therefore A^{-1}$ *exist*

Now, $A^3 - 6A^2 + 5A + 11I = O$

$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) + 11(IA^{-1}) = O$

$\Rightarrow AAI - 6AI + 5I + 11A^{-1} = O$

$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$

$$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{11}(-A^2 + 6A - 5I)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\left(\begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\left(\begin{bmatrix} -4 & -2 & -1 \\ 3 & -8 & 14 \\ -7 & 3 & -14 \end{bmatrix} + \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} -4+6-5 & -2+6-0 & -1+6-0 \\ 3+6-0 & -8+12-5 & 14-18-0 \\ -7+12-0 & 3-6-0 & -14+18-5 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

8. Solve system of linear equations, using matrix method,

$2x + y + z = 1$

$x - 2y - z = \frac{3}{2}$

$3y - 5z = 9$

Ans:

The given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & -4 & -2 \\ 0 & 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 1 \\ 2 & -4 & -2 \\ 0 & 3 & -5 \end{vmatrix} = 2(20+6) - 1(-10-0) + 1(6-0)$$

$= 52 + 10 + 6 = 68 \neq 0$

Thus, A is non-singular, Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$.

Cofactors of A are

$A_{11} = 20 + 6 = 26$,

$A_{12} = -(-10 + 0) = 10$,

$A_{13} = 6 + 0 = 6$

$A_{21} = -(-5 - 3) = 8$,

$A_{22} = -10 - 0 = -10$,

$A_{23} = -(6 - 0) = -6$

$A_{31} = (-2 + 4) = 2$,

$A_{32} = -(-4 - 2) = 6$,

$A_{33} = -8 - 2 = -10$

$$adj(A) = \begin{bmatrix} 26 & 10 & 6 \\ 8 & -10 & -6 \\ 2 & 6 & -10 \end{bmatrix}^T = \begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{68}\begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 26 & 8 & 2 \\ 10 & -10 & 6 \\ 6 & -6 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 26+24+18 \\ 10-30+54 \\ 6-18-90 \end{bmatrix} = \frac{1}{68}\begin{bmatrix} 68 \\ 34 \\ -102 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{-3}{2} \end{bmatrix}$$

$$Hence,\ x = 1,\ y = \frac{1}{2} \text{ and } z = \frac{-3}{2}$$

9. Solve system of linear equations, using matrix method,

$x - y + z = 4$

$2x + y - 3z = 0$

***x* + *y* + *z* = 2**

Ans:

The given system can be written as *AX* = *B*, where

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \ and \ \ B=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$$

$$Here, |A|=\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{vmatrix} = 1(1 + 3) - (-1)(2 + 3) + 1(2 - 1) = 4 + 5 + 1 = 10 \neq 0$$

Thus, *A* is non-singular, Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$.

Cofactors of *A* are

$A_{11} = 1 + 3 = 4$,

$A_{12} = -(2 + 3) = -5$,

$A_{13} = 2 - 1 = 1$,

$A_{21} = -(-1 - 1) = 2$,

$A_{22} = 1 - 1 = 0$,

$A_{23} = -(1 + 1) = -2$,

$A_{31} = 3 - 1 = 2$,

$A_{32} = -(-3 - 2) = 5$,

$A_{33} = 1 + 2 = 3$

$$adj(A)=\begin{bmatrix}4 & 5 & 1\\ 2 & 0 & -2\\ 2 & 5 & 3\end{bmatrix}^T=\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}16+0+4\\ -20+0+10\\ 4+0+6\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\ -10\\ 10\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

Hence, *x* = 2, *y* = − 1 and *z* = 1.

10. Solve system of linear equations, using matrix method,

2*x* + 3*y* +3 *z* = 5

***x* – 2*y* + *z* = – 4**

3*x* – *y* – 2*z* = 3

Ans:

The given system can be written as *AX* = *B*, where

$$A=\begin{bmatrix}2 & 3 & 3\\ 1 & -2 & 1\\ 3 & -1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \ and \ \ B=\begin{bmatrix}5\\ -4\\ 3\end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{vmatrix} = 2(4 + 1) - 3\,(-2 - 3) + 3\,(-1 + 6)$$

$= 10 + 15 + 15 = 40 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists. Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = 4 + 1 = 5$,

$A_{12} = -(-2 - 3) = 5$,

$A_{13} = (-1 + 6) = 5$,

$A_{21} = -(-6 + 3) = 3$,

$A_{22} = (-4 - 9) = -13$,

$A_{23} = -(-2 - 9) = 11$,

$A_{31} = 3 + 6 = 9$,

$A_{32} = -(2 - 3) = 1$,

$A_{33} = -4 - 3 = -7$

$$adj(A) = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1$, $y = 2$ and $z = -1$.

11. Solve system of linear equations, using matrix method,

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Ans:

The given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix} = 1(12 - 5) - (-1)\,(9 + 10) + 2\,(-3 - 8)$$

$= 7 + 19 - 22 = 4 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = 12 - 5 = 7$,

$A_{12} = -(9 + 10) = -19,$
$A_{13} = -3 - 8 = -11,$
$A_{21} = -(-3 + 2) = 1,$
$A_{22} = 3 - 4 = -1,$
$A_{23} = -(-1 + 2) = -1,$
$A_{31} = 5 - 8 = -3,$
$A_{32} = -(-5 - 6) = 11,$
$A_{33} = 4 + 3 = 7$

$$adj(A) = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2$, $y = 1$ and $z = 3$.

12. If A = $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$ find A^{-1}. Using A^{-1},Solve system of linear equations:

$2x - 3y + 5z = 11$
$3x + 2y - 4z = -5$
$x + y - 2z = -3$

Ans:

The given system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2(-4 + 4) - (-3)(-6 + 4) + 5(3 - 2)$$

$= 0 - 6 + 5 = -1 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = -4 + 4 = 0,$
$A_{12} = -(-6 + 4) = 2,$
$A_{13} = 3 - 2 = 1,$
$A_{21} = -(6 - 5) = -1,$
$A_{22} = -4 - 5 = -9,$
$A_{23} = -(2 + 3) = -5,$
$A_{31} = (12 - 10) = 2,$
$A_{32} = -(-8 - 15) = 23,$

$A_{33} = 4 + 9 = 13$

$$adj(A) = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1$, $y = 2$ and $z = 3$.

13. The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Ans:

Let the prices (per kg) of onion, wheat and rice be Rs. x, Rs. y and Rs. z, respectively then

$4x + 3y + 2z = 60$, $2x + 4y + 6z = 90$, $6x + 2y + 3z = 70$

This system of equations can be written as $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} = 4(12 - 12) - 3(6 - 36) + 2(4 - 24)$$

$= 0 + 90 - 40 = 50 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists. Therefore, the given system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are,

$A_{11} = 12 - 12 = 0$,

$A_{12} = -(6 - 36) = 30$,

$A_{13} = 4 - 24 = -20$,

$A_{21} = -(9 - 4) = -5$,

$A_{22} = 12 - 12 = 0$,

$A_{23} = -(8 - 18) = 10$,

$A_{31} = (18 - 8) = 10$,

$A_{32} = -(24 - 4) = -20$,

$A_{33} = 16 - 6 = 10$

$$adj(A) = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}^T = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore$ $x = 5, y = 8$ and $z = 8$.

Hence, price of onion per kg is Rs. 5, price of wheat per kg is Rs. 8 and that of rice per kg is Rs. 8.

14. Solve the system of equations:

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 4$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 1$$

$$\frac{6}{x} + \frac{9}{y} - \frac{20}{z} = 2$$

Ans:

Let $\frac{1}{x} = p,\ \frac{1}{y} = q\ \ and\ \ \frac{1}{z} = r$, then the given equations become

$2p + 3q + 10r = 4$, $4p - 6q + 5r = 1$, $6p + 9q - 20r = 2$

This system can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, X = \begin{bmatrix} p \\ q \\ r \end{bmatrix}\ \ and\ \ B = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{vmatrix} = 2(120-45) - 3(-80-30) + 10(36+36)$$

$= 150 + 330 + 720 = 1200 \neq 0$

Thus, A is non-singular. Therefore, its inverse exists.

Therefore, the above system is consistent and has a unique solution given by $X = A^{-1}B$

Cofactors of A are

$A_{11} = 120 - 45 = 75$,

$A_{12} = -(-80 - 30) = 110$,

$A_{13} = (36 + 36) = 72$,

$A_{21} = -(-60 - 90) = 150$,

$A_{22} = (-40 - 60) = -100$,

$A_{23} = -(18 - 18) = 0$,

$A_{31} = 15 + 60 = 75$,

$A_{32} = -(10 - 40) = 30$,

$A_{33} = -12 - 12 = -24$

$$adj(A) = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288+0-48 \end{bmatrix} = \frac{1}{1200}\begin{bmatrix} 600 \\ 400 \\ 240 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{5} \end{bmatrix}$$

$$\Rightarrow p = \frac{1}{2}, q = \frac{1}{3}, r = \frac{1}{5}$$

$$\Rightarrow \frac{1}{x} = \frac{1}{2}, \frac{1}{y} = \frac{1}{3}, \frac{1}{z} = \frac{1}{5}$$

$\Rightarrow x = 2, y = 3$ and $z = 5$.

15. Show that the matrix $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ satisfies the equation $A^2 - 4A + I = O$, where I is 2 × 2 identity matrix and O is 2 × 2 zero matrix. Using this equation, find A^{-1}.

Ans:

Given that $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$$A^2 = AA = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$$

$$Hence,\ A^2 - 4A + I = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - 4\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-8+1 & 12-12+0 \\ 4-4+0 & 7-8+1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$Now,\ A^2 - 4A + I = O$$

$$\Rightarrow AA - 4A = -I$$

$\Rightarrow AA(A^{-1}) - 4AA^{-1} = -IA^{-1}$ (Post multiplying by A^{-1} because $|A| \neq 0$)

$$\Rightarrow A(AA^{-1}) - 4I = -A^{-1}$$

$$\Rightarrow AI - 4I = -A^{-1}$$

$$\Rightarrow A^{-1} = 4I - A = 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4-2 & 0-3 \\ 0-1 & 4-2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

16. Solve the following system of equations by matrix method.

$3x - 2y + 3z = 8$
$2x + y - z = 1$
$4x - 3y + 2z = 4$

Ans:

The system of equation can be written as $AX = B$, where

$$A=\begin{bmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{ and } B=\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$$

$$Here, |A|=\begin{vmatrix}3 & -2 & 3\\ 2 & 1 & -1\\ 4 & -3 & 2\end{vmatrix}$$

= 3 (2 – 3) + 2(4 + 4) + 3 (– 6 – 4) = – 17 ≠ 0

Hence, A is nonsingular and so its inverse exists. Now

A_{11} = –1, A_{12} = – 8, A_{13} = –10

A_{21} = –5, A_{22} = – 6, A_{23} = 1

A_{31} = –1, A_{32} = 9, A_{33} = 7

$$adj(A)=\begin{bmatrix}-1 & -8 & 10\\ -5 & -6 & 1\\ -1 & 9 & 7\end{bmatrix}^T = \begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}$$

$$X = A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=-\frac{1}{17}\begin{bmatrix}-1 & -5 & -1\\ -8 & -6 & 9\\ -10 & 1 & 7\end{bmatrix}\begin{bmatrix}8\\ 1\\ 4\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=-\frac{1}{17}\begin{bmatrix}-17\\ -34\\ -51\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

Hence x = 1, y = 2 and z = 3.

17. Use product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$ **to solve the system of equations**

$x - y + 2z = 1$

$2y - 3z = 1$

$3x - 2y + 4z = 2$

Ans:

Consider the product $\begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$

$$=\begin{bmatrix}-2-9+12 & 0-2+2 & 1+3-4\\ 0+18-18 & 0+4-3 & 0-6+6\\ -6-18+24 & 0-4+4 & 3+6-8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$Hence, \begin{bmatrix}1 & -1 & 2\\ 0 & 2 & -3\\ 3 & -2 & 4\end{bmatrix}^{-1}=\begin{bmatrix}-2 & 0 & 1\\ 9 & 2 & -3\\ 6 & 1 & -2\end{bmatrix}$$

Now, given system of equations can be written, in matrix form, as follows

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0+2 \\ 9+2-6 \\ 6+1-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

Hence $x = 0$, $y = 5$ and $z = 3$

CHAPTER – 3: DETERMINANTS

MARKS WEIGHTAGE – 05 marks

Previous Years Board Exam (Important Questions & Answers)

1. Let A be a square matrix of order 3 × 3. Write the value of $|2A|$, where $|A| = 4$.

Ans:

Since $|2A| = 2^n|A|$ where n is order of matrix A.

Here $|A| = 4$ and $n = 3$

$\therefore\ |2A| = 2^3 \times 4 = 32$

2. Write the value of the following determinant: $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Ans:

Given that $\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Applying $R_1 \to R_1 - 6R_3$

$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix} = 0$ (Since R_1 is zero)

3. If A is a square matrix and $|A| = 2$, then write the value of $|AA'|$, where A' is the transpose of matrix A.

Ans:

$|AA|' = |A|.\ |A'| = |A|.\ |A| = |A|^2 = 2 \times 2 = 4$.

[since, $|AB| = |A|.|B|$ and $|A| = |A'|$, where A and B are square matrices.]

4. If A is a 3 × 3 matrix, $|A| \neq 0$ and $|3A| = k\,|A|$, then write the value of k.

Ans:

Here, $|3A| = k|A|$

$\Rightarrow 3^3|A| = k|A|$ [$\because$ $|kA| = kn|A|$ where n is order of A]

$\Rightarrow 27\,|A| = k|A|$

$\Rightarrow k = 27$

5. Evaluate: $\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

Ans:

$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}\pi = (a+ib)(a-ib)-(c+id)(-c+id)$

$= (a+ib)(a-ib)+(c+id)(c-id)$

$= a^2 - i^2b^2 + c^2 - i^2d^2$

$= a^2 + b^2 + c^2 + d^2$

6. If $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$, find the value of x.

Ans:

Given that $\begin{vmatrix} x+2 & 3 \\ x+5 & 4 \end{vmatrix} = 3$

$\Rightarrow$ $4x + 8 - 3x - 15 = 3$

$\Rightarrow x - 7 = 3$

$\Rightarrow x = 10$

7. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}.

Ans:

Minor of a_{23} = $\begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$ = 10 – 3 = 7.

8. Evaluate: $\begin{vmatrix} \cos 15^0 & \sin 15^0 \\ \sin 75^0 & \cos 75^0 \end{vmatrix}$

Ans:

Expanding the determinant, we get

cos 15° . cos 75° - sin 15° . sin 75°

= cos (15° + 75°) = cos 90° = 0

[since $\cos (A + B) = \cos A. \cos B - \sin A . \sin B$]

9. Write the value of the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

Ans:

Given determinant |A|= $\begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 6x & 9x & 12x \end{vmatrix}$

$= 3x \begin{vmatrix} 2 & 3 & 4 \\ 5 & 6 & 8 \\ 2 & 3 & 4 \end{vmatrix} = 0 (\because R_1 = R_3)$

10. Two schools P and Q want to award their selected students on the values of Tolerance, Kindness and Leadership. The school P wants to award Rs. x each, Rs. y each and Rs. z each for the three respective values to 3, 2 and 1 students respectively with a total award money of Rs. 2,200. School Q wants to spend Rs. 3,100 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as school P). If the total amount of award for one prize on each value is Rs. 1,200, using matrices, find the award money for each value. Apart from these three values, suggest one more value that should be considered for award.

Ans:

According to question,

$3x + 2y + z = 2200$

$4x + y + 3z = 3100$

$x + y + z = 1200$

The above system of equation may be written in matrix form as $AX = B$

$$A=\begin{bmatrix}3&2&1\\4&1&3\\1&1&1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}2200\\3100\\1200\end{bmatrix}$$

$$Here, |A|=\begin{vmatrix}3&2&1\\4&1&3\\1&1&1\end{vmatrix}=3(1-3)-2(4-3)+1(4-1)=-6-2+3=-5\neq 0$$

$\therefore$ A^{-1} exists.

Now, A_{11} = (1 – 3) = –2,

A_{12} = – (4 – 3) = –1,

A_{13} = (4 – 1) = 3,

A_{21} = – (2 – 1) = –1,

A_{22} = (3 – 1) = 2,

A_{23} = – (3 – 2) = –1

A_{31} = (6 – 1) = 5,

A_{32} = – (9 – 4) = –5,

A_{33} = (3 – 8) = –5

$$adj(A)=\begin{bmatrix}-2&-1&3\\-1&2&-1\\5&-5&-5\end{bmatrix}^T=\begin{bmatrix}-2&-1&5\\-1&2&-5\\3&-1&-5\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{-5}\begin{bmatrix}-2&-1&5\\-1&2&-5\\3&-1&-5\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2&1&-5\\1&-2&5\\-3&1&5\end{bmatrix}$$

Now, $X=A^{-1}B \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{5}\begin{bmatrix}2&1&-5\\1&-2&5\\-3&1&5\end{bmatrix}\begin{bmatrix}2200\\3100\\1200\end{bmatrix}$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{5}\begin{bmatrix}4400+3100-6000\\2200-6200+6000\\-6600+3100+6000\end{bmatrix}=\frac{1}{5}\begin{bmatrix}1500\\2000\\2500\end{bmatrix}=\begin{bmatrix}300\\400\\500\end{bmatrix}$$

$\Rightarrow$ $x = 300, y = 400, z = 500$

i.e., Rs. 300 for tolerance, Rs. 400 for kindness and Rs. 500 for leadership are awarded.

One more value like punctuality, honesty etc may be awarded.

11. 10 students were selected from a school on the basis of values for giving awards and were divided into three groups. The first group comprises hard workers, the second group has honest and law abiding students and the third group contains vigilant and obedient students. Double the number of students of the first group added to the number in the second group gives 13, while the combined strength of first and second group is four times that of the third group. Using matrix method, find the number of students in each group. Apart from the values, hard work, honesty and respect for law, vigilance and obedience, suggest one more value, which in your opinion, the school should consider for awards.

Ans:

Let no. of students in Ist, 2nd and 3rd group to *x, y, z* respectively.

From the statement we have

$$x + y + z = 10$$
$$2x + y = 13$$
$$x + y - 4z = 0$$

The above system of linear equations may be written in matrix form as $AX = B$ where

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 1 & 0\\ 1 & 1 & -4\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\ \ and\ \ B=\begin{bmatrix}10\\ 13\\ 0\end{bmatrix}$$

$$Here, |A|=\begin{vmatrix}1 & 1 & 1\\ 2 & 1 & 0\\ 1 & 1 & -4\end{vmatrix}=1(-4-0)-1(-8-0)+1(2-1)=-4+8+1=5\neq 0$$

$\therefore$ A^{-1} exists.

Now, A_{11} = – 4 – 0 = –4

A_{12} = – (–8 – 0) = 8

A_{13} = 2 – 1 = 1

A_{21} = –(–4 – 1) = 5

A_{22} = –4 – 1 = –5

A_{23} = –(1 – 1) = 0

A_{31} = 0 – 1 = –1

A_{32} = –(0 – 2) = 2

A_{33} = 1 – 2 = –1

$$adj(A)=\begin{bmatrix}-4 & 8 & 1\\ 5 & -5 & 0\\ -1 & 2 & -1\end{bmatrix}^T=\begin{bmatrix}-4 & 5 & -1\\ 8 & -5 & 2\\ 1 & 0 & -1\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{5}\begin{bmatrix}-4 & 5 & -1\\ 8 & -5 & 2\\ 1 & 0 & -1\end{bmatrix}$$

Now, $X=A^{-1}B\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-4 & 5 & -1\\ 8 & -5 & 2\\ 1 & 0 & -1\end{bmatrix}\begin{bmatrix}10\\ 13\\ 0\end{bmatrix}$

$$\Rightarrow\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{5}\begin{bmatrix}-40+65\\ 80-65\\ 10\end{bmatrix}=\frac{1}{5}\begin{bmatrix}25\\ 15\\ 10\end{bmatrix}=\begin{bmatrix}5\\ 3\\ 2\end{bmatrix}$$

$\Rightarrow x = 5, y = 3, z = 2$

12. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some others (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these values, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

Ans:

According to question

$x + y + z = 12$

$2x + 3y + 3z = 33$

$x – 2y + z = 0$

The above system of linear equation can be written in matrix form as $AX = B$ where

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 3\\ 1 & -2 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\ \ and\ \ B=\begin{bmatrix}12\\ 33\\ 0\end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{vmatrix}$$

= 1 (3 + 6) –1 (2–3) + 1 (– 4 –3) = 9 + 1 – 7 = 3

$\therefore$ A^{-1} exists.

$A_{11} = 9, A_{12} = 1, A_{13} = -7$

$A_{21} = -3, A_{22} = 0, A_{23} = 3$

$A_{31} = 0, A_{32} = -1, A_{33} = 1$

$$adj(A) = \begin{bmatrix} 9 & 1 & -7 \\ -3 & 0 & 3 \\ 0 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}\begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 108-99 \\ 12+0+0 \\ -84+99 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$\Rightarrow x = 3, y = 4, z = 5$

No. of awards for honesty = 3

No. of awards for helping others = 4

No. of awards for supervising = 5.

The persons, who work in the field of health and hygiene should also be awarded.

13. A school wants to award its students for the values of Honesty, Regularity and Hard work with a total cash award of Rs. **6,000. Three times the award money for Hardwork added to that given for honesty amounts to ` 11,000. The award money given for Honesty and Hardwork together is double the one given for Regularity. Represent the above situation algebraically and find the award money for each value, using matrix method. Apart from these values, namely, Honesty, Regularity and Hardwork, suggest one more value which the school must include for awards.**

Ans:

Let x, y and z be the awarded money for honesty, Regularity and hardwork.

From the statement

$x + y + z = 6000$ …(*i*)

$x + 3z = 11000$ …(*ii*)

$x + z = 2y \Rightarrow x - 2y + z = 0$ …(*iii*)

The above system of three equations may be written in matrix form as $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$$

$$Here, |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 0 & -2 & 1 \end{vmatrix} = 1(0+6) - 1(1-3) + 1(-2-0) = 6 + 2 - 2 = 6 \neq 0$$

Hence A^{-1} exist
If A_{ij} is co-factor of a_{ij} then
$A_{11} = 0 + 6 = 6$
$A_{12} = -(1 - 3) = 2$;
$A_{13} = (-2 - 0) = -2$
$A_{21} = -(1 + 2) = -3$
$A_{22} = 0$
$A_{23} = (-2 - 1) = -3$
$A_{31} = 3 - 0 = 3$
$A_{32} = -(3 - 1) = -2$;
$A_{33} = 0 - 1 = -1$

$$adj(A) = \begin{bmatrix} 6 & 2 & -2 \\ -3 & 0 & 3 \\ 3 & -2 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}$$

Now, $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 & -3 & 3 \\ 2 & 0 & -2 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 6000 \\ 11000 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 36000 - 33000 + 0 \\ 12000 + 0 + 0 \\ -12000 + 33000 + 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3000 \\ 12000 \\ 21000 \end{bmatrix} = \begin{bmatrix} 500 \\ 2000 \\ 3500 \end{bmatrix}$$

$\Rightarrow x = 500, y = 2000, z = 3500$
Except above three values, school must include discipline for award as discipline has great importance in student's life.

14. If $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, then write the value of *x*.

Ans:
Given that $\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$
$\Rightarrow (x + 1)(x + 2) - (x - 1)(x - 3) = 12 + 1$
$\Rightarrow x^2 + 2x + x + 2 - x^2 + 3x + x - 3 = 13$
$\Rightarrow 7x - 1 = 13$
$\Rightarrow 7x = 14$
$\Rightarrow x = 2$

15. Using matrices, solve the following system of equations:
$x - y + z = 4$; $2x + y - 3z = 0$; $x + y + z = 2$

Ans:
Given equations
$x - y + z = 4$
$2x + y - 3z = 0$
$x + y + z = 2$
We can write this system of equations as AX = B where

$$A=\begin{bmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \ and \ B=\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$$

$$Here, |A|=\begin{vmatrix}1 & -1 & 1\\ 2 & 1 & -3\\ 1 & 1 & 1\end{vmatrix}$$

= 1 (1 + 3) - (- 1) (2 + 3) + 1 (2 - 1) = 4 + 5 + 1 = 10

$\therefore$ A^{-1} exists.

$A_{11} = 4, A_{12} = -5, A_{13} = 1$

$A_{21} = 2, A_{22} = 0, A_{23} = -2$

$A_{31} = 2, A_{32} = 5, A_{33} = 3$

$$adj(A)=\begin{bmatrix}4 & -5 & 1\\ 2 & 0 & -2\\ 2 & 5 & 3\end{bmatrix}^T=\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}$$

$$\text{Now, } X = A^{-1}B \Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}4 & 2 & 2\\ -5 & 0 & 5\\ 1 & -2 & 3\end{bmatrix}\begin{bmatrix}4\\ 0\\ 2\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{10}\begin{bmatrix}16+0+4\\ -20+0+10\\ 4-0+6\end{bmatrix}=\frac{1}{10}\begin{bmatrix}20\\ -10\\ 10\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 1\end{bmatrix}$$

The required solution is

$\therefore$ $x = 2, y = - 1, z = 1$

16. If $A^{-1}=\begin{bmatrix}3 & -1 & 1\\ -15 & 6 & -5\\ 5 & -2 & 2\end{bmatrix}$ **and** $B=\begin{bmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{bmatrix}$**, find** $(AB)^{-1}$**.**

Ans:

For B^{-1}

$$|B|=\begin{vmatrix}1 & 2 & -2\\ -1 & 3 & 0\\ 0 & -2 & 1\end{vmatrix} = 1(3-0)-2(-1-0)-2(2-0) = 3+2-4 = 1 \neq 0$$

i.e., B is invertible matrix

$\Rightarrow B^{-1}$ exist.

$A_{11} = 3, A_{12} = 1, A_{13} = 2$

$A_{21} = 2, A_{22} = 1, A_{23} = 2$

$A_{31} = 6, A_{32} = 2, A_{33} = 5$

$$adj(B)=\begin{bmatrix}3 & 1 & 2\\ 2 & 1 & 2\\ 6 & 2 & 5\end{bmatrix}^T=\begin{bmatrix}3 & 2 & 6\\ 1 & 1 & 2\\ 2 & 2 & 5\end{bmatrix}$$

$$\therefore B^{-1} = \frac{1}{|B|}(adjb) = \frac{1}{1}\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now $(AB)^{-1} = B^{-1}. A^{-1}$

$$\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-30+30 & -3+12-12 & 3-10+12 \\ 3-15+10 & -1+6-4 & 1-5+4 \\ 6-30+25 & -2+12-10 & 2-10+10 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -3 & 5 \\ -2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. If $\begin{vmatrix} 2x & -1 \\ 4 & 2 \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 2 & 1 \end{vmatrix}$, then the – value of x is

(a) 3 (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $-\frac{1}{4}$

2. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$, then the – value of x is

(a) 3 (b) ± 6 (c) ± 3 (d) 6

3. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then the – value of x is

(a) 3 (b) –3 (c) ± 3 (d) 0

4. The value $\begin{vmatrix} 6 & 0 & -1 \\ 2 & 1 & 4 \\ 1 & 1 & 3 \end{vmatrix}$ is

(a) –7 (b) 7 (c) 8 (d) 10

5. If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then the value of $\Delta + \Delta_1$ is

(a) 0 (b) –1 (c) 1 (d) none of these

6. The value of $\begin{vmatrix} \cos ec^2 x & \cot^2 x & 1 \\ \cot^2 x & \cos ec^2 x & -1 \\ 42 & 40 & 2 \end{vmatrix}$ is

(a) 0 (b) –1 (c) 1 (d) none of these

7. The value of $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$ is

(a) 0 (b) –1 (c) 1 (d) none of these

8. If $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}, \Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$ then

(a) $\Delta_1 = -\Delta$ (b) $\Delta \neq \Delta_1$ (c) $\Delta - \Delta_1 = 0$ (d) none of these

9. If $\Delta = \begin{vmatrix} Ax^2 & x^3 & 1 \\ By^2 & y^3 & 1 \\ Cz^2 & z^3 & 1 \end{vmatrix}, \Delta_1 = \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ zy & zx & xy \end{vmatrix}$ then

(a) $\Delta_1 = -\Delta$ (b) $\Delta \neq \Delta_1$ (c) $\Delta - \Delta_1 = 0$ (d) $\Delta = x\Delta_1$

10. If x, y $\in$ R, then the determinant $\begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix}$ lies in the interval

(a) $\sqrt{2}, \sqrt{2}$ (b) [–1, 1] (c) $\sqrt{2}, 1$ (d) $1, \sqrt{2}$

11. The determinant $\Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$ is equal to

(a) 0 (b) –1 (c) 1 (d) none of these

12. The determinant $\Delta = \begin{vmatrix} \sqrt{37}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{74} & 5 & \sqrt{10} \\ 3+\sqrt{185} & \sqrt{15} & 5 \end{vmatrix}$ is equal to

(a) 0 (b) –1 (c) 1 (d) none of these

13. The determinant $\Delta = \begin{vmatrix} \sin^2 23^0 & \sin^2 67^0 & \cos 180^0 \\ -\sin^2 67^0 & -\sin^2 23^0 & \cos^2 180^0 \\ \cos 180^0 & \sin^2 23^0 & \sin^2 67^0 \end{vmatrix}$ is equal to

(a) 0 (b) –1 (c) 1 (d) none of these

14. If A = $\begin{vmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{vmatrix}$ and $A^{-1} = \begin{vmatrix} \frac{1}{2} & -4 & \frac{5}{2} \\ -\frac{1}{2} & 3 & -\frac{3}{2} \\ \frac{1}{2} & y & \frac{1}{2} \end{vmatrix}$, then the values of x and y are

(a) x = 0, y = 0 (b) x = 1, y = 1 (c) x = –1, y = 1 (d) x = 1, y = –1

15. The value of determinant $\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$ is

(a) $a^3 + b^3 + c^3$ (b) 3 bc (c) $a^3 + b^3 + c^3 - 3abc$ (d) none of these

16. The area of a triangle with vertices (–3, 0), (3, 0) and (0, k) is 9 sq. units. The value of k will be
(a) 9 (b) 3 (c) – 9 (d) 6

17. The determinant $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$ equals

(a) abc (b–c) (c – a) (a – b) (b) (b–c) (c – a) (a – b)
(c) (a + b + c) (b – c) (c – a) (a – b) (d) None of these

18. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$ is

(a) 0 (b) 2 (c) 1 (d) 3

19. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$ is equal to

(a) 0 (b) – 1 (c) 1 (d) None of these

20. If A, B and C are angles of a triangle, then the determinant $\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$ is equal to

(a) 0 (b) – 1 (c) 1 (d) None of these

21. Let f (t) = $\begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$, then $\lim_{t \to 0} \frac{f(t)}{t^2}$ is equal to

(a) 0 (b) – 1 (c) 2 (d) 3

22. Let f (x) = $\begin{vmatrix} \cos x & 2\sin x & \sin x \\ x & x & x \\ 1 & 2x & x \end{vmatrix}$, then $\lim_{x \to 0} \frac{f(x)}{x^2}$ is equal to

(a) 0 (b) – 1 (c) 2 (d) 3

23. The maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+\sin\theta & 1 \\ 1+\cos\theta & 1 & 1 \end{vmatrix}$ is (θ is real number)

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\sqrt{2}$ (d) $\frac{2\sqrt{3}}{4}$

24. If f (x) = $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$, then

(a) f (a) = 0 (b) f (b) = 0 (c) f (0) = 0 (d) f (1) = 0

25. If A = $\begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists if

(a) λ = 2 (b) λ ≠ 2 (c) λ ≠ – 2 (d) None of these

26. If A and B are invertible matrices, then which of the following is not correct?

(a) adj A = |A|. A^{-1} (b) $\det(a)^{-1} = [\det(a)]^{-1}$
(c) $(AB)^{-1} = B^{-1} A^{-1}$ (d) $(A + B)^{-1} = B^{-1} + A^{-1}$

27. If x, y, z are all different from zero and $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = 0$, then value of $x^{-1} + y^{-1} + z^{-1}$ is

(a) xyz (b) $x^{-1} y^{-1} z^{-1}$ (c) – x – y – z (d) –1

28. The value of the determinant $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$ is

(a) $9x^2 (x + y)$ (b) $9y^2 (x + y)$ (c) $3y^2 (x + y)$ (d) $7x^2 (x + y)$

29. There are two values of a which makes determinant, $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & -1 \\ 0 & 4 & 2a \end{vmatrix} = 86$, then sum of these number is

(a) 4 (b) 5 (c) – 4 (d) 9

30. If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and A_{ij} is cofactor of a_{ij}, then the value of Δ is given by

(a) $a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33}$ (b) $a_{11}A_{11} + a_{12}A_{21} + a_{13}A_{31}$
(c) $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ (d) $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

31. If A is a matrix of order 3 × 3, then |KA| = ______.

(a) 0 (b) –1 (c) 2 (d) 3

32. A and B are invertible matrices of the same order such that $|(AB)^{-1}| = 8$, If |A| = 2, then |B| is

(a) 16 (b) 4 (c) 6 (d) $\frac{1}{16}$

33. If a, b, c are all distinct, and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, then the value of abc is

(a) 0 (b) –1 (c) 3 (d) –3

34. Let A be a square matrix of order 2 × 2, then |KA| is equal to

(a) K|A| (b) $K^2|A|$ (c) $K^3|A|$ (d) 2K|A|

35. Let A be a non-angular square matrix of order 3 × 3, then |A . adj A| is equal to

(a) $|A|^3$ (b) $|A|^2$ (c) |A| (d) 3|A|

36. Let A be a square matrix of order 3 × 3 and k a scalar, then | kA | is equal to

(a) k | A | (b) | k | | A | (c) k^3 | A | (d) none of these

37. A is invertible matrix of order 3 × 3 and |A| = 9, then value of $|A^{-1}|$ is __________.

38. If area of a triangle with vertices (3, 2), (–1, 4) and (6, k) is 7 sq units, then possible values of k are___.

39. If A and B are invertible matrices of the same order $(AB)^{-1}$ is ___________.

40. If A is a matrix of order 3 × 3, then |3A| = _______ .

41. If A is invertible matrix of order 3 × 3, then $|A^{-1}|$ _______ .

42. If x, y, z ∈ R, then the value of determinant $\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$ is equal to _______.

43. If cos2θ = 0, then $\begin{vmatrix} 0 & \cos\theta & \sin\theta \\ \cos\theta & \sin\theta & 0 \\ \sin\theta & 0 & \cos\theta \end{vmatrix}^2$ = ________.

44. If A is a matrix of order 3 × 3, then $(A^2)^{-1}$ = ________.

45. If A is a matrix of order 3 × 3, then number of minors in determinant of A are ________.

46. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to _________.

47. If x = – 9 is a root of $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$, then other two roots are __________.

48. If A, B, C are the angles of a triangle, then $\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$ = __________.

49. The value of $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix}$ is _________.

50. Maximum value of $\Delta = \begin{vmatrix} 1 & 1 & 1+\cos\theta \\ 1 & 1+\sin\theta & 1 \\ 1 & 1 & 1 \end{vmatrix}$, where θ is a real number is ________.

CONTINUITY & DIFFERENTIABILITY

CHAPTER – 5: CONTINUITY AND DIFFERENTIABILITY

MARKS WEIGHTAGE – 08 marks

NCERT Important Questions & Answers

1. Find all points of discontinuity of *f*, where *f* is defined by $f(x)=\begin{cases}2x+3, & if\quad x\le 2\\ 2x-3, & if\quad x>2\end{cases}$.

Ans.

Here, $f(x)=\begin{cases}2x+3, & if\quad x\le 2\\ 2x-3, & if\quad x>2\end{cases}$.

At x =2, $LHL=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(2x+3)$

Putting $x = 2 - h$ has $x\to 2^-$ when $h\to 0$

$\therefore \lim_{x\to 2^-} f(x)=\lim_{h\to 0}(2(2-h)+3)=\lim_{h\to 0}(4-2h+3)=\lim_{h\to 0}(7-2h)=7$

At x =2, $RHL=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(2x-3)$

Putting $x = 2 + h$ as $x\to 2^+$ when $h\to 0$

$\therefore \lim_{x\to 2^+} f(x)=\lim_{h\to 0}(2(2+h)-3)=\lim_{h\to 0}(4+2h-3)=\lim_{h\to 0}(1+2h)=1$

$\therefore$ LHL ≠ RHL. Thus, $f(x)$ is discontinuous at $x = 2$.

2. Find all points of discontinuity of *f*, where *f* is defined by $f(x)=\begin{cases}\dfrac{|x|}{x}, & if\quad x\ne 0\\ 0, & if\quad x=0\end{cases}$

Ans. Here, $f(x)=\begin{cases}\dfrac{|x|}{x}, & if\quad x\ne 0\\ 0, & if\quad x=0\end{cases}$

$LHL=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\frac{|x|}{x}$

Putting $x = 0 - h$ as $x\to 0^-$ when $h\to 0$

$\therefore \lim_{x\to 0^-} f(x)=\lim_{h\to 0}\frac{|0-h|}{0-h}=\lim_{h\to 0}\frac{h}{-h}=-1$

$RHL=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\frac{|x|}{x}$

Putting $x = 0 + h$ as $x\to 0^+$; $h\to 0$

$\therefore \lim_{x\to 0^+} f(x)=\lim_{h\to 0}\frac{|0+h|}{0+h}=\lim_{h\to 0}\frac{h}{h}=1$

$\therefore$ LHL ≠ RHL. Thus, $f(x)$ is discontinuous at $x = 0$.

3. Find all points of discontinuity of *f*, where *f* is defined by $f(x)=\begin{cases}x^3-3, & if\quad x\le 2\\ x^2+1, & if\quad x>2\end{cases}$

Ans.

For $x < 2$, $f(x) = x^3 - 3$ and for $x > 2$, $f(x) = x^2 + 1$ is a polynomial function, so f is continuous in the above interval. Therefore, we have to check the continuity at $x = 2$.

$LHL = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^3 - 3)$

Putting $x = 2 - h$ has $x \to 2^-$ when $h \to 0$

$\therefore \lim_{x \to 2^-} f(x) = \lim_{h \to 0}((2-h)^3 - 3) = \lim_{h \to 0}(8 - 12h + 6h^2 - h^3 - 3)$

$= \lim_{h \to 0}(5 - 12h + 6h^2 - h^3) = 5$

$RHL = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + 1)$

Putting $x = 2 + h$ as $x \to 2^+$ when $h \to 0$

$\therefore \lim_{x \to 2^+} f(x) = \lim_{h \to 0}((2+h)^2 + 1) = \lim_{h \to 0}(4 + 4h + h^2 + 1)$

$= \lim_{h \to 0}(5 + 4h + h^2) = 5$

Also, $f(2) = (2)^3 - 3 = 8 - 3 = 5$ [since $f(x) = x^3 - 3$]

$\therefore$ LHL = RHL =f(2) . Thus, $f(x)$ is continuous at $x = 2$.

Hence, there is no point of discontinuity for this function $f(x)$.

4. Find the relationship between *a* and *b* so that the function *f* defined by $f(x) = \begin{cases} ax+1, & if \quad x \le 3 \\ bx+3, & if \quad x > 3 \end{cases}$ **is continuous at *x* = 3.**

Ans.

Here, $f(x) = \begin{cases} ax+1, & if \quad x \le 3 \\ bx+3, & if \quad x > 3 \end{cases}$

$LHL = \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax + 1)$

Putting $x = 3 - h$ has $x \to 3^-$ when $h \to 0$

$\therefore \lim_{x \to 3^-} f(x) = \lim_{h \to 0}(a(3-h) + 1) = \lim_{h \to 0}(3a - ah + 1) = 3a + 1$

$RHL = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (bx + 3)$

Putting $x = 3 + h$ as $x \to 3^+$ when $h \to 0$

$\therefore \lim_{x \to 3^+} f(x) = \lim_{h \to 0}(b(3+h) + 3) = \lim_{h \to 0}(3b + bh + 3) = 3b + 3$

Also, f(3) =3 a +1 [since $f(x) = ax + 1$]

Since, $f(x)$ is continuous at $x = 3$.

$\therefore$ LHL = RHL = f(3)

$\Rightarrow 3a + 1 = 3b + 3 \Rightarrow 3a = 3b + 2 \Rightarrow a = b + \frac{2}{3}$

5. For what value of λ is the function defined by $f(x) = \begin{cases} \lambda(x^2 - 2x), & if \quad x \le 0 \\ 4x+1, & if \quad x > 0 \end{cases}$ **continuous at *x* = 0? What about continuity at *x* = 1?**

Ans.

