ALMA MATER STUDIORUM · UNIVERSITÀ DI BOLOGNA

Scuola di Scienze
Corso di Laurea Magistrale in Fisica

Classical and quantum aspects of Black Hole dynamics

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Sessione II

Anno Accademico 2013/2014

Abstract

Il contenuto fisico della Relatività Generale è espresso dal Principio di Equivalenza, che sancisce L'EQUIVALENZA DI GEOMETRIA E GRAVITAZIONE. La teoria predice l'esistenza dei buchi neri, i più semplici oggetti macroscopici esistenti in natura: essi sono infatti descritti da pochi parametri, le cui variazioni obbediscono a leggi analoghe a quelle della termodinamica. La termodinamica dei buchi neri è posta su basi solide dalla meccanica quantistica, mediante il fenomeno noto come *radiazione di Hawking*. Questi risultati gettano una luce su una possibile teoria quantistica della gravitazione, ma ad oggi una simile teoria è ancora lontana.

In questa tesi ci proponiamo di studiare i buchi neri nei loro aspetti sia classici che quantistici. I primi due capitoli sono dedicati all'esposizione dei principali risultati raggiunti in ambito teorico: in particolare ci soffermeremo sui *singularity theorems*, le leggi della meccanica dei buchi neri e la radiazione di Hawking. Il terzo capitolo, che estende la discussione sulle singolarità, espone la teoria dei buchi neri non singolari, pensati come un modello effettivo di rimozione delle singolarità. Infine il quarto capitolo esplora le ulteriori conseguenze della meccanica quantistica sulla dinamica dei buchi neri, mediante l'uso della nozione di entropia di entanglement.

Contents

Introduction

General Relativity is the presently accepted theory of gravitation. Its physical content is contained in the Equivalence Principle:

Equivalence Principle. *Given a gravitational field, at any point in the spacetime it exists a* LOCAL *change of coordinates such that, in a sucfficently small neighborough of the point, the equations of the physical laws are those of Special Relativity, i.e. those valid in an inertial reference frame.*

The EP expresses the fact that gravity is locally equivalent to an inertial force. This is the reason why inertial mass and gravitational mass are the same quantity, as confirmed up to now by all the experiments. The principle induces to describe the precence of a gravitational field by replacing the Minkowski line element $ds^2 = \eta_{ab} dx^a dx^b$ with a more generic line element

$$ds^2 = g_{ab}(x) dx^a dx^b$$

such that the metric tensor field $g_{ab}(x)$ is locally equivalent, up to a dippheomorphism, to η_{ab}. If the equivalence holds globally, the spacetime is said to be *flat*: in this case there isn't any gravitational field, but one is simply describing things in a "wrong" reference frame. When a real gravitational field is present, the spacetime is said to be *curve*: gravity is nothing more than a curved spacetime.

What curves the spacetime? It is the matter itself which moves in it, and it does so according to Einstein's equations

$$R_{ab} - \frac{1}{2} R g_{ab} = \frac{8\pi G}{c^4} T_{ab}.$$

GR unifies geometry and physics: the gravitational field determines how the bodies move in the spacetime, and at the same time the distribution of matter determines how the gravitational field must be at each point. Observe that Einstein's equations are local: they preserve causality, i.e. two points of the spacetime can interact only if one is in the light cone of the other. However, the causal structure is not simple as in the flat case, because the spacetime curvature bends the light cones. We know for example that light is deflected when passes near a massive object.

A huge distortion of the light cones happens inside Black Holes. They are regions of spacetime, predicted by GR, where the gravitational field is so strong that even light cannot escape from there. Since nothing moves faster than light, nothing can escape. The existence of Black Holes has been accepted rather recently, when it was clarified that they form when a sufficiently massive body collapses under its own weight.

Parallely to their astrophysical study, black holes were also investigated from a theoretical point of view. A key result was the proof of the *singuarity theorems*: they showed that, when trapped surfaces form, they are always accompanied with singularities. A singularity is a point of the spacetime where Einstein's equations lose their predictive power, a fact intepreted as a breakdown of the theory itself. Analogous results state that the existence of the Big Bang singularity is very general, irrespective of the details of the way the universe is expanding. Singularity theorems are also notable for the methods used in the proof, the so called *global methods*: in essence, by the equivalence principle, gravity can be studied in a purely geometrical language, as the theory of the spacetime structure, with a little use of Einstein's equations[1]. In the forementioned situations, singularities occour when matter is compressed to high energy density. This circumstance suggests that when the densities are so high the theory should be modified. The need to modify General Relativity comes also from the fact that it is a classical theory, while all the other interactions are described by a quantum theory.

The suspect that General Relativity might emerge from a more fundamental reality became almost a certainty after the discovery of the laws of black hole mechanics. The dynamics of black holes is governed by laws analogous to that of thermodynamics; in particular, they possess an entropy and they emit particles at a temperature determined by purely geometrical factors. Remarkably, the particle emission is a consequence of the coupling of a quantum field with a time-asymmetric black hole geometry. However, if one takes seriously this prediction it seems that unitarity is manifestly violated, a fact known as *information loss paradox*.

The nature of information and entropy in black hole evolution has played a major role in the theoretical research of the last decades. In fact black holes constitute a playground for a synthesis of General Relativity, quantum mechanics and thermodynamics and they can shine a light on the deeper nature of the physical laws. In this work we want to review the main steps of black hole theory in General Relativity. We present both classical results (singularity theorems, laws of black hole mechanics) and quantum ones (Hawking radiation, entanglement entropy techniques). As a further discussion about singularity theorems, we review the theory of the socalled *nonsingular black holes*, which have been the subject of recent discussions in connection with the resolution of the information paradox.

[1] See Hawking and Ellis (1973).

Acknowledgments

I am grateful to Prof. Roberto Balbinot for having supported my university projects, with faith in my capacities and extremely good advices. Also, his teachings in General Relativity represented a fundamental background during the preparation of the present work.

I thank Prof. Carlo Rovelli, who welcomed me in his research group in Marseille, where I studied General Relativity and prepared this thesis under the excellent and irreplaceable supervision of Dott. Simone Speziale, from whom I hope to have learned how to develop my little ideas and organize my work in the most efficient way. I am also grateful to Hal Haggard and Goffredo Chirico for very useful discussions, and to Alejandro Perez both for discussions and for his advanced lectures on General Relativity.

I lived for six months in that stimulating atmosphere and I thank every permanent and visiting member of the group, because they made me feel in a big family.

My stay in Marseille was funded by the EU Erasmus Programme and by the University of Bologna. I thank the staff of all the libraries, offices and departments with which I interacted both in Bologna and in Marseille: in everything we do, there is a sign of their invisible but fundamental work.

I am grateful to Liliane Delmar, who rented out me a room in her house in Marseille, for her kindness and care that resembled those of my home.

Last but not least, I thank my family, my friends and those who are fond of me: they helped me not to lose my psychic equilibrium while studying physics.

Chapter 1

Black Holes

1.1 Schwarzschild Black Hole

1.1.1 The line element

Outside a static spherical self-gravitating astrophysical object, the spacetime is well approximately described by the line element (Hawking and Ellis, 1973, sec. 5.5)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \tag{1.1}$$

known as the *Schwarzschild metric*. Here $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ is the standard solid angle element.

The coordinates (t, r, θ, φ) have not to be interpeted literally, as the analogous ones in the Minkowski space, because General Relativity is invariant under dippheomorphisms and only observables have physical meaning. The system of coordinates in (1.1) is chosen in such a way to preserve the euclidean expression for the area of a two-sphere at fixed radial coordinate r:

$$A = 4\pi r^2. \tag{1.2}$$

Notice however that, for $M/r \ll 1$, (1.1) tends asymptotically to the flat Minkowski metric and then the parameter M is identified with the mass of the body measured at infinity: in this limit one approximately gives the standard physical meaning to the set of coordinates (t, r, θ, φ), thus making prediction of post-newtonian celestial mechanics (precession of perihelia, light deflection, etc.).

Typical astrophysical objects have radius R much greater then $2M$ (the so called Schwarzschild radius), so (1.1) describes the metric for $r > R$, and is matched in the region $r < R$ with a metric describing the interior of the star, planet or whatever it is. Nevertheless, we are interested in studying (1.1) as it covered the entire spacetime and the source of the gravitational field would be concentrated in the center.

The line element (1.1) is singular in $r = 0$ and $r = 2M$, but just the former is an essential singulartity, as revealed by the curvature invariant:

$$R_{abcd}R^{abcd} = \frac{48M^2}{r^6} \tag{1.3}$$

This suggests that it's possible to find an extended system of coordinates regularized in $r = 2M$. To this purpose, let's define the *tortoise coordinate*

$$r^* = r + 2M \log\left|\frac{r}{2M} - 1\right|. \tag{1.4}$$

Then

$$v = t + r^* \tag{1.5}$$

is the advanced null coordinate, while

$$u = t - r^* \tag{1.6}$$

is the retarded null one.

If one writes the metric (1.1) in terms of (v, r), instead of the pair (t, r), obtains the *advanced Eddington-Finkelstein extension*:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2. \tag{1.7}$$

This form of the metric is now regular in $r = 2M$, but the mixed term $dvdr$ generates a time-asimmetry. Actually, this asimmetry has a physical interpretation: for a time-like geodesic it must be $ds^2 < 0$, but for $r < 2M$ the inequality is satisfied only if $dr < 0$ [1]. So $r = 2M$ is a no-return surface: every observer who crosses it is condemned to fall towards the singularity $r = 0$, where he is scratched out by infinite tidal forces, and it can be shown that this happens in a finite proper time. For reasons to be clarified in a while, let's call this surface the *event horizon*.

What happens to light? For radially propagating light rays it must be

$$0 = ds^2 = -\left(1 - \frac{2M}{r}\right)dv^2 + 2dvdr$$

which has the two solutions

$$v = \text{const.} \qquad \text{ingoing null rays} \tag{1.8a}$$

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2M}{r}\right) \qquad \text{outgoing null rays} \tag{1.8b}$$

The first is trivial, since it's the way we defined the system of coordinates. From equation (1.8b) it's manifest that even an outgoing null ray cannot avoid to fall in for $r < 2M$, while on the surface $r = 2M$ it is frozen up. So, inside the event horizon, the light cone points inward and light cannot escape: no signal sent

[1] The temporal coordinate v grows along the affine parameter of the geodesic.

from the interior or from the horizon can reach a static external observer. For this reason, we will say that a Black Hole is present in the spacetime: the BH is the region $r < 2M$, while its boundary $r = 2M$ is called the event horizon, because it's the locus of the points from where no causal curves can reach the ouside.

One could have regularized the metric using the pair (u, r), thus obtaining the *retarded Eddington-Finkelstein extension*:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du^2 - 2du\,dr + r^2 d\Omega^2. \tag{1.9}$$

By repeating the same analysis as above, it results that (1.9) describes the opposite of a BH, i.e. a White Hole: in the WH region $r < 2M$ the light cones point outwards, so everything exits from a WH and, again, the boundary $r = 2M$ is called the event horizon. However, there is a crucial difference: in a BH the singularity is in the future and remains hidden to an observer who lives always outside; instead in a WH the singularity is in the past of every causal curve, so it afflicts seriously the predictability of the spacetime, like the Big Bang singularity. This is an unwanted feature and one would like to exclude the existence of such naked singularities.

1.1.2 Kruskal extension

So far we have extended the Schwarzschild solution using either the pair (v, r) or (u, r). We can use both the null coordinates (v, u) and obtain the line element

$$ds^2 = -\left(1 - \frac{2M}{r}\right)du\,dv + r^2 d\Omega^2 \tag{1.10}$$

with the inversion relations

$$t = \frac{u+v}{2} \tag{1.11a}$$

$$\mathrm{e}^{r/2M}\frac{r}{2M}\left|1 - \frac{2M}{r}\right| = \mathrm{e}^{(v-u)/4M} \tag{1.11b}$$

Let's define the *Kruskal coordinates*

$$V = 4M\mathrm{e}^{v/4M} \tag{1.12a}$$

$$U = \begin{cases} -4M\mathrm{e}^{-u/4M} & \text{if } r > 2M \\ +4M\mathrm{e}^{-u/4M} & \text{if } r < 2M \end{cases} \tag{1.12b}$$

from which

$$UV = 16M^2\mathrm{e}^{r/2M}\left(1 - \frac{r}{2M}\right) \tag{1.13}$$

and in terms of which the line element becomes

$$ds^2 = -\frac{2M}{r}\mathrm{e}^{-r/2M}dV\,dU + r^2 d\Omega^2. \tag{1.14}$$

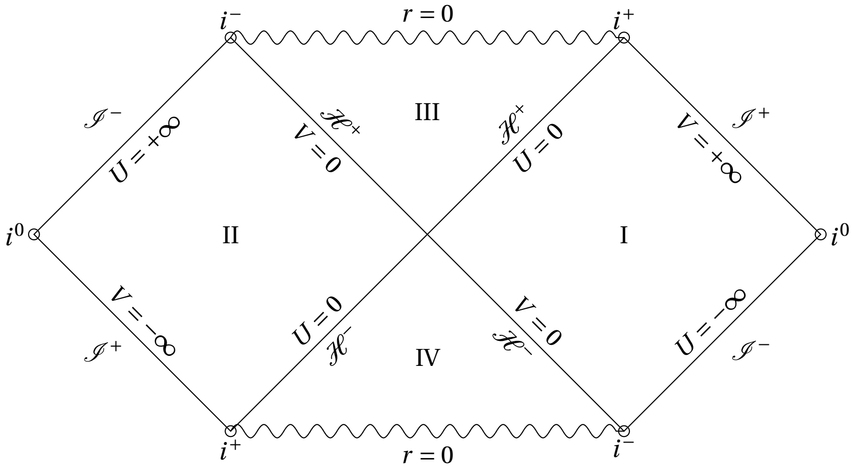

Figure 1.1: Conformal diagram of the Schwarzschild space-time.

From (1.11) and (1.12) the Kruskal coordinates are defined only in the range $V \in]0;+\infty[$ and $U \in]-\infty;0[$, but since the line element (1.14) is regular we can extend the range of definition of the coordinates to the full interval $V, U \in]-\infty;+\infty[$. The causal structure of the Kruskal line element is then represented in the conformal-compactified Penrose diagram (Wald, 1984; d'Inverno, 1998; Hawking and Ellis, 1973) of fig. 1.1.

The surface $r = 2M$ is null and corresponds to the lines $U = 0$ and $V = 0$: they consistutes the boundary of the BH (region III) and the WH (region IV). We shall distiguish two horizons:

- $\mathscr{H}^+$ is called the *future event horizon*, because it is the boundary of the past causal development of $\mathscr{I}^+$; every causal curve crossing $\mathscr{H}^+$ must fall into the singularity $r = 0$ of the BH;

- conversely, $\mathscr{H}^-$ is called the *past event horizon*, because it is the boundary of the future causal development of $\mathscr{I}^-$; every causal curve starting from the WH must cross $\mathscr{H}^-$ and can either reach the future infinity or fall into the BH.

While the advanced (resp. retarded) Eddington-Finkelstein extension contained only the BH (resp. WH) part of the spacetime, the Kruskal extension incorporates the whole causal structure and cannot be extended further, because every causal curve ends up at infinity or on an essential singularity. In fact, following (d'Inverno, 1998, p.230), we say that a spacetime $(\mathscr{M}, g_{ab})$ is *maximal* if every geodesic either can be extended for infinite values of the affine parameter in both directions or terminates on an essential singularity. Actually, the Kruskal system of coordinates (1.12) with the metric (1.14) is the only maximal extension of the Schwarzschild metric.

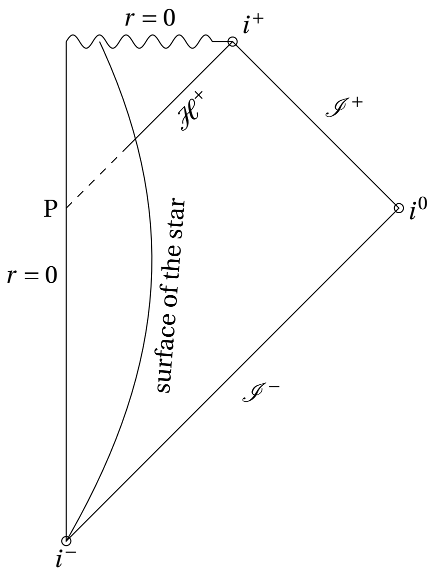

Figure 1.2: Conformal diagram of a collapsing star. Note that the event horizon forms at point P, i.e. before the surface of the star reaches the Schwarzschild radius.

1.1.3 Gravitational collapse

For a long time it was believed that stars with radius smaller than $2M$ couldn't exist, due to the strange behaviour of the BH region. The theoretical basis for the existence of BHs were given in the classical papers (Oppenheimer and Volkoff, 1939; Oppenheimer and Snyder, 1939), where the scenario is already presented in its essential features: a star is mantained in equlibrium by the positive pressure of the emitted radiation and the thermal energy of the burning nuclei, balanced by the negative pressure of the self-gravitational force; when the star exhausts its fuel, the only source of positive pressure is the Pauli exclusion principle; if the mass is greater than a critical limit, this pressure is non sufficient to counterbalance the gravitational force and the star collapses, under its own weight, until it reaches the Schwarzschild radius: at this point nothing can prevent the star to contracts indefinitely, and a BH forms[2].

The conformal diagram of a collapsing star is shown in fig. 1.2. Only the BH part of the original diagram is maintained, while the WH region is physically suppressed.

[2] For a better discussion on the lifecycle of a star see (Wald, 1984, p.132).

1.2 General properties

1.2.1 Introduction

So far we saw a specific solutions of the Einstein's equations describing a Black Hole, but BHs can be studied in a purely formal way without making reference to an explicit line element. This abstract approach led significant insights into BH theory, whose main achievements are:

1. the "no hair" theorem;
2. the BH singularity theorems;
3. the four laws of BH mechanics.

This section is focused on the latter two, though we are also going to spend some words on the first. Obviously, we don't pretend to be exaustive on the subject: our aim is just to offer a review with selected technical details[3].

First of all, we need a mathematical definition of what a BH is (Poisson, 2004; Wald, 1984). Consider a spacetime $(\mathcal{M}, g_{ab})$ and let's limit ourself to the case in which it is asymptotically flat. Intuitively, a BH is the locus of the points from wich a causal curve cannot escape to infinity: thus we say that a BH is present if $\mathcal{M}$ is not contained in $J^-(\mathscr{I}^+)$. The BH region is defined to be

$$\mathscr{B} = \mathcal{M} - J^-(\mathscr{I}^+)$$

and its boundary

$$\mathscr{H} = \dot{J}^-(\mathscr{I}^+) \cap \mathcal{M}$$

is called the *event horizon*. Observe that, since the event horizon is the boundary of the past of $\mathscr{I}^+$, it must be a null hypersurface.

From the defition it follows that $\mathscr{B}$ is a closed set: indeed, consider a point $q \in \mathscr{I}^+$ and another poit r lying after q w.r.t. the affine parametrization of $\mathscr{I}^+$; it is clear that for each $p \in J^-(q)$ one has $p \in I^-(r)$, so $\mathcal{M} \cap J^-(\mathscr{I}^+) \equiv \mathcal{M} \cap I^-(\mathscr{I}^+)$; it follows that $J^-(\mathscr{I}^+)$ is open, so $\mathscr{B}$ is closed. In particular $\mathscr{H} \subset \mathscr{B}$.

Before discussing the singularity theorems, let's give an account of the "no hair" theorem.

"No hair" theorem

"No hair" theorem concerns uniqueness of stationary axisymmetric BHs. A spacetime $\mathcal{M}$ is said to be *stationary* if it admits a Killing field ξ^a which is unit-timelike at infinity, while is said to be *axisymmetric* if admits a Killing field ϕ^a which

[3] For a review of the theoretical status of BH physics, we recommend the excellent Wald (2001). The "no hair" theorem is the main topic of Heusler (1996).

corresponds to rotations at infinity. One can strenghten stationarity, requiring staticity: a spacetime is said to be *static* if is stationary and, moreover, ξ^a is hypersurface orthogonal (this definition is equivalent to the statement that it exists a system of coordinate in which the metric g_{ab} both doesn't depend on the time coordinate t and is invariant under $t \to -t$).

A key result towards the statement of the theorem is that stationary BHs admit a Killing field χ^a which is normal to the event horizon; as a consequence χ^a must be null on the horizon because $\mathscr{H}$ is a null hypersurface: this fact is summarized by saying that the event horizon of a stationary BH is a *Killing horizon*. Moreover, if $\xi^a \neq \chi^a$, it is always possible to find a constant Ω_H such that

$$\chi^a = \xi^a + \Omega_H \phi^a \tag{1.15}$$

where Ω_H is said to be the angular velocity of the horizon. Hence a stationary BH must be axisymmetric (static in the limit $\Omega_H = 0$).

Then we can summarize the "no hair" theorem as follows (Poisson, 2004, p.205):

1. a vacuum static BH must be spherically symmetric, so by Birkhoff's theorem it must be a Schwarzschild BH;
2. a vacuum axysimmetric BH must belong to the Kerr family;
3. an electrovacuum axysimmetric BH must belong to the Kerr-Newmann family, which reduces to the Reissner-Nordstrom one in the static case.

Thus, if one neglects non-abelian fields, Kerr-Newmann BHs are the most general family of stationary BHs; actually, they are characterized by just three parameters: the mass M, the electric charge Q and the angular momentum J. However, we saw that a BH results from the gravitational collapse of a highly massive star, and in general a star can be highly nonspherical, so one can ask where the multipole moments - the "hair"- of the mass distribution end up after the collapse: the answer is that they are radiated away via gravitational and electromagnetic waves, so at the end what remains is a very simple object, pheraps the most simple macroscopic system existing in nature.

1.2.2 Singularity theorems

Before the proof of the singularity theorems, it was conceivable to think that singularities in General Relativity arise because of the high symmetry imposed to the solutions: for example, a future singularity emerges in the BH solutions, while a past singularity exists in the omogeneus and isotropic cosmology; one could suppose that, once more realistic asymmetric solutions had been considered, singularities would have disappeared. Now we know that this is not the case: singularity theorems show that both past and future singularities are not restricted to special classes of solutions, and they appear in very general situations.

In this thesis we are interested in future singularities inside BHs. We will proceed in this way: first we define in abstract what is meant by *singularity*; then we will introduce the tools of the so called *global methods*, i.e. equations and theorems about the global behaviour of causal curves in arbitrary spacetimes; finally we will proove Penrose's theorem, which shows the necessary occourence of singularities in gravitational collapse, and briefly discuss loopholes and generalizations.

