Classical Electrodynamics

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Abstract

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1 Introduction

1.1 The Field Concept

We are accustomed to think of matter as built up of particles, whose classical kinematics and dynamics are specified by coordinates $\boldsymbol{x}_k(t)$ and canonical momenta $\boldsymbol{p}_k(t)$. But electrodynamics requires a radically different description– Faraday's concept of field.

The electric and magnetic fields $\boldsymbol{E}(\boldsymbol{x}, t), \boldsymbol{B}(\boldsymbol{x}, t)$ are 6 dynamical variables that sit at each point in space and change with time. Since the initial fields at different points are independent we are dealing with a continuous infinity of degrees of freedom. This seems like a lot to swallow but it is the most efficient way to deal with the fact that in nature disturbances cannot propagate with infinite velocity.

The fields can be measured by observing their influence on a charged particle:

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}(\boldsymbol{x}, t) = q(\boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{x}, t)), \qquad \text{SI} \tag{1}$$

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}(\boldsymbol{x}, t) = q(\boldsymbol{E}(\boldsymbol{x}, t) + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}(\boldsymbol{x}, t)), \qquad \text{Gauss, Heaviside} \tag{2}$$

Here q is the charge carried by the particle and c is the universal speed of light. For arbitrary relativistic velocities, $\boldsymbol{p} = m\boldsymbol{v}/\sqrt{1 - v^2/c^2}$. In SI units electric and magnetic fields have different units. The SI unit of charge is the Coulomb (1C=1amp-sec=1A s). Currents are easier to control than charge so the standard definition of charge is via the amp ($1C = 1A \cdot s$) defined as that current in two long parallel wires that gives a force of 2×10^{-7}N/m when separated by 1 m.

1.2 Maxwell's equations: Field Equations of Motion.

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}, \qquad \text{SI} \tag{3}$$

$$\epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = \nabla \times \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{J}, \qquad \text{SI} \tag{4}$$

$$\nabla \cdot \boldsymbol{B} = 0, \qquad \epsilon_0 \nabla \cdot \boldsymbol{E} = \rho, \qquad \text{SI} \tag{5}$$

The SI units of $\epsilon_0 \boldsymbol{E}$ and $\boldsymbol{B}/\mu_0$ are Cm^{-2} and C(sm)^{-1} respectively. Since E and cB have the same units, it follows that $\epsilon_0 \mu_0 c^2$ is dimensionless. In fact Maxwell's equations imply that em waves travel at the speed $1/\sqrt{\epsilon_0 \mu_0}$, so $\epsilon_0 \mu_0 c^2 = 1$.

These equations are roughly parallel to the harmonic oscillator equations

$$\frac{dp}{dt} = -kx, \qquad m\frac{dx}{dt} = p \tag{6}$$

with the magnetic field analogous to the coordinates and the electric field analogous to the momentum.

Historically the fields induced by the sources were called $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} = \epsilon \boldsymbol{E}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M} = \boldsymbol{B}/\mu$, where $\boldsymbol{P}$ and $\mathbf{M}$ are the electric dipole moment density and the magnetic dipole moment density induced in the material by the presence of the fields. They were experimentally determined in materials where ϵ and μ varied from one material to another. The reason is that the sources on the right included only the "free" charges and currents which could be controlled in the lab, whereas materials were themselves made up of charged particles which moved internally causing currents. Then ϵ_0 and μ_0 were just the values of ϵ, μ in the vacuum.

1.3 Heaviside-Lorentz (HL) Units

We put hats on the (HL) quantities (q is charge and $\boldsymbol{m}$ is magnetic dipole moment):

$$\hat{E} = \sqrt{\epsilon_0}E, \qquad \hat{B} = B/\sqrt{\mu_0}, \qquad \hat{q} = q/\sqrt{\epsilon_0}, \qquad \hat{\boldsymbol{m}} = \boldsymbol{m}\sqrt{\mu_0} \tag{7}$$

$$\frac{1}{c}\frac{\partial \hat{\boldsymbol{B}}}{\partial t} = -\nabla \times \hat{\boldsymbol{E}}, \qquad \text{HL} \tag{8}$$

$$\frac{1}{c}\frac{\partial \hat{\boldsymbol{E}}}{\partial t} = \nabla \times \hat{\boldsymbol{B}} - \frac{1}{c}\hat{\boldsymbol{J}}, \qquad \text{HL} \tag{9}$$

$$\nabla \cdot \hat{\boldsymbol{B}} = 0, \qquad \nabla \cdot \hat{\boldsymbol{E}} = \hat{\rho}, \qquad \text{HL} \tag{10}$$

$$\boldsymbol{F} = \hat{q}\left(\hat{\boldsymbol{E}} + \frac{\boldsymbol{v}}{c} \times \hat{\boldsymbol{B}}\right), \qquad \text{HL} \tag{11}$$

In particle physics we go even further and choose units where $\hbar = c = 1$, which removes all coefficients from the equations.

1.4 Physical meaning of Maxwell's equations

We first turn to the divergence equations which do not involve time derivatives. These have no direct analogy in particle mechanics and represent constraints on the fields which hold independently at each time.

Constraint Equations (Gauss Laws)

Consider a region of space $\mathcal{R}$ and integrate both sides of the divergence equations over this region

$$Q_{\text{enclosed}} = \int_{\mathcal{R}} dV \rho = \epsilon_0 \int_{\mathcal{R}} dV \nabla \cdot \boldsymbol{E} = \epsilon_0 \int_{\partial \mathcal{R}} dS \hat{n} \cdot \boldsymbol{E} \equiv \epsilon_0 \Phi_E(\mathcal{R}) \tag{12}$$

$$0 = \int_{\mathcal{R}} dV \nabla \cdot \boldsymbol{B} = \int_{\partial \mathcal{R}} dS \hat{n} \cdot \boldsymbol{B} \equiv \Phi_B(\mathcal{R}) \tag{13}$$

The third equality in each of these equations is just one of the vector calculus analogues of the fundamental theorem of calculus ($x_1 = x, x_2 = y, x_3 = z$):

$$\int_{\mathcal{R}} d^3x \frac{\partial f}{\partial x_k} = \int_{\partial \mathcal{R}} dS n_k f \tag{14}$$

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(It's mathematics not physics!). Here dS is the element of surface area and $\boldsymbol{n}$ is the unit outward directed normal to the surface: $n_1 dS = dx_2 dx_3$, $n_2 dS = dx_3 dx_1$, $n_3 dS = dx_1 dx_2$.

The physics (Gauss) is the connection between the total charge enclosed by a closed surface and the flux of the corresponding field through the surface. In the magnetic case, there is no such thing as magnetic charge (magnetic monopoles don't seem to exist!), so the flux of the magnetic field through *any* closed surface is always zero. This linkage between charge and flux is the physical content of the two constraint Maxwell equations. Note that this linkage is valid at all times.

Dynamic equations

The remaining two Maxwell equations govern the time dependence of the fields, i.e. their dynamics. They are first order in time, but second order equations can be obtained by taking the curl of one and substituting the other:

$$\frac{\partial \nabla \times \boldsymbol{B}}{\partial t} = \mu_0 \epsilon_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu_0 \frac{\partial \boldsymbol{J}}{\partial t} = -\nabla \times (\nabla \times \boldsymbol{E}) = \nabla^2 \boldsymbol{E} - \nabla(\nabla \cdot \boldsymbol{E}) \tag{15}$$

$$\epsilon_0 \frac{\partial \nabla \times \boldsymbol{E}}{\partial t} = -\epsilon_0 \frac{\partial^2 \boldsymbol{B}}{\partial t^2} = \nabla \times \left(\nabla \times \frac{\boldsymbol{B}}{\mu_0} \right) - \nabla \times \boldsymbol{J} = -\nabla^2 \frac{\boldsymbol{B}}{\mu_0} - \nabla \times \boldsymbol{J} \tag{16}$$

Remembering $c^2 = 1/\epsilon_0\mu_0$ and Gauss law, a little rearrangement leads to

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \boldsymbol{E} = -\mu_0 \frac{\partial \boldsymbol{J}}{\partial t} - \nabla \frac{\rho}{\epsilon_0} \tag{17}$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \boldsymbol{B} = \mu_0 \nabla \times \boldsymbol{J} \tag{18}$$

We recognize these as wave equations, showing that electromagnetic fields can form waves traveling at speed c, the speed of light. In the oscillator analogy $-c^2\nabla^2$ plays the role of oscillating frequency. For a plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, this operator gives $c^2\boldsymbol{k}^2$. Thus frequency and wavelength are linked $\omega = ck = 2\pi c/\lambda$.

1.5   Charge conservation

A striking consequence of Maxwell's equations is that the charge and current sources are not independent. To see why, take the divergence of the second equation, using $\nabla \cdot (\nabla \times \boldsymbol{B}) = 0$ identically:

$$\epsilon_0 \frac{\partial \nabla \cdot \boldsymbol{E}}{\partial t} = \frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{J} \tag{19}$$

Integrating both sides over a region of space shows that the rate of decrease (increase) of the total charge in the region is exactly equal to the charge flowing out (in) through the boundary of the region.

Historically, the time derivative of $\boldsymbol{E}$ appearing on the left of the second equation was much too small to be seen experimentally. The empirically well-supported Ampere law

did not need the term. Maxwell realized, however, that without such a term, one would have $\nabla \cdot \boldsymbol{J} = 0$ implying the absurd conclusion that charge could never build up in a localized region. He resolved the absurdity by adding the time derivative term. This is a stunning example of world-class theoretical physics. Not only did it resolve an inconsistency of the equations, but it also implied the necessity of electromagnetic waves, which were experimentally confirmed only later.

1.6   Potentials and Gauge Invariance

Returning to the 4 Maxwell equations, we see that two of them (the first and third) do not involve sources. They are completely linear, and we should be able to solve them once and for all. We first note that if $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, then $\nabla \cdot \boldsymbol{B} = 0$ for any $\boldsymbol{A}$. The converse is also true: if $\nabla \cdot \boldsymbol{B} = 0$, then one can always find a *vector potential* $\boldsymbol{A}$ such that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. So by using $\boldsymbol{A}$ instead of $\boldsymbol{B}$ to describe the magnetic field, the third Maxwell equation will automatically be satisfied. Plugging this into the first Maxwell equation leads to

$$\nabla \times \left(\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{E} \right) = 0 \tag{20}$$

But any vector function with a vanishing curl can be expressed as the gradient of a scalar function, so we can write

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla \phi, \qquad \text{or} \qquad \boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla \phi \tag{21}$$

In summary we can put

$$\boldsymbol{B} = \nabla \times \mathbf{A}, \qquad \boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{22}$$

and then forget about the first and third Maxwell equations. This reduces the number of independent fields from 6 to 4. But the potentials are not unique: changing them by a *gauge* transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla \Lambda, \qquad \phi \to \phi - \frac{\partial \Lambda}{\partial t} \tag{23}$$

with Λ any function of space and time, leaves the fields unchanged. It is sometimes useful to fix this ambiguity by specifying a condition on the potentials, such as $\nabla \cdot \boldsymbol{A} = 0$ (Coulomb gauge), reducing the number of independent fields to 3. The two remaining Maxwell equations become

$$-\epsilon_0 \left(\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \nabla \frac{\partial \phi}{\partial t} \right) = \frac{-\nabla^2 \boldsymbol{A} + \nabla(\nabla \cdot \boldsymbol{A})}{\mu_0} - \boldsymbol{J} \tag{24}$$

$$-\epsilon_0 \left(\nabla^2 \phi - \nabla \cdot \frac{\partial \boldsymbol{A}}{\partial t} \right) = \rho \tag{25}$$

In Coulomb gauge there is a dramatic simplification:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = \frac{1}{c^2}\nabla\frac{\partial\phi}{\partial t} - \mu_0\boldsymbol{J} \tag{26}$$

$$-\epsilon_0\nabla^2\phi = \rho \tag{27}$$

which reveals that ϕ solves a constraint equation, and $\boldsymbol{A}$ solves a wave equation.

2   Electrostatics

For a particle moving in a potential, a static solution just has the particle sitting at rest at a relative minimum of the potential. But a static solution of Maxwell's equations is not so trivial: the fields won't depend on time, but they can have interesting dependence on the three spatial coordinates. If $\boldsymbol{E}$, $\boldsymbol{B}$ are time independent the 4 Maxwell equations reduce to two independent pairs:

$$-\nabla \times \boldsymbol{E} = 0, \qquad \epsilon_0\nabla \cdot \boldsymbol{E} = \rho \tag{28}$$

$$\nabla \times \frac{\boldsymbol{B}}{\mu_0} = \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{29}$$

It follows, of course, that ρ and $\boldsymbol{J}$ are also independent of time and that $\nabla \cdot \boldsymbol{J} = 0$. Note that the presence of a current implies that charges are moving, but such that the charge density does not change: *steady* currents give rise to static magnetic fields. An absolutely static solution would have zero currents.

Electrostatic problems involve solving the first pair of equations in different physical situations. The curl equation can be immediately solved in terms of the scalar potential by writing $\boldsymbol{E} = -\nabla\phi$, with ϕ solving Poisson's equations:

$$-\nabla^2\phi = \rho/\epsilon_0 \tag{30}$$

If $\rho = 0$, this is just the Laplace equation $-\nabla^2\phi = 0$, which has a large number of solutions. The simplest is the linear function $\phi = -\boldsymbol{a} \cdot \boldsymbol{r}$ which means a homogeneous electric field $\boldsymbol{E} = \boldsymbol{a}$. Quadratic functions like $xy, yz, xz, x^2 - y^2, y^2 - z^2$ also solve the equation. In fact one can find polynomial solutions of any order. Such solutions can always be added to any particular solution with $\rho \neq 0$. If the charge distribution is localized we can fix this ambiguity by requiring the fields to vanish at infinity.

2.1   Point charge and the Dirac delta function

Everyone knows that the potential for a point charge q sitting at the point $\boldsymbol{a}$ is

$$\phi = \frac{q}{4\pi\epsilon_0|\boldsymbol{r} - \boldsymbol{a}|} . \tag{31}$$

It is easy to show that the Laplacian $-\nabla^2$ applied to this function when $\boldsymbol{r}$ is away from $\boldsymbol{a}$ is 0. This is consistent with Poisson's equation since the charge density of a point charge is zero everywhere except its location. However, if we integrate the point charge density over any region that encloses $\boldsymbol{a}$, we should get the total charge q. Since the volume of that region can be arbitrarily small, the only way this could be is if $\rho(\boldsymbol{a}) = \infty$. Dirac introduced an "improper function" $\delta(\boldsymbol{r} - \boldsymbol{a})$ to describe this situation. It satisfies:

$$\delta(\boldsymbol{r} - \boldsymbol{a}) = 0, \quad \text{for all } \boldsymbol{r} \neq \boldsymbol{a}, \qquad \int_{\mathcal{R}} d^3 x \delta(\boldsymbol{r} - \boldsymbol{a}) = 1, \quad \text{for } \boldsymbol{a} \in \mathcal{R} \tag{32}$$

With this concept we can write the charge density of a point charge sitting at $\boldsymbol{a}$ as $\rho = q\delta(\boldsymbol{r} - \boldsymbol{a})$. Then for a "unit" point charge (by which we mean $q = \epsilon_0$)

$$-\nabla^2 \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{a}|} = \delta(\boldsymbol{r} - \boldsymbol{a}) \tag{33}$$

so that Poisson's equation is formally satisfied.

The potential for an arbitrary charge distribution is given by superposing the solutions for point charges:

$$\phi(\boldsymbol{r}) = \int d^3 x' \frac{\rho(\boldsymbol{r}')}{4\pi\epsilon_0|\boldsymbol{r} - \boldsymbol{r}'|} \tag{34}$$

which makes sense as long as $r^2\rho \to 0$ as $\boldsymbol{r} \to \infty$.

2.2   Interfaces between different materials

In materials Maxwell's equations assume the form

$$\nabla \cdot \boldsymbol{D} = \rho, \qquad \nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \times \boldsymbol{H} = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \tag{35}$$

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \to \epsilon \boldsymbol{E}, \qquad \boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M} \to \frac{1}{\mu}\boldsymbol{B} \tag{36}$$

The second forms for $\boldsymbol{D}, \boldsymbol{H}$ assume the material is homogeneous and isotropic. Applying Gauss's law (the first two equations) with a "pillbox" volume about a small area on the interface, one learns that

$$\hat{n} \cdot \boldsymbol{B}_2 = \hat{n} \cdot \boldsymbol{B}_1, \qquad \hat{n} \cdot \boldsymbol{D}_2 - \hat{n} \cdot \boldsymbol{D}_1 = \sigma \tag{37}$$

where $\hat{n}$ is the normal to the interface, directed from material 1 to material 2; and σ is the surface charge density at the interface. The conditions on tangential components are obtained using Stokes theorem:

$$\int dS\hat{n} \cdot (\nabla \times \boldsymbol{V}) = \oint dl \cdot \boldsymbol{V} \tag{38}$$

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Applying the last two equation with a rectangular loop that crosses the interface, shows that

$$\hat{n} \times (\boldsymbol{E}_2 - \boldsymbol{E}_1) = 0, \qquad \hat{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{K} \tag{39}$$

where $\boldsymbol{K}$ is the surface current density (units: amp/m). (If $\boldsymbol{H}_2$ points up and $\hat{n}$ points to the right, $\boldsymbol{K}$ points out of the paper.)

Specializing to electrostatics, we have that the tangential components of the electric field are continuous, and the discontinuity of the normal components of $\boldsymbol{D} = \epsilon \boldsymbol{E}$ are proportional to the surface charge density. Since the electric field vanishes inside a conductor this gives a direct connection between the field just outside a conductor and the surface charge density. The field is normal to the conductor and of magnitude σ/ϵ_0.

2.3   Uniqueness of electrostatic solutions, Green's theorem

Suppose ϕ_1, ϕ_2 are two solutions of Poisson's equation $-\nabla^2\phi = \rho/\epsilon_0$. Then $\Phi = \phi_2 - \phi_1$ is a solution of Laplace's equation $\nabla^2\Phi = 0$. One can find lots of solutions of Laplace's equation but they all have growing behavior at infinity. Consider the identity

$$\int_{\partial\mathcal{R}} dS\,\Phi\hat{n} \cdot \nabla\Phi = \int_{\mathcal{R}} dV\,\nabla \cdot (\Phi\nabla\Phi) = \int_{\mathcal{R}} dV\,\nabla\Phi \cdot \nabla\Phi. \tag{40}$$

If the left side vanishes (either because $\Phi \to 0$ fast enough at infinity or because Dirichlet of Neumann boundary conditions apply on the boundary of $\mathcal{R}$), Then Φ must be a constant, because the integrand on the right is positive definite implying $\nabla\Phi = 0$. This would imply that the electric fields of the two solutions would be identical. If the charge density ρ is localized and has finite total charge, it is reasonable to insist that the solution of the Poisson equation behave as $Q/(4\pi\epsilon_0 r)$ at large r, in which case Φ would vanish faster than $1/r$ at infinity. and being a constant would have to be identically zero.

More generally one has

$$\nabla \cdot (\psi\nabla\phi - \phi\nabla\psi) = \psi\nabla^2\phi - \phi\nabla^2\psi \tag{41}$$

which, upon integrating over a region $\mathcal{R}$, yields **Green's Theorem**

$$\int_{\mathcal{R}} dV(\psi\nabla^2\phi - \phi\nabla^2\psi) = \oint_{\partial\mathcal{R}} dS(\psi\hat{n} \cdot \nabla\phi - \phi\hat{n} \cdot \nabla\psi) \tag{42}$$

In the context of electrostatics, suppose ϕ is the potential associated with charge density $\rho = -\nabla^2\phi\epsilon_0$ and $\psi = \phi'$ is associated with charge density $\rho' = -\nabla^2\phi'\epsilon_0$, and furthermore that the boundary of $\mathcal{R}$ is a conducting surface with surface charge density $\sigma = -\hat{n}\cdot\boldsymbol{E}\epsilon_0 = \hat{n}\cdot\nabla\phi\epsilon_0$ in the first case and $\sigma' = \hat{n} \cdot \nabla\phi'\epsilon_0$ in the second case. Then Green's theorem becomes the **reciprocity theorem**:

$$\begin{aligned}
\int_{\mathcal{R}} dV(-\phi'\rho + \phi\rho') &= \oint_{\partial\mathcal{R}} dS(\phi'\sigma - \phi\sigma') \\
\int_{\mathcal{R}} dV\,\phi\rho' + \oint_{\partial\mathcal{R}} dS\,\phi\sigma' &= \int_{\mathcal{R}} dV\,\phi'\rho + \oint_{\partial\mathcal{R}} dS\,\phi'\sigma
\end{aligned} \tag{43}$$

2.4 Green functions

The typical electrostatic problem specifies a fixed charge distribution $\rho(\boldsymbol{r})$ and a number of surfaces on which the potential satisfies boundary conditions. Commonly these surfaces are the boundaries of conductors which are held at various fixed potentials, e.g. by batteries. Disregarding the surfaces a solution of Poisson's equation is just (34). This solution will not satisfy the boundary conditions: it is necessary to add a solution of Laplace's equation to it.

The Green function $G(\boldsymbol{r}, \boldsymbol{r}')$ is the solution of a boundary value problem on these surfaces where ρ/ϵ_0 is replaced by that of a point charge at $\boldsymbol{r}'$: $\delta(\boldsymbol{r} - \boldsymbol{r}')$. If there were no surfaces at all, G would just be the Coulomb potential $1/4\pi|\boldsymbol{r} - \boldsymbol{r}'|$ of a unit ($q = \epsilon_0$) point charge. The Dirichlet Green function G_D vanishes on all the surfaces, whereas the Neumann function has normal derivative is a constant on all surfaces. Since the Neumann function is the potential for a unit point charge, the total electric flux leaving the region must be 1, thus the normal derivative can be taken zero on **all** surfaces **only if some flux can escape to infinity**. Knowledge of the Green function for a given set of surfaces allows the explicit construction of the solution for an arbitrary charge distribution and any specified boundary values on the given surfaces. Unless the surfaces have very special geometries, though, it is virtually impossible to actually find the Green function!

This construction is a simple application of Green's theorem, with ϕ the potential we seek, and G the Green function:

$$-\epsilon_0 \nabla^2 \phi(\boldsymbol{r}) = \rho(\boldsymbol{r}), \qquad -\nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{44}$$

$$-\frac{1}{\epsilon_0} \int_{\mathcal{R}} dV \rho(\boldsymbol{r}) G(\boldsymbol{r}, \boldsymbol{r}') + \phi(\boldsymbol{r}') = \int_{\partial\mathcal{R}} dS\, G(\boldsymbol{r}, \boldsymbol{r}') \hat{n} \cdot \nabla \phi - \int_{\partial\mathcal{R}} dS\, \phi(\boldsymbol{r}) \hat{n} \cdot \nabla G(\boldsymbol{r}, \boldsymbol{r}') \tag{45}$$

If the value of ϕ is specified on the boundary, then we use the Dirichlet function which vanishes on the boundary:

$$\phi(\boldsymbol{r}') = \frac{1}{\epsilon_0} \int_{\mathcal{R}} dV \rho(\boldsymbol{r}) G_D(\boldsymbol{r}, \boldsymbol{r}') - \int_{\partial\mathcal{R}} dS\, \phi(\boldsymbol{r}) \hat{n} \cdot \nabla G_D(\boldsymbol{r}, \boldsymbol{r}') \tag{46}$$

On the other hand if the normal derivative of ϕ is specified on the boundary, we need the Neumann function. If the problem is such that the flux can escape to infinity, we can take the normal derivative of G_N to vanish on the finite surfaces and obtain

$$\phi(\boldsymbol{r}') = \frac{1}{\epsilon_0} \int_{\mathcal{R}} dV \rho(\boldsymbol{r}) G_N(\boldsymbol{r}, \boldsymbol{r}') + \int_{\partial\mathcal{R}} dS\, G_N(\boldsymbol{r}, \boldsymbol{r}') \hat{n} \cdot \nabla \phi(\boldsymbol{r}) \tag{47}$$

These formulas summarize a profound physics point: *the solution of an electrostatics problem is uniquely determined by the charge distribution and either Dirichlet or Neumann boundary conditions.* On the other hand they betray no hint of how to find that solution, beyond reducing the problem to finding the Green function. Developing techniques and tricks for doing that in various simple situations is the subject of the first three chapters of Jackson and the next three or four weeks of our course.

2.5 Electrostatic energy

A static configuration of charges stores energy: the charges are held in place by external forces. If those forces were released the charges would begin moving converting that energy to kinetic energy. We can evaluate the stored energy by computing the work required to bring all the charges in from infinite separation. As each charge is successively brought in work is done against the forces due to all the charges already in their final positions, $\Delta W = \phi \Delta Q$:

$$W_n = \sum_{i=1}^{n-1} \frac{q_n q_i}{4\pi\epsilon_0 |\boldsymbol{r}_n - \boldsymbol{r}_i|} \tag{48}$$

$$W_{\text{tot}} = \sum_{n=2}^{N} W_n = \frac{1}{8\pi\epsilon_0} \sum_{i \neq n} \frac{q_n q_i}{|\boldsymbol{r}_n - \boldsymbol{r}_i|} = \frac{1}{8\pi\epsilon_0} \sum_{i,n} \frac{q_n q_i}{|\boldsymbol{r}_n - \boldsymbol{r}_i|} - \frac{1}{8\pi\epsilon_0} \sum_i \frac{q_i^2}{|\boldsymbol{0}|} \tag{49}$$

$$\rightarrow \frac{1}{8\pi\epsilon_0} \int d^3x d^3y \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} - \frac{1}{8\pi\epsilon_0} \sum_i \frac{q_i^2}{|\boldsymbol{0}|} \tag{50}$$

The last term subtracts off the "self-energy" of the charges. It is more convenient to include these self-energies in the total energy:

$$E_{\text{tot}} \equiv W_{\text{tot}} + E_{\text{self}} = \frac{1}{8\pi\epsilon_0} \int d^3x d^3y \frac{\rho(\boldsymbol{x})\rho(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} = \frac{1}{2} \int d^3x \rho(\boldsymbol{x})\phi(\boldsymbol{x})$$

$$= \frac{\epsilon_0}{2} \int d^3x (-\nabla^2 \phi)\phi(\boldsymbol{x}) = \frac{\epsilon_0}{2} \int d^3x \boldsymbol{E}^2 \tag{51}$$

This last form makes no reference to the location and sizes of whatever charges produce the field. The energy can be attributed solely to the electric field itself. A nonzero electric field at a point adds $w = \epsilon_0 \boldsymbol{E}^2/2$ to the energy density at that point.

2.6 Capacitance

A very typical problem in electrostatics involves a set of conductors of various shapes and locations each held at some potential V_i. The potential everywhere outside these conductors is uniquely determined, which uniquely determines the field near the surface of each conductor, which uniquely determines the surface charge density on each conductor, so the total charge Q_i on each conductor is determined. Because of the linear relation between potential and charge density we can assert that

$$Q_i = \sum_j C_{ij} V_j \ . \tag{52}$$

The coefficients C_{ij} (which have dimension $\epsilon_0 \times \text{Length}$) depend on the geometry of the various conductors and are usually impossible to compute, though they can clearly be measured. C_{ii} is called the capacitance of conductor i and C_{ij} is the coefficient of induction of j on i. The

SI unit of capacitance is the farad (F). Thus in SI units we have $\epsilon_0 \approx 8.854 \times 10^{-12}$ F/m. Applying the reciprocity theorem for the two systems Q, V and Q', V' we find

$$0 \;=\; \sum_i (Q_i V_i' - Q_i' V_i) = \sum_{ij} (C_{ij} V_j V_i' - C_{ij} V_j' V_i = \sum_{ij} (C_{ij} - C_{ji}) V_j V_i' \tag{53}$$

which implies that $C_{ij} = C_{ji}$. Thus a system of N conductors has at most $N(N+1)/2$ independent C_{ij}. If the system has symmetry there may be even fewer independent components. For example, three identical conducting spheres at the vertices of an equilateral triangle would have $C_{11} = C_{22} = C_{33}$ and $C_{ij} = C_{12}$ for all $i \neq j$. Reciprocity and symmetry have reduced the 9 C_{ij} to only two independent ones in this case!

The electric energy of the system of conductors is

$$E = \frac{1}{2} \sum_i Q_i V_i = \frac{1}{2} \sum_{ij} C_{ij} V_i V_j = \frac{1}{2} \sum_{ij} C_{ij}^{-1} Q_i Q_j \tag{54}$$

Here C^{-1} is the inverse of the matrix C, $CC^{-1} = C^{-1}C = I$. Suppose we want to find the force on one of the conductors, say 1, exerted by all the others. Then after disconnecting the batteries, so that the chareges Q_i remain fixed, we move conductor 1 a small distance δx_k in the k direction. Then

$$F_i = -\frac{\partial E}{\partial x_i}\Big|_Q = -\frac{1}{2} \sum_{ij} \frac{\partial}{\partial x_k} C_{ij}^{-1}(x) Q_i Q_j = +\frac{1}{2} \sum_{ij} \frac{\partial}{\partial x_k} C_{ij}(x) V_i V_j = +\frac{\partial E}{\partial x_i}\Big|_V \tag{55}$$

The subscripts Q, V indicate which quantities are held fixed in taking the derivative with respect to x. *Note the sign difference*!

Finally, a very common special case is a simple capacitor, which has only two conductors, and further is electrically neutral ($Q_2 = -Q_1 \equiv Q$). Then the capacitor relations reduce to the single equation $Q = CV$ where $V = V_2 - V_1$. In terms of the C_{ij} of the general two conductor system, one finds

$$C = \frac{C_{11} C_{22} - C_{12}^2}{C_{11} + C_{22} + 2C_{12}} \tag{56}$$

3 Electrostatic Boundary-Value problems

3.1 Method of Images

This method exploits the fact that the potential of any system of charges satisfies Laplace's equation at all points in space where the charge density $\rho = 0$. Thus we can try to make an educated guess for a solution of a boundary value problem by placing various "image" charges behind the boundary, i.e. outside the region where we need to find ϕ. Frequently the symmetry of the boundaries helps guide the choice for the location of the image charges.

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Infinite Plane

An elementary example, where the choice is easy, is a charge above a conducting plane. The Dirichlet Green function for the region above the xy-plane is, writing $\boldsymbol{r} = \boldsymbol{\rho} + z\hat{z}$, and similarly for $\boldsymbol{r}'$:

$$G_D(\boldsymbol{r}, \boldsymbol{r}') \;=\; \frac{1}{4\pi}\left[\frac{1}{\sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (z - z')^2}} - \frac{1}{\sqrt{(\boldsymbol{\rho} - \boldsymbol{\rho}')^2 + (z + z')^2}}\right] \tag{57}$$

To solve the general Dirichlet problem we need

$$\hat{n} \cdot \nabla G \;=\; -\hat{z}\frac{\partial G}{\partial z}\Big|_{z=0}$$

$$=\; -\frac{1}{2\pi}\frac{z'}{\left((x - x')^2 + (y - y')^2 + z'^2\right)^{3/2}} \tag{58}$$

Here since the region of interest is the upper half plane, the *outward* normal to the bounding xy-plane points in the negative z- direction, i.e. $\hat{n} = -\hat{z}$. The general Dirichlet problem is to impose $\phi(x, y, 0) = V(x, y)$, Then, applying the Green formula, the potential above the plane is

$$\phi(\boldsymbol{r}') \;=\; \frac{1}{2\pi}\int dx\,dy\,V(x, y)\frac{z'}{\left((x - x')^2 + (y - y')^2 + z'^2\right)^{3/2}} \tag{59}$$

As a simple example, take $V = 0$ outside a disk of radius R and $V(x, y) = V_0$ inside the disk:

$$\phi(\boldsymbol{r}') \;=\; \phi(\rho', z') \;=\; \frac{V_0}{2\pi}\int_0^{2\pi} d\varphi \int_0^R \rho\,d\rho\frac{z'}{(\rho^2 + z'^2 + \rho'^2 - 2\rho\rho'\cos\varphi)^{3/2}} \tag{60}$$

The integral over ρ could be done leaving a complicated φ integral to do. However, when the observation point $\boldsymbol{r}'$ is on the z-axis, i.e. $\rho' = 0$, the integral simplifies dramatically

$$\phi(0, z') \;=\; \frac{V_0}{2\pi}\int_0^{2\pi} d\varphi \int_0^R \rho\,d\rho\frac{z'}{(\rho^2 + z'^2)^{3/2}}$$

$$=\; \frac{V_0 z'}{2}\int_0^{R^2} du(u + z'^2)^{-3/2} = V_0\left[1 - \frac{z'}{\sqrt{R^2 + z'^2}}\right] \tag{61}$$

Also, for general $V(x, y)$, it is easy to evaluate the behavior as $r' = \sqrt{\rho'^2 + z'^2} \gg R$:

$$\phi \;\sim\; \frac{z'}{2\pi r'^3}\int dx\,dy\,V(x, y) \to \frac{V_0 z' R^2}{2r'^3}, \qquad \text{for } r' \gg R \tag{62}$$

for the simple example.