Here, $f(x) = \begin{cases} \lambda(x^2 - 2x), & if \quad x \le 0 \\ 4x+1, & if \quad x > 0 \end{cases}$

At x = 0, $LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \lambda(x^2 - 2x)$

Putting $x = 0 - h$ as $x \to 0^-$ when $h \to 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{h \to 0} \lambda[(0-h)^2 - 2(0-h)] = \lim_{h \to 0} \lambda(h^2 + 2h) = 0$

At x = 0, $RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x+1)$

Putting $x = 0 + h$ as $x \to 0^+$; $h \to 0$

$\therefore \lim_{x \to 0^+} f(x) = \lim_{h \to 0} [4(0+h)+1] = \lim_{h \to 0} (4h+1) = 1$

$\therefore$ LHL ≠ RHL. Thus, $f(x)$ is discontinuous at $x = 0$ for any value of λ.

At x = 1, $LHL = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4x+1)$

Putting $x = 1 - h$ as $x \to 1^-$ when $h \to 0$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{h \to 0} (4(1+h)+1) = \lim_{h \to 0} (4+4h+1) = 5$

At x = 1, $RHL = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (4x+1)$

Putting $x = 1 + h$ as $x \to 1^+$; $h \to 0$

$\therefore \lim_{x \to 1^+} f(x) = \lim_{h \to 0} [4(1+h)+1] = \lim_{h \to 0} (4+4h+1) = 5$

$\therefore$ LHL = RHL. Thus, $f(x)$ is continuous at $x = 1$ for any value of λ.

6. Find all points of discontinuity of *f*, where $f(x) = \begin{cases} \frac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

Ans. Here, $f(x) = \begin{cases} \frac{\sin x}{x}, & if \quad x < 0 \\ x+1, & if \quad x \geq 0 \end{cases}$

At x = 0, $LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sin x}{x}$

Putting $x = 0 - h$ as $x \to 0^-$ when $h \to 0$

$\therefore \lim_{x \to 0^-} f(x) = \lim_{h \to 0} \frac{\sin(0-h)}{0-h} = \lim_{h \to 0} \frac{\sin(-h)}{-h} = \lim_{h \to 0} \frac{-\sin h}{-h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$

At x = 0, $RHL = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\sin x}{x}$

Putting $x = 0 + h$ as $x \to 0^+$; $h \to 0$

$\therefore \lim_{x \to 0^+} f(x) = \lim_{h \to 0} \frac{\sin(0+h)}{0+h} = \lim_{h \to 0} \frac{\sin h}{h} = 1$

Also, f(0) = 0 + 1 = 1

$\therefore$ LHL = RHL = f(0). Thus, $f(x)$ is continuous at $x = 0$.

When $x < 0$, sin x and x both are continuous. Therefore, $\frac{\sin x}{x}$ is also continuous.

When $x > 0$, $f(x) = x + 1$ is a polynomial. Therefore f is continuous.

Hence, there is no point of discontinuity for this function $f(x)$.

7. Determine if *f* defined by $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & if\quad x\neq 0\\ 0, & if\quad x=0\end{cases}$ **is a continuous function?**

Ans.

Here, $f(x)=\begin{cases} x^2\sin\frac{1}{x}, & if\quad x\neq 0\\ 0, & if\quad x=0\end{cases}$

At x = 0, $LHL=\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-} x^2\sin\frac{1}{x}$

Putting $x = 0 - h$ as $x\to 0^-$ when $h\to 0$

$\therefore \lim_{x\to 0^-} f(x)=\lim_{h\to 0}(0-h)^2\sin\frac{1}{0-h}=\lim_{h\to 0}\left(-h^2\sin\frac{1}{h}\right)=-0\times\sin\infty$

= 0 x value between –1 and 1 (since $-1\le \sin x\le 1$, for all values of $x\in R$)

At x = 0, $RHL=\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+} x^2\sin\frac{1}{x}$

Putting $x = 0 + h$ as $x\to 0^+$; $h\to 0$

$\therefore \lim_{x\to 0^+} f(x)=\lim_{h\to 0}(0+h)^2\sin\frac{1}{0+h}=\lim_{h\to 0}\left(h^2\sin\frac{1}{h}\right)=0\times\sin\infty$

= 0 x value between –1 and 1 (since $-1\le \sin x\le 1$, for all values of $x\in R$)

$\therefore$ LHL = RHL = f(0). Thus, $f(x)$ is continuous at $x = 0$.

8. Find the values of *k* so that the function *f* $f(x)=\begin{cases} \frac{k\cos x}{\pi-2x}, & if\quad x\neq\frac{\pi}{2}\\ 3, & if\quad x=\frac{\pi}{2}\end{cases}$ **is continuous at point** $x=\frac{\pi}{2}$

Ans.

Here, $f(x)=\begin{cases} \frac{k\cos x}{\pi-2x}, & if\quad x\neq\frac{\pi}{2}\\ 3, & if\quad x=\frac{\pi}{2}\end{cases}$

$LHL=\lim_{x\to \frac{\pi}{2}^-} f(x)=\lim_{x\to \frac{\pi}{2}^-}\frac{k\cos x}{\pi-2x}$

Putting $x=\frac{\pi}{2}-h$ as $x\to\frac{\pi}{2}^-$ when $h\to 0$

$\therefore \lim_{x\to \frac{\pi}{2}^-} f(x)=\lim_{h\to 0}\frac{k\cos\left(\frac{\pi}{2}-h\right)}{\pi-2\left(\frac{\pi}{2}-h\right)}=\lim_{h\to 0}\frac{ksinh}{2h}=\frac{k}{2}\times\lim_{h\to 0}\frac{sinh}{h}=\frac{k}{2}\times 1=\frac{k}{2}$

Since f(x) is continuous at $x=\frac{\pi}{2}$, therefore LHL = $f\left(\frac{\pi}{2}\right)$

Also, $f\left(\frac{\pi}{2}\right)=3\Rightarrow\frac{k}{2}=3\Rightarrow k=6$

9. Find the values of *k* so that the function *f* $f(x)=\begin{cases} kx+1, & if \quad x\le 5 \\ 3x-5, & if \quad x>5 \end{cases}$ **is continuous at point x = 5.**

Ans.

Here, $f(x)=\begin{cases} kx+1, & if \quad x\le 5 \\ 3x-5, & if \quad x>5 \end{cases}$

At x = 5, $LHL=\lim_{x\to 5^-} f(x)=\lim_{x\to 5^-}(kx+1)$

Putting $x = 5 - h$ has $x \to 5^-$ when $h \to 0$

$\therefore \lim_{x\to 5^-} f(x)=\lim_{h\to 0}(k(5-h)+1)=\lim_{h\to 0}(5k-kh+1)=5k+1$

At x = 5, $RHL=\lim_{x\to 5^+} f(x)=\lim_{x\to 5^+}(3x-5)$

Putting $x = 5 + h$ as $x \to 5^+$; $h \to 0$

$\therefore \lim_{x\to 5^+} f(x)=\lim_{h\to 0}(3(5+h)-5)=\lim_{h\to 0}(10+3h)=10$

Also, f(5) = 5k + 1

Since f(x) is continuous at x = 5, therefore LHL = RHL = f(5)

$\Rightarrow 5k+1=10 \Rightarrow 5k=9 \Rightarrow k=\frac{9}{5}$

10. Find the values of *a* and *b* such that the function defined by $f(x)=\begin{cases} 5, & if \quad x\le 2 \\ ax+b, & if \quad 2<x<10 \\ 21, & if \quad x\ge 10 \end{cases}$ **is a continuous function.**

Ans.

Here, $f(x)=\begin{cases} 5, & if \quad x\le 2 \\ ax+b, & if \quad 2<x<10 \\ 21, & if \quad x\ge 10 \end{cases}$

At x =2, $LHL=\lim_{x\to 2^-} f(x)=\lim_{x\to 2^-}(5)=5$

At x =2, $RHL=\lim_{x\to 2^+} f(x)=\lim_{x\to 2^+}(ax+b)$

Putting $x = 2 + h$ as $x \to 2^+$ when $h \to 0$

$\therefore \lim_{x\to 2^+} f(x)=\lim_{h\to 0}(a(2+h)+b)=\lim_{h\to 0}(2a+ah+b)=2a+b$

Also, f(2) = 5

Since $f(x)$ is continuous at $x = 2$, therefore LHL = RHL = f(2)

$\Rightarrow 2a+b=5$ ----------------- (1)

At x =10, $LHL=\lim_{x\to 10^-} f(x)=\lim_{x\to 10^-}(ax+b)$

Putting $x = 10 - h$ has $x \to 10^-$ when $h \to 0$

$\therefore \lim_{x\to 10^-} f(x)=\lim_{h\to 0}(a(10-h)+b)=\lim_{h\to 0}(10a-ah+b)=10a+b$

At x =2, $RHL = \lim_{x \to 10^+} f(x) = \lim_{x \to 10^+} (21) = 21$

Also, f(10) = 21

Since $f(x)$ is continuous at x = 10, therefore LHL = RHL = f(10)

Since, $f(x)$ is continuous at x = 10.

LHL = RHL = f(10)

$\Rightarrow$ 10a + b = 21 ………………(2)

Subtracting Eq. (1) from Eq. (2), we get 8a =16 $\Rightarrow$ a =2

Put a =2 in Eq. (1), we get 2 ×2 + b = 5 $\Rightarrow$ b =1

11. Prove that the function *f* given by *f* (*x*) = | *x* – 1 |, *x* ∈ R is not differentiable at *x* = 1.

Ans.

Given, $f(x) = |x-1| = \begin{cases} x-1, & if \quad x-1 \geq 0 \\ -(x-1), & if \quad x-1 < 0 \end{cases}$

We have to check the differentiability at x = 1

Here, $f(1) = 1 - 1 = 0$

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{1-(1-h)-0)}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h} = -1$$

and

$$Rf'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{(1+h)-1-0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = 1$$

$\therefore Lf'(1) \neq Rf'(1)$.

Hence, f(x) is not differentiable at x = 1

12. Find $\frac{dy}{dx}$ if $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Ans.

Let x = $\tan\theta$ $\Rightarrow \theta = \tan^{-1} x$, then we have

$$y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \cos^{-1}\cos 2\theta = 2\theta$$

$\Rightarrow y = 2\tan^{-1} x$

$$\Rightarrow \frac{dy}{dx} = 2 \times \frac{1}{1+x^2} = \frac{2}{1+x^2}$$

13. Find $\frac{dy}{dx}$ if $y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right), 0 < x < 1$

Ans.

Let x = $\tan\theta$ $\Rightarrow \theta = \tan^{-1} x$, then we have

$$y = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right) = \sin^{-1}\cos 2\theta = \sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$\Rightarrow y = \frac{\pi}{2} - 2\theta \Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = 0 - 2 \times \frac{1}{1+x^2} = \frac{-2}{1+x^2}$$

14. Find $\frac{dy}{dx}$ if $y = \cos^{-1}\left(\frac{2x}{1+x^2}\right), -1 < x < 1$

Ans.

Let x = $\tan\theta \Rightarrow \theta = \tan^{-1} x$, then we have

$$y = \cos^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \cos^{-1}\sin 2\theta = \cos^{-1}\cos\left(\frac{\pi}{2} - 2\theta\right)$$

$$\Rightarrow y = \frac{\pi}{2} - 2\theta \Rightarrow y = \frac{\pi}{2} - 2\tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = 0 - 2 \times \frac{1}{1+x^2} = \frac{-2}{1+x^2}$$

15. Find $\frac{dy}{dx}$ if $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), 0 < x < \frac{1}{\sqrt{2}}$

Ans.

Let $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$, then we have

$$y = \sec^{-1}\left(\frac{1}{2x^2-1}\right) = \sec^{-1}\left(\frac{1}{2\cos^2\theta - 1}\right) = \sec^{-1}\left(\frac{1}{\cos 2\theta}\right) = \sec^{-1}\sec 2\theta = 2\theta$$

$$\Rightarrow y = 2\cos^{-1} x = 2 \times \frac{-1}{\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

16. Differentiate sin ($\tan^{-1} e^{-x}$) with respect to x.

Ans.

Let y = sin ($\tan^{-1} e^{-x}$)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\sin(\tan^{-1} e^{-x})\right] = \cos(\tan^{-1} e^{-x}) \frac{d}{dx}(\tan^{-1} e^{-x})$$

$$= \cos(\tan^{-1} e^{-x}) \frac{1}{1+(e^{-x})^2} \frac{d}{dx}(e^{-x})$$

$$= \cos(\tan^{-1} e^{-x}) \frac{1}{1+e^{-2x}}(-e^{-x}) = -\frac{e^{-x}\cos(\tan^{-1} e^{-x})}{1+e^{-2x}}$$

17. Differentiate log (cos e^x) with respect to x.

Ans.

Let y = log (cos e^x)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\log(\cos e^x)\right] = \frac{1}{\cos e^x} \frac{d}{dx}(\cos e^x)$$

$$= \frac{1}{\cos e^x}(-\sin e^x) \frac{d}{dx}(e^x) = (-\tan e^x).e^x = -e^x \tan e^x$$

18. Differentiate cos (log x + e^x), $x > 0$ with respect to x.

Ans.

Let y = cos (log x + e^x)

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left[\cos(\log x + e^x)\right] = -\sin(\log x + e^x) \frac{d}{dx}(\log x + e^x)$$

$$= -\sin(\log x + e^x)\left(\frac{1}{x} + e^x\right) = -\sin(\log x + e^x)\left(\frac{1 + xe^x}{x}\right)$$

$$= \frac{-(1 + xe^x)\sin(\log x + e^x)}{x}$$

19. Find $\frac{dy}{dx}$ if $y^x + x^y + x^x = a^b$.

Ans.

Given that $y^x + x^y + x^x = a^b$

Putting $u = y^x$, $v = x^y$ and $w = x^x$, we get $u + v + w = a^b$

Therefore, $\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} = 0$ ------------------ (1)

Now, $u = y^x$. Taking logarithm on both sides, we have $\log u = x \log y$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log y) + \log y \frac{d}{dx}(x) = x\frac{1}{y}.\frac{dy}{dx} + \log y.1$$

$$\Rightarrow \frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) \text{ -------------------- (2)}$$

Also $v = x^y$

Taking logarithm on both sides, we have $\log v = y \log x$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = y\frac{d}{dx}(\log x) + \log x\frac{dy}{dx} = y\frac{1}{x} + \log x\frac{dy}{dx}$$

$$\Rightarrow \frac{dv}{dx} = v\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) = x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) \text{ ------------------ (3)}$$

Again $w = x^x$

Taking logarithm on both sides, we have $\log w = x \log x$.

Differentiating both sides w.r.t. x, we have

$$\frac{1}{w}\frac{dw}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x) = x\frac{1}{x} + \log x.1$$

$$\Rightarrow \frac{dw}{dx} = w(1 + \log x) = x^x(1 + \log x) \text{ -------------------- (4)}$$

From (1), (2), (3), (4), we have

$$y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) + x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + x^x(1 + \log x) = 0$$

$$(x.y^{x-1} + x^y.\log x)\frac{dy}{dx} = -x^x(1 + \log x) - y.x^{y-1} - y^x \log y$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\left[x^x(1 + \log x) + y.x^{y-1} + y^x \log y\right]}{x.y^{x-1} + x^y.\log x}$$

20. Differentiate $x^x - 2^{\sin x}$ with respect to x.

Ans.

Let $y = x^x - 2^{\sin x}$

Let $u = x^x$ and $v = 2^{\sin x}$ then we have $y = u - v$

Therefore, $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$ -------------------- (1)

Now, $u = x^x$

Taking logarithm on both sides, we have $\log u = x \log x$.

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x) = x\frac{1}{x} + \log x.1$$

$$\Rightarrow \frac{du}{dx} = u\left(1+\log x\right) = x^x\left(1+\log x\right) \quad \text{--------------------} \quad (2)$$

Again $v = 2^{sinx}$

Taking logarithm on both sides, we have log v = (sin$x$) log 2.

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = \cos x(\log 2) \Rightarrow \frac{dv}{dx} = v[\cos x(\log 2)] = 2^{\sin x}[\cos x(\log 2)]$$

From (1), (2) and (3), we get

$$\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx} = x^x\left(1+\log x\right) - 2^{\sin x}[\cos x(\log 2)]$$

21. Differentiate $(\log x)^x + x^{\log x}$ with respect to x.

Ans.

Let y = $(\log x)^x + x^{\log x}$

Let u = $(\log x)^x$ and v = $x^{\log x}$ then we have y = u + v

Therefore, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ -------------------- (1)

Now, $u = (\log x)^x$

Taking logarithm on both sides, we have log u = xlog(logx).

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}(x) = \frac{x}{\log x}\times\frac{1}{x} + \log(\log x)$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{1}{\log x} + \log(\log x)\right] = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] \quad \text{----------------------} \ (2)$$

Again $v = x^{logx}$

Taking logarithm on both sides, we have log v = (log$x$) logx = $(\log x)^2$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = 2\log x\frac{d}{dx}(\log x) = 2\log x \times \frac{1}{x}$$

$$\Rightarrow \frac{dv}{dx} = v\left[\frac{2\log x}{x}\right] = x^{\log x}\left[\frac{2\log x}{x}\right] \quad \text{----------------------} \ (3)$$

From (1), (2) and (3)

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + x^{\log x}\left[\frac{2\log x}{x}\right]$$

$$\Rightarrow \frac{dy}{dx} = (\log x)^{x-1}\left[1 + \log x \log(\log x)\right] + 2x^{\log x - 1}.\log x$$

22. Differentiate $(\sin x)^x + \sin^{-1}\sqrt{x}$ **with respect to x.**

Ans.

Let y = $(\sin x)^x + \sin^{-1}\sqrt{x}$

Let u = $(\sin x)^x$, v = $\sin^{-1}\sqrt{x}$ then we have y = u + v

Therefore, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ -------------------- (1)

Now, $u = (\sin x)^x$

Taking logarithm on both sides, we have log u = xlog(sin x).

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log(\sin x) + \log(\sin x)\frac{d}{dx}(x) = \frac{x}{\sin x}\times\cos x + \log(\sin x)$$

$$\Rightarrow \frac{du}{dx} = u\left[x\cot x + \log(\sin x)\right] = (\sin x)^x\left[x\cot x + \log(\sin x)\right] \quad \text{------------- (2)}$$

Again $v = \sin^{-1}\sqrt{x}$

Differentiating both sides w.r.t. x, we have

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}(\sqrt{x}) = \frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x-x^2}} \quad \text{---------------- (3)}$$

From (1), (2) and (3)

$$\frac{dy}{dx} = (\sin x)^x\left[x\cot x + \log(\sin x)\right] + \frac{1}{2\sqrt{x-x^2}}$$

23. Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x.

Ans.

Let $y = x^{\sin x} + (\sin x)^{\cos x}$

Let $u = x^{\sin x}$, $v = (\sin x)^{\cos x}$ then we have y = u + v

Therefore, $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ -------------------- (1)

Now, $u = x^{sinx}$

Taking logarithm on both sides, we have log u = (sinx) logx.

Differentiating both sides w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = \sin x\frac{d}{dx}\log x + \log x\frac{d}{dx}(\sin x) = \sin x\frac{1}{x} + \log x\cos x$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{\sin x}{x} + \log x\cos x\right] = x^{\sin x}\left[\frac{\sin x}{x} + \log x\cos x\right] \quad \text{---------------------- (2)}$$

Again $v = \sin x^{\cos x}$

Taking logarithm on both sides, we have log v = (cos$x$) log(sinx)

Differentiating both sides w.r.t. x, we have

$$\frac{1}{v}\frac{dv}{dx} = cosx\frac{d}{dx}\log(\sin x) + \log(\sin x)\frac{d}{dx}(\cos x) = \cos x\frac{1}{\sin x}\times\cos x + \log(\sin x)(-\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v\left[\cot x\cos x - \sin x\log(\sin x)\right] = \sin x^{\cos x}\left[\cot x\cos x - \sin x\log(\sin x)\right] \quad \text{----------- (3)}$$

From (1), (2) and (3)

$$\frac{dy}{dx} = x^{\sin x}\left[\frac{\sin x}{x} + \log x\cos x\right] + \sin x^{\cos x}\left[\cot x\cos x - \sin x\log(\sin x)\right]$$

24. Find $\frac{dy}{dx}$ if $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$.

Ans.

Given that $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1+\cos\theta), \frac{dy}{d\theta} = a(\sin\theta)$$

Therefore, $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a(\sin\theta)}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

25. Find $\frac{dy}{dx}$ if $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$

Ans.

Given that $x = \cos\theta - \cos 2\theta$, $y = \sin\theta - \sin 2\theta$
Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = -\sin\theta - (-\sin 2\theta)\times 2 = -\sin\theta + 2\sin 2\theta$$

$$\frac{dy}{d\theta} = \cos\theta - (\cos 2\theta)\times 2 = \cos\theta - 2\cos 2\theta$$

Therefore, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\sin\theta + 2\sin 2\theta}{\cos\theta - 2\cos 2\theta}$

26. Find $\frac{dy}{dx}$ if $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

Ans.
Given that $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$
Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta), \frac{dy}{d\theta} = a(0-\sin\theta) = -a\sin\theta$$

Therefore, $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \frac{-\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = -\cot\frac{\theta}{2}$

27. If $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$

Ans.
Given that $x = \sqrt{a^{\sin^{-1}t}}, y = \sqrt{a^{\cos^{-1}t}}$
Multiplying both we get,

$$xy = \sqrt{a^{\sin^{-1}t}}\sqrt{a^{\cos^{-1}t}} = \sqrt{a^{\sin^{-1}t}.a^{\cos^{-1}t}} = \sqrt{a^{\sin^{-1}t+\cos^{-1}t}} = \sqrt{a^{\pi/2}}$$

Differentiating both sides w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

28. If $y = 3e^{2x} + 2e^{3x}$, prove that $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

Ans.
Given that $y = 3e^{2x} + 2e^{3x}$
Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 6e^{2x} + 6^{3x} = 6(e^{2x} + e^{3x})$$

Again, Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = 6(2e^{2x} + 3e^{3x})$$

Now, $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 6(2e^{2x} + 3e^{3x}) - 5(6(e^{2x} + e^{3x})) + 6(3e^{2x} + 2e^{3x})$

$$= 12e^{2x} + 18e^{3x} - 30e^{2x} - 30e^{3x} + 18e^{2x} + 12e^{3x} = 0$$

29. If $y = 3\cos(\log x) + 4\sin(\log x)$, show that $x^2 y_2 + xy_1 + y = 0$

Ans.

Given that $y = 3 \cos (\log x) + 4 \sin (\log x)$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = y_1 = -3\sin(\log x)\frac{d}{dx}(\log x) + 4\cos(\log x)\frac{d}{dx}(\log x)$$

$$= -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x} = \frac{1}{x}\left[-3\sin(\log x) + 4\cos(\log x)\right]$$

$$\Rightarrow xy_1 = -3\sin(\log x) + 4\cos(\log x)$$

Again, Differentiating both sides w.r.t. x, we get

$$xy_2 + y_1.1 = -3\cos(\log x)\frac{d}{dx}(\log x) - 4\sin(\log x)\frac{d}{dx}(\log x)$$

$$= -3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x} = -\frac{1}{x}\left[3\cos(\log x) + 4\sin(\log x)\right] = -\frac{y}{x}$$

$$\Rightarrow x^2y_2 + xy_1 = -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

30. If $e^y (x + 1) = 1$, show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$

Ans.

Given that $e^y (x + 1) = 1$

$$\Rightarrow e^y = \frac{1}{x+1}$$

Differentiating both sides w.r.t. x, we get

$$e^y \frac{dy}{dx} = -\frac{1}{(x+1)^2} \Rightarrow \frac{1}{x+1}\frac{dy}{dx} = -\frac{1}{(x+1)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{1}{x+1}$$

Again, Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(-\frac{1}{x+1}\right)^2 = \left(\frac{dy}{dx}\right)^2$$

31. If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

Ans.

Given that $y = (\tan^{-1} x)^2$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1} x\frac{d}{dx}(\tan^{-1} x) = 2\tan^{-1} x \times \frac{1}{1+x^2}$$

$$\Rightarrow y_1 = \frac{2\tan^{-1} x}{1+x^2} \Rightarrow (1+x^2)y_1 = 2\tan^{-1} x$$

Again, Differentiating both sides w.r.t. x, we get

$$(1+x^2)y_2 + 2xy_1 = 2\times\frac{1}{1+x^2} \Rightarrow (1+x^2)^2 y_2 + 2x(1+x^2)y_1 = 2$$

32. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for , – 1 < x < 1, prove that $\frac{dy}{dx} = -\frac{1}{(1+x)^2}$

Ans.

Given that $x\sqrt{1+y} + y\sqrt{1+x} = 0 \Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides, we get

$$x^2(1+y) = y^2(1+x) \qquad \Rightarrow x^2 - y^2 + x^2y - y^2x = 0$$

$\Rightarrow (x-y)(x+y)+xy(x-y)=0 \quad \Rightarrow (x-y)(x+y+xy)=0$

$\Rightarrow x-y=0 \;\; or \;\; x+y+xy=0$

$\Rightarrow y=x \;\; or \;\; y(1+x)=-x \Rightarrow y=x \;\; or \;\; y=\frac{-x}{1+x}$

But y =x does not satisfy the given equation

So, we consider, $y=\frac{-x}{1+x}$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{-x}{x+1}\right)=-\frac{(1+x)\frac{d}{dx}(x)-x\frac{d}{dx}(1+x)}{(1+x)^2}=-\frac{(1+x)-x}{(1+x)^2}=-\frac{1}{(1+x)^2}$$

33. If cos *y* = *x* cos (*a* + *y*), with cos *a* ≠ ± 1, prove that $\frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$

Ans.

Given that cos *y* = *x* cos (*a* + *y*)$\Rightarrow x=\frac{\cos y}{\cos(a+y)}$

Differentiating both sides w.r.t. y, we get

$$\frac{dx}{dy}=\frac{d}{dy}\left(\frac{\cos y}{\cos(a+y)}\right)=\frac{\cos(a+y)(-\sin y)-\cos y[-\sin(a+y)]}{\cos^2(a+y)}$$

$$=\frac{\sin(a+y)\cos y-\cos(a+y)\sin y}{\cos^2(a+y)}=\frac{\sin(a+y-y)}{\cos^2(a+y)}=\frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos^2(a+y)}{\sin a}$$

34. If *x* = *a* (cos *t* + *t* sin *t*) and *y* = *a* (sin *t* – *t* cos *t*), find $\frac{d^2y}{dx^2}$

Ans.

Given that *x* = *a* (cos *t* + *t* sin *t*) and *y* = *a* (sin *t* – *t* cos *t*)

Differentiating both sides w.r.t. t, we get

$$\frac{dx}{dt}=a\left[\frac{d}{dt}\cos t+\left(t\frac{d}{dt}\sin t+\sin t\frac{d}{dt}(t)\right)\right]=a[-\sin t+(t\sin t+\sin t)]=at\cos t$$

and $\frac{dy}{dt}=a\left[\frac{d}{dt}\sin t+\left(t\frac{d}{dt}\cos t+\cos t\frac{d}{dt}(t)\right)\right]=a[\cos t-(-t\sin t+\cos t)]=at\sin t$

Now, $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{at\sin t}{at\cos t}=\tan t$

Again, Differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\tan t=\sec^2 t\times\frac{dt}{dx}=\frac{\sec^2 t}{at\cos t}=\frac{1}{at}\sec^3 t$$

35. Differentiate $\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)$ **w.r.t. x**

Ans.

Let y $=\tan^{-1}\left(\frac{\sin x}{1+\cos x}\right)=\tan^{-1}\left(\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right) = \tan^{-1}(\tan\frac{x}{2}) = \frac{x}{2}$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}$$

36. Differentiate $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ **w.r.t. x**

Ans.

Let y = $\sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right) = \sin^{-1}\left(\frac{2^x \times 2}{1+(2^x)^2}\right)$

Let $2^x = \tan\theta \Rightarrow \theta = \tan^{-1} 2^x$ then we have

$$y = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}(\sin 2\theta) = 2\theta = 2\tan^{-1} 2^x$$

Differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = 2 \times \frac{1}{1+(2^x)^2}\frac{d}{dx}(2^x) = \frac{2}{1+4^x}2^x \log 2 = \frac{2^{x+1}\log 2}{1+4^x}$$

37. Differentiate $\sin^2 x$ w.r.t. $e^{\cos x}$.

Ans.

Let u = $\sin^2 x$ and v = $e^{\cos x}$

Differentiating u and v w.r.t. x, we get

$$\frac{du}{dx} = 2\sin x \frac{d}{dx}(\sin x) = 2\sin x \cos x$$

and $\frac{dv}{dx} = e^{\cos x}\frac{d}{dx}(\cos x) = e^{\cos x}(-\sin x) = (-\sin x)e^{\cos x}$

Now, $\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2\sin x\cos x}{(-\sin x)e^{\cos x}} = -\frac{2\cos x}{e^{\cos x}}$

CHAPTER – 5: CONTINUITY AND DIFFERENTIABILITY

MARKS WEIGHTAGE – 08 marks

Previous Years Board Exam (Important Questions)

1. For what value of *k* is the following function continuous at *x* = 2?

$$f(x)=\begin{cases}2x+1 \ ; \ x<2\\ k \quad ; \ x=2\\ 3x-1 \ ; \ x>2\end{cases}$$

2. Discuss the continuity of the following function at *x* = 0 : $f(x)=\begin{cases}\dfrac{x^4+2x^3+x^2}{\tan^{-1}x}, & x\neq 0\\ 0 & , x=0\end{cases}$

3. If the function *f* (*x*) given by $f(x)=\begin{cases}3ax+b \ ; \ x>1\\ 11 \ ; \quad x=1\\ 5ax-2b \ ; \ x<1\end{cases}$ is continuous at *x* =1, find the values of *a* and *b*.

4. Find the relationship between '*a*' and '*b*' so that the function '*f*' defined by:

$f(x)=\begin{cases}ax+1, & x\le 3\\ bx+3, & x>3\end{cases}$ is continuous at *x* = 3 .

5. Show that the function *f*(*x*) = |*x* – 3| , $x\in R$, is continuous but not differentiable at *x* = 3.

6. Find the value of *k*, for which $f(x)=\begin{cases}\dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & -1\le x<0\\ \dfrac{2x+1}{x-1}, & 0\le x<1\end{cases}$ is continuous at *x* = 0.

7. Find the value of *k* so that the function *f*, defined by $f(x)=\begin{cases}kx+1, & x\le \pi\\ \cos x, & x>\pi\end{cases}$ is continuous at *x* = π.

8. Find the value of '*a*' for which the function *f* defined as

$f(x)=\begin{cases}a\sin\dfrac{\pi}{2}(x+1), & x\le 0\\ \dfrac{\tan x-\sin x}{x^3}, & x>0\end{cases}$ is continuous at *x* = 0.

9. Find all points of discontinuity of *f*, where *f* is defined as follows :

$$f(x)=\begin{cases}|x|+3 \ ; \ x\le -3\\ -2x \ ; \ -3<x<3\\ 6x+2 \ ; \ x\ge 3\end{cases}$$

10. Show that the function *f* defined as follows, is continuous at *x* = 2, but not differentiable:

$$f(x)=\begin{cases}3x-2 \ ; \ 0<x\le 1\\ 2x^2-x; \ 1<x\le 2\\ 5x-4 \ ; \ x>2\end{cases}$$

11. If $f(x)=\sqrt{\dfrac{\sec x-1}{\sec x+1}}$, then f'(x). Also find $f'\left(\dfrac{\pi}{2}\right)$.

12. Find $\dfrac{dy}{dx}$,if $(x^2+y^2)^2=xy$.

13. If sin $y = x$ sin($a + y$), prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$

14. If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

15. If $y=\frac{\sin^{-1}x}{\sqrt{1-x^2}}$, then prove that $(1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0$.

16. If $y = e^x(\sin x + \cos x)$, then show that $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$.

17. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, then find $\frac{d^2y}{dx^2}$.

18. If $\log(x^2+y^2)=2\tan^{-1}\left(\frac{y}{x}\right)$, then show that $\frac{dy}{dx}=\frac{x+y}{x-y}$

19. If $y = a \cos(\log x) + b \sin(\log x)$, then show that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$

20. If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, then show that $\frac{dy}{dx}=\sqrt{\frac{1-y^2}{1-x^2}}$

21. Find $\frac{dy}{dx}$,if $y=\sin^{-1}\left[x\sqrt{1-x}-\sqrt{x}\sqrt{1-x^2}\right]$

22. Find $\frac{dy}{dx}$,if $y=(\cos x)^x+(\sin x)^{1/x}$

23. Differentiate the following with respect to $x:\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$

24. If $y=\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right)$, find $\frac{dy}{dx}$

25. Differentiate the following function w.r.t. x : $(x)^{\cos x} + (\sin x)^{\tan x}$

26. If $y=e^{a\sin^{-1}x},-1\le x\le 1$, then show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$

27. If $y=\cos^{-1}\left(\frac{3x+4\sqrt{1-x^2}}{5}\right)$, find $\frac{dy}{dx}$

28. If $y = \text{cosec}^{-1} x$, $x > 1$. then show that $x(x^2-1)\frac{d^2y}{dx^2}+(2x^2-1)\frac{dy}{dx}=0$.

29. If $y=\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$ then show that $\frac{dy}{dx}=\sec x$. Also find $\frac{d^2y}{dx^2}$ at $x=\frac{\pi}{4}$.

30. Differentiate the following function with respect to x: $f(x)=\tan^{-1}\left(\frac{1-x}{1+x}\right)-\tan^{-1}\left(\frac{x+2}{1-2x}\right)$

31. If $x=a\left(\cos t+\log\tan\frac{t}{2}\right), y=a(1+\sin t)$, then find $\frac{d^2y}{dx^2}$.

32. If $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$, then find $\frac{d^2y}{dx^2}$.

33. Differentiate $x^{\cos x}+\frac{x^2+1}{x^2-1}$ w.r.t.x

34. If $x^y=e^{x-y}$, then show that $\frac{dy}{dx}=\frac{\log x}{[\log(xe)]^2}$

35. If $x = \tan\left(\frac{1}{a}\log y\right)$, then show that $(1+x^2)\frac{d^2y}{dx^2}+(2x-a)\frac{dy}{dx}=0$.

36. Prove that $\frac{d}{dx}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]=\sqrt{a^2-x^2}$

37. If $y=\log\left[x+\sqrt{x^2+1}\right]$ then show that $(x^2+1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

38. If $y=\log\left[x+\sqrt{x^2+a^2}\right]$ then show that $(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$

39. If $y=\sin^{-1}x$, then show that $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0$.

40. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}-1}{x}\right)$ w.r.t. x.

41. If $x = a$ (cos t + t sin t) and $y = a$ (sin t – t cos t), $0<t<\frac{\pi}{2}$, find $\frac{d^2x}{dt^2},\frac{d^2y}{dt^2},\frac{d^2y}{dx^2}$.

42. If $x^m y^m = (x+y)^{m+n}$, then show that $\frac{dy}{dx}=\frac{y}{x}$.

43. If $x^{16}y^9=(x^2+y)^{17}$, then show that $\frac{dy}{dx}=\frac{2y}{x}$.

44. If $x=a\sin t, y=a\left(\cos t+\log\tan\frac{t}{2}\right)$, then find $\frac{d^2y}{dx^2}$.

45. If $y^x=e^{y-x}$, then show that $\frac{dy}{dx}=\frac{(1+\log y)^2}{\log y}$.

46. If $x^y=e^{x-y}$, then show that $\frac{dy}{dx}=\frac{\log x}{(1+\log x)^2}$

47. Differentiate the following with respect to x : $\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$

48. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\frac{d^2y}{dx^2}$ at $\theta=\frac{\pi}{6}$.

49. If $x\sin(a + y) + \sin a\cos(a + y) = 0$, prove that $\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$.

50. If $y = \sin(\log x)$, then prove that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0$.

51. Show that the function $f(x) = 2x - |x|$ is continuous but not differentiable at $x = 0$.

52. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}\left(2x\sqrt{1-x^2}\right)$.

53. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

54. If $y=x^x$, then show that $\frac{d^2y}{dx^2}-\frac{1}{y}\left(\frac{dy}{dx}\right)^2-\frac{y}{x}=0$

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. The function f (x) = $\begin{cases} \frac{\sin x}{x} + \cos x, & if \quad x \neq 0 \\ k, & if \quad x = 0 \end{cases}$ is continuous at x = 0, then the value of k is

(a) 3 (b) 2 (c) 1 (d) 1.5

2. The function f (x) = [x], where [x] denotes the greatest integer function, is continuous at
(a) 4 (b) – 2 (c) 1 (d) 1.5

3. The number of points at which the function f (x) = $\frac{1}{x-[x]}$ is not continuous is

(a) 1 (b) 2 (c) 3 (d) none of these

4. The function given by f (x) = tanx is discontinuous on the set

(a) $\{n\pi : n \in Z\}$ (b) $\{2n\pi : n \in Z\}$ (c) $\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$ (d) $\left\{\frac{n\pi}{2} : n \in Z\right\}$

5. Let f (x)= |cosx|. Then,
(a) f is everywhere differentiable.
(b) f is everywhere continuous but not differentiable at x = nπ, n ∈Z .
(c) f is everywhere continuous but not differentiable at x = (2n + 1)$\frac{\pi}{2}$
(d) none of these.

6. The function f (x) = |x| + |x – 1| is
(a) continuous at x = 0 as well as at x = 1. (b) continuous at x = 1 but not at x = 0.
(c) discontinuous at x = 0 as well as at x = 1. (d) continuous at x = 0 but not at x = 1.

7. The value of k which makes the function defined by f(x) = $\begin{cases} \sin\frac{1}{x}, & if \quad x \neq 0 \\ k, & if \quad x = 0 \end{cases}$, continuous at x = 0 is

(a) 8 (b) 1 (c) –1 (d) none of these

8. The set of points where the functions f given by f (x) = |x – 3| cosx is differentiable is
(a) R (b) R – {3} (c) (0, ∞) (d) none of these

9. Differential coefficient of sec ($\tan^{-1}$x) w.r.t. x is

(a) $\frac{x}{\sqrt{1+x^2}}$ (b) $\frac{x}{1+x^2}$ (c) $x\sqrt{1+x^2}$ (d) $\frac{1}{\sqrt{1+x^2}}$

10. If u = $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ and v = $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then $\frac{du}{dv}$ is

(a) $\frac{1}{2}$ (b) x (c) $\frac{1-x^2}{1+x^2}$ (d) 1

11. A function f is said to be continuous for x ∈ R, if
(a) it is continuous at x = 0 (b) differentiable at x = 0
(c) continuous at two points (d) differentiable for x R

12. If f (x) = $\frac{\sin(e^{x-2}-1)}{\log(x-1)}$, x ≠ 2 and f(x) = k for x = 2, then value of k for which f is continuous is

(a) –2 (b) –1 (c) 0 (d) 1

13. If f (x) = 2x and g (x) = $\frac{x^2}{2}+1$, then which of the following can be a discontinuous function

(a) f (x) + g (x) (b) f (x) – g (x) (c) f (x) . g (x) (d) $\frac{g(x)}{f(x)}$

14. The function f (x) = $\frac{1-x^2}{4x-x^3}$ is

(a) discontinuous at only one point (b) discontinuous at exactly two points
(c) discontinuous at exactly three points (d) none of these

15. The set of points where the function f given by f (x) = |2x-1| sinx is differentiable is

(a) R (b) R – $\left\{\frac{1}{2}\right\}$ (c) (0, ∞) (d) none of these

16. The function f (x) = cot x is discontinuous on the set

(a) $\{n\pi : n \in Z\}$ (b) $\{2n\pi : n \in Z\}$ (c) $\left\{(2n+1)\frac{\pi}{2} : n \in Z\right\}$ (d) $\left\{\frac{n\pi}{2} : n \in Z\right\}$

17. Let f (x)= | sinx |. Then,

(a) f is everywhere differentiable.
(b) f is everywhere continuous but not differentiable at x = nπ, n ∈Z .
(c) f is everywhere continuous but not differentiable at x = (2n + 1)$\frac{\pi}{2}$, n ∈Z
(d) none of these.

18. The function f (x) = $e^{|x|}$ is

(a) continuous everywhere but not differentiable at x = 0
(b) continuous and differentiable everywhere
(c) not continuous at x = 0
(d) none of these.

19. If f (x) = $x^2 \sin\frac{1}{x}$, where x ≠ 0, then the value of the function f at x = 0, so that the function is continuous at x = 0, is

(a) 0 (b) – 1 (c) 1 (d) none of these

20. If f(x) = $\begin{cases} mx+1, & if \quad x \le \frac{\pi}{2} \\ \sin x + n, & if \quad x > \frac{\pi}{2} \end{cases}$, is continuous at x = $\frac{\pi}{2}$, then

(a) m = 1, n = 0 (b) m = $\frac{n\pi}{2}+1$ (c) n = $\frac{m\pi}{2}$ (d) m = n = $\frac{\pi}{2}$

21. If y = $\log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to ______.

(a) $\frac{4x^3}{1-x^4}$ (b) $\frac{-4x}{1-x^4}$ (c) $\frac{1}{4-x^4}$ (d) $\frac{-4x^3}{1-x^4}$

22. If y = $\sqrt{\sin x + y}$, then $\frac{dy}{dx}$ is equal to ______.

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$ (c) $\frac{\sin x}{2y-1}$ (d) $\frac{\sin x}{1-2y}$

23. The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1}x$ is

(a) 2 (b) $\frac{-1}{2\sqrt{1-x^2}}$ (c) $\frac{2}{x}$ (d) $1 - x^2$.