So, how can we define a singularity? The question presents a number of subtleties (Wald, 1984, sec. 9.1) that we don't care about here, and we just adopt the ortodox view: a singularity is an event of the spacetime from which a causal curve cannot be extended further in the past or in the future, where the future direction is the one along which the affine parameter increases. Essentially, this definition captures the idea that a causal curve suddendly begins (past singularity) or ends (future singularity). Given that, the trick to prove the occurrence of a singularity will be to show that at least one geodesic cannot be extended somewhere: in fact, the singularity theorems just state the existence of a singularity, without saying anything about its nature and location. Let us then introduce the tools that will allow us to draw such conclusions.

Raychaudhuri's equation

Consider an open set $O \in \mathscr{M}$. A *congruence* in O is a family of curves such that, for each $p \in O$, there passes one and only one curve: so the tangents to the curves generate a vector field and, conversely, every vector field generate a congruence. A congruence is said to be spacelike, timelike or null if the associated vector field is respectively spacelike, timelike or null. Since we want to study the behaviour of causal curves, we are interested in timelike and null congruences; moreover form now on we assume that the timelike vector field associated with a timelike congruence is normalized to -1.

Consider a congruence of timelike **geodesics**, and label by k^a the tangent to one of them. We ask: how does an infinitesimally close geodesic change, relatively to k^a, along the affine parameter? Such new geodesic has tangent vector $k^a + X^a$, where X^a is the displacement vector, and one has:

$$\frac{dX^a}{d\tau} = k^b \nabla_b X^a \overset{4}{=} X^b \nabla_b k^a = B^a_b X^b \tag{1.16}$$

thus X^a changes linearly by the action of the tensor $B_{ab} = \nabla_b k_a$. Eq. (1.16) is analogous to that of linear deformations in a continuous medium (Poisson, 2004, sec. 2.2), and in fact we can decompose B_{ab} in its irreducible components:

$$B_{ab} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} \tag{1.17}$$

where

$$\theta = B^{ab} g_{ab} \qquad \text{expansion} \tag{1.18a}$$

$$\sigma_{ab} = B_{(ab)} \qquad \text{shear} \tag{1.18b}$$

$$\omega_{ab} = B_{[ab]} \qquad \text{twist} \tag{1.18c}$$

and

$$h_{ab} = g_{ab} + k_a k_b$$

is the transverse metric tensor to k^a.

In particular, the expansion θ is the relative variation of the volume of a small neighborhood around k^a:

$$\theta = \frac{1}{V}\frac{dV}{d\tau}.$$

From the geodesic equation and the normalization of k^a it follows that

$$0 = B_{ab} k^a = B_{ab} k^b$$

and analogous equations for σ_{ab} and ω_{ab}.

All these quantities appear in the fundamental equation used in the proof of the singularity theorems, Raychaudhuri's equation:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} + \omega^{ab}\omega_{ab} - R_{ab}k^a k^b. \tag{1.19}$$

Eq. (1.19) can be simplified under the assumption that the congruence is hypersurface orthogonal: indeed it is a result of differential geometry that a vector field is hypersurface orthogonal if and only if $\omega_{ab} = 0$. Thus we obtain

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^{ab}\sigma_{ab} - R_{ab}k^a k^b. \tag{1.20}$$

Now, $\sigma^{ab}\sigma_{ab}$ is manifestly positive because the signature of the transverse metric is (+++); if we also require the strong energy condition (see Appendix B) we end up with the following inequality:

$$\frac{d\theta}{d\tau} \leq -\frac{1}{3}\theta^2. \tag{1.21}$$

[4] Recall: $\pounds_k X^a = 0$, because we are using the flows of k^a and X^a as independent coordinates.

Eq. (1.21) can be integrated and the result is

$$\theta(\tau) \leq \frac{3\theta_0}{3 + (\tau - \tau_0)\theta_0}. \tag{1.22}$$

If θ_0 is negative, we see that $\theta(\tau) \to -\infty$ at most when $(\tau - \tau_0) \to 3/|\theta_0|$, so we can state the following

Theorem 1.2.1. *Let k^a be the tangent vector field of a hypersurface orthogonal congruence of timelike geodesics, and let $R_{ab}k^a k^b \geq 0$, (as it is the case if matter satisfies the SEC); if the expansion assumes negative value θ_0 at some point, then $\theta(\tau) \to -\infty$ within finite proper time $\Delta\tau \leq 3/|\theta_0|$.*

These results can be generalized to a congruence of null geodesics: for our purposes, the only difference[5] is that the transverse manifold is 2-dimensional with signature (++), and in particular the expansion measures the relative change of the transverse area:

$$\theta = \frac{1}{A}\frac{dA}{d\tau}.$$

So we can extend the previous theorem to null geodesics:

Theorem 1.2.2. *Let k^a be thet tangent vector field of a hypersurface orthogonal congruence of null geodesics, and let $R_{ab}k^a k^b \geq 0$, (as it is the case if matter satisfies the SEC or the WEC); if the expansion assumes negative value θ_0 at some point, then $\theta(\tau) \to -\infty$ within finite affine parameter $\Delta\lambda \leq 2/|\theta_0|$.*

Conjugate points

In a flat manifold the geodesic connecting two points p and q, if it exists, is unique and extremizes the proper lenght; conversely, in a curved manifold there can be more than one geodesic connecting two points, and the one which extremizes the proper lenght doesn't necessarily exist: for example, the north and south pole of a sphere are connected by inifinitely many geodesics, all with the same proper lenght.

Two infinitesimally close geodesics connecting the points p and q define a deviation vector X^a, which vanishes at both p and q. When two points are connected by two (at least infinitesimally close) geodesics, they are said to be *conjugate points*. We can also define the notion of a point p conjugated with a smooth, embedded hypersurface Σ:

Definition 1.2.1. *Consider a smooth, embedded hypersurface Σ and a congruence of geodesics orthogonal to Σ. A point p is said to be cojugate to Σ along the geodesic γ of the congruence, if there exists an orthogonal deviation vector X^a to γ which is nonzero on Σ but vanishes at p.*

[5] See however (Wald, 1984, p.221).

A conjugate point is a focusing point for the geodesics, namely a point where $\theta \to -\infty$. From theorems 1.2.1 and 1.2.2 one would expect that, if the expansion is negative on Σ, then the congruence will necessairily develop conjugate points. Indeed this is the case, both for timelike and null congruences:

Theorem 1.2.3. *Let $(\mathscr{M}, g_{ab})$ be a spacetime such that $R_{ab}k^a k^b \geq 0$ for all timelike k^a. Let Σ be a smooth spacelike 3-dimensional hypersurface with negative expansion $\theta < K < 0$. Then every timelike geodesic γ orthogonal to Σ develops a conjugate point within proper time $\tau \leq 3/|K|$, assuming that γ can be extended that far.*

In the case of null geodesics we must make a distinction: a 2-dimensional spacelike hypersurface S possesses two null orthogonal vector, that we arbitrarily label as *ingoing* and *outgoing*; correspondingly there will be an ingoing expansion and an outgoing expansion. Given this caveat, one has the following

Theorem 1.2.4. *Let $(\mathscr{M}, g_{ab})$ be a spacetime such that $R_{ab}k^a k^b \geq 0$ for all null k^a. Let S be a smooth spacelike 2-dimensional hypersurface such that the expansion of, say, the outgoing null geodesic at a point $q \in S$ takes the negative value θ_0. Then within proper lenght $\lambda \leq 2/|\theta_0|$ there exists a point p conjugate to S along the outgoing null geodesic γ passing through q, assuming that γ can be extended that far.*

After having established under what conditions geodesics have conjugate points, let's return to the question whether there exists a geodesic which extremizes the proper lenght between two points. In fact the two things are related by the following theorems:

Theorem 1.2.5. *Let Σ be a smooth 3-dimensional spacelike hypersurface. The necessary and sufficient condition that a smooth timelike curve γ, connecting a point $q \in \Sigma$ with a point $p \in \mathscr{M}$, maximizes the proper time between p and q is that γ be a geodesic orthogonal to Σ in q, with no conjugate points between p and q.*

Theorem 1.2.6. *Let S be a smooth 2-dimensional spacelike hypersurface. The necessary and sufficient condition that a smooth causal curve γ, connecting a point $q \in S$ with a point $p \in \mathscr{M}$, cannot be smoothly deformed to a timelike curve is that γ be a null geodesic orthogonal to S in q, with no conjugate points between p and q.*

Theorem 1.2.5 just contains the conditions for the existence of a maximum proper time curve, but it doesn't say if it effectively exists: actually, a result of differential topology is that such a curve certainly exists if $(\mathscr{M}, g_{ab})$ is globally hyperbolic.

Global hyperbolicity

The causal structure that we build our intuition on is that of the Minkowski spacetime: here casual curves are regular and we can apply our naive feeling with a great confidence. The notion of global hyperbolicity allows to preserve many of the intuitive behaviour of causal curves even in curved spacetimes. It plays an important role in the proof of the Penrose's theorem and the discussion on nonsingular BHs, thus we want to remark what properties of global hyperbolicity are used. We could have done it during the respective discussions, but it seems easy to group all the comments in a single paragraph; it will be also the place where to give some useful terminology.

Let S be a closed and achronal set. The *edge* of S is the set of points $p \in S$ such that in a neighborhood of p there are $q \in I^+(S)$ and $r \in I^-(S)$ which can be connected by a timelike curve without intersecting S. A closed achronal set S with $\mathrm{edge}(S) = \emptyset$ is said a *slice*.

A spacetime $(\mathscr{M}, g_{ab})$ is globally hyeprbolic if there exists a slice Σ such that $\mathscr{M} = D^-(\Sigma) \cup D^+(\Sigma)$, i.e. $\mathscr{M}$ coincides with the causal domain of Σ. The slice Σ is said to be a Cauchy surface for $\mathscr{M}$.

It results that a globally hyperbolic spacetime has no closed timelike curves; moreover a global time function t can be choosen (more precisely, a global timelike vector field t^a)[6] such that hypersurfaces of constant t are Cauchy surfaces and the spacetime has topology $\mathscr{M} \cong \mathbb{R} \times \Sigma$, where Σ is any Cauchy surface. This corresponds to the idea that the history of the spacetime can be entirely predicted by an initial value surface at *constant time t*. Observe that in a globally hyperbolic spacetime THERE CANNOT BE A CHANGE OF TOPOLOGY, because the topology is once for all determined by any of the foliating slices Σ_t.

A property of global hyperbolic spacetimes, crucial for the proof of the Penrose's theorem, is the future causal simplicity. Given a point $p \in \mathscr{M}$, its future light cone $E^+(p)$ doesn't coincide with $\dot{I}^+(p)$, the boundary of its future causal development. In fact $E^+(p) \subseteq \dot{I}^+(p)$, because $I^+(p)$ is not necessarily closed. A spacetime is said to be *future causally simple* if $E^+(p) \equiv \dot{I}^+(p)$.

Actually, in a global hyperbolic spacetime $\dot{I}^+(p)$ is closed, so the spacetime is future causally simple. Note that it doesn't mean that $\dot{I}^+(p)$ is open and/or unbounded!

Future causally simplicity extends also to compact submanifolds and leads to the following

Theorem 1.2.7. *Let $(\mathscr{M}, g_{ab})$ be a globally hyperbolic spacetime and $K \subset \mathscr{M}$ a 2-dimensional compact spacelike hypersurface. Then every point $p \in \dot{I}^+(K)$ lies on*

[6] This fact is expressed by saying that the spacetime is *time orientable*.

a future directed null geodesic orthogonal to K with no conjugate points between K and p.

Proof. From the future causal simplicity every $p \in \dot{I}^+(K)$ lies on a null curve γ connecting p and K. If γ is not a null geodesic orthogonal to K or has conjugate points, then from Theorem 1.2.6 it is possible to deform γ in a timelike curve, i.e. $p \in I^+(K)$, which is a contraddiction. □

Penrose's theorem

As said before, the singularities appearing in BH solutions are not an artifact of the special symmetries imposed, but represent general features of BH theory. This was historically understood throughout Penrose's theorem (Penrose, 1965): it states, under certain reasonable assumptions, that a singularity necessairily forms when a body collapses under its own gravitational pressure. In order to use the tools of the previous paragraphs, we must formalize in geometrical terms the notion of a collapsing body.

Suppose that an event horizon has formed and inside it the surface of the star is contracting. In the BH region the gravitational field is so strong that even the "outgoing" light rays cannot avoid to fall towards the origin. So we are led to the following

Definition 1.2.2 (Trapping surface)**.** *A trapping surface is a smooth compact 2-dimensional spacelike surface such that both the ingoing and outgoing expansions of future directed null geodesics are negative:*

$$\theta^{in}, \theta^{out} < 0.$$

So a gravitational collapse is a phenomenon in which trapped surfaces form inside the event horizon. Now we are ready to formulate Penrose's theorem[7].

Theorem 1.2.8 (Penrose 1965)**.** *Let $(\mathscr{M}, g_{ab})$ be a spacetime containing a trapping surface $\mathscr{T}$. Then the following conditions cannot be simultaneously true:*

1. *$R_{ab}k^a k^b \geq 0$ for all null k^a (as it is the case if matter satisfies WEC or SEC);*
2. *there exists a non-compact Cauchy surface Σ;*
3. *the spacetime is null geodesically complete, i.e. every null geodesic is extendible to infinite values of the affine parameter.*

Proof. The proof proceeds in two steps: first it's shown that $\dot{I}^+(\mathscr{T})$ is compact, then this is shown to be incopatible with the non-compactness of Σ.

Let's define a function $\Psi^{\text{in}} : \mathscr{T} \times (0, a)$ such that $\Psi^{\text{in}}(p, b)$ is the point lying on the null future directed ingoing geodesic, starting from $p \in \mathscr{T}$, at an affine

[7] We follow Hawking and Ellis (1973).

distance b from p. Similarly, define Ψ^{out} for outgoing null geodesics. Finally, let be $\Psi(\mathscr{T}, a) = \Psi^{in}(\mathscr{T}, a) \cup \Psi^{out}(\mathscr{T}, a)$.

Since the spacetime is null geodesically complete, every future directed null geodesic starting from $p \in \mathscr{T}$, by Theorem 1.2.4, develops a conjugate point within proper distance $2/|\theta(p)|$. Hence, by Theorem 1.2.7, $\dot{I}^+(\mathscr{T}) \subset \Psi\big(\mathscr{T} \times \big[0, \frac{2}{|\theta_0|}\big]\big)$, where θ_0 is the minimum value of the expansion on $\mathscr{T}$. So $\dot{I}^+(\mathscr{T})$ is closed and bounded, i.e. is compact.

Let's show how this leads to a contraddiction. Indeed global hyperbolicity implies the existence of a timelike vector field t^a which, by definition, intersects Σ exactly once, while intersects $\dot{I}^+(\mathscr{T})$ at most once because $\dot{I}^+(\mathscr{T})$ is achronal. We can use t^a to map points of $\dot{I}^+(\mathscr{T})$ into points of Σ: let's call $S(\mathscr{T})$ the image of this map. $S(\mathscr{T})$ is compact.

The set $S(\mathscr{T})$ is closed if viewed as a subset of Σ. Nevertheless, a result of differential topology assures that in a time-orientable spacetime causal boundaries are C^0 sets: this implies that $S(\mathscr{T})$ is opened if viewed as a subset of Σ. Thus $S(\mathscr{T}) \equiv \Sigma$. But this is impossible because Σ in non-compact. □

Some comments are necessary. The theorem states the necessary occourrence of a singularity only if one accepts conditions 1 and 2. Condition 1 seems physically reasonable in a classical context, where either WEC or SEC are believed to hold. Condition 2 is more delicate. Non-compacteness of Σ seems to suggest that the theorem is valid only in an open universe; however, as pointed out in Hawking and Ellis (1973), it was used only to show that $S(\mathscr{T})$ is not the full Σ, so it is sufficient to require that it exists at least one timelike geodesic from Σ not intersecting $\dot{I}^+(\mathscr{T})$, i.e. escaping the BH: this is the case in a conventional BH scenario. So, what appears to be the most critical requirement is the existence of Σ itself. Thus the correct conclusion of Penrose's theorem is that, under assumptions 1-3, either a singularity forms or the spacetime is not globally hyperbolic.

The global hyperbolicity can be removed if one strenghtens condition 1 by requiring SEC for timelike vectors- see Hawking and Ellis (1973),Wald (1984). We will see that nonsingular BHs, if they exist, must necessairily violate SEC.

1.2.3 Laws of BH mechanics

The laws of BH mechanics were presented in Bardeen et al. (1973) and the authors recongized a strict analogy with the laws of thermodynamics of ordinary systems. As we shall see in the next chapter, this analogy has been established to be an identity, and the considerations involved to reach this conclusion make use of quantum field theory. So the laws of BH mechanics put a bridge between general relativity, quantum mechanics and thermodynamics in a very suggestive way. Rather than providing a technical proof of all the four laws, we prefer to concetrate on their physical meaning.

The zeroth law

We saw that any stationary BH admits a Killing field χ^a normal to the event horizon, eq. (1.15):

$$\chi^a = \xi^a + \Omega_H \phi^a$$

with vanishing norm on the horizon itself. It follows that also $\nabla^a(\chi^b\chi_b)$ is normal to the horizon, and then it exists a function κ on the horizon such that

$$\nabla_a(\chi^b\chi_b) \overset{\mathcal{H}}{=} -2\kappa\chi_a. \tag{1.23}$$

κ is called the *surface gravity* of the horizon. The name comes from the following reason: outside the horizon, the acceleration of an observer with 4-velocity

$$u^a = \frac{\chi^a}{\chi} \qquad \chi = (-\chi^a\chi_a)^{\frac{1}{2}}$$

is

$$a_b = u^a\nabla_a u_b = \frac{\nabla_b\chi}{\chi} \overset{\mathcal{H}}{\to} \frac{\kappa\chi_b}{\chi^2}$$

where in the last step the limit as one approaches the horizon is understood. Then

$$\kappa \overset{\mathcal{H}}{\to} \chi a. \tag{1.24}$$

In the stationary case, $\chi^a = \xi^a$, χ is the redshift factor and therefore χa is the force per unit mass that an observer at infinity must apply, to held a body at rest in a point of the space, i.e. the surface gravity of that point. So κ, in the stationary case, is the surface gravity of the horizon. In the non-stationary case this is not true, but we mantain the name "surface gravity".

κ is also related to the affine parametrization of the horizon. Using the defining equation of a Killing field, $\nabla_a\chi_b + \nabla_b\chi_a = 0$, eq. (1.23) becomes

$$\chi^b\nabla_b\chi_a = \kappa\chi_a. \tag{1.25}$$

This equation shows that the orbits of χ^a are not affinely parametrized geodesics on the horizon. Calling v the non-affine parameter defined by $\chi^a\nabla_a v = 1$, an affine parametrization is given by the vector

$$\left(\frac{\partial}{\partial V}\right)^a = k^a = \mathrm{e}^{-\Gamma}\chi^a \tag{1.26}$$

where $\chi^a \nabla_a \Gamma = \kappa$.

Now we are ready to give the statement of the zeroth law:

Theorem (Zeroth law)**.** *Given a stationary black hole, the surface gravity κ is constant over each connected region of the black hole horizon.*

In its original formulation, the proof of the zeroth law assumed the dominant energy condition; this restriction was subsequently removed by Rácz and Wald (1996).

Incidentally, the constancy of κ simplifies the affine parametrization of the horizon to

$$k^a = \mathrm{e}^{-\kappa v}\chi^a. \tag{1.27}$$

The surface gravity of a Schwarzschild BH is

$$\kappa = \begin{cases} \frac{1}{4M} & \text{on } \mathscr{H}^+ \\ -\frac{1}{4M} & \text{on } \mathscr{H}^- \end{cases}$$

so we see that the Kruskal coordinates are exactly the affinely parametrizing coordinates of the past and future horizons.

The first law

There are several derivations of the first law (Wald, 1994). The first to be performed was the *equilibrium state version*, in which the equilibrium configurations of two BHs, differing for infinitesimal variations of the parameters, are compared. Here we present the *physical process version*, in which one imagines to throw an infinitesimal amount of matter through the BH horizon. We suppose that, if the BH was originally at equilibrium, it settles to another equilibrium state after having absorbed the perturbation. For simplicity, suppose to perturbe a rotating BH in the vacuum, such that the zero order stress energy tensor is null: $T_{ab} = 0$. Then the perturbed stress energy tensor δT_{ab} satisfies $\nabla^a \delta T_{ab} = 0$. From the Killing vectors ξ^a and ϕ^a we identify the conserved currents

$$\mathscr{E}^a = -\delta T^{ab}\xi_b \qquad \text{mass-energy current} \tag{1.28a}$$

$$\mathscr{F}^a = \delta T^{ab}\phi_b \qquad \text{angular momentum current} \tag{1.28b}$$

Integrating them over the whole horizon we obtain the first-order changes in mass and angular momentum of the BH:

$$\delta M = \int_{\mathscr{H}} \mathscr{E}^a d\Sigma_a = \int_{\mathscr{H}} k^a \xi^b \delta T_{ab} dV dS \tag{1.29a}$$

$$\delta J = \int_{\mathscr{H}} \mathscr{F}^a d\Sigma_a = -\int_{\mathscr{H}} k^a \phi^b \delta T_{ab} dV dS \tag{1.29b}$$

where to first order in δT_{ab} we have neglected the changes in κ and Ω_H. We have chosen to parametrize the horizon with the affine parameter V; the affine normal vector k^a is related to the Killing field χ^a by $\chi^a = \kappa V k^a$.