Sphere

We next discuss a less trivial example: a point charge outside a spherical conductor held at 0 potential. (We assume the potential also vanishes at infinity.) Axial symmetry dictates

that any image charges lie on the line joining the center of the sphere to the charge outside. Let the radius of the sphere be a, and choose coordinates with origin at the center of the sphere and the charge outside on the positive y-axis at $(0, r', 0)$. A single image charge of $q' = -qa/r'$ and location $(0, a^2/r', 0)$ does the trick.

$$\phi(\boldsymbol{r}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\boldsymbol{r} - r'\hat{y}|} - \frac{a}{r'|\boldsymbol{r} - a^2\hat{y}/r'|} \right] \tag{63}$$

To see that ϕ vanishes on the sphere note that putting $\boldsymbol{r} = a\hat{r}$ gives:

$$|\boldsymbol{r} - r'\hat{y}|^2 = a^2 + r'^2 - 2r'\boldsymbol{r} \cdot \hat{y} \tag{64}$$

$$|\boldsymbol{r} - a^2\hat{y}/r'|^2 = a^2 + a^4/r'^2 - 2a^2\boldsymbol{r} \cdot \hat{y}/r' = \frac{a^2}{r'^2}(r'^2 + a^2 - 2r'\boldsymbol{r} \cdot \hat{y}) \tag{65}$$

This potential is proportional to the Dirichlet Green function for the exterior of a sphere. We just have to put the exterior charge $q = \epsilon_0$ at a general point $\boldsymbol{r}'$ and adjust the normalization:

$$G_D(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi} \left[\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} - \frac{a}{r'|\boldsymbol{r} - a^2\boldsymbol{r}'/r'^2|} \right] \tag{66}$$

This can be plugged into Eq.(46) to solve the arbitrary Dirichlet problem for a sphere. For this we need the normal derivative of G_D on the surface of the sphere:

$$\hat{n} \cdot \nabla G_D \Big|_{r=a} = -\frac{\partial G_D}{\partial r}$$

$$= \frac{a - r'^2/a}{4\pi(a^2 + r'^2 - 2a\hat{r} \cdot \boldsymbol{r}')^{3/2}} \tag{67}$$

Then the solution for the boundary condition $\phi(r = a, \theta, \varphi) = V(\theta, \varphi)$ is

$$\phi(r', \theta', \varphi') = -\int d\Omega V(\theta, \varphi) \frac{a(a^2 - r'^2)}{4\pi(a^2 + r'^2 - 2a\hat{r} \cdot \boldsymbol{r}')^{3/2}} \tag{68}$$

Here $d\Omega = \sin\theta d\theta d\varphi$, and $\hat{r} \cdot \boldsymbol{r}' = r'(\cos\theta \, \cos\theta' + \sin\theta \, \sin\theta' \, \cos(\varphi - \varphi'))$. While the method of images is very powerful for the simple geometry of a single sphere, it becomes less so with more complicated geometries. In the homework you will see this for the case of two conducting spheres, which require an infinite number of image charges.

Even the case of the region between two concentric spheres requires an infinite number of image charges, in spite of the spherical symmetry. This motivates the development of other approaches to electrostatic problems.

3.2   Method of Separation of Variables

Partial differential equations put conditions on functions of several variables. The separation of variables method first tries to find solutions that are products of functions of single variables. For instance

$$\phi(x, y, z) = X(x)Y(y)Z(z) \tag{69}$$

Plugging such a form into the diff eq reduces the problem to ordinary diff eqs in a single variable. After finding these one can find more general solutions by linear superposition of the factorized solutions. For this to work the differential operator of the original equation must be a sum of differential operators on single variables.

Cartesian Coordinates

In Cartesian coordinates $-\nabla^2 = -(\partial^2/\partial x^2) - (\partial^2/\partial y^2) - (\partial^2/\partial z^2)$, and Laplace's equation leads to

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} - \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0 \tag{70}$$

To solve this equation, each term must be a constant, say A, B, C respectively subject to $A + B + C = 0$. Thus we must solve three eigenvalue equations:

$$-\frac{\partial^2 X}{\partial x^2} = AX, \qquad -\frac{\partial^2 Y}{\partial y^2} = BY, \qquad -\frac{\partial^2 Z}{\partial z^2} = CZ = -(A+B)Z , \tag{71}$$

The solutions of these equations are just trigonometric and/or exponential functions. Two of the eigenvalues, say A, B are arbitrary.

Eigenvalue problems are familiar from quantum mechanics, in which dynamical variables are represented by linear operators whose eigenvalues are the possible results of measuring those variables. In this context the Laplacian is a sum of three commuting linear operators $-\partial^2/\partial x_i^2$. If those three operators are hermitian, one can find a basis of simultaneous eigenstates of those three operators.

But the Laplace equation requires the three eigenvalues to sum to zero, so we have only a two-fold basis. In Cartesian coordinates we can use these eigenfunctions to construct boundary value solutions for shapes made from planes parallel to the coordinate planes.

Consider a rectangular box of dimensions $a \times b \times c$, placed with three of the sides in the respective coordinate planes. The general Dirichlet problem would be to specify the potential arbitrarily on each of the 6 rectangular surfaces of the box. But we can build that solution by superposing solutions with the potential vanishing on 5 sides, but arbitrarily given on the sixth. Pick that side to be the one at $z = c$ where the potential is specified to be $V(x, y)$. Then we have

$$X_m(x) = \sin\frac{m\pi x}{a}, \qquad Y_n(y) = \sin\frac{n\pi y}{b}, \qquad Z_{mn}(z) = \sinh \pi z \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{72}$$

$$\phi(x, y, z) = \sum_{m.n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh \pi z \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{73}$$

This expresses the solution as a double Fourier series in x, y, with the A_{mn} determined by the last boundary condition

$$V(x, y) = \sum_{m,n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sinh \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \tag{74}$$

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Recall that the functions $\sin(mx\pi/a)$ are orthogonal on the interval $0 < x < a$

$$\int_0^a dx \sin \frac{mx\pi}{a} \sin \frac{nx\pi}{a} = \frac{a}{2}\delta_{mn} \tag{75}$$

This means we can solve for the A_{mn}

$$A_{mn} \sinh \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \frac{4}{ab} \int_0^a dx \int_0^b dy \sin \frac{mx\pi}{a} \sin \frac{ny\pi}{b} V(x,y) \tag{76}$$

For example, if V is a constant

$$A_{mn} \sinh \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} = \frac{4V}{mn\pi^2}(1 - (-)^m)(1 - (-)^n) = \frac{16V}{mn\pi^2}\delta_{m,\text{odd}}\delta_{n,\text{odd}} \tag{77}$$

and the potential reads

$$\phi(x,y,z) = \frac{16V}{\pi^2} \sum_{m,n=\text{odd}}^{\infty} \frac{1}{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \frac{\sinh \pi z \sqrt{m^2/a^2 + n^2/b^2}}{\sinh \pi c \sqrt{m^2/a^2 + n^2/b^2}} \tag{78}$$

Notice that as long as $z < c$, i.e. inside the box, the ratio of sinh's gives exponential convergence of the double sum. Although the individual terms of this expansion solve Laplace's equation outside the box, this sum will not give the proper outside solution because of bad behavior at large distances.

Near the boundaries of the box, the exponential convergence becomes less effective, and it is necessary to include ever more terms to maintain accuracy. Near corners and edges convergence is even worse, because fields and surface charge density can become singular there.

The representation of a constant by a truncated Fourier series (keeping the first N terms) of sin functions is always poor near the endpoints (Gibbs phenomenon). The error size doesn't improve for larger N, but the region over which it occurs does decrease with larger N. (See the figure.) Thus as long as x is away from the endpoints, one can approximate the constant arbitrarily accurately by including enough terms.

Gibbs Phenomenon: N=5,30
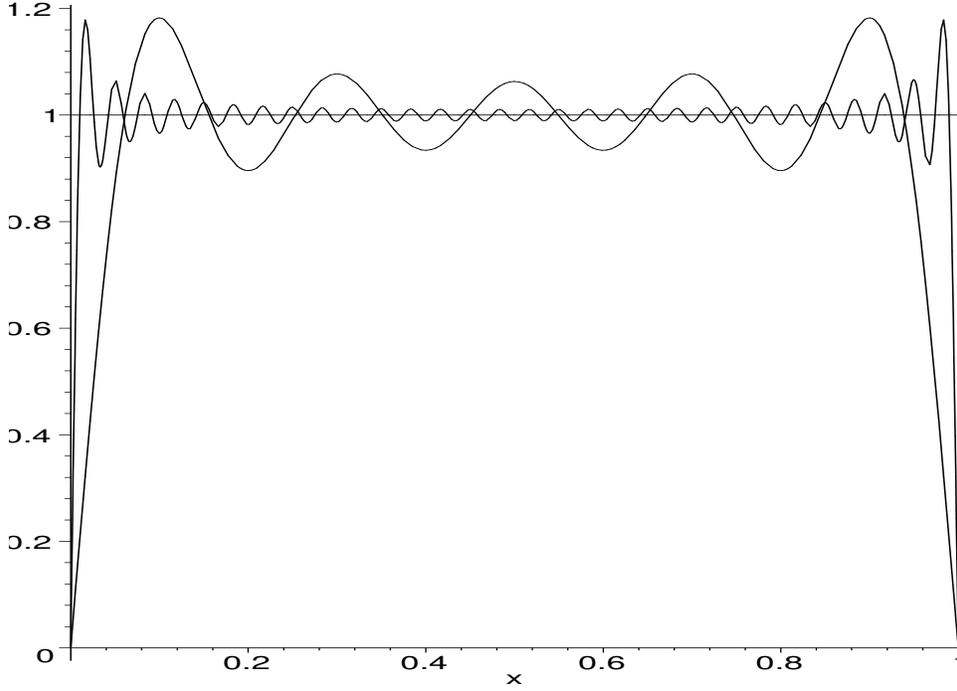
The singular behavior near corners and edges can be captured by replacing them by intersecting planes, which can be analyzed simply by the method of images. The Green function for the quadrant bounded by two perpendicular planes can be found using 3 image charges. That for the octant bounded by three perpendicular planes requires 7 image charges.

Spherical Coordinates

Spherical coordinates are r, θ, φ, the radius, polar angle from the z-axis, and azimuthal angle about the z-axis measured from the x axis: $(x, y, z) = r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

It is helpful to think of the Laplacian in terms of the quantum mechanical angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \nabla / i$:

$$
\begin{aligned}
\boldsymbol{L}^2 &= -\boldsymbol{r} \cdot (\nabla \times (\boldsymbol{r} \times \nabla)) = -r_i \nabla_j r_i \nabla_j + r_i \nabla_j r_j \nabla_i \\
&= -r^2 \nabla^2 - r\frac{\partial}{\partial r} + \left(r\frac{\partial}{\partial r}\right)^2 + 2r\frac{\partial}{\partial r} = -r^2\nabla^2 + r\frac{\partial^2}{\partial r^2}r
\end{aligned}
\tag{79}
$$

$$
-\nabla^2 = -\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{\boldsymbol{L}^2}{r^2}
\tag{80}
$$

Then, separation of variables involves factorizing $\phi(r, \theta, \varphi) = R(r)Y_{lm}(\theta, \varphi)$ where $Y_{lm}(\theta, \varphi)$, the spherical harmonic, is an eigenfunction of $\boldsymbol{L}^2, L_z$ with eigenvalues $l(l+1), m$ respectively where $l = 0, 1, 2 \cdots$ and $-l \le m \le l$. Then if ϕ satisfies Laplace's equation, we have

$$
\left(-\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{l(l+1)}{r^2}\right)R = 0
\tag{81}
$$

$$\mathbf{L}^2 Y_{lm} = l(l+1)Y_{lm}, \qquad L_z Y_{lm} = m Y_{lm} \tag{82}$$

The radial equation is easily solved with the most general solution

$$R_l(r) = A_l r^l + B_l \frac{1}{r^{l+1}} \tag{83}$$

The angular equations determine the spherical harmonics Y_{lm}. We can use our familiarity with quantum mechanics to efficiently analyze these solutions. For example we know that, defining $L_\pm \equiv L_x \pm i L_y$,

$$L_+ Y_{ll} = 0, \qquad L_- Y_{lm} = \sqrt{l(l+1) - m(m-1)} Y_{l,m-1} \tag{84}$$

To go further, we need the explicit form of the L's:

$$L_z = \frac{1}{i}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \qquad L_x = \frac{1}{i}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right), \qquad L_y = \frac{1}{i}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) \tag{85}$$

$$L_\pm = \pm z\left(\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y}\right) \mp (x \pm iy)\frac{\partial}{\partial z} \tag{86}$$

We see easily that $L_+(x+iy)^l = 0$ and $L_z(x+iy)^l = l(x+iy)^l$. So Y_{ll} is proportional to $(x+iy)^l = r^l \sin^l\theta e^{il\varphi}$

$$Y_{ll}(\theta,\varphi) = C\sin^l\theta e^{il\varphi} \tag{87}$$

We normalize the Y's to 1, i.e. $\int d\Omega |Y_{lm}|^2 = 1$, so

$$1 = |C|^2\int \sin\theta d\theta d\varphi \sin^{2l}\theta = 2\pi|C|^2\int_{-1}^{1} dz(1-z^2)^l = 2\pi|C|^2\int_0^2 du u^l(2-u)^l$$

$$= 2\pi|C|^2 2^{2l+1}\frac{(l!)^2}{(2l+1)!} \tag{88}$$

Then

$$Y_{ll}(\theta,\varphi) = \frac{(-)^l}{l!\sqrt{2\pi}}\sqrt{\frac{(2l+1)!}{2^{(2l+1)}}}\sin^l\theta e^{il\varphi} \tag{89}$$

The sign is the standard convention. By applying L_- repeatedly, one obtains all the Y_{lm} from $-l \le m \le +l$. Clearly, Y_{lm} has the φ dependence $e^{im\varphi}$, and the θ dependence is a polynomial in $\cos\theta, \sin\theta$.

A typical boundary value problem using spherical coordinates is to find the potential between two concentric spherical shells, say at $r = a, b$. Then the general solution of Laplace's equation between the shells is

$$\phi = \sum_{l=0}^\infty \sum_{m=-l}^l \left(A_{lm}r^l + \frac{B_{lm}}{r^{l+1}}\right)Y_{lm}(\theta,\varphi) \tag{90}$$

With the boundary conditions $\phi(a,\theta,\varphi) = V_a(\theta,\varphi)$, $\phi(b,\theta,\varphi) = V_b(\theta,\varphi)$, the coefficients are determined by

$$A_{lm}a^l + \frac{B_{lm}}{a^{l+1}} = \int d\Omega Y_{lm}^* V_a \tag{91}$$

$$A_{lm}b^l + \frac{B_{lm}}{b^{l+1}} = \int d\Omega Y_{lm}^* V_b \tag{92}$$

As an elementary example, if $V_{a,b}$ are constants, then only the $l = m = 0$ term survives, and the potential is isotropic

$$\phi = \frac{bV_b - aV_a}{b - a} + \frac{ab(V_a - V_b)}{r(b - a)} \tag{93}$$

3.3 Angle Differential Equations

We next express the L_i in terms of angle derivatives. According to the chain rule

$$\nabla = \nabla e\frac{\partial}{\partial r} + \nabla\theta\frac{\partial}{\partial\theta} + \nabla\varphi\frac{\partial}{\partial\varphi} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{\partial}{r\partial\theta} + \hat{\varphi}\frac{\partial}{r\sin\theta\partial\varphi} \tag{94}$$

where the hat over a variable means the unit vector in the direction of increasing variable. Then

$$\boldsymbol{r} \times \nabla = \hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{\partial}{\sin\theta\partial\varphi} \tag{95}$$

One can work out the components of various quantities:

$$\frac{\partial}{\partial x} \pm i\frac{\partial}{\partial y} = e^{\pm i\varphi}\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta} \pm \frac{i}{r\sin\theta}\frac{\partial}{\partial\varphi}\right), \qquad \frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta} \tag{96}$$

$$L_\pm = \pm e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\varphi}\right), \qquad L_z = \frac{1}{i}\frac{\partial}{\partial\varphi} \tag{97}$$

$$\boldsymbol{L}^2 = L_z^2 + \frac{1}{2}(L_+L_- + L_-L_+) = -(\boldsymbol{r} \times \nabla)^2$$

$$= -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \tag{98}$$

Then the equations satisfied by Y_{lm} are

$$\left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)Y_{lm} = l(l+1)Y_{lm}, \qquad \frac{1}{i}\frac{\partial}{\partial\varphi}Y_{lm} = mY_{lm} \tag{99}$$

These differential equations do not by themselves require that l, m be integers. Those conditions follow from the requirement that the Y_{lm} be globally defined in three dimensional space. First since $\varphi = 2\pi$ is the same point as $\varphi = 0$, then m must be an integer. It is a bit

more subtle to see that good behavior at $\theta = 0, \pi$ requires l to be a non-negative integer. It is well to keep in mind that boundary value problems that do not use all of space may not impose these quantization conditions.

Notice that the complex conjugate Y_{lm}^* satisfies the same differential equations as $Y_{l,-m}$. This implies they are multiples of one another. Since for $m > 0$, $Y_{lm} \propto L_+^m Y_{l0}$ and $Y_{l,-m} \propto L_-^m Y_{l0}$, it follows from $L_+^* = -L_-$ that

$$Y_{lm}^* = (-)^m Y_{l,-m} \tag{100}$$

We can factor off the φ dependence $Y_{lm} = \Theta_{lm}(\theta)e^{im\varphi}$ where Θ is proportional to the associated Legendre function $P_l^m(\theta)$ which satisfies

$$\left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta} \right) P_l^m = l(l+1)P_l^m \tag{101}$$

When a problem has azimuthal symmetry, only the Y_{l0} which are proportional to $P_l^0 \equiv P_l(\cos\theta)$, the Legendre polynomials, enter in the expansion of the solution of Laplace's equation. Changing variables to $x = \cos\theta$, the Legendre polynomials satisfy

$$-\frac{d}{dx}(1 - x^2)\frac{d}{dx}P_l(x) = l(l+1)P_l(x), \qquad P_l(1) = 1 \tag{102}$$

The Rodrigues formula

$$P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2 - 1)^l \tag{103}$$

gives an efficient way to generate the P_l. At $x = 1$ every factor must be differentiated showing that $P_l(1) = 1$

Actually, for l an integer, there are two solutions of the Legendre equation, only one of which is a polynomial. The non-polynomial one is called $Q_l(x)$. And when l is *not* an integer, neither solution is a polynomial. The Legendre polynomials are orthogonal

$$\int_{-1}^{1} dx\, P_l(x)P_{l'}(x) = \frac{2}{2l+1}\delta_{ll'}. \tag{104}$$

Given this normalization, we have $Y_{l0} = \sqrt{(2l+1)/4\pi}P_l(\cos\theta)$. More generally

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)e^{im\varphi} \tag{105}$$

By using the ladder angular momentum operators $L_\pm$ on the Y_{lm}, one can verify the explicit formula (for $m \geq 0$)

$$P_l^m(x) = (-)^m(1 - x^2)^{m/2}\frac{d^m}{dx^m}P_l(x), \qquad m \geq 0 \tag{106}$$

From this formula, it is clear that $P_l^m(1) = 0$ for $m > 0$, which implies $Y_{lm}(0, \varphi) = 0$ for $m \neq 0$.

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3.4 Problems with Azimuthal Symmetry

Solutions to electrostatic problems with azimuthal symmetry (no dependence on φ) only involve the Legendre polynomials

$$\phi(r,\theta) \;=\; \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right)P_l(\cos\theta) \tag{107}$$

$$a_l r^l + \frac{b_l}{r^{l+1}} \;=\; \frac{2l+1}{2}\int_{-1}^{1}dx\,\phi(r,\theta(x))P_l(x) \tag{108}$$

There are many such situations where it is easy to find the exact solution on the z-axis, but not off the axis. Since $P_l(1)=1$ the above expansion reduces at $\theta=0$ to

$$\phi(r,0) \;=\; \sum_{l=0}^{\infty}\left(a_l r^l + \frac{b_l}{r^{l+1}}\right) \tag{109}$$

Then one can determine the a_l, b_l by expanding the on axis solution in a power series. Then the angle dependence is obtained by inserting the Legendre polynomial.

As a useful and important example, the empty space Green function $1/4\pi|\boldsymbol{r}-\boldsymbol{r}'|$ viewed as a function of $\boldsymbol{r}$ has azimuthal symmetry about the line defined by $\boldsymbol{r}'$. Let θ be measured from $\boldsymbol{r}'$. Then

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \;=\; \frac{1}{4\pi\sqrt{r^2+r'^2-2rr'\cos\theta}}$$

$$= \sum_{l=0}^{\infty}R_l(r,r')P_l(\cos\theta) \tag{110}$$

To find R_l put $\theta=0$ and expand

$$\frac{1}{|r-r'|} = \begin{cases} \displaystyle\sum_{l=0}^{\infty}\frac{r^l}{r'^{l+1}} & r<r' \\[2ex] \displaystyle\sum_{l=0}^{\infty}\frac{r'^l}{r^{l+1}} & r>r' \end{cases} \tag{111}$$

A unified notation for the right side of this expansion is to define $r_<$ as the smaller of r, r' and $r_>$ as the larger of r, r'. Then $R_l(r,r') = r_<^l/4\pi r_>^{l+1}$ so

$$\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \;=\; \frac{1}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l(\cos\theta) \tag{112}$$

We can use this expansion to infer an expansion for the delta function in spherical coordinates:

$$\delta(\boldsymbol{r}-\boldsymbol{r}') \;=\; \frac{1}{r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi') \tag{113}$$

From the Green function equation we have

$$
\begin{aligned}
\delta(\boldsymbol{r} - \boldsymbol{r}') &= -\nabla^2 \frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \\
&= \frac{1}{4\pi} \sum_{l=0}^{\infty} \left(-\frac{1}{r}\frac{\partial^2}{\partial r^2} r + \frac{l(l+1)}{r^2} \right) \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \qquad (114)
\end{aligned}
$$

where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$. Now we use

$$
\begin{aligned}
\frac{r_<^l}{r_>^{l+1}} &= \theta(r - r')\frac{r'^l}{r^{l+1}} + \theta(r' - r)\frac{r^l}{r'^{l+1}} \\
\frac{\partial}{\partial r} r \frac{r_<^l}{r_>^{l+1}} &= \delta(r - r')\left(\frac{r'^l}{r^l} - \frac{r^{l+1}}{r'^{l+1}} \right) + \theta(r - r')\frac{\partial}{\partial r}\frac{r'^l}{r^l} + \theta(r' - r)\frac{\partial}{\partial r}\frac{r^{l+1}}{r'^{l+1}} \\
&= \theta(r - r')\frac{-lr'^l}{r^{l+1}} + \theta(r' - r)\frac{(l+1)r^l}{r'^{l+1}} \\
\frac{\partial^2}{\partial r^2} r \frac{r_<^l}{r_>^{l+1}} &= \delta(r - r')\left(\frac{-lr'^l}{r^{l+1}} - \frac{(l+1)r^l}{r'^{l+1}} \right) + \frac{l(l+1)}{r}\frac{r_<^l}{r_>^{l+1}} \\
\frac{1}{r}\frac{\partial^2}{\partial r^2} r \frac{r_<^l}{r_>^{l+1}} &= -\frac{2l+1}{r^2}\delta(r - r') + \frac{l(l+1)}{r^2}\frac{r_<^l}{r_>^{l+1}}
\end{aligned}
$$

from which we infer

$$
\begin{aligned}
\frac{1}{\sin\theta}\delta(\theta - \theta')\delta(\varphi - \varphi') &= \delta(\cos\theta - \cos\theta')\delta(\varphi - \varphi') \\
&= \frac{1}{4\pi}\sum_{l=0}^{\infty}(2l+1)\, P_l(\cos\gamma) = \sum_{lm} Y_{lm}(\theta, \varphi)Y_{lm}^*(\theta', \varphi') \quad (115)
\end{aligned}
$$

which expresses the completeness property of spherical harmonics.

Another example is the potential due to a ring of charge with radius a, parallel to the xy-plane and centered on the z-axis at $z = b$. The distance of the ring from the origin of coordinates is $c = \sqrt{a^2 + b^2}$. The potential on the z axis is

$$
\begin{aligned}
\phi(r, \theta = 0) &= \frac{q}{4\pi\epsilon_0(a^2 + (r - b)^2)^{1/2}} = \frac{q}{4\pi\epsilon_0(r^2 + c^2 - 2rc\cos\alpha)^{1/2}} \\
&= \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha) \qquad (116)
\end{aligned}
$$

where $\cos\alpha = b/c$ and $r_<, r_>$ are the smaller (larger) of r, c. To find the potential at any angle we just insert $P_l(\cos\theta)$:

$$
\phi(r, \theta) = \frac{q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}} P_l(\cos\alpha)P_l(\cos\theta) \qquad (117)
$$

3.5 Green function between two concentric spheres

We can adapt the expansion of the Coulomb potential in Legendre polynomials to construct the Green function for the region bounded by two concentric spheres of radii $a < b$. We just have to replace the function $r^l_</r^{l+1}_>$ by a solution of the radial Laplace equation that vanishes at $r = a, b$ and $r' = a, b$. This will be the case if it vanishes at $r_< = a$ and at $r_> = b$. The solution that does this is

$$
C\left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right) = C\left(\frac{r^l_<}{r^{l+1}_>} + \frac{a^{2l+1}}{b^{2l+1}}\frac{r^l_>}{r^{l+1}_<} - \frac{(rr')^l}{b^{2l+1}} - \frac{a^{2l+1}}{(rr')^{l+1}}\right)
$$
$$
= C\left(\left[1 - \frac{a^{2l+1}}{b^{2l+1}}\right]\frac{r^l_<}{r^{l+1}_>} + \frac{a^{2l+1}}{b^{2l+1}}\left[\frac{r^l_>}{r^{l+1}_<} + \frac{r^l_<}{r^{l+1}_>}\right] - \frac{(rr')^l}{b^{2l+1}} - \frac{a^{2l+1}}{(rr')^{l+1}}\right)
$$
$$
= C\left(\left[1 - \frac{a^{2l+1}}{b^{2l+1}}\right]\frac{r^l_<}{r^{l+1}_>} + \frac{a^{2l+1}}{b^{2l+1}}\left[\frac{r^l}{r'^{l+1}} + \frac{r'^l}{r^{l+1}}\right] - \frac{(rr')^l}{b^{2l+1}} - \frac{a^{2l+1}}{(rr')^{l+1}}\right) \tag{118}
$$

The constant C is determined by requiring that the discontinuity across $r = r'$ be that imposed by the delta function on the right side of the Green function equation. Inspection of the last line of the previous equation shows that $C = (1 - a^{2l+1}/b^{2l+1})^{-1}$, so we have

$$
G_D(\boldsymbol{r}, \boldsymbol{r}') = \frac{1}{4\pi}\sum_{l=0}^{\infty}\left(1 - \frac{a^{2l+1}}{b^{2l+1}}\right)^{-1}\left(r^l_< - \frac{a^{2l+1}}{r^{l+1}_<}\right)\left(\frac{1}{r^{l+1}_>} - \frac{r^l_>}{b^{2l+1}}\right)P_l(\cos\gamma) \tag{119}
$$

Where γ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}'$, $\boldsymbol{r}\cdot\boldsymbol{r}' = rr'\cos\gamma$. As special cases, note that $a \to 0$ yields the Green function inside a sphere of radius b; $b \to \infty$ yields the Green function outside a sphere of radius a; and $a \to 0, b \to \infty$ yields the Green function in all of space.

To use the Green function to solve the general Dirichlet boundary problem we need its normal derivative on the two boundaries:

$$
\hat{n}\cdot\nabla G_D\Big|_{r=b} = \frac{\partial G_D}{\partial r}\Big|_{r=b}
$$
$$
= -\frac{1}{4\pi}\sum_{l=0}^{\infty}\left(1 - \frac{a^{2l+1}}{b^{2l+1}}\right)^{-1}\left(r'^l - \frac{a^{2l+1}}{r'^{l+1}}\right)\frac{2l+1}{b^{l+2}}P_l(\cos\gamma) \tag{120}
$$
$$
\hat{n}\cdot\nabla G_D\Big|_{r=a} = -\frac{\partial G_D}{\partial r}\Big|_{r=a}
$$
$$
= -\frac{1}{4\pi}\sum_{l=0}^{\infty}\left(1 - \frac{a^{2l+1}}{b^{2l+1}}\right)^{-1}(2l+1)a^{l-1}\left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}}\right)P_l(\cos\gamma) \tag{121}
$$

The Addition Theorem for Spherical Harmonics Our Green function expansions involve the angle γ, which we would like to express in terms of the individual angles $\theta, \varphi, \theta', \varphi'$ of the two arguments of the Green function. This can be done through the addition theorem:

$$
P_l(\cos\gamma) = \frac{4\pi}{2l+1}\sum_{m=-l}^{+l}Y_{lm}(\theta, \varphi)Y^*_{lm}(\theta', \varphi') \tag{122}
$$

Our knowledge of quantum mechanics helps to see why this is true. We have the interpretation $Y_{lm}(\theta, \varphi) = \langle \theta, \varphi | lm \rangle$, so the right side involves

$$\langle \theta, \varphi | \left(\sum_{m=-l}^{l} |lm\rangle\langle lm| \right) |\theta', \varphi'\rangle \tag{123}$$

The key point is that the projector $\sum_m |lm\rangle\langle lm|$ is a rotational invariant:

$$U(R) \sum_m |lm\rangle\langle lm| U^\dagger(R) = \sum_m |lm\rangle\langle lm|.$$

Choosing the rotation that takes the direction θ', φ' to the z-axis establishes the result.