24. If $x = t^2$, $y = t^3$, then $\frac{d^2y}{dx^2}$ is

(a) $\frac{3}{2}$ (b) $\frac{3}{4t}$ (c) $\frac{3}{2t}$ (d) $\frac{3}{4}$

25. If the function $f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x}, & if \quad x \neq \frac{\pi}{2} \\ 3 \quad , & if \quad x = \frac{\pi}{2} \end{cases}$ be continuous at x = $\frac{\pi}{2}$, then k =

(a) 3 (b) 6 (c) 12 (d) none of these

26. In order that the function $f(x) = (x + 1)^{1/x}$ is continuous at x = 0, f(0) must be defined as

(a) f(0) = 0 (b) f(0) = e (c) f(0) = 1/e (d) f(0) = 1

27. Let $f(x) = \begin{cases} x^2 + k, & if \quad x \geq 0 \\ -x^2 - k, & if \quad x < 0 \end{cases}$. If the function f(x) be continuous at x = 0, then k =

(a) 0 (b) 1 (c) 2 (d) -2

28. The points at which the function $f(x) = \frac{x+1}{x^2 + x - 12}$ is discontinuous are

(a) -3, 4 (b) 3, -4 (c) -1, -3, 4 (d) -1, 3, 4

29. If f(x) = | x - b|, then function

(a) Is continuous at x = 1 (b) Is continuous at x = b

(c) Is discontinuous at x = b (d) None of these

30. If $f(x) = \begin{cases} 1 + x, & if \quad x \leq 2 \\ 5 - x, & if \quad x > 2 \end{cases}$, then

(a) f(x) is continuous at x = 2 (b) f(x) is discontinuous at x = 2

(c) f(x) is continuous at x = 3 (d) None of these

31. If function $f(x) = \begin{cases} \frac{x^2 - 1}{x - 1}, & if \quad x \neq 1 \\ k, & if \quad x = 1 \end{cases}$ is continuous at x = 1, then the value of k will be

(a) -1 (b) 2 (c) -3 (d) -2

32. For the function $f(x)=\begin{cases}\frac{\sin^2 ax}{x^2}, & if \quad x\neq 0\\ 1, & if \quad x=0\end{cases}$ which one is a true statement

(a) f(x) is continuous at x = 0
(b) f(x) is discontinuous at x = 0, when a $\neq \pm 1$
(c) f(x) is continuous at x = a
(d) None of these

33. If $f(x)=\begin{cases}\frac{\sin 2x}{5x}, & if \quad x\neq 0\\ k, & if \quad x=0\end{cases}$ is continuous at x = 0, then the value of k will be

(a) 1 (b) $\frac{2}{5}$ (c) $-\frac{2}{5}$ (d) None of these

34. If $f(x)=\begin{cases}\frac{|x-a|}{x-a}, & if \quad x\neq a\\ 1, & if \quad x=a\end{cases}$, then

(a) f(x) is continuous at x = a
(b) f(x) is discontinuous at x = a
(c) $\lim_{x\to a} f(x)=1$
(d) None of these

35. If $f(x)=\begin{cases}\frac{x^4-16}{x-2}, & if \quad x\neq 2\\ 16, & if \quad x=2\end{cases}$, then

(a) f(x) is continuous at x = 2
(b) f(x) is discontinuous at x = 2
(c) $\lim_{x\to 2} f(x)=16$
(d) None of these

36. The number of points at which the function f (x) = $\frac{1}{\log|x|}$ is discontinuous is ________.

37. If $f(x)=\begin{cases}ax+1, & if \quad x\geq 1\\ x+2, & if \quad x<1\end{cases}$ is continuous, then a should be equal to _______.

38. The derivative of $\log_{10}x$ w.r.t. x is ________.

39. If y = $\sec^{-1}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)$ + $\sin^{-1}\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\frac{dy}{dx}$ is equal to ______.

40. The deriative of sin x w.r.t. cos x is ________.

41. The function f (x) = $\frac{x+1}{1+\sqrt{1+x}}$ is continuous at x = 0 if f (0) is _________.

42. An example of a function which is continuous everywhere but fails to be differentiable exactly at two points is __________ .

43. Derivative of x^2 w.r.t. x^3 is _________.

44. If f (x) = |cosx|, then $f'\left(\frac{\pi}{4}\right)$ = _______ .

45. If f (x) = |cosx – sinx | , then $f'\left(\frac{\pi}{3}\right)$= _______.

46. For the curve $\sqrt{x}+\sqrt{y}=1$, $\frac{dy}{dx}$ at $\left(\frac{1}{4},\frac{1}{4}\right)$ is __________.

VECTOR ALGEBRA

CHAPTER – 10: VECTOR ALGEBRA

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. **Find the unit vector in the direction of the sum of the vectors, $\vec{a} = 2\hat{i} + 2\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + 3\hat{k}$**

Ans:

The sum of the given vectors is $\vec{a} + \vec{b}\ \ (= \vec{c}, say) = 4\hat{i} + 3\hat{j} - 2\hat{k}$

and $|\vec{c}| = \sqrt{4^2 + 3^2 + (-2)^2} = \sqrt{29}$

Thus, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{\sqrt{29}}(4\hat{i} + 3\hat{j} - 2\hat{k}) = \frac{4}{\sqrt{29}}\hat{i} + \frac{3}{\sqrt{29}}\hat{j} - \frac{2}{\sqrt{29}}\hat{k}$$

2. **Show that the points are $A(2\hat{i} - \hat{j} + \hat{k}), B(\hat{i} - 3\hat{j} - 5\hat{k}), C(3\hat{i} - 4\hat{j} - 4\hat{k})$ the vertices of a right angled triangle.**

Ans:

We have $\overrightarrow{AB} = (1-2)\hat{i} + (-3+1)\hat{j} + (-5-1)\hat{k} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC} = (3-1)\hat{i} + (-4+3)\hat{j} + (-4+5)\hat{k} = 2\hat{i} - \hat{j} + \hat{k}$

and $\overrightarrow{CA} = (2-3)\hat{i} + (-1+4)\hat{j} + (1+4)\hat{k} = -\hat{i} + 3\hat{j} + 5\hat{k}$

Then $|\overrightarrow{AB}|^2 = 41, |\overrightarrow{BC}|^2 = 6, |\overrightarrow{CA}|^2 = 35$

$\Rightarrow |\overrightarrow{AB}|^2 = |\overrightarrow{BC}|^2 + |\overrightarrow{CA}|^2$

Hence, the triangle is a right angled triangle.

3. **Find the direction cosines of the vector joining the points A(1, 2, –3) and B(–1, –2, 1), directed from A to B.**

Ans:

The given points are $A(1,2,-3)$ and $B(-1,-2,1)$.

Then $\overrightarrow{AB} = (-1-1)\hat{i} + (-2-2)\hat{j} + (1-(-3))\hat{k} = -2\hat{i} - 4\hat{j} + 4\hat{k}$

Now, $|\overrightarrow{AB}| = \sqrt{4+16+16} = \sqrt{36} = 6$

$\therefore$ unit vector along $\overrightarrow{AB} = \frac{1}{|\overrightarrow{AB}|}\overrightarrow{AB} = \frac{1}{6}(-2\hat{i} - 4\hat{j} + 4\hat{k}) = -\frac{1}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Hence direction cosines are $-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$

4. **Find the position vector of a point R which divides the line joining two points P and Q whose position vectors are $\hat{i} + 2\hat{j} - \hat{k}$ and $-\hat{i} + \hat{j} + \hat{k}$ respectively, in the ratio 2 : 1 (i) internally (ii) externally**

Ans:

The position vector of a point *R* divided the line segment joining two points *P* and *Q* in the ratio *m*: *n* is given by

Case I Internally $= \frac{m\vec{b} + n\vec{a}}{m+n}$

Case II Externally $= \frac{m\vec{b} - n\vec{a}}{m-n}$

Position vectors of *P* and *Q* are given as $\overrightarrow{OP} = \hat{i} + 2\hat{j} - \hat{k}, \overrightarrow{OQ} = -\hat{i} + \hat{j} + \hat{k}$

(i) Position vector of R [dividing (PQ) in the ratio *2 : 1* internally]

$$= \frac{m\overrightarrow{OQ} + n\overrightarrow{OP}}{m+n} = \frac{2(-\hat{i}+\hat{j}+\hat{k})+1(\hat{i}+2\hat{j}-\hat{k})}{2+1} = \frac{-\hat{i}+4\hat{j}+\hat{k}}{3} - \frac{1}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(i) Position vector of R [dividing (PQ) in the ratio *2 : 1* externally]

$$= \frac{m\overrightarrow{OQ} - n\overrightarrow{OP}}{m-n} = \frac{2(-\hat{i}+\hat{j}+\hat{k})-1(\hat{i}+2\hat{j}-\hat{k})}{2-1} = \frac{-3\hat{i}+0\hat{j}+3\hat{k}}{1} - 3\hat{i}+3\hat{k}$$

5. Find the position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2).

Ans:

Position vectors of *P* and *Q* are given as $\overrightarrow{OP} = 2\hat{i}+3\hat{j}+4\hat{k}, \overrightarrow{OQ} = 4\hat{i}+\hat{j}-2\hat{k}$

The position vector of the mid point of the vector joining the points P(2, 3, 4) and Q(4, 1, –2) is given by

Position Vector of the mid-point of (*PQ*) = $\frac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OP}\right) = \frac{1}{2}\left(4\hat{i}+\hat{j}-2\hat{k}+2\hat{i}+3\hat{j}+4\hat{k}\right)$

$$= \frac{1}{2}\left(6\hat{i}+4\hat{j}+2\hat{k}\right) = 3\hat{i}+2\hat{j}+\hat{k}$$

6. Show that the points A, B and C with position vectors, $\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}$, $\vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ and $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$ respectively form the vertices of a right angled triangle.

Ans:

Position vectors of points *A*, *B* and *C* are respectively given as

$\vec{a} = 3\hat{i}-4\hat{j}-4\hat{k}, \vec{b} = 2\hat{i}-\hat{j}+\hat{k}$ *and* $\vec{c} = \hat{i}-3\hat{j}-5\hat{k}$

Now, $\overrightarrow{AB} = \vec{b}-\vec{a} = 2\hat{i}-\hat{j}+\hat{k}-3\hat{i}+4\hat{j}+4\hat{k} = -\hat{i}+3\hat{j}+5\hat{k}$

$\Rightarrow |\overrightarrow{AB}|^2 = 1+9+25 = 35$

$\overrightarrow{BC} = \vec{c}-\vec{b} = \hat{i}-3\hat{j}-5\hat{k}-2\hat{i}+\hat{j}-\hat{k} = -\hat{i}-2\hat{j}-6\hat{k}$

$\Rightarrow |\overrightarrow{BC}|^2 = 1+4+36 = 41$

$\overrightarrow{CA} = \vec{a}-\vec{c} = 3\hat{i}-4\hat{j}-4\hat{k}-\hat{i}+3\hat{j}+5\hat{k} = 2\hat{i}-\hat{j}+\hat{k}$

$\Rightarrow |\overrightarrow{CA}|^2 = 4+1+1 = 6$

$\Rightarrow |\overrightarrow{BC}|^2 = |\overrightarrow{AB}|^2 + |\overrightarrow{CA}|^2$

Hence it form the vertices of a right angled triangle.

7. Find angle 'θ' between the vectors $\vec{a} = \hat{i}+\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$

Ans:

The angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

Now, $\vec{a}.\vec{b} = (\hat{i}+\hat{j}-\hat{k}).(\hat{i}-\hat{j}+\hat{k}) = 1-1-1 = -1$

Therefore, we have $\cos\theta = \frac{-1}{3} \Rightarrow \theta = \cos^{-1}\left(\frac{-1}{3}\right)$

8. If $\vec{a} = 5\hat{i}-\hat{j}-3\hat{k}$ and $\vec{b} = \hat{i}+3\hat{j}-5\hat{k}$, then show that the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular.

Ans:

We know that two nonzero vectors are perpendicular if their scalar product is zero.

Here, $\vec{a}+\vec{b} = 5\hat{i}-\hat{j}-3\hat{k}+\hat{i}+3\hat{j}-5\hat{k} = 6\hat{i}+2\hat{j}-8\hat{k}$

and $\vec{a}-\vec{b} = 5\hat{i}-\hat{j}-3\hat{k}-\hat{i}-3\hat{j}+5\hat{k} = 4\hat{i}-4\hat{j}+2\hat{k}$

Now, $(\vec{a}+\vec{b}).(\vec{a}-\vec{b}) = (6\hat{i}+2\hat{j}-8\hat{k}).(4\hat{i}-4\hat{j}+2\hat{k}) = 24-8-16=0$

Hence $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular.

9. Find $|\vec{a}-\vec{b}|$, if two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}|=2$, $|\vec{b}|=3$ and $\vec{a}.\vec{b}$ = 4.

Ans:

We have

$|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b}).(\vec{a}-\vec{b}) = \vec{a}.\vec{a}-\vec{a}.\vec{b}-\vec{b}.\vec{a}+\vec{b}.\vec{b}$

$=|\vec{a}|^2 - 2(\vec{a}.\vec{b}) + |\vec{b}|^2 = 2^2-2(4)+3^2 = 4-8+9=5$

$\Rightarrow |\vec{a}-\vec{b}| = \sqrt{5}$

10. Show that the points $A(-2\hat{i}+3\hat{j}+5\hat{k}), B(\hat{i}+2\hat{j}+3\hat{k}), C(7\hat{i}-\hat{k})$ are collinear.

Ans:

We have

$\overrightarrow{AB} = (1+2)\hat{i}+(2-3)\hat{j}+(3-5)\hat{k} = 3\hat{i}-\hat{j}-2\hat{k}$

$\overrightarrow{BC} = (7-1)\hat{i}+(0-2)\hat{j}+(-1-3)\hat{k} = 6\hat{i}-2\hat{j}-4\hat{k}$

$\overrightarrow{CA} = (7+2)\hat{i}+(0-3)\hat{j}+(-1-5)\hat{k} = 9\hat{i}-3\hat{j}-6\hat{k}$

Now, $|\overrightarrow{AB}|^2 = 14, |\overrightarrow{BC}|^2 = 56, |\overrightarrow{CA}|^2 = 126$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{14}, |\overrightarrow{BC}| = 2\sqrt{14}, |\overrightarrow{CA}| = 3\sqrt{14}$

$\Rightarrow |\overrightarrow{CA}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Hence the points A, B and C are collinear.

11. If $\vec{a},\vec{b},\vec{c}$ are unit vectors such that $\vec{a}+\vec{b}+\vec{c}=0$, find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

Ans:

Given that $|\vec{a}|=1, |\vec{b}|=1, |\vec{c}|=1, \vec{a}+\vec{b}+\vec{c}=0$

$(\vec{a}+\vec{b}+\vec{c})^2 = (\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c}) = 0$

$\Rightarrow \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}+\vec{b}.\vec{b}+\vec{b}.\vec{c}+\vec{b}.\vec{a}+\vec{c}.\vec{a}+\vec{c}.\vec{b}+\vec{c}.\vec{c}=0$

$\Rightarrow |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$

$\Rightarrow 1+1+1+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=0$

$\Rightarrow 2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=-3$

$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a} = \frac{-3}{2}$

12. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find $\angle$ ABC.

Ans:

We are given the points $A(1, 2, 3)$, $B(-1, 0, 0)$ and $C(0, 1, 2)$.

Also, it is given that $\angle ABC$ is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$

Now, $\overrightarrow{BA} = (\hat{i}+2\hat{j}+3\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k}) = 2\hat{i}+2\hat{j}+3\hat{k}$

$\Rightarrow |\overrightarrow{BA}| = \sqrt{4+4+9} = \sqrt{17}$

and $\overrightarrow{BC} = (0\hat{i}+\hat{j}+2\hat{k})-(-\hat{i}+0\hat{j}+0\hat{k}) = \hat{i}+\hat{j}+2\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{1+1+4} = \sqrt{6}$

$\overrightarrow{BA}.\overrightarrow{BC} = (2\hat{i}+2\hat{j}+3\hat{k}).(\hat{i}+\hat{j}+2\hat{k}) = 2+2+6=10$

$\cos\theta = \frac{\overrightarrow{BA}.\overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|} \Rightarrow \cos\angle ABC == \frac{10}{(\sqrt{17})(\sqrt{6})} = \frac{10}{\sqrt{102}}$

$\Rightarrow \angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$

13. Show that the points A(1, 2, 7), B(2, 6, 3) and C(3, 10, –1) are collinear.

Ans:

The given points are $A(1, 2, 7)$, B (2, 6, 3) and $C(3, 10, -1)$.

$\overrightarrow{AB} = (2\hat{i}+6\hat{j}+3\hat{k})-(\hat{i}+2\hat{j}+7\hat{k}) = \hat{i}+4\hat{j}-4\hat{k}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1+16+16} = \sqrt{33}$

$\overrightarrow{BC} = (3\hat{i}+10\hat{j}-\hat{k})-(2\hat{i}+6\hat{j}+3\hat{k}) = \hat{i}+4\hat{j}-4\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{1+16+16} = \sqrt{33}$

and $\overrightarrow{AC} = (3\hat{i}+10\hat{j}-\hat{k})-(\hat{i}+2\hat{j}+7\hat{k}) = 2\hat{i}+8\hat{j}-8\hat{k}$

$\Rightarrow |\overrightarrow{AC}| = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$

$\therefore |\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

Hence, the given points A, B and C are collinear.

14. Show that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ form the vertices of a right angled triangle.

Ans:

Let A= $2\hat{i}-\hat{j}+\hat{k}$ B = $\hat{i}-3\hat{j}-5\hat{k}$ and C = $3\hat{i}-4\hat{j}-4\hat{k}$

$\overrightarrow{AB} = (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k}) = -\hat{i}-2\hat{j}-6\hat{k}$

$\Rightarrow |\overrightarrow{AB}| = \sqrt{1+4+36} = \sqrt{41}$

$\overrightarrow{BC} = (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k}) = 2\hat{i}-\hat{j}+\hat{k}$

$\Rightarrow |\overrightarrow{BC}| = \sqrt{4+1+1} = \sqrt{6}$

and $\overrightarrow{AC} = (3\hat{i}-4\hat{j}-4\hat{k})-(2\hat{i}-\hat{j}+\hat{k}) = \hat{i}-3\hat{j}-5\hat{k}$

$\Rightarrow |\overrightarrow{AC}| = \sqrt{1+9+25} = \sqrt{35}$

$\therefore |\overrightarrow{AB}|^2 = |\overrightarrow{AC}|^2 + |\overrightarrow{BC}|^2$

Hence, ABC is a right angled triangle.

15. Find a unit vector perpendicular to each of the vectors $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$, where $\vec{a} = \hat{i}+\hat{j}+\hat{k}, \vec{b} = \hat{i}+2\hat{j}+3\hat{k}$.

Ans:

We have $\vec{a}+\vec{b} = 2\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{a}-\vec{b} = -\hat{j}-2\hat{k}$

A vector which is perpendicular to both $(\vec{a}+\vec{b})$ and $(\vec{a}-\vec{b})$ is given by

$$(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix} = -2\hat{i}+4\hat{j}-2\hat{k} (= \vec{c}, say)$$

Now, $|\vec{c}| = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$

Therefore, the required unit vector is

$$\hat{c} = \frac{1}{|\vec{c}|}\vec{c} = \frac{1}{2\sqrt{6}}(-2\hat{i}+4\hat{j}-2\hat{k}) = \frac{-1}{\sqrt{6}}\hat{i}+\frac{2}{\sqrt{6}}\hat{j}-\frac{2}{\sqrt{6}}\hat{k}$$

16. Find the area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

Ans:

We have $\overrightarrow{AB} = \hat{j}+2\hat{k}$ and $\overrightarrow{AC} = \hat{i}+2\hat{j}$.

The area of the given triangle is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 1 & 2\\ 1 & 2 & 0\end{vmatrix}=-4\hat{i}+2\hat{j}-\hat{k}$

Therefore, $|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{16+4+1}=\sqrt{21}$

Thus, the required area is $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{1}{2}\sqrt{21}$

17. Find the area of a parallelogram whose adjacent sides are given by the vectors $\vec{a}=3\hat{i}+\hat{j}+4\hat{k}$ and $\vec{b}=\hat{i}-\hat{j}+\hat{k}$.

Ans:

The area of a parallelogram with $\vec{a}$ and $\vec{b}$ as its adjacent sides is given by $|\vec{a}\times\vec{b}|$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 4\\ 1 & -1 & 1\end{vmatrix}=5\hat{i}+\hat{j}-4\hat{k}$

Therefore, $|\vec{a}\times\vec{b}|=\sqrt{25+1+16}=\sqrt{42}$

and hence, the required area is $\sqrt{42}$.

18. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Ans:

$\overrightarrow{AB}=(2\hat{i}+3\hat{j}+5\hat{k})-(\hat{i}+\hat{j}+2\hat{k})=\hat{i}+2\hat{j}+3\hat{k}$

$\overrightarrow{AC}=(\hat{i}+5\hat{j}+5\hat{k})-(\hat{i}+\hat{j}+2\hat{k})=4\hat{j}+3\hat{k}$

Now, $\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ 0 & 4 & 3\end{vmatrix}=-6\hat{i}-3\hat{j}+4\hat{k}$

$\Rightarrow|\overrightarrow{AB}\times\overrightarrow{AC}|=\sqrt{36+9+16}=\sqrt{61}$

Area of triangle ABC = $\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|=\frac{\sqrt{61}}{2}$ sq. units.

19. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Ans:

Adjacent sides of parallelogram are given by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=20\hat{i}+5\hat{j}-5\hat{k}$

$\Rightarrow|\vec{a}\times\vec{b}|=\sqrt{400+25+25}=\sqrt{450}=15\sqrt{2}$

Hence, the area of the given parallelogram is $15\sqrt{2}$ sq. units.

20. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, find the angle between $\vec{a}$ and $\vec{b}$.

Ans:

Given that vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$.

Also, $\vec{a}\times\vec{b}$ is a unit vector $\Rightarrow |\vec{a}\times\vec{b}|=1$

$\Rightarrow |\vec{a}|.|\vec{b}|\sin\theta=1 \Rightarrow 3\times\frac{\sqrt{2}}{3}\sin\theta=1$

$\Rightarrow \sin\theta=\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}$

21. If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overline{AB}$ and $\overline{CD}$. Deduce that $\overline{AB}$ and $\overline{CD}$ are collinear.

Ans:

Note that if θ is the angle between $\overline{AB}$ and $\overline{CD}$, then θ is also the angle between $\overline{AB}$ and $\overline{CD}$.

Now $\overline{AB}$ = Position vector of B – Position vector of A

$= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})=\hat{i}+4\hat{j}-\hat{k}$

Therefore, $|\overrightarrow{AB}|=\sqrt{1+16+1}=\sqrt{18}=3\sqrt{2}$

Similarly, $\overline{CD}=-2\hat{i}-8\hat{j}+2\hat{k} \Rightarrow |\overrightarrow{CD}|=\sqrt{4+64+4}=\sqrt{72}=6\sqrt{2}$

Thus, $\cos\theta=\frac{\overrightarrow{AB}.\overrightarrow{CD}}{|\overrightarrow{AB}||\overrightarrow{CD}|}=\frac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})}=-1$

Since 0 ≤ θ ≤ π, it follows that θ = π. This shows that $\overline{AB}$ and $\overline{CD}$ are collinear.

22. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}|=3, |\vec{b}|=4, |\vec{c}|=5$ and each one of them being perpendicular to the sum of the other two, find $|\vec{a}+\vec{b}+\vec{c}|$.

Ans:

Given that each one of them being perpendicular to the sum of the other two.

Therefore, $\vec{a}.(\vec{b}+\vec{c})=0, \vec{b}.(\vec{c}+\vec{a})=0, \vec{c}.(\vec{a}+\vec{b})=0$

Now, $|\vec{a}+\vec{b}+\vec{c}|^2=(\vec{a}+\vec{b}+\vec{c})^2=(\vec{a}+\vec{b}+\vec{c}).(\vec{a}+\vec{b}+\vec{c})$

$=\vec{a}.\vec{a}+\vec{a}.(\vec{b}+\vec{c})+\vec{b}.\vec{b}+\vec{b}.(\vec{c}+\vec{a})+\vec{c}.(\vec{a}+\vec{b})+\vec{c}.\vec{c}$

$=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2$

$=9+16+25=50$

Therefore, $|\vec{a}+\vec{b}+\vec{c}|=\sqrt{50}=5\sqrt{2}$

23. Find a vector of magnitude 5 units, and parallel to the resultant of the vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

Ans:

Given vectors $\vec{a}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+\hat{k}$.

Let $\vec{c}$ be the resultant vector $\vec{a}$ and $\vec{b}$ then

$\vec{c}=(2\hat{i}+3\hat{j}-\hat{k})+(\hat{i}-2\hat{j}+\hat{k})=3\hat{i}+\hat{j}+0\hat{k}$

$\Rightarrow |\vec{c}|=\sqrt{9+1+0}=\sqrt{10}$

$\therefore$ Unit vector in the direction of $\vec{c}$ = $\hat{c}=\frac{1}{|\vec{c}|}\vec{c}=\frac{1}{\sqrt{10}}(3\hat{i}+\hat{j})$

Hence, the vector of magnitude 5 units and parallel to the resultant of vectors $\vec{a}$ and $\vec{b}$ is

$\pm 5\hat{c}=\pm 5\frac{1}{\sqrt{10}}(3\hat{i}+\hat{j})=\pm\frac{3\sqrt{10}}{2}\hat{i}\pm\frac{\sqrt{10}}{2}\hat{j}$

24. The two adjacent sides of a parallelogram are $2\hat{i}-4\hat{j}+5\hat{k}$ and $\hat{i}-2\hat{j}-3\hat{k}$. Find the unit vector parallel to its diagonal. Also, find its area.

Ans:

Two adjacent sides of a parallelogram are given by $\vec{a}=2\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}-3\hat{k}$

Then the diagonal of a parallelogram is given by $\vec{c}=\vec{a}+\vec{b}$

$\therefore\ \vec{c}=\vec{a}+\vec{b}=2\hat{i}-4\hat{j}+5\hat{k}+\hat{i}-2\hat{j}-3\hat{k}=3\hat{i}-6\hat{j}+2\hat{k}$

$\Rightarrow|\vec{c}|=\sqrt{9+36+4}=\sqrt{49}=7$

Unit vector parallel to its diagonal = $\hat{c}=\frac{1}{|\vec{c}|}\vec{c}=\frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k})=\frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k}$

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -4 & 5\\ 1 & -2 & -3\end{vmatrix}=22\hat{i}+11\hat{j}+0\hat{k}$

Then the area of a parallelogram = $|\vec{a}\times\vec{b}|=\sqrt{484+121+0}=\sqrt{605}=11\sqrt{5}$ sq. units.

25. Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}, \vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{c}.\vec{d}=15$.

Ans:

The vector which is perpendicular to both $\vec{a}$ and $\vec{b}$ must be parallel to $\vec{a}\times\vec{b}$.

Now, $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 4 & 2\\ 3 & -2 & 7\end{vmatrix}=32\hat{i}-\hat{j}-14\hat{k}$

Let $\vec{d}=\lambda(\vec{a}\times\vec{b})=\lambda(32\hat{i}-\hat{j}-14\hat{k})$

Also $\vec{c}.\vec{d}=15\Rightarrow(2\hat{i}-\hat{j}+4\hat{k}).\lambda(32\hat{i}-\hat{j}-14\hat{k})=15$

$\Rightarrow 64\lambda+\lambda-56\lambda=15\Rightarrow 9\lambda=15\Rightarrow\lambda=\frac{15}{9}=\frac{5}{3}$

$\therefore$ Required vector $\vec{d}=\frac{5}{3}(32\hat{i}-\hat{j}-14\hat{k})$

26. The scalar product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ.

Ans: Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

Now, $\vec{b}+\vec{c}=2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\Rightarrow|\vec{b}+\vec{c}|=\sqrt{(2+\lambda)^2+36+4}=\sqrt{4+\lambda^2+4\lambda+40}=\sqrt{\lambda^2+4\lambda+44}$

Unit vector along $\vec{b}+\vec{c}$ is $\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}$

The scalar product of $\hat{i}+\hat{j}+\hat{k}$ with this unit vector is 1.

$\therefore(\hat{i}+\hat{j}+\hat{k}).\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}=1\Rightarrow(\hat{i}+\hat{j}+\hat{k}).\frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}}=1\Rightarrow\frac{\lambda+6}{\sqrt{\lambda^2+4\lambda+44}}=1$

$\Rightarrow\lambda+6=\sqrt{\lambda^2+4\lambda+44}\Rightarrow(\lambda+6)^2=\lambda^2+4\lambda+44$

$\Rightarrow\lambda^2+12\lambda+36=\lambda^2+4\lambda+44\Rightarrow 8\lambda=8\Rightarrow\lambda=1$

27. If with reference to the right handed system of mutually perpendicular unit vectors $\hat{i},\ \hat{j}\ and\ \hat{k}$, $\vec{\alpha} = 3\hat{i} - \hat{j}, \vec{\beta} = 2\hat{i} + \hat{j} - 3\hat{k}$, then express $\vec{\beta}$ □in the form $\vec{\beta} = \vec{\beta_1} + \vec{\beta_2}$, where $\vec{\beta_1}$ is parallel to $\vec{\alpha}$ and $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$ □

Ans:

Let $\vec{\beta_1} = \lambda\vec{\alpha}$, λ is a scalar, i.e. $\vec{\beta_1} = 3\lambda\hat{i} - \lambda\hat{j}$.

Now, $\vec{\beta_2} = \vec{\beta} - \vec{\beta_1} = (2-3\lambda)\hat{i} + (1+\lambda)\hat{j} - 3\hat{k}$

Now, since $\vec{\beta_2}$ is perpendicular to $\vec{\alpha}$, we should have $\vec{\alpha}$ □ $\vec{\beta_2}$ □□0. i.e.,

$$3(2-3\lambda) - (1+\lambda) = 0$$

$$\Rightarrow 6 - 9\lambda - 1 - \lambda = 0 \Rightarrow 5 - 10\lambda = 0$$

$$\Rightarrow 10\lambda = 5 \Rightarrow \lambda = \frac{5}{10} = \frac{1}{2}$$

Therefore, $\vec{\beta_1} = \frac{3}{2}\hat{i} - \frac{1}{2}\hat{j}\ \ and\ \ \vec{\beta_2} = \frac{1}{2}\hat{i} + \frac{3}{2}\hat{j} - 3\hat{k}$

28. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Ans:

Given that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors.

$\therefore\ \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

It is also given that $|\vec{a}| = |\vec{b}| = |\vec{c}|$

Let vector $\vec{a}+\vec{b}+\vec{c}$ be inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$ at angles α, β and γ respectively.

$$\cos\alpha = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{\vec{a}.\vec{a}+\vec{b}.\vec{a}+\vec{c}.\vec{a}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|^2+0+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{a}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{a}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\beta = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{b}|} = \frac{\vec{a}.\vec{b}+\vec{b}.\vec{b}+\vec{c}.\vec{b}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{0+|\vec{b}|^2+0}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{b}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{b}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

$$\cos\gamma = \frac{(\vec{a}+\vec{b}+\vec{c}).\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{c}|} = \frac{\vec{a}.\vec{c}+\vec{b}.\vec{c}+\vec{c}.\vec{c}}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{0+0+|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|}$$

$$= \frac{|\vec{c}|^2}{|\vec{a}+\vec{b}+\vec{c}||\vec{a}|} = \frac{|\vec{c}|}{|\vec{a}+\vec{b}+\vec{c}|}$$

Now as $|\vec{a}| = |\vec{b}| = |\vec{c}|$, therefore, cosα = cosβ = cosγ

∴ α = β = γ

Hence, the vector $\vec{a}+\vec{b}+\vec{c}$ is equally inclined to $\vec{a}$, $\vec{b}$ and $\vec{c}$.

CHAPTER – 10: VECTOR ALGEBRA

MARKS WEIGHTAGE – 05 marks

Previous Years Board Exam (Important Questions & Answers)

1. **Write the projection of vector $\hat{i}+\hat{j}+\hat{k}$ along the vector $\hat{j}$.**
   Ans:
   Required projection $=\dfrac{(\hat{i}+\hat{j}+\hat{k}).\hat{j}}{|\hat{j}|}=\dfrac{0+1+0}{\sqrt{0+1+0}}=\dfrac{1}{1}=1$

2. **Find a vector in the direction of vector $2\hat{i}-3\hat{j}+6\hat{k}$ which has magnitude 21 units.**
   Ans:
   Required vector $=21\left(\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{4+9+36}}\right)=21\left(\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{\sqrt{49}}\right)$
   $=21\left(\dfrac{2\hat{i}-3\hat{j}+6\hat{k}}{7}\right)=3(2\hat{i}-3\hat{j}+6\hat{k})=6\hat{i}-9\hat{j}+18\hat{k}$

3. **Write a unit vector in the direction of vector $\overrightarrow{PQ}$, where P and Q are the points (1, 3, 0) and (4, 5, 6) respectively.**
   Ans:
   $\overrightarrow{PQ}=(4-1)\hat{i}+(5-3)\hat{j}+(6-0)\hat{k}=3\hat{i}+2\hat{j}+6\hat{k}$
   $\therefore$ Required unit vector = $\dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{\sqrt{9+4+36}}=\dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{\sqrt{49}}=\dfrac{3\hat{i}+2\hat{j}+6\hat{k}}{7}=\dfrac{3}{7}\hat{i}+\dfrac{2}{7}\hat{j}+\dfrac{6}{7}\hat{k}$

4. **Write the value of the following : $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$**
   Ans:
   $\hat{i}\times(\hat{j}+\hat{k})+\hat{j}\times(\hat{k}+\hat{i})+\hat{k}\times(\hat{i}+\hat{j})$
   $=\hat{i}\times\hat{j}+\hat{i}\times\hat{k}+\hat{j}\times\hat{k}+\hat{j}\times\hat{i}+\hat{k}\times\hat{i}+\hat{k}\times\hat{j}$
   $=\hat{k}-\hat{j}+\hat{i}-\hat{k}+\hat{j}-\hat{i}=0$

5. **Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ and $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel.**
   Ans:
   Since given two vectors are parallel.
   $\Rightarrow\dfrac{3}{1}=\dfrac{2}{-2p}=\dfrac{9}{3}\Rightarrow\dfrac{3}{1}=\dfrac{2}{-2p}$
   $\Rightarrow -6p=2\Rightarrow p=-\dfrac{1}{3}$

6. **Find $\vec{a}.(\vec{b}\times\vec{c})$, if $\vec{a}=2\hat{i}+\hat{j}+3\hat{k},\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$.**
   Ans:
   Given that $\vec{a}=2\hat{i}+\hat{j}+3\hat{k},\vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}+2\hat{k}$
   $$\therefore \vec{a}.(\vec{b}\times\vec{c})=\begin{vmatrix}2&1&3\\-1&2&1\\3&1&2\end{vmatrix}=2(4-1)-1(-2-3)+3(-1-6)$$
   $=6+5-21=-10$

7. Show that the four points *A, B, C* and *D* with position vectors $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ are coplanar.

Ans:

Position vectors of *A*, *B*, *C* and *D* are

Position vector of *A* = $4\hat{i}+5\hat{j}+\hat{k}$

Position vector of *B* = $-\hat{j}-\hat{k}$

Position vector of *C* = $3\hat{i}+9\hat{j}+4\hat{k}$

Position vector of *D* = $4(-\hat{i}+\hat{j}+\hat{k}) = -4\hat{i}+4\hat{j}+4\hat{k}$

$\therefore \overrightarrow{AB} = -4\hat{i}-6\hat{j}-2\hat{k}, \overrightarrow{AC} = -\hat{i}+4\hat{j}+3\hat{k}, \overrightarrow{AD} = -8\hat{i}-\hat{j}+3\hat{k}$

Now, $\overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD}) = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = -4(12+3)+6(-3+24)-2(1+32)$

$= -60+126-66 = 0$

$\Rightarrow \overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD}) = 0$

Hence $\overrightarrow{AB}$, $\overrightarrow{AC}$ *and* $\overrightarrow{AD}$ are coplanar i.e. A, B, C and D are coplanar.

8. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with *x*-axis., $\frac{\pi}{2}$ with y-axis and an acute angle θ with *z*-axis.

Ans:

Direction cosines of required vector $\vec{a}$ are

$l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $m = \cos\frac{\pi}{2} = 0$ and $n = \cos\theta$

$\because l^2+m^2+n^2 = 1$

$\therefore \left(\frac{1}{\sqrt{2}}\right)^2 + 0 + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1-\frac{1}{2} = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow n = \frac{1}{\sqrt{2}}$

$\therefore$ Unit vector in the direction of $\vec{a} = \frac{1}{\sqrt{2}}\hat{i}+0\hat{j}+\frac{1}{\sqrt{2}}\hat{k}$

$\therefore \vec{a} = 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i}+0\hat{j}+\frac{1}{\sqrt{2}}\hat{k}\right) = 5\hat{i}+5\hat{k}$

9. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}|$ = 13 and $|\vec{a}|$ = 5 find the value of $|\vec{b}|$.

Ans:

Given $|\vec{a}+\vec{b}|$ = 13

$|\vec{a}+\vec{b}|^2 = 169 \Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 169$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 169$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 = 169 \quad \left[\because \vec{a}\perp\vec{b} \Rightarrow \vec{a}.\vec{b} = 0\right]$

$\Rightarrow |\vec{b}|^2 = 169 - |\vec{a}|^2 = 169-25 = 144$

$\Rightarrow |\vec{b}| = 12$

10. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$.

Ans:

Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b} = 2\hat{i}-3\hat{j}+6\hat{k}$

Projection of the vector $\vec{a}$ on $\vec{b} = \dfrac{\vec{a}.\vec{b}}{|\vec{b}|} = \dfrac{(\hat{i}+3\hat{j}+7\hat{k}).(2\hat{i}-3\hat{j}+6\hat{k})}{|2\hat{i}-3\hat{j}+6\hat{k}|}$

$= \dfrac{2-9+42}{\sqrt{4+9+36}} = \dfrac{35}{\sqrt{49}} = \dfrac{35}{7} = 5$

11. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.

Ans:

Given that $\vec{a}+\vec{b}$ is also a unit vector

$\therefore |\vec{a}+\vec{b}| = 1$

$|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 1^2 = 1$

$\Rightarrow 1+2\vec{a}.\vec{b}+1 = 1 \quad \left[\because |\vec{a}|=1, |\vec{b}|=1\right]$

$\Rightarrow 2\vec{a}.\vec{b} = -1 \Rightarrow \vec{a}.\vec{b} = -\dfrac{1}{2} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = -\dfrac{1}{2}$

$\Rightarrow 1\times1\times\cos\theta = -\dfrac{1}{2} \Rightarrow \cos\theta = -\dfrac{1}{2} \Rightarrow \cos\theta = \cos\dfrac{2\pi}{3}$

$\Rightarrow \theta = \dfrac{2\pi}{3}$

12. Vectors $\vec{a}, \vec{b}, \vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}$ = 0 and $|\vec{a}|=3,\ |\vec{b}|=5\ and\ |\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Ans:

$\vec{a}+\vec{b}+\vec{c} = 0 \Rightarrow \vec{a}+\vec{b} = -\vec{c}$

$\Rightarrow (\vec{a}+\vec{b})^2 = (-\vec{c})^2$

$|\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = \vec{c}.\vec{c}$

$\Rightarrow |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = |\vec{c}|^2 \Rightarrow 9+2\vec{a}.\vec{b}+25 = 49$

$\Rightarrow 2\vec{a}.\vec{b} = 49-25-9 = 15$

$\Rightarrow \vec{a}.\vec{b} = \dfrac{15}{2} \Rightarrow |\vec{a}||\vec{b}|\cos\theta = \dfrac{15}{2}$

$\Rightarrow 3\times5\times\cos\theta = \dfrac{15}{2} \Rightarrow \cos\theta = \dfrac{1}{2} \Rightarrow \cos\theta = \cos\dfrac{\pi}{3}$

$\Rightarrow \theta = \dfrac{\pi}{3}$

13. If $\vec{a}$ is a unit vector and $(\vec{x}-\vec{a})(\vec{x}+\vec{a}) = 24$, then write the value of $|\vec{x}|$.

Ans:

Given that $(\vec{x}-\vec{a})(\vec{x}+\vec{a}) = 24$

$\Rightarrow \vec{x}.\vec{x}+\vec{x}.\vec{a}-\vec{a}.\vec{x}-\vec{a}.\vec{a} = 24$

$\Rightarrow |\vec{x}|^2 - |\vec{a}|^2 = 24 \quad \left[\because \vec{x}.\vec{a} = \vec{a}.\vec{x}\right]$

$\Rightarrow |\vec{x}|^2 - 1 = 24 \Rightarrow |\vec{x}|^2 = 25 \Rightarrow |\vec{x}| = 5$

14. For any three vectors $\vec{a}, \vec{b}\ and\ \vec{c}$, write the value of the following:

$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

Ans:

$\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{c}+\vec{a})+\vec{c}\times(\vec{a}+\vec{b})$

$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{a}+\vec{c}\times\vec{b}$

$=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{c}-\vec{a}\times\vec{b}-\vec{a}\times\vec{c}-\vec{b}\times\vec{c}=0$

15. The magnitude of the vector product of the vector $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

Ans:

Let $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$

Now, $\vec{b}+\vec{c}=2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\Rightarrow|\vec{b}+\vec{c}|=\sqrt{(2+\lambda)^2+36+4}=\sqrt{4+\lambda^2+4\lambda+40}=\sqrt{\lambda^2+4\lambda+44}$

The vector product of $\hat{i}+\hat{j}+\hat{k}$ with this unit vector is $\sqrt{2}$.