Now we have

$$\begin{aligned} \delta M - \Omega_H \delta J &= \int_{\mathscr{H}} k^a (\xi^b + \Omega_H \phi^b) \delta T_{ab} dV dS \\ &= \int_{\mathscr{H}} k^a \chi^b \delta T_{ab} dV dS \\ &= \kappa \int_{\mathscr{H}} k^a k^b \delta T_{ab} V dV dS. \end{aligned} \tag{1.30}$$

From Raychaudhuri's equation, neglecting second order terms,

$$\frac{d\theta}{dV} = -R_{ab} k^a k^b = -8\pi \delta T_{ab} k^a k^a \tag{1.31}$$

and substituting in the previous integral we obtain

$$\begin{aligned} \delta M - \Omega_H \delta J &= -\frac{\kappa}{8\pi} \int_{\mathscr{H}} \frac{d\theta}{dV} V dV dS \\ &= \frac{\kappa}{8\pi} \int_{\mathscr{H}} \theta dV dS \end{aligned} \tag{1.32}$$

where in the last line we integrated by parts and used the fact that the BH is in equilibrium both at the beginning and at the end of the process (so θ vanishes at the extrema of integration for V). Since θ is the expansion of the horizon, its integration over $\mathscr{H}$ gives the change in the area δA. Therefore:

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J. \tag{1.33}$$

Eq. (1.33) is the first law of BH mechanics for a vacuum rotating black hole. The second therm on the r.h.s. is a work term: it gives the change of rotational energy. When the BH is not in the vacuum, one must modify the first law adding the work terms corresponding to the fields filling the spacetime. For example, the first law for a charged rotating BH is

$$\delta M = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J + \Phi_H \delta Q \tag{1.34}$$

where δQ is the change in the BH electric charge and Φ_H is the electrostatic potential on the horizon.

Observe that if the NEC holds ($\delta T_{ab} k^a k^b \geq 0$) then $\delta A \geq 0$. This is a particular case of the most general second law of BH mechanics.

The second law

Theorem (Second law)**.** *In any classical dynamical process, the area of the black hole horizon never decreases, provided the null energy condition holds.*

The second law is valid both if the BH is a single connected region and if it's composed by several disjoint connected regions. It can be violated in presence of negative energy densities.

The second law is related to the notion of *irreducible mass* of a black hole. In fact one can extract energy from a rotating black hole by means of classical processes: Penrose's process and superradiance- see Wald (1984). This energy is extracted at the expence of the rotational energy of the black hole, so the processes can be carried on until $J \neq 0$; when $J = 0$ the BH has no angular momentum and no more energy can be extracted. The fact that one cannot extract all the energy from a black hole is expressed by the existence af a quantity that never decreases, the irreducible mass: $\delta M_{\text{irr}} \geq 0$. Remarkably, M_{irr} turns out to be proportional to the area A of the black hole:

$$M_{\text{irr}} = \frac{A}{16\pi}.$$

Observe that the mass of a Scwarzschild black hole saturates M_{irr}, i.e. no energy can be gained from it.

The third law

The status of the third law is not as rigorous as the previous ones. It states that it is impossible, starting from a black hole with $\kappa \neq 0$, to reduce κ to 0 in a finite amount of time. Actually, it was just postulated but not proven in Bardeen et al. (1973). First evidences for this law were obtained through thought experiments (Wald, 1974): they consisted into throw into a charged rotating BH particles whose energy, charge and angular momentum were calibrated in such a way to reduce κ; as a result, the more κ goes to 0, the more the rate at wich it decreases becomes negligible. Israel (1986) gave a formal proof of the validity of the third law when the WEC is satisfyed.

However, WEC is violated in the quantum context: in fact, as we shall see in the next chapter, quantum particle creation near the horizon causes negative energy quanta to fall into the BH, consequently shrinking it: this mechanism can in principle reduce the surface gravity of a charged rotating BH to zero. Semiclassical analysis show that this dones't happen and that the third law holds even in the quantum case. But, as far I am aware, there isn't a conclusive proof ouside the semiclassical regime.

Some authors, motivated by the no hair theorem, state that the third law should be true in order to prevent naked singularities. Even if in principle this isn't logically wrong, we don't think that the third law has anything to do with naked singularities: indeed in the following we will construct a non-singular black hole and, by semiclassical analysis, will verify the validity of the third law (Appendix C.2).

Black hole thermodynamics Bardeen et al. (1973) noticed that the four laws are formally analogous to the laws of thermodynamics: the zeroth law corresponds to the fact that the temperature is constant at thermal equilibrium; the second law reminds the non-decreasing character of the entropy of isolated systems; the first law looks like the first principle of thermodynamics; finally the third law is similar to the Nerst-Simon law of thermodynamics. The following table displays these analogies:

Law	Black holes	Thermodynamics
Zeroth	$\kappa = \text{const.}$	$T = \text{const.}$
First	$\delta M = \frac{\kappa}{8\pi}\delta A + \delta W$	$\delta E = T\delta S + \delta W$
Second	$\delta A \geq 0$	$\delta S \geq 0$
Third	Impossibility of reaching $\kappa = 0$	Impossibility of reaching $T = 0$

In the original paper the authors were very cautious in underlying the correspondence between BH physics and thermodynamics. In fact, if one remains in classical physics, this correspondence ultimately fails. If one instead takes quantum mechanics into account, the analogy can be promoted to an equivalence! This is the subject of the next chapter.

Chapter 2

Quantum effects

2.1 Introduction

Although general relativity is a consistent theory of spacetime and appears to be in very good agreement with known experiments and experiences, it leaves us unsatisfied because it's not a quantum theory. The problem can be phrased in this way: General Relativity certainly describes gravitational phenomena at large scales, i.e. lenght and energy scales much larger than those studied in particle physics, but it seems inadequate to describe the spacetime structure at small scales. This inadequacy is due, for example, to the fact that the theory predicts singularities when matter is compressed at huge densities in a microscopic volume; a singularity can be viewed as a breakdown of the theory itself, so it seems the case that GR should be modified at small lenghts.

Moreover, quantization has became a paradigm in fundamental physics and the need for quantum gravity mainly comes from analogies with the other fundamental forces, which are successfully described by QFT in flat spacetime. However, there are no experiments showing evidences of a quantum structure of the spacetime, so up to now the motivations are purely theoretical. After almost ninety years of attempts, we still don't have any quantum theory of gravitation. The framework of QFT has been proven to be not naively appliable to GR, because the resulting quantum theory is not renormalizable. This result motivated alternative approaches, the main of which are: string theory, loop quantum gravity, euclidean path integral methods and renormalization goup approaches (see Kiefer (2006) and references therein for a brief review).

The scale at which the quantum nature of spacetime should manifest can be extimated by considering that the fundamental constants involved are G (gravitation), c (relativity) and $\hbar$ (quantum mechanics). They can be combined in only one way to obtain a lenght, namely the Planck lenght:

$$l_{\text{planck}} = \sqrt{\frac{G\hbar}{c^3}} \approx 10^{-35} m$$

and similarly to obtain a mass:

$$m_{\text{planck}} = \sqrt{\frac{\hbar c}{G}} \approx 10^{19} GeV/c^2.$$

However none of the theories of quantum gravity has been yet falsified nor confirmed by the experiments.

Parallely, another subject has been developed to investigate quantum effects of gravitation: quantum field theory in curved spacetimes. It consists on studying the dynamics of a matter field coupled to gravity, but while the field is treated quantum mechanically, gravity is treated classically as a background metric. This method is very similar to the atomic theory of Schrodinger, in which the electron is described by a wave function, while the electomagnetic field is approximated by the classical electrostatic potential. QFT in curved spacetimes proceeds in the same fashion. Obviously this is just an approximation valid when the spacetime curvature is much less than planckian: in this regime the quantum effects are small and one can neglect the dynamics induced on the spacetime itself. The main result of this approach is the Hawking radiation, which is the subject of this chapter: first, following Hawking (1975) and Wald (1975), we derive the Hawking radiation in a simple situation, then we deal with the information loss paradox. For QFT in curved spacetime we refer to (Birrell and Davies, 1984; Wald, 1994). For the Vaidya approximation we refer to Fabbri and Navarro-Salas (2005).

2.2 Hawking radiation

2.2.1 Black hole entropy

Bekenstein (1973) suggested that the analogy between the laws of black hole mechanics and those of thermodynamics was an identity and in particular he suggested that a BH has an entropy proportional to its area, up to a constant:

$$S_{BH} = \alpha A + \text{const.}$$

He observed that the irreducible mass M_{irr} of a BH (the energy which one cannot extract by means of classical processes) is analogous to the heat (the energy wich cannot be converted entirely into work), and M_{irr} turns out to be exactly proportional to the BH area. Since, at constant temperature, the variation of heat is $\delta Q = T\delta S$, this was a first hint.

Moreover, just like the entropy of an isolated system cannot decrease, the total area of a collection of BHs cannot decrease. Finally he observed that in principle a BH allows to violate the second law of thermodinamics just by dropping down it a system with non-zero entopy. This fact also led him to postulate the Generalized Second Law (GSL) of thermodynamics: *for an isolated system, the* *common entropy of the matter plus the entropy of the black holes never decreases*

$$\delta S_{\text{matt}} + \delta S_{BH} \geq 0.$$

If the GSL holds, a decrease of the common entropy of the universe is compensated by an increase of the area of the black holes.

Despite these convincing arguments, Bekenstein didn't suggest a mechanism for a BH to emit radiation. In fact, if one had to take seriously the proposal of Bekenstein, one should associate to a BH a temperature proportional to its surface gravity κ. However, as pointed out by Bardeen et al. (1973), a BH cannot be in thermal equilibrium with a black body radiation, because it can only absorb matter without emmitting anything: that's to say, the effective temperature of a BH is absolute zero. Therefore, unless one finds that someway a BH emits radiation at temperature $T = \alpha\kappa$, Bekenstein's ideas are wong.

The BH radiation was predicted to exist by Hawking (1974), after whom is now named *Hawking radiation*. As explained in much details in the following, he used the tools of QFT in curved spacetimes and found that, if a quantum field is coupled to the BH metric, the BH emits quanta of this field at temperature

$$T_H = \frac{\kappa}{2\pi k_B}$$

with a black body spectrum and the correct statistic (bosonic or fermionic, depending on the coupled field).

Hawking's result not only put on solid grounds the formulation of a thermodynamics of black holes, but more remarkably it opened a window on a yet unknown quantum theory of gravitation. The rest of this section is devoted to review the derivation for a massless scalar field.

2.2.2 QFT in flat spacetime

We start by revisiting QFT of a free scalar field in Minkowski spacetime: the equation of motion of such a field is

$$\nabla^a \nabla_a \phi - m^2 \phi = 0 \qquad \text{(Klein-Gordon equation)} \tag{2.1}$$

Given two solutions σ and ρ, their scalar product is defined as

$$\langle \sigma, \rho \rangle = i \int_\Sigma \left(\bar{\sigma} \nabla_a \rho - \rho \nabla_a \bar{\sigma} \right) n^a d\Sigma \tag{2.2}$$

where Σ is a Cauchy surface. To quantize the theory we want to promote ϕ to an operator acting on the Hilbert space of the solutions of the KG equation. Since $\langle \sigma, \rho \rangle$ is not positive-definite, we restrict ouself to positive frequency solutions, i.e. solutions $\phi(t, \vec{r})$ such that the Fourier expansion w.r.t. the time variable t

$$\tilde{\phi}(\omega, \vec{r}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(t, \vec{r}) \mathrm{e}^{i\omega t} dt \tag{2.3}$$

vanishes for $\omega < 0$. With this restriction the inner product becomes positive definite. We therefore define the *one-particle Hilbert space* $\mathscr{H}$ as the vector space of the positive frequency solutions σ of the KG equation, satisfying $\langle\sigma,\sigma\rangle < \infty$. The components of an element $\sigma \in \mathscr{H}$ in a given basis will be denoted by σ^a; similarly, the compnents of element $\bar{\sigma}$ in the dual vector space $\bar{\mathscr{H}}$ w.r.t. the dual basis will be denoted by $\bar{\sigma}_a$.

We construct the Fock space, defined as

$$\begin{aligned}\mathscr{F} &= \mathbb{C} \oplus \mathscr{H} \oplus (\mathscr{H} \otimes \mathscr{H})_S \oplus \cdots \oplus (\otimes^n \mathscr{H})_S \oplus \ldots \\ &= \bigoplus_{n=0}^{\infty} (\otimes^n \mathscr{H})_S\end{aligned} \tag{2.4}$$

where the subscript S denotes complete simmetrization (due to the bosonic statistics). The components of vector $\Psi \in \mathscr{F}$ are then expressed as

$$\Psi = \left(c, \sigma^a, \sigma^{ab}, \sigma^{abc}, \ldots\right). \tag{2.5}$$

The *vacuum state* is by definition

$$|0\rangle = (1, 0, 0, \ldots). \tag{2.6}$$

Let's define the following two operators:

1. ANNIHILATION OPERATOR

   For each $\bar{\rho} \in \bar{\mathscr{H}}$ the correspondent annihilation operator is a map $a : \mathscr{F} \to \mathscr{F}$ which acts on $\Psi \in \mathscr{F}$ as

   $$a(\bar{\rho})\Psi = \left(\sigma^a \bar{\rho}_a, \sqrt{2}\sigma^{ab}\bar{\rho}_b, \sqrt{3}\sigma^{abc}\bar{\rho}_c, \ldots\right). \tag{2.7}$$

   The operator $a(\bar{\rho})$ annihilates the vacuum: $a(\bar{\rho})\,|0\rangle = 0$.

2. CREATION OPERATOR

   For each $\rho \in \mathscr{H}$ the correspondent creation operator is a map $a^\dagger : \mathscr{F} \to \mathscr{F}$ which acts on $\Psi \in \mathscr{F}$ as

   $$a^\dagger(\rho) = \left(0, c\rho^a, \sqrt{2}\sigma^{(a}\rho^{b)}, \sqrt{3}\sigma^{(ab}\rho^{c)}, \ldots\right). \tag{2.8}$$

It easy to see that they satisfy the canonical commutation relations

$$\begin{aligned}&\left[a(\bar{\sigma}), a(\bar{\rho})\right] = 0 = \left[a^\dagger(\sigma), a^\dagger(\rho)\right] \\ &\left[a(\bar{\sigma}), a^\dagger(\rho)\right] = \langle\sigma, \rho\rangle\, \mathbb{1}\end{aligned} \tag{2.9}$$

Moreover the operator $N(\sigma) = a^\dagger(\sigma)a(\bar{\sigma})$ is the number operator relative to the one-particle state σ, i.e. their eigenvalues are vectors of $\mathscr{F}$ with a fixed number of σ-modes. Thus for each $\Psi \in \mathscr{F}$ the expectation value $\langle\Psi|\, N(\sigma)\,|\Psi\rangle$ yields the number of excitation in the one-particle state σ.

The general expression for the operator $\hat{\phi}$ is therefore

$$\hat{\phi}(x) = \sum_i \left[\sigma_i(x) a(\bar{\sigma}_i) + \bar{\sigma}_i(x) a^\dagger(\sigma_i)\right]. \tag{2.10}$$

2.2.3 Particle creation

Now suppose that the scalar field ϕ is not free, but it evolves in an effective potential $V(x)$. The equation of motion is then

$$\nabla_a \nabla^a \phi + m^2 \phi + V(x)\phi = 0 \tag{2.11}$$

and we also allow the spacetime to be curved (so, for example, $V(x)$ can represent the effects of the gravitational potential). The only restrictions that we impose is that the spacetime is globally hyperbolic and there exist two regions, the in-region in the distant past and the out-region in the distant future, where the spacetime is practically flat. Correspondingly, the quantum operator $\hat{\phi}$ can be decomposed both on the Cauchy surfaces Σ_{in} and Σ_{out}:

$$\hat{\phi}_{\text{in}}(x) = \sum_i \left[\alpha_i(x) a_{\text{in}}(\bar{\alpha}_i) + \bar{\alpha}_i(x) a^\dagger_{\text{in}}(\alpha_i) \right] \tag{2.12a}$$

$$\hat{\phi}_{\text{out}}(x) = \sum_i \left[\beta_i(x) a_{\text{out}}(\bar{\beta}_i) + \bar{\beta}_i(x) a^\dagger_{\text{out}}(\beta_i) \right] \tag{2.12b}$$

Sandwiching with α_k we obtain

$$\begin{aligned} a_{\text{in}}(\bar{\alpha}_k) &= \sum_i \left[\langle \alpha_k, \beta_i \rangle \, a_{\text{out}}(\bar{\beta}_i) + \langle \alpha_k, \bar{\beta}_i \rangle \, a^\dagger_{\text{out}}(\beta_i) \right] \\ &= a_{\text{out}}\left(\overline{\textstyle\sum_i \langle \beta_i, \alpha_k \rangle \, \beta_i}\right) + a^\dagger_{\text{out}}\left(\overline{\textstyle\sum_i \langle \bar{\beta}_i, \alpha_k \rangle \, \bar{\beta}_i}\right) \\ &= a_{\text{out}}(\overline{C\alpha_k}) - a^\dagger_{\text{out}}(\overline{D\alpha_k}) \end{aligned} \tag{2.13}$$

where $C\alpha_k$ and $D\alpha_k$ denote respectively the positive and negative frequency part of α_k IN THE FUTURE. Similarly we can decompose $a_{\text{out}}(\bar{\beta}_k)$ in the positive and negative frequency parts IN THE PAST:

$$\begin{aligned} a_{\text{out}}(\bar{\beta}_k) &= \sum_i \left[\langle \beta_k, \alpha_i \rangle \, a_{\text{in}}(\bar{\alpha}_i) + \langle \beta_k, \bar{\alpha}_i \rangle \, a^\dagger_{\text{in}}(\alpha_i) \right] \\ &= a_{\text{in}}\left(\overline{\textstyle\sum_i \langle \alpha_i, \beta_k \rangle \, \alpha_i}\right) + a^\dagger_{\text{in}}\left(\overline{\textstyle\sum_i \langle \bar{\alpha}_i, \beta_k \rangle \, \bar{\alpha}_i}\right) \\ &= a_{\text{in}}(\overline{A\beta_k}) - a^\dagger_{\text{in}}(\overline{B\beta_k}) \end{aligned} \tag{2.14}$$

and so on with the creation operators. At the end one has:

$$a_{\text{out}}(\bar{\beta}_k) = a_{\text{in}}(\overline{A\beta_k}) - a^\dagger_{\text{in}}(\overline{B\beta_k}) \tag{2.15a}$$

$$a^\dagger_{\text{out}}(\beta_k) = a^\dagger_{\text{in}}(A\beta_k) - a_{\text{in}}(B\beta_k) \tag{2.15b}$$

$$a_{\text{in}}(\bar{\alpha}_k) = a_{\text{out}}(\overline{C\alpha_k}) - a^\dagger_{\text{out}}(\overline{D\alpha_k}) \tag{2.15c}$$

$$a^\dagger_{\text{in}}(\beta_k) = a^\dagger_{\text{out}}(C\alpha_k) - a_{\text{out}}(D\alpha_k) \tag{2.15d}$$

Using the canonical commutation relations it is an exercise to prove the following identities:

$$A^\dagger A - B^\dagger B = \mathbb{1} \qquad C^\dagger C - D^\dagger D = \mathbb{1} \tag{2.16a}$$

$$A^\dagger \bar{B} = B^\dagger \bar{A} \qquad C^\dagger \bar{D} = D^\dagger \bar{C} \tag{2.16b}$$

$$A^\dagger = C \qquad B^\dagger = -\bar{D} \tag{2.16c}$$

Transformations (2.15), together with the conditions (2.16), are the socalled *Bogolubov transformations*.

Now suppose that the system is in the vacuum state $|0_{in}\rangle$ at Σ_{in} and ask what is the number of one-particle modes σ as measured by an observer on Σ_{out}. The answer is:

$$\begin{aligned}\langle 0_{in}|\, N_{out}(\sigma)\, |0_{in}\rangle &= \langle 0_{in}|\, a^{\dagger}_{out}(\sigma)\, a_{out}(\bar{\sigma})\, |0_{in}\rangle \\ &= \langle 0_{in}|\, a_{in}(\overline{B\sigma})\, a^{\dagger}_{in}(B\sigma)\, |0_{in}\rangle \\ &= \langle B\sigma, B\sigma\rangle = \|B\sigma\|^2.\end{aligned} \tag{2.17}$$

So the number of out-particles in a mode σ is the squared norm of the negative frequency part of σ IN THE PAST: we see that what in the past is seen as the vacuum, in the future is a state populated of particles according to (2.17). This can be interpreted as a phenomenon of *particle creation* by an external potential: the role of the potential is to evolve the modes of the field in such a way to mix positive and negative frequencies. So, unless $B\sigma = 0$ for all $\sigma \in \mathscr{H}_{out}$, the vacuum state $|0_{in}\rangle$ doesn't evolve in the vacuum state $|0_{out}\rangle$.

2.2.4 Gravitational collapse

To study particle creation by a Schwarzschild BH we have not to consider just the stationary spacetime (fig. 1.1), but the full process of gravitational collapse (fig. 1.2): indeed the Hawking radiation is due exactly to the global structure of a collapsing-body spacetime. Led by the extremely simple form of the laws of BH dynamics, which involve just the macroscopic parameters of a given BH and are insensitive to the whole history of the collapse, and in the spirit of the "no hair" theorem, we guess that the details of the collapse are not relevant for the bulk of the phenomenon, and what matters is just the mass of the resulting object. So we adopt a simplified model of gravitational collapse, ending in a Schwarzschild BH, described in advanced null coordinates by the following line element:

$$ds^2 = -\Big(1 - \frac{2M}{r}\theta(v - v_0)\Big)dv^2 + 2dv\,dr + r^2 d\Omega^2 \tag{2.18}$$

where $\theta(v - v_0)$ is the Heaviside theta distribution, assuming values 0 for $v < v_0$ and 1 for $v > v_0$. So, for $v < v_0$ the spacetime is Minkowskian, while for $v > v_0$ it is Schwarzschildian. The line $v = v_0$ represents a null sphere of energy M collapsing towards the center and forming a singularity, how can be seen by noticing that the only nonvanishing component of the stress-energy tensor is

$$T_{vv} = \frac{M}{4\pi r^2}\delta(v - v_0). \tag{2.19}$$

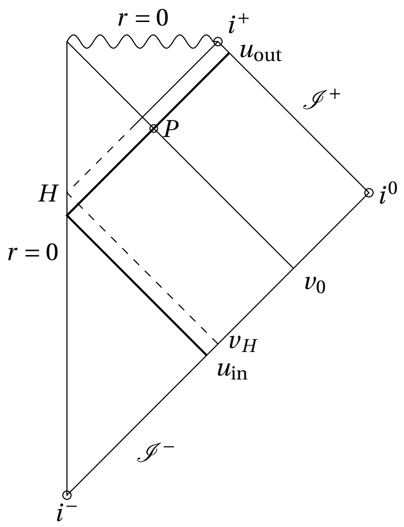

Figure 2.1: A collapsing null shell at $v = v_0$: the spacetime is Minkowski for $v < v_0$ and Schwarzschild for $v > v_0$. The thick line is the path followed by a positive frequency mode of a massless scalar field, in the geometric optics approximation.