Plugging the addition theorem into the Green function formulas yields

$$G_D(\boldsymbol{r}, \boldsymbol{r}') = \tag{124}$$
$$\sum_{l=0}^{\infty} \left(1 - \frac{a^{2l+1}}{b^{2l+1}} \right)^{-1} \left(r_<^l - \frac{a^{2l+1}}{r_<^{l+1}} \right) \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}} \right) \frac{1}{2l+1} \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi')$$

$$\hat{n} \cdot \nabla G_D \bigg|_{r=b} = -\sum_{l=0}^{\infty} \left(1 - \frac{a^{2l+1}}{b^{2l+1}} \right)^{-1} \left(r'^l - \frac{a^{2l+1}}{r'^{l+1}} \right) \frac{1}{b^{l+2}} \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \tag{125}$$

$$\hat{n} \cdot \nabla G_D \bigg|_{r=a} = -\sum_{l=0}^{\infty} \left(1 - \frac{a^{2l+1}}{b^{2l+1}} \right)^{-1} a^{l-1} \left(\frac{1}{r'^{l+1}} - \frac{r'^l}{b^{2l+1}} \right) \sum_{m=-l}^{+l} Y_{lm}(\theta, \varphi) Y_{lm}^*(\theta', \varphi') \tag{126}$$

3.6   Conductors with a Conical Singularity

Here we consider a conductor with the shape $\theta = \beta$. If $\beta = \pi/2$ it is the xy-plane, if $\beta < \pi/2$ it is a conical hole, and if $\beta > \pi/2$ it is a conical point. Separating variables and imposing azimuthal symmetry we try a solution $\phi = R(r)\Theta(\theta)$. R and Θ satisfy the same equations as before, but l is not required to be an integer, because the point $\cos\theta = -1$ is excluded from the region of interest. With l not an integer the Legendre function $P_l(x)$ is not a polynomial.

We can nonetheless use the differential equation to develop P_l in a power series about $x = 1$.

$$P_l(x) = \sum_{n=0}^{\infty} a_n \xi^n \tag{127}$$

where we put $\xi = (1 - x)/2$. The Legendre equation in the variable ξ reads

$$\frac{d}{d\xi} \xi(1 - \xi) \frac{d}{d\xi} P_l + l(l + 1)P_l = 0 \tag{128}$$

Plugging in the power series

$$0 = \sum_{n=0}^{\infty} a_n \{(n)[(n) - \xi(n + 1)]\xi^{n-1} + l(l + 1)\xi^n\}$$

$$= \sum_{n=0}^{\infty} \{(n+1)^2 a_{n+1} + [l(l+1) - n(n+1)]a_n\}\xi^n \tag{129}$$

Giving the recursion relation

$$a_{n+1} = \frac{n(n+1) - l(l+1)}{(n+1)^2}a_n \tag{130}$$

Thus the power series is completely determined. We immediately see that it terminates if l is an integer, and we would recover the usual Legendre polynomials. Otherwise $P_l(x)$ is a new analytic function called the Legendre function.

For the grounded conical shaped conductor, we wish to impose that $P_l(\cos\beta) = 0$. This is the quantization condition on l that replaces the previous requirement that it be an integer. The solutions of the radial Laplace equation are still r^l, r^{-l-1}, of which we would pick the first so that ϕ is finite at $r = 0$. We can organize the solutions for l as an ordered sequence: $0 < l_1 < l_2 < l_3 < \cdots$, and write the general solution

$$\phi(r,\theta) = \sum_{k=1}^{\infty} A_k r^{l_k} P_{l_k}(\cos\theta) \tag{131}$$

The precise values of A_k depend on the shape of the conductor away from the conical singularity. Generically, $A_1 \neq 0$ and the behavior near $r = 0$ is just given by the first term

$$\phi \sim A_1 r^{l_1} P_{l_1}(\cos\theta) \tag{132}$$

from which we learn that the fields and surface charge density behave as r^{l_1-1} at small r. Notice that if $l_1 < 1$ they blow up there.

To get a feeling for when $l_1 < 1$, notice that when $\beta = \pi/2$ the boundary surface is just the xy-plane. In this case the eigenvalue condition is just $P_l(0) = 0$ which implies that $l =$ odd, so $l_1 = 1$. Thus $\beta = \pi/2$ divides the case $l_1 > 1$ from the case $l_1 < 1$. Then inspection shows that $\beta < \pi/2$ corresponds to $l_1 > 1$ and $\beta > \pi/2$ to $l_1 < 1$. Thus the fields and surface charge density blow up at conical points: this is the principle behind the lightning rod.

3.7 Cylindrical Coordinates and Bessel functions

Cylindrical coordinate can be taken as z, $\rho = \sqrt{x^2 + y^2}$ and the azimuthal angle φ such that $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$. The Laplacian in these coordinates is obtained

$$\frac{\partial}{\partial\rho} = \cos\varphi\frac{\partial}{\partial x} + \sin\varphi\frac{\partial}{\partial y}, \qquad \frac{1}{\rho}\frac{\partial}{\partial\varphi} = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y}$$

$$\nabla^2 = \left(\cos\varphi\frac{\partial}{\partial\rho} - \sin\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^2 + \left(\sin\varphi\frac{\partial}{\partial\rho} + \cos\varphi\frac{1}{\rho}\frac{\partial}{\partial\varphi}\right)^2 + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} \tag{133}$$

The separation ansatz is $\phi = R(\rho)\Phi(\varphi)Z(z)$. Then

$$-\frac{\partial^2\Phi}{\partial\varphi^2} = m^2\Phi, \qquad -\frac{\partial^2 Z}{\partial z^2} = \gamma Z$$

$$-\frac{\partial^2 R}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial R}{\partial\rho} + \frac{m^2}{\rho^2}R + \gamma R = -\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\sqrt{\rho}R + \left(\gamma + \frac{4m^2-1}{4\rho^2}\right)R = 0 \qquad (134)$$

The solutions for Φ and Z are trig or exponential functions, depending on the sign of γ. If the full range of $0 \le \varphi \le 2\pi$, is in the region of interest, m must be a real integer. But γ can have either sign. We found for Cartesian and Spherical coordinates that the separation functions were elementary functions or polynomials of elementary functions (when the full range of angles $0 \le \theta \le \pi$ are included. When l was not an integer, of course, the Legendre functions were not polynomials and were new transcendental functions.

In the case of cylindrical coordinates the radial function R, will involve new transcendental functions, the Bessel functions, even when the full range of φ is included (i.e., when m is an integer). The radial equation is the Bessel equation, and it has qualitatively different solutions depending on the sign of γ. to see the difference, let's examine the equation at large ρ, when the m^2/ρ^2 term can be neglected. Then we have

$$-\frac{\partial^2}{\partial\rho^2}\sqrt{\rho}R \approx -\gamma\sqrt{\rho}R \qquad (135)$$

If $\gamma = -k^2$ is negative, we see that

$$R(\rho) \sim A\frac{\sin(k\rho+\delta)}{\sqrt{\rho}}, \qquad Z(z) = Be^{kz} + Ce^{-kz} \qquad (136)$$

whereas if $\gamma = \kappa^2$ is positive the asymptotic behavior is

$$R(\rho) \sim A\frac{e^{\kappa\rho}}{\sqrt{\rho}} + B\frac{e^{-\kappa\rho}}{\sqrt{\rho}}, \qquad Z(z) = C\cos\kappa z + D\sin\kappa z \qquad (137)$$

In the first case, the R has the character of a mildly damped sinusoid and in the second case R has exponential growth or decay. There was a similar distinction in Cartesian coordinates. This is paired with the opposite qualitative behavior in Z. In the first case the Bessel functions are denoted J_m, N_m and in the second case I_m, K_m. J, I have finite behavior at $\rho = 0$ and N, K blow up at $\rho = 0$. Just as in the case of Cartesian coordinates, two of the separation functions can be chosen to have oscillatory behavior but the third would necessarily have exponential behavior. In the cylindrical case Φ is necessarily oscillatory, so the exponential one has to be either R or Z. Notice that the sign of γ effectively changes when $\rho \to i\rho$, so we see that I, K should be expressible in terms of J, N with imaginary argument.

3.8 Mathematical Properties of Bessel Functions

To learn more about the Bessel functions, we solve the Bessel equation with power series, first with $\gamma = -k^2$, putting $R(\rho) = f(k\rho)$.

$$f(x) = \sum_{n=0}^{\infty} a_n x^{\nu+n}, \qquad f'' + \frac{f'}{x} - \left(\frac{m^2}{x^2} - 1\right) f = 0$$

$$0 = \sum_{n=0}^{\infty} a_n([(\nu+n)(\nu+n-1) + (\nu+n) - m^2]x^{\nu+n-2} + x^{\nu+n})$$

$$= \sum_{n=0}^{\infty} a_n([(\nu+n)(\nu+n) - m^2]x^{\nu+n-2} + x^{\nu+n}) \tag{138}$$

With $a_0 \neq 0$, the most singular power is $\nu - 2$ in the term with $n = 0$ and its coefficient must vanish: which determines $\nu = \pm m$. Then the coefficient of $x^{\nu-1}$ vanishes only if $a_1 = 0$. Then the diff eq implies the recursion formula

$$a_{n+2}[(\nu+n+2)^2 - m^2] = a_{n+2}(n+2)(n+2+2\nu) = -a_n,$$

$$a_{n+2} = \frac{-1}{(n+2)(n+2+2\nu)} a_n \tag{139}$$

Only the even $n = 2k$ are nonvanishing

$$a_{2k} = \frac{-1}{4k(k+\nu)} a_{2(k-1)} = \frac{(-)^k \nu!}{4^k (k)!(k+\nu)!} a_0 = \frac{(-)^k \Gamma(1+\nu)}{2^{2k} k! \Gamma(k+1+\nu)} a_0$$

$$J_{\pm m}(x) \equiv \sum_{k=0}^{\infty} \frac{(-)^k}{k! \Gamma(k+1\pm m)} \left(\frac{x}{2}\right)^{2k\pm m} \tag{140}$$

Here we have introduced Euler's gamma function $\Gamma(z)$, which has the property that $z\Gamma(z) = \Gamma(z+1)$ which makes it an extension of the factorial function from positive integers to arbitrary complex numbers. When $\mathrm{Re}\, z > 0$, Γ has the integral representation

$$\Gamma(z) = \int_0^{\infty} dt\, t^{z-1} e^{-t} \tag{141}$$

$\Gamma(z)$ has a simple pole for $z = -n$, $n = 0, 1, 2, \cdots$, which means that $1/\Gamma(z)$ has a zero at those points. $J_{\pm m}$ are independent only when m is not an integer. When m is a positive integer, $\Gamma(k+1-m) = \infty$ for $k = 0, \cdots, m-1$, so

$$J_{-m}(x) = \sum_{k=m}^{\infty} \frac{(-)^k}{k! \Gamma(k+1-m)} \left(\frac{x}{2}\right)^{2k-m}$$

$$= \sum_{k=0}^{\infty} \frac{(-)^{k+m}}{(k+m)! \Gamma(k+1)} \left(\frac{x}{2}\right)^{2k+m} = (-)^m J_m(x) \tag{142}$$

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When m is not an integer, we can replace the independent J_{-m} with the Neumann function defined as

$$N_m(x) \;=\; \frac{J_m(x)\cos m\pi - J_{-m}(x)}{\sin m\pi}, \qquad m \neq \text{integer} \tag{143}$$

When $m \to$ integer, the numerator and denominator of N_m both go to 0, and the limiting function remains independent of J_m. The leading term as $x \to 0$ is

$$-(x/2)^{-m}\frac{1}{\sin \pi m\, \Gamma(1-m)}.$$

. With the identity

$$\Gamma(z)\Gamma(1-z) \;=\; \frac{\pi}{\sin \pi z}, \qquad N_m(x) \sim -(x/2)^{-m}\frac{\Gamma(m)}{\pi}$$

which contributes even when m is an integer.

Then the two linearly independent solutions of Bessel's equation can be taken to be J_m, N_m. Their behavior at large x is

$$J_m(x) \;\sim\; \sqrt{\frac{2}{\pi}}\frac{\cos(x - m\pi/2 - \pi/4)}{\sqrt{x}}$$

$$N_m(x) \;\sim\; \sqrt{\frac{2}{\pi}}\frac{\sin(x - m\pi/2 - \pi/4)}{\sqrt{x}} \tag{144}$$

Sometimes it is useful to take linear combinations which have $e^{\pm ix}/\sqrt{x}$ asymptotic behavior. Clearly these are

$$H_m^{(1),(2)}(x) \;\equiv\; J_m \pm iN_m \;\sim\; \sqrt{\frac{2}{\pi x}}e^{\pm ix \mp im\pi/2 \mp i\pi/4} \tag{145}$$

and are called the Hankel functions.

When $\gamma = +\kappa^2$, the solutions of Bessel's equation are the above one at imaginary argument. They are conventionally taken to be

$$I_m(x) \;=\; i^{-m}J_m(ix), \qquad K_m(x) \;=\; \frac{\pi i^{m+1}}{2}H_m^{(1)}(ix) \tag{146}$$

so that I_m is finite or 0 at $x = 0$, and K_m decays exponentially as $x \to \infty$. In solving boundary value problems J_m, N_m play a role analogous to $\sin, \cos$ in Cartesian coordinates, whereas I_m, K_m play a role analogous to $\sinh, \cosh$.

Bessel functions of different orders are related by recursion formulas

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x}J_m(x), \qquad J_{m-1}(x) - J_{m+1}(x) = 2\frac{dJ_m(x)}{dx} \tag{147}$$

which are one line consequences of the power series expansions. The same formulas apply to $N_m, H_m^{(1)}, H_m^{(2)}$. When $m = 0$ the first equation is an identity, whereas the second one gives

$J_0' = -J_1$. The recursion relations for I_m, K_m are similar but there are some sign changes because of the powers of $i^{\pm m}$ in their relations to $J_m, H_m^{(1)}$:

$$I_{m-1}(x) - I_{m+1}(x) = \frac{2m}{x} I_m(x), \qquad I_{m-1}(x) + I_{m+1}(x) = 2\frac{dI_m(x)}{dx}$$

$$K_{m-1}(x) - K_{m+1}(x) = -\frac{2m}{x} K_m(x), \qquad K_{m-1}(x) + K_{m+1}(x) = -2\frac{dK_m(x)}{dx} \quad (148)$$

When $m = 0$, these relation reduce to $I_0' = I_1$ and $K_0' = -K_1$.

In Cartesian coordinates, we exploited the fact that $\sin(n\pi x/a)$ was a complete set of functions on the interval $[0, a]$. These functions all have zeroes at $x = 0, a$, which make them suited for solving Dirichlet boundary problems. In the cylindrical case J_m is analogous to sin, but we have a different function depending on the φ dependence. Because J_m oscillates about 0, it will have an infinite number of zeroes $x_{m_1}, x_{m_2}, \cdots$. The analog of $\sin(n\pi x/a)$ is then $J_m(x_{mn}\rho/a)$. We will take for granted that these are a complete set of functions on the interval $[0, a]$. However the proof that they are orthogonal for different n is instructive, paralleling the proof in quantum mechanics that eigenfunctions of hermitian operators with different eigenvalues are orthogonal.

$$\int_0^a \rho d\rho J_m(x_{ml}\rho/a) \left(\frac{m^2}{\rho^2} - \frac{x_{mn}^2}{a^2} \right) J_m(x_{mn}\rho/a) =$$

$$\int_0^a \rho d\rho J_m(x_{ml}\rho/a) \left(\frac{d^2 J_m(x_{mn}\rho/a)}{d\rho^2} + \frac{1}{\rho}\frac{dJ_m}{d\rho} \right) = -\int_0^a \rho d\rho \frac{dJ_m(x_{ml}\rho/a)}{d\rho} \frac{dJ_m(x_{mn}\rho/a)}{d\rho}$$

The right side is symmetric under $l \leftrightarrow n$, so it follows that

$$\left(\frac{x_{ml}^2}{a^2} - \frac{x_{mn}^2}{a^2} \right) \int_0^a \rho d\rho J_m(x_{ml}\rho/a) J_m(x_{mn}\rho/a) = 0 \quad (149)$$

So, if $x_{ml}^2 \neq x_{mn}^2$, orthogonality follows. To evaluate the normalization integral, note that

$$\frac{d}{d\rho}\left(\rho\frac{dJ_m}{d\rho} \right)^2 = 2\rho\left(\frac{dJ_m}{d\rho} \right)^2 + 2\rho^2\frac{dJ_m}{d\rho}\left(-\frac{1}{\rho}\frac{dJ_m}{d\rho} + \left(\frac{m^2}{\rho^2} - \frac{x_{mn}^2}{a^2} \right) J_m \right)$$

$$= m^2\frac{dJ_m^2}{d\rho} - \frac{\rho^2 x_{mn}^2}{a^2}\frac{dJ_m^2}{d\rho} = m^2\frac{dJ_m^2}{d\rho} - \frac{x_{mn}^2}{a^2}\frac{d(\rho^2 J_m^2)}{d\rho} + 2\rho\frac{x_{mn}^2}{a^2}J_m^2$$

$$2\rho\frac{x_{mn}^2}{a^2}J_m^2 = \frac{d}{d\rho}\left[\left(\rho\frac{dJ_m}{d\rho} \right)^2 - \left(m^2 - \frac{x_{mn}^2\rho^2}{a^2} \right) J_m^2 \right] \quad (150)$$

The quantity in brackets is $x_{mn}^2 J_m'^2(x_{mn})$ at $\rho = a$ and vanishes at $\rho = 0$. Thus

$$\int_0^a \rho d\rho J_m(x_{ml}\rho/a) J_m(x_{mn}\rho/a) = \frac{a^2}{2} J_m'^2(x_{mn})\delta_{ln} = \frac{a^2}{2} J_{m+1}^2(x_{mn})\delta_{ln} \quad (151)$$

The Fourier-Bessel series asserts that a function on the interval $[0, a]$ can be expanded

$$f(\rho) = \sum_n A_n J_m(x_{mn}\rho/a) \quad (152)$$

We shall assume this is true without proof.

If $a \to \infty$, the series becomes an integral and we speak of a Fourier-Bessel transform. Then orthogonality is expressed in terms of the dirac delta function. Recall the Cartesian representation of the delta function

$$\delta(x - x') = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} e^{i\lambda(x-x')} = \int_{0}^{\infty} \frac{d\lambda}{\pi} \cos \lambda(x - x') \qquad (153)$$

The normalization is determined by the large λ behavior of the integrand. For orthogonality of the $J_m(k\rho)$ with respect to ρ integration, consider the large ρ behavior of the Bessel functions:

$$\rho J_m(k\rho) J_m(k'\rho) \sim \frac{2}{\pi \sqrt{kk'}} \cos(k\rho - m\pi/2 - \pi/4) \cos(k'\rho - m\pi/2 - \pi/4)$$

$$\sim \frac{1}{\pi \sqrt{kk'}} \left[\cos(k - k')\rho + \cos((k + k')\rho - m\pi - \pi/2) \right] \qquad (154)$$

and we see that the first term has the large ρ behavior appropriate to $\delta(k - k')/k$, whereas the second is appropriate to a delta function contributing for $k + k' = 0$. If we understand that k, k' are positive, we can then conclude that the orthogonality conditions for Bessel functions on an infinite range are

$$\int_{0}^{\infty} \rho d\rho J_m(k\rho) J_m(k'\rho) = \frac{1}{k} \delta(k - k') \qquad (155)$$

Obviously the completeness relation, with repect to k integration is

$$\int_{0}^{\infty} k dk J_m(k\rho) J_m(k\rho') = \frac{1}{\rho} \delta(\rho - \rho') \qquad (156)$$

3.9 Boundary-value problems in cylindrical coordinates

The Dirichlet problem for a cylindrical bounding surface at $\rho = a$ with end faces at $z = 0, L$ is to solve Laplace's equation for $\phi(\rho, \varphi, z)$ in the interior with arbitrary values on the boundary: $\phi(a, \varphi, z) = V(\varphi, z)$, $\phi(\rho, \varphi, 0) = V_1(\rho, \varphi)$ and $\phi(\rho, \varphi, L) = V_2(\rho, \varphi)$. The general solution is a superposition of the three special cases where two of the V, V_1, V_2 are zero. We discuss the case $V = V_1 = 0$, leaving the other cases for exercises.

We assume a separated solution of the form

$$Z(z) = \sinh(kz), \qquad \Phi = e^{im\varphi}, \qquad R = J_{|m|}(k\rho) \qquad (157)$$

where we have not included a term with $N_{|m|}$ in R because it would be singular as $\rho \to 0$. The choice of sinh makes Z vanish at $z = 0$. To make R vanish at $\rho = a$, k is limited to the values $k_{mn} = x_{|m|n}/a$. Then we can expand

$$\phi = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_{|m|}(x_{|m|n}\rho/a) \sinh(x_{|m|n}z/a) e^{im\varphi} \qquad (158)$$

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The A_{mn} are determined by the last boundary condition

$$V(\rho, \varphi) = \sum_{m=-\infty}^{\infty} \sum_{n=1}^{\infty} A_{mn} J_{|m|}(x_{|m|n}\rho/a) \sinh(x_{|m|n}L/a) e^{im\varphi} \tag{159}$$

Then

$$A_{mn} \frac{a^2}{2} J_{|m|}'^2(x_{|m|n}) \sinh(x_{|m|n}L/a) = \int_0^{2\pi} \frac{d\varphi}{2\pi} e^{-im\varphi} \int_0^a \rho d\rho J_{|m|}(x_{|m|n}\rho/a) V(\rho, \varphi) \tag{160}$$

In the special case that V is a constant, this becomes

$$A_{mn} \frac{a^2}{2} J_{|m|}'^2(x_{|m|n}) \sinh(x_{|m|n}L/a) \ \to \ \delta_{m0} V \int_0^a \rho d\rho J_0(x_{0n}\rho/a) = \delta_{m0} V \frac{a^2}{x_{0n}^2} \int_0^{x_{0n}} x dx J_0(x)$$

From the sum of the two recursion relations for $m = 1$ we learn that $x J_0 = J_1 + x J_1' = (x J_1)'$ so the integral on the right is just $x_{0n} J_1(x_{0n}) = -x_{0n} J_0'(x_{0n})$. Then

$$\phi \ \to \ \sum_{n=1}^{\infty} A_{0n} J_0(x_{0n}\rho/a) \sinh(x_{0n}z/a) = -2V \sum_{n=1}^{\infty} \frac{J_0(x_{0n}\rho/a)}{x_{0n} J_0'(x_{0n})} \frac{\sinh(x_{0n}z/a)}{\sinh(x_{0n}L/a)} \tag{161}$$

3.10   Green functions in cylindrical coordinates

We begin by expanding the empty space Green function $1/4\pi|\boldsymbol{x} - \boldsymbol{x}'|$ in a complete set of functions in two of the coordinates. We have two choices: $e^{im\varphi}, e^{ikz}$, or $e^{im\varphi}, J_{|m|}(k\rho)$. In the first case we write

$$\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk \, g_m(k, \rho, \rho') e^{im(\varphi-\varphi')} e^{ik(z-z')} \tag{162}$$

Writing the delta function in cylindrical coordinates

$$\delta(\boldsymbol{x} - \boldsymbol{x}') = \frac{1}{\rho} \delta(\rho - \rho') \delta(\varphi - \varphi') \delta(z - z') \tag{163}$$

we see that g_m satisfies

$$-\frac{\partial^2 g_m}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial g_m}{\partial \rho} + \frac{m^2}{\rho^2} g_m + k^2 g_m = \frac{1}{\rho} \delta(\rho - \rho') \tag{164}$$

For $\rho \neq \rho'$, this is the Bessel equation for the exponentially behaved case, with the solutions $I_{|m|}(k\rho)$, $K_{|m|}(k\rho)$. Since I is well behaved at $\rho = 0$ and K is well behaved at $\rho = \infty$, we must have $g_m = A I_{|m|}(|k|\rho_<) K_{|m|}(|k|\rho_>)$, with A determined by the discontinuity condition

$$-\frac{1}{\rho'} = \left. \frac{\partial g_m}{\partial \rho} \right|_{\rho=\rho'+} - \left. \frac{\partial g_m}{\partial \rho} \right|_{\rho=\rho'-} = |k| A (I_{|m|} K_{|m|}' - K_{|m|} I_{|m|}') \Big|_{\rho=\rho'} = -\frac{A}{\rho'} \tag{165}$$

The last equality uses the fact that the Bessel equation implies that

$$\frac{d}{dx}\left[xI_m(x)K'_m(x) - xI'_m(x)K_m(x)\right] = 0 \tag{166}$$

$$\text{and} \quad xI_m(x)K'_m(x) - xI'_m(x)K_m(x) \sim -1, \qquad \text{as } x \to \infty \tag{167}$$

The quantity $W(A,B) = AB' - BA'$ is called the Wronskian of two solutions of a second order differential equation. What we have just learned is that $W(I_m, K_m) = -1/x$. The nonvanishing of the Wronskian of two solutions is a criterion that the two solutions are independent.

We conclude that $A = 1$, and the sought expansion reads

$$\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk I_{|m|}(|k|\rho_<) K_{|m|}(|k|\rho_>) e^{im(\varphi-\varphi')} e^{ik(z-z')} \tag{168}$$

Had we chosen $e^{im\varphi}, J_{|m|}(k\rho)$ as our complete set of functions, we would have found the expansion

$$\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{4\pi} \sum_{m=-\infty}^{\infty} \int_{0}^{\infty} dk J_{|m|}(k\rho) J_{|m|}(k\rho') e^{im(\varphi-\varphi')} e^{-k|z-z'|} \tag{169}$$

In this case, when the Laplacian is applied to the expansion, the delta function emerges from

$$-\frac{\partial^2}{\partial z^2} e^{-k|z-z'|} = \frac{\partial}{\partial z} k\epsilon(z-z') e^{-k|z-z'|} = 2k\delta(z-z') - k^2 e^{-k|z-z'|} \tag{170}$$

The factor of k makes the integral over k produce $\delta(\rho - \rho')/\rho$, and then the sum over m produces $\delta(\varphi - \varphi')$.

Starting with these expansions, it is easy to adjust the individual factors to accommodate cylindrical boundary conditions. For example, to impose Dirichlet conditions at $\rho = a$ in the first expansion, one would simply substitute $K_{|m|}(|k|\rho_>) \to K_{|m|}(|k|\rho_>) - K_{|m|}(|k|a)I_{|m|}(|k|\rho_>)/I_{|m|}(|k|a)$. For the second form of the expansion, in contrast, one would change the integral over k to a sum over n, restricting to $k = x_{|m|n}/a$ and adjusting the normalization so that the proper delta function emerges.

3.11 A little more wisdom about Green functions

We can think a little more systematically about Green functions by borrowing some terminology from quantum mechanics. The Green function equation can be written schematically as

$$DG(\boldsymbol{x}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{171}$$

where D is a linear differential operator. In quantum mechanics we think of D as a matrix $D_{\boldsymbol{x},\boldsymbol{x}'}$ and write the equation as

$$\int d^3y D_{\boldsymbol{x},\boldsymbol{y}} G(\boldsymbol{y}, \boldsymbol{x}') = \delta(\boldsymbol{x} - \boldsymbol{x}') $$

$$D_{\boldsymbol{x},\boldsymbol{y}} \equiv D\delta(\boldsymbol{x} - \boldsymbol{y}) \tag{172}$$

In this language the identity operator is $I_{\boldsymbol{x},\boldsymbol{y}} = \delta(\boldsymbol{x}-\boldsymbol{y})$. Then in abstract matrix notation the Green function equation is

$$DG = I, \qquad G = D^{-1} \tag{173}$$

which is to say that the matrix G is simply the inverse of the matrix D. A systematic way to construct a matrix inverse is to change basis to one in which D is diagonal. Then the inverse is just the diagonal matrix whose entries are the reciprocals of the entries of D. Of course we can't do this if D has some zero entries: we can only define D^{-1} on the subspace with non-zero eigenvalues of D. Going back to the differential equation, the dangerous zero eigenvalues of D are just the solutions of the homogeneous equation $D\psi = 0$, which in the case of electrostatics is Laplace's equation. In our work with Green functions, we eliminate the zero eigenvalue problem by properly specifying boundary conditions.

Let us apply this philosophy to the construction of the full space Green function for the Laplace equation. The eigenfunctions of $-\nabla^2$ are just the plane waves: $-\nabla^2 e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \boldsymbol{k}^2 e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$. So in $\boldsymbol{k}$ basis the green function is just $\delta(\boldsymbol{k}-\boldsymbol{k}')/\boldsymbol{k}^2$. Transforming back to coordinate basis is just the Fourier transform:

$$\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{(2\pi)^3}\int d^3k \frac{1}{\boldsymbol{k}^2} e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')} \tag{174}$$

This integral is well behaved near $k = 0$. This formula involves three integrals, in contrast to the expansions we have used thus far in the course which only involve two integrals. We may do the k_3 integration by contours. There are two poles at $k_3 = \pm i\sqrt{k_1^2 + k_2^2}$. The exponential factor $e^{ik_3(z-z')}$ is exponentially damped in the upper half k_3-plane when $z-z' > 0$. However, when $z-z' < 0$ it is exponentially damped in the lower half plane. In the first case we close the contour in the upper half plane picking up the pole at $k_3 = +i\sqrt{k_1^2 + k_2^2}$, and in the second case we close in the lower half plane picking up the pole at $k_3 = -i\sqrt{k_1^2 + k_2^2}$. The result is $2\pi e^{-|z-z'|\sqrt{k_1^2+k_2^2}}/2\sqrt{k_1^2 + k_2^2}$, and the formula can be written

$$\begin{aligned}
\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} &= \frac{1}{(2\pi)^2}\int \frac{d^2k}{2\sqrt{k_1^2 + k_2^2}} e^{i\boldsymbol{k}_\perp\cdot(\boldsymbol{x}-\boldsymbol{x}')_\perp} e^{-|z-z'|\sqrt{k_1^2+k_2^2}} \\
&= \frac{1}{(2\pi)^2}\int \frac{dk\,d\varphi}{2} e^{ik\sqrt{(x-x')^2+(y-y')^2}\cos\varphi} e^{-k|z-z'|} \\
&= \frac{1}{4\pi}\int_0^\infty dk\, J_0(k\sqrt{(x-x')^2+(y-y')^2}) e^{-k|z-z'|} \tag{175}
\end{aligned}$$

In this example we see the typical feature we have encountered in expansions of Green functions. The coordinate singled out for special treatment enters with a discontinuity of behavior, in contrast to the other coordinates.

3.12   Electrostatics in 2 Dimensions

In 2 dimensions we can make a powerful connection between electrostatics and the mathematics of analytic functions. First we can always think of the xy-plane as the complex

$z = x + iy$ plane. Next we recall that a function $f(z)$ is analytic in a region of the z-plane, if it has a well defined derivative, no matter in what direction it is taken.

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{176}$$

This condition leads to the Cauchy-Riemann equations. Resolve $f(x + iy) = f_1(x, y) + if_2(x, y)$ into its real (f_1) and imaginary (f_2) parts. Then we find

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_2}{\partial y}, \qquad \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial x} \tag{177}$$

Differentiating the first equation w.r.t. x and the second w.r.t y and adding the two equations shows that f_1 satisfy Laplace's equation, and reversing the roles of x, y shows similarly that f_2 satisfies the Laplace equation. Thus one can generate solutions of Laplace's equation in two dimensions by simply writing down analytic functions and taking the real or imaginary part.

Cauchy's Theorem If $f(z)$ is an analytic function in a region of the complex plane then, for any closed curve in this region $\oint dz f(z) = 0$. To prove this we write out

$$\oint dz f(z) = \oint (dx + idy)(f_1(x, y) + if_2(x, y)) = \oint (dx f_1 - dy f_2) + i\oint (dx f_2 + dy f_1)$$

$$= \oint d\boldsymbol{l} \cdot \boldsymbol{V}_1 + i\oint d\boldsymbol{l} \cdot \boldsymbol{V}_2 = \int dx dy (\nabla \times V_1)_z + i\int dx dy (\nabla \times V_2)_z \tag{178}$$

where $\boldsymbol{V}_1 = (f_1, -f_2)$, $\boldsymbol{V}_2 = (f_2, f_1)$, and the last equality is just Stoke's theorem. Then calculating

$$(\nabla \times V_1)_z = \frac{\partial f_1}{\partial y} - \frac{\partial(-f_2)}{\partial x} = 0, \qquad (\nabla \times V_2)_z = \frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial x} = 0 \tag{179}$$

by the Cauchy-Riemann equations.