$\therefore\ \left|\vec{a}\times\frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}\right|=\sqrt{2}\Rightarrow\left|\frac{\vec{a}\times(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}\right|=\sqrt{2}$

Now, $\vec{a}\times(\vec{b}+\vec{c})=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 2+\lambda & 6 & -2\end{vmatrix}=(-2-6)\hat{i}-(-2-2-\lambda)\hat{j}-(6-2-\lambda)\hat{k}$

$=-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}$

$\therefore\left|\frac{\vec{a}\times(\vec{b}+\vec{c})}{|\vec{b}+\vec{c}|}\right|=\sqrt{2}\Rightarrow\left|\frac{-8\hat{i}+(4+\lambda)\hat{j}+(4-\lambda)\hat{k}}{\sqrt{\lambda^2+4\lambda+44}}\right|=\sqrt{2}$

$\Rightarrow\frac{\sqrt{64+(4+\lambda)^2+(4-\lambda)^2}}{\sqrt{\lambda^2+4\lambda+44}}=\sqrt{2}$

$\frac{64+(4+\lambda)^2+(4-\lambda)^2}{\lambda^2+4\lambda+44}=2\Rightarrow\frac{64+16+\lambda^2+8\lambda+16+\lambda^2-8\lambda}{\lambda^2+4\lambda+44}=2\Rightarrow\frac{96+2\lambda^2}{\lambda^2+4\lambda+44}=2$

$\Rightarrow 96+2\lambda^2=2(\lambda^2+4\lambda+44)\Rightarrow 96+2\lambda^2=2\lambda^2+8\lambda+88$

$\Rightarrow 8\lambda=8\Rightarrow\lambda=1$

16. Find a unit vector perpendicular to each of the vectors $\vec{a}+2\vec{b}$ and $2\vec{a}+\vec{b}$, where $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$.

Ans:

Given that $\vec{a}=3\hat{i}+2\hat{j}+2\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$

$\therefore\vec{a}+2\vec{b}=3\hat{i}+2\hat{j}+2\hat{k}+2(\hat{i}+2\hat{j}-2\hat{k})$

$=3\hat{i}+2\hat{j}+2\hat{k}+2\hat{i}+4\hat{j}-4\hat{k}=5\hat{i}+6\hat{j}-2\hat{k}$

and $2\vec{a}+\vec{b}=2(3\hat{i}+2\hat{j}+2\hat{k})+\hat{i}+2\hat{j}-2\hat{k}$

$=6\hat{i}+4\hat{j}+4\hat{k}+\hat{i}+2\hat{j}-2\hat{k}=7\hat{i}+6\hat{j}+2\hat{k}$

Now, perpendicular vector of $\vec{a}+2\vec{b}$ and $2\vec{a}+\vec{b}$

$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 5 & 6 & -2\\ 7 & 6 & 2\end{vmatrix}=(12+12)\hat{i}-(10+14)\hat{j}+(30-42)\hat{k}$

$=24\hat{i}-24\hat{j}-12\hat{k}=12(2\hat{i}-2\hat{j}-\hat{k})$

$\therefore$ Required unit vector = $\pm\frac{12(2\hat{i}-2\hat{j}-\hat{k})}{12\sqrt{4+4+1}}=\pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{\sqrt{9}}$

$$= \pm\frac{2\hat{i}-2\hat{j}-\hat{k}}{3} = \pm\left(\frac{2}{3}\hat{i}-\frac{2}{3}\hat{j}-\frac{1}{3}\hat{k}\right)$$

17. If $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors.

Ans:

Given that $\vec{a}=\hat{i}-\hat{j}+7\hat{k}$ and $\vec{b}=5\hat{i}-\hat{j}+\lambda\hat{k}$

$\therefore\ \vec{a}+\vec{b}=\hat{i}-\hat{j}+7\hat{k}+5\hat{i}-\hat{j}+\lambda\hat{k}=6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}$

and $\vec{a}-\vec{b}=\hat{i}-\hat{j}+7\hat{k}-5\hat{i}+\hat{j}-\lambda\hat{k}=-4\hat{i}+(7-\lambda)\hat{k}$

Now, $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}-\vec{b})=0$

$\Rightarrow (6\hat{i}-2\hat{j}+(7+\lambda)\hat{k}).(-4\hat{i}+(7-\lambda)\hat{k}=0$

$\Rightarrow -24+0+(7+\lambda)(7-\lambda)=0$

$\Rightarrow -24+49-\lambda^2=0 \Rightarrow \lambda^2=25 \Rightarrow \lambda=\pm 5$

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a}.\vec{b}\ge 0$ only when

(a) $0<\theta<\frac{\pi}{2}$ (b) $0\le\theta\le\frac{\pi}{2}$ (c) $0<\theta<\pi$ (d) $0\le\theta\le\pi$

2. Let $\vec{a}$ and $\vec{b}$ be two unit vectors and θ is the angle between them. Then $\vec{a}+\vec{b}$ is a unit vector if θ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{2\pi}{3}$

3. The value of $\hat{i}.(\hat{j}\times\hat{k})+\hat{j}.(\hat{i}\times\hat{k})+\hat{k}.(\hat{i}\times\hat{j})$ is

(a) 0 (b) –1 (c) 1 (d) 3

4. If θ is the angle between any two vectors $\vec{a}$ and $\vec{b}$, then $|\vec{a}.\vec{b}|=|\vec{a}\times\vec{b}|$ when θ is equal to

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

5. The value of x for which $x(\hat{i}+\hat{j}+\hat{k})$ is a unit vector.

(a) 0 (b) $\pm\frac{1}{\sqrt{3}}$ (c) 1 (d) $\pm\sqrt{3}$

6. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

7. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}+\frac{1}{2}\hat{j}+4\hat{k}$, $\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$, respectively is

(a) 1/2 (b) 1 (c) 2 (d) 4

8. The area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$ is

(a) 15 (b) $15\sqrt{3}$ (c) $15\sqrt{2}$ (d) None of these

9. The area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5)

(a) $\sqrt{61}$ (b) $\frac{\sqrt{61}}{2}$ (c) $2\sqrt{61}$ (d) None of these

10. If $\vec{a}=\hat{i}-7\hat{j}+7\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+2\hat{k}$ then the value of $|\vec{a}\times\vec{b}|$ is

(a) 19 (b) $19\sqrt{3}$ (c) $19\sqrt{2}$ (d) None of these

11. The area of a triangle having the points A(1, 1, 1), B(1, 2, 3) and C(2, 3, 1) as its vertices.

(a) $\sqrt{21}$ (b) $2\sqrt{21}$ (c) $\frac{\sqrt{21}}{2}$ (d) None of these

12. If the points (–1, –1, 2), (2,m, 5) and (3,11, 6) are collinear, then the value of m is

(a) 8 (b) 4 (c) 2 (d) None of these

13. The magnitude of the vector $6\hat{i}+2\hat{j}+3\hat{k}$ is

(a) 5 (b) 7 (c) 12 (d) 1

14. The position vector of the point which divides the join of points with position vectors $\vec{a}+\vec{b}$ and $2\vec{a}-\vec{b}$ in the ratio 1 : 2 is

(a) $\frac{3\vec{a}+2\vec{b}}{3}$ (b) $\vec{a}$ (c) $\frac{5\vec{a}-\vec{b}}{3}$ (d) $\frac{4\vec{a}+\vec{b}}{3}$

15. The vector with initial point P (2, –3, 5) and terminal point Q(3, –4, 7) is

(a) $\hat{i}-\hat{j}+2\hat{k}$ (b) $5\hat{i}-7\hat{j}+12\hat{k}$ (c) $-\hat{i}+\hat{j}-2\hat{k}$ (d) None of these

16. The angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{3}$ (d) $\frac{5\pi}{6}$

17. The value of λ for which the two vectors $2\hat{i}-\hat{j}+2\hat{k}$ and $3\hat{i}+\lambda\hat{j}+\hat{k}$ are perpendicular is

(a) 2 (b) 4 (c) 6 (d) 8

18. The area of the parallelogram whose adjacent sides $\hat{i}+\hat{k}$ and $2\hat{i}+\hat{j}+\hat{k}$ are

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 3 (d) 4

19. If $|\vec{a}|=8, |\vec{b}|=3$ and $|\vec{a}\times\vec{b}|=12$, then value of $\vec{a}.\vec{b}$ is

(a) $6\sqrt{3}$ (b) $8\sqrt{31}$ (c) $2\sqrt{3}$ (d) None of these

20. The 2 vectors $\hat{j}+\hat{k}$ and $3\hat{i}-\hat{j}+4\hat{k}$ represents the two sides AB and AC, respectively of a ΔABC. The length of the median through A is

(a) $\frac{\sqrt{34}}{2}$ (b) $\frac{\sqrt{48}}{2}$ (c) $\sqrt{18}$ (d) None of these

21. The projection of vector $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ along $\vec{b}=\hat{i}+2\hat{j}+2\hat{k}$ is

(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\sqrt{6}$ (d) 2

22. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\sqrt{3}\vec{a}-\vec{b}$ to be a unit vector?

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

23. The unit vector perpendicular to the vectors $\hat{i}-\hat{j}$ and $\hat{i}+\hat{j}$ forming a right handed system is

(a) $\hat{k}$ (b) $-\hat{k}$ (c) $\frac{\hat{i}-\hat{j}}{\sqrt{2}}$ (d) $\frac{\hat{i}+\hat{j}}{\sqrt{2}}$

24. If $|\vec{a}|=3$ and $-1\le k\le 2$, then $|k\vec{a}|$ lies in the interval

(a) [0, 6] (b) [– 3, 6] (c) [3, 6] (d) [1, 2]

25. The vector in the direction of the vector $\hat{i}-2\hat{j}+2\hat{k}$ that has magnitude 9 is

(a) $\hat{i}-2\hat{j}+2\hat{k}$ (b) $3(\hat{i}-2\hat{j}+2\hat{k})$ (c) $9(\hat{i}-2\hat{j}+2\hat{k})$ (d) $\frac{\hat{i}-2\hat{j}+2\hat{k}}{3}$

26. The position vector of the point which divides the join of points with position vectors $2\vec{a}-3\vec{b}$ and $\vec{a}+\vec{b}$ in the ratio 3 : 1 is

(a) $\frac{3\vec{a}-2\vec{b}}{2}$ (b) $\frac{7\vec{a}-8\vec{b}}{4}$ (c) $\frac{3\vec{a}}{4}$ (d) $\frac{5\vec{a}}{4}$

27. The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is

(a) $-\hat{i}+12\hat{j}+4\hat{k}$ (b) $5\hat{i}+2\hat{j}-4\hat{k}$ (c) $-5\hat{i}+2\hat{j}+4\hat{k}$ (d) $\hat{i}+\hat{j}+\hat{k}$

28. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes $\sqrt{3}$ and 4, respectively, and $\vec{a}.\vec{b}=2\sqrt{3}$ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{\pi}{6}$

29. The value of λ for which the two vectors $3\hat{i}-6\hat{j}+\hat{k}$ and $2\hat{i}-4\hat{j}+\lambda\hat{k}$ are parallel is

(a) $\frac{3}{2}$ (b) $\frac{2}{3}$ (c) $\frac{5}{2}$ (d) $\frac{2}{5}$

30. The value of λ for which the two vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}+2\hat{j}+3\hat{k}$ are orthogonal is

(a) 0 (b) 1 (c) $\frac{3}{2}$ (d) $-\frac{5}{2}$

31. The vectors from origin to the points A and B are $\vec{a} = 2\hat{i} - 3\hat{j} + 2\hat{k}$ and $\vec{b} = 2\hat{i} + 3\hat{j} + \hat{k}$,respectively, then the area of triangle OAB is

(a) 340 (b) $\sqrt{25}$ (c) $\sqrt{229}$ (d) $\frac{1}{2}\sqrt{229}$

32. For any vector $\vec{a}$, the value of $(\vec{a} \times \hat{i})^2 + (\vec{a} \times \hat{j})^2 + (\vec{a} \times \hat{k})^2$ is equal to

(a) $\vec{a}^2$ (b) $3\vec{a}^2$ (c) $4\vec{a}^2$ (d) $2\vec{a}^2$

33. If $|\vec{a}| = 10, |\vec{b}| = 2$ and $\vec{a}.\vec{b} = 12$, then value of $|\vec{a} \times \vec{b}|$ is

(a) 5 (b) 10 (c) 14 (d) 16

34. The number of vectors of unit length perpendicular to the vectors $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{j} + \hat{k}$ is

(a) one (b) two (c) three (d) infinite

35. If $|\vec{a}| = 4$ and $-3 \le k \le 2$, then $|k\vec{a}|$ lies in the interval

(a) [0, 8] (b) [– 12, 8] (c) [0, 12] (d) [8, 12]

36. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 2, |\vec{b}| = 3, |\vec{c}| = 5$ then value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$ is

(a) 0 (b) 1 (c) – 19 (d) 38

37. If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ then the value of $\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}$ is

(a) 1 (b) 3 (c) $-\frac{3}{2}$ (d) None of these

38. The vectors $\lambda\hat{i} + \hat{j} + 2\hat{k}, \hat{i} + \lambda\hat{j} - \hat{k}, 2\hat{i} - \hat{j} + \lambda\hat{k}$ are coplanar if

(a) –2 (b) 0 (c) 1 (d) – 1

39. The vector $\vec{a} + \vec{b}$ bisects the angle between the non-collinear vectors $\vec{a}$ and $\vec{b}$ if ________

40. If $\vec{r}.\vec{a} = 0, \vec{r}.\vec{b} = 0, \vec{r}.\vec{c} = 0$ for some non-zero vector $\vec{r}$, then the value of $\vec{a}.(\vec{b} \times \vec{c})$ is ________

41. The vectors $\vec{a} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{b} = -\hat{i} - 2\hat{k}$ are the adjacent sides of a parallelogram. The acute angle between its diagonals is ________.

42. The values of k for which $|k\vec{a}| < |\vec{a}|$ and $k\vec{a} + \frac{1}{2}\vec{a}$ is parallel to $\vec{a}$ holds true are _______.

43. The value of the expression $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2$ is _______.

44. If $|\vec{a} \times \vec{b}|^2 + |\vec{a}.\vec{b}|^2 = 144$ and $|\vec{a}| = 4$, then $|\vec{b}|$ is equal to _______.

45. If $\vec{a}$ is any non-zero vector, then $(\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$ equals _______.

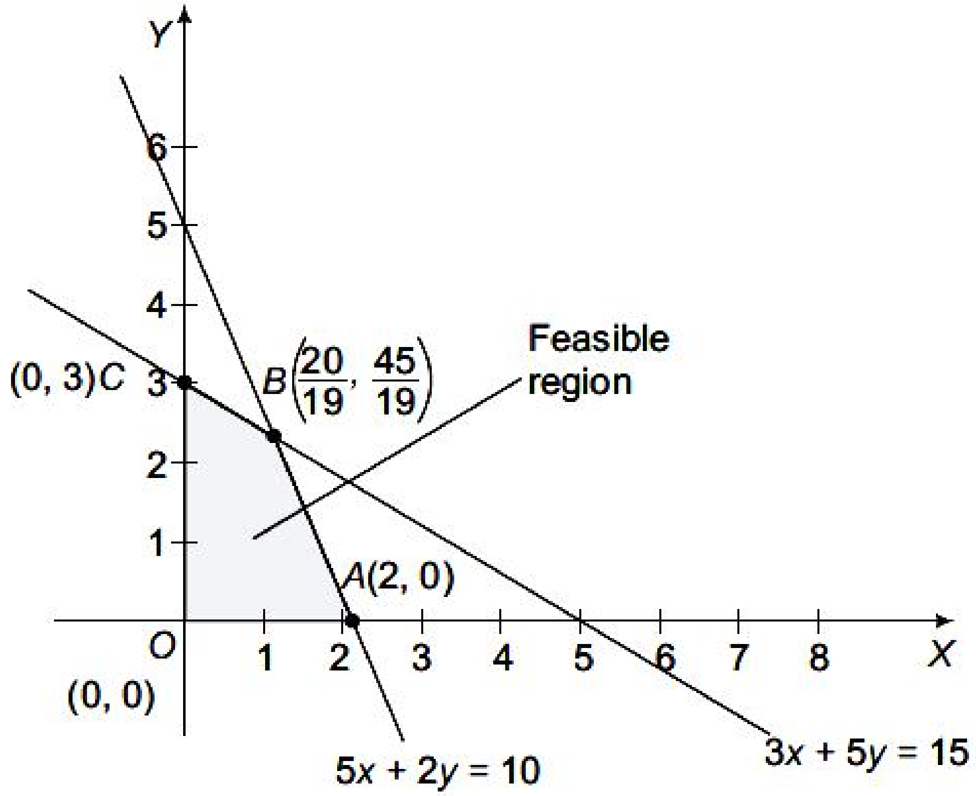
LINEAR PROGRAMMING

CHAPTER – 12: LINEAR PROGRAMMING

MARKS WEIGHTAGE – 05 marks

NCERT Important Questions & Answers

1. Solve the following Linear Programming Problems graphically:
Maximise Z = 5x + 3y subject to 3x + 5y ≤ 15, 5x + 2y ≤ 10, x ≥ 0, y ≥ 0.
Ans:
Our problem is to maximize $Z = 5x + 3y$ …(i)
Subject to constraints $3x + 5y \le 15$ …(ii)
$5x + 2y \le 10$ (iii)
$x \ge 0, y \ge 0$ …(iv)
Firstly, draw the graph of the line $3x + 5y = 15$
Secondly, draw the graph of the line $5x + 2y = 10$
On solving given equations $3x + 5y = 15$ and $5x + 2y = 10$, we get $x = \frac{20}{19}$, $y = \frac{45}{19}$
∴ Feasible region is *OABCO (see the below figure)*.

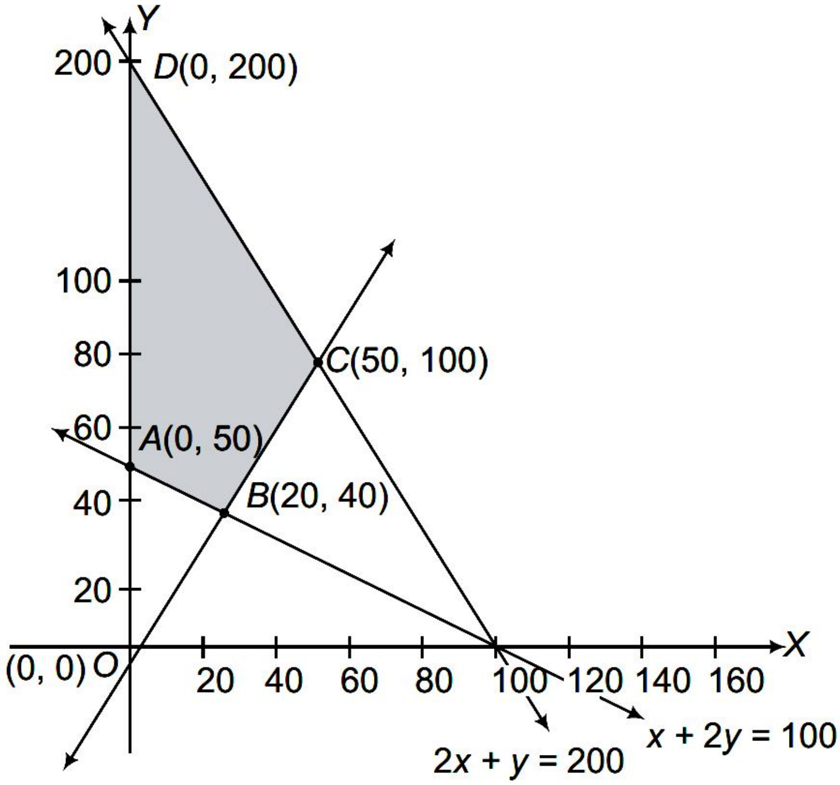


The corner points of the feasible region are O(0, 0), A(2, 0), B$\left(\frac{20}{19}, \frac{45}{19}\right)$ and $C(0, 3)$ The values of Z at these points are as follows:

Corner point	$Z = 5x + 3y$
$O(0, 0)$	0
$A(2, 0)$	10
$C(0, 3)$	9
$B\left(\frac{20}{19}, \frac{45}{19}\right)$	$\frac{235}{19}$ → Maximum

Therefore, the maximum value of Z is $\frac{235}{19}$ at the point $B\left(\frac{20}{19}, \frac{45}{19}\right)$.

2. Show that the minimum of Z occurs at more than two points.
Minimise and Maximise Z = $x + 2y$ subject to $x + 2y \geq 100$, $2x - y \leq 0$, $2x + y \leq 200$; $x, y \geq 0$.
Ans:
Our problem is to minimize and maximize
$Z = x + 2y$ …(i)
Subject to constraints are $x + 2y \geq 100$ …(ii)
$2x - y \leq 0$ …(iii)
$2x + y \leq 200$ …(iv)
$x \geq 0, y \geq 0$ …(v)
Firstly, draw the graph of the line $x + 2y = 100$
Secondly, draw the graph of line $2x - y = 0$
Thirdly, draw the graph of line $2x + y = 200$
On solving equations $2x - y = 0$ and $x + 2y = 100$, we get $B(20, 40)$ and on solving the equations $2x - y = 0$ and $2x + y = 200$, we get $C(50, 100)$.
$\therefore$ Feasible region is *ABCDA (see below figure)*



The corner points of the feasible region are $A(0, 50)$, $B(20, 40)$, $C(50, 100)$ and $D(0, 200)$. The values of Z at these points are as follows:

Corner point	$Z = x + 2y$
$A(0, 50)$	100 → Minimum
$B(20, 40)$	100 → Minimum
$C(50, 100)$	250
$D(0, 200)$	400 → Maximum

The maximum value of Z is 400 at $D(0, 200)$ and the minimum value of Z is 100 at all the points on the line segment joining $A(0, 50)$ and $B(20, 40)$.

3. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly

demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

Ans:

Let the manufacturer produces x pedestal lamps and y wooden shades everyday. We construct the following table :

Item	Number	Time on grinding/ cutting machine (in h)	Time on sprayer (in h)	Profit (in ₹)
A	x	2x	3x	5x
B	y	y	2y	3y
Total	x + y	2x + y	3x + 2y	5x + 3y
Availability		12	20	

The profit on a lamp is Rs. 5 and on the shades is Rs. 3.

Our problem is to maximize $Z = 5x + 3y$ …(i)

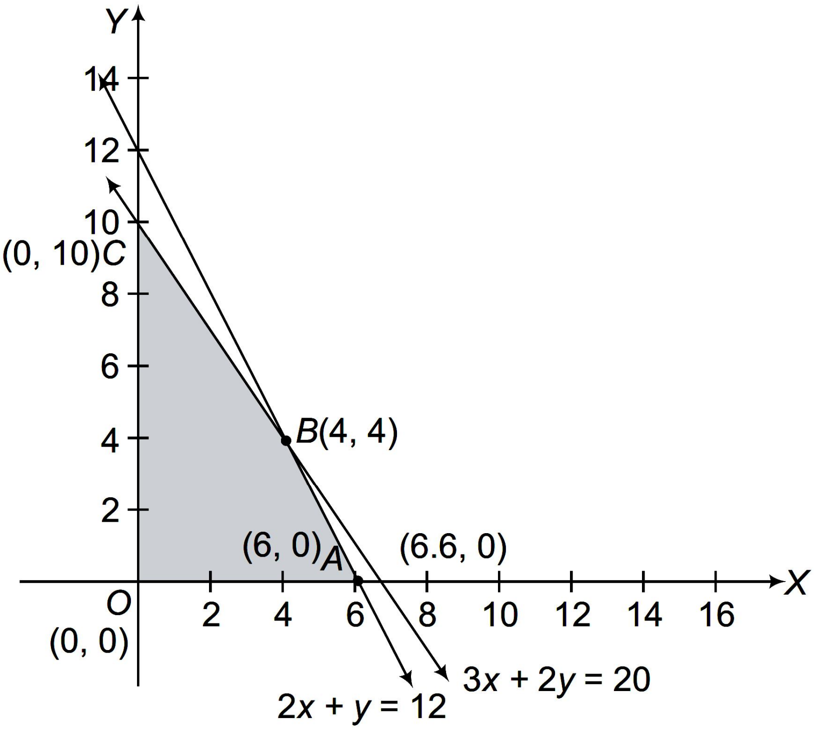
Subject to the constraints $2x + y \le 12$ …(ii) $3x + 2y \le 20$ …(iii) $x \ge 0, y \ge 0$ …(iv)

Firstly, draw the graph of the line $2x + y = 12$

Secondly, draw the graph of the line $3x + 2y = 20$

On solving equations $2x + y = 12$ and $3x + 2y = 20$, we get $B(4, 4)$.

$\therefore$ Feasible region is $OABCO$. (see below figure)



The corner points of the feasible region are $O(0, 0)$, $A(6, 0)$, $B(4, 4)$ and $C(0, 10)$. The values of Z at these points are as follows:

Corner point	Z = 5x + 3y
0(0, 0)	0
A(6, 0)	30
B(4, 4)	32 → Maximum
C(0, 10)	30

The maximum value of Z is Rs. 32 at $B(4, 4)$.

Thus, the manufacturer should produce 4 pedestal lamps and 4 wooden shades to maximize his profits.

4. A dietician has to develop a special diet using two foods P and Q. Each packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?

Ans:

Let x and y be the number of packets of food P and Q respectively. Obviously $x \geq 0$, $y \geq 0$. Mathematical formulation of the given problem is as follows:

Minimise $Z = 6x + 3y$ (vitamin A)

subject to the constraints

$12x + 3y \geq 240$ (constraint on calcium), i.e. $4x + y \geq 80$ ... (1)

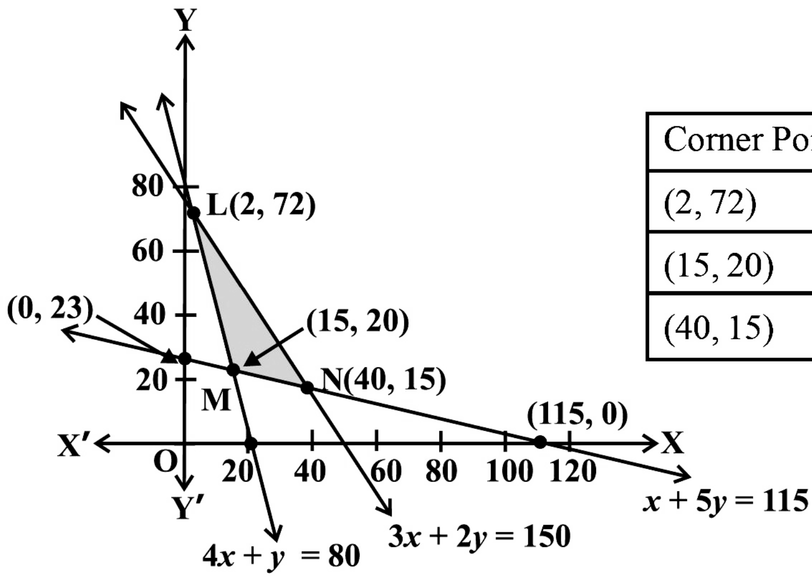
$4x + 20y \geq 460$ (constraint on iron), i.e. $x + 5y \geq 115$ ... (2)

$6x + 4y \leq 300$ (constraint on cholesterol), i.e. $3x + 2y \leq 150$ ... (3)

$x \geq 0$, $y \geq 0$ ... (4)

Let us graph the inequalities (1) to (4).

The feasible region (shaded) determined by the constraints (1) to (4) is shown in below figure and note that it is bounded. The coordinates of the corner points L, M and N are (2, 72), (15, 20) and (40, 15) respectively. Let us evaluate Z at these points:



Corner Point	$Z = 6x + 3y$	
(2, 72)	228	
(15, 20)	**150** ←	Minimum
(40, 15)	285	

From the table, we find that Z is minimum at the point (15, 20). Hence, the amount of vitamin A under the constraints given in the problem will be minimum, if 15 packets of food P and 20 packets of food Q are used in the special diet. The minimum amount of vitamin A will be 150 units.

5. A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	**Number of hours required on machines**		
	I	**II**	**III**
M	**1**	**2**	**1**
N	**2**	**1**	**1.25**

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

Ans:

Let x and y be the number of items M and N respectively.

Total profit on the production = Rs $(600\,x + 400\,y)$

Mathematical formulation of the given problem is as follows:

Maximise $Z = 600\,x + 400\,y$

subject to the constraints:

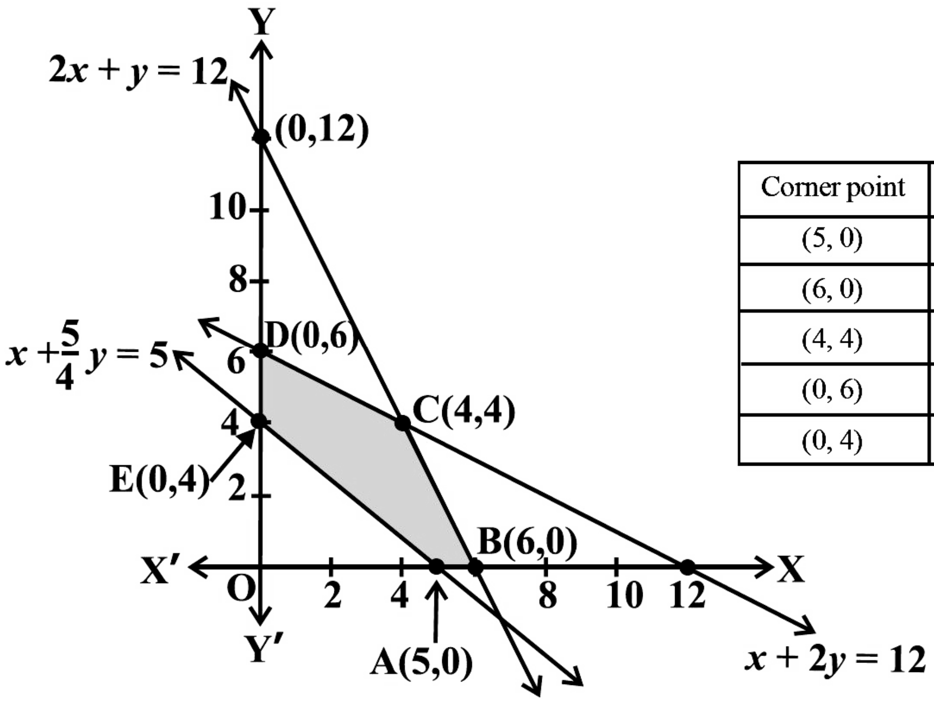
$x + 2y \le 12$ (constraint on Machine I) ... (1)

$2x + y \le 12$ (constraint on Machine II) ... (2)

$x + \frac{5}{4}y \ge 5$ (constraint on Machine III) ... (3)

$x \ge 0, y \ge 0$ ... (4)

Let us draw the graph of constraints (1) to (4). ABCDE is the feasible region (shaded) as shown in below figure determined by the constraints (1) to (4). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are (5, 0) (6, 0), (4, 4), (0, 6) and (0, 4) respectively. Let us evaluate $Z = 600\,x + 400\,y$ at these corner points.



Corner point	$Z = 600\,x + 400\,y$	
(5, 0)	3000	
(6, 0)	3600	
(4, 4)	**4000** ←	Maximum
(0, 6)	2400	
(0, 4)	1600	

We see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of Rs 4000.

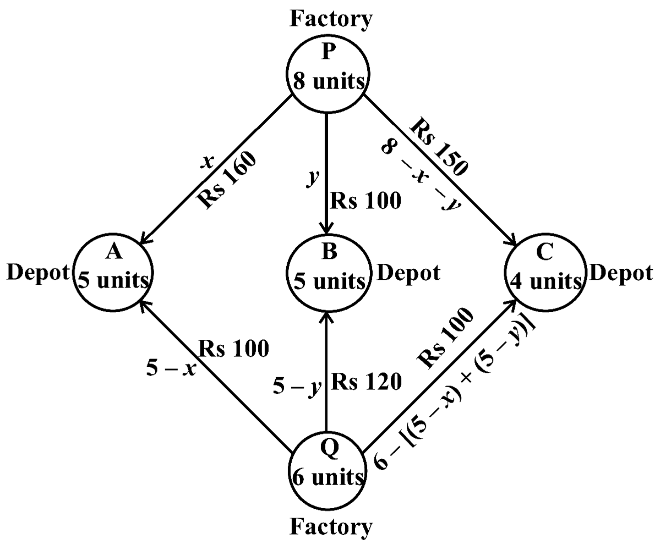
6. **There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:**

From/To	**Cost (in Rs.)**		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Ans:

Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then $(8 - x - y)$ units will be transported to depot at C



Hence, we have $x \geq 0$, $y \geq 0$ and $8 - x - y \geq 0$

i.e. $x \geq 0$, $y \geq 0$ and $x + y \leq 8$

Now, the weekly requirement of the depot at A is 5 units of the commodity. Since x units are transported from the factory at P, the remaining $(5 - x)$ units need to be transported from the factory at Q. Obviously, $5 - x \geq 0$, i.e. $x \leq 5$.

Similarly, $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory at Q to the depots at B and C respectively.

Thus, $5 - y \geq 0$, $x + y - 4 \geq 0$

i.e. $y \leq 5$, $x + y \geq 4$

Total transportation cost Z is given by

$Z = 160x + 100y + 100(5 - x) + 120(5 - y) + 100(x + y - 4) + 150(8 - x - y)$

$= 10(x - 7y + 190)$

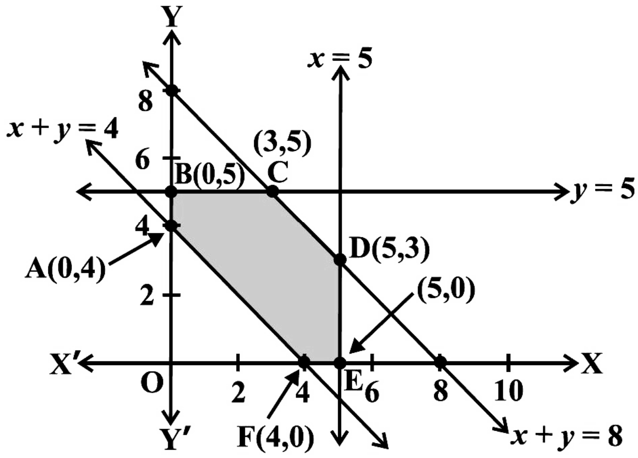
Therefore, the problem reduces to

Minimise $Z = 10(x - 7y + 190)$

subject to the constraints:

$x \geq 0, y \geq 0$ ... (1)
$x + y \leq 8$ ... (2)
$x \leq 5$ ... (3)
$y \leq 5$ ... (4)
and $x + y \geq 4$ ... (5)

The shaded region ABCDEF represented by the constraints (1) to (5) is the feasible region (see below figure). Observe that the feasible region is bounded. The coordinates of the corner points of the feasible region are (0, 4), (0, 5), (3, 5), (5, 3), (5, 0) and (4, 0).
Let us evaluate Z at these points.



Corner Point	Z = 10 (x – 7 y + 190)	
(0, 4)	1620	
(0, 5)	1550 ←	Minimum
(3, 5)	1580	
(5, 3)	1740	
(5, 0)	1950	
(4, 0)	1940	

From the table, we see that the minimum value of Z is 1550 at the point (0, 5).
Hence, the optimal transportation strategy will be to deliver 0, 5 and 3 units from the factory at P and 5, 0 and 1 units from the factory at Q to the depots at A, B and C respectively. Corresponding to this strategy, the transportation cost would be minimum, i.e., Rs 1550.

CHAPTER – 12: LINEAR PROGRAMMING

MARKS WEIGHTAGE – 05 marks

Previous Years Board Exam (Important Questions & Answers)

1. **A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs. 25 and that from a shade is Rs. 15. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit. Formulate an LPP and solve it graphically.**

Ans:

Let the manufacturer produces x padestal lamps and y wooden shades; then time taken by x pedestal lamps and y wooden shades on grinding/cutting machines = $(2x + y)$ hours and time taken on the sprayer = $(3x + 2y)$ hours.

Since grinding/cutting machine is available for at the most 12 hours.

$\therefore\ 2x + y \le 12$

and sprayer is available for at most 20 hours. Thus, we have

$\therefore\ 3x + 2y \le 20$

Now profit on the sale of x lamps and y shades is,

$Z = 25x + 15y$.

So, our problem is to find x and y so as to

Maximise $Z = 25x + 15y$ …(i)

Subject to the constraints:

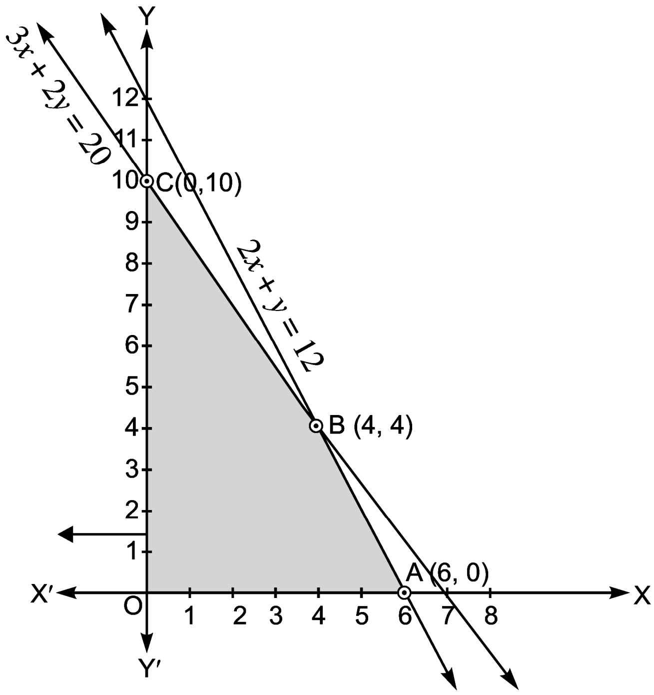
$3x + 2y \le 20$ …(ii)

$2x + y \le 12$ …(iii)

$x \ge 0$ …(iv)

$y \ge 0$ …(v)

The feasible region (shaded) $OABC$ determined by the linear inequalities (ii) to (v) is shown in the figure. The feasible region is bounded.



Let us evaluate the objective function at each corner point as shown below:

Corner Points	$Z = 25x + 15y$	
$O(0, 0)$	0	
$A(6, 0)$	150	
$B(4, 4)$	160	Maximum
$C(0, 10)$	150	

We find that maximum value of Z is Rs. 160 at $B(4, 4)$. Hence, manufacturer should produce 4 lamps and 4 shades to get maximum profit of Rs. 160.

2. A manufacturing company makes two types of teaching aids *A* and *B* of Mathematics for class XII. Each type of *A* requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of *B* requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of Rs. 80 on each piece of type *A* and Rs. 120 on each piece of type *B*. How many pieces of type *A* and type *B* should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week?