This is called the *Vaidya approximation*[1][2]. The conformal diagram is shown in fig. 2.1.

We want to apply the particle creation formalism developed in the previous subsection. Let's consider a massless scalar field ϕ minimally couped to the gravitational field:

$$\nabla^a \nabla_a \phi = 0. \tag{2.20}$$

Since the field is massless, we can give initial data on $\Sigma_{in} \equiv \mathscr{I}^-$. We ask if and with which spectrum an observer at $\mathscr{I}^+$ observes particle creation, if the field is in the vacuum state $|0_{in}\rangle$ at $\mathscr{I}^-$. But $\mathscr{I}^+$ is not a complete Cauchy surface, because initial data on it allow to predict only the portion of the spacetime external to the black hole; a suitable out-Cauchy surface can be chosen as the union of $\mathscr{I}^+$ and the BH horizon: $\Sigma_{out} = \mathscr{I}^+ \cup \mathscr{H}^+$. This is the most simple choice, but the final result doesn't depend on how we extend $\mathscr{I}^+$ to obtain a Cauchy surface. Then the field expands as:

$$\hat{\phi}_{in}(x) = \sum_i \left[\alpha_i(x) a_{in}(\bar{\alpha}_i) + \bar{\alpha}_i(x) a^\dagger_{in}(\alpha_i)\right] \tag{2.21a}$$

$$\hat{\phi}_{out}(x) = \sum_i \left[\beta_i(x) a_{out}(\bar{\beta}_i) + \bar{\beta}_i(x) a^\dagger_{out}(\beta_i) + \gamma_i(x) a_{bh}(\bar{\gamma}_i) + \bar{\gamma}_i(x) a^\dagger_{bh}(\gamma_i)\right] \tag{2.21b}$$

where the subscripts "out" and "bh" denote respectively the $\mathscr{I}^+$ part and the horizon part of Σ_{out}. The Bogolubov transformations from $\mathscr{I}^+$ to $\mathscr{I}^-$ then read:

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[1] The case of a very general collapse was treated by Hawking (1975).

[2] More rigorously, what is properly called *Vaidya spacetime* is the line element $ds^2 = -\big(1 - 2M(v)/r\big)dv^2 + 2dvdr + r^2 d\Omega^2$. Since we are only interested in the Heaviside theta case, we'll mantain our abuse of notation.

$$a_{\text{out}}(\bar{\beta}_k) = a_{\text{in}}(\overline{A\beta_k}) - a^{\dagger}_{\text{in}}(\overline{B\beta_k}) \tag{2.22a}$$

$$a^{\dagger}_{\text{out}}(\bar{\beta}_k) = a^{\dagger}_{\text{in}}(A\beta_k) - a_{\text{in}}(B\beta_k) \tag{2.22b}$$

Using the canonical commutation relations, the operators A and B satisfy again

$$A^{\dagger}A - B^{\dagger}B = \mathbb{1}. \tag{2.23}$$

The number of particles in the mode σ observed at $\mathscr{I}^+$ is given by the same previous formula:

$$\langle N(\sigma)\rangle = \|B\sigma\|^2. \tag{2.24}$$

To apply (2.24) we have to quantize $\hat{\phi}$ both on $\mathscr{I}^-$ and $\mathscr{I}^+$ and then express out-modes in terms of in-modes. First of all notice that, in our apprximation, the collapse is spherically symmetric, so we can expand the field in terms of spherical armonics:

$$\phi(t,\vec{r}) = \sum_{l,m} \frac{F_l(t,r)}{r} Y_{lm}(\theta,\varphi) \tag{2.25}$$

Eq. (2.20) reduces to

$$\Big(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2}\Big)F_l(t,r) = 0 \qquad \text{(Minkowski sector)} \tag{2.26a}$$

$$\Big(-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r_\star^2} - V_l(r)\Big)F_l(t,r) = 0 \qquad \text{(Schwarzschild sector)} \tag{2.26b}$$

where $V_l(r)$ is the gravitational effective potential

$$V_l(r) = \Big(1 - \frac{2M}{r}\Big)\Big(\frac{l(l+1)}{r^2} + \frac{2M}{r^3}\Big).$$

and $r_\star$ is the tortoise coordinate defined by eq. (1.4)

$$r^* = r + 2M\log\Big|\frac{r}{2M} - 1\Big|.$$

Some approximations are needed to simplify the calculations.

First of all, the spacetime appears more and more Minkowskian as one goes far from the horizon. Since particle creation happens because the dynamics of the spacetime mixes positive and negative frequences, we concentrate on the modes propagating close to the horizon when the BH forms. These modes reach $\mathscr{I}^+$ with a very high redshift: this is equivalent to say that modes received at $\mathscr{I}^+$ had a very high frequency when they began their path from $\mathscr{I}^-$: hence we use the geometric optics approximation from the horizon to $\mathscr{I}^-$, i.e. we follow null modes as they were light rays.

Moreover we expect the important physics to happen in the vicinity of the horizon, where $V_l(r)$ vanishes. Thus we can neglect the effect of the gravitational potential in the Schwarzschild sector, as it is enforced by assuming that the bulk of the transmitted rays are in the s-wake sector $l = 0$.

So the path of a null modes from $\mathscr{I}^+$ to $\mathscr{I}^-$ reduces to an outgoing ray from $\mathscr{I}^+$ to $r = 0$, where it is reflected on $\mathscr{I}^-$ as an ingoing ray (see fig. 2.1, thick line). With these simplifications, the equations of motion become

$$\Big(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r^2}\Big)F(t,r)=0 \qquad \text{(Minkowski sector)} \qquad (2.27a)$$

$$\Big(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial r_\star^2}\Big)F(t,r)=0 \qquad \text{(Schwarzschild sector)} \qquad (2.27b)$$

where we have dropped the subscript l since we set $l = 0$. They can be easily solved in terms of the double null coordinates

$$u_{\rm in}=t-r \qquad v_{\rm in}=t+r \qquad \text{(Minkowski sector)} \qquad (2.28a)$$

$$u_{\rm out}=t-r_\star \qquad v_{\rm out}=t+r_\star \qquad \text{(Schwarzschild sector)} \qquad (2.28b)$$

In both sectors, the phases of the ingoing and outgoing modes with frequency ω are

$$\phi\sim e^{-i\omega v} \qquad \text{(ingoing modes)} \qquad (2.29a)$$

$$\phi\sim e^{-i\omega u} \qquad \text{(outgoing modes)} \qquad (2.29b)$$

Consider an outgoing null mode of frequency ω received at $\mathscr{I}^+$. It has the form

$$\phi_\omega(u_{\rm out})=\phi_0 e^{-i\omega u_{\rm out}} \qquad (2.30)$$

where the time dependence is all in the phase and ϕ_0 is the time-independent prefactor. The coordinate $u_{\rm out}$ can be expressed in terms of the minkowskian coordinate $u_{\rm in}$ by imposing matching conditions at the point P, where the ray leaves the Schwarzschild sector and enters the Minkowski one. In fact, on the surface $v = v_0$ one must have

$$r(v_{\rm in},u_{\rm in})=r(v_{\rm out},u_{\rm out}) \qquad (2.31)$$

which, taking into account eq. (2.28), becomes

$$u_{\rm out}=u_{\rm in}-4M\log\Big(\frac{v_0-u_{\rm in}}{4M}-1\Big). \qquad (2.32)$$

Without any restriction we can put $v_0 = 0$, so that

$$u_{\rm out}=u_{\rm in}-4M\log\Big(-\frac{u_{\rm in}}{4M}-1\Big). \qquad (2.33)$$

Observe that the above equation is defined only for $-\infty < u_{\rm in} < -4M$. The physical meaning is evident from fig. 2.1: $\mathscr{I}^-$ is divided in two sets at the advanced coordinate $v = v_H$, in such a way that ingoing null rays with $-\infty < v < v_H$ are

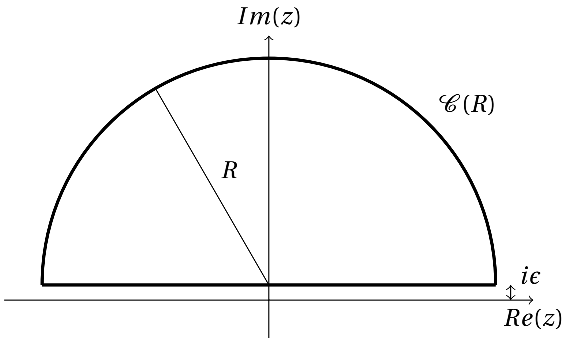


Figure 2.2: Contour of integration for the integral (2.36), where $R \to \infty$ and $\epsilon \to 0$ are understood. The contour is such that the log function has no branchpoints and the integrand vanishes on $\mathscr{C}(R)$ for $R \to \infty$.

reflected on $r = 0$ and trasmitted to $\mathscr{I}^+$ as outgoing modes, while the null rays with $v_H \le v < \infty$ fall into the BH region. We therefore deduce that $v_H = -4M$.

As we said, we are interested in null rays arriving in P at $r \simeq 2M$, which corresponds to $u_{\text{in}} \simeq -4M$, so we can approximately write

$$u_{\text{out}} \simeq -4M \log\Big(-\frac{u_{\text{in}}}{4M} - 1\Big). \tag{2.34}$$

We are ready to compute the negative frequency part of ϕ_ω on $\mathscr{I}^-$. Consider the Fourier transform of ϕ_ω w.r.t. to the postive frequency σ on $\mathscr{I}^-$:

$$\begin{aligned}\phi(\sigma) &\sim \int_{-\infty}^{-4M} \mathrm{e}^{i\omega 4M \log\left(\frac{-u_{\text{in}}-4M}{4M}\right)} \mathrm{e}^{i\sigma u_{\text{in}}} du_{\text{in}} \\ &= \mathrm{e}^{-i\sigma 4M} \int_{-\infty}^{0} \mathrm{e}^{i\omega 4M \log(-v/4M)} \mathrm{e}^{i\sigma v} dv\end{aligned} \tag{2.35}$$

with $\sigma > 0$. If we promote it to an integral over the complex variable z on the contour showed in fig. 2.2, then

$$\oint \mathrm{e}^{i\omega 4M \log(-z/4M)} \mathrm{e}^{i\sigma z} dz = 0 \tag{2.36}$$

for the integrand has no poles inside the contour of integration. It follows that

$$\begin{aligned}0 &= \int_{-\infty}^{0} \mathrm{e}^{i\omega\kappa^{-1}\log(-v\kappa^{-1})} \mathrm{e}^{i\sigma v} dv + \int_{0}^{\infty} \mathrm{e}^{i\omega\kappa^{-1}\log(-v\kappa^{-1}-i\epsilon)} \mathrm{e}^{i\sigma v} dv \\ &= \int_{-\infty}^{0} \mathrm{e}^{i\omega\kappa^{-1}\log(-v\kappa^{-1})} \mathrm{e}^{i\sigma v} dv - \int_{-\infty}^{0} \mathrm{e}^{i\omega\kappa^{-1}\log(v\kappa^{-1}-i\epsilon)} \mathrm{e}^{-i\sigma v} dv\end{aligned} \tag{2.37}$$

where $\kappa = 1/4M$ is the surface gravity of the BH. Observing that

$$\log(v\kappa^{-1} - i\epsilon) = \log|v\kappa^{-1}| - i\pi$$

one finally obtains

$$\phi(\sigma) = -\mathrm{e}^{\omega\pi\kappa^{-1}} \phi(-\sigma). \tag{2.38}$$

We see that the positive frequency part equals $-\mathrm{e}^{\omega\pi\kappa^{-1}}$ times the negative frequency part. From

$$A^\dagger A - B^\dagger B = \mathbb{1}$$

we deduce

$$1 = \langle\omega|\, A^\dagger A - B^\dagger B\,|\omega\rangle = \left(e^{2\omega\pi\kappa^{-1}} - 1\right)\|B\omega\|^2 \tag{2.39}$$

from which

$$\|B\omega\|^2 = \frac{1}{e^{2\omega\pi\kappa^{-1}} - 1}. \tag{2.40}$$

This is the desired result: it says that the expectation number $\langle N(\omega)\rangle$ on $\mathscr{I}^+$ coincides with that of a thermal distribution of bosons at the Hawking temperature

$$T_H = \frac{\kappa}{2\pi}. \tag{2.41}$$

By comparison with eq. (1.34) it allows to fix the BH entropy to be[3]

$$S_{BH} = k_B \frac{c^3 A}{4G\hbar} + \text{const.} \tag{2.42}$$

which is known as the *Bekenstein entropy*.

To be rigorous, eq. (2.40) is not sufficient to conclude that particles are emmited with a Planck spectrum. One should show that the different one-particle modes are uncorrelated and the expcetation values of obtaing different number of particles agrees with that of an exactly thermal spectrum. This was done by Wald (1975), who computed the reduced density matrix, obtained by tracking $|0_{\text{in}}\rangle$ with respect to the modes not reaching $\mathscr{I}^+$, and found that the probability of observing N particles with energy ω is

$$P(N,\omega) = \frac{e^{-\beta N\omega}}{\prod_\omega \left(1 - e^{-\beta\omega}\right)} \qquad \beta^{-1} = k_B T_H \tag{2.43}$$

in accordance with thermal emission.

Remarks The derivation of the Hawking radiation is purely mathematical, but one can ask what is the physical origin of the emitted quanta. One can interpret Hawking radiation as a spontaneus creation of particle-antiparticle pairs just outside the horizon, one with negative energy and one with positive energy with respect to infinity: the negative energy member of the pair falls into the BH, where negative energy states exist, while the other reaches spatial infinity. Therefore to each particle received at future infinity cooresponds a partner which falls into the BH. Since the partners have negative energy, they effectively reduce the BH mass, as measured at infinity.

Do we have any chance to observe the thermal radiation emitted by a BH? The magnitude of the Hawking temperature is

$$T_H \approx 6 \times 10^{-8} \left(\frac{M_\odot}{M}\right) K$$

[3] We have restored the physical constants.

so present-day BHs have a too much big mass to produce an observable effect. Moreover, T_H is dominated by the cosmic microwave background temperature ($T_{CMB} \simeq 2.7K$), therefore they absorb more energy than they emit. It has been speculated that primordial BHs can have a sufficiently low mass to emit an observable radiation, but no such radiation has been detected up to now.

2.3 Information loss paradox

Hawking radiation implies that a BH loses energy with time. The mass loss rate of a Schwarzschild BH can be extimated from the Stephan-Boltzmann law

$$\frac{dM}{dt} = -\sigma A T^4 \tag{2.44}$$

where σ is the Stephan-Boltzmann constant and A is the area of the BH event horizon. Inserting the expressions for T and A we find that

$$\frac{dM}{dt} = \frac{\alpha}{M^2} \qquad \text{where } \alpha \simeq 10^{-5} \frac{M_{\text{Planck}}}{t_{\text{Planck}}} \tag{2.45}$$

Naming t_{ev} the time after which the mass of the black hole is nearly planckian, integration of (2.45) gives

$$t_{\text{ev}} \approx M_0^3 \tag{2.46}$$

in Planck units. This is a crude extimation, relying on the assumption that semi-classical approximation holds up to the planck scale. If this is the case, when $t = t_{\text{ev}}$ the Hawking temperature is so high to unfreeze a high number of fields, causing a final explosions in which all the remaining mass of the BH is radiated away. So t_{ev} is also close to the evaporation time. The causal structure of a BH that completely evaporates is shown in 2.3. Hawking (1976) recognized that such a scenario has a dramatic consequence, the so called *information loss paradox*. Roughly speaking, it consists in the observation that one starts with a pure state $|0_{\text{in}}\rangle$ on $\mathscr{I}^-$ and, when the BH completely evaporates, ends up with a thermal -i.e. maximally mixed- state on $\mathscr{I}^+$. In terms of density matrices, it means that a pure density matrix $\hat{\rho}_{\text{in}}$ evolves in a mixed one $\hat{\rho}_{\text{out}}$. This can happen only if the evolution

$$\hat{\rho}_{\text{in}} \to \hat{\rho}_{\text{out}}$$

is not unitary, thus contraddicting one of the postulates of quantum mechanics. Unitarity assures the conservation of the probability current, or in other words that the information about the system is preserved by time evolution. When a BH evaporates information about the partners of the Hawking quanta is lost, vanished within the singularity.

Let's be just a bit more formal: the state $|0_{\text{in}}\rangle$ can be decomposed[4] on $\Sigma_{\text{out}} = \mathscr{I}^+ \cup \mathscr{H}^+$ as

$$|0_{\text{in}}\rangle = \frac{1}{\prod_\omega \sqrt{1 - e^{-\beta\omega}}} \sum_{\omega,N} e^{-\frac{\beta}{2}N\omega} |N,\omega\rangle_{\text{BH}} \otimes |N,\omega\rangle_{\text{out}} \tag{2.47}$$

[4]See for example Wald (1975).

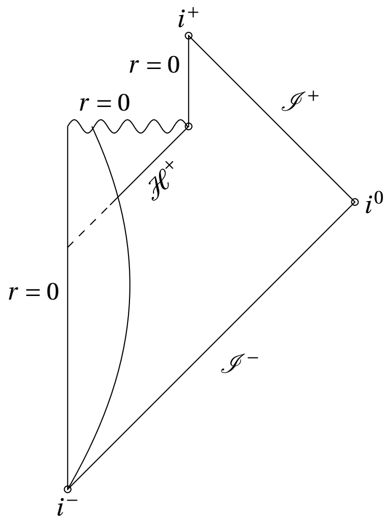


Figure 2.3: Penrose diagram of a collapsing star, forming a Schwarzschild clack hole and completely evaporating by Hawking radiation.

in Dirac notation, where the correlations between outgoing modes $|N,\omega\rangle_{\text{out}}$ and their ingoing partners $|N,\omega\rangle_{\text{BH}}$ appear explicitly. Thus we can write the density matrix $\hat{\rho}_{\text{in}}$ as

$$\begin{aligned}\hat{\rho}_{\text{in}} &= |0_{\text{in}}\rangle\langle 0_{\text{in}}| \\ &= \frac{1}{\prod_\omega\left(1-\mathrm{e}^{-\beta\omega}\right)}\sum_{\omega,N}\sum_{\omega',N'}\mathrm{e}^{-\frac{\beta}{2}(N\omega+N'\omega')}\,|N,\omega\rangle_{\text{BH}}\otimes|N,\omega\rangle_{\text{out}}\,\langle N',\omega'|_{\text{BH}}\otimes\langle N',\omega'|_{\text{out}}\end{aligned} \tag{2.48}$$

The final density matrix $\hat{\rho}_{\text{out}}$ is obtained by trackig w.r.t. the BH-modes:

$$\hat{\rho}_{\text{out}} = \frac{1}{\prod_\omega\left(1-\mathrm{e}^{-\beta\omega}\right)}\sum_{\omega,N}\mathrm{e}^{-\beta N\omega}\,|N,\omega\rangle_{\text{out}}\,\langle N,\omega|_{\text{out}} \tag{2.49}$$

We see that just "one half" of the state survives the evaporation, and the correlations between it and the rest are lost forever.

Lack of unitarity in Hawking radiation is cumbersome. As pointed out by Wald (1994), two wayouts are possible: i) the information is stored in a planck-size remnant, either stable or slowly evaporating after t_{ev}; ii) semiclassical approximation is violated well before the Planck scale and correlations find a way to escape the BH horizon during the evaporation, "riding" the Hawking quanta. Objections to the first option are that a planck-size remnant is too small to contain a huge entropy (approximately one half of the initial entropy of the BH), while the second option seems to imply a strong violation of macroscopic causality.

Hawking (1976) and Wald (1994) originally gave up and admit that unitarity is violated in quantum-gravitational processes. The recently discovered AdS-CFT correspondence brings in the direction of unitarity and has renewed the debate.

Chapter 3

Nonsingular Black Holes

3.1 Motivations

As we saw in sec. 1.2.2, singularities are predicted to occour in gravitational collapse if General Relativity holds together with suitable energy conditions: this denotes a breakdown of the theory, a conclusion justified by the fact that typical singularities occour in a huge density regime, when quantum effects are expected to dominate over classical ones. One expects that quantum gravitational effects "regularize" the singularity, but there isn't an accepted theory of quantum gravity to confront this expectation with.

A first result has been obtained in the different context of quantum cosmology by Ashtekar et al. (2006): using the machinery of Loop Quantum Gravity they showed that the Big Bang singularity is replaced by a quantum bounce, and the effective Friedmann equation becomes

$$\dot{a}^2 = \frac{8\pi G}{3}\left(1 - \frac{\rho}{\rho_{\text{crit}}}\right)\rho a^2. \tag{3.1}$$

where $\rho_{\text{crit}} \sim \rho_{\text{planck}}$.

While the common belief is that quantum gravitational effects arise when the scale of the phenomenon is of the order of the Planck lenght l_{planck}, eq. (3.1) shows that THEY MANIFEST AT THE PLANCK DENSITY ρ_{planck}. For example, in the case of a closed universe which recollapses, the effect is to make gravity repulsive causing a bounce, at which the size of the universe is still much larger than planckian.

If we accept this result, we can reasonably suppose that the same happens when a star collapses: instead of forming a singularity, the repulsive character of quantum gravity stops the contraction and causes a bounce. Now, since we are merely speculating, let us go just a little beyond. Ashtekar and Bojowald (2005) propose that the bounce transforms the BH region in a WH one and the star expands again outwards.

An alternative proposal is that the BH region remains such and the surface of the star reaches equlibrium between expanding quantum pressure and tendency to collapse: we call such a picture *Nonsigular Black Hole*. NSBHs are not a novelty: they are dated at least to Bardeen (1968) [12] and both abstract theory and explicit examples have been developing during the years. Our motivations follow the recent Rovelli and Vidotto (2014).

NSBHs have been considered in the literature as models to address the information loss paradox, i.e. the lack of unitarity consequent to the emission of the Hawking radiation. Unitarity is requested in standard quantum theories, so we expect that quantum gravity should solve this problem too. Approaches to quantum gravity can be very radical, as in fact most of them are, but it is still unclear if the information loss paradox requires the full quantum gravitational machinery or it can be faced with phenomenological models. The latter way is called a *conservative approach* in Hossenfelder and Smolin (2010): here they analize the main conservative approcaches and reach the conclusion that, in any effective model of unitarity recovering, nonsingularity is necessary. This is so because the precence of an horizon is only responsible of the quantum particle emission, but where information gets lost is inside the singularity. Further they advocate that, before starting to propose esotic solutions, one should try to do the possible with a conservative approach.