Suppose for the analytic function $f(x + iy) = f_1(x, y) + if_2(x, y)$ we choose $\phi = f_1$ as our solution of an electrostatics problem. Then the components of the electric field are

$$E_x = -\frac{\partial f_1}{\partial x} = -\frac{\partial f_2}{\partial y}, \qquad E_y = -\frac{\partial f_1}{\partial y} = +\frac{\partial f_2}{\partial x} \tag{180}$$

so we can recover the electric fields from either f_1 or f_2. If we introduce the alternating symbol $\epsilon_{kl} = -\epsilon_{lk}$ with $\epsilon_{12} = 1$, these equations can be written compactly as

$$E_k = -\frac{\partial f_1}{\partial x_k} = -\epsilon_{kl}\frac{\partial f_2}{\partial x_l} \tag{181}$$

with repeated indices summed.

The trivial analytic function $f(z) = z = x + iy$ has a real part vanishing at $x = 0$, so it obeys Dirichlet conditions on the y-axis. If we consider an analytic mapping such that

$z = f(z')$, the y-axis is mapped to the curve $z' = f^{-1}(iy)$, and the real part of $f(z')$ will be an electrostatic potential with Dirichlet conditions on this curve.

More generally, if we have a Green function $G(x, y; x', y')$ appropriate to one geometry, an analytic change of coordinates $z = x + iy = f(u + iv) = f_1(u, v) + if_2(u, v)$ gives a Green function for the transformed geometry. To see this we need to show that the Green function equation

$$-\nabla^2 G(z, z') = \delta(x - x')\delta(y - y') \tag{182}$$

is invariant in form under the coordinate change.

$$
\frac{\partial}{\partial u} = \frac{\partial f_1}{\partial u}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial u}\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial v} = \frac{\partial f_1}{\partial v}\frac{\partial}{\partial x} + \frac{\partial f_2}{\partial v}\frac{\partial}{\partial y}
$$

$$
\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = \nabla_u^2 f_1 \frac{\partial}{\partial x} + \nabla_u^2 f_2 \frac{\partial}{\partial y} + \left(\frac{\partial f_1}{\partial u}^2 + \frac{\partial f_1}{\partial v}^2\right)\frac{\partial^2}{\partial x^2} + \left(\frac{\partial f_2}{\partial u}^2 + \frac{\partial f_2}{\partial v}^2\right)\frac{\partial^2}{\partial x^2}
$$

$$
+ 2\left(\frac{\partial f_1}{\partial u}\frac{\partial f_1}{\partial v} + \frac{\partial f_2}{\partial u}\frac{\partial f_2}{\partial v}\right)\frac{\partial^2}{\partial x \partial y}
$$

$$
\nabla_u^2 = \left(\frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v} - \frac{\partial f_2}{\partial u}\frac{\partial f_1}{\partial v}\right)\nabla_x^2 = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\nabla_x^2 \tag{183}
$$

and from

$$
1 = \int d^2 u\, \delta(\boldsymbol{u} - \boldsymbol{u}') = \int d^2 x \left|\frac{\partial(u, v)}{\partial(x, y)}\right|\delta(\boldsymbol{u} - \boldsymbol{u}') = \int d^2 x\, \delta(\boldsymbol{x} - \boldsymbol{x}') \tag{184}
$$

we see that

$$
\delta(\boldsymbol{u} - \boldsymbol{u}') = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{185}
$$

which establishes the invariance of the Green function equation.

To illustrate the power of this conformal mapping technique, we solve a problem relevant to string theory. Consider the geometry
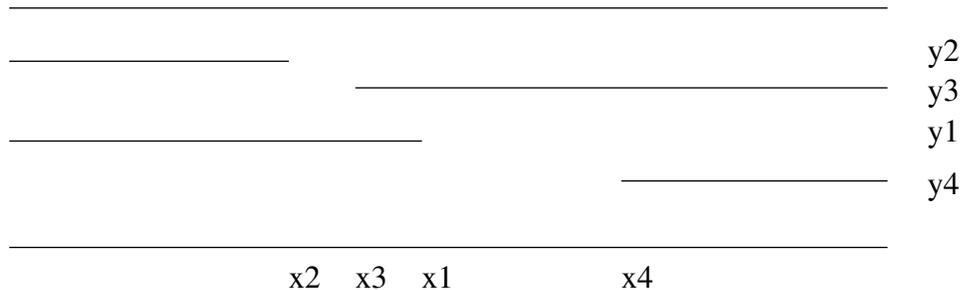
and we seek a Green function that vanishes on all the horizontal lines. The solution of this problem enables the calculation of the scattering of (in this case three incoming and three

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outgoing) strings. We can easily solve this problem by mapping $z = x + iy = f(w)$ the upper half w plane to this figure, with the real w axis mapping to the whole boundary of the figure.

Let us first consider the simple map

$$z = x + iy = \frac{L}{\pi} \ln w = \frac{L}{\pi} \ln(u + iv) \tag{186}$$

The positive real w axis, $0 < u$, $v = 0$ maps to the whole real z axis: $-\infty < x, \infty$, $y = 0$. To see what happens when w passes the origin, take a little detour into the upper half w-plane at $u = \epsilon$: $w = \epsilon e^{i\theta}$ and follow what happens when θ goes from 0 to π, when w goes to $-\epsilon$:

$$\frac{L}{\pi} \ln w = \frac{L}{\pi} \ln \epsilon + \frac{Li\theta}{\pi} \to \frac{L}{\pi} \ln \epsilon + iL = \frac{L}{\pi} \ln(-w) + iL \tag{187}$$

and we can now take w to $-\infty$ on the negative real axis while z follows the line $x + iL$, $-\infty < x < \infty$. Thus we have mapped the upper half w-plane to the strip $-\infty < x < \infty$, $0 < y < L$.

The map we seek is a generalization to several ln terms.

$$z = x + iy = \sum_{i=1}^{N} \frac{L_i}{\pi} \ln(w - u_i) \tag{188}$$

where we order the u's: $u_1 > u_2 > \cdots > u_N$. We now trace the image of the real w-axis, starting with $u_1 < u < \infty$. This range maps onto the whole real z axis. Performing a little detour around $u = u_1$ into the upper half w plane shows that the interval $u_2 < u < u_1$ maps onto a line at $y = \mathrm{Im} z = iL_1$, starting at $x = -\infty$ and x at first increases. Assuming $L_2 > 0$, as u approaches u_2 x returns to $-\infty$. This implies there is a value of u in the interval $u_2 < u < u_1$ when x stops increasing and starts to decrease. That point is where $dx/du = 0$. As u approaches u_2 a little detour around u_2 adds L_2 to the imaginary part of z, and x increases from $-\infty$. If $L_3 > 0$ there is another value of u where $dx/du = 0$, and x returns to $-\infty$, and a detour around u_3 adds L_3 to the imaginary part of z. This process repeats until we arrive at an $L_k < 0$. Then a detour around u_k *subtracts* L_k from the imaginary part of z. The boundary of the figure will be completely traced if the rest of the L_i's are negative and $\sum_{i=1}^{N} L_i > 0$.

To find the Dirichlet Green function for the figure, we start with the Green function on the upper half w plane, which is easily obtained from the method of images:

$$G_w(w, w') = -\frac{1}{2\pi}(\ln|w - w'| - \ln|w - w'^*|) \tag{189}$$

To find the Green function for the figure we first invert $z(w)$ to find $w(z)$ and then plug this into the formula for G:

$$G_z(z, z') = G_w(w(z), w(z')) \tag{190}$$

The simplest example is the strip (i.e. the case $N = 1$):

$$z = \frac{L}{\pi} \ln w, \qquad w = e^{\pi z / L}$$

$$G_z(z, z') = -\frac{1}{2\pi}(\ln|e^{\pi z/L} - e^{\pi z'/L}| - \ln|e^{\pi z/L} - e^{\pi z'^*/L}|) \tag{191}$$

Let's write out the arguments of the logarithms more explicitly by squaring them:

$$|e^{\pi z/L} - e^{\pi z'/L}|^2 = e^{2\pi x/L} + e^{2\pi x'/L} - 2e^{2\pi(x+x')/L}\cos\frac{\pi(y-y')}{L}$$

$$|e^{\pi z/L} - e^{\pi z'^*/L}|^2 = e^{2\pi x/L} + e^{2\pi x'/L} - 2e^{2\pi(x+x')/L}\cos\frac{\pi(y+y')}{L} \tag{192}$$

We can also write an expansion valid for $x - x' \to +\infty$.

$$\ln|e^{\pi z/L} - e^{\pi z'/L}|^2 = \frac{2\pi x}{L} + \ln(1 - e^{\pi(z'-z)/L}) + \ln(1 - e^{\pi(z'^*-z^*)/L})$$

$$= \frac{2\pi x}{L} - \sum_{n=1}^{\infty}\frac{1}{n}(e^{-n\pi(z-z')/L} + e^{-n\pi(z^*-z'^*)/L})$$

$$= \frac{2\pi x}{L} - \sum_{n=1}^{\infty}\frac{2}{n}e^{-n\pi(x-x')/L}\cos\frac{n\pi(y-y')}{L}$$

$$\ln|e^{\pi z/L} - e^{\pi z'^*/L}|^2 = \frac{2\pi x}{L} - \sum_{n=1}^{\infty}\frac{2}{n}e^{-n\pi(x-x')/L}\cos\frac{n\pi(y+y')}{L}$$

$$G(z, z') = \frac{1}{4\pi}\sum_{n=1}^{\infty}\frac{2}{n}e^{-n\pi(x-x')/L}\left(\cos\frac{n\pi(y-y')}{L} - \cos\frac{n\pi(y+y')}{L}\right)$$

$$= \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n\pi|x-x'|/L}\sin\frac{n\pi y}{L}\sin\frac{n\pi y'}{L} \tag{193}$$

where the absolute value signs make the expansion valid for both $x > x'$ and $x < x'$. We recover a familiar result. The case of $N = 2$ is the subject of a homework problem, where inverting $z(w)$ to find $w(z)$ is a little more involved.

4 The Multipole expansion and Dielectric Materials

4.1 Electric Multipoles

Electromagnetic fields provide one of the most important tools for experimentally probing the structure of matter, essentially by shining light on the system. By examining the behavior of systems in EM fields we can infer facts about the way charge is distributed in microscopic systems such as atoms and molecules. Since we are examining tiny systems from afar (the lab is huge compared to a molecule!), our experiments are usually not sensitive to the extremely fine details of the charge distribution. In the electrostatic realm, the multipole expansion takes advantage of this situation.

Suppose we have a charge density $\rho(\boldsymbol{r}')$ which is non-vanishing in a limited spatial region of size R. If the observation point $\boldsymbol{r}$ lies far outside of this region the electrostatic potential

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{194}$$

can be expanded in a power series in $1/r$.

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = (r^2 + r'^2 - 2\boldsymbol{r}\cdot\boldsymbol{r}')^{-1/2} = \frac{1}{r}\left(1 + \frac{r'^2}{r^2} - 2\hat{r}\cdot\frac{\boldsymbol{r}'}{r}\right)^{-1/2} = \frac{1}{r} + \hat{r}\cdot\frac{\boldsymbol{r}'}{r^2} + \cdots$$

$$\phi(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{r_{i_1}\cdots r_{i_n} Q^{i_1\cdots i_n}}{r^{2n+1}} \tag{195}$$

Here $Q^{i_1\cdots i_n}$ is an integral of $\rho(\boldsymbol{r}')$ times a homogeneous polynomial of order n in the components of $\boldsymbol{r}'$. Since $\boldsymbol{r}$ is outside the charge distribution, each term should satisfy Laplace's equation:

$$
\begin{aligned}
\nabla^2 \frac{r_{i_1}\cdots r_{i_n} Q^{i_1\cdots i_n}}{r^{2n+1}} &= \nabla_k\left[-(2n+1)r_k \frac{r_{i_1}\cdots r_{i_n} Q^{i_1\cdots i_n}}{r^{2n+3}} + n\frac{r_{i_2}\cdots r_{i_n} Q^{ki_2\cdots i_n}}{r^{2n+1}}\right]\\
&= \left[-3(2n+1)\frac{r_{i_1}\cdots r_{i_n} Q^{i_1\cdots i_n}}{r^{2n+3}} + (2n+1)(2n+3)\frac{r_{i_1}\cdots r_{i_n} Q^{i_1\cdots i_n}}{r^{2n+3}}\right.\\
&\qquad\left. -2n(2n+1)\frac{r_{i_1}r_{i_2}\cdots r_{i_n} Q^{i_1 i_2\cdots i_n}}{r^{2n+3}} + n(n-1)\frac{r_{i_3}\cdots r_{i_n} Q^{kki_3\cdots i_n}}{r^{2n+1}}\right]\\
&= n(n-1)\frac{r_{i_3}\cdots r_{i_n} Q^{kki_3\cdots i_n}}{r^{2n+1}} = 0 \tag{196}
\end{aligned}
$$

which implies $Q^{kki_3\cdots i_n} = 0$. The tensor $Q^{i_1\cdots i_n}$ which is completely symmetric in its indices and is traceless in all pairs of indices is the 2^n-multipole moment of the charge distribution.

A very efficient way to present the multipole expansion uses spherical coordinates, and the expansion of the Coulomb potential:

$$\frac{1}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \tag{197}$$

If $\boldsymbol{r}$ is completely outside the charge distribution, then plugging this expansion into the formula for ϕ, we can put $r_< = r'$ and $r_> = r$ to obtain

$$
\begin{aligned}
\phi(\boldsymbol{r}) &= \frac{1}{\epsilon_0} \int d^3 r' \rho(\boldsymbol{r}') \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}(\theta,\varphi) Y_{lm}^*(\theta',\varphi') \\
&= \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \int d^3 r' \rho(\boldsymbol{r}') r'^l Y_{lm}^*(\theta',\varphi') \\
&\equiv \frac{1}{\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{q_{lm}}{2l+1} \frac{Y_{lm}(\theta,\varphi)}{r^{l+1}} \\
q_{lm} &= \int d^3 r' \rho(\boldsymbol{r}') r'^l Y_{lm}^*(\theta',\varphi') \quad (198)
\end{aligned}
$$

The $2l+1$ components q_{lm} represent the 2^l pole moment. This matches the number of independent components of the tensor $Q^{i_1 \cdots i_l}$. To see this we first establish that a completely symmetric tensor of rank l has $(l+1)(l+2)/2$ independent components (exercise). Then the tracelessness condition places $(l-1)l/2$ conditions on these, and

$$
\frac{1}{2}[(l+1)(l+2) - l(l-1)] = \frac{1}{2}(4l+2) = 2l+1. \quad (199)
$$

We can construct the Cartesian tensor

$$
Q^{i_1 \cdots i_l} = A_l \int d^3 r \, r^{i_1} \cdots r^{i_l} \rho(\boldsymbol{r}) - [\text{Traces}] \quad (200)
$$

where $-[\text{Traces}]$ signifies terms with at least one factor δ_{i_k, i_n} which are determined to make Q traceless in all pairs of indices. We can determine A_l by comparing the q_{ll} contribution to the potential to the contribution of $Q^{-- \cdots -}$ where $V^\pm$ means $(V^x \pm iV^y)/\sqrt{2}$. With this understanding $\boldsymbol{V} \cdot \boldsymbol{W} = V^+ W^- + V^- W^+ + V^z W^z$. In particular for the position vector $\boldsymbol{r}$ we have $r^\pm = (x \pm iy)/\sqrt{2}$. Now

$$
\begin{aligned}
r^l Y_{ll} &= \sqrt{\frac{2l+1}{4\pi}} \frac{(-)^l \sqrt{(2l)!}}{2^l l!} (x+iy)^l \\
\frac{q_{ll} Y_{ll}}{(2l+1) r^{l+1}} &= \frac{1}{4\pi} \frac{(2l)!}{2^{2l}(l!)^2} \frac{(x+iy)^l}{r^{2l+1}} \int d^3 r' (x' - iy')^l \rho(\boldsymbol{r}') \\
&= \frac{1}{4\pi} \frac{(2l)!}{2^l(l!)^2} \frac{(r^+)^l}{r^{2l+1}} \frac{1}{A_l} Q^{-- \cdots -} \equiv \frac{1}{4\pi l!} \frac{r^{+l} Q^{-- \cdots -}}{r^{2l+1}} \quad (201)
\end{aligned}
$$

So we see that $A_l = (2l)!/2^l l!$. The first few are $A_0 = A_1 = 1$, $A_2 = 3$, $A_3 = 15$:

$$
Q = \int d^3 r \rho(\boldsymbol{r}), \qquad Q^i = \int d^3 r \, r^i \rho(\boldsymbol{r}), \qquad Q^{ij} = 3 \int d^3 r \left(r^i r^j - \frac{1}{3} \delta_{ij} r^2 \right) \rho(\boldsymbol{r})
$$

$$
Q^{ijk} = 15 \int d^3 r \left(r^i r^j r^k - \frac{1}{5} (\delta_{ij} r^k + \delta_{jk} r^i + \delta_{ik} r^j) r^2 \right) \rho(\boldsymbol{r}) \quad (202)
$$

The dipole $Q^i \equiv p^i$ is also called the electric dipole moment. Q^{ij} is the electric quadrapole moment. Note that the moments depend in general on the point about which they are evaluated. The lowest nonvanishing one however is translationally invariant.

A physical model of a monopole is simply a point charge, that of a dipole is equal and opposite point charges separated by an infinitesimal displacement $\boldsymbol{a}$ with $\boldsymbol{p} = q\boldsymbol{a}$. A quadrupole can be modelled as separated equal and opposite dipoles. Proceeding in this way we see that a model of a 2^l-pole requires at least 2^l point charges, arranged so all the multipole moments lower than 2^l are 0. In such a model if the separation of the charges is finite, there will in general be higher multipole moments of the charge distribution. These higher moments will become smaller as the distribution shrinks in size.

The first few terms of the multipole expansion for $\phi, \boldsymbol{E}$ read

$$
\begin{aligned}
\phi(\boldsymbol{r}) &= \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{r^3} + \frac{1}{2}\sum_{ij}\frac{Q_{ij}r^i r^j}{r^5} + \cdots\right] \\
&= \frac{1}{\epsilon_0}\left[\frac{q_{00}Y_{00}}{r} + \sum_{m=-1}^{+1}\frac{q_{1m}Y_{1m}}{3r^2} + \sum_{m=-2}^{+2}\frac{q_{2m}Y_{2m}}{5r^3} + \cdots\right] \\
E^i &= \frac{1}{4\pi\epsilon_0}\left[\frac{Qr^i}{r^3} + \frac{3r^i\boldsymbol{p}\cdot\boldsymbol{r} - p^i r^2}{r^5} + \cdots\right]
\end{aligned} \tag{203}
$$

These equations show the electric field produced by the multipole moments. We also need to know how the moments respond to external fields. A measure of this is the energy of a distribution of charges in an external field, $W = \int d^3r\rho(\boldsymbol{r})\phi(\boldsymbol{r})$. To isolate individual multipole contributions, we assume ϕ is slowly varying over the extent of the charge distribution, so it makes sense to expand

$$
\begin{aligned}
\phi(\boldsymbol{r}) &= \phi(0) + \boldsymbol{r}\cdot\nabla\phi + \frac{1}{2}r^i r^j\nabla_i\nabla_j\phi + \cdots \\
W &= Q\phi(0) - \boldsymbol{p}\cdot\boldsymbol{E} + \frac{1}{6}Q^{ij}\nabla_i\nabla_j\phi + \frac{1}{6}\nabla^2\phi\int r^2\rho + \cdots \\
&= Q\phi(0) - \boldsymbol{p}\cdot\boldsymbol{E} - \frac{1}{6}Q^{ij}\nabla_i E_j + \cdots
\end{aligned} \tag{204}
$$

where we used $\nabla^2\phi = 0$ which holds since any sources of the external field ϕ are far from the charge distribution. We see, for example, that an electric dipole likes to be aligned with the external field. If the external field is itself produced by a distant dipole $\boldsymbol{p}_2$, the interaction energy with a dipole $\boldsymbol{p}_1$ is

$$
W_{12} = \frac{\boldsymbol{p}_1\cdot\boldsymbol{p}_2 - 3\hat{r}_{12}\cdot\boldsymbol{p}_1\hat{r}_{12}\cdot\boldsymbol{p}_2}{4\pi\epsilon_0|\boldsymbol{r}_1 - \boldsymbol{r}_2|^3} \tag{205}
$$

This formula shows that the attractive or repulsive character of the interaction depends on the orientation of the dipoles with respect to each other and to the direction $\hat{r}_{12}$ from one dipole to the other.

4.2 Electrostatics in Dielectric Materials

So far the only materials we have considered have been conductors in which, by definition, charges move freely. In the presence of external fields the charges in the bulk of the conductor rearrange until $\boldsymbol{E} = 0$ in the bulk, and all charge resides on the surface of the conductor. Thus the electrostatics of conductors amounts to solving Laplace's equation with Dirichlet boundary conditions.

In contrast dielectrics are insulators: charge is not free to move about in response to electric fields. But the atoms and molecules that make up the material are bound states of charged particles, and the electric field will deform the charge distribution. If the atoms have zero dipole moment with no field present, the application of the field will induce a non zero dipole moment (and indeed nonzero higher multipole moments as well.) These induced dipole moments will produce an electric field that adds to the external field. The collective effect of these induced moments throughout the material can be accounted for in an average way by material dependent modifications in the vacuum Maxwell equations. We do this by distinguishing the free charge density (controlled by the experimenter) and the bound charge density, which can be influenced only indirectly by the presence of applied fields.

We assume that the bound charge density is due to a dipole moment density $\boldsymbol{P}(\boldsymbol{x})$, the polarization, that is the coarse grained average of the dipole moments of the individual atoms/molecules. We allow it to vary with $\boldsymbol{x}$ both because the material itself may not be homogeneous and also because the fields that induce them will vary from point to point. We need to associate a charge density with $\boldsymbol{P}$ to know how it will enter Maxwell's equations. To find this we examine the electric potential due to a distribution of dipoles:

$$
\begin{aligned}
\phi_{dipoles} &= \int d^3 r' \frac{\boldsymbol{P}(\boldsymbol{r}') \cdot (\boldsymbol{r} - \boldsymbol{r}')}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|^3} = \int d^3 r' \boldsymbol{P}(\boldsymbol{r}') \cdot \nabla' \frac{1}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|} \\
&= \int d^3 r' \frac{-\nabla' \cdot \boldsymbol{P}}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|} + \oint dS \frac{\hat{n} \cdot \boldsymbol{P}}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}'|}
\end{aligned}
\tag{206}
$$

from which we see that the associated bulk charge density is $\rho_{\text{bound}} = -\nabla \cdot \boldsymbol{P}$, and there is a surface charge density of $\hat{\boldsymbol{n}} \cdot \boldsymbol{P}$ on the boundary of the material. So we write the total charge density $\rho_{\text{total}} = \rho + \rho_{\text{bound}}$ where ρ is the free charge density (under experimental control), we can write the relevant Maxwell equation as

$$
\begin{aligned}
\epsilon_0 \nabla \cdot \boldsymbol{E} &= \rho - \nabla \cdot \boldsymbol{P} \\
\nabla \cdot \boldsymbol{D} &\equiv \nabla \cdot (\epsilon_0 \boldsymbol{E} + \boldsymbol{P}) = \rho
\end{aligned}
\tag{207}
$$

We see that the so-called displacement field $\boldsymbol{D} \equiv \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$ has the free charge density ρ as its source. It can therefore be regarded as the applied (external) field, as distinguished from $\boldsymbol{E}$ which is the total field (due to all sources). The other electrostatic equation $\nabla \times \boldsymbol{E} = 0$ is not modified by the material. The electrostatic equations

$$
\nabla \cdot \boldsymbol{D} = \rho, \qquad \nabla \times \boldsymbol{E} = 0
\tag{208}
$$

therefore need to be completed with a relation between $\boldsymbol{D}$ and $\boldsymbol{E}$, or equivalently a relation between $\boldsymbol{P}$ and $\boldsymbol{E}$. Because different fields enter the curl and divergence equations,

the boundary conditions across the interface of two different materials are different for the transverse and normal components. The parallel components of $\boldsymbol{E}$ are continuous whereas the normal components of $\boldsymbol{D}$ have a discontinuity proportional to the free surface charge density σ:

$$(\boldsymbol{D}_2 - \boldsymbol{D}_1) \cdot \hat{n}_{12} = \sigma$$

where $\hat{n}_{12}$ is the unit normal vector directed into region 2.

The relation between $\boldsymbol{E}$ and $\boldsymbol{P}$ can be quite complicated. We shall mostly assume a linear response of the material to applied fields, meaning that

$$P^i = \epsilon_0 \chi_{ij} E_j \tag{209}$$

where χ is called the electric susceptibility and is assumed to be independent of the field. More generally the response could be nonlinear, meaning that χ would itself depend upon $\boldsymbol{E}$. the above hypothesis also presumes that $\boldsymbol{P} = 0$ in the absence of an applied field, i.e. that the materials we consider with have no permanent polarization. For simplicity we shall also only consider isotropic materials, for which $\chi_{ij}^e = \chi_e \delta_{ij}$. Then

$$\boldsymbol{P} = \chi_e \epsilon_0 \boldsymbol{E}, \qquad \boldsymbol{D} = \epsilon_0(1 + \chi_e)\boldsymbol{E} \equiv \epsilon \boldsymbol{E} \tag{210}$$

and $\epsilon(\boldsymbol{r})$ may vary with position.

4.3 Energy and Forces on Dielectrics

Dielectric objects will feel forces when introduced into electric fields. In general, this is a complicated issue, which we won't delve into. We shall limit our discussion to simple questions of energetics, from which we can infer forces in simple situations. To get expressions for the energy of a system in the presence of materials, we can calculate the work done to establish a free charge density in the system. Since the electric force is proportional to $\boldsymbol{E} = -\nabla\phi$ changing the charge density by $\delta\rho$ requires the work $\delta W = \int \phi \delta\rho$. Of course the change in ρ induces a change in $\boldsymbol{D}$ via $\nabla \cdot \delta\boldsymbol{D} = \delta\rho$, so the work can be rewritten

$$\delta W = \int \phi \nabla \cdot \delta\boldsymbol{D} = \int \boldsymbol{E} \cdot \delta\boldsymbol{D} \tag{211}$$

after an integration by parts. However to find the total work we need more information, relating $\boldsymbol{D}$ to $\boldsymbol{E}$. We just consider the simplest homogeneous linear response $D^i = \int d^3y \epsilon_{ij}(\boldsymbol{x}, \boldsymbol{y}) E^j(\boldsymbol{y})$, with $\epsilon_{ij}(\boldsymbol{x}, \boldsymbol{y}) = \epsilon_{ji}(\boldsymbol{y}, \boldsymbol{x})$ independent of $\boldsymbol{E}$. Keeping ϵ fixed as we build up the charge distribution we have

$$\delta \int d^3x \boldsymbol{D} \cdot \boldsymbol{E} = \delta \int d^3x d^3y \epsilon_{ij}(\boldsymbol{x}, \boldsymbol{y}) E^i(\boldsymbol{x}) E^j(\boldsymbol{y}) = 2\delta \int d^3x d^3y \epsilon_{ij}(\boldsymbol{x}, \boldsymbol{y}) E^i(\boldsymbol{x}) \delta E^j(\boldsymbol{y})$$

$$= 2 \int d^3x \boldsymbol{E} \cdot \delta\boldsymbol{D} = 2\delta W \tag{212}$$

So, in the case of linear response (and only then!) we conclude that the total work done in establishing the charge distribution ρ is

$$W = \frac{1}{2}\int d^3x \boldsymbol{D}\cdot\boldsymbol{E} = \frac{1}{2}\int d^3x \rho\phi \tag{213}$$

and we can identify W with the energy of the system. When $\epsilon \to \epsilon_0$ it reduces to the vacuum expression.

We can infer the force on a part of the system by displacing that part of the system an amount $\delta\xi$. Then *provided that the system is completely isolated from external sources of energy,* we can identify the force as

$$F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_{\text{isolated}}, \tag{214}$$

Here isolated means among other things that all electrodes are disconnected from batteries, so the total charge on each electrode in the system is held fixed.

Since it is frequently simpler to evaluate the energy at fixed potentials on the individual electrodes, it is extremely useful to know that the change in energy under the displacement $\delta\xi$ for the isolated system is simply the negative of the change in energy at constant potentials. To see this, let the system contain a number of electrodes where the free charge resides. Then since each electrode is an equipotential we can write

$$W = \frac{1}{2}\sum_i Q_i V_i. \tag{215}$$

Under the displacement of a dielectric body of the isolated system the dielectric properties change but since the charges are fixed we have

$$\delta W_{\text{isolated}} = W = \frac{1}{2}\sum_i Q_i \delta V_i \tag{216}$$

Now, holding the dielectrics in place, we can reconnect the batteries to restore the original potentials $\delta_2 V_i = -\delta V_i$. But in this second step the charges change, so

$$\delta_2 W = \frac{1}{2}\sum_i (\delta_2 Q_i V_i + Q_i \delta_2 V_i) = \sum_i Q_i \delta_2 V_i = -\sum_i Q_i \delta V_i = -2\delta W_{\text{isolated}} \tag{217}$$

where we used the fact that for linear response at fixed dielectric properties, $\sum_i \delta_2 Q_i V_i = \sum_i Q_i \delta_2 V_i$). The total change in W at fixed potential is thus

$$\delta W_{\text{fixedpotentials}} = \delta W_{\text{isolated}} + \delta_2 W = -\delta W_{\text{isolated}} \tag{218}$$

4.4 Boundary value problems with dielectrics

Let us now solve a couple of boundary problems, turning first to a dielectric sphere in a uniform external electric field $\boldsymbol{E} = E\hat{z}$. That field corresponds to a potential $\phi_e = -Ez = -Er\cos\theta$ in spherical coordinates. Because only the $l = 1$ Legendre polynomial appears in the external field, spherical symmetry assures that the solution of Laplace's equation for our problem will be of the form $f(r)\cos\theta$, where $f(r)$ is a (different inside and out) linear combination of r and $1/r^2$. The potential outside and inside will then be

$$\phi_{\text{out}} = \left(-Er + \frac{b}{r^2}\right)\cos\theta, \qquad \phi_{\text{in}} = ar\cos\theta \tag{219}$$

where the $1/r^2$ term inside is forbidden by good behavior at $r = 0$ and the coefficient of the r term outside is chosen to produce a uniform field at large r. Continuity of D_n at $r = R$ then imposes $\epsilon a = \epsilon_0(-E - 2b/R^3)$ and continuity of E_t imposes $aR = -ER + b/R^2$. Solving these equations,

$$b = ER^3\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}, \qquad a = -\frac{3E}{2 + \epsilon/\epsilon_0}$$

$$\phi_{\text{out}} = -E\left(r - \frac{R^3}{r^2}\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)\cos\theta, \qquad \phi_{\text{in}} = -\frac{3Er}{2 + \epsilon/\epsilon_0}\cos\theta \tag{220}$$

The inside fields are given by

$$\boldsymbol{E}_{\text{in}} = \frac{3E}{2 + \epsilon/\epsilon_0}\hat{z}, \qquad \boldsymbol{D}_{\text{in}} = \frac{3E\epsilon}{2 + \epsilon/\epsilon_0}\hat{z}, \qquad \boldsymbol{P} = \frac{3E(\epsilon - \epsilon_0)}{2 + \epsilon/\epsilon_0}\hat{z} \tag{221}$$

which are all uniform parallel to the z-axxis within the sphere. Notice that for $\epsilon > \epsilon_0$ the polarization points in the same direction as the external field in accord with the intuition of screening: the external field pushes positive charge in its direction and negative charge in the opposite direction. If we could have $\epsilon < \epsilon_0$, *we would have anti-screening*. This anti-intuitive result is the basic reason that all materials have $\epsilon > \epsilon_0$. However, if we immersed the sphere in another dielectric with $\epsilon' > \epsilon$ we would simulate this antiscreening phenoomenon.

