Part1B Advanced Physics

Classical Dynamics

Lecture Handout: 1

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Contents

Synopsis

Revision of Newtonian mechanics and non-inertial frames: Revision of Newton's laws, momentum, angular momentum, work and energy, conservative forces, central forces. Rotating frames, centrifugal and Coriolis forces.

Normal modes: Analysis of motion of many-particle systems in terms of normal modes. Degrees of freedom of a system, matrix notation, orthogonality of the eigenvectors, zero frequency and degenerate modes. Use of symmetries to find normal modes. Modes of molecules and continuous systems (standing waves).

Orbits: Kepler's laws. Effective potentials and the radial equation. Circular and elliptic orbits. Escape velocity, transfer orbits and gravitational slingshot of space probes. Parabolic and hyperbolic orbits, classical treatment of Rutherford scattering.

Rigid body dynamics: Angular velocity and angular momentum as vectors, conservation of angular momentum. Moment of inertia tensor, principal axes. Free precession. Forced precession, gyroscopes. Lamour precession.

Elastic media: Hooke's Law. Stress and strain tensors, principal stresses and strains. Elastic moduli for isotropic solids and relations between them. Poisson's ratio. Bending of beams, bending moments.

Books

Two good books that cover much of the course but at a somewhat too elementary level are:
Introduction to Classical Mechanics, French A.P. and Ebison M.G. (Chapman & Hall 1986): Particle dynamics, orbits, elementary rigid body dynamics.

Vibrations and Waves, French A.P. (Chapman & Hall 1971): Normal modes.

More comprehensive are:
Principles of Dynamics, Greenwood D.T. (2nd edn Prentice & Hall 1988).
Classical Mechanics, Barger V.D. and Olsson M.G. (McGaw-Hill 1995).
Mechanics, Landau L.D. and Lifshitz E.M. (3rd edn Butterworth-Heinemann 1976)

For elasticity:
Lectures on Physics, Feynman R.P. *et. al.* (Addison-Wesley 1963), Vol2: two useful chapters.

For the relevant maths:
Mathematical Methods for Physics and Engineering, Riley R.F., Hobson M.P. and Bence S.J. (CUP 1997)

Acknowledgments

The lecture handouts for this course are based in part on those of the previous Dynamics lecturers, in particular those of Prof. Webber and Dr's Mackay and Kenderdine.

1 Review of Newtonian Mechanics and Non-Inertial Frames

1.1 Review of Newtonian Mechanics

1.1.1 Newtonian Mechanics is:

- Non- Relativistic, i.e. $v << c = 3x10^8 \text{ms}^{-1}$ (speed of light)

- Classical, i.e. $Et >> h = 6.6x10^{-34}\text{Js}$ (Planck's constant)

and so assumes that:
- the mass of objects is independent of velocity, time or frame of reference,

- measurements of length and time are independent of the frame of reference and, like measurements of mass, are made by comparison with an arbitrary standard,

- all parameters can be known precisely.

1.1.2 The use of vectors

- Vectors have both *magnitude* and *direction*.

- If a physical quantity has both magnitude and direction and adds, resolves and behaves under coordinate transformations like a vector, it can be usefully represented by a vector.

- Obvious vectors: position, velocity, acceleration, force, momentum

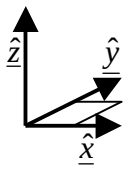
- Not such obvious vectors: quantities that 'happen' to behave as vectors and so can be represented as vectors such as angular velocity, angular acceleration, torque, and angular momentum.

1.1.3 Vector basis sets and frames of reference

- Vectors 'exist' in space and have properties that are independent of any basis set used to represent them. The same is true of functions of vectors, such as dot and cross products, and derivatives of such as div, grad and curl.

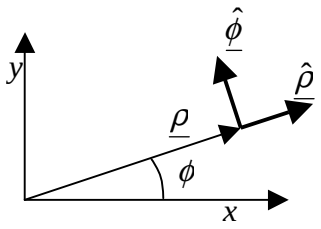
- Vectors may be represented with various basis sets, e.g:

Cartesian coordinates (basis set of unit vectors independent of position:)



Cartesian basis set

$$\underline{r} = (x, \quad y, \quad z), \quad \text{where} \quad \underline{r} = x\underline{\hat{x}} + y\underline{\hat{y}} + z\underline{\hat{z}}, \qquad \underline{v} = (\dot{x}, \quad \dot{y}, \quad \dot{z})$$



Cylindrical polars

Cylindrical polar coordinates (basis set varies with position, $\underline{r}$):

$$\underline{r} = (\rho, \quad z), \qquad \text{where} \qquad \underline{r} = \rho\,\underline{\hat{\rho}} + z\,\underline{\hat{z}}$$
$$\underline{v} = (\dot{\rho}, \quad \rho\dot{\phi}, \quad \dot{z}) = \dot{\rho}\,\underline{\hat{\rho}} + \rho\dot{\phi}\,\underline{\hat{\phi}} + \dot{z}\,\underline{\hat{z}}$$

- Coordinate axes (e.g. $\underline{\hat{x}}$, $\underline{\hat{y}}$, $\underline{\hat{z}}$) and a clock provide a frame of reference, with respect to which every event can be labelled by its space and time coordinates ($\underline{r}$, t).

- A frame of reference in which Newton's laws hold is an *inertial frame*.

- If a frame S (with coordinates ($\underline{r}$, t)) is an inertial frame and frame S' (with coordinates ($\underline{r}$', t')) is in *uniform* motion relative to S, then S' is also an inertial frame, because accelerations are the same in both frames. The equations relating the coordinates of the two frames are know as the Galilean transformation:

$$t' = t$$
$$\underline{r}' = \underline{r} - \underline{R}$$
$$\underline{R} = \underline{R}_0 + \underline{u}t$$

where $\underline{R}$ is the position of the origin of the S' frame in the S frame at time t and $\underline{u}$ is the velocity of S' with respect to S.
The velocities and accelerations are related by:

$$\underline{\dot{r}}' = \underline{\dot{r}} - \underline{\dot{R}} = \underline{\dot{r}} - \underline{u}$$
$$\Rightarrow \qquad \underline{\ddot{r}}' = \underline{\ddot{r}} = \frac{\underline{F}}{m} \qquad \Rightarrow \text{Newton's Laws hold in S' if true in S.}$$

1.1.4 Mechanics

= Statics (absence of motion)
+ Kinematics (description of motion, using vectors for position and velocity)
+ Dynamics (prediction of motion, and involves forces and/or energy)

1.1.5 Statics

- Forces and torques balance.

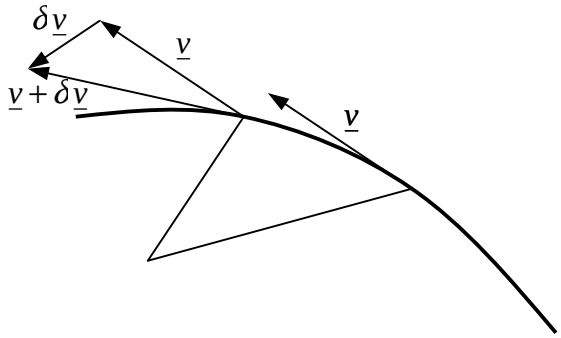
1.1.6 Kinematics

- motion and be described with respect to different frames of reference, and ‘kinematics’ includes a treatment of the ‘fictitious’ forces that need to be introduced to describe motion in an accelerating frame.