In this chapter we review the theory of NSBHs, and we see that they effectively constitute a paradigm, rather than a mere collection of examples.

3.2 An explicit model: Hayward metric

In costructing NSBHs we are explicitely assuming that the three classical BH metrics predicted by the "no hair" theorem are wrong, and have to be to altered in such a way that they are nonsingular. For simplicity we limit ourself to static uncharged BHs and we just study the changes to the Scwarzschild metric.

It is very reasonable to require the modifications gradually decreasing with the radial distance, such that they become negligible in the small curvature regime. So all the quantum effects described in Ch. 2 are practically unchanged and we can think NSBHs to be evolving objects with thermodynamical properties (at least when the weak field approximation is valid). The Hawking radiation, the process which leads the evaporation, is modeled as a small perturbation of a stationary BH background. Analogously we don't start directly with a model of evaporating NSBH: we first look for a stationary metric and then turn to dynamical considerations.

How much free are we to write a stationary nonsingular metric? The singularity theorems rely on some energy conditions which are believed to hold in ordinary situations. If we want to escape the conclusions of the singularity theorems, we can in principle violate the energy conditions, but they are fundamental to make the global properties of Einstein's equations non-trivial. In

[1] The Bardeen's original paper has gone quite lost, or at least is very difficult to be found.
[2] For an historical review on NSBHs see Ansoldi (2008).

a conservative approach, at least one energy condition is recommended and it is reasonable to keep the WEC (positivity of the energy). Actually, the "no hair" theorem is proved under the assumption of NEC, but NEC is implied by WEC, so the Schwarzschlid BH is the only static solution in vacuum: it follows that a static NSBH must be a solution of the Einstein's equations with a suitable stress energy-tensor. This distribution of energy in the space might represent the polarization of the vacuum by the gravitational field: this is the interpretation proposed in Poisson and Israel (1988). They guess that the effective vacuum polarization tensor is proportional to the curvature invariant $K = R^{\mathrm{abcd}} R_{\mathrm{abcd}}$, so that Einstein's equations become

$$G^a_b = 8\pi < T^a_b > \propto K^2 \delta^a_b. \tag{3.2}$$

Following this suggestion we can offer an argument for a particular form of the metric. Let the line element be

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2 \tag{3.3}$$

where

$$F(r) = 1 - \frac{2M(r)}{r}. \tag{3.4}$$

Far from the center, the function $M(r)$ is the ADM energy contained in a sphere with radius r, which is settling up to the asymptotic value M. Then the curvature square is approximately

$$K \simeq \frac{48M^2(r)}{r^6}.$$

On the other hand

$$T^1_1 = -\frac{M'(r)}{4\pi r^2}$$

so we obtain the equation

$$M'(r) \sim \frac{M^2(r)}{r^4}$$

which yields

$$M(r) = \frac{Ar^3}{r^3 + B} \qquad A, B = \text{const.} \tag{3.5}$$

The constant A must be equal to M, in order to ensure $M(r) \xrightarrow{\infty} M$. The constant B cannot be determined from the classical asymptotic behaviour: it becomes important at small distances, so it's related to the quantum effects. Although we derived (3.5) just far from the center, we can extend it to the whole spacetime as an effective metric describing the quantum corrections to the Schwarzschild solution. In fact, following Hayward (2006), let us fix B so that

$$M(r) = \frac{Mr^3}{r^3 + 2ML^2} \tag{3.6}$$

where L is a free parameter with dimensions of a lenght. Then the metric becomes

$$\begin{aligned} ds^2 &= -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2 \\ F(r) &= 1 - \frac{2Mr^2}{r^3 + 2ML^2} \end{aligned} \tag{3.7}$$

The line element (3.7) is the *Hayward metric*. Originally suggested in Hayward (2006), it has been recently reconsidered in (Rovelli and Vidotto, 2014; Frolov, 2014; Bardeen, 2014).

Let's analize the stationary properties of the Hayward metric.

The spacetime is flat both at spatial infinity, where we recover the Schwarzschild behaviour

$$\lim_{r\to\infty} F(r) = 1 - \frac{2M}{r} \tag{3.8}$$

and in the centre, where

$$\lim_{r\to 0} F(r) = 1 - \frac{r^2}{L^2}\,. \tag{3.9}$$

Expression (3.9) corresponds to a de Sitter spacetime with cosmological constant $\Lambda = 3/L^2$; in fact the stress-energy tensor in the origin goes as

$$\lim_{r\to 0} T^a_b = -\frac{3}{8\pi L^2}\delta^a_b \tag{3.10}$$

thus reproducing the equation of state for a vacuum fluid $\rho = -p$. As a consequence the curvature square in the origin is regular: $R^{\mathrm{abcd}}R_{\mathrm{abcd}} = 24/L^4$. This can be interpeted by saying that there is a central core with negative pressure preventing the collapse, modeled by an effective cosmological constant. Thus the parameter L measures the effective quantum pressure of the core.

The spacetime (3.7) contains a trapped region, the boundary of which is given implicitly by the points satisfying $F(r) = 0$: this equation can be solved for M, giving

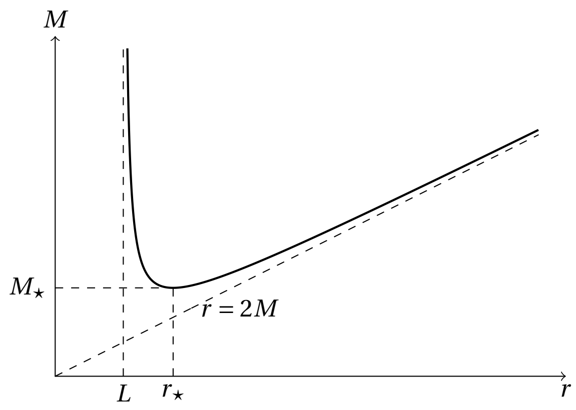
$$M = \frac{1}{2}\frac{r^3}{r^2 - L^2}\,. \tag{3.11}$$

A simple analysis (fig. 3.1) reveals that $F(r)$ has zeroes only when M is grater or equal than a critical mass $M_\star = 3\sqrt{3}L/4$: for $M > M_\star$ there are two zeroes r_+ and r_-, such that $r_- > r_+$ and, when $M \gg L$, $r_- \to L$ and $r_+ \to 2M$; for $M = M_\star$ the two zeroes degenerate into a single one at the critical radius $r_\star = \sqrt{3}L$. When $M < M_\star$, $F(r)$ has no zeroes and therefore there isn't any trapped surface: in this case (3.7) doesn't describe a BH.

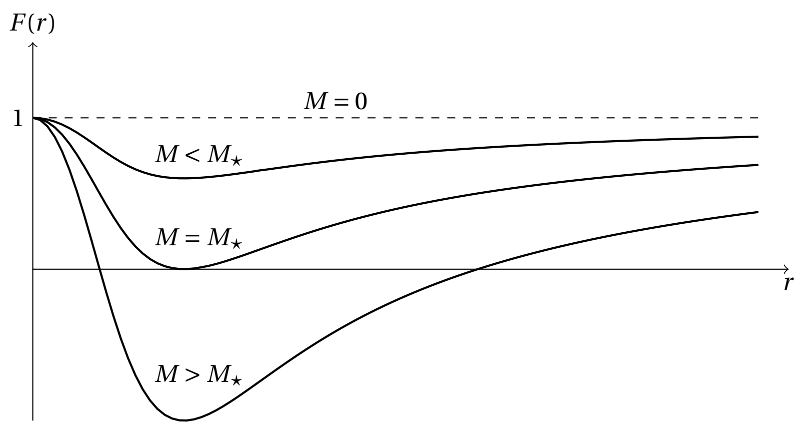
Actually, the zeroes of $F(r)$ can be found explicitly in a parametric form:

$$r_+ = \frac{2M}{3}\left[1 + 2\cos\left(\frac{x}{3}\right)\right] \tag{3.12a}$$

$$r_- = \frac{2M}{3}\left[1 - 2\cos\left(\frac{x+\pi}{3}\right)\right] \tag{3.12b}$$



(a)



(b)

Figure 3.1: (a) The relation between mass and horizons. Notice that $L < r_- \leq \sqrt{3}L$ and $\sqrt{3}L \leq r_+ < 2M$. (b) The three possible regimes of $F(r)$.

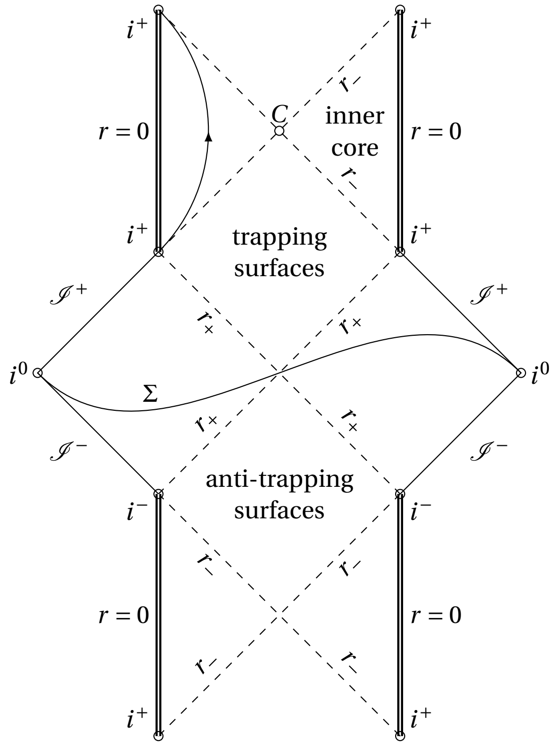


Figure 3.2: Conformal diagram of the non-degenerate Hayward spacetime. The diagram repeats itself infinitely many times in both up and down directions.

where

$$\cos x = 1 - \frac{27L^2}{8M^2} \qquad x \in]0, \pi]. \tag{3.13}$$

The conformal diagram of the non-degenerate case is shown in 3.2: we see that the surface $r = r_+$ is an event horizon, delimiting a BH region of trapping surfaces and a WH region of anti-trapping surfaces. Remarkably, the spacetime is not globally hyperbolic, there being timelike geodesics (e.g. the arrow line) that cannot be predicted from a generic slice Σ in the outer region. In fact, the future causal domain of Σ is delimited by future infinity and by the two wedges of r_- bifurcating in C: for this reason, $r = r_-$ is called a *Cauchy horizon*.

The degenerate conformal diagram is shown in fig. 3.3: the spacetime is again non-globally hyperbolic, but this time the event horizon and the Cauchy horizon coincide in $\mathscr{H} : r = r_\star$.

Now we can understand why the Hayward BH doesn't violate the singularity theorems. First of all, from the fact that the equation of state of de Sitter vacuum is $\rho = -p$, we see immediately that SEC is violated at least in a neighborhood of the centre, so only Penrose's theorem remains. We saw at the beginning of this section that at least the WEC should be respected, but in fact we never imposed it in the derivation of the metric. Let's show that the Hayward metric is a solu-

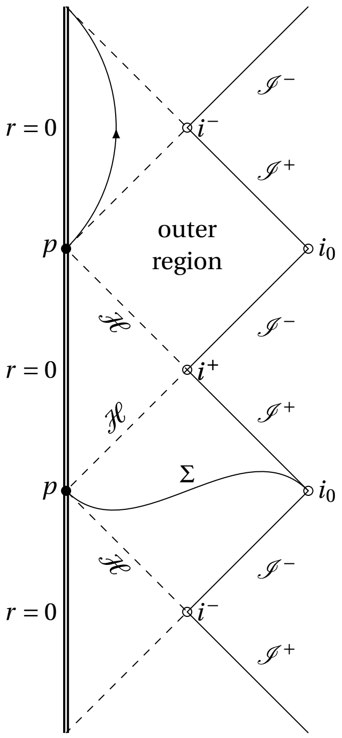


Figure 3.3: Conformal diagram of the degenerate Hayward spacetime. The points p are exceptional points at $r = \infty$.

tion of Einstein's equations with a stress-energy tensor satisfying the WEC. The components of the stress-energy tensor are

$$T^t_t = T^r_r = -\frac{3L^2M^2}{2\pi(2L^2M + r^3)^2} \tag{3.14a}$$

$$T^\theta_\theta = T^\phi_\phi = -\frac{3L^2M^2(L^2M - r^3)}{\pi(2L^2M + r^3)} \tag{3.14b}$$

To identify the energy density we have to distinguish three regions:

1. INNER CORE: $r < r_-$;
2. TRAPPING ZONE: $r_- \leq r \leq r_+$;
3. EXTERIOR: $r > r_+$.

The energy density is $\rho = -T^n_n$, where n is the timelike entry of the metric. So in principle one should separate the case 1 and 3, where the timelike entry is t, from the case 2, where it is r: but in practice $T^t_t = T^r_r$ and one can verify WEC all at once. Thus we identify

$$\rho = -T^t_t \qquad p_r = T^r_r \qquad p_\perp = T^\theta_\theta = T^\phi_\phi$$

from which

$$\rho + p_r = 0 \qquad \rho + p_\perp = \frac{9L^2M^2r^3}{2\pi(2L^2M + r^3)^3}.$$

Since they are manifestly non-negative, the WEC is satisfied. It follows that the only reason for which the Penrose's theorem doesn't apply to the Hayward spacetime is the lack of global hyperbolicity. In sec. 3.3.2 we will clarify the way this non-global hyperbolicity occours.

3.3 General properties

3.3.1 Dymnikova's theorem

We present a theorem, originally proved in Dymnikova (2002), stating that NSBH metrics can be classified in two universal classes: we will see that Hayward metric is not an isolated case, but it is just a member of the most simple of the two classes, and all the key properties on which we based our previous discussion are shared by all the members of the same class. Therefore our discussion about Hayward's metric is for many aspects paradigmatic in the context of NSBHs.

Let's start with the most generic stationary spherically symmetric line element (which must be static for the Birkhoff's theorem):

$$ds^2 = -e^{\mu(r)}dt^2 + e^{\nu(r)}dr^2 + r^2 d\Omega^2. \tag{3.15}$$

Einstein's equations become

$$8\pi T_t^t = -8\pi\rho = -e^{-\nu}\left(\frac{\nu'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{3.16a}$$

$$8\pi T_r^r = 8\pi p_r = e^{-\nu}\left(\frac{\mu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \tag{3.16b}$$

$$8\pi T_\theta^\theta = 8\pi T_\phi^\phi = 8\pi p_\perp = e^{-\nu}\left(\frac{\mu''}{2} + \frac{\mu'^2}{4} + \frac{(\mu' - \nu')}{2r} - \frac{\mu'\nu'}{4}\right) \tag{3.16c}$$

We committed an abuse of notation in naming $T_t^t = -\rho$ and $T_r^r = p_r$, because if $e^\nu < 0$ the roles of T_t^t and T_r^r are inverted. Nevertheless we continue to use this notation to avoid confusion in the reader, and we'll specify where things are different when it will be the case.

From (3.16) we have

$$p_\perp = p_r + \frac{r}{2}p_r' + (\rho + p_r)\frac{M(r) + 4\pi r^3 p_r}{2\big(r - 2M(r)\big)} \tag{3.17}$$

which is a generalization of the Tolman-Oppenheimer-Volkoff equation (Wald, 1984, pag. 127) in the case of different pressures, and

$$-T_t^t + T_r^r = \rho + p_r = \frac{e^{-\nu}(\mu' + \nu')}{8\pi r}. \tag{3.18}$$

Integration of (3.16b) gives

$$e^{-\nu} = 1 - \frac{2M(r)}{r} \qquad M(r) = 4\pi\int_0^r \rho r^2 dr. \tag{3.19}$$

If $\rho(r) = M\delta(r)$ and all the pressures are zero we recover the Schwarzschild solution. Now we want to modify the Schwarzschild solution in such a way that it is nonsingular. In order to do this, we make some assumptions:

1. the Dominant Energy Condition (DEC) holds[3];
2. the metric is regular in $r = 0$[4];
3. asymptotic flatness;
4. finiteness of the ADM mass.

Let's explore the consequences of these assumption. Most of the information is taken from the behaviour at spatial infinity and in the center.

Infinity The ADM mass is $M = 4\pi\int_0^r \rho r^2 dr$: its finiteness implies $e^{\nu(\infty)} = 1$, i.e. $\nu(\infty) = 0$; moreover, ρ must vanish at infinity quicker than r^{-3}. Since DEC implies

$$|p_i| \leq \rho \qquad (i = 1,2,3) \tag{3.20}$$

it follows that all the pressures vanish at inifinity. Finally, asymptotic flatness implies $\mu(\infty) = 0$.

Center From (3.19), $\nu(0) = 0$ (we don't have any such requirement on $\mu(0)$): it means that $r = 0$ is timelike.

Regularity of the density, $\rho(0) < \infty$, and DEC, $p_i \leq \rho$, lead to regularity of pressures. This in turn implies, by (3.18), that $(\mu'+\nu')(0) = 0$. In general, however, $(\mu' + \nu')(r) \geq 0$: this follows immediately from $\rho + p_i \geq 0$ and eq. (3.18). Observe that, when e^ν is negative, (3.18) becomes

$$-T_t^t + T_r^r = -(\rho + p_r) = \frac{e^{-\nu}(\mu' + \nu')}{8\pi r}$$

and $(\mu' + \nu')(r) \geq 0$ still holds.

The function $\Phi = \mu + \nu$ grows from $\mu(0)$ in the center to $\mu(\infty) + \nu(\infty) = 0$ at infinity, so $\mu(0) \leq 0$. The value of $\mu(0)$ is free and plays the role of a family parameter.

Special class The choice $\mu(0) = 0$ selects a special class of nonsingular metrics: with this choice the function Φ is null everywhere and the line element becomes

$$ds^2 = -\left(1 - \frac{2M(r)}{r}\right)dt^2 + \left(1 - \frac{2M(r)}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.21}$$

[3] We hope to have sufficiently argumented that not imposing any energy condition hasn't too much sense.

[4] We mean that the metric, the stress-energy tensor and the curvature scalar must be regular.

Since $p_r = -\rho$, eq. (3.18) is simplified:

$$p_\perp = -\rho - \frac{r}{2}\rho'. \tag{3.22}$$

By DEC, $p_i + \rho \geq 0$, it follows that $\rho' < 0$, i.e. ρ has a monotonically decreasing profile. Regularity of pressures implies $\rho' < \infty$, and therefore $p_\perp \to -\rho$ as $r \to 0$. As a result the equation of state in the center is $p_i = -\rho$, which is the equation of a de Sitter vacuum. The only regular solution is:

$$\lim_{r\to 0} ds^2 = -\left(1 - \frac{r^2}{L^2}\right)dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{3.23}$$

So we have shown that, under the assumptions of regularity, asymptotic flatness and DEC, there exits a special class of nonsingular black holes such that the line element is Schwartzschild at $r \to \infty$ and de Sitter at $r \to 0$. It is important to point out the role of the DEC: DEC splits in two conditions, $p_i \leq \rho$ and $p_i + \rho \geq 0$; the first has been used just to show that pressures are bounded; since the second is contained also in the WEC, WE CAN REPLACE DEC by requiring WEC and regularity of pressures; if we require regularity of all the component of the stress-energy tensor, not just the pressures, then even NEC is sufficient.

Since the metric is timelike both in the center and at spatial infinity, the existence of a trapping region requires at least the presence of two horizons or a degenerate horizon, but in principle more horizons are possible. In any case a Cauchy horizon is necessary to violate global hyperbolicity, in accordance with the Penrose's theorem.

We can think to the Hayward metric as the simplest choice among the member of the above-mentioned special class. For completeness we give a little survey of other members appeared in the literature. They are all of the form

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2$$

so we limit to give the function $F(r)$.

1. BARDEEN

$$F(r) = 1 - \frac{2Mr^2}{\left(r^2 + a^2\right)^{\frac{3}{2}}} \tag{3.24}$$

   It was the first NSBH: it appeared in Bardeen (1968) and was obtained in the context of nonlinear electrodynamics.

2. DYMNIKOVA

$$F(r) = 1 - \frac{2M\left(1 - \exp\left(-\frac{r^3}{2ML^2}\right)\right)}{r} \tag{3.25}$$

   It was obtained in Dymnikova (1992) by adopting the simple density profile

$$\rho(r) = \rho_0 \exp\left(-\frac{r^3}{2ML^2}\right)$$

   where $\rho_0 = \frac{3}{8\pi L^2}$ is the energy density of the de Sitter core.

3. NICOLINI

   In Nicolini (2005) the author assumes that the mass density distribution is a gaussian with standard deviation of order $\sigma \approx L$:

$$\rho(r) = \frac{M}{(4\pi L^2)^{\frac{3}{2}}} \exp\left(-\frac{r^2}{4L^2}\right).$$

   As a consequence

$$F(r) = 1 - \frac{4M}{r\sqrt{\pi}} \gamma\left(\frac{3}{2}; \frac{r^2}{4L^2}\right) \tag{3.26}$$

   where

$$\gamma\left(\frac{3}{2}; \frac{r^2}{4L^2}\right) = \int_0^{\frac{r^2}{4L^2}} t^{\frac{1}{2}} e^{-t} dt$$

   is the incomplete gamma function.

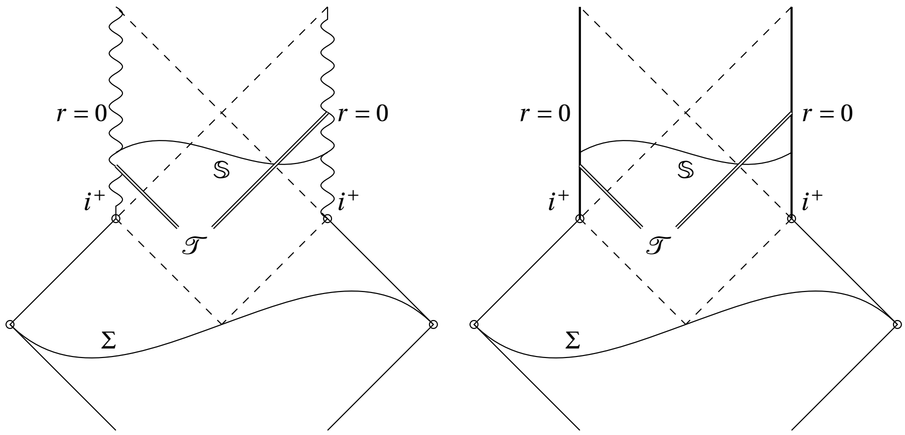
These metrics all have two horizons and share the same global properties with Hayward's metric; moreover they exhibit the same qualitative surface gravity profile (see fig. 3.6). For example, a study of the thermodinamical properties of Dymnikova's metric is presented in Dymnikova (1997): by numerical analysis, the surface gravity has a maximum for $M_{\text{crit}} \approx 1.1L$ and becomes null for $M_\star \approx 0.86L$; note that the ratio $M_{\text{crit}}/M_\star \approx 1.27$ is close to the Hayward value $M_{\text{crit}}/M_\star \approx 1.3$ and to the Nicolini value $M_{\text{crit}}/M_\star \approx 1.26$. Even if we haven't a strong argument, we can at least suppose that, for a wide class of metrics, the maximum occours for a mass M_{crit} of the same order of $M_\star$.