Ans: Let x and y be the number of pieces of type A and B manufactured per week respectively. If Z be the profit then,

Objective function, $Z = 80x + 120y$

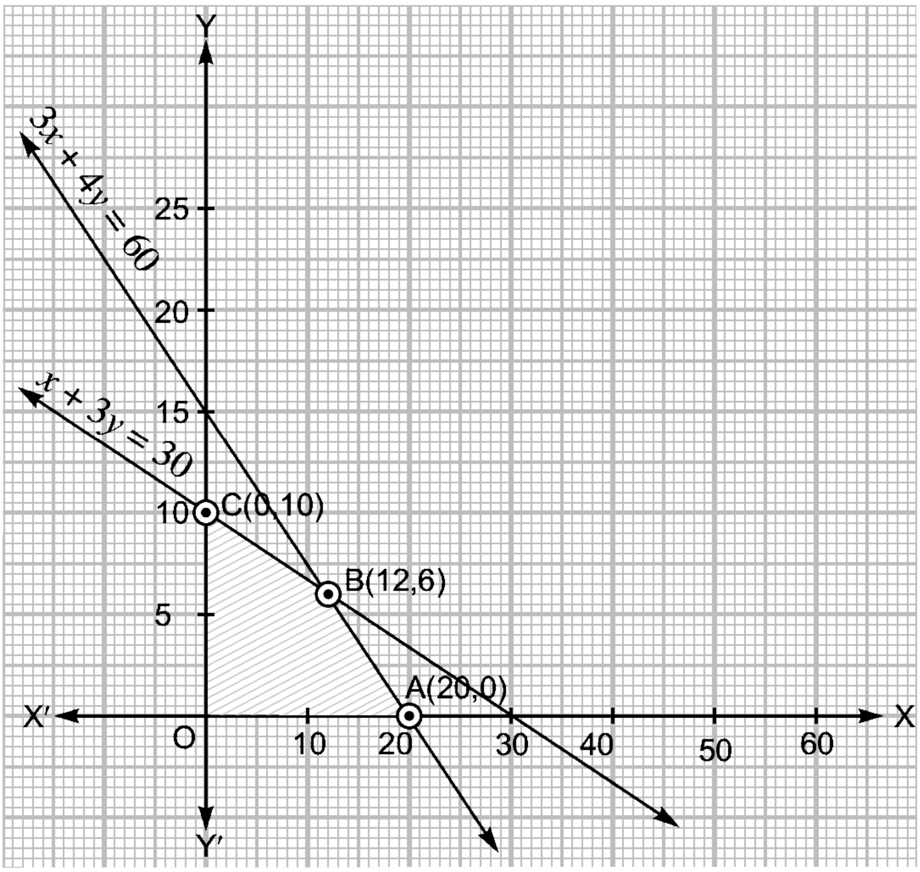
We have to maximize Z, subject to the constraints

$9x + 12y \leq 180 \Rightarrow 3x + 4y \leq 60$ ...(*i*)

$x + 3y \leq 30$ ...(*ii*)

$x \geq 0, y \geq 0$ ...(*iii*)

The graph of constraints are drawn and feasible region $OABC$ is obtained, which is bounded having corner points $O(0, 0)$, $A(20, 0)$, $B(12, 6)$ and $C(0, 10)$



Now the value of objective function is obtained at corner points as

Corner point	$Z = 80x + 120y$
O (0, 0)	0
A (20, 0)	1600
B (12, 6)	1680 ← Maximum
C (0, 10)	1200

Hence, the company will get the maximum profit of Rs. 1,680 by making 12 pieces of type A and 6 pieces of type B of teaching aid.

Yes, teaching aid is necessary for teaching learning process as

(*i*) it makes learning very easy.

(*ii*) it provides active learning.

(*iii*) students are able to grasp and understand concept more easily and in active manner.

3. A dealer in rural area wishes to purchase a number of sewing machines. He has only Rs. 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him Rs. 360 and a manually operated sewing machine Rs. 240. He can sell an electronic sewing machine at a profit of Rs. 22 and a manually operated sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximise his profit? Make it as a LPP and solve it graphically.

Ans:

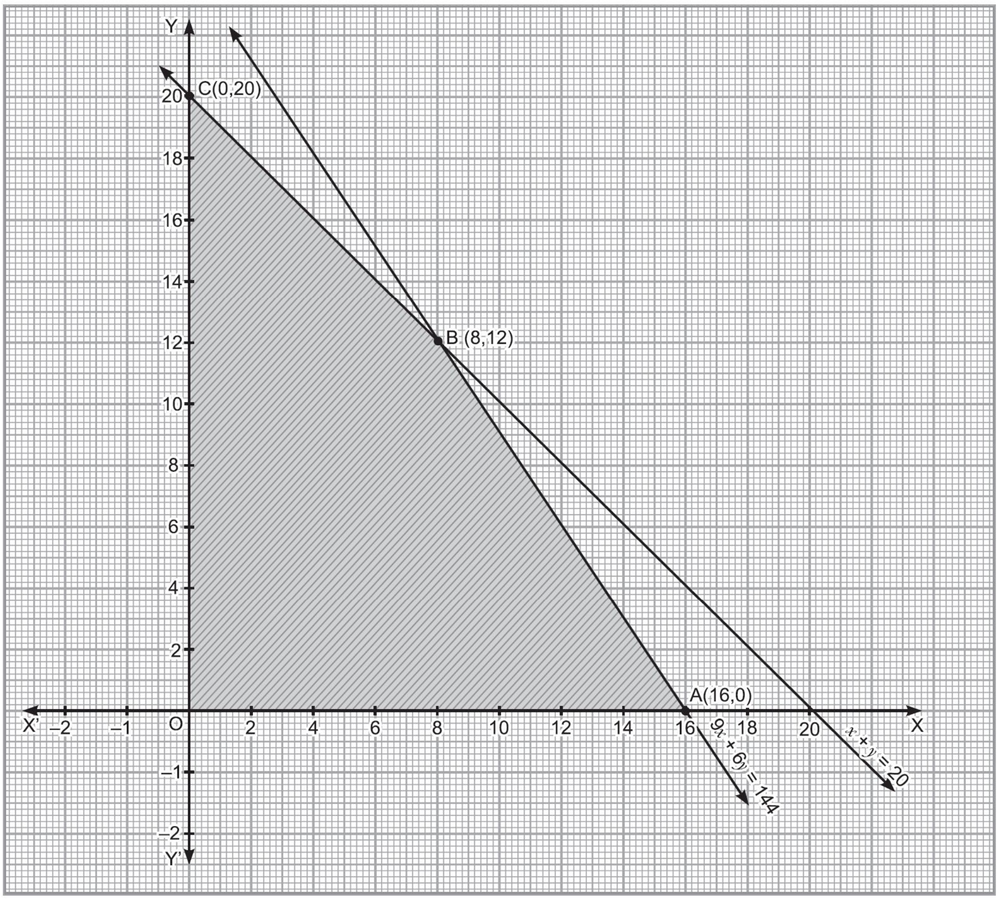
Suppose dealer purchase x electronic sewing machines and y manually operated sewing machines. If Z denotes the total profit. Then according to question

(Objective function) $Z = 22x + 18y$

Also, $x + y \leq 20$

$360x + 240y \leq 5760 \Rightarrow 9x + 6y \leq 144$

$x \geq 0, y \geq 0.$



We have to maximise Z subject to above constraint.
To solve graphically, at first we draw the graph of line corresponding to given inequations and shade the feasible region $OABC$.
The corner points of the feasible region $OABC$ are $O(0, 0)$, $A(16, 0)$, $B(8, 12)$ and $C(0, 20)$.
Now the value of objective function Z at corner points are obtained in table as

Corner points	$Z = 22x + 18y$	
$O(0, 0)$	$Z = 0$	
$A(16, 0)$	$Z = 22 \times 16 + 18 \times 0 = 352$	
$B(8, 12)$	$Z = 22 \times 8 + 18 \times 12 = 392$	→ Maximum
$C(0, 20)$	$Z = 22 \times 0 + 18 \times 20 = 360$	

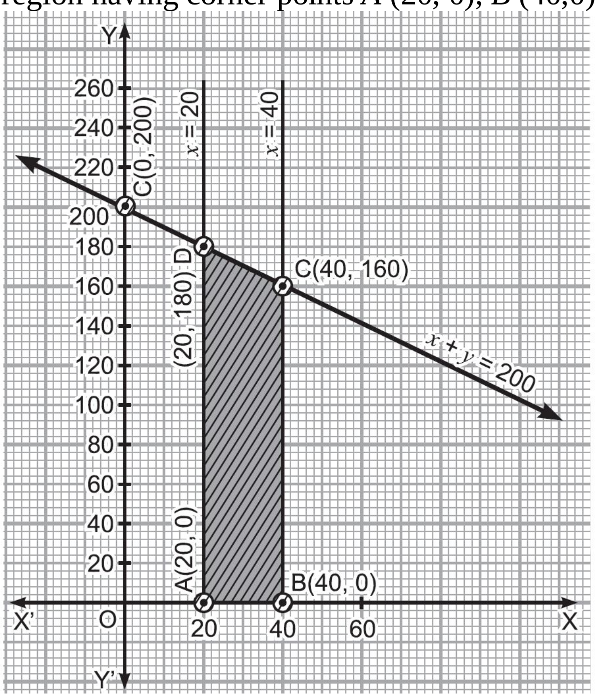
From table, it is obvious that Z is maximum when $x = 8$ and $y = 12$.
Hence, dealer should purchase 8 electronic sewing machines and 12 manually operated sewing machines to obtain the maximum profit ` 392 under given condition.

4. An aeroplane can carry a maximum of 200 passengers. A profit of `500 is made on each executive class ticket out of which 20% will go to the welfare fund of the employees. Similarly a profit of `400 is made on each economy ticket out of which 25% will go for the improvement of facilities provided to economy class passengers. In both cases, the remaining profit goes to the airline's fund. The airline reserves at least 20 seats for executive class. However at least four times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the net profit of the airline. Make the above as an LPP and solve graphically. Do you think, more passengers would prefer to travel by such an airline than by others?

Ans:

Let there be x tickets of executive class and y tickets of economy class. Let Z be net profit of the airline. Here, we have to maximise z. Now $Z = 500x \times \frac{80}{100} + y \times \frac{75}{100}$

$Z = 400x + 300y$ ....(i)
According to question $x \geq 20$ ....(ii)
Also $x + y \leq 200$ ....(iii)
$\Rightarrow x + 4x \leq 200 \Rightarrow 5x \leq 200 \Rightarrow x \leq 40$ ....(iv)
Shaded region is feasible region having corner points A (20, 0), B (40,0) C (40, 160), D (20,180)



Now value of Z is calculated at corner point as

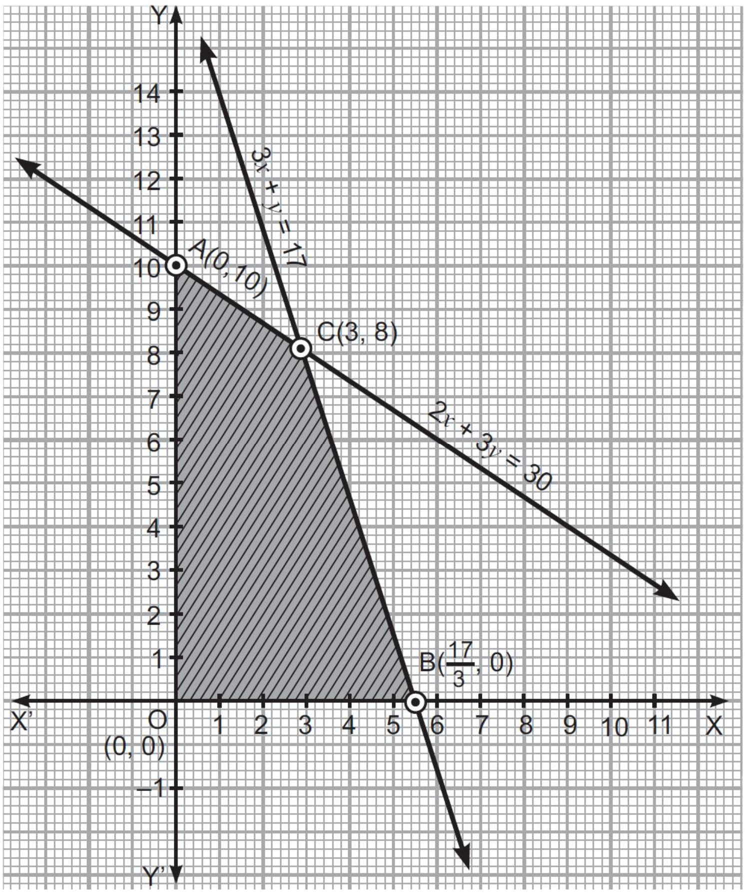
Corner points	$Z = 400x + 300y$
(20, 0)	8,000
(40, 0)	16,000
(40, 160)	64,000 ← Maximum
(20, 180)	60,000

Hence, 40 tickets of executive class and 160 tickets of economy class should be sold to maximise the net profit of the airlines.
Yes, more passengers would prefer to travel by such an airline, because some amount of profit is invested for welfare fund.

5. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3 workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at Rs. 100 and Rs. 120 per unit respectively, how should he use his resources to maximise the total revenue? Form the above as an LPP and solve graphically. Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate?

Ans:

Let x, y unit of goods A and B are produced respectively.
Let Z be total revenue
Here $Z = 100x + 120y$ ....(i)
Also $2x + 3y \leq 30$ ....(ii) $3x + y \leq 17$ ....(iii)
$x \geq 0$ ....(iv) $y \geq 0$ ....(v)
On plotting graph of above constants or inequalities (ii), (iii), (iv) and (v). We get shaded region as feasible region having corner points A, O, B and C.



For co-ordinate of 'C'
Two equations (*ii*) and (*iii*) are solved and we get coordinate of $C = (3, 8)$
Now the value of Z is evaluated at corner point as:

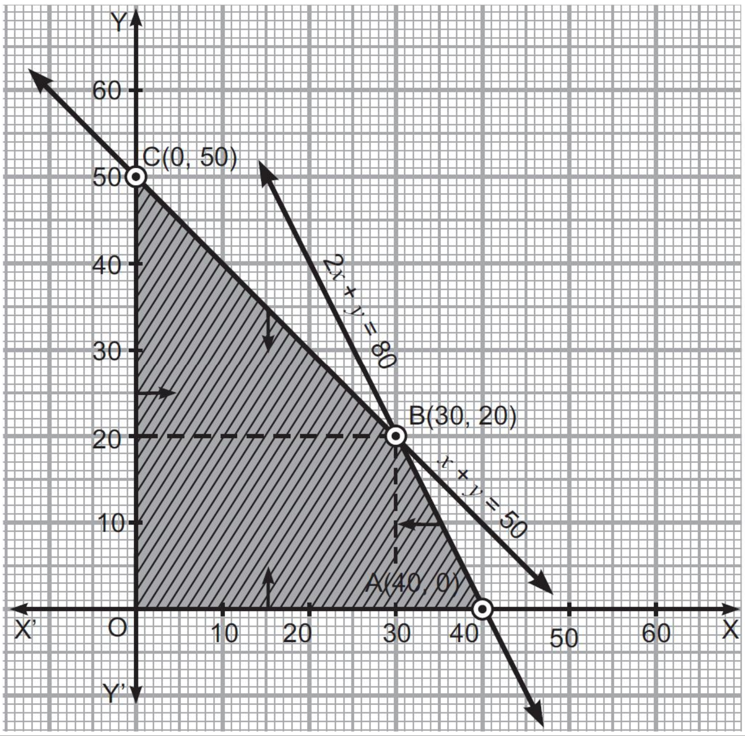
Corner point	$Z = 100x + 120y$
(0, 10)	1200
(0, 0)	0
$\left(\frac{17}{3}, 0\right)$	$\frac{1700}{3}$
(3, 8)	1260 ⟵ Maximum

Therefore maximum revenue is Rs. 1,260 when 2 workers and 8 units capital are used for production. Yes, although women workers have less physical efficiency but it can be managed by her other efficiency.

6. A cooperative society of farmers has 50 hectares of land to grow two crops A and B. The profits from crops A and B per hectare are estimated as `10,500 and `9,000 respectively. To control weeds, a liquid herbicide has to be used for crops A and B at the rate of 20 litres and 10 litres per hectare, respectively. Further not more than 800 litres of herbicide should be used in order to protect fish and wildlife using a pond which collects drainage from this land. Keeping in mind that the protection of fish and other wildlife is more important than earning profit, how much land should be allocated to each crop so as to maximize the total profit? Form an LPP from the above and solve it graphically. Do you agree with the message that the protection of wildlife is utmost necessary to preserve the balance in environment?

Ans:

Let x and y hectare of land be allocated to crop A and B respectively. If Z is the profit then
$Z = 10500x + 9000y$ …(*i*)
We have to maximize Z subject to the constraints
$x + y \le 50$ …(*ii*)
$20x + 10y \le 800 \Rightarrow 2x + y \le 80$ …(*iii*) $x \ge 0, y \ge 0$ …(*iv*)
The graph of system of inequalities (ii) to (iv) are drawn, which gives feasible region OABC with corner points O (0, 0), A (40, 0), B (30, 20) and C (0, 50).
Firstly, draw the graph of the line $x + y = 50$
Secondly, draw the graph of the line $2x + y = 80$



Feasible region is bounded.
Now value of Z is calculated at corner point as

Corner point	$Z = 10500x + 9000y$
O (0, 0)	0
A (40, 0)	420000
B (30, 20)	495000 ← Maximum
C (0, 50)	450000

Hence the co-operative society of farmers will get the maximum profit of Rs. 4,95,000 by allocating 30 hectares for crop A and 20 hectares for crop B.
Yes, because excess use of herbicide can make drainage water poisonous and thus it harm the life of water living creature and wildlife.

7. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of grinding/cutting machine and a sprayer. It takes 2 hours on the grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes one hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs. 5 and that from a shade is Rs. 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit? Make an L.P.P. and solve it graphically.

Ans:

Let the number of padestal lamps and wooden shades manufactured by cottage industry be x and y respectively.
Here profit is the objective function Z.

$\therefore\ Z = 5x + 3y \ldots (i)$

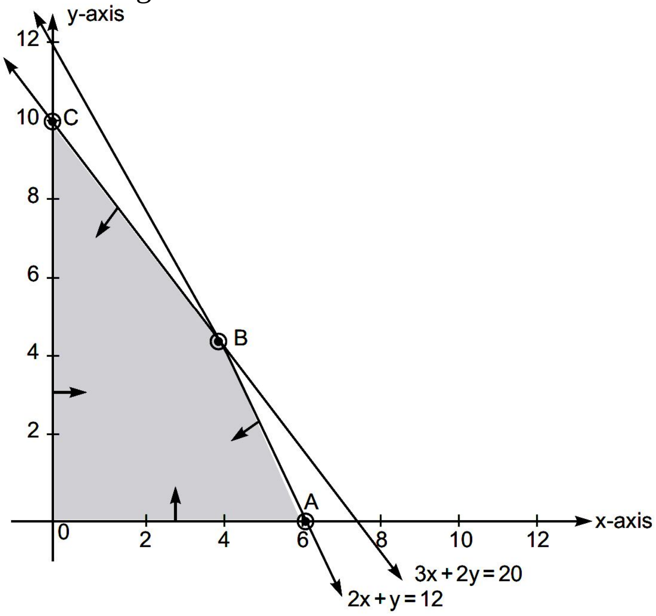
We have to maximise Z subject to the constrains

$2x + y \leq 12 \ldots (ii)$

$3x + 2y \leq 20 \ldots (iii)$

$x \geq 0$ and $y \geq 0 \ldots (iv)$

On plotting graph of above constraints or inequalities (*ii*), (*iii*) and (*iv*) we get shaded region having corner point A, B, C as feasible region.

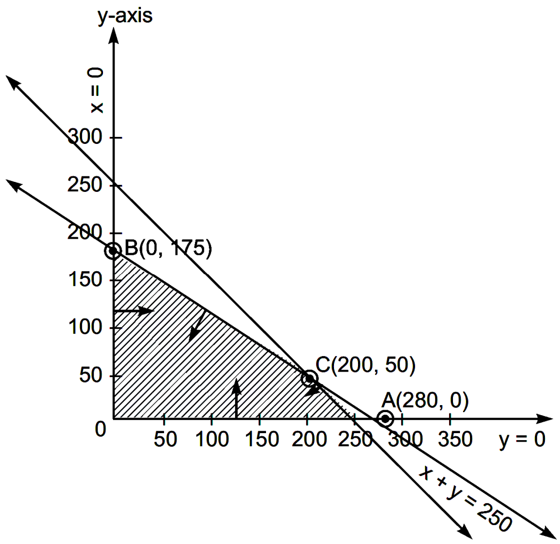


Since (0, 0) Satisfy $3x + 2y \leq 20$
$\Rightarrow$ Graph of $3x + 2y \leq 20$ is that half plane in which origin lies.
The shaded area *OABC* is the feasible region whose corner points are *O*, *A*, *B* and *C*.
For coordinate B.
Equation $2x + y = 12$ and $3x + 2y = 20$ are solved as
$3x + 2(12 - 2x) = 20$
$\Rightarrow 3x + 24 - 4x = 20 \Rightarrow x = 4$
$\therefore \Rightarrow y = 12 - 8 = 4$
Coordinate of $B = (4, 4)$
Now we evaluate objective function Z at each corner.

Corner points	$Z = 5x + 3y$
0 (0, 0)	0
A (6, 0)	30
B (4, 4)	32 ← maximum
C (0, 10)	30

Hence maximum profit is ` 32 when manufacturer produces 4 lamps and 4 shades.

8. A merchant plans to sell two types of personal computers — a desktop model and a portable model that will cost Rs. 25,000 and Rs. 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs. 70 lakhs and his profit on the desktop model is Rs. 4,500 and on the portable model is Rs. 5,000. Make an L.P.P. and solve it graphically.
Ans:
Let the number of desktop and portable computers to be sold be x and y respectively.
Here, Profit is the objective function Z.
$\therefore Z = 4500x + 5000y$ …(i)
we have to maximise z subject to the constraints
$x + y \leq 250$ …(ii) (Demand Constraint)
$25000x + 40000y \leq 70,00,000$ …(iii) (Investment constraint)
$\Rightarrow 5x + 8y \leq 1400$
$x \geq 0, y \geq 0$ …(iv) (Non-negative constraint)
On plotting graph of above constraints or inequalities, we get shaded region having corner point A, B, C as feasible region.



For coordinates of *C*, equation $x + y = 250$ and $5x + 8y = 1400$ are solved and we get $x = 200$, $y = 50$
Now, we evaluate objective function *Z* at each corner

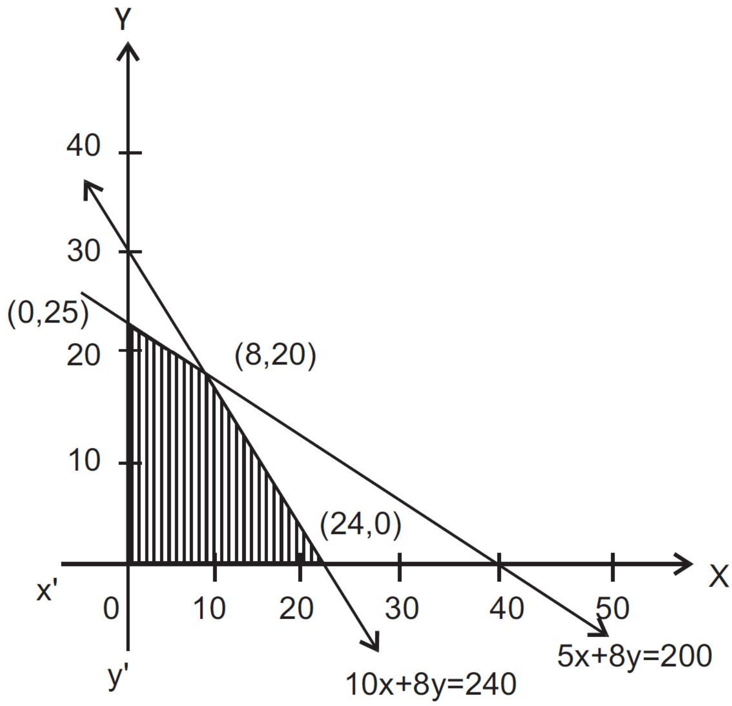
Corner Point	$Z = 4500x + 5000y$
$O(0, 0)$	0
$A(250, 0)$	1125000
$C(200, 50)$	1150000 ← maximum
$B(0, 175)$	875000

Maximum profit is Rs. 11,50,000 when he plan to sell 200 unit desktop and 50 portable computers.

9. A factory makes two types of items *A* and *B*, made of plywood. One piece of item *A* requires 5 minutes for cutting and 10 minutes for assembling. One piece of item *B* requires 8 minutes for cutting and 8 minutes for assembling. There are 3 hours and 20 minutes available for cutting and 4 hours for assembling. The profit on one piece of item *A* is Rs 5 and that on item *B* is Rs 6. How many pieces of each type should the factory make so as to maximise profit? Make it as an L.P.P. and solve it graphically.

Ans:

Let the factory makes *x* pieces of item *A* and *B* by pieces of item.
Time required by item *A* (one piece)
cutting = 5 minutes, assembling = 10 minutes
Time required by item *B* (one piece)
cutting = 8 minutes, assembling = 8 minutes
Total time cutting = 3 hours & 20 minutes, assembling = 4 hours
Profit on one piece item *A* = Rs 5, item *B* = Rs 6
Thus, our problem is maximized $Z = 5x + 6y$
Subject to $x \geq 0$, $y \geq 0$
$5x + 8y \leq 200$
$10x + 8y \leq 240$
On plotting graph of above constraints or inequalities, we get shaded region.

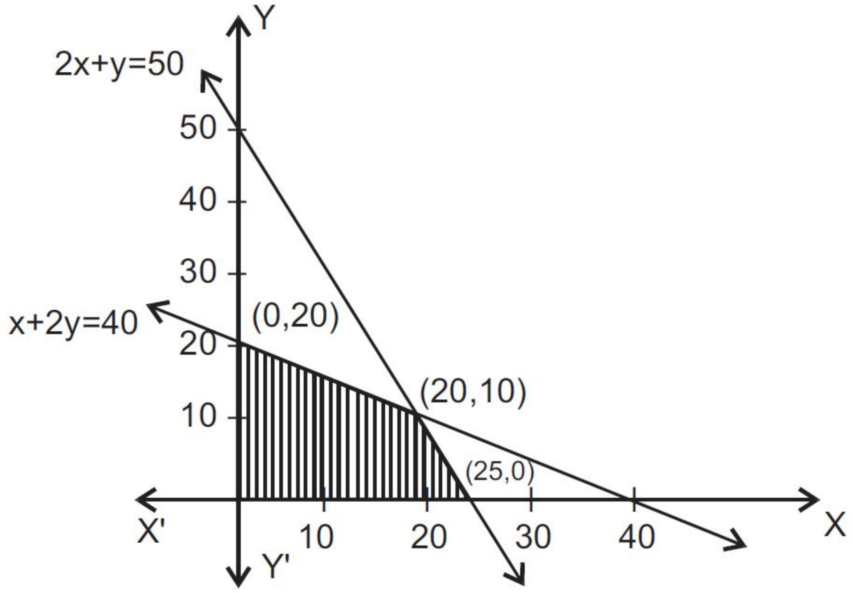


From figure, possible points for maximum value of *z* are at (24, 0), (8, 20), (0, 25).
at (24, 0), $z = 120$
at (8, 20), $z = 40 + 120 = 160$ (maximum)

at (0, 25), $z = 150$
∴ 8 pieces of item A and 20 pieces of item B produce maximum profit of Rs 160.

10. One kind of cake requires 300 g of flour and 15 g of fat, another kind of cake requires 150 g of flour and 30 g of fat. Find the maximum number of cakes which can be made from 7 × 5 kg of flour and 600 g of fat, assuming that there is no shortage of the other ingredients used in making the cakes. Make it as an L.P.P. and solve it graphically.

Ans:
Let number of first kind and second kind of cakes that can be made be x and y respectively
Then, the given problem is
Maximize, $z = x + y$

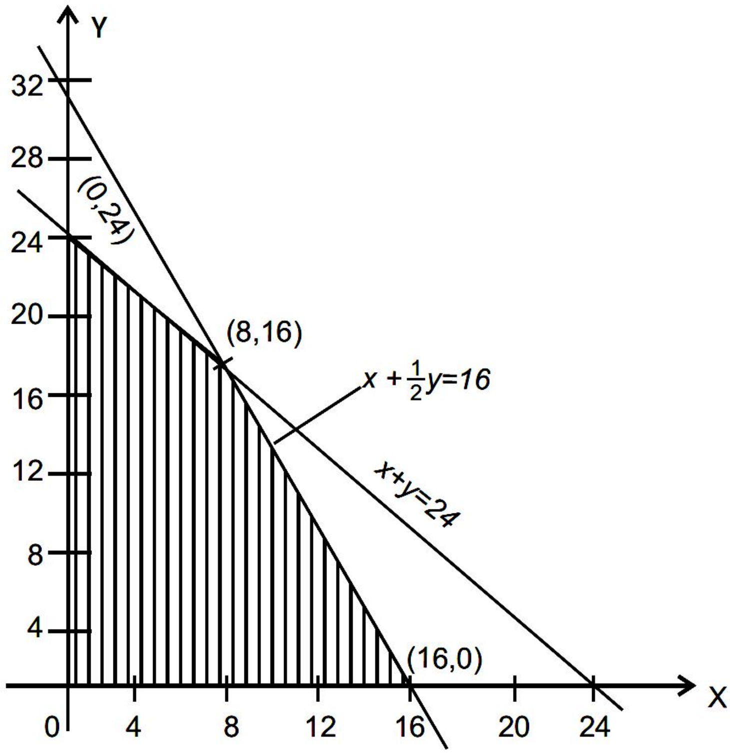


Subjected to $x \geq 0$, $y \geq 0$
$300x + 150y \leq 7500 \Rightarrow 2x + y \leq 50$
$15x + 30y \leq 600 \Rightarrow x + 2y \leq 40$
On plotting graph of above constraints or inequalities, we get shaded region.
From graph, three possible points are (25, 0), (20, 10), (0, 20)
At (25, 0), $z = x + y = 25 + 0 = 25$
At (20, 10), $z = x + y = 20 + 10 = 30 \leftarrow$ Maximum
At (0, 20), $z = 0 + 20 = 20$
As Z is maximum at (20, 10), *i.e.*, $x = 20$, $y = 10$.
∴ 20 cakes of type I and 10 cakes of type II can be made.

11. A small firm manufactures gold rings and chains. The total number of rings and chains manufactured per day is atmost 24. It takes 1 hour to make a ring and 30 minutes to make a chain. The maximum number of hours available per day is 16. If the profit on a ring is Rs. 300 and that on a chain is Rs 190, find the number of rings and chains that should be manufactured per day, so as to earn the maximum profit. Make it as an L.P.P. and solve it graphically.

Ans:
Total no. of rings & chain manufactured per day = 24.
Time taken in manufacturing ring = 1 hour
Time taken in manufacturing chain = 30 minutes

One time available per day = 16
Maximum profit on ring = Rs 300
Maximum profit on chain = Rs 190
Let gold rings manufactured per day = x
Chains manufactured per day = y
L.P.P. is maximize $Z = 300x + 190y$



Subject to $x \geq 0$,
$y \geq 0$,
$x + y \leq 24$
and $x + \frac{1}{2}y \leq 16$
On plotting graph of above constraints or inequalities, we get shaded region.

Possible points for maximum Z are (16, 0), (8, 16) and (0, 24).

At (16, 0), $Z = 4800 + 0 = 4800$
At (8, 16), $Z = 2400 + 3040 = 5440 \leftarrow$ Maximum
At (0, 24), $Z = 0 + 4560 = 4560$
Z is maximum at (8, 16).
$\therefore$ 8 gold rings & 16 chains must be manufactured per day.

PROBABILITY

CHAPTER – 13: PROBABILITY

MARKS WEIGHTAGE – 08 marks

NCERT Important Questions & Answers

1. **A die is thrown three times. Events A and B are defined as below: A : 4 on the third throw; B : 6 on the first and 5 on the second throw. Find the probability of A given that B has already occurred.**
Ans:
The sample space has 216 outcomes.
Now A =
(1,1,4) (1,2,4) ... (1,6,4) (2,1,4) (2,2,4) ... (2,6,4)
(3,1,4) (3,2,4) ... (3,6,4) (4,1,4) (4,2,4) ...(4,6,4)
(5,1,4) (5,2,4) ... (5,6,4) (6,1,4) (6,2,4) ...(6,6,4)
B = {(6,5,1), (6,5,2), (6,5,3), (6,5,4), (6,5,5), (6,5,6)}
and A ∩ B = {(6,5,4)}.
Now, $P(B)=\frac{6}{216}$ *and* $P(A\cap B)=\frac{1}{216}$
Then $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{\frac{1}{216}}{\frac{6}{216}}=\frac{1}{6}$

2. **A die is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?**
Ans:
Let E be the event that 'number 4 appears at least once' and F be the event that 'the sum of the numbers appearing is 6'.
Then, E = {(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (1,4), (2,4), (3,4), (5,4), (6,4)}
and F = {(1,5), (2,4), (3,3), (4,2), (5,1)}
We have $P(E)=\frac{11}{36}$ *and* $P(E)=\frac{5}{36}$
Also E∩F = {(2,4), (4,2)}
Therefore $P(E\cap F)=\frac{2}{36}$
Hence, the required probability, $P(E/F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{2}{36}}{\frac{5}{36}}=\frac{2}{5}$

3. **A black and a red dice are rolled. (a) Find the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5. (b) Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.**
Ans:
Let the first observation be from the black die and second from the red die.
When two dice (one black and another red) are rolled, the sample space
$S = 6 \times 6 = 36$ (equally likely sample events)
(i) Let E : set of events in which sum greater than 9 and F : set of events in which black die resulted in a 5
E = {(6,4), (4,6), (5, 5), (5,6), (6, 5), (6,6)} $\Rightarrow n(E) = 6$
and F = {(5, 1), (5,2), (5, 3), (5, 4), (5, 5), (5,6)} $\Rightarrow n(F) = 6$
$\Rightarrow E \cap F$ = {(5, 5), (5,6)} $\Rightarrow n(E \cap F) = 2$

The conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5, is given by P(E/F)

$P(E) = \frac{6}{36}$ and $P(F) = \frac{6}{36}$

Also, $P(E \cap F) = \frac{2}{36}$

$\therefore\ P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{2}{6} = \frac{1}{3}$

(ii) Let E : set of events having 8 as the sum of the observations,

F : set of events in which red die resulted in a (in any one die) number less than 4

$\Rightarrow E$ = {(2,6), (3, 5), (4,4), (5, 3), (6,2)} $\Rightarrow n(E) = 5$

and F = {(1, 1), (1, 2),….. (3, 1), (3, 2)…… (5, 1), (5, 2),……….} $\Rightarrow n(F) = 18$

$\Rightarrow E \cap F$ = {(5, 3), (6, 2)} $\Rightarrow n(E \cap F) = 2$

The conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5, is given by P(E/F)

$P(E) = \frac{5}{36}$ and $P(F) = \frac{18}{36}$

Also, $P(E \cap F) = \frac{2}{36}$

$\therefore\ P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{2}{36}}{\frac{18}{36}} = \frac{2}{18} = \frac{1}{9}$

4. An instructor has a question bank consisting of 300 easy True / False questions, 200 difficult True / False questions, 500 easy multiple choice questions and 400 difficult multiple choice questions. If a question is selected at random from the question bank, what is the probability that it will be an easy question given that it is a multiple choice question?

Ans:

Total number of questions = 300 + 200 + 500 + 400 = 1400.

Let E be the event that selected question is an easy question

Then, $n(E) = 500 + 300 = 800$

$\therefore\ P(E) = \frac{800}{1400}$

Let F be the event that selected question is a multiple choice question.

Then, $n(F) = 500 + 400 = 900$

$\therefore P(F) = \frac{900}{1400}$

Also, $P(E \cap F) = \frac{500}{1400}$

$\Rightarrow P(E/F) = \frac{P(E \cap F)}{P(F)} = \frac{\frac{500}{1400}}{\frac{900}{1400}} = \frac{500}{900} = \frac{5}{9}$

5. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement. What is the probability that both drawn balls are black?

Ans:

Let E and F denote respectively the events that first and second ball drawn are black. We have to find P(E ∩ F) or P (EF).

Now P(E) = P (black ball in first draw) = $\frac{10}{15}$

Also given that the first ball drawn is black, i.e., event E has occurred, now there are 9 black balls and five white balls left in the urn. Therefore, the probability that the second ball drawn is black, given that the ball in the first draw is black, is nothing but the conditional probability of F given that E has occurred.

i.e. P(F|E) = $\frac{9}{14}$

By multiplication rule of probability, we have

P (E ∩ F) = P(E) P(F|E) = $\frac{10}{15}\times\frac{9}{14}=\frac{3}{7}$

6. Two balls are drawn at random with replacement from a box containing 10 black and 8 red balls. Find the probability that (i) both balls are red. (ii) first ball is black and second is red. (iii) one of them is black and other is red.

Ans:

Total number of balls = 18, number of red balls = 8 and number of black balls = 10

$\therefore$ Probability of drawing a red ball = $\frac{8}{18}$

Similarly, probability of drawing a black ball = $\frac{10}{18}$

(i) Probability of getting both red balls = *P* (both balls are red)

= *P* (a red ball is drawn at first draw and again a red ball at second draw) = $\frac{8}{18}\times\frac{8}{18}=\frac{16}{81}$

(ii) *P* (probability of getting first ball is black and second is red) = $\frac{10}{18}\times\frac{8}{18}=\frac{20}{81}$

(iii) Probability of getting one black and other red ball = *P*(first ball is black and second is red) + *P* (first ball is red and second is black) = $\frac{10}{18}\times\frac{8}{18}+\frac{8}{18}\times\frac{10}{18}=\frac{20}{81}+\frac{20}{81}=\frac{40}{81}$

7. Probability of solving specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem.

Ans:

Probability of solving the problem by *A*, *P*(a) = $\frac{1}{2}$

Probability of solving the problem by *B*, *P*(b) = $\frac{1}{3}$

Probability of not solving the problem by *A* = *P*(*A*') = 1 – *P*(a) = $1-\frac{1}{2}=\frac{1}{2}$

and probability of not solving the problem by *B* = *P*(B') = 1 – *P*(b) = $1-\frac{1}{3}=\frac{2}{3}$

(i) *P* (the problem is solved) = 1 – *P*(none of them solve the problem) =

$=1-P(A'\cap B')=1-P(A')P(B')$

(since *A* and *B* are independent *A*' and *B*' are independent)

$=1-\left(\frac{1}{2}\times\frac{2}{3}\right)=1-\frac{1}{3}=\frac{2}{3}$

(ii) *P* (exactly one of them solve the problem) = *P*(a) *P*(*B*') + *P*(*A*') *P*(b)

$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}=\frac{1}{3}+\frac{1}{6}=\frac{2+1}{6}=\frac{3}{6}=\frac{1}{2}$

8. In a hostel, 60% of the students read Hindi news paper, 40% read English news paper and 20% read both Hindi and English news papers. A student is selected at random. (a) Find the probability that she reads neither Hindi nor English news papers. (b) If she reads Hindi news paper, find the probability that she reads English news paper. (c) If she reads English news paper, find the probability that she reads Hindi news paper.

Ans:

Let H : Set of students reading Hindi newspaper and E : set of students reading English newspaper.

Let $n(S)$ = 100 Then, $n(H)$ = 60

$n(E)$ = 40 and $n(H \cap E)$ = 20

$$\therefore P(H) = \frac{60}{100} = \frac{3}{5}, P(E) = \frac{40}{100} = \frac{2}{5} \text{ and } P(H \cap E) = \frac{20}{100} = \frac{1}{5}$$

(i) Required probability = P (student reads neither Hindi nor English newspaper) =

$$= P(H' \cap E') = P(H \cup E)' = 1 - P(H \cup E)$$

$$= 1 - [P(H) + P(E) - P(H \cap E)] = 1 - \left[\frac{3}{5} + \frac{2}{5} - \frac{1}{5}\right] = 1 - \frac{4}{5} = \frac{1}{5}$$

(ii) Required probability = P(a randomly chosen student reads English newspaper, if he/she reads Hindi newspaper) = $P(E/H) = \dfrac{P(E \cap H)}{P(H)} = \dfrac{\frac{1}{5}}{\frac{3}{5}} = \dfrac{1}{3}$

(iii) Required probability = P (student reads Hindi newspaper when it is given that reads English newspaper) = $P(H/E) = \dfrac{P(H \cap E)}{P(E)} = \dfrac{\frac{1}{5}}{\frac{2}{5}} = \dfrac{1}{2}$

9. Bag I contains 3 red and 4 black balls while another Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from Bag II.

Ans:

Let E_1 be the event of choosing the bag I, E_2 the event of choosing the bag II and A be the event of drawing a red ball.