- definition of velocity and acceleration

Velocity: $\underline{v} = \lim_{\delta t \to 0} \frac{\delta \underline{r}}{\delta t}$

Acceleration: $\underline{a} = \lim_{\delta t \to 0} \frac{\delta \underline{v}}{\delta t}$



1.1.7 Dynamics

- motion can be described with respect to different frames of reference, and ‘kinematics’ includes a treatment of the ‘fictitious’ forces that need to be introduced to describe motion in an accelerating frame.

- basic principle of dynamics, Newton’s second law [N2]:

$$m\underline{a} = \underline{F}$$

mass of body — applied force

1.1.8 Force

- N2 is not a definition of force – force is defined as a push or pull that changes or tends to change a body’s state of rest or relative motion. Force can be measured by various means, the simplest being to compare it (using a balance of some kind) with the force of gravity on a set of standard weights.

1.1.9 Mass

- N2 is, however, a definition of inertial mass, which was shown by experiment (summarised in Newton’s Law of Gravitation) to be proportional to gravitational mass.

1.1.10 Momentum

- Momentum is defined as the product of mass and velocity:

$$\underline{p} = m\underline{v}$$

$$\dot{\underline{p}} = \frac{d\,\underline{p}}{dt} = \underline{F} \qquad \text{(N2 again)}$$

- If a force acts on a particle from time t_i to t_f, then the time integral of the force (the impulse) is equal to the change in momentum:

$$\underline{p}_f - \underline{p}_i = \int_{t_f}^{t_f} \underline{F}(t)dt$$

- The impulse of a force can be extremely useful in problems such as collisions where the force varies rapidly in an unknown way, but all that is of interest is the 'net effect' of the force.

1.1.11 Motion of a collection of particles: the centre of mass

- The centre of mass $\underline{R}$ of a collection of particles (positions $\underline{r}_i$) is defined as:

$$M\underline{R} = \sum_i m_i \underline{r}_i$$

where the total mass M is given by:

$$M = \sum_i m_i$$

Using N2, the acceleration of the centre of mass is given by:

$$M\ddot{\underline{R}} = \sum_i m_i \ddot{\underline{r}}_i = \sum_i \underline{F}_i = \sum_i \underline{F}_{io} + \sum_i \sum_j \underline{F}_{ij}$$

where $\underline{F}_i$ is the total force on the i^{th} particle, $\underline{F}_{io}$ is the external force on the i^{th} particle and $\underline{F}_{ij}$ is the force on the i^{th} particle from the j^{th} . Since $\underline{F}_{ij} = -\underline{F}_{ji}$ by Newton's 3rd law, the term $\sum_i \sum_j \underline{F}_{ij}$ sums to zero, and *the acceleration of the centre of mass is given by the sum of the external forces* $\underline{F}_o$:

$$M\ddot{\underline{R}} = \sum_i m_i \ddot{\underline{r}}_i = \sum_i \underline{F}_{io} = \underline{F}_o$$

Since $M\underline{R} = \underline{P}_o$, the total momentum, in the absence of an external force *momentum is conserved* ($\underline{F}_o = \dot{\underline{P}}_o = 0$).

1.1.12 Motion of a collection of particles: rotations about the centre of Mass

- The motion of a collection of particles can be considered in two parts – the motion *of* the centre of mass, and the movement (or rotation if the particles make a rigid body) of the collection of particles *about* the centre of mass. To help handle motion about a point, it is useful to define the quantities torque and angular momentum.

1.1.13 Torque

- The 'effectiveness' of a force at producing rotations about a particular point, A, is given by the *moment* (or *torque*) $\underline{G}$ of the force about that point:

$$\underline{G} = \underline{r} \times \underline{F}$$

where $\underline{r}$ is the position of the point of action of the force relative to the reference point, A. Since torque is defined as the cross product of two vectors, it itself is automatically a vector.

1.1.14 Angular momentum

- The angular momentum of a particle about a particular point, A, is given by the moment of the linear momentum about that point:

$$\underline{J} = \underline{r} \times \underline{p}$$

1.1.15 Relation between $\underline{G}$ and $\underline{J}$

- For a single particle, (index *i*)direct application of N2 to the definitions of $\underline{G}$ and $\underline{J}$ gives:

$$\dot{\underline{J}}_i = \dot{\underline{r}}_i \times \underline{p}_i + \underline{r}_i \times \dot{\underline{p}}_i = \underline{v}_i \times m\underline{v}_i + \underline{r}_i \times \underline{F}_i = 0 + \underline{G}_i = \underline{G}_i$$

i.e. torque is the rate of change of angular momentum

- For a collection of particles with no forces between them, the rate of change of angular momentum of each particle can be summed and equated to the sum of the external torques acting on each particle.

$$\dot{\underline{J}} = \sum_i \dot{\underline{J}}_i = \sum_i \underline{G}_i = \underline{G}_o$$

where $\underline{J}$ is the total angular velocity of the system and $\underline{G}_o$ is the total external couple on the system.

- If now two of the particles have are interacting, then the contribution to the total couple will be:

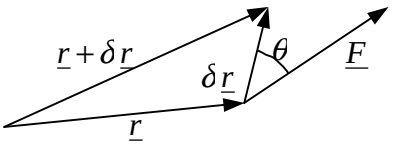
$$\Delta \underline{G} = \underline{r}_1 \times \underline{F}_{12} + \underline{r}_2 \times \underline{F}_{21} = (\underline{r}_1 - \underline{r}_2) \times \underline{F}_{12}$$

($\underline{F}_{12} = -\underline{F}_{21}$ by N3). If we assume that the force between the two particles is directed along the line between them then $\underline{F}_{12}$ is parallel to $(\underline{r}_1 - \underline{r}_2)$ and $\Delta \underline{G} = 0$. It follows that the internal forces do not affect the result that the rate of change of total angular momentum of a system is given by the sum of the externally applied torques.

- Angular momentum was 'invented' because it is a useful tool in rotational mechanics. The relation $\dot{\underline{J}} = \underline{G}_o$ enables forces applied to a body to be related to the its rotational acceleration, and hence solve the dynamics.

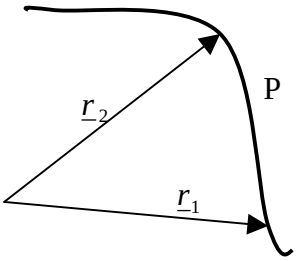
1.1.16 Work and Energy

- Work done by a force = force x (distance moved parallel to force)



$$\delta W = \underline{F}.\delta \underline{r}$$

- Work done by a force moving along path P



(using N2)

$$W = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}.d\underline{r} = \int_{t_1}^{t_2} \underline{F}.\underline{v}\ dt$$

$$= m\int_{t_1}^{t_2} \frac{d\underline{v}}{dt}.\underline{v}\ dt = \frac{1}{2}m\int_{t_1}^{t_2} \frac{d}{dt}\left(\underline{v}^2\right) dt$$

$$= \frac{1}{2}m\underline{v}_2^2 - \frac{1}{2}m\underline{v}_1^2$$

= increase in **KINETIC ENERGY** (T)

1.1.17 Potential Energy and Conservative Forces

- If the work done, *W*, by a force in moving along a path from a point $\underline{r}_1$ to a point $\underline{r}_2$ is independent of the path taken, then the force is know as a *conservative force* and it is possible to relate the work done to a function $U(\underline{r})$ which takes the form of a potential energy:

$$W = U(\underline{r}_1) - U(\underline{r}_2)$$

(The equation defines *U* so that if work is done by the force, *U* decreases)

- equating expressions for the work done in terms of kinetic and potential energies gives:

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = U(\underline{r}_1) - U(\underline{r}_2)$$

$$\Rightarrow \qquad U(\underline{r}_1) + \frac{1}{2}mv_1^2 = U(\underline{r}_2) + \frac{1}{2}mv_2^2 \qquad \textbf{Conservation of Energy } (E = T + V)$$

- 'Conservation of energy' is simply the integral form of N2. Other forms of 'energy' (heat, electostatic *etc*) are 'fixed' so as to fit in with this simple idea.