Remark The main limitation of Dymnikova's theorem is assuming General Relativity to be valid: other causal structures are possible if one modifies GR (see Bronnikov et al. (2007) and references therein).

3.3.2 Topology change

Lack of global hyperbolicity is essential in NSBH metrics, because it allows to escape the Penrose's theorem, but it can be obtained in several ways: for example, one can artificially remove points from the spacetime. Of course, this isn't what happens in the Hayward metric. Borde (1997, 1994) clarified how global hyperbolicity is evaded in NSBHs: throughout topology change.

Fig. 3.4b shows a portion of the conformal diagram of a non-extremal NSBH: in the inner core all the slices are compact and the spacetime has topology $\mathcal{M} \simeq \mathbb{R} \times S_3$, i.e. the inner core behaves as a closed universe; vice versa, in the global hyperbolic region the spacetime has topology $\mathcal{M} \simeq \mathbb{R} \times \Sigma$, where Σ is a non-compact Cauchy surface, so it is manfest that topology changes from one region to the other. Obviously such a spacetime cannot be global hyperbolic, otherwise it would be possible to map a compact slice on a non-compact one using the "time" vector field.



(a) (b)

Figure 3.4: (a) Reissner-Nordstrom black hole: the spatial slices Σ and $\mathbb{S}$ have the same topology $\mathbb{R} \times S_2$; double lines represent future directed light rays emanating from a trapped surface $\mathscr{T}$; (b) nonsingular Black Hole: here the slice $\mathbb{S}$ has topology S_3, i.e. the universe inside is closed; the future light cone of $\mathscr{T}$ turns around the universe across the $r = 0$ lines.

When spatial slices are compact, light rays wrap around the universe and, as a result, the future (or past) light cone emanating from a given point is a compact sheet (Borde, 1994): the same happens in the core of a NSBH, where light rays can proceed beyond the origin $r = 0$; on the contrary, in the corresponding confromal section of a Reissner-Nordstrom BH, light rays hit the singularity and the topology is everywhere $\mathscr{M} \simeq \mathbb{R} \times \Sigma$, though the spacetime is not globally hyperbolic too, fig. 3.4a.

Then it appears that the singularity is avoided not simply because the spacetime is not globally hyperbolic, but rather because it is such VIA TOPOLOGY CHANGE. In Borde (1997) the proof of Penrose's theorem is reversed and the author deonstrates that compactness of future light cones, emanating from events in the trapping region, is necessary in NSBHs. More precisely, the following theorem is proven:

Theorem 3.3.1. *Let $(\mathscr{M}, g_{ab})$ be a spacetime containing a trapping surface $\mathscr{T}$, such that the null energy condition $R_{ab}k^a k^b \geq 0$ holds for every null vector k^a. If the following conditions*

1. *the spacetime is future causally simple, with $E^+(K) \neq \emptyset \ \forall K \subset \mathscr{M}$*
2. *the spacetime is null geodesically complete*

are satisfied, then there is a compact slice in the causal future of $\mathscr{T}$.

The two conditions above are "regularity requirements", since are expected to be valid in every sufficiently regular spacetime; in particular, condition 2 says that there are no singularities. In Penrose's theorem such regularity was assured by the global hyperbolicity condition, which cannot be true in NSBHs.

Proof. The first part of the proof is identical to the one of Penrose's theorem, so we don't repeat it: it shows that $\dot{I}^+(\mathscr{T})$ is compact. Future causally simplicity implies $\dot{I}^+(\mathscr{T}) = E^+(\mathscr{T})$. Now observe that $\dot{I}^+(\mathscr{T})$, by definition, has no edge. Then $E^+(\mathscr{T})$ is the desired compact slice. □

Notice that the theorem doesn't apply to extremal nonsingular black holes, where no trapping zone is present. However it is commonly assumed that a gravitational collapse doesn't end up with an extremal configuration, and that a trapping surface must form. Hence extremal NSBHs appear only during the dynamical evolution of non-extremal ones.

3.4 Shortcomings of the Hayward metric

The deSitter vacuum manifestly violates the Strong Energy Condition, which expresses attractiveness of gravity. This can be easily seen from the deSitter equation of state

$$p_i = -\rho \qquad i = 1,2,3$$

that implies

$$\rho + \sum_i p_i = -2\rho < 0$$

in contraddiction with eq. (B.5c). It follows from Dymnikova's theorem that the Hayward metric, and more generally all the metrics belonging to the special class selected by $\mu(0) = 0$, violate SEC at least in a neighborough of the origin. This is compatible with our motivation adducted in sec. 3.1, where we argued that an effective description of quantum gravitational effects introduces a repulsive behaviour.

Despite of this favorable conclusion, and in spite of the wide use made in the literature of "$\mu(0) = 0$" NSBHs, they suffer of a physical shortcoming, in that a clock in the centre is not delayed with respect to a clock at infinity. Recall that the relation between the time interval δt_∞ ticked by a clock at infinity and the corresponding time interval δt_0 ticked by a clock in the origin is (see eq. (A.5)):

$$\delta t_0 = \sqrt{g_{00}}\delta t_\infty = \sqrt{1+2\phi_0}\delta t_\infty.$$

Now the deSitter metric has $g_{00} = 1$, therefore $\delta t_0 = \delta t_\infty$ as we claimed. This is an unpalatable feature because we expect the central core of the star to possess physical properties different from those of flat spacetime. It is interesting to ask whether it can be modified, without exiting from this class of models. Effectively this difficulty can be overcomed if we don't restrict to the case $\mu(0) = 0$, but we rather make the choice $\mu(0) = \log\epsilon < 0$: making reference to the proof of Dymnikova's theorem, this in turn implies $\log\epsilon = \Phi(0) < 0$; moreover the function $\Phi(r)$ is monotonically increasing from $\Phi(0) = \log\epsilon$ to $\Phi(\infty) = 0$ and therefore is always negative. We can thus parametrize the metric as

$$\begin{aligned} ds^2 &= -e^{\Phi(r)}F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2 \\ &= -G(r)F(r)dt^2 + F^{-1}(r)dr^2 + r^2 d\Omega^2. \end{aligned} \tag{3.27}$$

With this choice we can arbitrarily lower the delay factor $G(0) = \epsilon$ between δt_0 and δt_∞.

At this stage the function $G(r)$ is completely arbitrary. One can begin to put some constraints on its form by imposing that, near the centre, the equation of state is still that of a deSitter fluid: this condition is achieved if

$$\lim_{r\to 0} ds^2 = -\left(1 - \frac{r^2}{L^2}\right)\epsilon dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{3.28}$$

where the constant ϵ can be absorbed into a redefinition of t. Expanding around $r = 0$ we have

$$\begin{aligned} g_{00} &= -G(r)F(r) \\ &\to \big(\epsilon + G'(0)r + \frac{1}{2}G''(0)r^2\big)\big(1 - \frac{r^2}{L^2}\big) \\ &= \big(\epsilon + G'(0)r + \frac{1}{2}G''(0)r^2 - \epsilon\frac{r^2}{L^2}\big) \end{aligned} \tag{3.29}$$

from which we see that it must be

$$G'(0) = G''(0) = 0. \tag{3.30}$$

The form of $G(r)$ can be further constrained by observing that $\mu(0) = 0$ NSBHs, in addition to the clock delay problem, suffer of another inconsistency: they fail to reproduce the first quantum correction to Newton's law, as computed in (Bjerrum-Bohr et al., 2003), namely

$$\lim_{r\to\infty}\phi(r) = -\frac{M}{r}\left(1 + \beta\frac{l^2_{\text{planck}}}{r^2}\right) \qquad \beta = \frac{41}{10\pi} \tag{3.31}$$

The Hayward metric gives corrections of order $o(r^{-4})$, therefore we require

$$\lim_{r\to\infty} G(r) = 1 - 2\beta\frac{Ml^2_{\text{planck}}}{r^3} \tag{3.32}$$

which, together with (3.30), constraints $G(r)$ both in the centre and at infinity. A suitable choice is

$$G(r) = 1 - \frac{k}{r^3 + \tau} \qquad k = 2\beta M l^2_{\text{planck}} \quad \tau = \frac{k}{1-\epsilon} \tag{3.33}$$

Analytical and numerical investigations show that (3.33) causes a violation of the WEC and, even worse, the curvature invariant $K = R^{abcd}R_{abcd}$ exceeds by many orders of magnitude the planckian curvature! This extreme trans-planckian behaviour hinders the value of this metric as an effective low-energy description of quantum gravity effects. An approach to the question is take a general G(r), and require that it still satisfies the WEC, so let us write down the corresponding conditions. Unlike the Hayward case, we must distinguish the inner core and the exterior, where $\rho = -T^0_0$ and $p_r = T^1_1$, from the trapping zone where $\rho = -T^1_1$ and $p_r = T^0_0$. Notice, howevere, that the position of the horizons is unchanged, because it is still determined by $F(r) = 0$.

Let's first focus on the "radial part" of the stress energy tensor:

$$T^0_0 = \frac{-1 + F(r) + rF'(r)}{8\pi r^2} \tag{3.34a}$$

$$T^1_1 = \frac{G(r)\big(-1 + F(r) + rF'(r)\big) + rF(r)G'(r)}{8\pi r^2 G(r)} \tag{3.34b}$$

The WEC condition, eq. (B.5a), becomes:

$$\frac{-1+F(r)+rF'(r)}{8\pi r^2} \geq 0 \qquad \text{for } r<r_- \text{ and } r>r_+ \tag{3.35a}$$

$$\frac{G(r)\big(-1+F(r)+rF'(r)\big)+rF(r)G'(r)}{8\pi r^2 G(r)} \geq 0 \qquad \text{for } r_-<r<r_+ \tag{3.35b}$$

which corresponds to $\rho \geq 0$, and

$$\frac{F(r)G'(r)}{8\pi r G(r)} \geq 0 \qquad \text{for } r<r_- \text{ and } r>r_+ \tag{3.36a}$$

$$-\frac{F(r)G'(r)}{8\pi r G(r)} \geq 0 \qquad \text{for } r_-<r<r_+ \tag{3.36b}$$

coresponding to $\rho + p_r \geq 0$.

From the sign of $F(r)$ and from $\Phi'(r) \geq 0$ we see that (3.36) is automatically satisfied. Eq. (3.35a) is also true because it has the same form of the $\Phi = 0$ case. Hence the only non-trivial requirement is eq. (3.35b). The condition $\rho + p_\perp \geq 0$ yields the more complicated equations

$$\frac{1}{32\pi r^2 G(r)^2}\Big[-r^2F(r)G'(r)^2 + G(r)^2\big(4-4F(r)+2r^2F''(r)\big) + rG(r)\big(3rF'(r)G'(r)+2F(r)G'(r)+2rF(r)G''(r)\big)\Big] \quad \text{for } r<r_- \text{ and } r>r_+ \tag{3.37a}$$

$$\frac{1}{32\pi r^2 G(r)^2}\Big[-r^2F(r)G'(r)^2 + G(r)^2\big(4-4F(r)+2r^2F''(r)\big) + rG(r)\big(3rF'(r)G'(r)+2F(r)G'(r)-2rF(r)G''(r)\big)\Big] \quad \text{for } r_-<r<r_+ \tag{3.37b}$$

These are conditions as differential equations on a function, and it is very hard to find explicit solutions which also fulfil the properties listed above. Actually we found that violations of WEC comes from the trasversal part (3.37), even for $G(r)$ not satisfying (3.30) and (3.32).In fact, we will see in the next chapter another reason why it is important to consider the $\mu(0) < 0$ models, in connection with the evaporation process: therefore we presume that an effective description of Black Hole evaporation via NSBHs, if valid, should incorporate the time delay factor $e^{\Phi(r)}$.

3.5 Dynamics

We supposed that the negative quantum pressure has the effect of settling the BH on a stationary nonsingular configuration. We expect the Hawking radiation to take place, in analogy with the Schwarzschild case, and to be thus the only source of dynamics. Thus we want to use the Hyward metric to include back-reaction from Hawking radiation. The effect of the Hawking radiation on the metric is encoded by a time-dependent ADM mass M, rather than a static one. It's more convenient to write the dynamical metric in advanced coordinates:

$$\begin{aligned} ds^2 &= -F(v,r)dv^2 + 2dvdr + r^2 d\Omega^2 \\ F(v,r) &= 1 - \frac{2M(v,r)}{r} \end{aligned} \tag{3.38}$$

where

$$M(v,r) = \frac{M(v)r^3}{r^3 + 2M(v)L^2}. \tag{3.39}$$

In particular we expect $M'(v) < 0$.

While in the static case $r = r_+$ is null, in the dynamical case it's a timelike surface and can be crossed by causal geodesics. As a consequence, it cannot be an event horizon anymore; nevertheless it mantains a key property of the event horizon: it is the boudary of the trapping surfaces. To see this, observe that by (3.38) the outer future-propagating null rays have tangent vector

$$k_a = \big(-F(v,r), 2, 0, 0\big) \qquad k^a = \big(2, F(v,r), 0, 0\big) \tag{3.40}$$

As seen in sec. 1.2.3 the affinely parametrized tangent vector is

$$\tilde{k}^a = e^{-\Gamma} k^a$$

where $k^a \nabla_a \Gamma = \kappa$[5]. Then the outer null expansion is

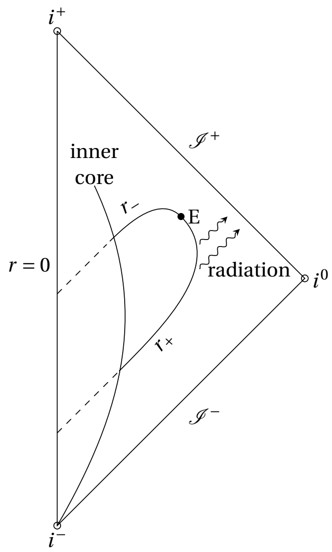
$$\theta_+ = \tilde{k}^a_{;a} = \frac{e^{-\Gamma}}{r^2}\big(r - 2M(v,r)\big). \tag{3.41}$$

The trapping region is the one where $\theta_+ < 0$, i.e. $r < 2M(v,r)$. Its boundary, called *trapping horizon*, is $r = 2M(v,r)$, whose solution is $r = r_+(v)$.

Two evaporation scenarios are possible (see fig. 3.5):

1. As the the mass decreases the horizons get closer and closer until they reach the extremal configuration and then disappear (fig.3.5a)(Hayward, 2006; Frolov, 2014). The evaporation leaves a remnant of mass $< M_\star$, the fate of which can be either to be stable or, more likely, to explode under

[5] κ is the *inaffinity*. It coincides with the surface gravity when evaluated on the null generator of an event horizon.



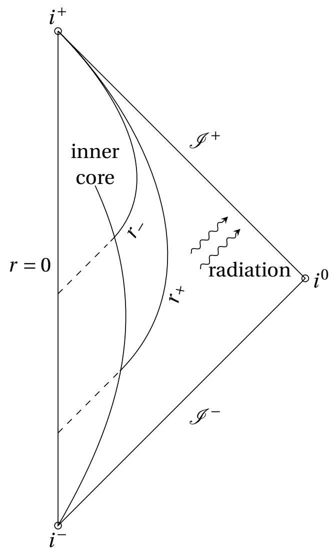


Figure 3.5: (a) Conformal diagram of a NSBH evaporating in a finite amount of time. The point E lies on the orbit $r = r_\star$; (b) Conformal diagram of an asymptotically evaporating NSBH.

the quatum-induced inflation: we have left this point undetermined in the diagram, replaced by the writing "inner core";[6]

2. The horizons approach each other asymptotically in infinite time (fig.3.5b). This is the scenario advocated in (Hossenfelder et al., 2010; Alesci and Modesto, 2014) on the basis of a different NSBH model.

In any case restoration of unitarity is possible: in case 1 information fallen into the BH can finally return to infinity after the evaporation (Frolov, 2014); in case 2 information can never escape the BH and remains trapped forever, but, since the BH never disappears, correlations are always there though unaccessible to an observer at infinity (Alesci and Modesto, 2014).

We want to understand which of the two scenarios is the most physically reliable. Remind that the Hawking temperature is related to the surface gravity by the relation

$$T = \frac{\kappa}{2\pi}.$$

[6]For completeness we refer to the classical paper Roman and Bergmann (1983), in which the authors argue that Hawking radiation may not feed the BH with a sufficient amount of negative energy to close the apparent horizon.

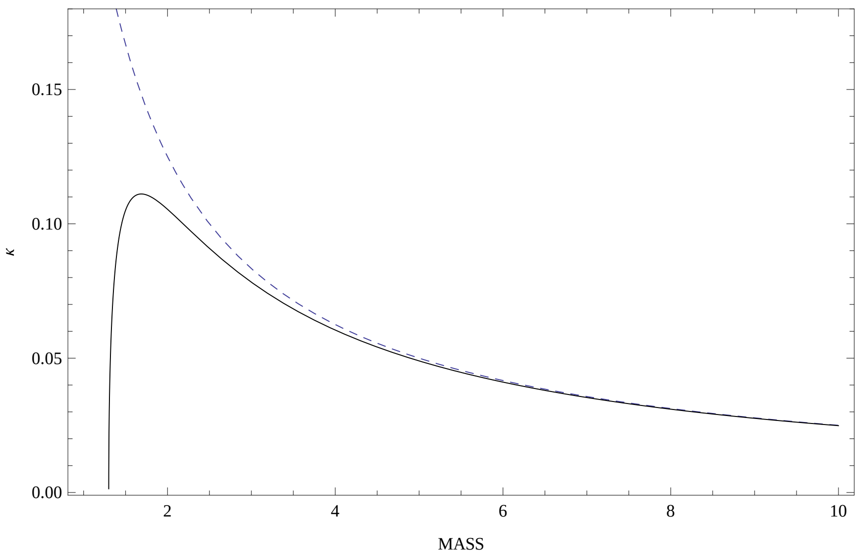


Figure 3.6: The surface gravity κ as a function of the mass ($L = 1$). The curve has a maximum for $M_{\rm crit} = \frac{27}{16}L$. The dashed line is the classical surface gravity $\kappa = \frac{1}{4M}$.

From eq. (3.3)-(3.4) the surface gravity is

$$\kappa = \frac{M(r_+) - r_+ M'(r_+)}{r_+^2}$$

which, using $r_+ = 2M(r_+)$, can be rewritten as

$$\kappa = \frac{1}{2r_+} - \frac{M'(r_+)}{r_+}$$

and finally, by the explicit expression (3.6), we obtain

$$\kappa = \frac{3}{4M} - \frac{1}{r_+}. \tag{3.42}$$

The behaviour of κ is shown in fig. 3.6. For large M it coincides with the classical one, but it differs significantly as the mass approaches the critical value: instead of growing to infinity, it reaches a maximum and then drops to $\kappa = 0$ at the critical mass $M_\star = 3\sqrt{3}L/4$. The maximum occours at $M_{\rm crit} = 27L/16$, as shown in Appendix C. When $M < M_\star$ there is no BH and the surface gravity is not defined.

If one takes seriously the whole curve, then the BH evaporates when $\kappa = 0$; but, from the third law of BH mechanics, the extremal configuration cannot be reached in a finite amount of time (as confirmed by an explicit calculation in Appendix C). So it seems that we are left with scenario (b).

In reality, we must be catious in drawing such a conclusion: what we have done is to assume that dynamics is the succession of the static *pictures* taken at any instant of time [7], but we know that this is not true. The first obvious reason is that we neglected backreaction effects, which can become important in the later stages of the evaporation, causing large deviations from the quasi-static behaviour. But, even in the semiclassical regime things can be rather different: in Balbinot (1986) it is shown that, under certain conditions, a Reissner-Nordstrom black hole presents a period of negative energy flux at infinity when the mass is still large. In the next chapter we introduce recently developed tecniques (Smerlak and Bianchi, 2014) based on entanglement entropy to analyse quantum effects in curved spacetimes: they reveal that negative energy fluxes are not peculiar of special cases, but emerge in a wide class of evolving spacetimes. To sum up, evaporation cannot simply be addressed in a quasi-static fashion and requires careful discussions, since the way in which quantum mechanics affects spacetime evolution is unclear: this will be the subject of the next chapter.

[7] We refer to this approach as *quasi-static*, while we use the expression *semiclassical regime* to indicate the early stages of the evaporation, when the BH mass is large compared to the Planck mass and QFT in curved spacetime is a good approximation.

Chapter 4

Entanglement entropy

4.1 Introduction

In Ch. 2 we saw that a black hole, or at least a semiclassical evolving one, has an associated entropy

$$S_{BH} = k_B \frac{c^3 A}{4G\hbar} + \text{const.}$$

which contributes as an additional entropy term in the Generalized Second Law of thermodynamics

$$\delta S_{\text{matter}} + \delta S_{BH} \geq 0.$$

Today there isn't yet an explanation of the physical mechanism behind the Bekenstein entropy (Bekenstein, 1994): by analogy with statistical mechanics we expect

$$S_{BH} = k_B \log\big(\#\ \text{degrees of freedom}\big) \tag{4.1}$$

but the unsolved question is *What are the relevant degrees of freedom?*

A first logical possibility is that one should count the number of internal states associated with a macroscopical BH configuration, i.e. with given values of mass, charge and angular momentum; in fact we know by the "no hair" theorem that many different initial conditions can lead to undstinguishable black holes with the same macroscopic parameters. This is a *bulk* interpretation of the BH entropy.

An alternative is to consider the horizon itself as our thermodynamical system and to count the number of quantum states compatible with it: this is an *holographic* interpretation.