The electric field outside the dielectric sphere can be seen to be the sum of the external field $E\hat{z}$ plus the electric field of a dipole with dipole moment $\boldsymbol{p} = \boldsymbol{P}V_{sphere} = 4\pi R^3\boldsymbol{P}/3$. It is amusing to consider two extreme limits. If $\epsilon \to \infty$ we regain the outside field of a spherical conductor in the external field.

$$\phi_{\text{out}} \to -E\left(r - \frac{R^3}{r^2}\right)\cos\theta \tag{222}$$

so the surface is an equipotential with $\phi = 0$. The corresponding inside electric field goes to zero.

$$\phi_{\text{in}} \to 0, \qquad \epsilon \to \infty \tag{223}$$

Next consider the (unphysical) limit $\epsilon \to 0$:

$$\phi_{\text{out}} \quad \to \quad -E\left(r + \frac{R^3}{2r^2}\right)\cos\theta, \qquad \epsilon = 0,$$

$$\phi_{\text{in}} \quad \to \quad -\frac{3E}{2}z, \qquad \epsilon = 0 \tag{224}$$

In this case the surface is not an equipotential, but rather the normal component of the electric field $E_n = -\partial\phi/\partial r$ vanishes on it. The sphere seems to repel electric field lines. At first sight this seems incompatible with the fact that $\boldsymbol{E}_{\text{in}} \neq 0$ in the limit. But, of course $\boldsymbol{D}_{\text{in}} \to 0$. Thus we can say that an $\epsilon = 0$ dielectric would expel $\boldsymbol{D}$ lines from its interior. This is exactly what a superconductor does to $\boldsymbol{B}$ field lines.

For one more example we study a planar interface between dielectrics ϵ_1, ϵ_2. Choose coordinates so the interface is the xy-plane, with ϵ_1 assigned to $z > 0$, and place a charge q on the z-axis at $z = D$. To calculate ϕ for $z > 0$, we try an image charge q' on the z-axis at $z = -D$. To calculate ϕ for $z < 0$ we try an image charge q'' at $z = +D$.

$$\phi_> \quad = \quad \frac{1}{4\pi\epsilon_1}\left[\frac{q}{|\boldsymbol{r} - D\hat{z}|} + \frac{q'}{|\boldsymbol{r} + D\hat{z}|}\right]$$

$$\phi_< \quad = \quad \frac{1}{4\pi\epsilon_2}\frac{q''}{|\boldsymbol{r} - D\hat{z}|} \tag{225}$$

Continuity of D_z, E_x at $z = 0$:

$$\frac{q''(-D)}{(x^2 + y^2 + D^2)^{3/2}} \quad = \quad \frac{q(-D)}{(x^2 + y^2 + D^2)^{3/2}} + \frac{q'D}{(x^2 + y^2 + D^2)^{3/2}}$$

$$\frac{q''x}{\epsilon_2(x^2 + y^2 + D^2)^{3/2}} \quad = \quad \frac{qx}{\epsilon_1(x^2 + y^2 + D^2)^{3/2}} + \frac{q'x}{\epsilon_1(x^2 + y^2 + D^2)^{3/2}}$$

$$q' \quad = \quad q\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}, \qquad q'' = q\frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}$$

$$\phi_> \quad = \quad \frac{q}{4\pi\epsilon_1}\left[\frac{1}{|\boldsymbol{r} - D\hat{z}|} + \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\frac{1}{|\boldsymbol{r} + D\hat{z}|}\right]$$

$$\phi_< \quad = \quad \frac{2q}{4\pi(\epsilon_1 + \epsilon_2)|\boldsymbol{r} - D\hat{z}|} \tag{226}$$

Again, for $\epsilon_2 \to \infty$ we see perfect screening (as in a conductor. And for $\epsilon_2 \to 0$ we see perfect anti-screening.

4.5 Models for χ_e

Dilute Materials

A realistic description of susceptibility must use quantum mechanics. For example the energy eigenstates of a one electron atom satisfy

$$H|E_n\rangle \quad = \quad \left(\frac{\boldsymbol{p}^2}{2m} + V(x)\right)|E_n\rangle = E_n|E_n\rangle \tag{227}$$

If we put the atom in an external field $\boldsymbol{E} = -\nabla\phi$, the term $q\phi = -e\phi$ must be added to H (The charge of the electron is $q = -e$!) We are interested in calculating the dipole moment $\boldsymbol{p}$ induced in the atom by this field. The linear response approximation, means we can use first order perturbation theory in ϕ. The perturbation in the ground state satisfies

$$
\begin{aligned}
(H - E_G)\delta|G\rangle &= -q\phi|G\rangle + |G\rangle\delta E_G = -q\phi|G\rangle + |G\rangle\langle G|q\phi|G\rangle && (228) \\
\langle\boldsymbol{p}\rangle &\equiv (\langle G| + \delta\langle G|)\boldsymbol{p}(|G\rangle + \delta|G\rangle) \approx \langle G|\boldsymbol{p}|G\rangle + \langle G|\boldsymbol{p}\delta|G\rangle + (\delta\langle G|)\boldsymbol{p}|G\rangle
\end{aligned}
$$

If the ground state is **nondegenerate** and parity is a good symmetry, then $\langle G|\boldsymbol{p}|G\rangle = 0$ which is why atoms with permanent dipole moments are rare. Since we can always add a multiple of $|G\rangle$ to $\delta|G\rangle$, we can choose $\delta|G\rangle$ orthogonal to the ground state and write :

$$
\langle\boldsymbol{p}\rangle = \sum_{E_n > E_G} \frac{\langle G|\boldsymbol{p}|n\rangle\langle n| - q\phi|G\rangle + \langle G| - q\phi|n\rangle\langle n|\boldsymbol{p}|G\rangle}{E_n - E_G} \tag{229}
$$

If we assume the field is uniform over the atom, we can put $\phi = -\boldsymbol{r}\cdot\boldsymbol{E}$, and of course $\boldsymbol{p} = q\boldsymbol{r}$, so we can express the result as

$$
\langle p^i\rangle = q^2\left[\sum_{E_n > E_G} \frac{\langle G|r^i|n\rangle\langle n|r^j|G\rangle + \langle G|r^j|n\rangle\langle n|r^i|G\rangle}{E_n - E_G}\right] E^j \equiv \epsilon_0\gamma_{ij}E^j \tag{230}
$$

Rotational invariance of the ground state implies that the quantity in brackets is proportional to δ_{ij}, so we can write $\gamma_{ij} = \gamma_e\delta_{ij}$ with, putting $q = -e$

$$
\epsilon_0\gamma_e = \frac{2e^2}{3}\sum_{E_n > E_G} \frac{\langle G|\boldsymbol{r}|n\rangle \cdot \langle n|\boldsymbol{r}|G\rangle}{E_n - E_G} \geq 0 \tag{231}
$$

An ultrasimple model is to take a harmonic oscillator potential $V = m\omega^2\boldsymbol{r}^2/2$. Then $\boldsymbol{r}$ connects the ground state only to the first excited state, and no other. Thus we can put $E_n - E_G = \hbar\omega$, and sum over all n to conclude that

$$
\begin{aligned}
\epsilon_0\gamma_e &= \frac{2e^2}{3\hbar\omega}\langle G|\boldsymbol{r}\cdot\boldsymbol{r}|G\rangle = \frac{4e^2}{3m\hbar\omega^3}\langle G|V|G\rangle = \frac{2e^2}{3m\hbar\omega^3}E_G = \frac{2e^2}{3m\hbar\omega^3}\frac{3\hbar\omega}{2} \\
&= \frac{e^2}{m\omega^2} = \epsilon_0\frac{4\pi\alpha\hbar c}{m\omega^2}
\end{aligned} \tag{232}
$$

Notice that γ_e has dimensions of (Length)^3 or Volume. Then an estimate for susceptibility of the bulk material is $\chi_e = N\gamma_e$ where N is the number density of atoms. This estimate can be trusted when the number density is small enough so the individual atoms can be treated in isolation, as for example with a dilute gas.

Denser Materials

For denser materials, an important effect that must be accounted for is that the self field of each individual atom is not active in inducing its dipole moment, but it definitely

contributes to the averaged electric field $\boldsymbol{E}$: the dipole moment of each atom is induced by the field of all the *other* atoms. For dense materials the self field can be comparable to the field due to the other atoms, so a satisfactory estimate must remove it explicitly. If the field from a dipole is averaged over a volume V that contains it, the result [Jackson (4.18)] is $\langle \boldsymbol{E}_{\text{dipole}} \rangle = -\boldsymbol{p}/3V\epsilon_0$. To see this, we calculate

$$\int_V d^3x \boldsymbol{E} = \int d^3x d^3x' \frac{\rho(\boldsymbol{r}')}{4\pi\epsilon_0}(-\nabla)\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \int d^3x' \frac{\rho(\boldsymbol{r}')}{4\pi\epsilon_0} \oint_{\partial V} dS \frac{-\boldsymbol{n}}{|\boldsymbol{r}-\boldsymbol{r}'|} \quad (233)$$

If V is a sphere of radius R,

$$\oint_{\partial V} dS \frac{-\boldsymbol{n}}{|\boldsymbol{r}-\boldsymbol{r}'|} = R^2 \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \int d\Omega(-\hat{r}) P_l(\hat{r}\cdot\hat{r}') = -R^2 \frac{r_<}{r_>^2} \int d\Omega \hat{r}\hat{r}\cdot\hat{r}' = -\frac{4\pi R^2}{3}\frac{r_<}{r_>^2}\hat{r}'$$

If the charge distribution is completely within the sphere, $r_< = r', r_> = R$ and we have

$$\langle \boldsymbol{E} \rangle = \frac{1}{V}\int_V d^3x \boldsymbol{E} = -\frac{1}{3V\epsilon_0}\int d^3x' \boldsymbol{r}'\rho(\boldsymbol{r}') = -\frac{\boldsymbol{p}}{3V\epsilon_0} \quad (234)$$

If there is one such dipole in each such volume V, the right side is just $-\boldsymbol{P}/3\epsilon_0$. If the total average field, including the applied external field is $\boldsymbol{E}$ then the field active in inducing polarization of each atom is $\boldsymbol{E} - \boldsymbol{E}_{\text{self}} \approx \boldsymbol{E} + \boldsymbol{P}/3\epsilon_0$. Then

$$\boldsymbol{P} = N\boldsymbol{p}_{\text{atom}} = N\gamma\epsilon_0(\boldsymbol{E} + \boldsymbol{P}/3\epsilon_0)$$
$$\boldsymbol{P} = \frac{N\gamma}{1-N\gamma/3}\epsilon_0\boldsymbol{E} = \chi_e\epsilon_0\boldsymbol{E} \quad (235)$$

This rough estimate shows that χ_e is not necessarily linear in the density of the material. One can invert the relation

$$N\gamma = \frac{\chi_e}{1+\chi_e/3} = 3\frac{\epsilon-\epsilon_0}{2\epsilon_0+\epsilon} \quad (236)$$

giving the Clausius-Mossotti relation.

Permanent Dipoles and Temperature Dependence

Even though parity symmetry would forbid permanent electric dipole moments, there are definitely molecules called polar molecules that have an effectively permanent electric dipole moment. This happens when energy eigenstaes $|E_\pm\rangle$ with opposite parity have nearly degenerate energy eigenvalues. In that case the states $|E_+\rangle \pm |E_-\rangle$ which possess a dipole moment are relatively long-lived compared to the collison time. If there were nothing to disrupt their response they would immediately line up with an applied electric field. However thermal fluctuations inhibit this aligning so that the response is more like a normal dielectric. We can give an elementary account of this effect using the canonical ensemble. A permanent

dipole moment contributes $-\boldsymbol{p} \cdot \boldsymbol{E} = -pE\cos\theta$, for $\boldsymbol{E} = \hat{z}E$, to the energy. The thermal average of the dipole moment, taking only the θ degree of freedom into account, is then

$$
\begin{aligned}
\langle \boldsymbol{p} \rangle &= \frac{\int d\Omega\, p(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\exp\{pE\cos\theta/kT\}}{\int d\Omega \exp\{pE\cos\theta/kT\}} \\
&= p\hat{z}\frac{\int_{-1}^{1} d\cos\theta \cos\theta \exp\{pE\cos\theta/kT\}}{\int_{-1}^{1} d\cos\theta \exp\{pE\cos\theta/kT\}} \\
&= p\hat{z}\frac{\partial}{\partial\alpha}\ln\frac{2\sinh\alpha}{\alpha}\Big|_{\alpha=pE/kT} = p\hat{z}\left(\coth\frac{pE}{kT} - \frac{kT}{pE}\right) \\
&\rightarrow \frac{p^2}{3kT}\boldsymbol{E}, \qquad \frac{pE}{kT} \ll 1
\end{aligned}
\tag{237}
$$

Notice, by the way, that before taking the weak field limit, the induced dipole moment showed non-linear dependence on the electric field!

In contrast, as $T \rightarrow zero$ the coth goes to 1 and $\langle \mathbf{p} \rangle \rightarrow p\hat{z}$ showing what happens if thermal fluctuations are negligible.

5 Magnetostatics

We next study problems involving static magnetic fields. We have seen that specializing Maxwell's equations to static fields decouples the electric and magnetic field equations. For simplicity we assume $\boldsymbol{E} = 0$, though it is not necessary. The equations for the static magnetic field in vacuum are

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{238}$$

The current density must of course be static also, but taking the divergence of both sides of the curl equation shows that the current density must satisfy the constraint $\nabla \cdot \boldsymbol{J} = 0$. This means that charge cannot accumulate in any region:

$$I_{\text{out}} = \oint dA \hat{n} \cdot \boldsymbol{J} = \int d^3 x \nabla \cdot \boldsymbol{J} = 0 \tag{239}$$

The first (inhomogeneous) equation embodies Amp'ere's law. Integrating both sides over a surface S bounded by a closed curve C, one obtains

$$
\begin{aligned}
\mu_0 I_{\text{S}} &= \int_{\text{S}} d\boldsymbol{S} \cdot \mu_0 \boldsymbol{J} = \int_{\text{S}} d\boldsymbol{S} \cdot \nabla \times \boldsymbol{B} \\
&= \oint_{\text{C}} d\boldsymbol{l} \cdot \boldsymbol{B}
\end{aligned} \tag{240}
$$

By definition I_{S} is the total current through the surface S, which can also be characterized as the total current enclosed by the closed curve C. When symmetry considerations limit the field orientation, Ampère's law is frequently enough to determine the strength of the field.

The second (homogeneous) equation can be solved, once and for all, by introducing the vector potential $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Plugging this into the first equation leads to

$$\nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} \tag{241}$$

This equation reduces to a Poisson equation in *Coulomb Gauge* $\nabla \cdot \boldsymbol{A} = 0$. We can then write the solution in vacuum as

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \boldsymbol{B} = \frac{\mu_0}{4\pi} \int d^3 r' \frac{\boldsymbol{J}(\boldsymbol{r}') \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \tag{242}$$

For this to solve the original equation the Coulomb gauge constraint $\nabla \cdot \boldsymbol{A} = 0$ must be satisfied. And it is, because $\nabla |\boldsymbol{r} - \boldsymbol{r}'|^{-1} = -\nabla' |\boldsymbol{r} - \boldsymbol{r}'|^{-1}$ and an integration by parts produces $\nabla' \cdot \boldsymbol{J}(\boldsymbol{r}') = 0$.

Specializing $\boldsymbol{J}$ to a closed current carrying loop of thin wire, we can write $d^3 r' \boldsymbol{J} \rightarrow I d\boldsymbol{l}$, where $d\boldsymbol{l}$ is a directed element of length parallel to the tangent vector pointed in the direction of current flow. Then we arrive at the Biot-Savart law:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{I \mu_0}{4\pi} \oint \frac{d\boldsymbol{l}}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \boldsymbol{B}(\boldsymbol{r}) = \frac{I \mu_0}{4\pi} \oint \frac{d\boldsymbol{l} \times (\boldsymbol{r} - \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|^3} \tag{243}$$

5.1 Circular Current Loop

A simple application of this formula is to a circular current loop of radius R. Centering the loop on the origin so that it lies in the xy plane, we have $\boldsymbol{r}' = R(\hat{x}\cos\phi' + \hat{y}\sin\phi')$, $d\boldsymbol{l} = Rd\phi'(-\hat{x}\sin\phi' + \hat{y}\cos\phi')$, and $(\boldsymbol{r} - \boldsymbol{r}')^2 = r^2 + R^2 - 2rR\sin\theta\cos(\phi' - \phi)$. Then

$$\boldsymbol{A} = \frac{IR\mu_0}{4\pi}\int_0^{2\pi} d\phi' \frac{-\hat{x}\sin\phi' + \hat{y}\cos\phi'}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos(\phi' - \phi)}} \tag{244}$$

writing the numerator as

$$-\hat{x}\sin\phi' + \hat{y}\cos\phi' = (-\hat{x}\cos\phi - \hat{y}\sin\phi)\sin(\phi' - \phi) + (-\hat{x}\sin\phi + \hat{y}\cos\phi)\cos(\phi' - \phi)$$

we see that the first term produces a total derivative in the integrand, which integrates to zero. The second term is just $\hat{\phi}\cos(\phi' - \phi)$ so

$$\boldsymbol{A} = \frac{IR\mu_0}{4\pi}\hat{\phi}\int_0^{2\pi} d\phi' \frac{\cos\phi'}{\sqrt{r^2 + R^2 - 2rR\sin\theta\cos\phi'}} \tag{245}$$

The integral is not elementary–it involves complete elliptic integrals. However it is easy enough to expand in powers of $2rR\sin\theta\cos\phi'/(r^2 + R^2)$. Since only even powers of $\cos\phi'$ have a nonvanishing integral, the first nonvanishing term is

$$\boldsymbol{A} \sim \frac{IR\mu_0}{4\pi}\hat{\phi}\int_0^{2\pi} d\phi' \frac{rR\sin\theta\cos^2\phi'}{(r^2 + R^2)^{3/2}} = \frac{\pi IR^2\mu_0 \hat{z} \times \boldsymbol{r}}{4\pi(r^2 + R^2)^{3/2}} \equiv \frac{\mu_0 \boldsymbol{m} \times \boldsymbol{r}}{4\pi(r^2 + R^2)^{3/2}} \tag{246}$$

$$\boldsymbol{m} \equiv I\pi R^2\hat{z} = IA\hat{z} \tag{247}$$

where we have defined the magnetic dipole moment $\boldsymbol{m}$. The approximation here is $2rR\sin\theta\cos\phi' \ll (r^2 + R^2)$, which is valid if $\boldsymbol{r}$ is on or close to the z-axis ($\sin\theta \ll 1$) or for $r \gg R$ at any θ. In particular the approximation is exact on the z-axis. In the approximation $r \gg R$ the magnetic field is then

$$\boldsymbol{B} \equiv \nabla \times \boldsymbol{A}$$
$$\approx \frac{\mu_0}{4\pi}\left[-\frac{3}{r^5}\boldsymbol{r} \times (\boldsymbol{m} \times \boldsymbol{r}) + \frac{1}{r^3}\boldsymbol{\nabla} \times (\boldsymbol{m} \times \boldsymbol{r})\right] = \frac{\mu_0}{4\pi}\frac{3\boldsymbol{r}\boldsymbol{m}\cdot\boldsymbol{r} - \boldsymbol{m}r^2}{r^5} \tag{248}$$

which is entirely analogous to the electric dipole field.

Actually, it is worth noting a subtle delta function difference between the electric and magnetic dipole fields. To see this we rewrite the potentials for the two cases as derivatives:

$$\phi_{\text{dipole}} = \frac{\boldsymbol{p}\cdot\boldsymbol{r}}{4\pi\epsilon_0 r^3} = -\boldsymbol{p}\cdot\nabla\frac{1}{4\pi\epsilon_0 r}$$

$$E^i_{\text{dipole}} = -\nabla\phi_{\text{dipole}} = (\nabla^i\nabla^j - \nabla^2\frac{\delta_{ij}}{3})p^j\frac{1}{4\pi\epsilon_0 r} - \frac{p^i}{3\epsilon_0 r}\delta(\boldsymbol{r}) \tag{249}$$

$$\boldsymbol{A}_{\text{dipole}} = -\boldsymbol{m}\times\nabla\frac{\mu_0}{4\pi r}$$

$$B^i_{\text{dipole}} = (\nabla \times \boldsymbol{A}_{\text{dipole}})^i = (-\delta_{ij}\nabla^2 + \nabla^i\nabla^j)m^j\frac{\mu_0}{4\pi r}$$

$$= (\nabla^i\nabla^j - \nabla^2\frac{\delta_{ij}}{3})m^j\frac{\mu_0}{4\pi r} + \frac{2m^i\mu_0}{3r}\delta(\boldsymbol{r}) \tag{250}$$

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The $+2/3$ in the magnetic case versus $-1/3$ in the electric case reflects the different microscopic origins of the respective dipole moments.

5.2 Magnetic Multipoles

The vector potential of a localized current distribution can be expanded in powers of $1/r$:

$$
\begin{aligned}
\boldsymbol{A} &= \mu_0 \sum_{lm} \frac{\boldsymbol{m}_{lm} Y_{lm}(\theta, \phi)}{(2l+1)r^{l+1}} \\
\boldsymbol{m}_{lm} &= \int d^3r' \boldsymbol{J}(\boldsymbol{r}') r'^l Y_{lm}^*(\theta', \phi')
\end{aligned}
\tag{251}
$$

One can also represent the moments in Cartesian basis, with

$$
\boldsymbol{A} = \frac{\mu_0}{4\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{r^{i_1} \cdots r^{i_n} \boldsymbol{m}_{i_1 \cdots i_n}}{r^{2n+1}}
\tag{252}
$$

Just as in the electric case the LaPlace equation requires that $\boldsymbol{m}_{i_1 \cdots i_l}$ is traceless in all pairs of indices. This condition implies that the total number of components is $3(2l+1)$, which is a factor of three too many to describe a 2^l pole moment. The factor of three is because $\boldsymbol{m}$ is a thre vector.

However the constraint $\nabla \cdot \boldsymbol{J} = 0$ implies a number of relations among them:

$$
\begin{aligned}
0 &= \int d^3r \nabla \cdot (\boldsymbol{J} r^{i_1} \cdots r^{i_n}) \\
&= \sum_{k=1}^{n} \int d^3r J^{i_k} r^{i_1} \cdots \hat{r}^{i_k} \cdots r^{i_n}
\end{aligned}
\tag{253}
$$

where the $\hat{r}$ factor is deleted. [2]

---

[2]To count the number of conditions imposed, we first note tha the sum over k makes the integrand a completely symmetric tensor with $n+1$ components. As we have seen such a tensor has $(n+2)(n+3)/2$ components. However, they are not all independent because of the traceless conditions already imposed. We can't count these twice! To correct for this, take the trace over a pair of indices. Write the indices of the jth component of m_l as $m_{j;i_1 \cdots i_l}$ so the condition is:

$$
\sum_{k=1}^{l+1} m_{i_k; i_1 \cdots i_{k-1}, i_{k+1}, \cdots i_{l+1}} = 0
\tag{254}
$$

Taking the trace over, say, i_1, i_2, and remembering that a trace of a pair of indices after the semicolon is already zero, the trace of the condition becomes

$$
m_{k; k i_3 \cdots i_{l+1}} = 0.
\tag{255}
$$

This condition was not taken into account before. However a second trace is only over indices after the semicolon and *has* been taken into account, so we have to remove these. After the second trace there will be

For $n = 0, 1$ the conditions reduce to

$$\int d^3 r \boldsymbol{J} = 0, \qquad \int d^3 r J^i r^j = -\int d^3 r J^j r^i \equiv -\epsilon_{ijk} m_k \qquad (257)$$

Then the first nonvanishing term in the multipole expansion is the magnetic dipole term:

$$\boldsymbol{A} \sim \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \boldsymbol{r}}{r^3}, \qquad \boldsymbol{m} = \frac{1}{2} \int d^3 r' \boldsymbol{r}' \times \boldsymbol{J} \qquad (258)$$

For a current loop the expression for the dipole moment reduces to

$$\boldsymbol{m} = \frac{I}{2} \oint \boldsymbol{r}' \times \boldsymbol{dl}' \to I \boldsymbol{S} \qquad (259)$$

for a planar loop, where $\boldsymbol{S}$ is the directed area of the loop, pointing in the direction of the normal to the plane. The sign is determined by the right hand rule: if the fingers of the right hand point in the direction of the current the thumb points in the direction of $\boldsymbol{S}$.

The magnetic moment of a system of moving point particles follows from recognizing that the current density is just $\boldsymbol{J} = \sum_i q \boldsymbol{v}_i \delta(\boldsymbol{r} - \boldsymbol{r}_i(t))$ Then

$$\boldsymbol{m} = \frac{1}{2} \sum_i q_i (\boldsymbol{r}_i \times \boldsymbol{v}_i) = \sum_i \frac{q_i}{2m_i} \boldsymbol{L}_i \qquad (260)$$

where $\boldsymbol{L}_i$ is the angular momentum of the ith particle. If the particles are identical, the magnetic moment is proportional to the total angular momentum of the system. In quantum mechanics we are also familiar with magnetic moments due to intrinsic spin, e.g. for the electron, it is $-g e \boldsymbol{S}/2m_e$, with $g \approx 2$.

The magnetic force and torque on a general current distribution are

$$\boldsymbol{F} = \int d^3 r \boldsymbol{J} \times \boldsymbol{B}, \qquad \boldsymbol{N} = \int d^3 r \boldsymbol{r} \times (\boldsymbol{J} \times \boldsymbol{B}). \qquad (261)$$

For a field varying slowly over the current distribution the torque formula can be approximated

$$\boldsymbol{N}^i \approx B^j \int d^3 r r^j J^i - B^i \int d^3 r r^j J^j = -\epsilon_{ijk} B^j m_k = (\boldsymbol{m} \times \boldsymbol{B})^i \qquad (262)$$

where we used the fact determined above that the first term is antisymmetric in $i \leftrightarrow j$, which implies that the second term is 0. The force on the dipole in a uniform field is zero because $\int d^3 x \boldsymbol{J} = 0$.

$l - 3$ free indices for a count of $(l-2)(l-1)/2$. thus the number of new conditions arising from $\nabla \cdot \boldsymbol{J} = 0$ is

$$\frac{(l+2)(l+3)}{2} - \frac{(l-2)(l-1)}{2} = \frac{8l+4}{2} = 2(2l+1). \qquad (256)$$

This reduces $3(2l+1)$ to $2l+1$, which is the correct number of components for a 2^l magnetic multipole.

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5.3 Magnetic Fields in Magnetic Materials

Just as with dielectric materials, we try to deal with the response of magnetic materials to applied fields in an average way by introducing a magnetic dipole density (magnetization) $\boldsymbol{M}$, which will in general depend on the applied fields. The first thing to figure out is how the magnetization modifies the vacuum Maxwell equations. We do this by considering the vector potential produced by an an infinitesimal element $\boldsymbol{M}d^3x$, which acts like a point dipole

$$
\begin{aligned}
\boldsymbol{A} &= \frac{\mu_0}{4\pi}\int d^3x' \frac{\boldsymbol{M}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3} = \frac{\mu_0}{4\pi}\int d^3x' \boldsymbol{M}(\boldsymbol{r}')\times\nabla'\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \\
&= \frac{\mu_0}{4\pi}\int d^3x' \frac{\nabla'\times\boldsymbol{M}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}
\end{aligned}
\tag{263}
$$

This shows that the magnetization produces an effective current density $\boldsymbol{J}_M = \nabla\times\boldsymbol{M}$. From now on we use $\boldsymbol{J}$ to denote the free current density, so the total current density in the presence of magnetization is $\boldsymbol{J}+\nabla\times\boldsymbol{M}$, so the Maxwell equations become

$$
\begin{aligned}
\nabla\times\frac{\boldsymbol{B}}{\mu_0} &= \boldsymbol{J}+\nabla\times\boldsymbol{M}, \qquad \nabla\cdot\boldsymbol{B}=0 \\
\nabla\times\boldsymbol{H} &= \boldsymbol{J}, \qquad \nabla\cdot\boldsymbol{B}=0, \qquad \boldsymbol{B}=\mu_0(\boldsymbol{H}+\boldsymbol{M})
\end{aligned}
\tag{264}
$$

The established terminology is that $\boldsymbol{H}$ is called the "magnetic field" and $\boldsymbol{B}$ is called the "magnetic induction". Comparison with the electrostatic equations

$$
\nabla\times\boldsymbol{E} = 0, \qquad \nabla\cdot\boldsymbol{D}=\rho, \qquad \boldsymbol{D}=\epsilon_0\boldsymbol{E}+\boldsymbol{P}
\tag{265}
$$

shows that in the absence of free sources ($\rho=\boldsymbol{J}=0$) there is a precise mathematical analogy between $\boldsymbol{H}$ and $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{D}$, and $\mu_0\boldsymbol{M}$ and $\boldsymbol{P}$. The analogy is spoiled by the sources. Nonetheless any source free problem in electrostatics can be mapped to a source free problem in magnetostatics.

The magnetization depends on the magnetic field. For weak enough fields a linear dependence $\boldsymbol{M}=\chi_m\boldsymbol{H}$ is usually valid, in which case we define the magnetic permeability μ via $\boldsymbol{B}=\mu_0(1+\chi_m)\boldsymbol{H}\equiv\mu\boldsymbol{H}$. However ferromagnetism, which is not uncommon, is a striking and important exception to such a linear approximation. We are all quite familiar from childhood with permanent magnets which have a stable magnetization in the complete absence of applied fields.

5.4 Boundary conditions

At the interface between two different magnetic materials, $\hat{n}\cdot\boldsymbol{B}$ is continuous and $\hat{n}\times\boldsymbol{H}$ has a discontinuity determined by the surface current $\boldsymbol{K}$, in the absence of which $\boldsymbol{H}_t$ is continuous. When the magnetic response is linear we can express the boundary conditions in terms of $\boldsymbol{B}$ or $\boldsymbol{H}$ alone:

$$
\begin{aligned}
\hat{n}\cdot(\boldsymbol{B}_2-\boldsymbol{B}_1)=0, &\qquad \hat{n}\times(\frac{1}{\mu_2}\boldsymbol{B}_2-\frac{1}{\mu_1}\boldsymbol{B}_1)=\boldsymbol{K} \\
\hat{n}\cdot(\mu_2\boldsymbol{H}_2-\mu_1\boldsymbol{H}_1)=0, &\qquad \hat{n}\times(\boldsymbol{H}_2-\boldsymbol{H}_1)=\boldsymbol{K}
\end{aligned}
\tag{266}
$$

Notice that, from the point of view of material 2, if $\boldsymbol{K} = 0$ and $\mu_1 \gg \mu_2$, then $\boldsymbol{H}_2$ is nearly normal to the interface. This is analogous to the electric field near a conductor, which makes good sense because $\mu \to \infty$ is analogous to $\epsilon \to \infty$. If $\mu_1 = 0$ material 1 expels $\boldsymbol{B}$ lines as in a superconductor.

5.5   Examples of Magnetic Boundary value Problems

Permeable spherical Shell in a Uniform Magnetic field.