- The force may be recovered from the potential energy by differentiating:

1D $\qquad F(x) = -\dfrac{dU(x)}{dx}$

3D $\qquad \underline{F}(\underline{r}) = -\underline{\nabla} U(\underline{r})$

($\underline{\nabla}$ is the gradient vector operator which is given in Cartesian coordinates by

$$\underline{\nabla} U = \left(\frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial y}, \quad \frac{\partial U}{\partial z} \right) \qquad)$$

- Central forces are conservative. Suppose:

$$\underline{F}(\underline{r}) = f(r)\hat{\underline{r}}$$

and since $\qquad \hat{\underline{r}}.d\underline{r} = dr$

we have $\qquad \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}.d\underline{r} = \int_{r_1}^{r_2} f(r)\,dr$

since the work done by the force in going from $\underline{r}_1$ to $\underline{r}_2$ is a scalar integral of $f(r)$, the work done is independent of path and the force is conservative

- How can we know if a force is conservative, without performing integrals over all paths? A force is conservative if:

$$\int \underline{F}.d\underline{r} = 0$$

for any closed path. Stokes theorem states that integral of a vector round a closed loop is equal to the integral of the curl of that vector across any surface bounded by that loop. Since for a conservative force $\int \underline{F}.d\underline{r} = 0$ for all loops then $\int \underline{\nabla} \times \underline{F}.d\underline{\mathbf{S}} = 0$ for any integration over any area and so for a conservative field:

$$\underline{\nabla} \times \underline{F} = 0$$

So, if the curl of a force is everywhere zero, the force is conservative and there exists a scalar function $\phi(\underline{r})$ such that:

$$\underline{F} = -\underline{\nabla}\phi$$

1.2 Non-Inertial Frames

1.2.1 Transformations from stationary to rotating frames

- If a point on a rigid body has a position vector $\underline{r}$ measured with respect to the axis of rotation, and if the body is rotating with and angular velocity $\underline{\omega}$, then the velocity of the point with respect to stationary coordinates is given by:

$$\underline{v}_0 = \underline{\omega} \times \underline{r}$$

(vectors with the suffix $_0$ are with respect to the stationary frame, S_0, those without a suffix are with respect to the rotating frame, S, and for convenience the equations are written for the instant in time when the coordinate axes of S and S_0 coincide, so $\underline{r}_0 = \underline{r}$ and $\underline{\omega}_0 = \underline{\omega}$)

- If the point is also moving on the rigid body with a velocity $\underline{v}$ with respect to coordinates that rotate with the body, then this extra velocity must be added into the velocity $\underline{v}_0$:

$$\underline{v}_0 = \underline{v} + \underline{\omega} \times \underline{r}$$

- similarly the time derivative of any vector in S_0 is related to that of the same vector quantity, but measured with respect to S by:

$$\left[\frac{d\underline{A}}{dt}\right]_{S_0} = \left[\frac{d\underline{A}}{dt}\right]_S + \underline{\omega} \times \underline{A}$$

1.2.2 Derivation of fictitious forces

- Newton's Laws only apply in inertial frames. It is, however, often convenient to solve dynamical problems from the point of view of a rotating (non-inertial) frame such as the Earth. ***If*** you are making observations in such a frame, ***and if*** you insist that you want to use Newton's laws to describe what you see, then you have to allow for 'fictitious' forces that seem to push objects around in mysterious ways. In dynamics these 'forces' are introduced as a 'fix' to make Newton's laws work in a frame where they are actually invalid, because solving the problem in an 'proper' inertial frame is harder than worrying about the fictitious ('fix') forces in the non-inertial one. They are also introduced because to an observer forces are more apparent than accelerations, and if something appears to accelerate, we have enough of an intuitive feel for Newton's laws to ask what force is accelerating them. If you want to hold an object still in a rotating frame, technically what you are doing is accelerating the object towards the axis of rotation. However, what you see is a stationary object and what you feel is the very real centrifugal force (actually a N3 reaction to you pushing and accelerating the object inwards) that tends to fling the object outwards, and if you let go of that object, it is a fair question to ask what force produces the apparent acceleration outwards that ensues.

- In order to derive an expression for the acceleration of an object as seen in a rotating frame, we can use the formula given above that rates the time derivative of an vector in stationary and rotating frames. Firstly for velocity:

$$\left[\frac{d\underline{r}_0}{dt}\right]_{S_0} = \left[\frac{d\underline{r}}{dt}\right]_S + \underline{\omega}\times\underline{r}, \qquad i.e. \quad \underline{v}_0 = \underline{v} + \underline{\omega}\times\underline{r}$$

The subtle bit of this derivation now follows. The true acceleration of an object (i.e. measured in an inertial frame such as S_0) is the time derivative of the true velocity, i.e. $\underline{v}_0$ if you measure it in S_0 and $\underline{v}+\underline{\omega}\times\underline{r}$ if you measure it in S. (note the observer in S thinks the velocity is just $\underline{v}$, but once he is told that actually it is $\underline{v}+\underline{\omega}\times\underline{r}$ he can then use this to work out the true acceleration in S coordinates.) So, the expression for the true acceleration, $\underline{a}_0$, in terms of vectors measured in S is:

$$\underline{a}_0 = \left[\frac{d\underline{v}_0}{dt}\right]_{S_0} = \left[\frac{d\underline{v}_0}{dt}\right]_S + \underline{\omega}\times\underline{v}_0 = \left[\frac{d(\underline{v}+\underline{\omega}\times\underline{r})}{dt}\right]_S + \underline{\omega}\times(\underline{v}+\underline{\omega}\times\underline{r})$$
$$= \underline{a} + \underline{\omega}\times\underline{v} + \underline{\omega}\times\underline{v} + \underline{\omega}\times(\underline{\omega}\times\underline{r})$$
$$= \underline{a} + 2\underline{\omega}\times\underline{v} + \underline{\omega}\times(\underline{\omega}\times\underline{r})$$

Now, an observer in the rotating frame measures an acceleration $\underline{a}$, and as far as he is concerned, since he wishes to believe that Newton's laws apply in his non-inertial frame, there must therefore be a total force acting on the object given by $m\underline{a}$:

$$m\underline{a} \quad = \quad \underline{F} \quad - 2m\underline{\omega}\times\underline{v} \quad - m\underline{\omega}\times(\underline{\omega}\times\underline{r})$$

Total Apparent force True force $= m\underline{a}_0$ Coriolis force centrifugal force

- An alternative relation between $\underline{a}$ and $\underline{a}_0$ can also be derived by expressing the position vector of a point, $\underline{r}_0$ explicitly in terms of the coordinates (x, y, z) and rotating basis set, ($\underline{\hat{x}}$, $\underline{\hat{y}}$, $\underline{\hat{z}}$) in S and taking derivatives w.r.t. time, noting that: $\dot{\underline{\hat{x}}} = \underline{\omega}\times\underline{\hat{x}}$, $\ddot{\underline{\hat{x}}} = \underline{\omega}\times(\underline{\omega}\times\underline{\hat{x}})$ etc.