The connection between this subject and the notion of entanglement entropy[1] came after the papers by Bombelli et al. (1986) and Srednicki (1993). They considered (3+1)-dimensional QFT of a massless scalar field in flat spacetime, at zero temperature, in its nondegenerate ground state $|0\rangle$. The von Neumann entropy associated with the density matrix $\hat{\rho} = |0\rangle\langle 0|$ is null, because the system is

[1] We will define entanglement entropy in a while.

in a pure state:

$$S = -Tr[\hat{\rho}\log\hat{\rho}] \equiv 0.$$

They imagined to decompose the space in two regions, by means of a spherical two-surface of area A and then considered the reduced density matrix $\hat{\rho}_{\text{red}}$, obtained tracking $\hat{\rho}$ over the degrees of freedom inside the sphere. They were able to show that the leading contribution to the von Neumann entropy associated with $\hat{\rho}_{\text{red}}$ is

$$S_{\text{red}} = c_1 \frac{A}{\epsilon^2} + (\text{subleading terms}) \tag{4.2}$$

where c_1 is a constant and ϵ is an UV cutoff. The reason why S_{red} is nonzero is that empty space contains coarse grained correlations of the form

$$< \Phi(x)\Phi(y) > \sim \frac{1}{|d(x,y)|^2}$$

and then the outside of the sphere shares half of its correlations with the inside. If one thinks to the spherical boundary as the BH horizon, the analogy with the entropy of a black hole is apparent, although (4.2) was obtained in a flat background.

The proportionality between S_{red} and the area of the boundary led to suppose that Bekenstein entropy could be explained via the von Neumann entropy of the reduced density matrix of the fields coupled with gravity. Such entropy is called *entanglement entropy*, since it's a measure of the correlations between two subsystems. The proposal suffers of two serious problems:

1. the UV cutoff shoul be fine tuned to give the $\frac{1}{4}$ factor of the Bekenstein formula;

2. S_{red} scales linearly with the number of fields which are progressively excited as the temperature grows: it means that the UV cutoff knows the structure of the IR physics. This unpleasant feature is called the *species problem*.

So it doesn't seem appropriate to identify Bekenstein entropy with the entanglement entropy of fields. We can be modest and say that at least part of the entropy of the black hole is explanable as entanglement entropy, but this is really too modest if compared to the original question. Fortunately this is not the end of the story, for entanglement entropy reveals to be a powerful and promising tool in the study of the quantum aspects of BH physics.

The best starting point for a profane is the living review Solodukhin (2011). In this chapter we certainly don't pretend to be exhaustive: our exposition, and the subsequent applications, reflect both the environment in which this work has been prepared, and the limited time at our disposal with respect to the vastity of the subject.

4.1.1 Definitions and results

Consider a quantum system in a pure state $|\Psi\rangle \in \mathscr{H}$, with reduced density matrix $\hat{\rho} = |\Psi\rangle\langle\Psi|$. Let $\mathscr{H}_A \subset \mathscr{H}$ be the Hilbert space of an observer A who measures a subset A of the whole system, and $\mathscr{H}_B \subset \mathscr{H}$ the Hilbert space of an observer B who measures the remainder. The density matrix associated to the subsystem A is $\hat{\rho}_A = Tr_B\hat{\rho}$. The *entanglement entropy* (EE) of A is just the von Neumann entropy of $\hat{\rho}_A$:

$$S_A = -Tr_A\left[\hat{\rho}_A \log \hat{\rho}_A\right]. \tag{4.3}$$

The same construction applies if $\hat{\rho}$ is not a pure state ($\hat{\rho}^2 = \hat{\rho}$) but a mixed one ($\hat{\rho} \neq \hat{\rho}^2$).

Entanglement entropy has the following properties:

1. NON-EXTENSIVITY

   If $\hat{\rho}$ is a pure state, then $S_A = S_B$. Then EE is not necessarily extensive.

   The previous equality follows from the fact that every pure state $|\Psi\rangle \in \mathscr{H}$, such that $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$, can be decomposed in a suitable basis as

   $$|\Psi\rangle = \sum_i \sqrt{p_i}\,|i\rangle_A\,|i'\rangle_B \qquad \text{(Schmidt decomposition)}$$

   where $p_i \in [0,1[$ are such that $\sum p_i = 1$.

2. UPPER BOUND

   Given $\hat{\rho}_A = \sum p_i\,|i\rangle_A\,\langle i|_A$, the associated von Neumann entropy is bounded by

   $$S_A \leq \log m \qquad m = \dim \mathscr{H}_A \tag{4.4}$$

   the equality holding when the state is maximally mixed, i.e. when $p_i = \frac{1}{m}$ for all i.

3. SUBADDITIVITY

   $$S_A + S_B \geq S_{A\cup B} \tag{4.5}$$

What is the physical meaning of the entanglement entropy? It measures the amount of quantum correlations between A and B: the higher the EE, the more the correlations. A complementary point of view is to say that EE measures *information.*

Let $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$ such that $N = \dim \mathscr{H}$, $m = \dim \mathscr{H}_A$ and $n = \dim \mathscr{H}_B$. The amount of information contained in the subsystem A is defined to be the discrepancy of its entanglement entropy from the maximum:

$$I(A) = \log m - S_A \tag{4.6}$$

and similarly

$$I(B) = \log n - S_B \tag{4.7}$$

$$I(A \cup B) = \log N - S_{A \cup B}. \tag{4.8}$$

Then the information shared by A and B is

$$\begin{aligned} I(A, B) &= I(A \cup B) - I(A) - I(B) \\ &= S_A + S_B - S_{A \cup B}. \end{aligned} \tag{4.9}$$

This is the socalled *mutual information*, which by subadditivity is non-negative. Notice that if $\hat{\rho}$ is a pure state the mutual information is $I(A, B) = 2S_A = 2S_B$.

(1+1)-Conformal Field Theory Robust results have been obtained in the context of (1+1)-dimensional conformal field theory: due to the high symmetry imposed, it is possible to compute the exact expression of entanglemet entropy in a large variety of cases. The first was found by Holzhey et al. (1994) and later reobtained by Calabrese and Cardy (2004). They showed that the EE of a linear segment of lenght l, thought as a subset of an infinitely long 1-dimensional quantum system at zero temperature, is given by the simple expression

$$S = \frac{c}{3} \log \frac{l}{\epsilon} + \text{const.} \tag{4.10}$$

where c is a topological invariant of the theory called "central charge", which depends on the particular unitary representation of the symmetry group[2]. As it is apparent, EE is not extensive. The fact that the UV cutoff ϵ appears inside the logarithm is remarkable, since it makes the variation of EE insensitive to the cutoff, i.e. finite and well defined.

Calabrese and Cardy (2004) studied EE of QFTs in (1+1)-dimensions[3] and found that, using an appropriate conformal map, eq. (4.10) could be used to compute the EE of a segment in a thermal mixed state with temperature β^{-1}:

$$S(\beta) = \frac{c}{3} \log\Big(\frac{\beta}{\pi\epsilon} \sinh \frac{\pi l}{\beta}\Big) + \text{const.} \tag{4.11}$$

The asymptotic behaviour of (4.11) is

$$S(\beta) \sim \begin{cases} \frac{c}{3} \log \frac{l}{\epsilon} & l \ll \beta \\ \frac{c\pi l}{3\beta} & l \gg \beta \end{cases} \tag{4.12}$$

Then EE measures quantum correlations only for low temperatures, where the flat case formula is recovered, while for high temperatures it's just the ordinary measure of thermal correlations.

[2] In the case of a massless scalar field $c = 1$.

[3] See also Calabrese and Cardy (2009) and Calabrese and Cardy (2006).

4.2 Page's argument

Page (1993b) used entanglement entropy to analyze the outcome of the Hawking radiation. He reasoned as follows: he assumed that at least part of the BH entropy is due to the correlations with the Hawking radiation, and that the global state (BH + radiation) is pure; if one wants the final state of the radiation to be pure after the BH completely evaporates, then one should require its EE to vanish after the evaporation. Nevertheless, as radiation is emitted, more and more correlation are set up with the BH system and one doesn't see how EE can decrease to zero. He concluded that, if unitarity has to be preserved, then semiclassical analysis breaks up at a certain stage of the evaporation, and there is a turning point- the socalled *Page time*- at which correlations begin to escape the BH horizon to purify the radiation: from the Page time on, the picture of an entangled pair production at the horizon fades, because the BH reemits the previously fallen-in correlations, instead of creating new ones.

Remarkably, he was able to compute the turning point: it occours when the BH has emitted one half of its original BH entropy. For a Schwarzschild BH this corresponds to a residual mass $M_{\text{Page}} = M_0/\sqrt{2} \gg M_{\text{Planck}}$: semiclassical analysis fails when the mass is still large and the curvature on the horizon is small, i.e. when semiclassical arguments **sould work!**

To estimate the Page time we first give the following naive argument: from the Generalized Second Law of thermodynamics

$$\delta S_{BH} + \delta S_{\text{rad}} \geq 0.$$

At the beginning of the evaporation, say $t = 0$, the entropy is just that of the BH:

$$S_0 = S_{BH}(0) + S_{\text{rad}}(0) \equiv S_{BH}(0)$$

while at an intermediate time t both the contributions are present:

$$S_t = S_{BH}(t) + S_{\text{rad}}(t)$$

hence from the GSL

$$S_{BH}(t) + S_{\text{rad}}(t) \geq S_{BH}(0).$$

Now recall that for a bipartite pure state $S_A^{\text{ent}} = S_B^{\text{ent}}$, so the BH entropy is at least equal to, if not greater of, the entropy of the radiation[4]:

$$S_{BH}(t) \geq S_{\text{rad}}(t).$$

Therefore it is straightforward to get

$$S_{BH}(t) \geq \frac{S_{BH}(0)}{2}$$

[4]Of course we are assuming all the entropy of the radiation to be entanglement entropy, which is a good approximation in the semiclassical regime when the radiation is nearly thermal.

so we deduce that the hypothesis of perpetual creation of new correlations is certainly false after the Page time, and after then the number of outcoming correlations must be greater that the number of incoming ones.

The original argument by Page is more precise. It relies on a previous conjecture, proposed in Page (1993a) and later proved to be valid by Foong and Kanno (1994):

Theorem 4.2.1 (Page, Foong and Kanno)**.** *Consider a quantum system in a pure state of a Hilbert space* $\mathscr{H}$ *of dimension* $\dim\mathscr{H} = N$*. Divide the system in two subsystems A and B such that* $\mathscr{H} = \mathscr{H}_A \otimes \mathscr{H}_B$*, and let* $\dim\mathscr{H}_A = m$ *and* $\dim\mathscr{H}_B = n$*, so* $N = mn$*. Finally suppose* $m \leq n$*. If the system is in a random pure state* $|\Psi\rangle \in \mathscr{H}$*, then the average entanglement entropy is*

$$\langle S_{m,n}\rangle = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n} \tag{4.13}$$

which in case $1 \ll m \leq n$ *reduces to*

$$\langle S_{m,n}\rangle \simeq \log m - \frac{m}{2n} \,. \tag{4.14}$$

The consequences of Page's theorem go beyon the BH physics, so let's proceed in abstract. For our purposes it will be sufficient to use eq. (4.14). The first immediate consequence is that the smaller subsystem contains less that one half bit of information:

$$I(A) = \log m - \langle S_{m,n}\rangle = \frac{m}{2n} \tag{4.15}$$

so it doesn't show any sign that the overall state is pure: an observer restricted to A would measure a nearly thermal state. On the other hand, the information of the larger subsystem and the mutual information are:

$$I(B) = \log\frac{n}{m} + \frac{m}{2n} \tag{4.16}$$

$$I(A,B) = \log m^2 - \frac{m}{n} \tag{4.17}$$

and the three correctly sum up to the total information $I_{\text{tot}} = \log N = \log(mn)$. There are two limit cases:

$$I_{\text{tot}} \sim \begin{cases} I(B) & m \ll n \\ I(A,B) & m \simeq n \end{cases} \tag{4.18}$$

so most of the information is stored in the larger subsystem or in the correlations between the two, depending on wether B is much greater than A or they are comparable.

Now let's apply these results to BH evaporation. In the early stages of the evaporation the BH is the larger subsystem and the radiation is the smaller one, so the radiation shows no sign of the global purity and is effectively perceived as a thermal mixture. When the situation is reversed, i.e. the BH has emitted for a very long time, the radiation is a much larger subsystem and so it stores almost all the information, thus becoming purer and purer. Again we have reached the conclusion that information must necessairily come out of the BH, via quantum correlations, if the evolution is unitary.

The turning point occours when $I(A,B)$ is maximum, for $m = \sqrt{N}$. Page assumes that when the BH is still large the Bekenstein entropy approximately is

$$S_{BH} \simeq \log\left(\dim \mathscr{H}_{BH}\right)$$

where $\mathscr{H}_{BH}$ is the Hilbert space compatible with the "no hair" theorem. Then at the beginning of the Hawking process $S_{BH}(0) \simeq \log N$ while at the Page time $S_{BH}(t_{\text{Page}}) \simeq \log m = \frac{1}{2}\log N$, and the relation between Page time and entropy is retrieved.

Observe that the validity of Page's argument relies only on unitarity, it's not necessary that the BH completely evaporates and thus it applies also to non-singular black hole scenarios. It leaves us a fascinating explanation of unitarity: information flows from the BH to the environment because geometry converts itself into particles. But it also leaves us with unpleasant problems: what is the mechanism behind the flow of correlations from the inside to the outside? Does it imply some form of non-locality? Recently the subject become more intriguing when AMPS (Almheiri et al., 2013) claimed that, if the Page argument is correct, one must choice between unitarity and the equivalence priciple.

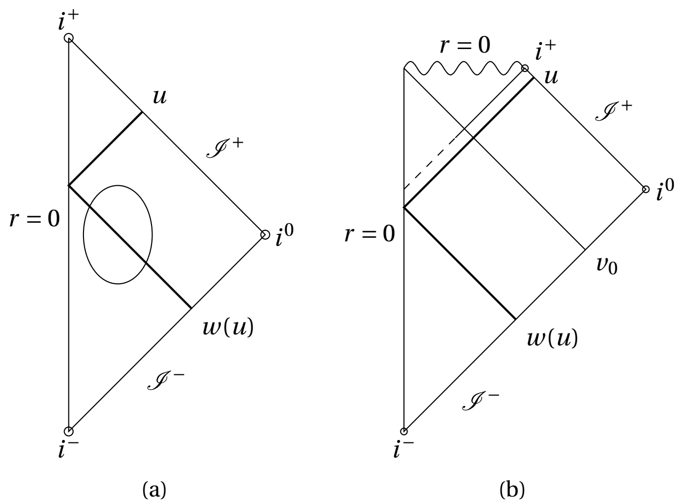


Figure 4.1: (a) A spacetime with the global structure of Minkowski: both $\mathscr{I}^- \cup i^-$ and $\mathscr{I}^+ \cup i^+$ are Cauchy surfaces; the ellipse is an exemplyfying region where the gravitational field is non-vanishing; (b) Global structure of a Vaidya collapsing null shell forming a Schwarzschild black hole.

4.3 Smeared entanglement entropy

Consider a two-dimensional asymptotically flat spcetime $\mathscr{M}$ with line element

$$ds^2 = -\Omega^2(v,u)dv^2du^2$$

where (v,u) are double null coordinates corresponding respectively to ingoing and outgoing modes. Since we are addressing the unitarity problem, we are mainly interested in spacetimes with the global structure of Minkowski (fig. 4.1a), but in principle the results of this section are applicable also to more complicated global structures such as a collapsing star (fig. 4.1b). In both cases there exists a region $\mathscr{G}$ from which outgoing light rays can be traced back to (at least a connected part of) $\mathscr{I}^-$, by means of the reflecting surface $r=0$. So there is a function $w : \mathscr{G} \to \mathscr{I}^-$ mapping the u-coordinate of an event on $\mathscr{G}$ into the corresponding v-coordinate on $\mathscr{I}^-$. Observe that: i) $\mathscr{I}^+$ belongs to $\mathscr{G}$; ii) for a given event (v,u), $w(u) \leq v$; iii) $w(u)$ is normalized as $\dot{w}(u) \to 1$ when $u \to -\infty$.

We have just faced the function $w(u)$ during the computation of the Hawking spectrum. There we interpreted null rays as the trajectories of a massless field modes in the geometric optics approximation. Indeed we saw that asymptotic flatness allows to well-define QFT both on $\mathscr{I}^-$ and $\mathscr{I}^+$, and $w(u)$ provides a sort of *scattering map* between the two theories: in particular, we derived the expansion of the vacuum $|0_-\rangle$ of $\mathscr{I}^-$ in terms of the excited modes of $\mathscr{I}^+$.

Following Bianchi and Smerlak (2014) we analyze things from a different but complementary perspective. Consider a CFT with central charge c defined on

$\mathscr{M}$, and let $|0_-\rangle$ and $|0_+\rangle$ be the vacuum states respectively on $\mathscr{I}^-$ and $\mathscr{I}^+$. The entanglement entropy of an interval $[u_0, u_1] \in \mathscr{I}^+$ of lenght $|u_0 - u_1| = l$ in the vacuum state at zero temperature is provided by formula (4.10) (up to a constant):

$$S^l_{|0_+\rangle} = \frac{c}{6} \log \frac{(u_1 - u_0)^2}{\epsilon^2} \tag{4.19}$$

and an analogous expression holds for an interval in the vacuum state on $\mathscr{I}^-$.

From QFT in curved in curved spacetimes, we know that a system in the vacuum state $|0_-\rangle$ is not necessairily perceived as a system in $|0_+\rangle$, when it is let to evolve from $\mathscr{I}^-$ to $\mathscr{I}^+$. So we expect that, if a single interval of lenght l evolve from $\mathscr{I}^-$ to $\mathscr{I}^+$, the difference

$$S^l_{|0_-\rangle} - S^l_{|0_+\rangle},$$

i.e. the excess of entanglement entropy measured by an observer in the vacuum state on $\mathscr{I}^+$, is different from zero. To compute this difference we need to express EE in a cutoff independent way: to this purpose we introduce the notion of *smeared entanglement entropy*.

Consider again the domain $A \equiv [u_0, u_1] \in \mathscr{I}^+$ of leght l, and introduce the *smeared* domain $A \cup \Delta \equiv [u_0 - \delta u_0, u_1 + \delta u_1]$, where $\Delta \equiv [u_0 - \delta u_0, u_0] \cup [u_1, u_1 + \delta u_1]$. Then the smeared entanglement entropy of A is defined as

$$S^l_{|0_+\rangle} = \lim_{\delta u_0, \delta u_1 \to 0} \frac{S(A) + S(A \cup \Delta) - S(\Delta)}{2} \tag{4.20}$$

and the resulting expression[5] is cutoff independent:

$$S^l_{|0_+\rangle} = \frac{c}{6} \log \frac{(u_1 - u_0)^2}{\delta u_0 \delta u_1}. \tag{4.21}$$

Projecting the domain A on $\mathscr{I}^-$ through the map $w(u)$, we obtain the entanglement entropy

$$\begin{aligned} S^l_{|0_-\rangle} &= \frac{c}{6} \log \frac{(w_1 - w_0)^2}{\delta w_0 \delta w_1} \\ &= \frac{c}{6} \log \frac{(w_1 - w_0)^2}{\dot{w}_0 \dot{w}_1 \delta u_0 \delta u_1} \end{aligned} \tag{4.22}$$

where the dot means a derative w.r.t. u. Now we can compute the excess of entanglement entropy, and the result is:

$$S(l) = \frac{c}{6} \log \frac{(w_1 - w_0)^2}{\dot{w}_0 \dot{w}_1 (u_1 - u_0)^2}. \tag{4.23}$$

[5] In the limit $\delta u_0, \delta u_1 \to 0$, $S(\Delta)$ coincides with $S\big([u_0 - \delta u_0, u_0]\big) + S\big([u_1, u_1 + \delta u_1]\big)$.

Example: collapsing star Consider the two-dimensional collapsing star diagram in fig. 4.1b: in the Vaidya approximation (see Ch. 2) the function $w(u)$ can be computed implicitely:

$$u = w - 4M\log\left(-\frac{w}{4M} - 1\right); \tag{4.24}$$

in the limit $t_\star \to \infty$ we have

$$w(u) \simeq -4M\left(\mathrm{e}^{-u/4M} + 1\right) \qquad \dot{w}(u) \simeq \mathrm{e}^{-u/4M}. \tag{4.25}$$

Putting (4.25) into (4.23) we obtain

$$S(l) = \frac{c}{3}\log\left(\frac{\sinh \pi l T_H}{\pi l T_H}\right) \qquad T_H = \frac{1}{8\pi M} \tag{4.26}$$

which is precisely the excess EE of an interval l at temperature T_H.

Let's return to the general case. From (4.23) we get the excess EE of the whole $\mathscr{I}^+$ up to an advanced time u:

$$S(u) = -\frac{c}{6}\log \dot{w}(u) \tag{4.27}$$

where we have taken the limit $\lim_{u_0\to-\infty} S(l)$ and used the fact that $\lim_{u\to-\infty}\dot{w}(u) = 1$. Expression (4.27) can be remarkably related to the energy flux at future null infinity. Indeed, while the flux of energy at $\mathscr{I}^+$ in the vacuum $|0_+\rangle$ is null, the vacuum $|0_-\rangle$ has a nonvanishing value of the energy flux at $\mathscr{I}^+$ given by the Davies-Fulling-Unruh formula (Davies et al., 1976; Birrell and Davies, 1984)

$$F(u) = -\frac{c}{24\pi}\left(\frac{\dddot{w}(u)}{\dot{w}(u)} - \frac{3}{2}\frac{\ddot{w}^2(u)}{\dot{w}^2(u)}\right) \tag{4.28}$$

and using (4.27) it reduces to

$$F(u) = \frac{1}{4\pi}\left(\frac{3}{c}\dot{S}^2(u) + \ddot{S}(u)\right). \tag{4.29}$$

Thus the energy flux at infinity is completely determined by the excess of entanglement entropy. Defining

$$\dot{S}(u) = \frac{c}{3}\frac{\dot{\psi}(u)}{\psi(u)} \qquad V(u) = \frac{12\pi}{c}F(u) \tag{4.30}$$

eq. (4.29) can be rewritten as

$$\ddot{\psi}(u) = V(u)\psi(u). \tag{4.31}$$

The asymptotic behaviour of $S(u)$ is not totally free. First of all, by construction, $S(-\infty) = 0$. $S(+\infty)$ can instead be different from zero, e.g. if the BH doesn't completely evaporate[6]. Moreover we expect that $S(u)$ has a smooth profile as $u \to \pm\infty$, in such a way that

$$\lim_{u\pm\infty} \dot{S(u)} = 0.$$

Then integration of (4.31) yields

$$\int_{\mathscr{I}^+} V(u)\psi(u)du = 0. \tag{4.32}$$

The function $\psi(u)$ obeys

$$\psi(u_1) = \exp\left[\frac{3}{c}(S_1 - S_0)\right]\psi(u_0)$$

so it has a constant sign and, if we require $S(u)$ to be bounded from below, it never vanishes. So eq. (4.32) is equivalent to

$$\int_{\mathscr{I}^+} F(u)e^{3S(u)/c} du = 0. \tag{4.33}$$

Bianchi and Smerlak (2014) claim that (4.33) is the most important result of their analysis: it indeed implies that $F(u)$ must be negative somewhere! In conclusion: any two-dimensional conformal field theory, such that the global structure of the spacetime is non trivial, predicts negative-energy fluxes to $\mathscr{I}^+$.