Let a, b be respectively the inner and outer radii of the shell which is assumed to have uniform permeability μ. This problem is tractable because a uniform field $\boldsymbol{H}_0 = H_0 \hat{z}$ is derivable from a potential, $\phi_0 = -H_0 z = -H_0 r \cos\theta = -r H_0 P_1(\cos\theta)$, which only contains the $l = 1$ Legendre polynomials. Since $\nabla \times \boldsymbol{H} = 0$, we can write for the total field $\boldsymbol{H} = -\nabla\phi$, and keeping only $l = 1$ modes we have in the three regions

$$
\begin{aligned}
\phi &= A_1 r \cos\theta, & 0 < r < a \\
\phi &= (A_2 r + B_2/r^2)\cos\theta, & a < r < b \\
\phi &= (-r H_0 + B_3/r^2)\cos\theta, & b < r < \infty
\end{aligned} \tag{267}
$$

Notice that in each of the theee regions the field $\boldsymbol{H}$ is a (different) linear combination of a magnetic dipole field, with moment parallel to $\boldsymbol{H}_0$, plus a uniform field parallel to $\boldsymbol{H}_0$. Continuity of the tangential component of $\boldsymbol{H}$ at the interfaces imposes the conditions

$$
A_1 a = A_2 a + B_2/a^2, \qquad A_2 b + B_2/b^2 = -b H_0 + B_3/b^2 \tag{268}
$$

and continuity of the normal component of $B_n = \mu \partial\phi/\partial r$ in the second region and $B_n = \mu_0 \partial\phi/\partial r$ in the first and third regions gives

$$
\mu_0 A_1 = \mu A_2 - 2\mu B_2/a^3, \qquad \mu A_2 - 2\mu B_2/b^3 = -\mu_0 H_0 - 2\mu_0 B_3/b^3. \tag{269}
$$

From the equations at $r = a$, we learn that

$$
A_2 = \frac{2\mu + \mu_0}{3\mu} A_1, \qquad B_2 = a^3 \frac{\mu - \mu_0}{3\mu} A_1 \tag{270}
$$

Plugging this into the equations at $r = b$ yields a pair of equations that determine A_1, B_3:

$$
\begin{aligned}
A_1 &= \frac{-9\mu\mu_0}{(\mu + 2\mu_0)(2\mu + \mu_0) - 2(\mu - \mu_0)^2 a^3/b^3} H_0 \\
B_3 &= \frac{(\mu - \mu_0)(2\mu + \mu_0)(b^3 - a^3)}{(\mu + 2\mu_0)(2\mu + \mu_0) - 2(\mu - \mu_0)^2 a^3/b^3} H_0
\end{aligned} \tag{271}
$$

Let us first consider the field in the cavity

$$
\begin{aligned}
\boldsymbol{H}_{\text{cavity}} &= -A_1 \hat{z} = \frac{9\mu\mu_0}{(\mu + 2\mu_0)(2\mu + \mu_0) - 2(\mu - \mu_0)^2 a^3/b^3} \boldsymbol{H}_0 \\
&\sim \begin{cases} \frac{9\mu_0 b^3}{2\mu(b^3 - a^3)} \boldsymbol{H}_0, & \mu \gg \mu_0 \\ \frac{9\mu b^3}{2\mu_0(b^3 - a^3)} \boldsymbol{H}_0, & \mu_0 \gg \mu. \end{cases}
\end{aligned} \tag{272}
$$

We see that a highly permeable material shields its interior from external magnetic fields. This is physically reasonable when we remember that in the electric case $\epsilon \to \infty$ corresponds to a conductor and conductors shield their interiors from external electric fields. Note that we also have shielding in the opposite limit $\mu \to 0$ for which the shell is a superconductor.

The magnetic field outside the shell is the external uniform field $\boldsymbol{H}_0$ plus the field of a magnetic dipole with dipole moment $\boldsymbol{m}$:

$$\boldsymbol{H}_{\text{out}} = \boldsymbol{H}_0 + \frac{3\boldsymbol{r}\boldsymbol{m}\cdot\boldsymbol{r} - r^2\boldsymbol{m}}{4\pi r^5}, \qquad \phi_{\text{out}} = -\boldsymbol{H}_0 \cdot \boldsymbol{r} + \frac{\boldsymbol{m}\cdot\boldsymbol{r}}{4\pi r^3} \tag{273}$$

$$\boldsymbol{m} = 4\pi B_3 \hat{z} = \frac{(\mu - \mu_0)(2\mu + \mu_0)}{(\mu + 2\mu_0)(2\mu + \mu_0) - 2(\mu - \mu_0)^2 a^3/b^3} 4\pi (b^3 - a^3)\boldsymbol{H}_0 \tag{274}$$

In the limit $\mu \to \infty$, $B_3 \to b^3 H_0$ and we see that the outer surface $r = b$ becomes an equipotential, in accord with the conductor analogy. In the opposite limit $\mu \to 0$, $B_3 \to -b^3 H_0/2$ and we see that $H_n = -\partial\phi/\partial r$ vanishes at $r = b$, in accord with the Meissner effect for a superconductor.

Within the shell,

$$A_2 = \frac{-3\mu_0(2\mu + \mu_0)}{(\mu + 2\mu_0)(2\mu + \mu_0) - 2(\mu - \mu_0)^2 a^3/b^3} H_0 \sim -\frac{3\mu_0 b^3}{\mu(b^3 - a^3)} H_0$$

$$B_2 = a^3 \frac{-3\mu_0(\mu - \mu_0)}{(\mu + 2\mu_0)(2\mu + \mu_0) - 2(\mu - \mu_0)^2 a^3/b^3} H_0 \sim -a^3 \frac{3\mu_0 b^3}{2\mu(b^3 - a^3)} H_0 \tag{275}$$

so $\boldsymbol{H} \to$ *zero* within the shell at high permeability, again in accord with the conductor analogy. But $\boldsymbol{B} = \mu\boldsymbol{H}$ does *not* vanish within the shell at high permeability. Indeed, the $\boldsymbol{B}$ field lines cannot terminate anywhere and in fact must be closed loops. In contrast when $\mu \to 0$ the $\boldsymbol{B}$ field does vanish within the shell ($\boldsymbol{B}$ field lines are expelled from a superconductor!

The limit $a \to 0$ reduces this problem to an exact analogue of the dielectric sphere. As a check on our calculation we can compare the two solutions. In the limit we have

$$B_3 \to \frac{\mu - \mu_0}{\mu + 2\mu_0} b^3 H_0, \qquad A_2 \to -\frac{3\mu_0}{\mu + 2\mu_0} H_0, \qquad B_2 \to 0$$

$$\boldsymbol{H}_{\text{inside}} = \frac{3\mu_0}{\mu + 2\mu_0}\boldsymbol{H}_0, \qquad \boldsymbol{M} = \left(\frac{\mu}{\mu_0} - 1\right)\boldsymbol{H}_{\text{inside}} = \frac{3(\mu - \mu_0)}{\mu + 2\mu_0}\boldsymbol{H}_0 \tag{276}$$

By expressing the solution in terms of $\boldsymbol{M}$ instead of μ, we obtain the solution for a permanently magnetized sphere in an external field:

$$\boldsymbol{H}_{\text{out}} = \boldsymbol{H}_0 + \frac{b^3 M(3\boldsymbol{r}z - r^2\hat{z})}{3r^5} = \boldsymbol{H}_0 + \frac{(3\boldsymbol{r}\boldsymbol{m}\cdot\boldsymbol{r} - r^2\boldsymbol{m})}{4\pi r^5}, \qquad \boldsymbol{m} = \frac{4\pi}{3} b^3 \boldsymbol{M} \tag{277}$$

$$\boldsymbol{H}_{\text{in}} = \boldsymbol{H}_0 + \left(\frac{3\mu_0}{\mu + 2\mu_0} - 1\right)\boldsymbol{H}_0 = \boldsymbol{H}_0 - \frac{1}{3}\boldsymbol{M}$$

$$\boldsymbol{B}_{\text{in}} = \mu_0(\boldsymbol{H}_{\text{in}} + \boldsymbol{M}) = \mu_0\boldsymbol{H}_0 + \frac{2\mu_0}{3}\boldsymbol{M} \tag{278}$$

Taking $H_0 \to 0$ at fixed $\boldsymbol{M}$ shows that the outside field of a uniformly magnetized sphere is thatof a magnetic dipole at the center of the sphere.

Problems with Prescribed Magnetization

We have just seen how to get the field of a permanently magnetized sphere with uniform $\boldsymbol{M}$ by taking a limit of a permeable sphere. More generally we might be given a different shape and space dependent $\boldsymbol{M}(\boldsymbol{r})$. This is a very common situation for magnetostatics because of the existence of permanent magnets. The (less common) electrostatic analogue would be prescribed $\boldsymbol{P}(\boldsymbol{r})$.

If $\boldsymbol{M}(\boldsymbol{r})$ is fixed we have a choice in setting up the field equations, expressing them either in terms of $\boldsymbol{H}$ or $\boldsymbol{B}$:

$$\nabla \times \boldsymbol{H} = 0, \qquad \nabla \cdot \boldsymbol{H} = -\nabla \cdot \boldsymbol{M} \equiv \rho_M \tag{279}$$
$$\nabla \cdot \boldsymbol{B} = 0, \qquad \nabla \times \boldsymbol{B} = \mu_0 \nabla \times \boldsymbol{M} \equiv \mu_0 \boldsymbol{J}_M \tag{280}$$

For magnetized objects of finite extent the magnetization, which is 0 outside the boundary, could be nonzero just inside the boundary. The discontinuity then implies either a surface magnetic charge density σ_M or a surface current density $\boldsymbol{K}_M$, depending on which point of view is taken. Using the usual arguments with a Gaussian pill box or a Stokesian loop, leads to the identification

$$\sigma_M = \boldsymbol{n} \cdot \boldsymbol{M}, \qquad \boldsymbol{K}_M = -\boldsymbol{n} \times \boldsymbol{M} \tag{281}$$

For example, a uniformly magnetized cylinder with $\boldsymbol{M}$ parallel to its axis can either be thought of as parallel oppositely charged plates at its ends or as a current sheet (or stack of current loops) wrapping its curved boundary.

5.6   Energy and Magnetic Materials

The nature of the electromagnetic force on a charged particle

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{282}$$

shows that the magnetic force does no work because it is always perpendicular to the particle's velocity. The power delivered to the particle, the rate at which work is done on the particle by the field, is

$$\frac{dW}{dt} = \boldsymbol{v} \cdot \boldsymbol{F} = q\boldsymbol{v} \cdot \boldsymbol{E} \tag{283}$$

and depends only on the electric field. Thus to find the work done in establishing non-zero static magnetic fields, we have to take into account the electric field generated by a changing magnetic field implied by the Maxwell equation embodying Faraday's Law:

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{284}$$

We can then identify this work with the energy stored in the magnetic field. When currents flow in the presence of electromagnetic fields the work done on the fields in a short time δt is

$$
\begin{aligned}
\delta W & = -\delta t \int d^3x \boldsymbol{J} \cdot \boldsymbol{E} = -\delta t \int d^3x (\nabla \times \boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t}) \cdot \boldsymbol{E} \\
& = \delta t \int d^3x \frac{\partial \boldsymbol{D}}{\partial t} \cdot \boldsymbol{E} - \delta t \int d^3x \boldsymbol{H} \cdot \nabla \times \boldsymbol{E} \\
& = \int d^3x \delta \boldsymbol{D} \cdot \boldsymbol{E} + \delta t \int d^3x \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \\
& = \int d^3x \boldsymbol{E} \cdot \delta \boldsymbol{D} + \int d^3x \boldsymbol{H} \cdot \delta \boldsymbol{B} = \delta W_E + \delta W_M
\end{aligned}
\tag{285}
$$

The first term is the work done on the electric field, which we have already discussed in electrostatics, and the second term is the work done on the magnetic field, which we have been seeking. The magnetic/electric analogies are apparent in this formula. However, when we express the works in terms of sources and potentials, we see a reversal of roles:

$$
\begin{aligned}
\delta W_E & = -\int d^3x \boldsymbol{\nabla}\phi \cdot \delta \boldsymbol{D} = \int d^3x \phi \delta \rho \\
\delta W_M & = \int d^3x \boldsymbol{H} \cdot \nabla \times \delta \boldsymbol{A} = \int d^3x \boldsymbol{J} \cdot \delta \boldsymbol{A}
\end{aligned}
\tag{286}
$$

This reversal makes good physical sense, because the condition for zero work on the magnetic field is that $\boldsymbol{B}$ (or $\boldsymbol{A}$) doesn't change. The zero work condition on the electric field is that the charges on the various capacitor plates are unchanged. The internal energy U, identified by $\delta U = \delta W_E + \delta W_M$, can be used to calculate the electromagnetic force on an object as

$$
F_\xi = -\frac{\partial U}{\partial \xi}\Big|_{Q,\boldsymbol{A}}
\tag{287}
$$

where $\delta \xi$ is a virtual displacement of the body and the subscripts denote that the derivative is evaluated at constant charge and magnetic potential. This result can be neatly summarized

$$
\begin{aligned}
dU & = -Fd\xi + \int d^3x (\boldsymbol{E} \cdot \delta \boldsymbol{D} + \boldsymbol{H} \cdot \delta \boldsymbol{B}) \\
& = -Fd\xi + d\int d^3x (\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) - \int d^3x \delta \boldsymbol{E} \cdot \boldsymbol{D} - \int d^3x \delta \boldsymbol{H} \cdot \boldsymbol{B} \\
d\tilde{U} & = -Fd\xi - \int d^3x \delta \boldsymbol{E} \cdot \boldsymbol{D} - \int d^3x \delta \boldsymbol{H} \cdot \boldsymbol{B} \\
\tilde{U} & = U - \int d^3x (\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})
\end{aligned}
\tag{288}
$$

From a practical point of view it is easier both theoretically and experimentally to deal with systems with fixed electric potentials and currents. The above relations show for example

that

$$F_\xi = -\frac{\partial \tilde{U}}{\partial \xi}\bigg|_{\phi, \boldsymbol{J}} \tag{289}$$

When we have linear response, $\boldsymbol{D} = \epsilon \boldsymbol{E}$, $\boldsymbol{B} = \mu \boldsymbol{H}$, one can explicitly evaluate

$$U = \frac{1}{2}\int d^3x(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{H}\cdot\boldsymbol{B}) = -\tilde{U} \tag{290}$$

5.7 Models of χ_m

As for the electric susceptibility we begin by studying the magnetic dipole moment induced by applying a magnetic field to an atomic system, which for simplicity we take to be a single charged particle of mass M bound in a potential $V(\boldsymbol{r})$. The magnetic field enters the Hamiltonian through the vector potential in a way dictated by the Lagrangian

$$L = \frac{1}{2}M\dot{\boldsymbol{r}}^2 - V + q\dot{\boldsymbol{r}}\cdot\boldsymbol{A} \tag{291}$$

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = M\dot{\boldsymbol{r}} + q\boldsymbol{A} \tag{292}$$

$$H = \frac{1}{2M}(\boldsymbol{p} - q\boldsymbol{A})^2 + V = \frac{\boldsymbol{p}^2}{2M} - \frac{q}{2M}(\boldsymbol{p}\cdot\boldsymbol{A} + \boldsymbol{A}\cdot\boldsymbol{p}) + \frac{q^2\boldsymbol{A}^2}{2M} + V \tag{293}$$

A key difference from the electric field is that H has a term quadratic in the field. For a similar reason the current density operator and the magnetic dipole moment operator have linear terms in the field

$$\boldsymbol{J} = q\dot{\boldsymbol{r}}\delta(\boldsymbol{r} - \boldsymbol{r}(t)) = \frac{q}{M}(\boldsymbol{p} - q\boldsymbol{A})\delta(\boldsymbol{r} - \boldsymbol{r}(t)) \tag{294}$$

$$\boldsymbol{m} = \frac{q}{2M}\boldsymbol{r}\times(\boldsymbol{p} - q\boldsymbol{A}) = \frac{q\boldsymbol{L}}{2M} - \frac{q^2\boldsymbol{r}\times\boldsymbol{A}}{2M}$$

$$\rightarrow -\frac{\partial H}{\partial \boldsymbol{B}}, \qquad \text{for} \quad \boldsymbol{A} = \boldsymbol{B}\times\boldsymbol{r}/2 \tag{295}$$

where the last line holds for a uniform field $\boldsymbol{A} = \boldsymbol{B}\times\boldsymbol{r}/2$. For reasons similar to the electric susceptibility case, the first term makes a positive contribution to the magnetic susceptibility. However the second term, which had no analogue in the electric case, contributes negatively. To see this, specialize to a uniform magnetic field for which we can put $\boldsymbol{A} = \boldsymbol{B}\times\boldsymbol{r}/2$.

$$-\langle G|\frac{q^2\boldsymbol{r}\times\boldsymbol{A}}{2M}|G\rangle \rightarrow -\frac{q^2}{4M}\langle G|(r^2\boldsymbol{B} - (\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{r})|G\rangle = -\frac{q^2}{6M}\langle r^2\rangle\boldsymbol{B} \tag{296}$$

in a rotationally invariant ground state. In a rotationally invariant ground state the first term $(q\boldsymbol{L}/M)$ in $\boldsymbol{m}$ gives a zero contribution, even in second order perturbation theory, because $\boldsymbol{L}|G\rangle = 0$ means $\langle E_n|\boldsymbol{L}|G\rangle = 0$ for all n! Thus the ground state is intrinsically diamagnetic. On the other hand, in a state with angular momentum, the first term in $\boldsymbol{m}$

gives the system a permanent magnetic moment, which at finite temperature contributes a positive (paramagnetic) contribution to the susceptibility, in a way entirely analogous to the contribution of polar molecules to the electric susceptibility. The electron's intrinsic spin also produces a magnetic moment which contributes paramagnetically.

The dominance of the diamagnetic $\boldsymbol{A}$ term in the dipole moment relied on the very special nature of the ground state–it is not only rotationally invariant but it is separated by an energy gap from excited states. Thus at sufficiently low temperatures $kT \ll$ Gap so all atoms are effectively in their ground state and hence rotationally invariant (which forces the $\boldsymbol{L}$ term to be zero). In contrast, consider a free charged particle in a magnetic field, where the unperturbed system has no gap. On the one hand it would seem that the diamagnetic term could get huge because $\langle \boldsymbol{r}^2 \rangle$ is unbounded. But if we consider the system at finite temperature, the *classical* partition function is completely independent of the magnetic field.

$$
\begin{aligned}
Z &= \int \frac{d^3r \, d^3p}{(2\pi\hbar)^3} \exp\left\{-\frac{\beta}{2M}(\boldsymbol{p} - q\boldsymbol{A})^2\right\} = \int \frac{d^3r \, d^3p}{(2\pi\hbar)^3} \exp\left\{-\frac{\beta}{2M}\boldsymbol{p}^2\right\} \\
&= \frac{V}{(2\pi\hbar)^3}\left(\frac{2\pi M}{\beta}\right)^{3/2}
\end{aligned}
\tag{297}
$$

by a shift of the momentum integration variable. As far as **classical** statistical mechanics goes the effects of the two terms in $\boldsymbol{m}$ cancel!

However, quantum mechanically this cancellation is incomplete, so a weak diamagnetic behavior remains, known as Landau diamagnetism. A nice treatment of the quantum mechanics is given in the Landau-Lifshitz course textbook on quantum mechanics. A uniform magnetic field in the z direction can be represented by the vector potential $A_y = Bx, A_x = A_z = 0$, so $(\boldsymbol{p} - q\boldsymbol{A})^2 = p_x^2 + p_z^2 + (p_y - qBx)^2$, which shows that the x variable feels a harmonic oscillator potential with spring constant $k = q^2B^2/m$ and equilibrium position $x_0 = p_y/qB$. The frequency of the oscillator is $\omega = \sqrt{k/M} = qB/M$. Thus the energy eigenvalues are simply

$$
E_n(p_z) = (n + 1/2)\hbar\omega + p_z^2/2M.
\tag{298}
$$

Each of these levels has a degeneracy labelled by p_y. To put the system in a heat bath, we also need to put it in a finite volume, say a cube of length L. Then $0 \leq x_0 \leq L$ implies $0 \leq p_y \leq qBL$ so the degeneracy is $\int dy \, dp_y/(2\pi\hbar) = qBL^2/(2\pi\hbar)$. Then

$$
\begin{aligned}
Z &= \frac{qBL^2}{(2\pi\hbar)} \sum_{n=0}^{\infty} \int \frac{L \, dp_z}{2\pi\hbar} e^{-\beta((n+1/2)\hbar\omega + p_z^2/2M)} = \frac{qBL^3}{(2\pi\hbar)^2}\sqrt{\frac{2\pi M}{\beta}} \frac{1}{2\sinh(\beta\hbar qB/2M)} \\
&\sim \frac{qBL^3}{(2\pi\hbar)^2}\sqrt{\frac{2\pi M}{\beta}} \frac{M}{qB\beta\hbar}\left(1 - \frac{\beta^2 q^2 B^2 \hbar^2}{24M^2}\right) = \frac{L^3}{(2\pi\hbar)^3}\left(\frac{2\pi M}{\beta}\right)^{3/2}\left(1 - \frac{\beta^2 q^2 B^2 \hbar^2}{24M^2}\right) \\
F &\equiv -\frac{1}{\beta}\ln Z \sim F_0 + \frac{\beta q^2 B^2 \hbar^2}{24M^2} = F_0 + \frac{q^2\hbar^2}{24M^2 kT}B^2
\end{aligned}
\tag{299}
$$

A nifty thing to know about statistical systems is that the induced magnetic moment can be obtained by differentiating the free energy in the presence of a magnetic field with respect to the magnetic field. This is based on the identity

$$\boldsymbol{m} = -\frac{\partial H}{\partial \boldsymbol{B}} \tag{300}$$

where H is the Hamiltonian for a particle moving in a uniform magnetic field, for which $\boldsymbol{A} = \boldsymbol{B} \times \boldsymbol{r}/2$:

$$
\begin{aligned}
H &= \frac{\boldsymbol{p}^2}{2m} - \frac{q}{2M}\boldsymbol{L}\cdot\boldsymbol{B} + \frac{q^2}{8M}(r^2\boldsymbol{B}^2 - (\boldsymbol{r}\cdot\boldsymbol{B})^2) \\
\frac{\partial H}{\partial \boldsymbol{B}} &= -\frac{q}{2M}\boldsymbol{L} + \frac{q^2}{4M}(r^2\boldsymbol{B} - \boldsymbol{r}\cdot\boldsymbol{B}\boldsymbol{r}) = -\boldsymbol{m}
\end{aligned} \tag{301}
$$

Then the thermal average

$$\langle\boldsymbol{m}\rangle = \frac{1}{Z}\mathrm{Tr}\,\boldsymbol{m}e^{-\beta H} = \frac{1}{\beta Z}\frac{\partial Z}{\partial \boldsymbol{B}} = -\frac{\partial F}{\partial \boldsymbol{B}} = -\frac{q^2\hbar^2}{12M^2 kT}\boldsymbol{B} \tag{302}$$

the minus sign here shows that the system is diamagnetic ($\mu < \mu_0$) A similar formula applies to the induced electric dipole moment, except of course the sign is positive $\epsilon > \epsilon_0$. This feature brings new insight to the theorem that $\epsilon > \epsilon_0$ whereas μ can be larger or smaller than μ_0. The electric field appears linearly in the Hamiltonian, the $\boldsymbol{E}^2$ in the free energy comes entirely from second order perturbation theory. Second order shifts in the ground state energy are always negative, so the induced electric dipole moment will be parallel to the field, i.e. $\epsilon > \epsilon_0$. In contrast, there are both linear and quadratic magnetic field terms in the Hamiltonian. The quadratic term is explicitly positive (diamagnetic), whereas the quadratic correction that comes from second order perturbation of the linear term is negative (paramagnetic). These two terms are in competition and which one wins depends on the material.

5.8   Faraday's Law

Faraday's law of induction for a stationary circuit is a direct consequence of Stoke's theorem applied to the Maxwell equation involving $\nabla \times \boldsymbol{E}$:

$$\mathrm{Emf} = \oint_C d\boldsymbol{l}\cdot\boldsymbol{E} = \int dS\hat{n}\cdot\nabla\times\boldsymbol{E} = -\int dS\hat{n}\cdot\frac{\partial\boldsymbol{B}}{\partial t} = -\frac{d\Phi_B}{dt} \tag{303}$$

where the closed contour C coincides with the circuit and the surface integral is over any surface spanning C. Here Emf stands for electromotive force, which by definition is the effective potential difference that drives currents between two points. In other words if the circuit has resistance R, it will carry a current $I = \mathrm{Emf}/R$.

Written in the form

$$\mathrm{Emf} = -\frac{d\Phi_B}{dt} \tag{304}$$

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Faraday's law is generally valid even when parts of the circuit are in motion. In that case

$$
\begin{aligned}
\frac{d\Phi_B}{dt} &= \int dS\hat{n} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \oint (\boldsymbol{v} \times d\boldsymbol{l}) \cdot \boldsymbol{B} \\
&= \int dS\hat{n} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \oint d\boldsymbol{l} \cdot (\boldsymbol{v} \times \boldsymbol{B}) \\
\text{Emf} &= -\int dS\hat{n} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \oint d\boldsymbol{l} \cdot (\boldsymbol{v} \times \boldsymbol{B}) \qquad (305)
\end{aligned}
$$

The second term which is motional Emf is just the contribution due to the Lorentz force law. Alternatively we can think of it in terms of the effective electric field in the rest frame of the moving circuit, $\boldsymbol{E}' \approx \boldsymbol{v} \times \boldsymbol{B}$. This second point of view will be more transparent when we discuss the Lorentz invariance of Maxwell's equations.

The minus sign on the right of Faraday's law is a statement of Lenz's law: induced currents due to a changing magnetic field are always in the direction which opposes the change.

5.9   Inductance

Faraday's law will play an essential role in our understanding of electromagnetic waves. But there are also practical implications for ac circuits which we briefly touch on here. Time varying currents produce time varying magnetic fields which inevitably induce Emf 's in the circuits carrying the currents. To deal with this effect one introduces the inductor as a basic circuit element.
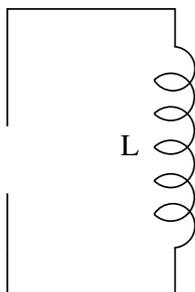
Such an element produces an Emf $= -LdI/dt$, where L is called the inductance of the circuit element. The sense of the Emf is best decided on the basis of Lenz's Law: it must drive current in the direction that opposes change. We will not dwell on circuit problems here, but rather focus on insights we can bring to bear on underlying physics of inductance.

A good way to understand how to calculate inductance is to exploit its contribution to the energy stored in magnetic fields. We have already developed formulas for magnetic energy in terms of fields. The definition of inductance shows that the power delivered to an inductor is

$$
\frac{dU}{dt} = I\text{Emf} = LI\frac{dI}{dt} = \frac{d}{dt}\frac{1}{2}LI^2, \qquad U = \frac{1}{2}LI^2 \qquad (306)
$$

More generally, suppose we have a set of circuits carrying the currents I_k. We know that the magnetic field is linear in the currents and the energy must therefore be quadratic in the currents. Thus we can write

$$U = \frac{1}{2} \sum_{j,k} L_{jk} I_j I_k \tag{307}$$

which defines the inductance matrix of the set of circuits. The diagonal element $L_{jj} \equiv L_j$ is called the self inductance of circuit j and the off diagonal elements $L_{ij} \equiv M_{ij}$ is called the mutual inductance of circuit j on circuit i (or vice versa, since $M_{ij} = M_{ji}$.

The most systematic approach to calculating the inductance matrix is to solve for the fields produced by all the currents and then plug the fields into the formula for the field energy which reduces to a bilinear form in the currents from which all the inductances can be read off. If each circuit is simply a current carrying loop C_j we can write

$$
\begin{aligned}
U &= \frac{1}{2} \int d^3x \, \boldsymbol{H} \cdot \boldsymbol{B} = \frac{1}{2} \int d^3x \, \boldsymbol{J} \cdot \boldsymbol{A} \\
&= \frac{1}{2} \sum_j I_j \oint_{C_j} d\boldsymbol{l} \cdot \boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{8\pi} \sum_j I_j \cdot \oint_{C_j} d\boldsymbol{l} \sum_k I_k \oint_{C_k} d\boldsymbol{l}' \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|}
\end{aligned} \tag{308}
$$

$$L_j = \frac{\mu_0}{4\pi} \oint_{C_j} \oint_{C_j} \frac{d\boldsymbol{l} \cdot d\boldsymbol{l}'}{|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad M_{jk} = \frac{\mu_0}{4\pi} \oint_{C_j} d\boldsymbol{l} \cdot \oint_{C_k} \frac{d\boldsymbol{l}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{309}$$

We see that there is a ln divergence in the self inductance from the part of the integrand when the points coincide. This divergence is a defect of the thin wire approximation and requires that the wire not have zero thickness. The mutual inductance has a well defined thin wire approximation, assuming the two current loops are disjoint from one another. We can write it alternatively as

$$M_{jk} = \frac{1}{I_k} \oint_{C_j} d\boldsymbol{l} \cdot \boldsymbol{A}_k = \frac{1}{I_k} \int_{S_j} dS \hat{n} \cdot \boldsymbol{B}_k = \frac{1}{I_k} \Phi_{jk} \tag{310}$$

Where $\boldsymbol{A}_k, \boldsymbol{B}_k$ are the potential and field produced by loop C_k, and Φ_{jk} is the magnetic flux of the field produced by C_k enclosed by C_j.

To get a meaningful calculation of the self inductance we need to consider a wire of finite radius b. So we consider a current loop of size a in a wire of radius $b << a$. The last problem in set 11 steps you through an explicit analysis for a circular loop. Here we are only concerned with analyzing the $\ln(b/a)$ dependence of the inductance, which is universal for all smooth loops. We start by considering a long straight wire of radius b, whose field is easily determined by Ampere's law (Problem 28)

$$\boldsymbol{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\phi}, \quad \rho > b, \qquad \boldsymbol{B} = \frac{\mu_0 I \rho}{2\pi b^2} \hat{\phi}, \quad \rho < b \tag{311}$$

by directly integrating $\nabla \times \boldsymbol{A} = \boldsymbol{B}$, we find $A_x = A_y = 0$, and

$$A_z = -\frac{\mu_0 I}{2\pi} \ln \frac{\rho}{\rho_0}, \quad \rho > b; \qquad A_z = \frac{\mu_0 I}{4\pi} \left(1 - \frac{\rho^2}{b^2}\right) - \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho_0}, \quad \rho < b \tag{312}$$

We have chosen integration constants so that A_z is continuous at $\rho = b$. For the infinitely long straight wire the constant ρ_0 is arbitrary. However, suppose the wire is actually in a smooth closed loop of typical size a. Then,

$$\boldsymbol{A(r)} \;=\; \frac{\mu_0}{4\pi} \int d^3 x' \frac{\boldsymbol{J(r')}}{|\boldsymbol{r} - \boldsymbol{r'}|} \tag{313}$$

would have no such arbitrary integration constant, since it would be uniquely fixed by requiring $\boldsymbol{A} \to 0$ at large distances from the loop. If we have found $\boldsymbol{A}$ for this loop, then if we examine it near the wire it must behave (since $b << a$) as it does in the straight wire case but with ρ_0 uniquely determined, however possibly having different (unique) values $\rho_0 = af(s)$ at different points s of the loop. For the circular loop in the assigned problem, symmetry determines ρ_0 to be the same for all points on the loop.