Then $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(A|E_1)$ = P(drawing a red ball from Bag I) = $\frac{3}{7}$

and $P(A|E_2)$ = P(drawing a red ball from Bag II) = $\frac{5}{11}$

Now, the probability of drawing a ball from Bag II, being given that it is red, is $P(E_2|A)$

By using Bayes' theorem, we have

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)} = \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{1}{2} \times \frac{3}{7} + \frac{1}{2} \times \frac{5}{11}} = \frac{35}{68}$$

10. Given three identical boxes I, II and III, each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in the box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Ans:

Let E_1, E_2 and E_3 be the events that boxes I, II and III are chosen, respectively.

Then $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, let A be the event that 'the coin drawn is of gold'

Then $P(A|E_1)$ = P(a gold coin from bag I) = 2/2 =1

$P(A|E_2)$ = P(a gold coin from bag II) = 0

$P(A|E_3)$ = P(a gold coin from bag III) = $\frac{1}{2}$

Now, the probability that the other coin in the box is of gold

= the probability that gold coin is drawn from the box I.

= $P(E_1|A)$

By Bayes' theorem, we know that

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times 0 + \frac{1}{3}\times\frac{1}{2}} = \frac{2}{3}$$

11. In a factory which manufactures bolts, machines A, B and C manufacture respectively 25%, 35% and 40% of the bolts. Of their outputs, 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B?

Ans:

Let events B_1, B_2, B_3 be the following :

B_1 : the bolt is manufactured by machine A

B_2 : the bolt is manufactured by machine B

B_3 : the bolt is manufactured by machine C

Clearly, B_1, B_2, B_3 are mutually exclusive and exhaustive events and hence, they represent a partition of the sample space.

Let the event E be 'the bolt is defective'.

The event E occurs with B_1 or with B_2 or with B_3. Given that,

$P(B_1) = 25\% = 0.25$, $P(B_2) = 0.35$ and $P(B_3) = 0.40$

Again $P(E|B_1)$ = Probability that the bolt drawn is defective given that it is manufactured by machine A = 5% = 0.05

Similarly, $P(E|B_2) = 0.04$, $P(E|B_3) = 0.02$.

Hence, by Bayes' Theorem, we have

$$P(B_2/E) = \frac{P(B_2)P(E/B_2)}{P(B_1)P(E/B_1) + P(B_2)P(E/B_2) + P(B_3)P(E/B_3)}$$

$$= \frac{0.35\times 0.04}{0.25\times 0.05 + 0.35\times 0.04 + 0.40\times 0.02} = \frac{0.0140}{0.0345} = \frac{28}{69}$$

12. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ *and* $\frac{1}{2}$, if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

Ans:

Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter, and other means of transport respectively.

Then $P(T_1) = \frac{3}{10}$, $P(T_2) = \frac{1}{5}$, $P(T_3) = \frac{1}{10}$ and $P(T_4) = \frac{2}{5}$

$P(E|T_1)$ = Probability that the doctor arriving late comes by train = $\frac{1}{4}$

Similarly, $P(E|T_2) = \frac{1}{3}$, $P(E|T_3) = \frac{1}{12}$ and $P(E|T_4) = 0$, since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E)$ = Probability that the doctor arriving late comes by train

$$\Rightarrow P(T_1/E) = \frac{P(T_1)P(E/T_1)}{P(T_1)P(E/T_1) + P(T_2)P(E/T_2) + P(T_3)P(E/T_3) + P(T_4)P(E/T_4)}$$

$$= \frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0} = \frac{3}{40}\times\frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$

13. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Ans:

Let E be the event that the man reports that six occurs in the throwing of the die and let S_1 be the event that six occurs and S_2 be the event that six does not occur.

Then $P(S_1)$ = Probability that *six* occurs = $\frac{1}{6}$

$P(S_2)$ = Probability that *six* does not occur = $\frac{5}{6}$

$P(E|S_1)$ = Probability that the man reports that *six* occurs when *six* has actually occurred on the die

= Probability that the man speaks the truth = $\frac{3}{4}$

$P(E|S_2)$ = Probability that the man reports that *six* occurs when *six* has not actually occurred on the die

= Probability that the man does not speak the truth = $1 - \frac{3}{4} = \frac{1}{4}$

Thus, by Bayes' theorem, we get

$P(S_1|E)$ = Probability that the report of the man that *six* has occurred is actually a *six*

$$\Rightarrow P(S_1/E) = \frac{P(S_1)P(E/S_1)}{P(S_1)P(E/S_1)+P(S_2)P(E/S_2)} = \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4}+\frac{5}{6}\times\frac{1}{4}} = \frac{1}{8}\times\frac{24}{8} = \frac{3}{8}$$

Hence, the required probability is $\frac{3}{8}$.

14. A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Ans:

Let E_1 : first bag is selected, E_2 : second bag is selected

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$P(E_1) = P(E_2) = \frac{1}{2}$

Let E : ball drawn is red.

P(E/E$_1$) = P(drawing a red ball from first bag) = $\frac{4}{8}=\frac{1}{2}$

P(E/E$_2$) = P(drawing a red ball from second bag) = $\frac{2}{8}=\frac{1}{4}$

By using Baye's theorem,

Required probability = $P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$

$$=\frac{\frac{1}{2}\times\frac{1}{2}}{\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{1}{4}+\frac{1}{8}}=\frac{\frac{1}{4}}{\frac{2+1}{8}}=\frac{\frac{1}{4}}{\frac{3}{8}}=\frac{2}{3}$$

15. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain A grade and 20% of day scholars attain A grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an A grade, what is the probability that the student is a hostlier?

Ans:

Let E_1 : the event that the student is residing in hostel and E_2 : the event that the student is not residing in the hostel.

Let E : a student attains A grade,

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$P(E_1)$ = 60% = $\frac{60}{100}=\frac{3}{5}$ and $P(E_2)$ = 40% = $\frac{40}{100}=\frac{2}{5}$

Then P(E/E$_1$) = 30% = $\frac{30}{100}=\frac{3}{10}$ and P(E/E$_2$) = 20% = $\frac{20}{100}=\frac{2}{10}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}=\frac{\frac{3}{5}\times\frac{3}{10}}{\frac{3}{5}\times\frac{3}{10}+\frac{2}{5}\times\frac{2}{10}}=\frac{9}{9+4}=\frac{9}{13}$$

16. In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{4}$. What is the probability that the student knows the answer given that he answered it correctly?

Ans:

Let E_1 : the event that the student knows the answer and E_2 : the event that the student guesses the answer.

Then, E_1 and E_2 are mutually exclusive and exhaustive. Moreover,

$P(E_1)$ = $\frac{3}{4}$ and $P(E_2)$ = $\frac{1}{4}$

Let E : the answer is correct.

The probability that the student answered correctly, given that he knows the answer, is 1 *i.e., P* $P(E/E_1)$ = 1

Probability that the students answered correctly, given that the he guessed, is $\frac{1}{4}$

i.e., $P(E/E_2)$ = $\frac{1}{4}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{3}{4}\times 1}{\frac{3}{4}\times 1+\frac{1}{4}\times\frac{1}{4}}=\frac{\frac{3}{4}}{\frac{3}{4}+\frac{1}{16}}=\frac{\frac{12}{16}}{\frac{12+1}{16}}=\frac{12}{13}$$

17. There are three coins. One is a two headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the time and third is an unbiased coin. One of the three coins is chosen at random and tossed, it shows heads, what is the probability that it was the two headed coin ?

Ans:

Let E_1 : the event that the coin chosen is two headed, E_2 : the event that the coin chosen is biased and E_3 : the event that the coin chosen is unbiased

$\Rightarrow E_1, E_2, E_3$ are mutually exclusive and exhaustive events. Moreover,

$\Rightarrow P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Let E : tosses coin shows up a head,

$\therefore$ $P(E/E_1)$ = P(coin showing heads, given that it is a two headed coin) = 1

$P(E/E_2)$ = P(coin showing heads, given that it is a biased coin) = $75\%=\frac{75}{100}=\frac{3}{4}$

$P(E/E3)$ = P(coin showing heads, given that it is an unbiased coin) = $\frac{1}{2}$

The probability that the coin is two headed, given that it shows head, is given by $P(E_1/E)$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{2}}=\frac{1}{1+\frac{3}{4}+\frac{1}{2}}=\frac{1}{\frac{4+3+2}{4}}=\frac{4}{9}$$

18. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accidents are 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

Ans:

There are 2000 scooter drivers, 4000 car drivers and 6000 truck drivers.

Total number of drivers = 2000 + 4000 + 6000 = 12000

Let E_1 : the event that insured person is a scooter driver, E_2 : the event that insured person is a car driver and E_3 : the event that insured person is a truck driver.

Then, E_1, E_2, E_3 are mutually exclusive and exhaustive events. Moreover,

$P(E_1) = \frac{2000}{12000}=\frac{1}{6}$, $P(E_2) = \frac{4000}{12000}=\frac{1}{3}$ and $P(E_3) = \frac{6000}{12000}=\frac{1}{2}$

Let E : the events that insured person meets with an accident,

$\therefore$ $P(E/E_1)$ = P(scooter driver met with an accident) = $0.01=\frac{1}{100}$

$P(E/E_2)$ = P(car driver met with an accident) = $0.03=\frac{3}{100}$

$P(E/E3)$ = P(truck driver met with an accident) = $0.15=\frac{15}{100}$

The probability that the driver is a scooter driver, given he met with an accident, is given by $P(E_1/E)$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{\frac{1}{100}\times\frac{1}{6}}{\frac{1}{100}\times\frac{1}{6}+\frac{3}{100}\times\frac{1}{3}+\frac{15}{100}\times\frac{1}{2}}=\frac{\frac{1}{6}}{\frac{1}{6}+1+\frac{15}{2}}=\frac{1}{1+6+45}=\frac{1}{52}$$

19. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% produced by machine B were defective. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by machine B?

Ans:

Let E_1 : the event that the item is produced by machine *A* and E_2 : the event that the item is produced by machine *B*.

Then, E_1 and E_2 are mutually exclusive and exhaustive events. Moreover,

$P(E_1)$ = 60% $=\frac{60}{100}=\frac{3}{5}$ and $P(E_2)$ = 40% = $\frac{40}{100}=\frac{2}{5}$

Let *E* : the event that the item chosen is defective,

$\therefore$ $P(E/E_1)$ = *P*(machine *A* produced defective items) = $2\%=\frac{2}{100}$

$P(E/E_2)$ = P(machine *B* produced defective items) = $1\%=\frac{1}{100}$

The probability that the randomly selected item was from machine *B*, given that it is defective, is given by $P(E_2/E)$

By using Baye's theorem, we obtain

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{2}{5}\times\frac{1}{100}}{\frac{3}{5}\times\frac{2}{100}+\frac{2}{5}\times\frac{1}{100}}=\frac{\frac{2}{500}}{\frac{6}{500}+\frac{2}{500}}=\frac{2}{6+2}=\frac{2}{8}=\frac{1}{4}$$

20. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die?

Ans:

Let E_1 : the event that 5 or 6 is shown on die and E_2 : the event that 1, 2, 3, or 4 is shown on die.

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

and $n(E_1)$ = 2, $n(E_2)$ = 4

Also, $n(S)$ = 6

$P(E_1)$ = $\frac{2}{6}=\frac{1}{3}$ and $P(E_2)$ = $\frac{4}{6}=\frac{2}{3}$

Let *E* : The event that exactly one head show up,

$\therefore$ $P(E/E_1)$ = *P*(exactly one head show up when coin is tossed thrice) = *P*{HTT, THT, TTH} = $\frac{3}{8}$

$P(E/E_2)$ = P(head shows up when coin is tossed once) = $\frac{1}{2}$

The probability that the girl threw, 1, 2, 3 or 4 with the die, if she obtained exactly one head, is given by $P(E_2/E)$

By using Baye's theorem, we obtain

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}}=\frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}}=\frac{8}{3+8}=\frac{8}{11}$$

21. A manufacturer has three machine operators A, B and C. The first operator A produces 1% defective items, where as the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B is on the job for 30% of the time and C is on the job for 20% of the time. A defective item is produced, what is the probability that it was produced by A?

Ans:

Let E_1 : the event that item is produced by machine A, E_2 : the event that item is produced by machine B and E_3 : the event that item is produced by machine C

Here, E_1, E_2 and E_3 are mutually exclusive and exhaustive events.

Moreover,

$P(E_1) = 50\% = \frac{50}{100}$

$P(E_2) = 30\% = \frac{30}{100}$

and $P(E_3) = 20\% = \frac{20}{100}$

Let E : The event that item chosen is found to be defective',

$\therefore P(E/E_1) = \frac{1}{100}$, $P(E/E_2) = \frac{5}{100}$, $P(E/E_3) = \frac{7}{100}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)}$$

$$=\frac{\frac{50}{100}\times\frac{1}{100}}{\frac{50}{100}\times\frac{1}{100}+\frac{30}{100}\times\frac{5}{100}+\frac{20}{100}\times\frac{7}{100}}=\frac{50}{50+150+140}=\frac{50}{340}=\frac{5}{34}$$

22. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be both diamonds. Find the probability of the lost card being a diamond.

Ans:

Let E_1 : the event that lost cards is a diamond $\Rightarrow n(E_1) = 13$

E_2 : lost cards is not a diamond $\Rightarrow n(E_2) = 52 - 13 = 39$

And, $n(S) = 52$

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

$\therefore P(E_1) = \frac{13}{52}=\frac{1}{4}$ and $P(E_2) = \frac{39}{52}=\frac{3}{4}$

Let E : the events that two cards drawn from the remaining pack are diamonds,

When one diamond card is lost, there are 12 diamond cards out of 51 cards.

The cards can be drawn out of 12 diamond cards in ${}^{12}C_2$ ways.

Similarly, 2 diamond cards can be drawn out of 51 cards in ${}^{51}C_2$ ways. The probability of getting two cards, when one diamond card is lost, is given by P(E/E_1)

$$\therefore P(E/E_1) = \frac{{}^{12}C_2}{{}^{51}C_2} = \frac{\frac{12\times 11}{1\times 2}}{\frac{51\times 50}{1\times 2}} = \frac{12\times 11}{51\times 50} = \frac{132}{2550}$$

$$\text{and } P(E/E_2) = \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{\frac{13\times 12}{1\times 2}}{\frac{51\times 50}{1\times 2}} = \frac{13\times 12}{51\times 50} = \frac{156}{2550}$$

By using Baye's theorem, we obtain

$$P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$= \frac{\frac{1}{4}\times\frac{132}{2550}}{\frac{1}{4}\times\frac{132}{2550}+\frac{3}{4}\times\frac{156}{2550}} = \frac{132}{132+468} = \frac{132}{600} = \frac{11}{50}$$

23. Two cards are drawn successively with replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces.

Ans:

The number of aces is a random variable. Let it be denoted by X. Clearly, X can take the values 0, 1, or 2.

Now, since the draws are done with replacement, therefore, the two draws form independent experiments.

Therefore, P(X = 0) = P(non-ace and non-ace) = P(non-ace) × P(non-ace) $= \frac{48}{52}\times\frac{48}{52} = \frac{144}{169}$

P(X = 1) = P(ace and non-ace or non-ace and ace)

= P(ace and non-ace) + P(non-ace and ace)

= P(ace). P(non-ace) + P (non-ace) . P(ace) $= \frac{4}{52}\times\frac{48}{52}+\frac{48}{52}\times\frac{4}{52} = \frac{24}{169}$

and P(X = 2) = P (ace and ace) $= \frac{4}{52}\times\frac{4}{52} = \frac{1}{169}$

Thus, the required probability distribution is

X	0	1	2
P(X)	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

24. Find the probability distribution of number of doublets in three throws of a pair of dice.

Ans:

Let X denote the number of doublets. Possible doublets are

(1,1) , (2,2), (3,3), (4,4), (5,5), (6,6)

Clearly, X can take the value 0, 1, 2, or 3.

Probability of getting a doublet $= \frac{6}{36} = \frac{1}{6}$

Probability of not getting a doublet $= 1 - \frac{1}{6} = \frac{5}{6}$

Now P(X = 0) = P (no doublet) $= \frac{5}{6}\times\frac{5}{6}\times\frac{5}{6} = \frac{125}{216}$

P(X = 1) = P (one doublet and two non-doublets) $= \frac{1}{6}\times\frac{5}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

$= 3\left(\frac{1}{6}\times\frac{5^2}{6^2}\right) = \frac{75}{216}$

P(X = 2) = P (two doublets and one non-doublet) $= \frac{1}{6}\times\frac{1}{6}\times\frac{5}{6}+\frac{5}{6}\times\frac{1}{6}\times\frac{1}{6}+\frac{1}{6}\times\frac{5}{6}\times\frac{1}{6}$

$= 3\left(\frac{1}{6^2}\times\frac{5}{6}\right) = \frac{15}{216}$

and P(X = 3) = P (three doublets) $= \frac{1}{6}\times\frac{1}{6}\times\frac{1}{6} = \frac{1}{216}$

Thus, the required probability distribution is

X	0	1	2	3
P(X)	$\frac{125}{216}$	$\frac{75}{216}$	$\frac{15}{216}$	$\frac{1}{216}$

25. Find the probability distribution of (i) number of heads in two tosses of a coin. (ii) number of tails in the simultaneous tosses of three coins. (iii) number of heads in four tosses of a coin.

Ans:

(i) When one coin is tossed twice, the sample space is $S = \{HH, HT, TH, TT\}$.

Let X denotes, the number of heads in any outcome in S,

$X(HH) = 2$, $X(HT) = 1$, $X(TH) = 1$ and $X(TT) = 0$

Therefore, X can take the value of 0, 1 or 2. It is known that

$P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$

$\therefore P(X = 0) = P$ (tail occurs on both tosses) $= P(\{TT\}) = \frac{1}{4}$

$P(X = 1) = P$ (one head and one tail occurs) $= P(\{TH, HT\}) = \frac{2}{4} = \frac{1}{2}$

and $P(X = 2) = P$ (head occurs on both tosses) $= P(\{HH\}) = \frac{1}{4}$

Thus, the required probability distribution is as follows

X	0	1	2
P(X)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(ii) When three coins are tossed thrice, the sample space is $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ which contains eight equally likely sample points.

Let X represent the number of tails. Then, X can take values 0, 1, 2 and 3.

$P(X = 0) = P$ (no tail) $= P(\{HHH\}) = \frac{1}{8}$,

$P(X = 1) = P$ (one tail and two heads show up) $= P(\{HHT, HTH, THH\}) = \frac{3}{8}$,

$P(X = 2) = P$ (two tails and one head show up) $= P(\{HTT, THT, TTH\}) = \frac{3}{8}$

and $P(X = 3) = P$ (three tails show up) $= P(\{TTT\}) = \frac{1}{8}$

Thus, the probability distribution is as follows

X	0	1	2	3
P (X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(iii) When a coin is tossed four times, the sample space is
S = {HHHH, HHHT, HHTH, HTHT, HTTH, HTTT, THHH, HTHH, THHT, THTH, HHTT, TTHH, TTHT, TTTH, THTT, TTTT} which contains 16 equally likely sample points.
Let *X* be the random variable, which represents the number of heads. It can be seen that *X* can take the value of 0, 1, 2, 3 or 4.

$P(X = 0) = P$(no head shows up) $= P\{TTTT\} = \frac{1}{16}$,

$P(X = 1) = P$ (one head and three tails show up) $= P(HTTT, THTT, TTHT, TTTH) = \frac{4}{16} = \frac{1}{4}$,

$P(X = 2) = P$ (two heads and two tails show up) $= P(\{HHTT, HTHT, HTTH, THHT, THTH, TTHH\})$
$= \frac{6}{16} = \frac{3}{8}$,

$P(X = 3) = P$ (three heads and one tail show up) $= P(\{HHHT, HHTH, HTHH, THHH\}) = \frac{4}{16} = \frac{1}{4}$

and $P(X = 4) = P$ (four heads show up) $= P(\{HHHH\}) = \frac{1}{16}$

Thus, the probability distribution is as follows:

X	0	1	2	3	4
P (X)	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

26. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as (i) number greater than 4 (ii) six appears on at least one die

Ans:

When a die is tossed two times, we obtain (6 x 6) = 36 number of sample points.

(i) Let *X* be the random variable which denotes the number greater than 4 in two tosses of a die. So *X* may have values 0, 1 or 2.

Now, $P(X = 0) = P$ (number less than or equal to 4 on both the tosses) $= \frac{4}{6} \times \frac{4}{6} = \frac{16}{36} = \frac{4}{9}$,

$P(X = 1) = P$ (number less than or equal to 4 on first toss and greater than 4 on second toss) + P(number greater than 4 on first toss and less than or equal to 4 on second toss)

$= \frac{4}{6} \times \frac{2}{6} + \frac{4}{6} \times \frac{2}{6} = \frac{8}{36} + \frac{8}{36} = \frac{16}{36} = \frac{4}{9}$

$P(X = 2) = P$ (number greater than 4 on both the tosses) $= \frac{2}{6} \times \frac{2}{6} = \frac{4}{36} = \frac{1}{9}$

Probability distribution of *X*, *i.e.*, number of successes is

X	0	1	2
P(X)	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(ii) Let *X* be the random variable which denotes the number of six appears on atleast one die. So, *X* may have values 0 or 1.

$P(X = 0) = P$ (six does not appear on any of the die) $= \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$P(X = 1) = P$ (six appears on atleast one of the die) = $\frac{11}{36}$

Thus, the required probability distribution is as follows

X	0	1
P(X)	$\frac{25}{36}$	$\frac{11}{36}$

27. From a lot of 30 bulbs which include 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.

Ans:

It is given that out of 30 bulbs, 6 are defective.

Number of non-defective bulbs = 30 – 6 = 24

4 bulbs are drawn from the lot with replacement.

Let $p = P$(obtaining a defective bulb when a bulb is drawn) = $\frac{6}{30} = \frac{1}{5}$

and $q = P$(obtaining a non-defective bulb when a bulb is drawn) = $\frac{24}{30} = \frac{4}{5}$

Using Binomial distribution, we have

$P(X = 0) = P$ (no defective bulb in the sample) = ${}^4C_0 p^0 q^4 = \left(\frac{4}{5}\right)^4 = \frac{256}{625}$

$P(X = 1) = P$ (one defective bulb in the sample) = ${}^4C_1 p^1 q^3 = 4\left(\frac{1}{5}\right)^1\left(\frac{4}{5}\right)^3 = \frac{256}{625}$

$P(X = 2) = P$ (two defective and two non-defective bulbs are drawn) = ${}^4C_2 p^2 q^2 = 6\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2 = \frac{96}{625}$

$P(X = 3) = P$ (three defective and one non-defective bulbs are drawn) =

${}^4C_3 p^3 q^1 = 4\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^1 = \frac{16}{625}$

$P(X = 4) = P$ (four defective bulbs are drawn) = ${}^4C_4 p^4 q^0 = \left(\frac{1}{5}\right)^4 = \frac{1}{625}$

Therefore, the required probability distribution is as follows.

X	0	1	2	3	4
P (X)	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

28. A coin is biased so that the head is 3 times as likely to occur as tail. If the coin is tossed twice, find the probability distribution of number of tails.

Ans:

Let X denotes the random variable which denotes the number of tails when a biased coin is tossed twice.

So, X may have value 0, 1 or 2.

Since, the coin is biased in which head is 3 times as likely to occur as a tail.

$\therefore P\{H\} = \frac{3}{4}$ and $P\{T\} = \frac{1}{4}$

$P(X = 1) = P\{HH\} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

$P(X = 1) = P$ (one tail and one head) = $P\{HT,TH\} = P\{HT\} + P\{TH\} + P\{H\}P\{T\} + P\{T\}P\{H\}$

$= \frac{3}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{3}{4}=\frac{3}{16}+\frac{3}{16}=\frac{6}{16}=\frac{3}{8}$

$P(X = 2) = P$ (two tails) $= P\{TT\} = P\{T\}\, P\{T\} = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

Therefore, the required probability distribution is as follows

X	0	1	2
P(X)	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

29. A random variable X has the following probability distribution:

X	0	1	2	3	4	5	6	7
P(X)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Determine
(i) *k* (ii) P(X < 3)
(iii) P(X > 6) (iv) P(0 < X < 3)
Ans:

(i) It is known that the sum of a probability distribution of random variable is one *i.e.*, $\sum P(X) = 1$, therefore

$P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$

$\Rightarrow 0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$\Rightarrow 10k^2+9k-1=0$

$\Rightarrow (k+1)(10k-1)=0$

$\Rightarrow k=-1 \;\; or \;\; k=\frac{1}{10}$

k = – 1 is not possible as the probability of an event is never negative.

$\therefore k=\frac{1}{10}$

(ii) $P(X < 3) = P(0) + P(1) + P(2) = 0+k+2k=3k=\frac{3}{10}$

(iii) $P(X > 6) = P(7) = 7k^2+k=\frac{7}{100}+\frac{1}{10}=\frac{7+10}{100}=\frac{17}{100}$

(iv) $P(0 < X < 3) = P(1) + P(2) = k+2k=3k=\frac{3}{10}$

30. The random variable X has a probability distribution P(X) of the following form, where *k* is some number :

$$P(X)=\begin{cases} k \;\; if \;\; x=0 \\ 2k \;\; if \;\; x=1 \\ 3k, \;\; if \;\; x=2 \\ 0, otherwise \end{cases}$$

(a) Determine the value of *k*.
(b) Find P (X < 2), P (X ≤ 2), P(X ≥ 2).
Ans:

Given distribution of *X* is

X	0	1	2	otherwise
P(X)	k	$2k$	$3k$	0

(a) Since, $\sum P(X) = 1$, therefore $P(0) + P(1) + P(2) + P$ (otherwise) = 1

$\Rightarrow k + 2k + 3k + 0 = 1 \Rightarrow 6k = 1 \Rightarrow k = \frac{1}{6}$

(b) $P(X = 2) = P(0) + P(1) = k + 2k = 3k = 3 \times \frac{1}{6} = \frac{1}{2}$

$P(X \leq 2) = P(0) + P(1) + P(2) = k + 2k + 3k = 6k = 6 \times \frac{1}{6} = 1$

and $P(X \geq 2) = P(2) + P(\text{otherwise}) = 3k + 0 = 3k = 3 \times \frac{1}{6} = \frac{1}{2}$

31. If a fair coin is tossed 10 times, find the probability of (i) exactly six heads (ii) at least six heads (iii) at most six heads.

Ans:

The repeated tosses of a coin are Bernoulli trials. Let X denote the number of heads in an experiment of 10 trials.

Clearly, X has the binomial distribution with $n = 10$ and $p = \frac{1}{2}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2, ..., n$

Here $n = 10$,

$p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore $P(X = x) = {}^{10}C_x \left(\frac{1}{2}\right)^{10-x} \left(\frac{1}{2}\right)^x = {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

Now (i) $P(X = 6) = {}^{10}C_6 \left(\frac{1}{2}\right)^{10} = \frac{10!}{6! \times 4!} \times \frac{1}{2^{10}} = \frac{105}{512}$

(ii) P(at least six heads) = $P(X \geq 6)$

$= P(X = 6) + P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10)$

$= {}^{10}C_6 \left(\frac{1}{2}\right)^{10} + {}^{10}C_7 \left(\frac{1}{2}\right)^{10} + {}^{10}C_8 \left(\frac{1}{2}\right)^{10} + {}^{10}C_9 \left(\frac{1}{2}\right)^{10} + {}^{10}C_{10} \left(\frac{1}{2}\right)^{10}$

$= \left[\left(\frac{10!}{6! \times 4!}\right) + \left(\frac{10!}{7! \times 3!}\right) + \left(\frac{10!}{8! \times 2!}\right) + \left(\frac{10!}{9! \times 1!}\right) + \left(\frac{10!}{10!}\right)\right] \frac{1}{2^{10}} = \frac{193}{512}$

(iii) P(at most six heads) = $P(X \leq 6)$

$= P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$

$= {}^{10}C_0 \left(\frac{1}{2}\right)^{10} + {}^{10}C_1 \left(\frac{1}{2}\right)^{10} + {}^{10}C_2 \left(\frac{1}{2}\right)^{10} + {}^{10}C_3 \left(\frac{1}{2}\right)^{10} + {}^{10}C_4 \left(\frac{1}{2}\right)^{10} + {}^{10}C_5 \left(\frac{1}{2}\right)^{10} + {}^{10}C_6 \left(\frac{1}{2}\right)^{10}$

$= \frac{848}{1024} = \frac{53}{64}$

32. A die is thrown 6 times. If 'getting an odd number' is a success, what is the probability of (i) 5 successes? (ii) at least 5 successes? (iii) at most 5 successes?

Ans:

The repeated tosses of a die are Bernoulli trials. Let X denote the number of successes of getting odd numbers in an experiment of 6 trials.

$p = P$ (success) = P (getting an odd number in a single throwof a die)

$\therefore p = \frac{3}{6} = \frac{1}{2}$ and $q = P$ (failure) $= 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^{n-r}\, q^r$, where $r = 0, 1, 2,...,n$

$$P(X=r) = {}^6C_r \left(\frac{1}{2}\right)^{6-r} \left(\frac{1}{2}\right)^r = {}^6C_r \left(\frac{1}{2}\right)^6$$

(i) P (5 successes) = ${}^6C_5 \left(\frac{1}{2}\right)^6 = 6 \times \frac{1}{2^6} = \frac{6}{64} = \frac{3}{32}$

(ii) P (atleast 5 successes) = P (5 successes) + P (6 successes) = ${}^6C_5 \left(\frac{1}{2}\right)^6 + {}^6C_6 \left(\frac{1}{2}\right)^6 = \frac{6}{64} + \frac{1}{64} = \frac{7}{64}$

(iii) P (atmost 5 successes) = 1 – P (6 successes) = $1 - {}^6C_6 \left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}$

33. There are 5% defective items in a large bulk of items. What is the probability that a sample of 10 items will include not more than one defective item?

Ans:

Let X denote the number of defective items in a sample of 10 items drawn successively. Since, the drawing is done with replacement, the trials are Bernoulli trials.

$p = P$ (success) = 5% = $\frac{5}{100} = \frac{1}{20}$

and $q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

X has a binomial distribution with $n = 10$ and $p = \frac{1}{20}$ and $q = \frac{19}{20}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, where $r = 0, 1, 2,...,n$

$$P(X=r) = {}^{10}C_r \left(\frac{1}{20}\right)^r \left(\frac{19}{20}\right)^{10-r}$$

Required probability = P (not more than one defective item)

$= P(0) + P(1) = {}^{10}C_0 p^0 q^{10} + {}^{10}C_1 p^1 q^9 = q^{10} + 10pq^9$

$$= q^9(q + 10p) = \left(\frac{19}{20}\right)^9 \left(\frac{19}{20} + 10 \times \frac{1}{20}\right) = \frac{29}{20}\left(\frac{19}{20}\right)^9$$

34. Five cards are drawn successively with replacement from a well-shuffled deck of 52 cards. What is the probability that (i) all the five cards are spades? (ii) only 3 cards are spades? (iii) none is a spade?

Ans:

Let X represent the number of spade cards among the five cards drawn. Since, the drawing card is with replacement, the trials are Bernoulli trials.

In a well-shuffled deck of 52 cards, there are 13 spade cards.

$p = P$ (success) = P (a spade card is drawn) = $\frac{13}{52} = \frac{1}{4}$

and $q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$

X has a binomial distribution with $n = 5$, $p = \frac{1}{4}$ and $q = \frac{3}{4}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, where $r = 0, 1, 2,...,n$

$$P(X=r) = {}^5C_r \left(\frac{1}{4}\right)^r \left(\frac{3}{4}\right)^{5-r}$$

(i) P (all the five cards are spades) = $P(X = 5) = {}^5C_5 p^5 q^0 = p^5 = \left(\frac{1}{4}\right)^5 = \frac{1}{1024}$

(ii) P (only three cards are spades) = $P(X = 3) = {}^5C_3 p^3 q^2 = 10\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{1024} = \frac{45}{512}$

(iii) P (none is a spade) = $P(X = 0) = {}^5C_0 p^0 q^5 = \left(\frac{3}{4}\right)^5 = \frac{243}{1024}$

35. It is known that 10% of certain articles manufactured are defective. What is the probability that in a random sample of 12 such articles, 9 are defective?

Ans:

The repeated selections of articles in a random sample space are Bernoulli trials. Let X denotes the number of times of selecting defective articles in a random sample space of 12 articles.

Here, $p = 10\% = \frac{10}{100} = \frac{1}{10}$

and $q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

Clearly, X has a binomial distribution with $n = 12$, $p = \frac{1}{10}$ and $q = \frac{9}{10}$

Therefore, by Binomial distribution

$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$, where $r = 0, 1, 2, ..., n$

$$P(X = r) = {}^{12}C_r \left(\frac{1}{10}\right)^r \left(\frac{9}{10}\right)^{12-r}$$

Required probability = P (9 items are defective) = $P(X = 9) = {}^{12}C_3 p^9 q^3 = {}^{12}C_3 \left(\frac{1}{10}\right)^9 \left(\frac{9}{10}\right)^3$

$$= \frac{12\times11\times10}{1\times2\times3}.\frac{9^3}{10^{12}} = \frac{22\times9^3}{10^{11}}$$

36. The probability of a shooter hitting a target is $\frac{3}{4}$. How many minimum number of times must he/she fire so that the probability of hitting the target at least once is more than 0.99?

Ans:

Let the shooter fire n times. Obviously, n fires are n Bernoulli trials. In each trial, p = probability of hitting the target = $\frac{3}{4}$ and q = probability of not hitting the target = $\frac{1}{4}$.

Therefore P(X = x) = ${}^nC_x q^{n-x} p^x$, $x = 0, 1, 2, ..., n$

$$= {}^nC_x \left(\frac{1}{4}\right)^{n-x}\left(\frac{3}{4}\right)^x = {}^nC_x \frac{3^x}{4^n}$$

Now, given that,

P(hitting the target at least once) > 0.99

i.e. P($x \geq 1$) > 0.99

Therefore, 1 – P (x = 0) > 0.99

$$\Rightarrow 1 - {}^nC_0 \frac{1}{4^n} > 0.99 \Rightarrow {}^nC_0 \frac{1}{4^n} < 0.01$$

$$\Rightarrow \frac{1}{4^n} < 0.01 \Rightarrow 4^n > \frac{1}{0.01} = 100$$

The minimum value of n to satisfy the inequality (1) is 4.

Thus, the shooter must fire 4 times.

37. A and B throw a die alternatively till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if A starts first.

Ans:

Let S denote the success (getting a '6') and F denote the failure (not getting a '6').

Thus, P(S) = $\frac{1}{6}$, P(F) = $\frac{5}{6}$

P(A wins in the first throw) = P(S) = $\frac{1}{6}$

A gets the third throw, when the first throw by A and second throw by B result into failures.

Therefore, P(A wins in the 3rd throw) = P(FFS) = P(F)P(F)P(S) = $\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}=\left(\frac{5}{6}\right)^2\times\frac{1}{6}$

P(A wins in the 5th throw) = P (FFFFS) = $\left(\frac{5}{6}\right)^4\times\frac{1}{6}$ and so on.

Hence, P(A wins) = $\frac{1}{6}+\left(\frac{5}{6}\right)^2\times\frac{1}{6}+\left(\frac{5}{6}\right)^4\times\frac{1}{6}+.....=\frac{\frac{1}{6}}{1-\frac{25}{36}}=\frac{6}{11}$

P(B wins) = 1 – P (A wins) = $1-\frac{6}{11}=\frac{5}{11}$

38. Suppose that 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

Ans:

Let E_1 : the event that selected person is a male and E_2 : the event that selected person is a female.

E_1 and E_2 are mutually exclusive and exhaustive events. Moreover,

$P(E_1) = P(E_2) = \frac{1}{2}$

Let E : the event that selected person is grey haired.

Then P(E/E$_1$) = $\frac{5}{100}=\frac{1}{20}$ and P(E/E$_2$) = $\frac{0.25}{100}=\frac{1}{400}$

By using Baye's theorem, we obtain

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}=\frac{\frac{1}{2}\times\frac{1}{20}}{\frac{1}{2}\times\frac{1}{20}+\frac{1}{2}\times\frac{1}{400}}=\frac{\frac{1}{20}}{\frac{1}{20}+\frac{1}{400}}=\frac{\frac{1}{20}}{\frac{21}{400}}=\frac{20}{21}$$

39. Suppose that 90% of people are right-handed. What is the probability that at most 6 of a random sample of 10 people are right-handed?

Ans:

A person can be either right handed or left handed. It is given that 90% of the people are right handed.

Therefore p = $\frac{90}{100}=\frac{9}{10}$

and $q = 1-p = 1-\frac{9}{10}=\frac{1}{10}$, n = 10

Clearly, X has a binomial distribution with n = 10, p = $\frac{9}{10}$ and q = $\frac{1}{10}$

Therefore P(X = x) = ${}^nC_x q^{n-x} p^x$, x = 0, 1, 2,...,n

$$P(X=r)={}^{10}C_r\left(\frac{9}{10}\right)^r\left(\frac{1}{10}\right)^{10-r}$$

Required probability, $P(X \le 6)$

$= 1 - P$ {more than 6 are right handed $(7 \le X \le 10)$]

$$= 1-\sum_{r=7}^{10}{}^{10}C_r\left(\frac{9}{10}\right)^r\left(\frac{1}{10}\right)^{10-r}$$

40. An urn contains 25 balls of which 10 balls bear a mark 'X' and the remaining 15 bear a mark 'Y'. A ball is drawn at random from the urn, its mark is noted down and it is replaced. If 6 balls are drawn in this way, find the probability that (i) all will bear 'X' mark. (ii) not more than 2 will bear 'Y' mark. (iii) at least one ball will bear 'Y' mark. (iv) the number of balls with 'X' mark and 'Y' mark will be equal.

Ans:

It is case of Bernoulli trials with $n = 6$. Let success be defined as drawing a ball marked X.

$p = P$ (a success in a single draw) = $\frac{10}{25}=\frac{2}{5}$

and $q = 1 - p = 1-\frac{2}{5}=\frac{3}{5}$

Clearly, Z has a binomial distribution with $n = 6$, $p = \frac{2}{5}$ and $q = \frac{3}{5}$

Therefore P(Z = r) = ${}^nC_x q^{n-x}p^x$, $x = 0, 1, 2,...,n$

$$P(Z=r)={}^6C_r\left(\frac{2}{5}\right)^r\left(\frac{3}{5}\right)^{6-r}$$

(i) P (all bear mark X) = P (6 success) = ${}^6C_6 p^6 q^0 = \left(\frac{2}{5}\right)^6$

(ii) P (not more than 2 bear mark Y)

= P (not less than 4 bear mark X) = P (atleast 4 successes)

= $P(4) + P(5) + P(6)$ = ${}^6C_4 p^4 q^2 + {}^6C_5 p^5 q^1 + {}^6C_6 p^6 q^0$

$$=15\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^2+6\left(\frac{2}{5}\right)^5\left(\frac{3}{5}\right)^1+\left(\frac{2}{5}\right)^6$$

$$=\left(\frac{2}{5}\right)^4\left[\frac{27}{5}+\frac{36}{25}+\frac{4}{25}\right]=\left(\frac{2}{5}\right)^4\left[\frac{135+36+4}{25}\right]$$

$$=\left(\frac{2}{5}\right)^4\times\frac{175}{25}=7\left(\frac{2}{5}\right)^4$$

(iii) P (atleast one ball will bear mark Y) = P (atleast one failure)

= P (atmost five successes) = $1-P(6)=1-\left(\frac{2}{5}\right)^6$

(iv) Required probability = P (three successes and three failures)

$$=P(3)={}^6C_3p^3q^3=20\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^3=20\times\frac{8}{125}\times\frac{27}{125}=\frac{864}{3125}$$

41. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is $\frac{5}{6}$. What is the probability that he will knock down fewer than 2 hurdles?

Ans:

It is a case of Bernoulli trials, where success is crossing a hurdle successfully without knocking it down and $n = 10$.

$p = P\text{ (success)} = \frac{5}{6}$

$q = 1 - p = 1 - \frac{5}{6} = \frac{1}{6}$

Let X be the random variable that represents the number of times the player will knock down the hurdle.

Clearly, X has a binomial distribution with $n = 10$ and $p = \frac{5}{6}$

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2, ..., n$

$$P(X = r) = {}^{10}C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{10-r}$$

P (player knocking down less than 2 hurdles) = $P(x < 2)$

$= P(0) + P(1) = {}^{10}C_0 p^0 q^{10} + {}^{10}C_1 p^1 q^9$

$$= \left(\frac{5}{6}\right)^{10} + 10\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^9 = \left(\frac{5}{6}\right)^9\left[\frac{5}{6} + \frac{10}{6}\right] = \left(\frac{5}{6}\right)^9 \times \frac{15}{6} = \frac{5}{2} \times \left(\frac{5}{6}\right)^9 = \frac{5^{10}}{2 \times 6^9}$$

42. A die is thrown again and again until three sixes are obtained. Find the probability of obtaining the third six in the sixth throw of the die.