4.4 Applications to black holes

Black holes are four-dimensional objects, but the bulk of the evaporation is emitted in the s-wave sector. For this reason, it doesn't seem unreasonable to study a 2-dimensional toy model of black hole. In this spirit, we want to apply the previous analysis to the Hawking evaporation and explore the consequences. Two main results are reached: (i) the emitted radiation has a negative energy phase; (ii) the purification time has an extremely long lower bound.

Negative energy phase Let's consider the entanglement entropy formula found by Page

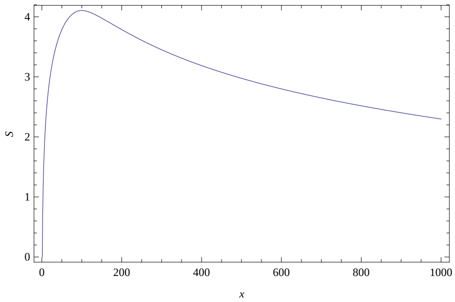
$$S \simeq \log m - \frac{m}{2n}$$

where $mn = N$ is the dimension of the total Hilbert space and $1 \ll m \leq n$ is assumed. Since m is always the dimension of the smaller subsystem, the EE of the emitted radiation is

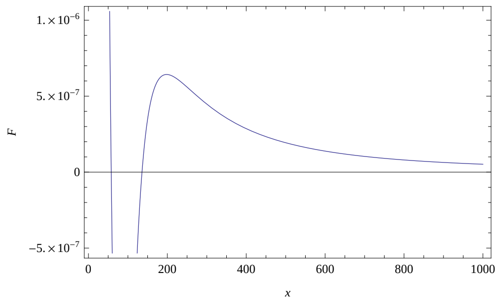
$$S_{\text{rad}} = \begin{cases} \log x - \frac{x^2}{2N} & \text{for } x < \sqrt{N} \\ \log \frac{N}{x} - \frac{N}{2x^2} & \text{for } x > \sqrt{N} \end{cases} \tag{4.34}$$

where $x = \dim \mathscr{H}_{\text{rad}}$. As discussed above, it has a maximum for $x_{\text{Page}} = \sqrt{N}$. It

[6] It doesn't mean that evolution is non-unitary, because information can remain trapped in some form of remnant and reach i^+ rather than $\mathscr{I}^+$.



(a)



(b)

Figure 4.2: (a) Entanglement entropy of the radiation S_{rad} as a function of $x = \dim \mathscr{H}_{\text{rad}}$; (b) the corresponding energy flux F at infinity as computed from eq. (4.29). F is negative in a neighbourhood of the maximum of S.

follows from eq. (4.29) that the $F(u)$ is negative at the Page time. Actually, there is a negative energy phase around t_{Page}, as shown in fig. 4.2. As a result the Bondi mass

$$M(u) = M_0 - \int_{-\infty}^{u} F(u)du$$

is not monotonically increasing, but there is a temporary decrease just before the beginning of the purification phase.

Purification time [7] The purification phase is the period in which the early radiation, which is in a nearly maximally mixed state, begins to be purified by the late radiation. IF WE ASSUME that information is fully purified at sufficiently late times, i.e. if we assume no stable remnants, then the purification time is bounded from below. This can be seen as follows:

Let $u_\star$ be the value of the affine coordinate u on $\mathscr{I}^+$ where the entanglement entropy is maximal, and let u_{f} be the value of u where the entanglement entropy and its first derivative is practically indistinguishable from zero:

$$S(u) \leq S(u_\star) \qquad S(u_{\text{f}}), \dot{S}(u_{\text{f}}) = 0$$

Then we define the purification time as

$$\Delta u = u_{\text{f}} - u_\star \tag{4.35}$$

Assume also that the semiclassical approximation is valid in the interval $u \in]-\infty, u_\star]$. Combining (4.25), (4.27) and (4.28) we obtain

$$\dot{S}(u) \simeq \frac{\kappa}{6} \qquad \dot{M}(u) \simeq -\frac{\kappa^2}{48\pi} \qquad \text{for } u \in]-\infty, u_\star] \tag{4.36}$$

[7]This and the following paragraph are based on a private discussion with E. Bianchi, M. Smerlak, C. Rovelli, S. Speiale and T. De Lorenzo.

The energy emitted in the purification phase is

$$E_{\rm f} = \int_{u_\star}^{u_{\rm f}} F(u)\,du = \frac{3}{4\pi}\int_{u_\star}^{u_{\rm f}} \dot{S}(u)^2\,du \tag{4.37}$$

where we in the second equality we used $\dot{S}(u_\star) = \dot{S}(u_{\rm f}) = 0$. Eq. (4.37) is formally the lagrangian of a free point in two dimensions, so it is minimized by $\dot{S}(u) =$ const.:

$$E_{\rm f} \geq \frac{3}{4\pi}\frac{S(u_\star)^2}{\Delta u}. \tag{4.38}$$

Conservation of the energy requires $E_{\rm f} \leq M_\star$, where $M_\star$ is the value f the BH mass at $u = u_\star$, so

$$\Delta u \geq \frac{3}{4\pi}\frac{S(u_\star)^2}{M_\star}. \tag{4.39}$$

But from (4.36) one has $S(u_\star) \simeq 16\pi(M_0^2 - M_\star^2)$, therefore

$$\Delta u \geq 192\pi\frac{\left(M_0^2 - M_\star^2\right)^2}{M_\star}. \tag{4.40}$$

This is the desired result:

If the semiclassical approximation is valid until the mass becomes planckian, $M_\star \approx M_{\rm planck}$, then $\Delta u_{\rm min} \sim M_0^4$, in accordance with previous results obtained by Carlitz and Willey (1987). The remnant should emit a huge purifying entropy, but with a very small energy at its disposal: the only way is to release particles with long waveleght in a large interval of time. The late radiation is nearly thermal, like the early one: correlations cannot be measured locally because they are spread in time.

If instead, as suggested by Page's argument, semiclassical approximation breaks down when the mass is still macroscopic, $M_\star \approx$ a fraction of M_0, then $\Delta u_{\rm min} \sim M_0^3$. Again the correlations are distributed over large time intervals. Whatever the case, the purification time turns out to be extremely long.

Nonsingular Black Holes The inner horizon of the Hayward BH, or that of a Reissner-Nordstrom BH, plays the role of an attractor for outgoing null rays propagating in its vicinity: as a consequence the freqency of a photon is exponentially blueshifted. As observed in Poisson and Israel (1990), this causes the Cauchy horizon to be unstable under small perturbation and to be replaced by an effective singularity. If however we allow the NSBH to be a transient phase wich evaporates, leaving a remnant without any horizon, then no effective singularity forms; nevertheless the blue-shift tendency manifests its catastrophic effects in the form of a huge energy burst emitted just after the evaporation. This is really unpleasant, for it means that the Bondi mass of the remnant can become arbitrarily negative! Incidentally observe that, if we correct the Hayward metric with the time delay prefactor $G(r)$, as proposed in sec. 3.3.2, and assume that its value is still small over r_-, then the squeezing effect of the Cauchy horizon is compesated by a peeling effect due to the redshift induced by $G(r)$.

One can ask how much the evaporation scenario is modifies by the insertion of such $G(r)$. Actually, not so much: indeed the surface gravity of the outer horizon r_+ changes just by a multiplicative factor $\sqrt{G(r_+)}$:

$$\kappa = \sqrt{G(r_+)}\left(\frac{M(r_+) - r_+ M'(r_+)}{r_+^2}\right). \tag{4.41}$$

Now it is reasonable to assume that in regions of low curvature the metric is not sensibly modified, i.e. we expect $G(r_+) \approx 1$ and thus the evaporation scenario to be quite the same. Notice, however, that this holds just in the semiclassical regime, because, when the NSBH evaporates, $r_+ \approx r_-$ and the effects of $G(r)$ can become important even on the outer horizon.

In Bianchi et al. (2014) the implications of EE entropy techniques on the evaporation of Black Holes are analyzed, by means of the eqs. (4.27) and (4.29). In particular, the authors show how to compute the purification time for a given evaporation scenario, but they don't perform the explicit calculation for a NSBH. It seems however[8] that the purification time turns out to be too much short, in comparison with the lower bounds found in the previous paragraph. In this regards, we observe that the time delay factor dilates neighbouring null rays "over which" the information is transported in the geometric optics approximation, and consequently lenghtens the purification time. Therefore, as previously anticipated, we find more than one reason to believe that the time delay factor $G(r)$ should be considered in a more realistic effective model of Nonsingular Black Hole evaporation.

[8]M. Smerlak, private communication.

Conclusions

In this work we have reviewed the theory of Black Holes, as predicted by General Relativity and Quantum Mechanics.

In Ch. 1, after a brief account of the Schwarzschild solution, we introduced the global methods and showed how GR predicts the occourrence of singularities in a wide range of situations. Next we saw that BHs obey a set of four simple laws, analogous to those of thermodynamics, a fact leading to speculate that they underlie a deeper complexity, despite their apparent simplicity.

In Ch. 2, following classical papers by Hawking and Wald, we combined GR and QM in a semiclassical approximation and derived the Hawking radiation, a phenomenon of spontaneous particle creation by BHs. Then we stressed a paradoxical implication, namely the information loss paradox (loss of unitarity in pure states evolution).

In Ch. 3 we reviewed the theory of Nonsingular Black Holes, proposed as a possible effective description of unitarity restoration. We presented two fundamental results: (i) NSBHs can be grouped in two universality classes, a classification which allows to predict many global properties even before providing an explicit line element; (ii) NSBHs are accompanied by topology change. We also made a particular choice of the line element and studied its properties, which in view of the previous results should be considered paradigmatic.

In Ch. 4 we returned on the connection between BHs and thermodynamics, according to which BHs possess an entropy proportional to their area. We investigated a possible quantum explanation of BH entropy, throughout the notion of entanglement entropy. We found that entanglement entropy puts severe constraints on the unitary evolution of a BH, and implies departures from GR even in low curvature regimes, if one waits a sufficient time (Page time). The most remarkable feature is that Hawking radiation has a period of negative energy emission.

Among the results of our analysis, we found more than one reason to suggest that modelling Black Hole evaporation via effective nonsingular metrics requires the introduction of a *time delay factor* in the metric: this additional term was originally introduced in Ch. 3 to delay a clock at the centre with respect to one at infinity, but at the end of Ch. 4 we observed that it may also address the energy conservation problem and the purification time inconsistency, exhibited by the classical NSBH scenarios considered in the literature.

Appendix A

Killing fields

Let

$$x'^a = x^a + \chi^a \tag{A.1}$$

be a change of coordinate generated by the infinitesimal vector displacement χ^a. The field χ^a is said to be a *Killing field* if the transformation (A.1) preserves the form of the metric, i.e. if

$$g'_{ab}(x) - g_{ab}(x) = 0.$$

This is equivalent to require that the Lie derivative of g_{ab} w.r.t. χ^a is null:

$$\pounds_\chi g_{ab}(x) = 0$$

which expands to

$$\nabla_a \chi_b + \nabla_b \chi_a = 0. \tag{A.2}$$

On the other hand, if (A.2) holds, the form of the metric is preserved, and therefore eq. (A.2) is a necessary and sufficient condition for $g_{ab}(x)$ being invariant under (A.1).

Suppose that the spacetime admits a Killing field χ^a and let u^a be the tangent vector to an affinely parametrized geodesic:

$$u^a \nabla_a u_b = 0.$$

Then the quantity $\mathscr{E} = -\chi^b u_b$ is conserved along the geodesic. Indeed

$$u^a \nabla_a (\chi^b u_b) = \chi^b u^a \nabla_a u_b + u^a u^b \nabla_a \chi_b = 0$$

where the first term vanishes by the geodesic equation, and the second because $\nabla_a \chi_b$ is an antisymmetric tensor. $\mathscr{E}$ is called the *Killing energy* of u^a w.r.t. χ^a.

Now consider a particle travelling along the geodesic u^a. If χ^a is timelike, there exists a special family of observers defined by the orbits of χ^a. The local energy of the particle, as measured by such observers, is defined as

$$E = -\frac{\chi^b}{\chi} u_b \qquad \chi = (-\chi^b \chi_b)^{1/2} \tag{A.3}$$

and we see that the quantity $E\chi$ is constant along the geodesic of the particle. In particular, if the spacetime is stationary, we can consider the family of the observers at rest w.r.t. to the time vector field $\chi^a = (1,0,0,0)$. The relation between the local energy measured at $\vec{r} = \vec{r}_1$ and that measured at $\vec{r} = \vec{r}_2$ is

$$E_1 = \frac{\chi_2}{\chi_1} E_2 = \left(\frac{g_{00}(\vec{r}_2)}{g_{00}(\vec{r}_1)} \right)^{1/2} E_2. \tag{A.4}$$

This formula enable us to compute the time delay between a clock at the space point $\vec{r}$ and a clock at infinity: infact, in Einstein's fashion, we can model a clock as a photon emitter; by Einstein's formula $E = h\nu$ we get[1]

$$\nu = \frac{1}{\sqrt{g_{00}(\vec{r})}} \nu_\infty$$

and finally, putting $\nu = 1/\delta t$, we obtain the desired relation

$$\delta t = \sqrt{g_{00}(\vec{r})} \delta t_\infty. \tag{A.5}$$

.

[1]We restric to asymptotically flat spacetimes, where $g_{00}(\infty) = 1$.

Appendix B

Energy conditions

Einstein's equations are

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab} \tag{B.1}$$

or equivalently

$$R_{ab} = 8\pi \Big(T_{ab} - \frac{1}{2} T g_{ab} \Big). \tag{B.2}$$

They imply the conservation equation

$$\nabla_b T^b_a = 0. \tag{B.3}$$

Let k^a be the tangent vector to a timelike geodesic. Then $-T^a_b k^b$ is the energy-momentum flux as seen by an observer moving along that geodesic, while $T_{ab} k^a k^b$ is the energy density measured by him. In general they don't obey any conservation law, but if the spacetime is stationary there is a timelike Killig vector field ξ^a associated with time translations identifying a special family of stationary observers, w.r.t. which the following continuity equation holds:

$$\nabla_a (T^a_b \xi^b) = 0. \tag{B.4}$$

Eq. (B.4) expresses energy-momentum conservation, as expected in presence of a time translation symmetry.

In general the stress-energy tensor is the sum of the contributions of each field composing the macroscopic object under study, so it can be difficult to give a simple expression for it. Nevertheless, under physical considerations, the behaviour of the stress-energy tensor can be constrained with some inequalities called *energy conditions*. Energy conditions are invoked to prove all the singularity theorems and many other important results in general relativity: we invite the reader to consult the recent review Curiel (2014). There are three main energy conditions:

1. Weak Energy Condition (WEC)

$$T_{ab}k^a k^b \geq 0 \qquad \text{for all timelike } k^a$$

It expresses the requirement that the energy density be non negative.

2. Dominant Energy Condition (DEC)

The DEC consists in the requirement that the energy flux never exceeds the speed of light, so it is equivalent to the condition that $-T^a_b k^b$ is a causal vector for all timelike k^a.

3. Strong Energy Condition (SEC)

$$\left(T_{ab} - \frac{1}{2}T g_{ab}\right)k^a k^b \geq 0 \qquad \text{for all timelike } k^a$$

The physical interpretation is not immediate. First observe that by eq. (B.2) it is equivalent to $R_{ab}k^a k^b \geq 0$. Looking at the Raychaudhuri's equation (1.19) we see that SEC gives a positive contribution to the convergence of a congruence and can be interpreted as a local tendency of gravity to be attractive.

For continuity, all the conditions are still valid in case of null k^a: in this case WEC and SEC coincide and are grouped under the name Null Energy Condition (NEC). Observe also that WEC is implied by DEC but not by SEC.

We are particularly interested in stress-energy tensors of the form

$$T^a_b = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{pmatrix}$$

where $-\rho$ is the entry corresponding to the timelike direction. Then it can be shown that the energy conditions are equivalent to the following inequalities:

$$\text{WEC:} \quad \rho \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \quad (i = 1,2,3) \tag{B.5a}$$

$$\text{DEC:} \quad \rho \geq 0 \quad \text{and} \quad |p_i| \leq \rho \quad (i = 1,2,3) \tag{B.5b}$$

$$\text{SEC:} \quad \rho + \sum_{i=1}^{3} p_i \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \quad (i = 1,2,3) \tag{B.5c}$$

Appendix C

Exact results

In this appendix we show that: (i) the profile of the surface gravity κ for the Hayward metric has a maximum when $M = 27L/16$; (ii) if the evaporation is always quasi-static, then the Hayward BH evaporates asymptotically at future infinity.

C.1 Maximum of κ

The surface gravity of the Hayward spacetime is

$$\kappa = \frac{3}{4M} - \frac{1}{r_+} \tag{C.1}$$

where r_+ is given in parametric form:

$$r_+ = \frac{2M}{3}\left(1 + 2\cos\frac{x}{3}\right) \tag{C.2a}$$

$$\cos x = 1 - \frac{27L^2}{8M^2} \qquad \text{with } x \in]0;\pi] \tag{C.2b}$$

The point $x \to 0$ corresponds to the case $r_+ \to 2M$, while on the contrary the point $x = \pi$ corresponds to the extremal configuration $M = M_\star = 3\sqrt{3}L/4$. We see that

$$\frac{4M}{3} \leq r_+ < 2M$$

from which $\kappa \geq 0$, the equality sign holding only at the extremal point $x = \pi$. We are going to show that κ has one and only one extremum: it follows that it must be a maximum.

First we notice that the function $f = L\kappa$ depends on M and L only through the ratio M/L. Since multiplication by L is just a dilatation, it will be the same to find the maximum of f w.r.t. the variable $z = M/L$. So we have

$$f = \frac{3}{4z} - \frac{3}{2z}\frac{1}{\left(1 + 2\cos\frac{x}{3}\right)} \tag{C.3a}$$

$$\cos x = 1 - \frac{27}{8z^2} \tag{C.3b}$$

We need the *triple cosine formula*:

$$\cos x = \cos\frac{x}{3}\left(4\cos^2\frac{x}{3} - 3\right). \tag{C.4}$$

By differentiation

$$\frac{27}{4z^3} = \frac{d\cos x}{dz} = 3\left(4\cos^2\frac{x}{3} - 1\right)\frac{d\cos\frac{x}{3}}{dz}. \tag{C.5}$$

It follows that

$$\frac{df}{dz} = \frac{3}{2z^2}\left[\frac{9}{2z^2}\frac{1}{\left(1+2\cos\frac{x}{3}\right)^2\left(4\cos^2\frac{x}{3}-1\right)} + \frac{1}{\left(1+2\cos\frac{x}{3}\right)} - \frac{1}{2}\right]. \tag{C.6}$$

We have to solve the equation $\frac{df}{dz} = 0$, which reduces to

$$4\cos^4\frac{x}{3} - 2\cos^2\frac{x}{3} + \left(\frac{1}{2} - \frac{9}{4z^2}\right) = 0 \tag{C.7}$$

whose solution is

$$\cos\frac{x}{3} = \frac{1}{2}\sqrt{1 \pm \frac{3}{z}}. \tag{C.8}$$

Substituting into (C.4) we obtain

$$\cos^2 x = 1 - \frac{27}{4z^2} \pm \frac{27}{4z^3}. \tag{C.9}$$

Finally, using (C.3b) it's easy to see that consistency selects the solution with the + sign, and that

$$z = \frac{27}{16} \tag{C.10}$$

which was the desired result.

C.2 Evaporation time

The evaporation time can be computed by applying the Stephan-Boltzmann law

$$\frac{dM}{dt} = -\sigma A T^4 \tag{C.11}$$

where σ is the Stephan-Boltzmann constant and $A = 4\pi r_+^2$ is the area of the black hole. The Hawking temperature is

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi}\left(\frac{3}{4M} - \frac{1}{r_+}\right) = \frac{1}{2\pi}\frac{3r_+ - 4M}{4Mr_+}. \tag{C.12}$$

Using eq.(C.2a) we obtain

$$\frac{dM}{dt} = -\frac{\beta}{M^2} \frac{\left(2\cos\frac{x}{3} - 1\right)^4}{\left(2\cos\frac{x}{3} + 1\right)^2} \tag{C.13}$$

where β is a positive constant.

Again, we prefer to use the rescaled variable $m = M/L$, and we redefine $X = \cos\frac{x}{3}$:

$$\frac{dm}{dt} = -\frac{\beta'}{m^2} \frac{(2X-1)^4}{(2X+1)^2}. \tag{C.14}$$

From the triple cosine formula

$$\cos x = 1 - \frac{27}{8m^2} = 4X^3 - 3X \tag{C.15}$$

so

$$\frac{dX}{dt} = -\gamma \frac{(2X-1)^4 (1+3X-4X^3)^{\frac{5}{2}}}{(2X+1)^2 (4X^2-1)} \tag{C.16}$$

where γ is a positive constant.

The evaporation starts at an undetermined value $m = m_0$ and ends when $m = m_1 = 3\sqrt{3}/4$: this corresponds to $x_1 = \pi$, i.e. $X_1 = \frac{1}{2}$. Equation (C.16) can then be integrated for t:

$$t_1 - t_0 = -\frac{1}{\gamma} \int_{X_0}^{X_1} dX \frac{(2X+1)^2 (4X^2-1)}{(2X-1)^4 (1+3X-4X^3)^{\frac{5}{2}}}. \tag{C.17}$$

The integral on the r.h.s. can be solved exactly and the result goes to $-\infty$ when evaluated in X_1: then $t_1 - t_0 \to +\infty$, as we claimed.

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