$$\underline{r}_0 = x\underline{\hat{x}} + y\underline{\hat{y}} + z\underline{\hat{z}}$$

$$\dot{\underline{r}}_0 = \dot{x}\underline{\hat{x}} + x\dot{\underline{\hat{x}}} + ... + ... \qquad (y \text{ and } z \text{ terms similar})$$
$$= \dot{x}\underline{\hat{x}} + x(\underline{\omega}\times\underline{\hat{x}}) + ... + ... \qquad (y \text{ and } z \text{ terms similar})$$

$$\ddot{\underline{r}}_0 = \ddot{x}\underline{\hat{x}} + 2\dot{x}(\underline{\omega}\times\underline{\hat{x}}) + x[\underline{\omega}\times(\underline{\omega}\times\underline{\hat{x}})] ... + ... \qquad (y \text{ and } z \text{ terms similar})$$

$$\underline{a}_0 = \underline{a} + 2\underline{\omega}\times\underline{v} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) \qquad \text{(as previously)}$$

1.2.3 Centrifugal force

- Using cylindrical polar coordinates (ρ, ϕ, z) (see section 1.1.3) with the axis of rotation along the z axis:

$\underline{r} = \rho\,\underline{\hat{\rho}} + z\,\underline{\hat{z}}$, $\quad \underline{\omega} = \omega\,\underline{\hat{z}}$, $\quad \rho$ is the distance from the axis or rotation

$$\Rightarrow \quad \underline{\omega}\times\underline{r} = \omega\rho\left(\underline{\hat{z}}\times\underline{\hat{\rho}}\right) = \omega\rho\,\underline{\hat{\phi}}$$

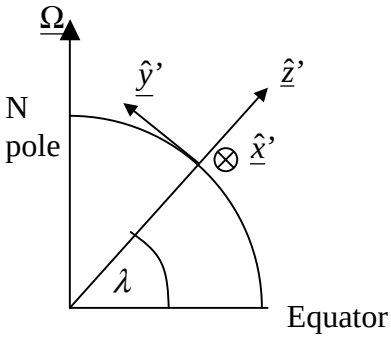
$$\Rightarrow \quad -\underline{\omega}\times\left(\underline{\omega}\times\underline{r}\right) = -\omega^2\rho\left(\underline{\hat{z}}\times\underline{\hat{\phi}}\right) = \omega^2\rho\,\underline{\hat{\rho}}$$

i.e. Centrifugal force = $m\omega^2\times$ (distance from axis of rotation)
and is directed away from the axis of rotation

Example: rotation of Earth at equator

$$\Omega^2 R_e = \left(\frac{2\pi}{1\,day}\right)^2 6400\,\text{km} \approx 0.003\text{g}$$

1.2.4 Coriolis force on Earth's surface



$$\underline{F}_{Cor} = -2m\underline{\Omega}\times\underline{v}$$

$\underline{\Omega}$ = Earth's angular velocity
$\underline{v}$ = velocity in frame S′, fixed to Earth

Coordinate axes in S′:
$\hat{x}$' pointing East
$\hat{y}$' pointing North
$\hat{z}$' pointing radially outwards at latitude λ.

$$\underline{\Omega} = \Omega\cos\lambda\,\underline{\hat{y}}' + \Omega\sin\lambda\,\underline{\hat{z}}'$$

for $\underline{v}$ in the plane of the Earth's surface $\quad \underline{v} = v_x\,\underline{\hat{x}}' + v_y\,\underline{\hat{y}}'$

$$\Rightarrow \quad \underline{F}_{Cor} = 2m\Omega\sin\lambda\left(v_y\,\underline{\hat{x}}' - v_x\,\underline{\hat{y}}'\right) \quad \text{sideways force}$$

$$\Rightarrow \quad + 2m\Omega\cos\lambda\,v_x\,\underline{\hat{z}}' \quad \text{vertical force}$$

- The sideways force is always perpendicular to $\underline{v}$ in surface plane with a magnitude of $2m\Omega v\sin\lambda$, independent of the direction of $\underline{v}$ in the surface plane, and acts to the right in the northern hemisphere, and to the left in the southern hemisphere.

2 Normal Modes

2.1 Definition

- In general no analytical solution is possible for the motion of a many body system.

- BUT there is a large class of systems which are close to mechanical equilibrium for which the inter-body potentials between components of the system may be expanded about a local minimum value as a Taylor expansion with only the quadratic term being of importance for small displacements:

$$V(\underline{x}) = V_0 + \sum_i \frac{\partial V}{\partial x_i} x_i + \frac{1}{2} \sum_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j} x_i x_j + \ldots$$

where $\frac{\partial V}{\partial x_i} = 0$ for all *i* since the potential is being expanded about a local minimum at $\underline{x} = \underline{0}$

A potential which does not have higher order terms in its expansion that the 2nd order ones is said to be *harmonic*.

- In 1D we would then have:

$$V(x) = \frac{1}{2} k x^2$$

where x is the displacement of the particle from its equilibrium position, and the equation of motion is:

$$m\ddot{x} = -\frac{d^2 V}{dx^2} x = -kx$$

and the solution can be written in one the well known forms:

Real number forms

$x = x_0 \cos(\omega t + \varepsilon)$ initial position: $x_0 \cos(\varepsilon)$ initial velocity: $-\omega x_0 \sin(\varepsilon)$

$x = A\cos(\omega t) + B\sin(\omega t)$ initial position: A initial velocity: ωB

Complex number forms

$x = \mathrm{Re}(X) = \mathrm{Re}[x_0 \exp(i(\omega t + \varepsilon))]$

initial position: $x_0 \cos(\varepsilon)$ initial velocity: $-\omega x_0 \sin(\varepsilon)$

(x_0 is a real number)

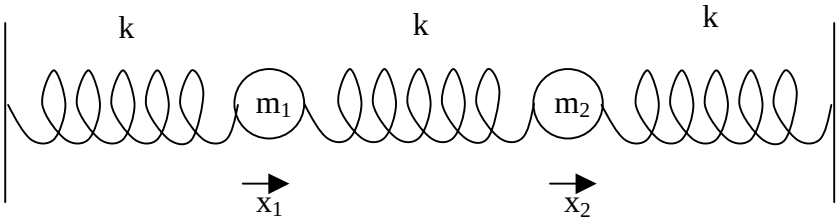
$x = \mathrm{Re}(X) = \mathrm{Re}[X_0 \exp(i\omega t)]$

$X_0 = x_0 \exp(i\varepsilon)$ initial position: $\mathrm{Re}(X_0)$ initial velocity: $-\omega \,\mathrm{Im}(X_0)$

where the angular frequency of the simple harmonic motion is $\omega = \sqrt{k/m}$

- What happens when we go to more than one dimension? We find that provided the inter-particle potential approximates to a harmonic (quadratic) form, we can resolve the motion of a general system for small displacements about its equilibrium into a number of Normal Modes. In each of these normal modes of vibration the whole system oscillates together with one frequency, and a general motion of the system can be written as the sum of various amounts of each of these modes. If *N* coordinates are required to describe the system, then there will be *N* normal modes.