One can now use the formula

$$
\begin{aligned}
U \;&=\; \frac{1}{2} \int d^3 x \boldsymbol{J} \cdot \boldsymbol{A} \approx \frac{I^2}{2\pi b^2} \oint ds\, 2\pi \int_0^b \rho d\rho \left[\frac{\mu_0}{4\pi} \left(1 - \frac{\rho^2}{b^2} \right) - \frac{\mu_0}{2\pi} \ln \frac{b}{af(s)} \right] \\
&\approx\; \frac{\mu_0 I^2}{4\pi} \oint ds \left[\left(\frac{1}{2} - \frac{1}{4} \right) - \ln \frac{b}{af(s)} \right] = \frac{\mu_0 I^2}{4\pi} \left[C \left(\frac{1}{4} + \ln \frac{a}{b} \right) + \oint ds \ln f(s) \right] \\
L \;&=\; \frac{\mu_0 C}{2\pi} \left[\ln \frac{a}{b} + \frac{1}{4} + \frac{1}{C} \oint ds \ln f(s) \right]
\end{aligned}
\tag{314}
$$

where C is the circumference of the current loop. The approximation here is $b << a$ where a is the smallest length scale characterizing the loop, among the size and radii of curvature. The term involving $f(s)$ is a finite constant depending on the geometry of the loop. The coefficient of $\ln(a/b)$ is universal for all shapes with no length scale smaller than a.

5.10   Conductivity and the Quasi-static approximation

We have so far only considered conductors in the context of electrostatics, where we assume that any motion of charges has ceased. But in magnetostatics we have been assuming steady currents in which charges keep up a steady motion which is the source of magnetic fields. We now turn briefly to the physics underlying the production of steady currents in conductors.

In an elementary context we are all familiar with Ohm's Law $I = V/R$ which relates the current through a resistive material to the applied voltage. If electrons in a metal were truly free, an applied electric field would cause them to continually accelerate, leading to an increasing current. However, normal metals have imperfections which scatter electrons, with the effect that an electron accelerating anti-parallel to an electric field is scattered into a new direction, losing its correlation with the field. Thus one has a limiting drift velocity $\boldsymbol{v} = -e\boldsymbol{E}t_{\mathrm{f}}$, where t_f is the mean time the electron travels freely before a collision, that determines an effective steady current. The electrons are actually moving at much higher speeds but in random directions. Phenomenologically, the effect is described by a generalization of Ohm's Law $\boldsymbol{J} = \sigma\boldsymbol{E}$ where σ is called the conductivity of the material.

Insulators are of course characterized by $\sigma = 0$. if a uniform current density is traveling through a block with cross sectional area A and length L, the total current is

$$I \;\; = \;\; JA = \sigma A E = \frac{\sigma A}{L} V, \qquad R = \frac{L}{\sigma A} \equiv \rho \frac{L}{A} \tag{315}$$

where $\rho = 1/\sigma$ is called the resistivity.

With the concept of conductivity, we can now incorporate this new feature in the full Maxwell equations. The one involving the current becomes

$$\nabla \times \boldsymbol{H} \;\; = \;\; \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} = \sigma \boldsymbol{E} + \frac{\partial \boldsymbol{D}}{\partial t} \tag{316}$$

We will see later that the conductivity can be related to the imaginary part of the frequency dependent dielectric constant.[3]

Here we just follow the implications of conductivity in the quasi-static approximation, which amounts to dropping the $\partial \boldsymbol{D}/\partial t$ term in the Maxwell equations. To understand the significance of the approximation, let $L(T)$ be a typical distance (time) over which the fields vary significantly. Then the Faraday equation shows that $E = O(L/T)B$. Plugging this into the Ampere equation shows that $\partial \boldsymbol{D}/\partial t = O(\epsilon L/T^2)B$ whereas the curl term of the same equation is $O(1/\mu L)B$. The ratio of the former to the latter terms is thus $O(\epsilon \mu L^2/T^2)$. Thus the time derivative term is negligible provided $L/T << 1/\sqrt{\epsilon \mu} = c$ the speed of light. Alternatively we may say that the fields are essentially static over the time it takes light to travel across the system. It is often an excellent approximation because the speed of light is so large. It breaks down if one can drive the fields at frequencies comparable to c/L. For $L = 1$m this is 3×10^8Hz$= 300$MHz!

Returning to the approximation, we solve the homogeneous equations by introducing potentials in Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$

$$\boldsymbol{B} \;\; = \;\; \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{318}$$

which involve no approximation and plug these into the inhomogeneous equations, assuming $\rho = 0$ and dropping $\partial \boldsymbol{D}/\partial t$,

$$-\nabla^2 \phi \;\; = \;\; 0, \qquad -\nabla^2 \boldsymbol{A} = -\sigma \mu \frac{\partial \boldsymbol{A}}{\partial t} \tag{319}$$

Let's drive the fields at frequency ω, so we put $\boldsymbol{A} = \boldsymbol{A}_0(\boldsymbol{r})e^{-i\omega t}$, so $\boldsymbol{A}_0$ satisfies

$$(\nabla^2 + i\omega\mu\sigma)\boldsymbol{A}_0 = 0 \tag{320}$$

---

[3]Very roughly we can see this here by putting $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \boldsymbol{r} e^{-i\omega t}$ and evaluating

$$\frac{\partial \boldsymbol{D}}{\partial t} = -i\omega\epsilon(\omega)\boldsymbol{E} \tag{317}$$

A static conductivity would emerge as the $\omega \to 0$ limit $-i\omega\epsilon(\omega) \to \sigma$. The sophisticated point is that a non-zero limit would imply that $\epsilon(\omega)$ has a pole at $\omega = 0$. More on this later!

For simple example, we assume only z dependence so $-\boldsymbol{A}_0'' = i\omega\mu\sigma\boldsymbol{A}_0$, or

$$\boldsymbol{A}_0 = \boldsymbol{C}\exp\frac{\pm z i\sqrt{\omega\mu\sigma}(1+i)}{\sqrt{2}} = \boldsymbol{C}\exp\frac{z\sqrt{\omega\mu\sigma}(\mp 1\pm i)}{\sqrt{2}}$$

The solution which describes a conducting material filling the half space $z > 0$ and driven by a magnetic field at $z = 0$ with time dependence $\boldsymbol{C}\cos\omega t = \operatorname{Re}\boldsymbol{C}e^{-i\omega t}$, is

$$\boldsymbol{A} = \boldsymbol{C}\operatorname{Re}\left(e^{-z\sqrt{\omega\mu\sigma}(1-i)/\sqrt{2}-i\omega t}\right) = \boldsymbol{C}e^{-z\sqrt{\omega\mu\sigma/2}}\cos\left(\omega t - z\sqrt{\omega\mu\sigma/2}\right) \qquad (321)$$

where we rejected the exponentially increasing solution since the driving field is localized at $z = 0$. Clearly, the field is exponentially damped in z, penetrating only a distance $\delta = \sqrt{2/\omega\mu\sigma}$, called the skin depth. Note the frequency dependence: at low frequency the fields penetrate farther. The square root dependence on the time scale is typical of diffusion.

6 Maxwell's Equations

6.1 Ampere-Maxwell Equation in electromagnetic materials

We finally confront the physics of the full set of Maxwell's equations, including the famous "displacement current" term added to the right side of the ampere equation. In vacuum the equation reads

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}_{\text{tot}} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{322}$$

When we modified the Ampere equation to account for a material with induced magnetization we neglected this term, so let us first repair this omission. The procedure is to break up the current into free plus bound parts $\boldsymbol{J}_{\text{tot}} = \boldsymbol{J} + \boldsymbol{J}_{\text{bound}}$. We identified $\nabla \times \boldsymbol{M}$ as the contribution of the magnetization to the bound current density. But there is also a contribution to the bound current density from the polarization $\boldsymbol{P}$ when it depends on time. We can infer this contribution from current conservation by remembering that the polarization contributes $\rho_{\text{bound}} = -\nabla \cdot \boldsymbol{P}$ to the bound charge density. Then

$$\frac{\partial \rho_{\text{bound}}}{\partial t} = -\nabla \cdot \frac{\partial \boldsymbol{P}}{\partial t} \equiv -\nabla \cdot \boldsymbol{J}_{\text{bound}} \tag{323}$$

Thus the total bound current density is $\boldsymbol{J}_{\text{bound}} = \nabla \times \boldsymbol{M} + \partial \boldsymbol{P}/\partial t$. Then remembering the definitions $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$, we arrive at the complete set of Maxwell equations in materials

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{324}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \nabla \cdot \boldsymbol{D} = \rho \tag{325}$$

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}), \qquad \boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{326}$$

We also remember that for isotropic linear response we have $\boldsymbol{D} = \epsilon \boldsymbol{E}, \boldsymbol{B} = \mu \boldsymbol{H}$. And the equations involving the sources imply charge conservation $\dot{\rho} + \nabla \cdot \boldsymbol{J} = 0$. The homogeneous Maxwell equations can be solved once and for all by setting

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t} \tag{327}$$

Then for materials that exhibit isotropic linear response, the Maxwell Equations reduce to

$$\nabla \cdot \left(-\epsilon \nabla \phi - \epsilon \frac{\partial \boldsymbol{A}}{\partial t} \right) = \rho$$

$$\nabla \times \left(\frac{1}{\mu} \nabla \times \boldsymbol{A} \right) = \boldsymbol{J} + \frac{\partial}{\partial t} \left(-\epsilon \nabla \phi - \epsilon \frac{\partial \boldsymbol{A}}{\partial t} \right) \tag{328}$$

In vacuum, or materials where ϵ and μ are constants, we can get a dramatic simplification by imposing a convenient gauge condition. there are two popular choices. First, the Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, for which we have

$$-\nabla^2 \phi = \frac{\rho}{\epsilon}, \qquad \left(-\nabla^2 + \epsilon\mu\frac{\partial^2}{\partial t^2}\right) \boldsymbol{A} = \mu\left(\boldsymbol{J} - \epsilon\nabla\frac{\partial\phi}{\partial t}\right) \tag{329}$$

The first equation determines ϕ completely in terms of ρ:

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon}\int d^3x' \frac{\rho(\boldsymbol{r}', t)}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{330}$$

and taking the divergence of both sides of the second equation shows that charge must be conserved:

$$0 = \nabla \cdot \boldsymbol{J} + \frac{\partial\rho}{\partial t} \tag{331}$$

Clearly $\boldsymbol{A}$ satisfies a wave equation with wave speed $c' = 1/\sqrt{\epsilon\mu}$. In vacuum this is the speed of light $c = 1/\sqrt{\epsilon_0\mu_0}$.

The second popular choice is the Lorenz gauge:

$$\nabla \cdot \boldsymbol{A} + \mu\epsilon\frac{\partial\phi}{\partial t} = \nabla \cdot \boldsymbol{A} + \frac{1}{c'^2}\frac{\partial\phi}{\partial t} = 0 \tag{332}$$

In the Lorenz gauge Maxwell's equations read

$$\left(-\nabla^2 + \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\phi = \frac{\rho}{\epsilon}, \qquad \left(-\nabla^2 + \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\boldsymbol{A} = \mu\boldsymbol{J} \tag{333}$$

so both ϕ and $\boldsymbol{A}$ satisfy wave equations.

6.2 Energy and Momentum and Their Conservation

We begin by retracing the work energy theorem with due care keeping surface terms:

$$\begin{aligned}
\boldsymbol{J} \cdot \boldsymbol{E} &= (\nabla \times \boldsymbol{H}) \cdot \boldsymbol{E} - \boldsymbol{E} \cdot \frac{\partial\boldsymbol{D}}{\partial t} \\
&= \nabla_i(\boldsymbol{H} \times \boldsymbol{E})^i + \boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) - \boldsymbol{E} \cdot \frac{\partial\boldsymbol{D}}{\partial t} \\
&= \nabla \cdot (\boldsymbol{H} \times \boldsymbol{E}) - \boldsymbol{H} \cdot \frac{\partial\boldsymbol{B}}{\partial t} - \boldsymbol{E} \cdot \frac{\partial\boldsymbol{D}}{\partial t} \tag{334} \\
&\to -\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \frac{\partial u}{\partial t} \tag{335}
\end{aligned}$$

where $u = (\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B})/2$ is the energy density stored in the fields, and the last form assumes linear response, $\boldsymbol{D} = \epsilon\boldsymbol{E}$, $\boldsymbol{B} = \mu\boldsymbol{H}$ with ϵ, μ independent of time. Since the left side

is the density of work per unit time that the fields do on the system, we see that $\boldsymbol{S} \equiv \boldsymbol{E} \times \boldsymbol{H}$ has the interpretation as the flux density of energy (in exact parallel to the interpretation of $\boldsymbol{J}$ as the flux (current) density of charge). In other words the surface integral $\oint dS\hat{n} \cdot \boldsymbol{S}$ is the rate of total energy flowing through the surface. In particular, if this rate stays finite as the enclosing surface is taken to infinity, it gives the rate at which energy is radiated from the system.

A similar set of manipulations on the force density leads to an identification of the field momentum:

$$
\begin{aligned}
\rho\boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} &= \boldsymbol{E}\nabla \cdot \boldsymbol{D} + (\nabla \times \boldsymbol{H}) \times \boldsymbol{B} - \frac{\partial \boldsymbol{D}}{\partial t} \times \boldsymbol{B} \\
&= -\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) - \boldsymbol{D} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{E}\nabla \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \nabla \boldsymbol{H} - B_i \nabla H_i \\
&= -\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \boldsymbol{D} \cdot \nabla \boldsymbol{E} - D_i \nabla E_i + \boldsymbol{E}\nabla \cdot \boldsymbol{D} + \boldsymbol{B} \cdot \nabla \boldsymbol{H} - B_i \nabla H_i \\
&= -\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \nabla_i(D_i\boldsymbol{E} + B_i\boldsymbol{H}) - \boldsymbol{H}\nabla \cdot \boldsymbol{B} - D_i \nabla E_i - B_i \nabla H_i \\
&= -\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \nabla_i(D_i\boldsymbol{E} + B_i\boldsymbol{H}) - D_i \nabla E_i - B_i \nabla H_i \\
&\to -\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \nabla_i(D_i\boldsymbol{E} + B_i\boldsymbol{H}) - \frac{1}{2}\nabla(\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H}) \qquad (336)
\end{aligned}
$$

where the last form assumes linear response, with ϵ, μ independent of $\boldsymbol{r}$. Since the left side is the force density that the fields exert on the rest of the system, we see that $\boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B}$ is the momentum density contained in the fields. Notice that for linear response $\boldsymbol{g} = \epsilon\mu\boldsymbol{E} \times \boldsymbol{H} = \boldsymbol{S}/c'^2$. This is exactly as expected if the energy density is $|c'\boldsymbol{g}|$ and travels at speed c'! We also see that $T^{ij} = E^i D^j + H^i B^j - \delta_{ij}u$ is the stress tensor or minus the jth component of the flux density of the ith component of momentum. Rewriting the force equation as

$$
\nabla_j T^{ij} = \rho E^i (+\boldsymbol{J} \times \boldsymbol{B})^i + \frac{\partial g^i}{\partial t} \qquad (337)
$$

We recognize the right side as the total force density: the first two terms give the force density on charged particles and last term gives the time rate of change of the momentum density carried by the fields. Integrating both sides oover a volume V, the right side is the ratee of change of the total momentum carried by particles and fields. Therefore by Gauss' theorem the left side gives the force the fields exert on the volume V as

$$
F^i = \oint_{\partial V} dS n_j T^{ij} \qquad (338)
$$

which justifies the interpretation of T as the stress tensor.

In summary we now have identified the energy and momentum stored in the fields

$$
U = \frac{1}{2}\int d^3x(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}), \quad \boldsymbol{P} = \int d^3x \boldsymbol{D} \times \boldsymbol{B} = \int d^3x \epsilon\mu\boldsymbol{E} \times \boldsymbol{H} \qquad (339)
$$

where we have assumed linear response. We also bring together the conservation laws we have obtained for the dentities ρ, u, g^k of these charge, energy and momentum respectively:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{J} = 0 \tag{340}$$

$$\frac{\partial u}{\partial t} + \nabla \cdot \boldsymbol{S} = -\boldsymbol{E} \cdot \boldsymbol{J}, \qquad \boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{341}$$

$$\frac{\partial g^k}{\partial t} - \nabla_l T^{kl} = -\rho E^k - (\boldsymbol{J} \times \boldsymbol{B})^k, \qquad T^{kl} = E^k D^l + H^k B^l - \delta_{kl} u \tag{342}$$

The non-zero right sides of the last two equations simply reflect the changing energy and momentum of the mechanical particles in the system.

6.3   Solving Maxwell's equations with Green Functions

Following the Green function method we used in statics, we seek to build a solution of the wave equation with an arbitrary source

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) \psi = J(\boldsymbol{r}, t) \tag{343}$$

from a Green function solution with a delta function source

$$\left(-\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \right) G(\boldsymbol{r}, t; \boldsymbol{r}', t') = \delta(\boldsymbol{r} - \boldsymbol{r}')\delta(t - t')$$

$$\psi(\boldsymbol{r}, t) = \int d^3r' dt' G(\boldsymbol{r}, t; \boldsymbol{r}', t') J(\boldsymbol{r}', t') \tag{344}$$

The trick is to find a useful form for G. Here we find the Green function for empty space, i.e. with no surfaces on which we must impose boundary conditions.

Unlike the Laplace equation, the wave equation involves a hyperbolic differential operator– the time and space derivatives enter with opposite signs. This means that there are interesting solutions of the homogeneous equation that stay bounded at infinity. In particular one can define several interesting Green functions. For example, if we solve for G by Fourier transformation we find

$$G(\boldsymbol{r}, t; \boldsymbol{r}', t') = \int \frac{d^3k\, d\omega}{(2\pi)^4} \frac{e^{i\boldsymbol{k} \cdot (\boldsymbol{r} - \boldsymbol{r}') - i\omega(t - t')}}{\boldsymbol{k}^2 - \omega^2/c^2} \tag{345}$$

and we see that the denominator can vanish making the integral ill-defined. One popular way to make it well-defined is to add a small imaginary part $\pm i\epsilon$ to the denominator, while keeping the integration variables real.

But we can also find an explicit solution analogous to the Coulomb potential for the Laplace equation. We start by doing a Fourier transform in time only.

$$G(\boldsymbol{r}, t; \boldsymbol{r}', t') = \int \frac{d\omega}{2\pi} G(\boldsymbol{r}, \boldsymbol{r}'; \omega) e^{-i\omega(t - t')} \tag{346}$$

$$\left(-\nabla^2 - \frac{\omega^2}{c^2} \right) G(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \delta(\boldsymbol{r} - \boldsymbol{r}') \tag{347}$$

If ω were zero we know that $1/4\pi|\boldsymbol{r} - \boldsymbol{r}'|$ would be the answer. It is not hard to show that either of the simple modifications

$$G_{\pm}(\boldsymbol{r}, \boldsymbol{r}'; \omega) = \frac{e^{\pm i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c}}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \qquad (348)$$

takes care of the ω^2/c^2 term[4]. Feeding this back into the Fourier transform we discover the remarkable formulas

$$G_{\pm}(\boldsymbol{r}, t; \boldsymbol{r}', t') = \frac{\delta(\pm|\boldsymbol{r} - \boldsymbol{r}'|/c - (t - t'))}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \qquad (349)$$

The delta function in the numerator says that a flash at $\boldsymbol{r}', t'$ will be observed at $\boldsymbol{r}$ at a different time $t = t' \pm |\boldsymbol{r} - \boldsymbol{r}'|/c$. G_+ describes our usual notion of causality since the observation is after the flash. It is called the retarded Green function. G_- is the time-reversed version which says the observation precedes the flash. It is called the advanced Green function. They each have their uses. But for now we prefer the more intuitive retarded Green function, which will always be understood if the $+$ subscript is deleted.

The solution of the wave equation with arbitrary source can now be written

$$\psi(\boldsymbol{r}, t) = \psi_0(\boldsymbol{r}, t) + \int d^3x' \frac{J(\boldsymbol{r}', t - |\boldsymbol{r} - \boldsymbol{r}'|/c)}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \equiv \psi_0(\boldsymbol{r}, t) + \int d^3x' \frac{J_{\text{ret}}}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \qquad (350)$$

where the t' integral was done using the delta function. In the last form we have introduce the shorthand $\{\}_{\text{ret}}$ to indicate that the source is evaluated at the retarded time. The source dependence is very reminiscent of the Coulomb potential except that its time argument is $|\boldsymbol{r} - \boldsymbol{r}'|/c$ earlier than t. This time shift is just the time it would take light to travel from the source point to the observation point: the retardation is very reasonable from a causal point of view. For the case of a harmonically varying source $J \propto e^{-i\omega t}$, the retardation shift just provides the phase $e^{-i\omega|\boldsymbol{r}-\boldsymbol{r}'|/c}$ in the integrand. Notice that this phase would be very small in the quasi-static approximation. It makes a big qualitative difference in the fields however. Without it fields would behave as $1/r^2$ at large distances. But the gradient of the retardation phase provides a contribution to the field that behaves as ω/cr, a slower fall off. Thus at large enough distances $r >> c/\omega$ that contribution dominates. The existence of $1/r$ behavior in the fields implies a $1/r^2$ behavior in the energy density and the flow of energy density: indeed it is precisely the falloff needed to allow for radiation of energy to long distances (as we shall soon see).

We can use this Green function to find the source dependence of any quantity that satisfies the wave equation. In particular, in Lorenz gauge:

$$\phi = \int d^3x' \frac{\rho_{\text{ret}}/\epsilon}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}, \qquad \boldsymbol{A} = \int d^3x' \frac{\mu \boldsymbol{J}_{\text{ret}}}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|} \qquad (351)$$

---

[4]It is easiest to use a coordinate system with $\boldsymbol{r}' = 0$. Then $-\nabla^2(e^{ikr}/4\pi r) = -k^2(e^{ikr}/4\pi r) + \delta(\boldsymbol{r})$

We could either use these expressions to derive $\boldsymbol{E}, \boldsymbol{B}$, or alternatively use the fact that the electric and magnetic fields also satisfy wave equations of their own:

$$\left(-\nabla^2 + \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\boldsymbol{E} = -\nabla\frac{\rho}{\epsilon} - \mu\frac{\partial\boldsymbol{J}}{\partial t}, \qquad \left(-\nabla^2 + \epsilon\mu\frac{\partial^2}{\partial t^2}\right)\boldsymbol{B} = \mu\nabla\times\boldsymbol{J} \qquad (352)$$

Care must be taken when dealing with an expression like $[\nabla' f]_{\mathrm{ret}}$:

$$\nabla'[f]_{\mathrm{ret}} = \nabla'[f(\boldsymbol{r}', t - |\boldsymbol{r}-\boldsymbol{r}'|/c)] = [\nabla' f]_{\mathrm{ret}} + \frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|c}\left[\frac{\partial f}{\partial t'}\right]_{\mathrm{ret}} \qquad (353)$$

6.4 Fields with Harmonic Time Dependence

When the fields vary harmonically in time, it is most convenient to represent them as complex fields, after which we may simply take their real parts:

$$\boldsymbol{A}(\boldsymbol{r}, t) \rightarrow \boldsymbol{A}(\boldsymbol{r})e^{-i\omega t}, \qquad \boldsymbol{B}(\boldsymbol{r}, t) \rightarrow \boldsymbol{B}(\boldsymbol{r})e^{-i\omega t}, \qquad \boldsymbol{E}(\boldsymbol{r}, t) \rightarrow \boldsymbol{E}(\boldsymbol{r})e^{-i\omega t} \qquad (354)$$

Many physical questions about harmonically varying fields can be adequately answered by taking a time average over a period of oscillation. Of course, the time average of any of the fields listed above is zero. But the time average of a quantity that is bilinear in these fields is non zero. Call F, G any pair of the fields and consider

$$\begin{aligned} \mathrm{Re}F \times \mathrm{Re}G &= \frac{1}{4}(F(\boldsymbol{r})e^{-i\omega t} + F^*(\boldsymbol{r})e^{+i\omega t})(G(\boldsymbol{r})e^{-i\omega t} + G^*(\boldsymbol{r})e^{+i\omega t}) \\ &= \frac{1}{4}(F(\boldsymbol{r})G^*(\boldsymbol{r}) + F^*(\boldsymbol{r})G(\boldsymbol{r}) + F(\boldsymbol{r})G(\boldsymbol{r})e^{-2i\omega t} + F*(\boldsymbol{r})G^*(\boldsymbol{r})e^{+2i\omega t}) \\ &= \frac{1}{2}\mathrm{Re}(F(\boldsymbol{r})G^*(\boldsymbol{r}) + F(\boldsymbol{r})G(\boldsymbol{r})e^{-2i\omega t}) \end{aligned} \qquad (355)$$

Clearly the time average of the last term is zero, so we may write

$$\langle\mathrm{Re}F \times \mathrm{Re}G\rangle_t = \frac{1}{2}\mathrm{Re}[F(\boldsymbol{r})G^*(\boldsymbol{r})] \qquad (356)$$

Since energy, momentum, the Poynting vector, and the stress tensor are all bilinear in the fields, we may painlessly read off the time averages of these quantities without keeping track of the detailed time dependence.

As an example of this procedure, which we will study in much more detail in the future, suppose we wish to know the time averaged power radiated by a harmonically varying source. for this we need $\mathrm{Re}(\boldsymbol{E}\times\boldsymbol{H}^*)/2$ evaluated at large distances from the source. So we assume complex sources $\boldsymbol{J}(\boldsymbol{r})e^{-i\omega t}$, $\rho(\boldsymbol{r})e^{-i\omega t}$, which by current conservation are related by $\rho = \nabla\cdot\boldsymbol{J}/i\omega$. Then in Lorenz gauge, we have

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\int d^3x'\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}e^{-i\omega(t-|\boldsymbol{r}-\boldsymbol{r}'|/c)} \sim \frac{\mu_0}{4\pi r}e^{i\omega(r/c-t)}\int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}$$

$$\phi \sim \frac{1}{4\pi\epsilon_0 r}\frac{e^{i\omega(r/c-t)}}{i\omega}\int d^3x'\nabla'\cdot \boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c} = \frac{e^{i\omega(r/c-t)}}{4\pi\epsilon_0 rc}\hat{r}\cdot \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}$$

$$\sim \frac{\mu_0 c e^{i\omega(r/c-t)}}{4\pi r}\hat{r}\cdot \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}$$

$$\boldsymbol{B} \sim \frac{i\omega\mu_0}{4\pi rc}e^{i\omega(r/c-t)}\hat{r}\times \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}$$

$$\boldsymbol{E} \sim \frac{i\omega\mu_0}{4\pi r}e^{i\omega(r/c-t)}\int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c} - \frac{i\omega\mu_0\hat{r}}{4\pi r}e^{i\omega(r/c-t)}\hat{r}\cdot \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}$$

$$\sim \frac{i\omega\mu_0 e^{i\omega(r/c-t)}}{4\pi r}\left[\int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c} - \hat{r}\hat{r}\cdot \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}\right]$$

$$\sim -\frac{i\omega\mu_0 e^{i\omega(r/c-t)}}{4\pi r}\hat{r}\times \left(\hat{r}\times \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\omega\hat{r}\cdot\boldsymbol{r}'/c}\right) = -c\hat{r}\times \boldsymbol{B} \qquad (357)$$

It is now convenient to define the Fourier transform

$$\tilde{\boldsymbol{J}}(\boldsymbol{k}) \equiv \int d^3x'\boldsymbol{J}(\boldsymbol{r}')e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \qquad (358)$$

so we can write succinctly

$$\boldsymbol{E}\times \boldsymbol{H}^* \sim \hat{r}\frac{\omega^2\mu_0}{16\pi^2 r^2 c}\left[\tilde{\boldsymbol{J}}(\frac{\omega}{c}\hat{r})\cdot \tilde{\boldsymbol{J}}^*(\frac{\omega}{c}\hat{r}) - \hat{r}\cdot \tilde{\boldsymbol{J}}(\frac{\omega}{c}\hat{r})\hat{r}\cdot \tilde{\boldsymbol{J}}^*(\frac{\omega}{c}\hat{r})\right]$$

$$dP = dS\hat{n}\cdot \frac{1}{2}\mathrm{Re}\boldsymbol{E}\times \boldsymbol{H}^* = r^2 d\Omega\hat{r}\cdot \frac{1}{2}\mathrm{Re}\boldsymbol{E}\times \boldsymbol{H}^*$$

$$\frac{dP}{d\Omega} \sim \frac{\omega^2\mu_0}{32\pi^2 c}\left[\tilde{\boldsymbol{J}}(\frac{\omega}{c}\hat{r})\cdot \tilde{\boldsymbol{J}}^*(\frac{\omega}{c}\hat{r}) - \hat{r}\cdot \tilde{\boldsymbol{J}}(\frac{\omega}{c}\hat{r})\hat{r}\cdot \tilde{\boldsymbol{J}}^*(\frac{\omega}{c}\hat{r})\right] \qquad (359)$$

The last line gives the power radiated into solid angle $d\Omega$. In the low frequency limit $\tilde{\boldsymbol{J}}$ is simply the volume integral of the current density, which vanishes for magnetostatics. When there is time dependence we have the identity

$$\nabla\cdot (r^k\boldsymbol{J}) = \boldsymbol{J} - r^k\frac{\partial\rho}{\partial t} \to \boldsymbol{J}^k + i\omega r^k\rho \qquad (360)$$

for harmonic time variation. Thus, as $\omega \to 0$, $\tilde{\boldsymbol{J}}(\omega/c) \sim -i\omega\int d^3x\boldsymbol{r}\rho = -i\omega\boldsymbol{p}$ where $\boldsymbol{p}$ is the electric dipole moment of the charge distribution. In this limit

$$\frac{dP}{d\Omega} \sim \frac{\omega^4\mu_0}{32\pi^2 c}[\boldsymbol{p}\cdot \boldsymbol{p}^* - \hat{r}\cdot \boldsymbol{p}\hat{r}\cdot \boldsymbol{p}^*]$$

$$\sim \frac{\omega^4\mu_0}{32\pi^2 c}|\boldsymbol{p}|^2\sin^2\theta \qquad (361)$$

where the last line only holds if $\boldsymbol{p}$ is parallel to $\boldsymbol{p}^*$, when θ is the angle between $\hat{r}$ and $\boldsymbol{p}$. If $\boldsymbol{p}\times \boldsymbol{p}^* \neq 0$, then one has to use the expression on the next to the last line. In this "electric

dipole approximation" the total power radiated is easily obtained

$$
\begin{aligned}
P &= \frac{\omega^4 \mu_0}{32\pi^2 c} \int d\Omega(|\boldsymbol{p}|^2 - \hat{r} \cdot \boldsymbol{p}\hat{r} \cdot \boldsymbol{p}^*) = \frac{\omega^4 \mu_0}{32\pi^2 c}|\boldsymbol{p}|^2 4\pi(1 - 1/3) \\
&= \frac{\omega^4 \mu_0}{12\pi c}|\boldsymbol{p}|^2 = \frac{\omega^4}{12\pi\epsilon_0 c^3}|\boldsymbol{p}|^2 \to \frac{\alpha\omega^4\hbar}{3c^2}\langle \boldsymbol{r}\rangle^2
\end{aligned}
\tag{362}
$$

where $\alpha = e^2/4\pi\epsilon_0\hbar c$ is the dimensionless fine structure constant which has the approximate value $1/137$.

6.5   The Dirac Monopole

There is no evidence at all that magnetic monopoles exist in nature. But Dirac has shown us how to describe them, if they should one day be found. To start we recognize that even without monopoles, we can create the magnetic field of a monopole $\boldsymbol{B} = g\boldsymbol{r}/4\pi r^3$ almost everywhere by introducing an infinitely long and thin magnet or solenoid which ends at the origin. Such a field can be represented by a vector potential $\boldsymbol{A}$ everywhere outside the magnet.

$$
\nabla \times \boldsymbol{A} = \frac{g\boldsymbol{r}}{4\pi r^3}, \qquad \theta \neq \pi
\tag{363}
$$

where we assumed the magnet is along the negative z-axis.