Ans:

When a die is rolled once, probability of obtaining a six is $\frac{1}{6}$ and that of not obtaining a six is

$1 - \frac{1}{6} = \frac{5}{6}$

Let $p = \frac{1}{6}$ and $q = \frac{5}{6}$

Clearly, X has a binomial distribution.

Therefore $P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2, ..., n$

$$P(X = r) = {}^5C_r \left(\frac{1}{6}\right)^r \left(\frac{5}{6}\right)^{5-r}$$

$\therefore$ P (obtaining third six in the sixth throw)

= P(obtaining two sixes in first five throws and a six in the sixth throw)

= P (obtaining two sixes in first five throws) $\times \frac{1}{6}$

$$= {}^5C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 \times \frac{1}{6} = 10 \times \frac{1}{36} \times \frac{125}{216} = \frac{625}{23328}$$

43. Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduce the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga?

Ans:

Let E_1 : the event that the patient follows meditation and yoga and E_2 : the event that the patient uses drug.

Therefore E_1 and E_2 are mutually exclusive events and

$P(E_1) = P(E_2) = \frac{1}{2}$

$\therefore\ P(E/E_1) = \frac{40}{100}\left(1-\frac{30}{100}\right)=\frac{28}{100}$

$P(E/E_2) = \frac{40}{100}\left(1-\frac{25}{100}\right)=\frac{30}{100}$

Let E : the event that the selected patient suffers a heart attack

By using Baye's theorem, we obtain

P (patient who suffers heart attack follows meditation and yoga) =

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{28}{100}}{\frac{1}{2}\times\frac{30}{100}+\frac{1}{2}\times\frac{28}{100}}=\frac{\frac{14}{100}}{\frac{15}{100}+\frac{14}{100}}=\frac{14}{15+14}=\frac{14}{29}$$

44. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. One ball is transferred from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred ball is black.

Ans:

Let E_1 : red ball is transferred from bag I to bag II

and E_2 : black ball is transferred from bag I to bag II

$\therefore$ E_1 and E_2 are mutually exclusive and exhaustive events.

$P(E_1) = \frac{3}{3+4}=\frac{3}{7}$

$P(E_2) = \frac{4}{3+4}=\frac{4}{7}$

$\therefore\ P(E/E_1) = \frac{4+1}{(4+1)+5}=\frac{5}{10}=\frac{1}{2}$

$P(E/E_2) = \frac{4}{4+(5+1)}=\frac{4}{10}=\frac{2}{5}$

Let E be the event that the ball drawn is red. When a red ball is transferred from bag I to II.

When a black ball is transferred from bag I to II.

By using Baye's theorem, we obtain

$$P(E_2/E)=\frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$=\frac{\frac{4}{7}\times\frac{2}{5}}{\frac{3}{7}\times\frac{1}{2}+\frac{4}{7}\times\frac{2}{5}}=\frac{\frac{8}{35}}{\frac{3}{14}+\frac{8}{35}}=\frac{\frac{8}{35}}{\frac{305+112}{14\times 35}}=\frac{8\times 14}{217}=\frac{16}{31}$$

CHAPTER – 13: PROBABILITY

MARKS WEIGHTAGE – 08 marks

Previous Years Board Exam (Important Questions & Answers)

1. **There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?**

Ans:

Let E_1 ,E_2 ,E_3 and A be events defined as

E_1 = selection of two-headed coin

E_2 = selection of biased coin that comes up head 75% of the times.

E_3 = selection of biased coin that comes up tail 40% of the times.

A = getting head.

$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

$P(A/E_1) = 1$, $P(A/E_2) = \frac{3}{4}$ and $P(A/E_3) = \frac{3}{5}$

By using Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{3}\times\frac{3}{5}} = \frac{\frac{1}{3}}{\frac{1}{3}+\frac{1}{4}+\frac{1}{5}} = \frac{\frac{1}{3}}{\frac{20+15+12}{20}} = \frac{1}{3}\times\frac{60}{47} = \frac{20}{47}$$

2. **Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.**

Ans:

First six positive integers are 1, 2, 3, 4, 5, 6

If two numbers are selected at random from above six numbers then sample space S is given by

$$S = \{(1,2)\ (1,3), (1,4), (1,5), (1,6), (2,1), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,4), (3,5), (3,6), (4,1), (4,2), (4,3), (4,5), (4,6), (5,1), (5,2), (5,3), (5,4), (5,6), (6,1), (6,2), (6,3)\ (6,4)\ (6,5)\}$$

$n(s) = 30$.

Here, X is random variable, which may have value 2, 3, 4, 5 or 6.

Therefore, required probability distribution is given as

$P(X = 2)$ = Probability of event getting (1, 2), (2, 1) = $\frac{2}{30}$

$P(X = 3)$ = Probability of event getting (1, 3), (2, 3), (3, 1), (3, 2) = $\frac{4}{30}$

$P(X = 4)$ = Probability of event getting (1, 4), (2, 4), (3, 4), (4, 1), (4, 2), (4, 3) = $\frac{6}{30}$

$P(X = 5)$ = Probability of event getting (1, 5), (2, 5), (3, 5), (4, 5), (5, 1), (5, 2), (5, 3), (5, 4) = $\frac{8}{30}$

$P(X = 6)$ = Probability of event getting (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5) = $\frac{10}{30}$

It is represented in tabular form as

X	2	3	4	5	6
$P(X)$	$\frac{2}{30}$	$\frac{4}{30}$	$\frac{6}{30}$	$\frac{8}{30}$	$\frac{10}{30}$

Required mean = $E(x)$ = $\sum p_i x_i = 2\times\frac{2}{30}+3\times\frac{4}{30}+4\times\frac{6}{30}+5\times\frac{8}{30}+6\times\frac{10}{30}$

$= \frac{4+12+24+40+60}{30} = \frac{140}{30} = \frac{14}{3} = 4\frac{2}{3}$

3. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes.

Ans:

An experiment succeeds thrice as often as it fails.

$\Rightarrow p = P$(getting success) =

and $q = P$(getting failure) =

Here, number of trials = $n = 5$

By binomial distribution, we have

$P(X = x) = {}^nC_x q^{n-x} p^x$, $x = 0, 1, 2,...,n$

$P(X = r) = {}^5C_r \left(\frac{3}{4}\right)^r \left(\frac{1}{4}\right)^{5-r}$

Now , P(getting at least 3 success) = $P(X = 3) + P(X = 4) + P(X = 5)$

$= {}^5C_3\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + {}^5C_4\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + {}^5C_5\left(\frac{3}{4}\right)^5$

$= \left(\frac{3}{4}\right)^3\left[10\times\frac{1}{16}+15\times\frac{1}{16}+9\times\frac{1}{16}\right] = \frac{27}{64}\times\frac{34}{16} = \frac{459}{512}$

4. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade.

Ans:

Let E_1, E_2 , E_3 ,E_4 and A be event defined as

E_1 = the lost card is a spade card.

E_2 = the lost card is a heart card.

E_3 = the lost card is a club card.

E_4 = the lost card is diamond card.

and A = Drawing three spade cards from the remaining cards.

$P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$

$P(A/E_1) = \frac{{}^{12}C_3}{{}^{51}C_3} = \frac{220}{20825}$, $P(A/E_2) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}$

$P(A/E_3) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}$ $P(A/E_4) = \frac{{}^{13}C_3}{{}^{51}C_3} = \frac{286}{20825}$

By using Baye's theorem, we have

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)+P(E_4)P(A/E_4)}$$

$$= \frac{\frac{1}{4}\times\frac{220}{20825}}{\frac{1}{4}\times\frac{220}{20825}+\frac{1}{4}\times\frac{286}{20825}+\frac{1}{4}\times\frac{286}{20825}+\frac{1}{4}\times\frac{286}{20825}} = \frac{220}{220+286+286+286} = \frac{220}{1078} = \frac{10}{49}$$

5. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls given that (*i*) the youngest is a girl? (*ii*) atleast one is a girl?

Ans:

A family has 2 children, then Sample space = $S = \{BB, BG, GB, GG\}$, where B stands for Boy and G for Girl.

(*i*) Let A and B be two event such that

A = Both are girls = $\{GG\}$

B = the youngest is a girl = $\{BG, GG\}$

Now, $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}$

(*ii*) Let C be event such that

C = at least one is a girl = $\{BG, GB, GG\}$

Now, $P(A/C) = \frac{P(A \cap C)}{P(C)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$

6. Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is actually a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society?

Ans:

Let E_1, E_2 and E be three events such that

E_1 = six occurs

E_2 = six does not occurs

E = man reports that six occurs in the throwing of the dice.

Now $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{5}{6}$

$P(E/E_1) = \frac{4}{5}$, $P(E/E_2) = 1-\frac{4}{5} = \frac{1}{5}$

By using Baye's theorem, we obtain

$$P(E_1/E) = \frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)}$$

$$= \frac{\frac{1}{6}\times\frac{4}{5}}{\frac{1}{6}\times\frac{4}{5}+\frac{5}{6}\times\frac{1}{5}} = \frac{4}{4+5} = \frac{4}{9}$$

7. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively. Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time.

Ans:

Let E_1 and E_2 be two events such that

E_1 = A coming to the school in time.
E_2 = B coming to the school in time.

Here $P(E_1) = \frac{3}{7}$ and $P(E_2) = \frac{5}{7}$

$\Rightarrow P(\overline{E_1}) = \frac{4}{7}, P(\overline{E_2}) = \frac{2}{7}$

P (only one of them coming to the school in time) = $P(E_1)P(\overline{E_2}) + P(\overline{E_1})P(E_2)$

$= \frac{3}{7} \times \frac{2}{7} + \frac{5}{7} \times \frac{4}{7} = \frac{6+20}{49} = \frac{26}{49}$

Coming to school in time i.e., punctuality is a part of discipline which is very essential for development of an individual.

8. In a hockey match, both teams *A* and *B* scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find their respective probabilities of winning the match and state whether the decision of the referee was fair or not.

Ans:

Let E_1, E_2 be two events such that
E_1 = the captain of team '*A*' gets a six.
E_2 = the captain of team '*B*' gets a six.

Here $P(E_1) = \frac{1}{6}$, $P(E_2) = \frac{1}{6}$

$P(E_1') = 1 - \frac{1}{6} = \frac{5}{6},\ P(E_2') = 1 - \frac{1}{6} = \frac{5}{6}$

Now P (winning the match by team *A*) = $\frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \ldots\ldots.$

$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} + \ldots..$

$= \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$

$\therefore P$ (winning the match by team B) = $1 - \frac{6}{11} = \frac{5}{11}$

The decision of refree was not fair because the probability of winning match is more for that team who start to throw dice.

9. A speaks truth in 60% of the cases, while B in 90% of the cases. In what percent of cases are they likely to contradict each other in stating the same fact? In the cases of contradiction do you think, the statement of B will carry more weight as he speaks truth in more number of cases than A?

Ans:

Let E_1 be the event that A speaks truth and E_2 be the event that B speaks truth. Then E and f are independent events such that

$P(E_1) = \frac{60}{100} = \frac{3}{5}, P(E_1) = \frac{90}{100} = \frac{9}{10}$

$\Rightarrow P(\overline{E_1}) = \frac{2}{5}, P(\overline{E_2}) = \frac{1}{10}$

P (A and B contradict each other) = $P(E_1)P(\overline{E_2}) + P(\overline{E_1})P(E_2)$

$= \frac{3}{5} \times \frac{1}{10} + \frac{2}{5} \times \frac{9}{10} = \frac{3+18}{50} = \frac{21}{50}$

Yes, the statement of *B* will carry more weight as the probability of B to speak truth is more than that of *A*.

10. Assume that the chances of a patient having a heart attack is 40%. Assuming that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chance by 25%. At a time a patient can choose any one of the two options with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. Find the probability that the patient followed a course of meditation and yoga. Interpret the result and state which of the above stated methods is more beneficial for the patient.

Ans:

Let E_1, E_2 , A be events defined as

E_1 = treatment of heart attack with Yoga and meditation

E_2 = treatment of heart attack with certain drugs.

A = Person getting heart attack.

$P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{2}$

$P(A/E_1) = 40\% - \left(40 \times \frac{30}{100}\right)\% = 40\% - 12\% = 28\% = \frac{28}{100}$

$P(A/E_2) = 40\% - \left(40 \times \frac{25}{100}\right)\% = 40\% - 10\% = 30\% = \frac{30}{100}$

By using Baye's theorem, we have

$$P(E_1 / A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2} \times \frac{28}{100}}{\frac{1}{2} \times \frac{28}{100} + \frac{1}{2} \times \frac{30}{100}} = \frac{28}{28+30} = \frac{28}{58} = \frac{14}{29}$$

The problem emphasises the importance of Yoga and meditation.

Treatment with Yoga and meditation is more beneficial for the heart patient.

11. In a certain college, 4% of boys and 1% of girls are taller than 1.75 metres. Furthermore, 60% of the students in the college are girls. A student is selected at random from the college and is found to be taller than 1.75 metres. Find the probability that the selected student is a girl.

Ans:

Let E_1, E_2, A be events such that

E_1 = student selected is girl

E_2 = student selected is Boy

A = student selected is taller than 1.75 metres.

$P(E_1) = \frac{60}{100} = \frac{3}{5}$, $P(E_2) = \frac{40}{100} = \frac{2}{5}$

$P(A/E_1) = \frac{1}{100}$, $P(A/E_2) = \frac{4}{100}$

By using Baye's theorem, we have

$$P(E_1 / A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{3}{5}\times\frac{1}{100}}{\frac{3}{5}\times\frac{1}{100}+\frac{2}{5}\times\frac{4}{100}} = \frac{3}{3+8} = \frac{3}{11}$$

12. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin 3 times and notes the number of heads. If she gets 1,2,3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1,2,3, or 4 with the die?

Ans:

Consider the following events:

E_1 = Getting 5 or 6 in a single throw of a die.

E_2 = Getting 1, 2, 3, or 4 in a single throw of a die.

A = Getting exactly one head.

$P(E_1) = \frac{2}{6} = \frac{1}{3}$, $P(E_2) = \frac{4}{6} = \frac{2}{3}$

P(A/E$_1$) = Probability of getting exactly one head when a coin is tossed three times

$= {}^3C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 = 3\times\frac{1}{2}\times\frac{1}{4} = \frac{3}{8}$

P(A/E$_2$) = Probability of getting exactly one head when a coin is tossed once only = $\frac{1}{2}$

By using Baye's theorem, we have

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$= \frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}} = \frac{1}{3}\times\frac{24}{11} = \frac{8}{11}$$

13. How many times must a man toss a fair coin, so that the probability of having at least one head is more than 80%?

Ans:

Let no. of times of tossing a coin be n.

Here, Probability of getting a head in a chance = $p = \frac{1}{2}$

Probability of getting no head in a chance = $q = 1-\frac{1}{2} = \frac{1}{2}$

Now, P (having at least one head) = $P(X \geq 1)$

$= 1 - P(X = 0) = 1 - {}^nC_0 p^0 q^{n-0} = 1 - 1.1.\left(\frac{1}{2}\right)^n = 1-\left(\frac{1}{2}\right)^n$

From question

$1-\left(\frac{1}{2}\right)^n > \frac{80}{100}$

$\Rightarrow 1-\left(\frac{1}{2}\right)^n > \frac{8}{10} \Rightarrow 1-\frac{8}{10} > \frac{1}{2^n}$

$\Rightarrow \frac{1}{5} > \frac{1}{2^n} \Rightarrow 2^n > 5 \Rightarrow n \geq 3$

A man must have to toss a fair coin 3 times.

14. Of the students in a college, it is known that 60% reside in hostel and 40% day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain 'A' grade and 20% of day scholars attain 'A' grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an 'A' grade, what is the probability that the student is a hosteler?

Ans:

Let E_1 , E_2 and A be events such that

E_1 = student is a hosteler

E_2 = student is a day scholar

A = getting A grade.

$P(E_1) = \frac{60}{100} = \frac{6}{10}$, $P(E_2) = \frac{40}{100} = \frac{4}{10}$

$P(A/E_1) = \frac{30}{100} = \frac{3}{10}$, $P(A/E_2) = \frac{20}{100} = \frac{2}{10}$

By using Baye's theorem, we have

$$P(E_1 / A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{6}{10} \times \frac{3}{10}}{\frac{6}{10} \times \frac{3}{10} + \frac{4}{10} \times \frac{2}{10}} = \frac{18}{18+8} = \frac{18}{26} = \frac{9}{13}$$

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. Let A and B be two events. If P (A) = 0.2, P (B) = 0.4, P (A$\cup$B) = 0.6, then P (A | B) is equal to
(a) 0.8 (b) 0.5 (c) 0.3 (d) 0

2. Let A and B be two events such that P (A) = 0.6, P (B) = 0.2, and P (A | B) = 0.5. Then P (A' | B') equals
(a) $\frac{1}{10}$ (b) $\frac{1}{30}$ (c) $\frac{3}{8}$ (d) $\frac{6}{7}$

3. If A and B are independent events such that 0 < P (A) < 1 and 0 < P (B) < 1, then which of the following is not correct?
(a) A and B are mutually exclusive (b) A and B' are independent
(c) A' and B are independent (d) A' and B' are independent

4. Let X be a discrete random variable. The probability distribution of X is given below:

X	30	10	–10
P(X)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{2}$

Then E (X) is equal to
(a) 6 (b) 4 (c) 3 (d) – 5

5. If A and B are two events such that P(A) ≠ 0 and P(B | A) = 1, then
(a) $A \subset B$ (b) $B \subset A$ (c) $B = \varphi$ (d) $A = \varphi$

6. If P(A|B) > P(A), then which of the following is correct :
(a) P(B|A) < P(B) (b) P(A ∩ B) < P(A) . P(B)
(c) P(B|A) > P(B) (d) P(B|A) = P(B)

7. If A and B are any two events such that P(A) + P(B) – P(A and B) = P(A), then
(a) P(B|A) = 1 (b) P(A|B) = 1
(c) P(B|A) = 0 (d) P(A|B) = 0

8. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective is
(a) $\frac{1}{10}$ (b) $\left(\frac{1}{2}\right)^5$ (c) $\left(\frac{9}{10}\right)^5$ (d) $\frac{9}{10}$

9. The probability that a student is not a swimmer is $\frac{1}{5}$. Then the probability that out of five students, four are swimmers is
(a) ${}^5C_4\left(\frac{4}{5}\right)^4\frac{1}{5}$ (b) $\left(\frac{4}{5}\right)^4\frac{1}{5}$ (c) ${}^5C_1\frac{1}{5}\left(\frac{4}{5}\right)^4$ (d) None of these

10. The mean of the numbers obtained on throwing a die having written 1 on three faces, 2 on two faces and 5 on one face is
(a) 1 (b) 2 (c) 5 (d) $\frac{8}{3}$

11. Suppose that two cards are drawn at random from a deck of cards. Let X be the number of aces obtained. Then the value of E(X) is
(a) $\frac{37}{221}$ (b) $\frac{5}{13}$ (c) $\frac{1}{13}$ (d) $\frac{2}{13}$

12. Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed. A reports that a head appears. The probability that actually there was head is

(a) $\frac{4}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$

13. The probability of obtaining an even prime number on each die, when a pair of dice is rolled is

(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{12}$ (d) $\frac{1}{36}$

14. If P (A) = $\frac{1}{2}$, P(B) = 0, then P (A|B) is

(a) 0 (b) $\frac{1}{2}$ (c) not defined (d) 1

15. Two events A and B will be independent, if

(a) A and B are mutually exclusive
(b) P(A′B′) = [1 – P(A)] [1 – P(B)]
(c) P(A) = P(B)
(d) P(A) + P(B) = 1

16. If A and B are two events such that A B and P(B) ≠ 0, then which of the following is correct?

(a) $P(A|B) = \frac{P(B)}{P(A)}$ (b) P(A|B) < P(A) (c) P(A|B) ≥ P(A) (d) None of these

17. If P(A) = $\frac{4}{5}$, and P(A ∩ B) = $\frac{7}{10}$, then P(B | A) is equal to

(a) $\frac{1}{10}$ (b) $\frac{1}{8}$ (c) $\frac{7}{8}$ (d) $\frac{17}{20}$

18. If P(A ∩ B) = $\frac{7}{10}$ and P(B) = $\frac{17}{10}$, then P (A | B) equals

(a) $\frac{14}{17}$ (b) $\frac{17}{20}$ (c) $\frac{7}{8}$ (d) $\frac{1}{8}$

19. If P(A) = $\frac{3}{10}$, P (B) = $\frac{2}{5}$ and P(A ∪ B) = $\frac{3}{5}$, then P (B | A) + P (A | B) equals

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{5}{12}$ (d) $\frac{7}{2}$

20. If P(A) = $\frac{2}{5}$, P(B) = $\frac{3}{10}$ and P (A ∩ B) = $\frac{1}{5}$, then P(A' | B').P(B' | A ') is equal to

(a) $\frac{5}{6}$ (b) $\frac{5}{7}$ (c) $\frac{25}{42}$ (d) 1

21. If P(A) = 0.4, P(B) = 0.8 and P(B | A) = 0.6, then P(A ∪ B) is equal to

(a) 0.24 (b) 0.3 (c) 0.48 (d) 0.96

22. If A and B are two events and $A \neq \phi$, $B \neq \phi$, then

(a) $P(A \mid B) = P(A).P(B)$ (b) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

(c) $P(A \mid B).P(B \mid A) = 1$ (d) $P(A \mid B) = P(A) \mid P(B)$

23. A family has two children. What is the probability that both the children are boys given that at least one of them is a boy ?

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{5}$

24. A and B are events such that P(A) = 0.4, P(B) = 0.3 and $P(A \cup B) = 0.5$. Then $P(B' \cap A)$ equals

(a) $\frac{2}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{10}$ (d) $\frac{1}{5}$

25. If A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A/B) = \frac{1}{4}$, then $P(A' \cap B')$ equals

(a) $\frac{1}{12}$ (b) $\frac{3}{4}$ (c) $\frac{1}{4}$ (d) $\frac{3}{16}$

26. You are given that A and B are two events such that $P(B) = \frac{3}{5}$, $P(A \mid B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then P(A) equals

(a) $\frac{3}{10}$ (b) $\frac{1}{5}$ (c) $\frac{1}{2}$ (d) $\frac{3}{5}$

27. If $P(B) = \frac{3}{5}$, $P(A \mid B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then $P(A \cup B)' + P(A' \cup B) =$

(a) $\frac{1}{5}$ (b) $\frac{4}{5}$ (c) $\frac{1}{2}$ (d) 1

28. Let $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$. Then $P(A' \mid B)$ is equal to

(a) $\frac{6}{13}$ (b) $\frac{4}{13}$ (c) $\frac{4}{9}$ (d) $\frac{5}{9}$

29. If A and B are such events that $P(A) > 0$ and $P(B) \neq 1$, then $P(A' \mid B')$ equals.

(a) $1 - P(A \mid B)$ (b) $1 - P(A' \mid B)$ (c) $\dfrac{1 - P(A \cup B)}{P(B')}$ (d) $P(A') \mid P(B')$

30. If A and B are two independent events with $P(A) = \frac{3}{5}$ and $P(B) = \frac{4}{9}$, then $P(A' \cap B')$ equals

(a) $\frac{4}{15}$ (b) $\frac{8}{45}$ (c) $\frac{1}{3}$ (d) $\frac{2}{9}$

31. If two events are independent, then

(a) they must be mutually exclusive (b) the sum of their probabilities must be equal to 1

(c) (A) and (B) both are correct (d) None of the above is correct

32. Let A and B be two events such that P(A) = $\frac{3}{8}$, P(B) = $\frac{5}{8}$ and P(A ∪ B) = $\frac{3}{4}$. Then P(A | B).P(A' | B) is equal to

(a) $\frac{2}{5}$ (b) $\frac{3}{8}$ (c) $\frac{3}{20}$ (d) $\frac{6}{25}$

33. If the events A and B are independent, then P(A ∩ B) is equal to

(a) P (A) + P (B) (b) P(A) – P(B) (c) P (A) . P(B) (d) P(A) | P(B)

34. Two events E and F are independent. If P(E) = 0.3, P(E ∪ F) = 0.5, then P(E | F)–P(F | E) equals

(a) $\frac{2}{7}$ (b) $\frac{3}{35}$ (c) $\frac{1}{70}$ (d) $\frac{1}{7}$

35. A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement, the probability of getting exactly one red ball is

(a) $\frac{45}{196}$ (b) $\frac{135}{392}$ (c) $\frac{15}{56}$ (d) $\frac{15}{29}$

36. A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement, the probability that exactly two of the three balls were red, the first ball being red, is

(a) $\frac{1}{3}$ (b) $\frac{4}{7}$ (c) $\frac{15}{28}$ (d) $\frac{5}{28}$

37. Three persons, A, B and C, fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is

(a) 0.024 (b) 0.188 (c) 0.336 (d) 0.452

38. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{7}$

39. A die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $\frac{3}{4}$

40. A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

(a) $\frac{3}{28}$ (b) $\frac{2}{21}$ (c) $\frac{1}{28}$ (d) $\frac{167}{168}$

41. A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, the probability that both are dead is

(a) $\frac{33}{56}$ (b) $\frac{9}{64}$ (c) $\frac{1}{14}$ (d) $\frac{3}{28}$

42. Eight coins are tossed together. The probability of getting exactly 3 heads is

(a) $\frac{1}{256}$ (b) $\frac{7}{32}$ (c) $\frac{5}{32}$ (d) $\frac{3}{32}$

43. Two dice are thrown. If it is known that the sum of numbers on the dice was less than 6, the probability of getting a sum 3, is

(a) $\frac{1}{18}$ (b) $\frac{5}{18}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$

44. Which one is not a requirement of a binomial distribution?

(a) There are 2 outcomes for each trial
(b) There is a fixed number of trials
(c) The outcomes must be dependent on each other
(d) The probability of success must be the same for all the trials

45. Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability, that both cards are queens, is

(a) $\frac{1}{13}\times\frac{1}{13}$ (b) $\frac{1}{13}+\frac{1}{13}$ (c) $\frac{1}{13}\times\frac{1}{17}$ (d) $\frac{1}{13}\times\frac{4}{51}$

46. The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

(a) $\frac{7}{64}$ (b) $\frac{7}{128}$ (c) $\frac{45}{1024}$ (d) $\frac{7}{41}$

47. You are given that A and B are two events such that $P(B) = \frac{3}{5}$, $P(A \mid B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then $P(B \mid A')$ equals

(a) $\frac{1}{5}$ (b) $\frac{3}{10}$ (c) $\frac{1}{2}$ (d) $\frac{3}{5}$

48. The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

(a) ${}^5C_4\,(0.7)^4\,(0.3)$ (b) ${}^5C_1\,(0.7)\,(0.3)^4$ (c) ${}^5C_4\,(0.7)\,(0.3)^4$ (d) $(0.7)^4\,(0.3)$

49. The probability distribution of a discrete random variable X is given below:

X	2	3	4	5
P(X)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

The value of k is

(a) 8 (b) 16 (c) 32 (d) 48

50. For the following probability distribution:

X	– 4	– 3	– 2	– 1	0
P(X)	0.1	0.2	0.3	0.2	0.2

E(X) is equal to :

(a) 0 (b) –1 (c) –2 (d) –1.8

51. For the following probability distribution

X	1	2	3	4
P(X)	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

$E(X^2)$ is equal to

(a) 3 (b) 5 (c) 7 (d) 10

52. Suppose a random variable X follows the binomial distribution with parameters n and p, where $0 < p < 1$. If $P(x = r) / P(x = n-r)$ is independent of n and r, then p equals

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{5}$ (d) $\frac{1}{7}$

53. In a college, 30% students fail in physics, 25% fail in mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in physics if she has failed in mathematics is

(a) $\frac{1}{10}$ (b) $\frac{2}{5}$ (c) $\frac{9}{10}$ (d) $\frac{1}{3}$

54. A and B are two students. Their chances of solving a problem correctly are $\frac{1}{3}$ and $\frac{1}{4}$, respectively. If the probability of their making a common error is, $\frac{1}{20}$ and they obtain the same answer, then the probability of their answer to be correct is

(a) $\frac{1}{12}$ (b) $\frac{1}{40}$ (c) $\frac{13}{120}$ (d) $\frac{10}{13}$

55. A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(a) $\left(\frac{9}{10}\right)^5$ (b) $\frac{1}{2}\left(\frac{9}{10}\right)^4$ (c) $\frac{1}{2}\left(\frac{9}{10}\right)^5$ (d) $\left(\frac{9}{10}\right)^5+\frac{1}{2}\left(\frac{9}{10}\right)^4$

56. If A and B are independent events such that P (A) = p, P (B) = 2p and P (Exactly one of A, B) = $\frac{5}{9}$, then p = ___________

57. If A and B' are independent events then P (A'$\cup$B) = 1 – ________

58. A die is thrown twice and the sum of the numbers appearing is observed to be 6. The conditional probability that the number 4 has appeared at least once is ______

59. Given that E and F are events such that P(E) = 0.6, P(F) = 0.3 and P(E ∩ F) = 0.2, then P (E|F) = ________

60. If P(B) = 0.5 and P (A ∩ B) = 0.32, then P(A|B) = _________

61. If P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4, then P(A ∩ B) = _________

62. If P (A) = 0.8, P (B) = 0.5 and P (B|A) = 0.4, then P(A|B) = _________

63. A black and a red dice are rolled, then the conditional probability of obtaining a sum greater than 9, given that the black die resulted in a 5 is ______

64. A fair die is rolled. Consider events E = {1,3,5}, F = {2,3} and G = {2,3,4,5} then P(E|F) = _____

65. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, then the conditional probability that both are girls given that the youngest is a girl is _____

OTHER CHAPTERS NCERT MOST IMP QUESTIONS & OBJECTIVE TYPE QUESTIONS

CHAPTER – 6: APPLICATION OF DERIVATIVES

MARKS WEIGHTAGE – 09 marks

NCERT Important Questions

EXERCISE 6.2
- ☞ Q5
- ☞ Q6
- ☞ Q7
- ☞ Q8
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- ☞ Q15
- ☞ Q16

EXERCISE 6.3
- ☞ Q7
- ☞ Q8
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- ☞ Q18
- ☞ Q19
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EXERCISE 6.5
- ☞ **Q**17
- ☞ **Q**18
- ☞ Q19
- ☞ Q20
- ☞ Q21
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MISC. EXERCISE.
- ☞ Q7
- ☞ Q8
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SOLVED EXAMPLES.
- ☞ 8 (Pg 201)
- ☞ 11 (Pg 202)
- ☞ 12 (Pg 203)
- ☞ 13 (Pg 204)
- ☞ 17 (Pg 209)
- ☞ 18 (Pg 209)
- ☞ 20 (Pg 210)
- ☞ 29 (Pg 222)
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- ☞ 37 (Pg 226)
- ☞ 38 (Pg 227)
- ☞ 39(Pg 230)
- ☞ 41 (Pg 231)
- ☞ 43 (Pg 235)
- ☞ 50 (Pg 240)

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. The abscissa of the point on the curve $3y = 6x - 5x^3$, the normal at which passes through origin is:

(a) 1 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

2. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$

(a) touch each other (b) cut at right angle

(c) cut at an angle $\frac{\pi}{3}$ (d) cut at an angle $\frac{\pi}{4}$

3. The tangent to the curve given by $x = e^t . \cos t$, $y = e^t . \sin t$ at $t = \frac{\pi}{4}$ makes with x-axis an angle:

(a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

4. The equation of the normal to the curve y = sinx at (0, 0) is:

(a) x = 0 (d) y = 0 (c) x + y = 0 (d) x – y = 0

5. The point on the curve $y^2 = x$, where the tangent makes an angle of $\frac{\pi}{4}$ with x-axis is

(a) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (c) (4, 2) (d) (1, 1)

6. The curve $y = x^{\frac{1}{5}}$ has at (0, 0)

(a) a vertical tangent (parallel to y-axis) (b) a horizontal tangent (parallel to x-axis)

(c) an oblique tangent (d) no tangent

7. The equation of normal to the curve $3x^2 - y^2 = 8$ which is parallel to the line x + 3y = 8 is

(a) 3x – y = 8 (b) 3x + y + 8 = 0 (c) x + 3y ± 8 = 0 (d) x + 3y = 0

8. If the curve $ay + x^2 = 7$ and $x^3 = y$, cut orthogonally at (1, 1), then the value of a is:

(a) 1 (b) 0 (c) – 6 (d) 0.6

9. If $y = x^4 - 10$ and if x changes from 2 to 1.99, what is the change in y

(a)0.32 (b) 0.032 (c) 5.68 (d) 5.968

10. The equation of tangent to the curve $y (1 + x^2) = 2 - x$, where it crosses x-axis is:

(a) x + 5y = 2 (b) x – 5y = 2 (c) 5x – y = 2 (d) 5x + y = 2

11. The points at which the tangents to the curve $y = x^3 - 12x + 18$ are parallel to x-axis are:

(a) (2, –2), (–2, –34) (b) (2, 34), (–2, 0) (c) (0, 34), (–2, 0) (d) (2, 2), (–2, 34)

12. The tangent to the curve $y = e^2x$ at the point (0, 1) meets x-axis at:

(a) (0, 1) (b) $\left(-\frac{1}{2}, 0\right)$ (c) (2, 0) (d) (0, 2)

13. The slope of tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2, –1) is:

(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) $\frac{-6}{7}$ (d) –6

14. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ intersect at an angle of

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

15. The interval on which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing is:

(a) $[-1, \infty)$ (b) $[-2, -1]$ (c) $(-\infty, -2]$ (d) $[-1, 1]$

16. Let the $f : R \to R$ be defined by $f(x) = 2x + \cos x$, then f :

(a) has a minimum at $x = \pi$ (b) has a maximum, at $x = 0$

(c) is a decreasing function (d) is an increasing function

17. $y = x(x - 3)^2$ decreases for the values of x given by :

(a) $1 < x < 3$ (b) $x < 0$ (c) $x > 0$ (d) $0 < x < \frac{3}{2}$

18. The function $f(x) = 4\sin^3 x - 6\sin^2 x + 12\sin x + 100$ is strictly

(a) increasing in $\left(\pi, \frac{3\pi}{2}\right)$ (b) decreasing in $\left(\frac{\pi}{2}, \pi\right)$

(c) decreasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) decreasing in $\left[0, \frac{\pi}{2}\right]$

19. Which of the following functions is decreasing on $\left(0, \frac{\pi}{2}\right)$?

(a) sin2x (b) tanx (c) cosx (d) cos 3x

20. The function $f(x) = \tan x - x$

(a) always increases (b) always decreases

(c) never increases (d) sometimes increases and sometimes decreases.

21. If x is real, the minimum value of $x^2 - 8x + 17$ is

(a) –1 (b) 0 (c) 1 (d) 2

22. The smallest value of the polynomial $x^3 - 18x^2 + 96x$ in [0, 9] is

(a) 126 (b) 0 (c) 135 (d) 160

23. The function $f(x) = 2x^3 - 3x^2 - 12x + 4$, has

(a) two points of local maximum (b) two points of local minimum

(c) one maxima and one minima (d) no maxima or minima

24. The maximum value of sin x . cos x is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\sqrt{2}$ (d) $2\sqrt{2}$

25. At $x = \frac{5\pi}{6}$, $f(x) = 2\sin 3x + 3\cos 3x$ is:

(a) maximum (b) minimum (c) zero (d) neither maximum nor minimum.