2.2 An example of a 2 coordinate normal mode system



- consider the case where $m_1 = m_2 = m$

- if the tensions in the springs, labelled left to right are, F_1, F_2, and F_3 then we have:

$$F_1 = kx_1$$
$$F_2 = k(x_2 - x_1)$$
$$F_3 = -kx_3$$

and the equations of motion for the two masses are:

$$m\ddot{x}_1 = F_2 - F_1 = -2kx_1 + kx_2 \qquad 2.2.1$$
$$m\ddot{x}_2 = F_2 - F_3 = \quad kx_1 - 2kx_2$$

If we now look for normal mode solutions in which both coordinates oscillate with the same frequency, i.e.

$$\underline{x} = \underline{x}_0 \cos(\omega t + \varepsilon) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \cos(\omega t + \varepsilon) \qquad 2.2.2$$

Substituting this trial solution into equations 2.2.1 gives:

$$-\omega^2 m x_{10} \cos(\omega t + \varepsilon) = (-2kx_{10} + kx_{20}) \cos(\omega t + \varepsilon)$$
$$-\omega^2 m x_{20} \cos(\omega t + \varepsilon) = (\quad kx_{10} - 2kx_{20}) \cos(\omega t + \varepsilon)$$

and dividing through by $\cos(\omega t + \varepsilon)$ gives:

$$-\omega^2 m x_{10} = -2kx_{10} + kx_{20} \qquad 2.2.3$$
$$-\omega^2 m x_{20} = kx_{10} - 2kx_{20}$$

and rearranging equations 2.2.3 gives:

$$\frac{x_{10}}{x_{20}} = \frac{k}{2k - m\omega^2} \qquad 2.2.4$$
$$\frac{x_{10}}{x_{20}} = \frac{2k - m\omega^2}{k}$$

Hence a solution of the form of equation 2.2.2 is only possible if both equations 2.2.4 are true. To find the values of ω for which this is true we can equate the two equations 2.2.4:

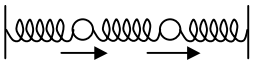
$$\frac{k}{2k - m\omega^2} = \frac{2k - m\omega^2}{k}$$

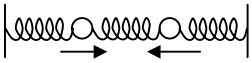
$$\Rightarrow \qquad k^2 = \left(2k - m\omega^2\right)^2$$

and we have two possible frequencies: $\omega_- = \sqrt{\frac{k}{m}}$ and $\omega_+ = \sqrt{\frac{3k}{m}}$

(note – the two solutions $\omega_- = \sqrt{\frac{k}{m}}$ and $\omega_- = -\sqrt{\frac{k}{m}}$ give identical normal modes since $\cos(\omega_- t) = \cos(-\omega_- t)$, so are counted as one)

Substituting back into equations 2.2.4 gives:

for $\omega = \omega_-$ $\quad \frac{x_{10}^-}{x_{20}^-} = 1 \quad \underline{x}_0^- = \frac{x_0^-}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = x_0^- \underline{e}^-$ in phase: 

and for $\omega = \omega_+$ $\quad \frac{x_{10}^+}{x_{20}^+} = -1 \quad \underline{x}_0^+ = \frac{x_0^+}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix} = x_0^+ \underline{e}^+$ out of phase: 

(the unit vectors $\underline{e}^- = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $\underline{e}^+ = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$ are used because they are orthonormal, i.e. orthogonal to each other and normalised $\underline{e}^+.\underline{e}^+ = \underline{e}^-.\underline{e}^- = 1$, $\underline{e}^+.\underline{e}^- = 0$. We will show later in this section that this orthonomal property is very useful when it comes to fitting the initial conditions. In general such we will show in sections 2.4 and 2.6 that one can always find such orthonormal sets and indeed that is why the modes are called *normal* modes)

A general motion of the system can be written as a linear combination of the two modes

$$\underline{x}(t) = \underline{x}_0^- \cos(\omega_- t + \varepsilon_-) + \underline{x}_0^+ \cos(\omega_+ t + \varepsilon_+)$$

or:

$$x_1(t) = \frac{1}{\sqrt{2}}\left[x_0^- \cos(\omega_- t + \varepsilon_-) + x_0^+ \cos(\omega_+ t + \varepsilon_+)\right]$$

$$x_2(t) = \frac{1}{\sqrt{2}}\left[x_0^- \cos(\omega_- t + \varepsilon_-) - x_0^+ \cos(\omega_+ t + \varepsilon_+)\right]$$

Suppose we have the starting (initial) condition that one of the masses is displaced from equilibrium by and amount *a*, i.e. we have:

$$x_1 = 0\,, \quad x_2 = a\,, \quad \dot{x}_1 = 0\,, \quad \ddot{x}_2 = 0$$

writing the general solution in the form:

$$\underline{x}(t) = \; x_0^- \underline{e}^- \cos(\omega_- t + \varepsilon_-) + x_0^+ \underline{e}^+ \cos(\omega_+ t + \varepsilon_+)$$

we have:

$$\dot{\underline{x}}(t) = \; -\omega_- x_0^- \underline{e}^- \sin(\omega_- t + \varepsilon_-) - \omega_+ x_0^+ \underline{e}^+ \sin(\omega_+ t + \varepsilon_+)$$

and at $t = 0$ we have:

$$\underline{x}(0) = \; x_0^- \underline{e}^- \cos(\varepsilon_-) + x_0^+ \underline{e}^+ \cos(\varepsilon_+) = \begin{pmatrix} 0 \\ a \end{pmatrix} \qquad 2.2.5$$

$$\dot{\underline{x}}(0) = \; -\omega_- x_0^- \underline{e}^- \sin(\varepsilon_-) - \omega_+ x_0^+ \underline{e}^+ \sin(\varepsilon_+) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2.2.6$$

if we take equation 2.2.6 first and take the dot product of both sides with $\underline{e}^-$ we get

$$\dot{\underline{x}}(0).\underline{e}^- = \; -\omega_- x_0^- \underline{e}^- .\underline{e}^- \sin(\varepsilon_-) - \omega_+ x_0^+ \underline{e}^+ .\underline{e}^- \sin(\varepsilon_+) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \underline{e}^- = 0$$

and using the orthogonality conditions $\underline{e}^- .\underline{e}^- = 1$, $\underline{e}^+ .\underline{e}^- = 0$ we get

$$x_0^- \sin(\varepsilon_-) = 0$$

and since $x_0^- \neq 0$, we have $\sin(\varepsilon_-) = 0$ and $\varepsilon_- = 0$

similarly taking the dot product of both sides of equation 2.2.6 with $\underline{e}^+$ we get $\varepsilon_+ = 0$

Equation 2.2.5 now becomes:

$$\underline{x}(0) = \; x_0^- \underline{e}^- + x_0^+ \underline{e}^+ = \begin{pmatrix} 0 \\ a \end{pmatrix}$$

and again taking the dot product with $\underline{e}^-$ and $\underline{e}^+$ gives:

$$\underline{x}(0).\underline{e}^- = x_0^- \underline{e}^-.\underline{e}^- + x_0^+ \underline{e}^+.\underline{e}^- = \begin{pmatrix} 0 \\ a \end{pmatrix} \underline{e}^-$$

$$\Rightarrow \qquad x_0^- = \begin{pmatrix} 0 \\ a \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{a}{\sqrt{2}}$$

and

$$\underline{x}(0).\underline{e}^+ = x_0^- \underline{e}^-.\underline{e}^+ + x_0^+ \underline{e}^+.\underline{e}^+ = \begin{pmatrix} 0 \\ a \end{pmatrix} \underline{e}^+$$

$$\Rightarrow \qquad x_0^+ = \begin{pmatrix} 0 \\ a \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\frac{a}{\sqrt{2}}$$

giving the solution:

$$x_1(t) = \frac{1}{2} a[\cos \omega_- t + \cos \omega_+ t]$$

$$x_2(t) = \frac{1}{2} a[\cos \omega_- t - \cos \omega_+ t]$$

2.3 Matrix notation for a 2 coordinate system

- we can write equations 2.2.1 in the form

$$m \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad 2.3.1$$

- substituting in equation 2.3.1 for a normal mode $\underline{x} = \underline{x}_0 \cos(\omega t + \varepsilon) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \cos(\omega t + \varepsilon)$ and dividing by $\cos(\omega t + \varepsilon)$ gives:

$$-m\omega^2 \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} -m\omega^2 & 0 \\ 0 & -m\omega^2 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$\Rightarrow \qquad \begin{pmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{pmatrix} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad 2.3.1$$

- Since the right and side of equation 2.3.1 is zero, the matrix is not invertable, i.e. it is singular, and its determinant must be zero. Again, this places a restriction on permitted values of ω.

$$\begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix} = 0$$

$\Rightarrow \quad \left(2k - m\omega^2\right)^2 - k^2 = 0$ as previously

- We can write: $\begin{pmatrix} 2k & -k \\ -k & 2k \end{pmatrix} = \underline{\underline{K}}$ and $\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} = \underline{\underline{M}}$ or for different masses $\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = \underline{\underline{M}}$

- Now the normal mode equation looks like:

$$-\omega^2 \underline{\underline{M}} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = -\underline{\underline{K}} \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix}$$

$$\Rightarrow \quad \left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right) \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad \det\left(\underline{\underline{K}} - \omega^2 \underline{\underline{M}}\right) = 0$$

- The matrix $\underline{\underline{M}}$ is invertable, $\underline{\underline{M}}^{-1} = \begin{pmatrix} m_1^{-1} & 0 \\ 0 & m_2^{-1} \end{pmatrix}$

$\Rightarrow \quad \det\left(\underline{\underline{M}}^{-1} \underline{\underline{K}} - \omega^2 \underline{\underline{I}}\right) = 0$ $\underline{\underline{I}}$ is the unit matrix

- The normal mode frequencies squared are therefore the **eigenvalves** of the matrix $\underline{\underline{M}}^{-1} \underline{\underline{K}}$ and the unit vectors $\underline{e}^-$ and $\underline{e}^+$ introduced above as normal mode displacement patterns turn out to be the normalised **eigenvectors** of the matrix $\underline{\underline{M}}^{-1} \underline{\underline{K}}$.

2.4 General solution of a N normal mode system

- For a system with *N* coordinates the matrices $\underline{\underline{K}}$ and $\underline{\underline{M}}$ will be $N \times N$ matrices, where the matrix element K_{ij} is the restoring force on particle *i* for unit displacement of particle *j*.

- Writing the total potential energy of the system as a Taylor expansion about the equilibrium position, all first order terms vanish, leaving the 2nd order terms as the most significant ones.

$$V(\underline{x}) = V_0 + \frac{1}{2}\sum_{ij}\frac{\partial^2 V}{\partial x_i \partial x_j}x_i x_j$$

and $\frac{\partial^2 V}{\partial x_i \partial x_j} = K_{ij}$. Since $\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i}$, $K_{ij} = K_{ji}$ and the $\underline{\underline{K}}$ matrix is symmetrical.

- The **Secular Equation**:

$$\det\left(\underline{\underline{M}}^{-1}\underline{\underline{K}} - \omega^2 \underline{\underline{I}}\right) = 0$$

now as N solutions – with N frequencies and N normal modes, and the squared frequencies are eigenvalues of $\underline{\underline{M}}^{-1}\underline{\underline{K}}$.

- In general when a matrix acts on a vector, it changes both its direction and magnitude. However, for a N dimensional matrix one can find N *eigenvectors* which only change their magnitude when operated on by the matrix, and not their direction. Thus an eigenvector of a matrix $\underline{\underline{A}}$ is one which obeys the relation:

$$\begin{aligned} & \underline{\underline{A}}\underline{e} = \lambda \underline{e} \\ \Rightarrow \quad & \left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{e} = 0 \\ \Rightarrow \quad & \det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = 0 \end{aligned}$$

where λ is the eigenvalue associated with eigenvector $\underline{e}$.

- If $\underline{\underline{A}}$ is a real symmetric matrix, its eigenvalues are all real and its eigenvectors are orthogonal, and can be normalised so that.

$$\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$$

examples of these orthonormal eigenvectors and their use was given above in section 2.2.

- The general solution to an N dimensional normal mode problem, *i.e.* how the values of all the coordinates of the system vary with time, can then be expressed as a sum over the N normal modes.

$$\underline{x}(t) = \sum_{j=1}^{N} x_0^{(j)} \underline{e}_j \cos\left(\omega_j t + \varepsilon_j\right)$$

or in complex notation

$$\underline{x}(t) = \mathrm{Re}\sum_{j=1}^{N} X_0^{(j)} \underline{e}_j \exp\left(i\omega_j t\right)$$

2.5 Zero frequency modes

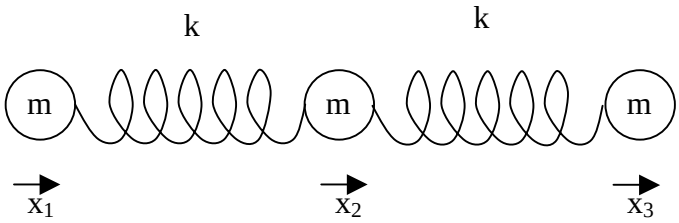
- Zero frequency modes correspond to free motion (translation or rotation) with no restoring force.

- Zero frequency modes must be written as:

$$\underline{x}_t(t) = \underline{x}_{t0}(A + Bt)$$

where $A\underline{x}_{t0}$ gives where the motion starts at $t = 0$ and $B\underline{x}_{t0}$ gives the velocities.

-Consider 3 equal masses connected by 2 springs, with the masses free to move along the line joining the centres of the masses:



$$m\ddot{x}_1 = k(x_2 - x_1)$$
$$m\ddot{x}_2 = k(x_1 - x_2) + k(x_3 - x_2)$$
$$m\ddot{x}_3 = k(x_2 - x_3)$$

and

$$\underline{\underline{M}}^{-1} \cdot \underline{\underline{K}} = \frac{k}{m}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

normal mode frequencies $\omega^2 = \alpha k/m$ where $\begin{vmatrix} 1-\alpha & -1 & 0 \\ -1 & 2-\alpha & -1 \\ 0 & -1 & 1-\alpha \end{vmatrix} = 0$ and:

$$\Rightarrow \quad (1-\alpha)^2(2-\alpha) - 2(1-\alpha) = 0$$

$$\Rightarrow \quad (1-\alpha)(\alpha-3)\alpha = 0$$

$$\Rightarrow \quad \alpha = 0,\ 1,\ 3$$

and the normal mode frequencies are

$$\omega_1 = 0\,, \qquad \omega_2 = \sqrt{k/m}\,, \quad \text{and } \omega_3 = \sqrt{3k/m}$$

the corresponding eigenvectors are found by substituting the frequencies back into the eigenvector equation:

$$\left(\underline{\underline{M}}^{-1}\underline{\underline{K}} - \omega_j^2 \underline{\underline{I}}\right)\underline{e}_j = 0$$

i.e.