26. Maximum slope of the curve $y = -x^3 + 3x^2 + 9x - 27$ is:

(a) 0 (b) 12 (c) 16 (d) 32

27. $f(x) = x^x$ has a stationary point at

(a) x = e (b) x = $\frac{1}{e}$ (c) x = 1 (d) x = $\sqrt{e}$

28. The maximum value of $\left(\frac{1}{x}\right)^x$ is

(a) e (b) e^e (c) $e^{\frac{1}{e}}$ (d) $\left(\frac{1}{e}\right)^{\frac{1}{e}}$

29. The total revenue in Rupees received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$. The marginal revenue, when x = 15 is
(a) 116 (b) 96 (c) 90 (d) 126

30. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?
(a) (0,1) (b) $\left(\frac{\pi}{2}, \pi\right)$ (c) $\left(0, \frac{\pi}{2}\right)$ (d) None of these

31. The interval in which $y = x^2 e^{-x}$ is increasing is
(a) $(-\infty, \infty)$ (b) $(-2, 0)$ (c) $(2, \infty)$ (d) $(0, 2)$

32. The slope of the normal to the curve $y = 2x^2 + 3 \sin x$ at x = 0 is
(a) 3 (b) $\frac{1}{3}$ (c) –3 (d) $-\frac{1}{3}$

33. The line y = x + 1 is a tangent to the curve y2 = 4x at the point
(a) (1, 2) (b) (2, 1) (c) (1, – 2) (d) (– 1, 2)

34. The point on the curve $x^2 = 2y$ which is nearest to the point (0, 5) is
(a) $(2\sqrt{2}, 4)$ (b) $(2\sqrt{2}, 0)$ (c) (0, 0) (d) (2, 2)

35. For all real values of x, the minimum value of $\frac{1-x+x^2}{1+x+x^2}$ is

(a) 0 (b) 1 (c) 3 (d) $\frac{1}{3}$

36. The maximum value of $[x(x-1)+1]^{\frac{1}{3}}$, $0 \le x \le 1$ is

(a) $\left(\frac{1}{3}\right)^{\frac{1}{3}}$ (b) $\frac{1}{2}$ (c) 1 (d) 0

37. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic metre per hour. Then the depth of the wheat is increasing at the rate of
(a) 1 m^3/h (b) 0.1 m^3/h (c) 1.1 m^3/h (d) 0.5 m^3/h

38. The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point (2,– 1) is
(a) $\frac{22}{7}$ (b) $\frac{6}{7}$ (c) $\frac{7}{6}$ (d) $\frac{-6}{7}$

39. The line y = mx + 1 is a tangent to the curve $y^2 = 4x$ if the value of m is

(a) 1 (b) 2 (c) 3 (d) $\frac{1}{2}$

40. The normal at the point (1,1) on the curve $2y + x^2 = 3$ is
(a) $x + y = 0$ (b) $x - y = 0$ (c) $x + y + 1 = 0$ (d) $x - y = 0$

41. The normal to the curve $x^2 = 4y$ passing (1,2) is
(a) $x + y = 3$ (b) $x - y = 3$ (c) $x + y = 1$ (d) $x - y = 1$

42. The points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes are
(a) $\left(4, \pm\frac{8}{3}\right)$ (b) $\left(4, -\frac{8}{3}\right)$ (c) $\left(4, \pm\frac{3}{8}\right)$ (d) $\left(\pm 4, \frac{3}{8}\right)$

43. The values of a for which $y = x^2 + ax + 25$ touches the axis of x are______.

44. If f (x) = $\frac{1}{4x^2 + 2x + 1}$, then its maximum value is _______.

45. Let f have second deriative at c such that f '(c) = 0 and f□"(c) > 0, then c is a point of ______.

46. Minimum value of f if f (x) = sinx in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is _____.

47. The maximum value of sinx + cosx is _____.

48. The curves $y = 4x^2 + 2x - 8$ and $y = x^3 - x + 13$ touch each other at the point_____.

49. The equation of normal to the curve y = tanx at (0, 0) is ________.

50. The values of a for which the function f (x) = sinx – ax + b increases on R are ______.

51. The function f (x) = $\frac{2x^2 - 1}{x^4}$, x > 0, decreases in the interval _______.

52. The least value of the function f (x) = ax + $\frac{b}{x}$ (a > 0, b > 0, x > 0) is ______.

53. The angle θ, $0 < \theta < \frac{\pi}{2}$, which increases twice as fast as its sine is ________

CHAPTER – 7: INTEGRALS

MARKS WEIGHTAGE – 08 marks

NCERT Important Questions

EXERCISE 7.1
- ☞ Q10
- ☞ Q12
- ☞ Q13
- ☞ Q18
- ☞ Q20

EXERCISE 7.2
- ☞ Q9
- ☞ Q10
- ☞ Q14
- ☞ Q15
- ☞ Q19
- ☞ Q20
- ☞ Q28
- ☞ Q31
- ☞ Q32
- ☞ Q33
- ☞ Q34
- ☞ Q36
- ☞ Q37

EXERCISE 7.3
- ☞ Q3
- ☞ Q5
- ☞ Q10
- ☞ Q13
- ☞ Q14
- ☞ Q16
- ☞ Q20
- ☞ Q22

EXERCISE 7.4
- ☞ Q7
- ☞ Q14
- ☞ Q16
- ☞ Q19
- ☞ Q22

EXERCISE 7.5 .
- ☞ Q3
- ☞ Q7
- ☞ Q8
- ☞ Q9
- ☞ Q15
- ☞ Q16
- ☞ Q17
- ☞ Q18
- ☞ Q19
- ☞ Q20
- ☞ Q21

EXERCISE 7.6
- ☞ Q6
- ☞ Q7
- ☞ Q10
- ☞ Q11
- ☞ Q17
- ☞ Q18
- ☞ Q20
- ☞ Q21

EXERCISE 7.9
- ☞ Q14
- ☞ Q16

EXERCISE 7.10
- ☞ Q3
- ☞ Q5
- ☞ Q6

EXERCISE 7.11
- ☞ Q2
- ☞ Q5
- ☞ Q8
- ☞ Q10
- ☞ Q12
- ☞ Q15
- ☞ Q16

MISC. EXERCISE.
Q7
Q10
Q11

MISC. EXERCISE.
Q14
Q19
Q20
Q21
Q22
Q24
Q26
Q28
Q30
Q31
Q32
Q33
Q44

SOLVED EXAMPLES.
- ☞ Example 6 Pg (203)
- ☞ Example 15 (Pg 320)
- ☞ Example 32 (Pg 344)
- ☞ Example 34 (Pg 345)
- ☞ Example 35 (Pg 345)
- ☞ Example 36 (Pg 346)
- ☞ Example 39 (Pg 348)
- ☞ Example 40 (Pg 349)
- ☞ Example 41 (Pg 350)
- ☞ Example 44 (Pg 351)

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. $\int e^{x}(\cos x-\sin x)dx$ is equal to

(a) $e^x \cos x + C$ (b) $e^x \sin x + C$ (c) $-e^x \cos x + C$ (d) $-e^x \sin x + C$

2. $\int \frac{dx}{\sin^2 x\cos^2 x}$ is equal to

(a) tanx + cotx + C (b) (tan x + cotx)^2 + C (c) tanx – cotx + C (d) (tanx – cotx)^2 + C

3. If $\int \frac{x^3}{\sqrt{1+x^2}}dx = a(1+x^2)^{\frac{3}{2}}+b\sqrt{1+x^2}+C$, then

(a) $a=\frac{1}{3},b=1$ (b) $a=\frac{-1}{3},b=1$ (c) $a=\frac{-1}{3},b=-1$ (d) $a=\frac{1}{3},b=-1$

4. If $\int \frac{3e^x-5e^{-x}}{4e^x+5e^{-x}}dx$ = ax + b log $|4e^x + 5e^{-x}|$ + C, then

(a) $a=\frac{-1}{8},b=\frac{7}{8}$ (b) $a=\frac{1}{8},b=\frac{7}{8}$ (c) $a=\frac{-1}{8},b=\frac{-7}{8}$ (d) $a=\frac{1}{8},b=\frac{-7}{8}$

5. If $\int \frac{dx}{(x+2)(x^2+1)} = a\log|1+x^2|+b\tan^{-1}x+\frac{1}{5}\log|x+2|+C$, then

(a) $a=\frac{-1}{10},b=\frac{-2}{5}$ (b) $a=\frac{1}{10},b=\frac{-2}{5}$ (c) $a=\frac{-1}{10},b=\frac{2}{5}$ (d) $a=\frac{1}{10},b=\frac{2}{5}$

6. $\int \frac{x^3}{x+1}dx$ is equal to

(a) $x+\frac{x^2}{2}+\frac{x^3}{3}-\log|1-x|+C$ (b) $x+\frac{x^2}{2}-\frac{x^3}{3}-\log|1-x|+C$

(c) $x-\frac{x^2}{2}-\frac{x^3}{3}-\log|1+x|+C$ (d) $x-\frac{x^2}{2}+\frac{x^3}{3}-\log|1+x|+C$

7. $\int \frac{x^9}{(4x^2+1)^6}dx$ is equal to

(a) $\frac{1}{5x}\left(4+\frac{1}{x^2}\right)^{-5}+C$ (b) $\frac{1}{5}\left(4+\frac{1}{x^2}\right)^{-5}+C$

(c) $\frac{1}{10x}\left(\frac{1}{x^2}+4\right)^{-5}+C$ (d) $\frac{1}{10}\left(\frac{1}{x^2}+4\right)^{-5}+C$

8. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ is equal to

(a) $\sin(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$ (b) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$

(c) $\operatorname{cosec}(b-a)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$ (d) $\sin(b-a)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$

9. $\int \frac{x+\sin x}{1+\cos x}dx$ is equal to

(a) $\log|1+\cos x|+C$ (b) $\log|x+\sin x|+C$ (c) $x-\tan\frac{x}{2}+C$ (d) $x.\tan\frac{x}{2}+C$

10. $\int \frac{dx}{\cos(x-a)\cos(x-b)}$ is equal to

(a) $\operatorname{cosec}(a-b)\log\left|\frac{\sin(x-a)}{\sin(x-b)}\right|+C$ (b) $\operatorname{cosec}(a-b)\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right|+C$

(c) $\operatorname{cosec}(a-b)\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+C$ (d) $\operatorname{cosec}(a-b)\log\left|\frac{\cos(x-b)}{\cos(x-a)}\right|+C$

11. $\int \tan^{-1}\sqrt{x}dx$ is equal to

(a) $(x+1)\tan^{-1}\sqrt{x}-\sqrt{x}+C$ (b) $x\tan^{-1}\sqrt{x}-\sqrt{x}+C$

(c) $\sqrt{x}-x\tan^{-1}\sqrt{x}+C$ (d) $\sqrt{x}-(x+1)\tan^{-1}\sqrt{x}+C$

12. $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$ is equal to

(a) $\frac{e^x}{1+x^2}+C$ (b) $\frac{-e^x}{1+x^2}+C$ (c) $\frac{e^x}{(1+x^2)^2}+C$ (d) $\frac{-e^x}{(1+x^2)^2}+C$

13. $\int \frac{dx}{\sqrt{1+x}+\sqrt{x}}$ is equal to

(a) $\frac{2}{3}(1+x)^{\frac{2}{3}}-\frac{2}{3}x^{\frac{2}{3}}+C$ (b) $\frac{3}{2}(1+x)^{\frac{2}{3}}+\frac{3}{2}x^{\frac{3}{2}}+C$

(c) $\frac{3}{2}(1+x)^{\frac{3}{2}}+\frac{3}{2}x^{\frac{3}{2}}+C$ (d) $\frac{2}{3}(1+x)^{\frac{3}{2}}-\frac{2}{3}x^{\frac{3}{2}}+C$

14. $\int \frac{dx}{(e^x+1)(2e^x+3)}$ is equal to

(a) $x+\log\left|e^x+1\right|-\frac{2}{3}\log\left|2e^x+3\right|+C$ (b) $x-\frac{2}{3}\log\left|e^x+1\right|+\log\left|2e^x+3\right|+C$

(c) $\frac{3}{2}(1+x)^{\frac{3}{2}}+\frac{3}{2}x^{\frac{3}{2}}+C$ (d) None of these

15. $\int \frac{dx}{x(x^n+1)}$ is equal to

(a) $\log\left|\frac{x^n}{1+x^n}\right|+C$ (b) $\frac{1}{n}\log\left|\frac{x^n+1}{x^n}\right|+C$

(c) $\log\left|\frac{x^n+1}{x^n}\right|+C$ (d) $\frac{1}{n}\log\left|\frac{x^n}{1+x^n}\right|+C$

16. The value of $\int \frac{1+x+x^2}{1+x^2}e^{\tan^{-1}x}dx$ is equal to

(a) $x^2e^{\tan^{-1}x}$ (b) $e^{\tan^{-1}x}+C$ (c) $xe^{\tan^{-1}x}+C$ (d) None of these

17. The value of $\int \frac{\cos^3 x dx}{\sin^2 x + \sin x}$ is equal to

(a) log sin x − sin x + c (b) log |sin x| – sin x + c

(c) log |sin x| + c (d) None of these

18. $\int \frac{\tan dx}{\sqrt{\cos x}}$ is equal to

(a) $\frac{2}{\sqrt{\sin x}} + C$ (b) $\frac{2}{\sqrt{\cos x}} + C$

(c) $\frac{2}{\sqrt{\tan x}} + C$ (d) $\frac{2}{(\sin x)^{3/2}} + C$

19. $\int \frac{dx}{x \log x \log(\log x)}$ is equal to

(a) $\log|\log(\log x)| + C$ (b) $|\log x| + C$

(c) $\log\left|\log\left(\frac{1}{x}\right)\right| + C$ (d) $\log|\log x| + C$

20. $\int e^x \left(\frac{1+\sqrt{1-x^2}\sin^{-1} x}{\sqrt{1-x^2}} d \right) x$ is equal to

(a) $e^x \sin^{-1} x + C$ (b) $\frac{e^x}{\sqrt{1-x^2}} + C$

(c) $e^x \sqrt{1-x^2} + C$ (d) $\sqrt{1-x^2} \sin^{-1} x + C$

21. The value of $\int \frac{x^3}{1+x^8} dx$ is equal to

(a) $\frac{1}{4}\tan^{-1} x^4 + C$ (b) $\frac{1}{2}\tan^{-1} x^4 + C$ (c) $\frac{1}{4}\cot^{-1} x^2 + C$ (d) None of these

22. $\int_{a+c}^{b+c} f(x)dx$ is equal to

(a) $\int_a^b f(x)dx$ (b) $\int_a^b f(x-c)dx$ (c) $\int_a^b f(x+c)dx$ (d) $\int_{a-c}^{b-c} f(x)dx$

23. If f and g are continuous functions in [0, 1] satisfying f (x) = f (a – x) and g (x) + g (a – x) = a, then $\int_0^a f(x).g(x)dx$ is equal to

(a) $\frac{a}{2}$ (b) $\frac{a}{2}\int_0^a f(x)dx$ (c) $\int_0^a f(x)dx$ (d) $a\int_0^a f(x)dx$

24. If $\int_0^y \frac{dt}{\sqrt{1+9t^2}}$ and $\frac{d^2y}{dx^2}$ = ay, then a is equal to

(a) 3 (b) 6 (c) 9 (d) 1

25. $\int_{-1}^{1} \frac{x^3+|x|+1}{x^2+2|x|+1}dx$ is equal to

(a) log 2 (b) 2 log 2 (c) $\frac{1}{2}$log 2 (d) 4 log 2.

26. If $\int_0^1 \frac{e^t dt}{1+t} = a$ then $\int_0^1 \frac{e^t dt}{(1+t)^2}$ is equal to

(a) $a-1+\frac{e}{2}$ (b) $a+1-\frac{e}{2}$ (c) $a-1-\frac{e}{2}$ (d) $a+1+\frac{e}{2}$

27. The value of $\int xe^x dx$ is equal to

(a) $xe^x + e^x + c$ (b) $xe^x - e^x + c$ (c) $-xe^x + e^x + c$ (d) None of these

28. $\int \frac{dx}{\sqrt{2x-x^2}}$ is equal to

(a) $\sin^{-1}(1-x)+C$ (b) $-\cos^{-1}(1-x)+C$
(c) $\sin^{-1}(x-1)+C$ (d) $\cos^{-1}(x-1)+C$

29. $\int_{-2}^{2} |x\cos\pi x|\ dx$ is equal to

(a) $\frac{8}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{2}{\pi}$ (d) $\frac{1}{\pi}$

30. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos\theta}dx$ is equal to

(a) 2(sinx + xcosθ) + C (b) 2(sinx – xcosθ) + C
(c) 2(sinx + 2xcosθ) + C (d) 2(sinx – 2x cosθ) + C

31. $\int e^{\log\sin x}dx$ is equal to

(a) sin x + c (b) −cos x + c (c) $e^{\log\cos x}$ + c (d) None of these

32. $\int \frac{\cos 2x}{\cos x}dx$ is equal to

(a) 2sin x + log |(sec x − tan x)| + c (b) 2sin x − log |(sec x − tan x)| + c
(c) 2sin x + log |(sec x + tan x)| + c (d) 2sin x − log |(sec x + tan x)| + c

33. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{1+\cos 2x}$ is equal to

(a) 1 (b) 2 (c) 3 (d) 4

34. $\int \frac{1}{4\cos^3 x - 3\cos x}dx$ is equal to

(a) $\frac{1}{3}\log|\sec 3x - \tan 3x| + C$ (b) $\frac{1}{3}\log|\sec 3x + \tan 3x| + C$

(c) $\frac{1}{4}\log|\sec 3x + \tan 3x| + C$ (d) $\frac{1}{4}\log|\sec 3x - \tan 3x| + C$

35. $\int \sin(\log x)dx$ is equal to

(a) $\frac{x}{2}\left[\sin(\log x)+\cos(\log x)\right]+C$ (b) $\frac{1}{3}\log|\sec 3x+\tan 3x|+C$

(c) $\frac{x}{2}\left[\cos(\log x)-\sin(\log x)\right]+C$ (d) $x\left[\sin(\log x)-\cos(\log x)\right]+C$

36. $\int_0^{\frac{\pi}{2}} \sqrt{1-\sin 2x}\ \ dx$ is equal to

(a) $2\sqrt{2}$ (b) $2(\sqrt{2}+1)$ (c) 2 (d) $2(\sqrt{2}-1)$

37. $\int \frac{\sec x \cos ecx}{\log \tan x} dx$ is equal to

(a) log (tan x) + c (b) cot (log x) + c

(c) log (log tan x) + c (d) tan (log x) + c

38. $\int x^2 \sin x dx$ is equal to

(a) x^2 sin x − 2x cos x + c (b) x^2 sin x + c

(c) − x^2 cos x + 2x sin x + 2 cos x + c (d) − x^2 sin x − 2x cos x + sin x + c

39. $\int (e^{\log x}+\sin x)\cos x dx$ is equal to

(a) x sin x + cos x − $\sin^2$ x + c (b) x cos x − $\sin^2$ x + c

(c) x sin x + cos x − ($\cos^2$ x)/2 + c (d) x^2 sin x + cos x − $\sin^3$ x + c

40. $\int \left[\sin(\log x)+\cos(\log x)\right]dx$ is equal to

(a) x sin (log x) + c (b) x cos (log x) + c

(c) x log (sin x) + c (d) x log (cos x) + c

41. $\int \frac{dx}{\cos^6 x+\sin^6 x}$ is equal to

(a) log |tan x − cot x| + c (b) log|cot x – tan x| + c

(c) $\tan^{-1}$ (tan x − cot x) + c (d) $\tan^{-1}$ (2 cot 2 x) + c

42. $\int_0^{\frac{\pi}{2}} \cos x.e^{\sin x}dx$ is equal to _____________

43. If $\int_0^{a} \frac{1}{1+4x^2}dx = \frac{\pi}{8}$, then a is equal to ________

44. The value of $\int_{-\pi}^{\pi} \sin^3 x\cos^2 x dx$ is ___________.

45. $\int \frac{\sin x}{3+4\cos^2 x}dx =$ ___________

46. $\int \frac{x+3}{(x+4)^2} e^x dx =$ __________

47. $\int \frac{\sin^6 x}{\cos^8 x} dx$ = _______

48. $\int_{-a}^{a} f(x)dx = 0$, if f is an _______ function.

49. $\int_{0}^{2a} f(x)dx = 2\int_{0}^{a} f(x)dx$, if f (2a – x) = _______.

50. $\int_{0}^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x} dx =$ ________

CHAPTER – 8: APPLICATION OF THE INTEGRALS

MARKS WEIGHTAGE – 06 marks

NCERT Important Questions

EXERCISE 8.1
- ☞ Q3
- ☞ Q4
- ☞ Q6
- ☞ Q7
- ☞ Q9
- ☞ Q10
- ☞ Q11

EXERCISE 8.2
- ☞ Q3
- ☞ Q4
- ☞ Q5

SOLVED EXAMPLES.
- ☞ Example 2 (Pg 362)
- ☞ Example 4 (Pg 364)
- ☞ Example 6 (Pg 368)
- ☞ Example 7 (Pg 368)
- ☞ Example 8 (Pg 369)
- ☞ Example 9 (Pg 370)
- ☞ Example 10 (Pg 370)
- ☞ Example 13 (Pg 373)

MISC EXERCISE.
- ☞ Q1
- ☞ Q2
- ☞ Q3
- ☞ Q5
- ☞ Q6
- ☞ Q10
- ☞ Q11
- ☞ Q13
- ☞ Q14

CHAPTER – 9: DIFFERENTIAL EQUATIONS

MARKS WEIGHTAGE – 07 marks

NCERT Important Questions

EXERCISE 9.1
- Q4
- Q11

EXERCISE 9.2
- Q4
- Q7
- Q9
- Q10

EXERCISE 9.4
- Q3
- Q4
- Q5
- Q6
- Q10
- Q12

EXERCISE 9.5
- Q1
- Q3
- Q4
- Q5
- Q6
- Q8
- Q9
- Q10
- Q13
- Q15

EXERCISE 9.6
- Q2
- Q5
- Q7
- Q8
- Q9
- Q13
- Q14
- Q15

MISC. EXERCISE.
- Q6
- Q9
- Q10
- Q13
- Q14

SOLVED EXAMPLES.
- Example 10 (Pg 393)
- Example 16 (Pg 401)
- Example 17 (Pg 403)
- Example 19 (Pg 410)
- Example 20 (Pg 411)
- Example 27 (Pg 417)
- Example 28 (Pg 418)

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. The degree of the differential equation $\left(1+\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$ is

(a) 1 (b) 2 (c) 3 (d) 4

2. The degree of the differential equation $\frac{d^2y}{dx^2}+3\left(\frac{dy}{dx}\right)^2 = x^2 \log\left(\frac{d^2y}{dx^2}\right)$ is

(a) 1 (b) 2 (c) 3 (d) not defined

3. The order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = \frac{d^2y}{dx^2}$ respectively, are

(a) 1, 2 (b) 2, 2 (c) 2, 1 (d) 4, 2

4. The order of the differential equation of all circles of given radius a is:
(a) 1 (b) 2 (c) 3 (d) 4

5. The solution of the differential equation $2x.\frac{dy}{dx} - y = 3$ represents a family of

(a) straight lines (b) circles (c) parabolas (d) ellipses

6. The integrating factor of the differential equation $\frac{dy}{dx}(x\log x) + y = 2\log x$ is

(a) e^x (b) log x (c) log (log x) (d) x

7. A solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x.\frac{dy}{dx} + y = 0$ is

(a) y = 2 (b) y = 2x (c) y = 2x – 4 (d) y = 2x2 – 4

8. Which of the following is not a homogeneous function of x and y.

(a) $x^2 + 2xy$ (b) 2x – y (c) $\cos^2\left(\frac{y}{x}\right) + \frac{y}{x}$ (d) sinx – cosy

9. Solution of the differential equation $\frac{dx}{x} + \frac{dy}{y} = 0$ is

(a) $\frac{1}{x} + \frac{1}{y} = c$ (b) logx . logy = c (c) xy = c (d) x + y = c

10. The solution of the differential equation $x\frac{dy}{dx} + 2y = x^2$ is

(a) $y = \frac{x^2 + c}{4x^2}$ (b) $y = \frac{x^2}{4} + c$ (c) $y = \frac{x^4 + c}{x^2}$ (d) $y = \frac{x^4 + c}{4x^2}$

11. The degree of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = x\sin\frac{dy}{dx}$ is

(a) 1 (b) 2 (c) 3 (d) not defined

12. The degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = \frac{d^2y}{dx^2}$ is

(a) 4 (b) $\frac{3}{2}$ (c) not defined (d) 2

13. The order and degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^{\frac{1}{4}} + x^{\frac{1}{5}} = 0$ respectively, are

(a) 2 and not defined (b) 2 and 2 (c) 2 and 3 (d) 3 and 3

14. If y = e^{-x} (Acosx + Bsinx), then y is a solution of

(a) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=0$ (b) $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+2y=0$

(c) $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=0$ (d) $\frac{d^2y}{dx^2}+2y=0$

15. The differential equation for y = Acos ax + Bsin ax, where A and B are arbitrary constants is

(a) $\frac{d^2y}{dx^2}-\alpha^2y=0$ (b) $\frac{d^2y}{dx^2}+\alpha^2y=0$

(c) $\frac{d^2y}{dx^2}+\alpha y=0$ (d) $\frac{d^2y}{dx^2}-\alpha y=0$

16. Solution of differential equation xdy – ydx = 0 represents :

(a) a rectangular hyperbola (b) parabola whose vertex is at origin

(c) straight line passing through origin (d) a circle whose centre is at origin

17. Integrating factor of the differential equation $\cos x.\frac{dy}{dx}+y\sin x=1$ is :

(a) cosx (b) tanx (c) secx (d) sinx

18. Solution of the differential equation tany sec^2x dx + tanx sec^2ydy = 0 is :

(a) tanx + tany = k (b) tanx – tany = k

(c) $\frac{\tan x}{\tan y}=k$ (d) tanx . tany = k

19. Family y = Ax + A^3 of curves is represented by the differential equation of degree:

(a) 1 (b) 2 (c) 3 (d) 4

20. Integrating factor of $\frac{xdy}{dx}-y=x^4-3x$ is:

(a) x (b) logx (c) $\frac{1}{x}$ d) – x

21. Solution of $\frac{dy}{dx}-y=1$, y (0) = 1 is given by

(a) xy = – e^x (b) xy = – e^{-x} (c) xy = – 1 (d) y = 2e^x – 1

22. The number of solutions of $\frac{dy}{dx}=\frac{y+1}{x-1}$ when y (1) = 2 is :

(a) none (b) one (c) two (d) infinite

23. $\tan^{-1}$ x + $\tan^{-1}$ y = c is the general solution of the differential equation:

(a) $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ (b) $\frac{dy}{dx}=\frac{1+x^2}{1+y^2}$

(c) (1 + x^2) dy + (1 + y^2) dx = 0 (d) (1 + x^2) dx + (1 + y^2) dy = 0

24. Integrating factor of the differential equation is

(a) – x (b) $\frac{x}{1+x^2}$ (c) $\sqrt{1-x^2}$ (d) $\frac{1}{4}\log(1-x^2)$

25. Which of the following is a second order differential equation?

(a) $(y')^2 + x = y^2$ (b) $y'y' + y = \sin x$ (c) $y' + (y')^2 + y = 0$ (d) $y' = y^2$

26. The differential equation $y\frac{dy}{dx} + x = c$ represents :

(a) Family of hyperbolas (b) Family of parabolas

(c) Family of ellipses (d) Family of circles

27. The general solution of $e^x \cos y\, dx - e^x \sin y\, dy = 0$ is :

(a) $e^x \cos y = k$ (b) $e^x \sin y = k$ (c) $e^x = k \cos y$ (d) $e^x = k \sin y$

28. The degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 6y^5 = 0$ is:

(a) 1 (b) 2 (c) 3 (d) 5

29. The solution of $\frac{dy}{dx} + y = e^{-x}$, y(0) = 0 is :

(a) $y = e^x (x - 1)$ (b) $y = xe^{-x}$ (c) $y = xe^{-x} + 1$ (d) $y = (x + 1)e^{-x}$

30. Integrating factor of the differential equation $\frac{dy}{dx} + y\tan x - \sec x = 0$ is:

(a) cosx (b) secx (c) $e^{\cos x}$ (d) esecx

31. The solution of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ is

(a) $y = \tan^{-1}x$ (b) $y - x = k(1 + xy)$ (c) $x = \tan^{-1}y$ (d) tan (xy) = k

32. The integrating factor of the differential equation $\frac{dy}{dx} + y = \frac{1+y}{x}$ is:

(a) $\frac{x}{e^x}$ (b) $\frac{e^x}{x}$ (c) xex (d) ex

33. $y = ae^{mx} + be^{-mx}$ satisfies which of the following differential equation?

(a) $\frac{dy}{dx} + my = 0$ (b) $\frac{dy}{dx} - my = 0$

(c) $\frac{d^2y}{dx^2} - m^2y = 0$ (d) $\frac{d^2y}{dx^2} + m^2y = 0$

34. The solution of the differential equation cosx siny dx + sinx cosy dy = 0 is :

(a) $\frac{\sin x}{\sin y} = c$ (b) sinx siny = c (c) sinx + siny = c (d) cosx cosy = c

35. The differential equation of the family of curves $x^2 + y^2 - 2ay = 0$, where a is arbitrary constant, is:

(a) $(x^2 - y^2)\frac{dy}{dx} = 2xy$ (b) $2(x^2 + y^2)\frac{dy}{dx} = xy$

(c) $2(x^2 - y^2)\frac{dy}{dx} = xy$ (d) $(x^2 + y^2)\frac{dy}{dx} = 2xy$

36. The solution of $x\frac{dy}{dx}+y=e^x$ is:

(a) $y=\frac{e^x}{x}+\frac{k}{x}$ (b) $y=xe^x+cx$ (c) $y=xe^x+k$ (d) $x=\frac{e^y}{y}+\frac{k}{y}$

37. Family y = Ax + A^3 of curves will correspond to a differential equation of order

(a) 3 (b) 2 (c) 1 (d) not defined

38. The general solution of $\frac{dy}{dx}=2xe^{x^2-y}$ is:

(a) $e^{x^2-y}=c$ (b) $e^{-y}+e^{x^2}=c$ (c) $e^y=e^{x^2}+c$ (d) $e^{x^2+y}=c$

39. The curve for which the slope of the tangent at any point is equal to the ratio of the abscissa to the ordinate of the point is :

(a) an ellipse (b) parabola (c) circle (d) rectangular hyperbola

40. The general solution of the differential equation $\frac{dy}{dx}=e^{\frac{x^2}{2}}+xy$ is:

(a) $y=ce^{\frac{-x^2}{2}}$ (b) $y=ce^{\frac{x^2}{2}}$ (c) $y=(x+c)e^{\frac{x^2}{2}}$ (d) $y=(c-x)e^{\frac{x^2}{2}}$

41. The solution of the equation (2y – 1) dx – (2x + 3)dy = 0 is :

(a) $\frac{2x-1}{2y+3}=k$ (b) $\frac{2y+1}{2x-3}=k$ (c) $\frac{2x+3}{2y-1}=k$ (d) $\frac{2x-1}{2y-1}=k$

42. The differential equation for which y = acosx + bsinx is a solution, is :

(a) $\frac{d^2y}{dx^2}+2y=0$ (b) $\frac{d^2y}{dx^2}-y=0$

(c) $\frac{d^2y}{dx^2}+(a+b)y=0$ (d) $\frac{d^2y}{dx^2}+(a-b)y=0$

43. The solution of $\frac{dy}{dx}+y=e^{-x}$,y (0) = 0 is:

(a) $y = e^{-x}(x-1)$ (b) $y = xe^x$ (c) $y = xe^{-x}+1$ (d) $y = xe^{-x}$

44. The order and degree of the differential equation $\left(\frac{d^3y}{dx^3}\right)^2-3\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^4=y^4$ are :

(a) 1, 4 (b) 3, 4 (c) 2, 4 (d) 3, 2

45. The order and degree of the differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]=\frac{d^2y}{dx^2}$ are :

(a) 2, 3/2 (b) 2, 3 (c) 2, 1 (d) 3, 4

46. The differential equation of the family of curves y^2 = 4a (x + a) is :

(a) $y^2=4\frac{dy}{dx}\left(x+\frac{dy}{dx}\right)$ (b) $2y\frac{dy}{dx}=4a$

(c) $y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0$ (d) $2x\frac{dy}{dx}+y\left(\frac{dy}{dx}\right)^2=y$

47. Which of the following is the general solution of $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=0$?

(a) y = (Ax + B)e^x (b) y = (Ax + B)e^{-x}
(c) y = Ae^x + Be^{-x} (d) y = Acosx + Bsinx

48. General solution of $\frac{dy}{dx}+y\tan x=\sec x$ is :

(a) y secx = tanx + c (b) y tanx = secx + c
(c) tanx = y tanx + c (d) x secx = tany + c

49. Solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=\sin x$ is:

(a) x (y + cosx) = sinx + c (b) x (y – cosx) = sinx + c
(c) xy cosx = sinx + c (d) x (y + cosx) = cosx + c

50. The general solution of the differential equation (e^x + 1) ydy = (y + 1) e^xdx is:
(a) (y + 1) = k (ex + 1) (b) y + 1 = ex + 1 + k
(c) y = log {k (y + 1) (ex + 1)} (d) $y=\log\left\{\frac{e^x+1}{y+1}\right\}+k$

51. The solution of the differential equation is :

(a) y = e^{x-y} – x^2 e^{-y} + c (b) e^y – e^x = $\frac{x^3}{3}$ + c

(c) e^x + e^y = $\frac{x^3}{3}$ + c (d) e^x – e^y = $\frac{x^3}{3}$ + c

52. The solution of the differential equation $\frac{dy}{dx}+\frac{2xy}{1+x^2}=\frac{1}{(1+x^2)^2}$ is :

(a) y (1 + x^2) = c + $\tan^{-1}$x (b) $\frac{y}{1+x^2}=c+\tan^{-1}x$
(c) y log (1 + x^2) = c + $\tan^{-1}$x (d) y (1 + x^2) = c + $\sin^{-1}$x

53. Order of the differential equation representing the family of parabolas y^2 = 4ax is __________ .

54. The degree of the differential equation $\left(\frac{dy}{dx}\right)^3+\left(\frac{d^2y}{dx^2}\right)^2=0$ is ________ .

55. The number of arbitrary constants in a particular solution of the differential equation tan x dx + tan y dy = 0 is __________ .

56. F (x, y) = $\frac{\sqrt{x^2+y^2}+y}{x}$ is a homogeneous function of degree__________ .

57. An appropriate substitution to solve the differential equation $\frac{dx}{dy}=\frac{x^2\log\left(\frac{x}{y}\right)-x^2}{xy\log\left(\frac{x}{y}\right)}$ is__________ .

58. Integrating factor of the differential equation $x\frac{dy}{dx}-y=\sin x$ is __________ .

59. The general solution of the differential equation $\frac{dy}{dx}=e^{x-y}$ is __________ .

60. The general solution of the differential equation $\frac{dy}{dx}+\frac{y}{x}=1$ is ___________ .

61. The differential equation representing the family of curves y = Asinx + Bcosx is ___________ .

62. $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dy}{dx}=1(x\neq 0)$ when written in the form $\frac{dy}{dx}+Py=Q$, then P = ___________ .

63. The number of arbitrary constants in the general solution of a differential equation of order three is __________.

64. The solution of differential equation coty dx = xdy is __________.

65. The solution of the differential equation ydx + (x + xy)dy = 0 is _______.

CHAPTER – 11: THREE DIMENSIONAL GEOMETRY

MARKS WEIGHTAGE – 11 marks

NCERT Important Questions

EXERCISE 11.1
- ☞ Q4
- ☞ Q5

EXERCISE 11.2
- ☞ Q6
- ☞ Q9
- ☞ Q12
- ☞ Q14
- ☞ Q15
- ☞ Q16
- ☞ Q17

EXERCISE 11.3
- ☞ Q5
- ☞ Q6
- ☞ Q9
- ☞ Q10
- ☞ Q11

MISC. EXERCISE.
- ☞ Q7
- ☞ Q9
- ☞ Q11
- ☞ Q12
- ☞ Q13
- ☞ Q14
- ☞ Q15
- ☞ Q17
- ☞ Q18
- ☞ Q19
- ☞ Q20

SOLVED EXAMPLES.
- ☞ Example 5 (Pg 467)
- ☞ Example 6 (Pg 469)
- ☞ Example 11 (Pg 476)
- ☞ Example 12 (Pg 476)
- ☞ Example 14 (Pg 480)
- ☞ Example 16 (Pg 481)
- ☞ Example 20 (Pg 486)
- ☞ Example 21 (Pg 488)
- ☞ Example 25 (Pg 492)
- ☞ Example 27 (Pg 495)
- ☞ Example 28 (Pg 495)
- ☞ Example 30 (Pg 497)

OBJECTIVE TYPE QUESTIONS (1 MARK)

1. The coordinates of the foot of the perpendicular drawn from the point (2, 5, 7) on the x-axis are given by
   (a) (2, 0, 0) (b) (0, 5, 0) (c) (0, 0, 7) (d) (0, 5, 7)

2. P is a point on the line segment joining the points (3, 2, –1) and (6, 2, –2). If x co-ordinate of P is 5, then its y co-ordinate is
   (a) 2 (b) 1 (c) –1 (d) –2

3. If α, β, γ are the angles that a line makes with the positive direction of x, y, z axis, respectively, then the direction cosines of the line are.
   (a) $\sin\alpha, \sin\beta, \sin\gamma$ (b) $\cos\alpha, \cos\beta, \cos\gamma$
   (c) $\tan\alpha, \tan\beta, \tan\gamma$ (d) $\cos^2\alpha, \cos^2\beta, \cos^2\gamma$

4. The distance of a point P (a, b, c) from x-axis is
   (a) $\sqrt{a^2+c^2}$ (b) $\sqrt{a^2+b^2}$ (c) $\sqrt{b^2+c^2}$ (d) b^2+c^2

5. The equations of x-axis in space are
   (a) x = 0, y = 0 (b) x = 0, z = 0 (c) x = 0 (d) y = 0, z = 0

6. A line makes equal angles with co-ordinate axis. Direction cosines of this line are

(a) $\pm(1,1,1)$ (b) $\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$ (c) $\pm\left(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right)$ (d) $\pm\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

7. Distance of the point (α, β, γ) from y-axis is
(a) β (b) $|\beta|$ (c) $|\beta|+|\gamma|$ (d) $\sqrt{\alpha^2+\gamma^2}$

8. If the directions cosines of a line are k, k, k, then
(a) k>0 (b) 0<k<1 (c) k=1 (d) $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$

9. The distance of the plane $\vec{r}.(\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k})=1$ from the origin is
(a) 1 (b) 7 (c) $\frac{1}{7}$ (d) None of these

10. The reflection of the point (α, β, γ) in the xy– plane is
(a) (α, β, 0) (b) (0,0, γ) (c) (–α,– β, γ) (d) (α, β,– γ)

11. The area of the quadrilateral ABCD, where A(0,4,1), B (2, 3, –1), C(4, 5, 0) and D (2, 6, 2), is equal to
(a) 9 sq. units (b) 18 sq. units (c) 27 sq. units (d) 81 sq. units

12. The locus represented by xy + yz = 0 is
(a) A pair of perpendicular lines (b) A pair of parallel lines
(c) A pair of parallel planes (d) A pair of perpendicular planes

13. The equation of a straight line parallel to the x-axis is given by
(a) $\frac{x-a}{1}=\frac{y-b}{1}=\frac{z-c}{1}$ (b) $\frac{x-a}{0}=\frac{y-b}{1}=\frac{z-c}{1}$
(c) $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$ (d) $\frac{x-a}{1}=\frac{y-b}{0}=\frac{z-c}{0}$

14. The plane 2x – 3y + 6z – 11 = 0 makes an angle $\sin^{-1}(\alpha)$ with x-axis. The value of α is equal to
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{7}$ (d) $\frac{3}{7}$

15. Which one of the following is best condition for the plane ax + by + cz + d = 0 to intersect the x- and y-axis at equal angle?
(a) |a| = |b| (b) a = -b (c) a = b (d) $a^2 + b^2 = 1$

16. If P(2, 3, -6) and Q(3, -4, 5) are two points, the direction cosines of the line PQ are
(a) $-\frac{1}{\sqrt{171}},-\frac{7}{\sqrt{171}},-\frac{11}{\sqrt{171}}$ (b) $\frac{1}{\sqrt{171}},-\frac{7}{\sqrt{171}},\frac{11}{\sqrt{171}}$
(c) $\frac{1}{\sqrt{171}},\frac{7}{\sqrt{171}},-\frac{11}{\sqrt{171}}$ (d) $-\frac{7}{\sqrt{171}},-\frac{1}{\sqrt{171}},\frac{11}{\sqrt{171}}$

17. The ratio in which yz-plane divides the line joining the points A(3, 1, -5) and B(1, 4, -6) is
(a) -3 : 1 (b) 3 : 1 (c) -1 : 3 (d) 1 : 3

18. A straight line is inclined to the axes of x and z at angles 45° and 60°, respectively, then the inclination of the line to the y-axis is
(a) 30° (b) 45° (c) 60° (d) 90°

19. The points (4, 7, 8), (2, 3, 4), (-1, -2, 1) and (1, 2, 5) are
(a) The vertices of a parallelogram (b) Collinear
(c) The vertices of a trapezium (d) Concyclic

20. The equation of the plane parallel to the plane 4x - 3y + 2z + 1 = 0 and passing through the point (5, 1, - 6) is
(a) 4x − 3y + 2z − 5 = 0 (b) 3x − 4y + 2z − 5 = 0
(c) 4x − 3y + 2z + 5 = 0 (d) 3x − 4y + 2z + 5 = 0

21. A plane is passed through the middle point of the segment A(-2, 5, 1) and B(6, 1, 5) and is perpendicular to this line. Its equation is
(a) 2x − y + z = 4 (b) 2x + y + z = 4
(c) x − 3y + z = 5 (d) x − 4y + 2z = 5

22. The sum of the direction cosines of a straight line is
(a) Zero (b) One (c) Constant (d) None of these

23. The equation of the plane that contains the line of intersection of the planes x + y + z - 6 = 0 and 2x + 3y + z + 5 = 0 and perpendicular to the xy-plane is
(a) x - 2y + 11 = 0 (b) x + 2y + 11 = 0
(c) x + 2y - 11 = 0 (d) x - 2y - 11 = 0

24. The planes: 2x – y + 4z = 5 and 5x – 2.5y + 10z = 6 are
(a) Perpendicular (b) Parallel (c) intersect y-axis (d) passes through $\left(0, 0, \frac{5}{4}\right)$

25. The image of the point (1, 3, 4) in the plane 2x - y + z + 3 = 0 is
(a) (-3, 5, 2) (b) (3, 2, 5) (c) (-5, 3, -2) (d) (-2, 5, 3)

26. Distance between the planes 2x + 3y + 4z = 4 and 4x + 6y + 8z =12 is
(a) 4 units (b) 8 units (c) $\frac{2}{\sqrt{29}}$ units (d) 2 units

27. If direction ratios of a line are proportional to 1, 2, –3; then its direction cosines are
(a) $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}$ (b) $-\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}$
(c) $-\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ (d) $\frac{1}{\sqrt{14}}, -\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}$

28. If a line makes and angle $\frac{\pi}{3}$ and $\frac{\pi}{4}$ with x-axis and z-axis respectively, then the angle made by the line with y-axis is
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{5\pi}{12}$

29. The lines $\frac{x-2}{1} = \frac{y+4}{2} = \frac{z-3}{3}$ and $\frac{x}{2} = \frac{y-1}{4} = \frac{z+3}{6}$ are
(a) skew (b) Parallel (c) intersecting (d) coincident

30. The straight line $\frac{x-3}{3}=\frac{y-2}{1}=\frac{z-1}{0}$ is

(a) parallel to x-axis (b) parallel to y-axis
(c) parallel to z-axis (c) perpendicular to z-axis

31. The equation of the plane which cuts equal intercepts of unit length on the coordinate axes is

(a) x + y – z = 1 (b) x + y + z = 0 (c) x + y + z = 2 (d) x + y + z = 1

32. The intercepts made by the plane 2x – 3y + 4z = 12 on the coordinate axes are

(a) 6, –4, 3 (b) 2, –3, 4 (c) $\frac{1}{6},-\frac{1}{4},\frac{1}{3}$ (d) 1, 1, 1

33. If a line has direction ratios 2, –1, –2 then its direction cosines are

(a) $\frac{2}{3},\frac{1}{3},\frac{2}{3}$ (b) $\frac{2}{3},\frac{1}{3},-\frac{2}{3}$ (c) $\frac{2}{3},-\frac{1}{3},-\frac{2}{3}$ (d) $-\frac{2}{3},-\frac{1}{3},-\frac{2}{3}$

34. If a line makes angles α, β, γ with the positive direction of co-ordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is

(a) 2 (b) 1 (c) –1 (d) –2

35. If a line makes angles 90^0, 60^0 and θ with x, y and z-axis respectively, where θ is acute angle, the value of θ is

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{6}$

36. If a line makes angles $\frac{\pi}{2},\frac{3\pi}{4}$ *and* $\frac{\pi}{4}$ with x, y, z axis, respectively, then its direction cosines are _______

37. If a line makes angles α, β, γ with the positive directions of the coordinate axes, then the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ is _______

38. The vector equation of the line passing through the points (3,5,4) and ________ is $\vec{r}=3\hat{i}+5\hat{j}+4\hat{k}+\lambda(2\hat{i}+3\hat{j}+7\hat{k})$

39. A plane passes through the points (2,0,0) (0,3,0) and (0,0,4). The equation of plane is __________.

40. The direction cosines of the vector $(2\hat{i}+2\hat{j}-\hat{k})$ are __________.

41. The vector equation of the line $\frac{x-5}{3}=\frac{y+4}{7}=\frac{z-6}{2}$ is __________.

42. The vector equation of the line through the points (3,4,–7) and (1,–1,6) is __________.

43. The cartesian equation of the plane $\vec{r}.(\hat{i}+\hat{j}-\hat{k})=2$ is __________.