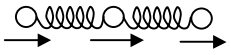
$$\begin{pmatrix} 1-\alpha_j & -1 & 0 \\ -1 & 2-\alpha_j & -1 \\ 0 & -1 & 1-\alpha_j \end{pmatrix}\begin{pmatrix} e_{j1} \\ e_{j2} \\ e_{j3} \end{pmatrix} = 0$$

hence, after normalising the eigenvectors are:

$$\underline{e}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \underline{e}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \underline{e}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

and we have three modes:

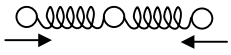
Mode 1



$\omega_1 = 0$

No restoring forces, translation

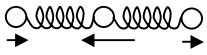
Mode 2



$\omega_2 = \sqrt{k/m}$

Central mass stationary

Mode 3



$\omega_3 = \sqrt{3k/m}$

Max. spring extension, max. ω

2.6 Orthogonality of normal modes

- We can show that for a symmetric matrix the eigenvectors are orthogonal by considering the action of the matrix on two of its eigenvectors

$$\underline{\underline{A}}\underline{e}^{(a)} = \lambda^{(a)}\underline{e}^{(a)} \qquad\qquad \underline{\underline{A}}\underline{e}^{(b)} = \lambda^{(b)}\underline{e}^{(b)}$$

pre-multiplying each equation by the other eigenvector gives:

$$\underline{e}^{(b)^T}\,\underline{\underline{A}}\underline{e}^{(a)} = \lambda^{(a)}\underline{e}^{(b)^T}\,\underline{e}^{(a)} \qquad\qquad \underline{e}^{(a)^T}\,\underline{\underline{A}}\underline{e}^{(b)} = \lambda^{(b)}\underline{e}^{(a)^T}\,\underline{e}^{(b)}$$

taking the transpose of the first equation gives $\underline{e}^{(a)T}\,\underline{\underline{A}}^{T}\,\underline{e}^{(b)} = \lambda^{(a)}\underline{e}^{(a)T}\,\underline{e}^{(b)}$ and since $\underline{\underline{A}}^{T} = \underline{\underline{A}}$ we can equate the left hand side of both equations, to give:

$$0 = \left(\lambda^{(a)} - \lambda^{(b)}\right)\underline{e}^{(a)T}\,\underline{e}^{(b)}$$

so if $\lambda^{(a)} \neq \lambda^{(b)}$ then $\underline{e}^{(a)T}\,\underline{e}^{(b)} = 0$ i.e. the eigenvectors are orthogonal.

- The matrix $\underline{\underline{M}}^{-1}\underline{\underline{K}}$ in the secular equation, $\det\left(\underline{\underline{M}}^{-1}\underline{\underline{K}} - \omega^2\,\underline{\underline{I}}\right) = 0$, is only symmetric if all the masses of the interacting particles are equal. If the masses of the interacting particles are not equal, then whilst the basic properties of the normal modes are unaffected, the eigenvectors are no longer orthogonal. However we can derive an orthogonality relation. Consider again two eigenvectors (the eigenvalues are written with λ again rather than ω^2 to simplify the algebra).

$$\underline{\underline{K}}\,\underline{e}^{(a)} = \lambda^{(a)}\underline{\underline{M}}\,\underline{e}^{(a)} \qquad\qquad \underline{\underline{K}}\,\underline{e}^{(b)} = \lambda^{(b)}\underline{\underline{M}}\,\underline{e}^{(b)}$$

pre-multiplying each equation by the other eigenvector gives:

$$\underline{e}^{(b)T}\,\underline{\underline{K}}\,\underline{e}^{(a)} = \lambda^{(a)}\underline{e}^{(b)T}\,\underline{\underline{M}}\,\underline{e}^{(a)} \qquad\qquad \underline{e}^{(a)T}\,\underline{\underline{K}}\,\underline{e}^{(b)} = \lambda^{(b)}\underline{e}^{(a)T}\,\underline{\underline{M}}\,\underline{e}^{(b)}$$

Taking the transpose of the first equation gives $\underline{e}^{(a)T}\,\underline{\underline{K}}^{T}\,\underline{e}^{(b)} = \lambda^{(a)}\underline{e}^{(a)T}\,\underline{\underline{M}}^{T}\,\underline{e}^{(b)}$ and since $\underline{\underline{K}}^{T} = \underline{\underline{K}}$ and $\underline{\underline{M}}^{T} = \underline{\underline{M}}$ we can equate the left hand sides of both equations to give:

$$0 = \left(\lambda^{(a)} - \lambda^{(b)}\right)\underline{e}^{(a)T}\,\underline{\underline{M}}\,\underline{e}^{(b)}$$

so if $\lambda^{(a)} \neq \lambda^{(b)}$ then $\underline{e}^{(a)T}\,\underline{\underline{M}}\,\underline{e}^{(b)} = 0$ and if we have a general displacement given as sum of eigenvectors $\underline{x} = \sum_j c_j\,\underline{e}^{(j)}$ then pre-multiplying by $\underline{e}^{(k)T}\,\underline{\underline{M}}$ extracts the coefficients c_j thus:

$$\underline{e}^{(k)T}\,\underline{\underline{M}}\,\underline{x} = \sum_j c_j\,\underline{e}^{(k)T}\,\underline{\underline{M}}\,\underline{e}^{(j)} = \underline{e}^{(k)T}\,\underline{\underline{M}}\,\underline{e}^{(k)}c_k$$

($\underline{e}^{(k)T}\,\underline{\underline{M}}\,\underline{e}^{(k)}$ can easily be calculated). Given the more complicated form of the orthogonality relation, there is no point in normalising the eigenvectors, as was done for the case where all the masses of the particles is the same.

- Take the case of a two masses connected by a spring in 1d, with $m_2 = 2m_1$ and $\underline{\underline{M}} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = m_1 \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, the un-normalised eigenvectors are $\underline{e}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (translation) and $\underline{e}^{(2)} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ (vibration- to find the eigenvector consider the fact that the centre of mass has no momentum for the vibrational case). If we have a general position of the system given by:

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c^{(1)}\underline{e}^{(1)} + c^{(2)}\underline{e}^{(2)} = c^{(1)}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c^{(2)}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

we can find the values of the coefficients $c^{(1)}$ and $c^{(2)}$ by first pre-multiplying by the M matrix:

$$\underline{\underline{M}}\,\underline{x} = \begin{pmatrix} m_1 x_1 \\ m_2 x_2 \end{pmatrix} = m_1 \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix}$$

and then pre-multiply by $\underline{e}^{(1)}$ to give:

$$\underline{e}^{(1)T}\,\underline{\underline{M}}\,\underline{x} = m_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix} = c^{(1)}\underline{e}^{(1)T}\,\underline{\underline{M}}\,\underline{e}^{(1)}$$

and since $\underline{e}^{(1)T}\,\underline{\underline{M}}\,\underline{e}^{(1)} = m_1 + m_2$ we have: $c^{(1)} = \dfrac{m_1(x_1 + 2x_2)}{m_1 + m_2}$

- Taking the proof above that $\underline{e}^{(a)T}\,\underline{\underline{M}}\,\underline{e}^{(b)} = 0$ and repeating it, but starting with $\dfrac{1}{\lambda^{(a)}}\underline{\underline{K}}\,\underline{e}^{(a)} = \underline{\underline{M}}\,\underline{e}^{(a)}$ that gives:

$$0 = \left(\frac{1}{\lambda^{(a)}} - \frac{1}{\lambda^{(b)}}\right)\underline{e}^{(a)T}\,\underline{\underline{K}}\,\underline{e}^{(b)}$$

a second orthogonality relation $\underline{e}^{(a)T}\,\underline{\underline{K}}\,\underline{e}^{(b)} = 0$

2.7 Energy of normal modes

- Kinetic energy given by $T = \sum_j \frac{1}{2} m_j \dot{x}_j^2 = \frac{1}{2} \underline{\dot{x}} \underline{\underline{M}} \underline{\dot{x}}$

- When the system is displaced to a position $\underline{x}$, then the force of the j^{th} component is given by:

$$F_j = -\sum_k K_{jk} x_k$$

If we wish to find the potential energy stored in the system when it is at a displacement $\underline{x}$ then we can start with the system at equilibrium and increase the displacement linearly to $\underline{x}$ by setting the displacement to be equal to $\underline{\chi} = \mu \underline{x}$ and increasing μ from 0 to 1. The work done on the system for a small change in μ is then:

$$dW = -\underline{F} \cdot d\underline{x} = -\underline{F} \cdot \underline{x} \, d\mu = \sum_j \sum_k K_{jk} \mu \, x_k x_j d\mu$$

and integrating gives:

$$U = \int dW = \int_0^1 \sum_j \sum_k K_{jk} \mu \, x_k x_j d\mu = \frac{1}{2} \sum_j \sum_k K_{jk} x_k x_j = \frac{1}{2} \underline{x}^T \underline{\underline{K}} \underline{x}$$

The total energy of a system is then given by:

$$E = T + U = \frac{1}{2} \underline{\dot{x}} \underline{\underline{M}} \underline{\dot{x}} + \frac{1}{2} \underline{x}^T \underline{\underline{K}} \underline{x}$$

- Consider the energy stored in a single normal mode or amplitude $x_0^{(j)}$ we have:

$$\underline{x} = x_0^{(j)} \cos\left(\omega_j t + \varepsilon_j\right) \underline{e}_j$$
$$\underline{\dot{x}} = -x_0^{(j)} \omega_j \sin\left(\omega_j t + \varepsilon_j\right) \underline{e}_j$$

and

$$T_j = \frac{1}{2} x_0^{(j)^2} \omega_j^2 \sin^2\left(\omega_j t + \varepsilon_j\right) \underline{e}_j^T \underline{\underline{M}} \underline{e}_j$$
$$U_j = \frac{1}{2} x_0^{(j)^2} \cos^2\left(\omega_j t + \varepsilon_j\right) \underline{e}_j^T \underline{\underline{K}} \underline{e}_j$$

but we have: $\left(\underline{\underline{K}} - \omega_j^2 \underline{\underline{M}}\right) \underline{e}_j = \underline{0}$

and so $\underline{e}_j^T\left(\underline{\underline{K}}-\omega^2\underline{\underline{M}}\right)\underline{e}_j=0$ and $\underline{e}_j^T\underline{\underline{K}}\underline{e}_j=\underline{e}_j^T\omega^2\underline{\underline{M}}\underline{e}_j$

and the total energy stored in the j^{th} mode is given by:

$$E_j=T_j+U_j=\frac{1}{2}x_0^{(j)^2}\omega_j^2\underline{e}_j^T\underline{\underline{M}}\underline{e}_j=\frac{1}{2}x_0^{(j)^2}\underline{e}_j^T\underline{\underline{K}}\underline{e}_j$$

and average values are given by:

$$\overline{T}_j=\overline{U_j}=\frac{E_j}{2}$$

$\Rightarrow$ equipartition between kinetic and potential energies for oscillating system.

- now consider a superposition of normal modes:

$$\underline{x}=\sum_j x_0^{(j)}\cos\left(\omega_j t+\varepsilon_j\right)\underline{e}_j$$

$$\dot{\underline{x}}=-\sum_j x_0^{(j)}\omega_j\sin\left(\omega_j t+\varepsilon_j\right)\underline{e}_j$$

$$T=\frac{1}{2}\dot{\underline{x}}\underline{\underline{M}}\dot{\underline{x}}=\frac{1}{2}\sum_{j,k}x_0^{(j)}x_0^{(k)}\omega_j\omega_k\sin\left(\omega_j t+\varepsilon_j\right)\sin\left(\omega_k t+\varepsilon_k\right)\underline{e}_k^T\underline{\underline{M}}\underline{e}_j$$

but if $k\neq j$ $\underline{e}_k^T\underline{\underline{M}}\underline{e}_j=0$, so the expression for T becomes

$$T=\frac{1}{2}\sum_j x_0^{(j)^2}\omega_j^2\sin^2\left(\omega_j t+\varepsilon_j\right)\underline{e}_j^T\underline{\underline{M}}\underline{e}_j$$

$$=\sum_j T_j$$

and similarly:

$$U=\frac{1}{2}\sum_j x_0^{(j)^2}\cos^2\left(\omega_j t+\varepsilon_j\right)\underline{e}_j^T\underline{\underline{K}}\underline{e}_j$$

$$=\sum_j U_j$$

and so:

$$E=\sum_j E_j$$

i.e. the total energy of the system can be written in terms of a sum of the energies stored in each individual normal mode. Since normal modes oscillate independently of each other, their amplitude remains constant in time and no energy is transferred from one mode to another.

2.8 Degenerate modes

- For systems of high symmetry, some modes have the same frequency, and in which case the modes are said to be degenerates.

- The proof's that eigenvectors were orthogonal (or orthogonal when using the $\underline{e}^{(a)T}\,\underline{\underline{M}}\,\underline{e}^{(b)}=0$ and $\underline{e}^{(a)T}\,\underline{\underline{K}}\,\underline{e}^{(b)}=0$ relations) all applied when the eigenvalues (normal mode frequencies) were different.

- if two modes have the same frequency, then any linear combination of them will also be a normal mode of the same frequency. In terms of eigenvalues of a matrix $\underline{\underline{A}}$, this can be seen as follows:

if $\underline{\underline{A}}\underline{e}^{(a)}=\lambda\underline{e}^{(a)}$ and $\underline{\underline{A}}\underline{e}^{(b)}=\lambda\underline{e}^{(b)}$ then $\underline{\underline{A}}\left(\alpha\,\underline{e}^{(a)}+\underline{e}^{(b)}\right)=\lambda\left(\alpha\,\underline{e}^{(a)}+\underline{\beta e}^{(b)}\right)$

If we have a pair of normalised eigenvectors $\underline{e}_0^{(a)}$ and $\underline{e}_0^{(b)}$ that are degenerate and are not orthogonal, it is always possible to derive a pair of eigenvectors from them that are orthogonal to each other. For example such a pair ($\underline{e}^{(a)}$ and $\underline{e}^{(b)}$, say) can be produced by the following method:

$$\underline{e}^{(a)}=\underline{e}_0^{(a)}$$
$$\underline{e}^{(b)}=\underline{e}_0^{(b)}-\left(\underline{e}_0^{(a)}\cdot\underline{e}_0^{(b)}\right)\underline{e}_0^{(a)}$$

(this is orthogonal to $\underline{e}^{(a)}$ but not normalised)

2.9 Use of symmetries to find modes/Guessing displacement patterns

- It is very important to use the symmetry of a particular system, and previous experience to GUESS normal mode displacement patterns. Often quite complex systems can be completely solved in this manner – and it is a lot quicker than setting up and solving the secular equation.

- Key points are:

(1) If you have *N* coordinates there will be *N* normal modes

(2) Displacement patterns have to be orthogonal to each other for non-degenerate normal modes. The first thing to check with a new guessed normal mode is if it is orthogonal to the ones already established – if not – either guess again, or use a relation like the one at the end of section 2.9 to orthogonalise the new mode with respect to the ones already discovered.

(3) For a free system – remember the translation and rotation modes

2.10 Modes of molecules

- If there are M atoms in a molecule, then there will be $3M$ normal modes

- There will always be 3 zero frequency translational modes.

- If the molecule is linear the there will be 2 rotational zero frequency modes, otherwise it will have three rotational modes.

- There are, therefore, $3M$-6 vibrational modes for a non linear molecule and $3M$-5 for a linear ones.

2.11 Continuous systems: normal modes and standing waves

- If one considers a piece of string with N masses on it –one can sketch out displacement patterns of the N masses

- As the number N rises and the separation becomes smaller and small, the system eventually goes over to one better described by a continuous model – and the normal mode has the form of a standing wave on a continuous system.