The role of the magnet in this situation is to avoid a contradiction with Gauss' law for a monopole which reads

$$
\oint dS\hat{n} \cdot \boldsymbol{B} = g
\tag{364}
$$

if the surface encloses the monopole. On the other hand $\oint dS\hat{n}\cdot\nabla\times\boldsymbol{A} = 0$. The contradiction is resolved because there is a uniform magnetic field down the magnet whose flux cancels the flux g from the monopole field.

So the trick of describing a monopole is to remove the physical effects of the magnet (or the return flux). That is we write

$$
\boldsymbol{B}_{\text{monopole}} = \nabla \times \boldsymbol{A} + \boldsymbol{B}_{\text{Dirac string}}
\tag{365}
$$

where the second term is chosen to precisely cancel the return flux, which can be arranged to follow any path from the origin to ∞.

A way to mathematically characterize the subtlety is to use Stokes theorem to write

$$
\oint_C d\boldsymbol{l} \cdot \boldsymbol{A} = \int dS\hat{n} \cdot \nabla \times \boldsymbol{A}
\tag{366}
$$

where the surface is any one that spans the closed curve C. The line integral on the left is sensitive to the return flux in the magnet since it is picked up whenever the surface is

pierced by the magnet. This is true no matter how far the curve C stays from the magnet. The question is whether this line integral can be measured. Classical charged particles do not see $\boldsymbol{A}$ directly–they only respond to $\boldsymbol{B}$. However $\boldsymbol{A}$ does directly appear in the Schrodinger equation, and we can see that the line integral can be measured by a subtle quantum interference experiment (Aharanov-Bohm effect).

As a first step, consider the Schrodinger equation with $\boldsymbol{A} = \nabla \Lambda$ a pure gauge. Then

$$(\boldsymbol{p} - q\boldsymbol{A})\psi = (-\hbar i \nabla - q\nabla\Lambda)\psi = -i\hbar\nabla(e^{-iq\Lambda/\hbar}\psi) \tag{367}$$

so $\psi_\Lambda = e^{-iq\Lambda/\hbar}\psi$ satisfies the free Schrodinger equation. If $\boldsymbol{B} = 0$ everywhere we can solve $\nabla\Lambda = \boldsymbol{A}$ with a single valued Λ. But suppose $\boldsymbol{B} = 0$ outside a thin tube which contains magnetic flux Φ. Then when we try to remove $\boldsymbol{A}$ from the Schrodinger equation by solving $\nabla\Lambda = \boldsymbol{A}$ we will find that Λ is not single valued because $\Lambda(2\pi) - \Lambda(0) = \oint d\boldsymbol{l} \cdot \boldsymbol{A} = \Phi$ by Stokes theorem. This means that $\psi_\Lambda(2\pi) = e^{-iq\Phi/\hbar}\psi_\Lambda(0)$. This phase would be measured in a quantum interference experiment. However we see that there will be no measurable effect if Φ is quantized according to

$$\Phi = \frac{2\pi N\hbar}{q} \tag{368}$$

The flux carried by the Dirac string is $\Phi = g$, so it will be invisible to such an experiment provided the Dirac quantization holds:

$$\frac{gq}{4\pi} = \frac{N\hbar}{2}. \tag{369}$$

6.6   Symmetries of Maxwell Equations

It is generally useful to understand the symmetries of the dynamics of a system. We implicitly exploit the symmetries under translations and rotations when we pick the most convenient coordinate system in the solution of problems.

Translation Symmetry in Space and Time

A symmetry is identified by finding transformations of the dynamical variables that leave the equations invariant in form. First consider translations in space $\boldsymbol{r} \rightarrow \boldsymbol{r} + \boldsymbol{a}$:

$$\boldsymbol{E}_a(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r} - \boldsymbol{a}, t), \qquad \boldsymbol{B}_a(\boldsymbol{r}, t) = \boldsymbol{B}(\boldsymbol{r} - \boldsymbol{a}, t) \tag{370}$$

In vacuum or in materials in which ϵ, μ are constants in space, $\boldsymbol{D}$ and $\boldsymbol{H}$ are similarly transformed. Then because the Maxwell equations involve no spatially dependent coefficients they are solved by the translated fields if they are solved by the original ones. When we have translational invariance we can choose any point as the origin of our coordinate system. When there are spatially dependent external sources or fields, translation invariance in the dynamical fields alone is broken. Invariance of the dynamics under translations in time

$$\boldsymbol{E}_a(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, t - a), \qquad \boldsymbol{B}_a(\boldsymbol{r}, t) = \boldsymbol{B}(\boldsymbol{r}, t - a) \tag{371}$$

will similarly hold if μ, ϵ are time independent and there are no time dependent sources.

Rotation Symmetry

Invariance under rotations is a bit more subtle, because the fields and ∇ are rotational vectors. This means that their components rotate among themselves along with the coordinates. A rotation of coordinates can be represented as $r^{i\prime} = R_{ij}r^j$, where the matrix R satisfies $R_{ik}R_{jk} = \delta_{ij}$. In matrix notation we can write this as $RR^T = I$, where T signifies transpose. The significance of this constraint on R is that it ensures that scalar products of vectors are rotational invariants:

$$\boldsymbol{v}' \cdot \boldsymbol{w}' = v'^i w'^i = R_{ij}R_{ik}v^j w^k = \delta_{jk}v^j w^k = v^j w^j = \boldsymbol{v} \cdot \boldsymbol{w} \tag{372}$$

The cross product of two vectors also transforms as a vector, which we see as follows

$$
\begin{aligned}
(\boldsymbol{v}' \times \boldsymbol{w}')^i &= \epsilon_{ijk}v'^j w'^k = \epsilon_{ijk}R_{jl}R_{km}v^l w^m = R_{ir}R_{nr}\epsilon_{njk}R_{jl}R_{km}v^l w^m \\
&= R_{ir}\epsilon_{rlm}\det R v^l w^m = \det R R_{ir}(\boldsymbol{v} \times \boldsymbol{w})^r \\
&= R_{ir}(\boldsymbol{v} \times \boldsymbol{w})^r, \qquad \det R = 1
\end{aligned}
\tag{373}
$$

From $RR^T = I$ it follows that $(\det R)^2 = 1$. So $\det R = \pm 1$. We call rotations with $\det R = 1$ proper rotations, which are those that can be continuously deformed to the identity. They form the group $SO(3)$. Those with $\det R = -1$ can be expressed as a proper rotation times $-I$, inversion. Also note that

$$\nabla'_k = \nabla_l \partial r^l / \partial r'^k = (R^{-1})_{lk}\nabla_l = R_{kl}\nabla_l \tag{374}$$

which shows that ∇ transforms like a vector under rotations. Then the fields transform as follows under proper rotations

$$
\begin{aligned}
E_R^i(\boldsymbol{r}', t) &= R_{ij}E^j(\boldsymbol{r}, t) = R_{ij}E^j(R^{-1}\boldsymbol{r}', t) \\
B_R^i(\boldsymbol{r}', t) &= R_{ij}B^j(\boldsymbol{r}, t) = R_{ij}B^j(R^{-1}\boldsymbol{r}', t)
\end{aligned}
\tag{375}
$$

From all these relations it follows that Maxwell's equations are invariant under proper rotations, provided ρ transforms as a rotational scalar field and $\boldsymbol{J}$ transforms as a vector field.

Parity (Inversion) Invariance

Instead of discussing all improper rotations it is enough to discuss the discrete inversion transformation $\boldsymbol{r}' = -\boldsymbol{r}$. This operation is also known as parity. It is equivalent to reflection in a mirror followed by a rotation. Take a mirror in the xy-plane. Then reflection in it means $z' = -z$, $(x, y)' = (x, y)$. If we now rotate π radians about the z-axis we arrive at full inversion $\boldsymbol{r}' = -\boldsymbol{r}$. Vectors which flip sign under parity are called polar vectors. Vectors which do not flip sign under parity are called axial vectors or pseudovectors. The cross product of two polar vectors is an axial vector. Clearly ∇ is a polar vector. Then examination of the Faraday Maxwell equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{376}$$

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shows that it will be invariant under parity, provided $\boldsymbol{E}$ and $\boldsymbol{B}$ transform oppositely under parity. Assuming that charge doesn't flip sign when reflected in a mirror, we should choose $\boldsymbol{E}$ to be a polar vector and $\boldsymbol{B}$ an axial vector. This is consistent with the Ampere Maxwell equation because $\nabla \times \boldsymbol{H}$ is then a polar vector as is $\partial \boldsymbol{D}/\partial t$ and $\boldsymbol{J}$. With all these assignments Maxwell's equations are invariant under parity:

$$\boldsymbol{E}'(\boldsymbol{r}',t) = -\boldsymbol{E}(\boldsymbol{r},t) = -\boldsymbol{E}(-\boldsymbol{r}',t), \qquad \boldsymbol{B}'(\boldsymbol{r}',t) = +\boldsymbol{B}(\boldsymbol{r},t) = +\boldsymbol{B}(-\boldsymbol{r}',t) \qquad (377)$$

Note that should magnetic monopoles exist, magnetic charge would have to flip sign when reflected in a mirror. More precisely, if you apply the parity transform on the magnetic monopole solution, you create a monopole solution for the opposite monopole charge. If, in addition, you flip the monopole charge you get a new solution for the original sign of the charge. Parity invariance means that this new solution should be related to the original one by a gauge transformation (see Problem 47 (J,6.19)).

Time Reversal Invariance

Finally we come to time reversal $t \to -t$. By time reversal invariance we simply mean that if a motion is a solution of the dynamics, so is its time reversal. This is true of Newton's equations simply because it is second order in time derivatives. Maxwell's equations involve single time derivatives so the discussion is more subtle. Again looking at the Faraday equation, we see that invariance will hold if $\boldsymbol{E}$ and $\boldsymbol{B}$ transform oppositely under time reversal. It is natural to assign $\boldsymbol{B}$ to be odd under time reversal.

$$\boldsymbol{E}'(\boldsymbol{r},t') = \boldsymbol{E}(\boldsymbol{r},-t'), \qquad \boldsymbol{B}'(\boldsymbol{r},t') = -\boldsymbol{B}(\boldsymbol{r},-t') \qquad (378)$$

That will make the Lorentz force even under time reversal, so Newton's equation stays invariant. It also jibes with the fact that time reversal would flip the direction of all currents so the induced magnetic fields would also flip. Thus we see that Maxwell's equations, with these assignments, are invariant under time reversal.

When we bring in quantum mechanics parity and time reversal invariance can have powerful implications. As a simple example consider the conditions under which a point particle with spin can have a permanent electric or magnetic dipole moment. A spinless particle can have neither kind of dipole moment by rotational invariance. The dipole moment, being a vector, must be proportional to the spin operator. But angular momentum is even under parity and odd under time reversal. The magnetic dipole moment has precisely these properties, so an elementary particle with spin can have a permanent magnetic dipole moment consistently with both parity and time reversal invariance. On the other hand the electric dipole moment is odd under parity and even under time reversal. Thus a permanent dipole moment can occur only if both parity and time reversal invariance are violated. In fact both are violated in Nature but time reversal violation is much tinier than parity violation, so an electric dipole moment of the neutron has yet to be detected.

Boost Invariance

In addition to invariance under translations, rotations, parity, and time reversal, Maxwell's equations are also invariant under a change of inertial frames (boost transformations). These

are the famous Lorentz transformations, e.g. a boost in the x direction is given by

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \qquad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \qquad y' = y \qquad z' = z \qquad (379)$$

Maxwell's equations will be invariant under these boost transformations if the fields transform as

$$
\begin{aligned}
E^{y\prime} &= \frac{E^y - vB^z}{\sqrt{1 - v^2/c^2}}, \qquad E^{z\prime} = \frac{E^z + vB^y}{\sqrt{1 - v^2/c^2}}, \qquad E^{x\prime} = E^x, \qquad B^{x\prime} = B^x \\
B^{y\prime} &= \frac{B^y + vE^z/c^2}{\sqrt{1 - v^2/c^2}}, \qquad B^{z\prime} = \frac{B^z - vE^y/c^2}{\sqrt{1 - v^2/c^2}}
\end{aligned} \qquad (380)
$$

The clash between these boost transformation rules and the corresponding boost transformations in Newton's mechanics is what led Einstein to invent relativity. We defer a more detailed discussion of special relativity to next semester (Jackson, Chapter 11).

7   Electromagnetic Plane Waves

When we discussed electrostatics, the electric field seemed little more than a nifty way to describe the force between charged particles. But Maxwell's equations reveal that the electric and magnetic fields are dynamical entities in their own right, carrying their own energy and momentum, which can be transferred to electromagnetically active particles. Total energy and momentum are conserved, provided that those of the fields are included in the accounting.

The most direct way to focus on the dynamical properties of the fields themselves is to study solutions of the sourceless Maxwell equations. As we shall now make clear the simplest such solutions are plane waves, by which we mean that the space and time dependence of the fields is of the form $e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}$. The use of complex exponentials rather than sines and cosines is a great convenience. We will therefore study complex solutions from which real solutions can be obtained by superposing a complex solution with its complex conjugate. The dynamical equations determine a relation between ω and $\boldsymbol{k}$ which we express by writing $\omega(\boldsymbol{k})$.

An exact plane wave is, of course, an idealization that is never realized in the real world. This means that realistic solutions will actually be a superposition of plane waves (a wave packet), expressed as a Fourier transform:

$$\psi(\boldsymbol{r},t) \;=\; \int d^3k f(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega(\boldsymbol{k})t} \tag{381}$$

where ψ represents any of the fields of interest. To approximate a plane wave, we choose f to be very narrowly peaked about a specific wave number $\boldsymbol{k}_0$. Then it is meaningful to expand $\omega(\boldsymbol{k})$ in a Taylor series about $\boldsymbol{k}_0$:

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k}_0) + (\boldsymbol{k}-\boldsymbol{k}_0)\cdot\nabla_k\omega(\boldsymbol{k}_0) + O((\boldsymbol{k}-\boldsymbol{k}_0)^2) \tag{382}$$

If we neglect quadratic terms and higher we obtain the approximation

$$\begin{aligned}
\psi(\boldsymbol{r},t) &\approx e^{-i(\omega_0+\boldsymbol{k}_0\cdot\nabla\omega(\boldsymbol{k}_0))t} \int d^3k f(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\nabla\omega(\boldsymbol{k}_0)t)} \\
&\approx e^{-i(\omega_0+\boldsymbol{k}_0\cdot\nabla\omega(\boldsymbol{k}_0))t}\psi(\boldsymbol{r}-\nabla\omega(\boldsymbol{k}_0)t,0)
\end{aligned} \tag{383}$$

so in this approximation the packet shape moves without deformation at the *group velocity* $\boldsymbol{v}_g = \nabla\omega(\boldsymbol{k}_0)$. Including the quadratic and higher terms leads to the phenomenon of dispersion in which the shape deforms as it travels along. In particular it will tend to spread at very late times.

Now let's look specifically at electromagnetic plane waves:

$$\boldsymbol{E}(\boldsymbol{r},t) = \boldsymbol{E}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t}, \qquad \boldsymbol{B}(\boldsymbol{r},t) = \boldsymbol{B}_0 e^{i\boldsymbol{k}\cdot\boldsymbol{r}-i\omega t} \tag{384}$$

Then the sourceless Maxwell equations reduce to

$$\boldsymbol{k}\cdot\boldsymbol{E}_0 = \boldsymbol{k}\cdot\boldsymbol{B}_0 = 0, \qquad i\boldsymbol{k}\times\boldsymbol{E}_0 = i\omega\boldsymbol{B}_0, \qquad i\boldsymbol{k}\times\boldsymbol{B}_0 = -\epsilon\mu i\omega\boldsymbol{E}_0 \tag{385}$$

$$\boldsymbol{B}_0 = \frac{\boldsymbol{k}\times\boldsymbol{E}_0}{\omega}, \qquad \epsilon\mu\omega^2\boldsymbol{E}_0 = -\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}_0) = \boldsymbol{k}^2\boldsymbol{E}_0, \qquad \omega = \frac{|\boldsymbol{k}|}{\sqrt{\epsilon\mu}} = c'|\boldsymbol{k}| \tag{386}$$

We see that $\boldsymbol{k}$ and $\boldsymbol{E}_0$ completely determine the solution. Here $c' = c/n$, with the index of refraction $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$, is the speed of light in the material. When n is independent of frequency, this is also the group velocity. When n depends on ω the group velocity differs from c' which is then called the phase velocity.

The direction of $\boldsymbol{E}$ defines the polarization of the wave. Polarization is always transverse to the direction of wave propagation. If the two components of $\boldsymbol{E}_0$ are relatively real $\mathrm{Re}\boldsymbol{E}$ will point in this fixed direction, and the wave is said to be linearly polarized. If there is a relative phase between the two components, the wave is elliptically polarized. For example, take $\boldsymbol{k} = k\hat{z}$, and $\boldsymbol{E}_0 = \hat{x} + re^{i\chi}\hat{y}$ Then

$$
\begin{aligned}
\mathrm{Re}\boldsymbol{E} &= \hat{x}\cos(kz - \omega t) + r\hat{y}\cos(\chi + kz - \omega t) \\
&= (\hat{x} + r\cos\chi\hat{y})\cos(kz - \omega t) - r\hat{y}\sin\chi\sin(kz - \omega t). \quad (387)
\end{aligned}
$$

The time averaged energy flow of a plane wave is obtained from the Poynting vector

$$
\langle\boldsymbol{S}\rangle = \frac{1}{2}\mathrm{Re}\boldsymbol{E}\times\boldsymbol{H}^* = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\mathrm{Re}\boldsymbol{E}_0\times(\hat{k}\times\boldsymbol{E}_0^*) = \hat{k}\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|\boldsymbol{E}_0|^2 = \hat{k}c'\langle u\rangle \quad (388)
$$

Because the Poynting vector measures energy flow, it is a natural measure of the intensity of the plane wave.

7.1   Reflection and Refraction at a Plane Interface

The behavior of a plane wave impinging on an interface between two materials is determined by the same continuity conditions we have used in electro and magneto statics. The normal components of $\boldsymbol{B}, \boldsymbol{D}$ are continuous and the parallel components of $\boldsymbol{E}, \boldsymbol{H}$ are continuous, provided there are no surface currents or surface charge density. Applying these conditions will determine the reflected and refracted wave in terms of the incident wave.

Consider a single planar interface spanning the xy-plane, $z = 0$. The materials below and above the plane have properties $\epsilon, \mu, n = c\sqrt{\epsilon\mu}$ and $\epsilon', \mu', n' = c\sqrt{\epsilon'\mu'}$ respectively. We make the plane wave ansatz, with common frequency $\omega = |\boldsymbol{k}|c/n$.

$$
\boldsymbol{E} = \begin{cases} \left(\boldsymbol{E}_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \boldsymbol{E}_r e^{i\boldsymbol{k}_r\cdot\boldsymbol{r}}\right)e^{-i\omega t} & z < 0 \\ \boldsymbol{E}_t e^{i\boldsymbol{k}_t\cdot\boldsymbol{r} - i\omega t} & z > 0 \end{cases} \quad (389)
$$

$$
= e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r} - i\omega t}\begin{cases} \boldsymbol{E}_i e^{ikz} + \boldsymbol{E}_r e^{-ikz} & z < 0 \\ \boldsymbol{E}_t e^{ik'z} & z > 0 \end{cases} \quad (390)
$$

$$
\boldsymbol{B} = \begin{cases} \left(\boldsymbol{B}_i e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + \boldsymbol{B}_r e^{i\boldsymbol{k}_r\cdot\boldsymbol{r}}\right)e^{-i\omega t} & z < 0 \\ \boldsymbol{B}_t e^{i\boldsymbol{k}_t\cdot\boldsymbol{r}}e^{-i\omega t} & z > 0 \end{cases} \quad (391)
$$

$$
= e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r} - i\omega t}\begin{cases} \boldsymbol{B}_i e^{ikz} + \boldsymbol{B}_r e^{-ikz} & z < 0 \\ \boldsymbol{B}_t e^{ik'z} & z > 0 \end{cases} \quad (392)
$$

$$
\boldsymbol{B}_i = \frac{\boldsymbol{k}\times\boldsymbol{E}_i}{\omega}, \qquad \boldsymbol{B}_r = \frac{\boldsymbol{k}_r\times\boldsymbol{E}_r}{\omega}, \qquad \boldsymbol{B}_t = \frac{\boldsymbol{k}_t\times\boldsymbol{E}_t}{\omega} \quad (393)
$$

Translation invariance in time and in the xy plane allows the common factor $e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r}_\perp - i\omega t}$ in all components of the plane wave: $\boldsymbol{k} = \boldsymbol{k}_\perp + k\hat{z}$, $\boldsymbol{k}_r = \boldsymbol{k}_\perp - k\hat{z}$, $\boldsymbol{k}_t = \boldsymbol{k}_\perp + k'\hat{z}$. From $|\boldsymbol{k}| = n\omega/c$

and $|\boldsymbol{k}_t| = n'\omega/c$, we see that $k = \sqrt{n^2\omega^2/c^2 - \boldsymbol{k}_\perp^2}$ and $k' = \sqrt{n'^2\omega^2/c^2 - \boldsymbol{k}_\perp^2}$. Measuring the angles of incidence i, reflection r, and transmission t relative to the normal to the interface, the equality of the transverse components of $\boldsymbol{k}$ for all three components together with the propotionality of their magnitudes to the index of refraction $n = \sqrt{\epsilon\mu}c$ or $n' = \sqrt{\epsilon'\mu'}c$ implies

$$n\sin r \;\; = \;\; n\sin i = n'\sin t, \qquad r = i, \qquad \sin t = \frac{n}{n'}\sin i \tag{394}$$

The last of these is just Snell's law of refraction. Since the incident wave vector $\boldsymbol{k}$ has real components, $\boldsymbol{k}_\perp^2 \leq n^2\omega^2/c^2$. But if $n' < n$, there can be angles where k' is imaginary (total internal reflection).

The reason that it is correct to interpret the respective terms in the superposition of plane waves as incident, reflected, and transmitted has to do with how those terms behave inside a wave packet with narrowly defined $\boldsymbol{k}$. So we choose $f(\boldsymbol{k})$ to be sharply peaked about $\boldsymbol{k}_0$, multiply the plane wave solution by f and integrate over $\boldsymbol{k}$:

$$\boldsymbol{E} \;\; \approx \;\; \begin{cases} \left(\boldsymbol{E}_i F(z - v_g^z t) + \boldsymbol{E}_r(\boldsymbol{k}_{0r}) F(-z - v_g^z t)\right) & z < 0 \\ \boldsymbol{E}_t(\boldsymbol{k}_{0t}) F(zn'/n - v_g^z t) & z > 0 \end{cases} \tag{395}$$

Here F is the narrow packet approximation to $\int d^3k\, f(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot\boldsymbol{r} - i\omega(\boldsymbol{k})t}$. We we only display the $z - v_g t$ dependence since that is the only coordinate that varies from one term to the other. If we focus on this motion in the z direction, we compare the modulating factors

$$F(z - v_g^z t), \qquad F(-z - v_g^z t), \qquad F(zn'/n - v_g^z t) \tag{396}$$

which apply in the regions $z < 0$, $z < 0$, $z > 0$ respectively. We set up the experiment by F having support when its argument is near $-L$. Then at $t = 0$ only the first factor contributes. The second and third factor then have support near $z = L$ and $zn'/n = -L$, where neither are part of the solution. But then for $v_g^z t > 2L$ the first factor's support is in the region $z > 0$ where it is no longer a part of the solution. On the other hand the second factor then has support for $z < -L$ and the third has support at $zn'/n > L$, where they are both part of the solution. Thus at early times only the $\boldsymbol{E}_i$ term contributes and at late times only the $\boldsymbol{E}_{r,t}$ terms contribute, exactly as we wished to interpret them.

At $z = 0$ we have the boundary conditions that the tangential components of $\boldsymbol{E}, \boldsymbol{H}$ are continuous and the normal components of $\boldsymbol{D}, \boldsymbol{B}$ are continuous. Since tangential and normal components satisfy different continuity conditions, it is convenient to work them out for the two cases where $\boldsymbol{E}$ or $\boldsymbol{B}$ is parallel to the interface (perpendicular to the plane of incidence, which we take to be the yz-plane: $\boldsymbol{k}_\perp = k_\perp \hat{y}$). A general polarization can be obtained from these two independent special cases by linear superposition.

First take the case $\boldsymbol{E} = E\hat{x}$ parallel to the interface. Continuity of $\boldsymbol{E}$ is then simply $E_i + E_r = E_t$. Continuity of the $B^z = E(\boldsymbol{k}_\perp \times \hat{x})^z/\omega$ is an automatic consequence of the continuity of E and the equality of the transverse components of $\boldsymbol{k}_{i,r,t}$. Continuity of the tangential component of $\boldsymbol{H} = H^y\hat{y} + H^z\hat{z}$ then reads

$$B_i^y + B_r^y = \frac{\mu}{\mu'}B_t^y, \tag{397}$$

The y component of $\boldsymbol{k} \times \boldsymbol{E}$ is $k^z E$ since $\boldsymbol{E} = E\hat{x}$. Then we use $k_r^z = -k$, $k_t^z = k' = kn'\cos t/n\cos i$ to write the two continuity equations as

$$E_i + E_r = E_t, \qquad E_i - E_r = \frac{\mu}{\mu'}\frac{k'}{k}E_t = \frac{\mu}{\mu'}\frac{n'\cos t}{n\cos i}E_t$$

$$E_t = E_i\frac{2n\mu'\cos i}{n\mu'\cos i + n'\mu\cos t}, \qquad E_r = E_i\frac{n\mu'\cos i - n'\mu\cos t}{n\mu'\cos i + n'\mu\cos t}$$

$$n'\cos t = \sqrt{n'^2 - n'^2\sin^2 t} = \sqrt{n'^2 - n^2\sin^2 i} \tag{398}$$

In the second case it is $\boldsymbol{H}$ that is parallel to the interface. Now $\boldsymbol{E} = E^y\hat{y} + E^z\hat{z}$, and the $\boldsymbol{B} = B\hat{x}$. Continuity of tangential $\boldsymbol{H}$ is $B_i + B_r = B_t\frac{\mu}{\mu'}$. Then $\boldsymbol{E} = -\boldsymbol{k} \times \boldsymbol{B}/\omega\epsilon\mu$ or $\boldsymbol{D} = -\boldsymbol{k} \times \boldsymbol{H}/\omega$. D^z is now automatically continuous, and the continuity of tangential $\boldsymbol{E}$ together with the first equation then read

$$B_i - B_r = B_t\frac{n}{n'}\frac{\cos t}{\cos i}, \qquad B_i + B_r = B_t\frac{\mu}{\mu'}$$

$$B_t = B_i\frac{2\mu'n'\cos i}{\mu n'\cos i + \mu'n\cos t}, \qquad B_r = B_i\frac{\mu n'\cos i - \mu'n\cos t}{\mu n'\cos i + \mu'n\cos t}, \tag{399}$$

By remembering that $|\boldsymbol{B}| = |\boldsymbol{E}|n/c$ we see that these equations are, up to possible sign differences, the same as Jackson's Eq. 7.41. In fact our signs are correct whereas Jackson's are either confusing or wrong. The difference is sharpest for the case of the reflected wave at normal incidence. Our formulation shows that if $n' > n$, the electric field of the reflected wave is opposite to that in the incident wave, while the magnetic field doesn't change sign. If $n < n'$ the the sign flipping fields are reversed. If one takes Jackson's signs literally, his Eq (7.41) is in direct contradiction with his Eq. 7.39 at normal incidence. Our corresponding equations are consistent and correct. The electric fields for i, r, t in the second case are pointed in the respective directions:

$$-\boldsymbol{k} \times \hat{x} = -(k^y\hat{y} + k^z\hat{z}) \times \hat{x} = k^y\hat{z} - k^z\hat{y} = k(\hat{z}\sin i - \hat{y}\cos i)$$

$$-\boldsymbol{k}_r \times \hat{x} = -(k^y\hat{y} - k^z\hat{z}) \times \hat{x} = k^y\hat{z} + k^z\hat{y} = k(\hat{z}\sin i + \hat{y}\cos i)$$

$$-\boldsymbol{k}_t \times \hat{x} = -(k^y\hat{y} + k^z\hat{z}n'\cos t/n\cos i) \times \hat{x} = k(\hat{z}\sin i - \hat{y}(n'/n)\cos t) \tag{400}$$

from which we see that the second situation agrees with the first at normal incidence.

For completeness we write out the fields for general polarization:

$$\boldsymbol{E}_{i,r} = E_{i,r}\hat{x} - B_{i,r}\frac{c^2\boldsymbol{k}_{i,r} \times \hat{x}}{n^2\omega}, \qquad \boldsymbol{E}_t = E_t\hat{x} - B_t\frac{c^2\boldsymbol{k}_t \times \hat{x}}{n'^2\omega}$$

$$\boldsymbol{B}_{i,r} = B_{i,r}\hat{x} + E_{i,r}\frac{\boldsymbol{k}_{i,r} \times \hat{x}}{\omega}, \qquad \boldsymbol{B}_t = B_t\hat{x} + E_t\frac{\boldsymbol{k}_t \times \hat{x}}{\omega} \tag{401}$$

where $E_{i,r,t}$ are defined in (398) and $B_{i,r,t}$ are defined in (399).

7.2 Brewster's Angle

There is an angle of incidence (Brewster's angle) where the reflected amplitude for polarization in the plane of incidence vanishes. In this case the reflected wave will be completely polarized in a direction perpendicular to the plane of incidence. This happens when

$$
\begin{aligned}
\mu n' \cos i &= \mu' n \cos t = \frac{\mu' n}{n'} \sqrt{n'^2 - n^2 \sin^2 i} \\
\cos^2 i &= \frac{\mu'^2 n^2 (n'^2 - n^2)}{\mu^2 n'^4 - \mu'^2 n^4} \rightarrow \frac{n^2}{n'^2 + n^2}
\end{aligned}
\tag{402}
$$

where the last form is for $\mu' = \mu$, when $\tan i = n'/n$. In contrast, if we search for an angle where the reflected wave with polarization perpendicular to the plane of incidence vanishes, we find the condition

$$
n^2 \mu'^2 \cos^2 i = \mu^2 n'^2 - \mu^2 n^2 \sin^2 i = \mu^2 (n'^2 - n^2) + n^2 \mu^2 \cos^2 i
\tag{403}
$$

which has no solution when $\mu' = \mu$. Since most materials have $\mu \approx \mu_0$ to tremendous accuracy, the Brewster effect is practically limited to polarization perpendicular to the plane of incidence.

7.3 Total Internal Reflection

Another important phenomenon, mentioned briefly above, occurs when $n > n'$. Then $\sin t > \sin i$ so there is an angle $i_0 < \pi/2$ where $t = \pi/2$ so the refracted wave doesn't travel into the region $z > 0$. This is the angle of total internal reflection $\sin i_0 = n'/n$. Any $i > i_0$ would mean $\sin t > 1$ and $\cos t = i \sqrt{\sin^2 t - 1}$ would be pure imaginary. The significance of this is that $k_t^z = k' = (\omega/c)\sqrt{n'^2 - n^2 \sin^2 i}$ is pure imaginary and the z dependence of the fields for $z > 0$ is exponentially damped $\sim e^{-z|k^z|}$. The fields are nonzero for $z > 0$ but sharply attenuated. The conclusion that reflection is total is confirmed by noting that $|E_r|^2 = |E_i|^2$, so the energy flow in the reflected wave equals the energy flow in the incident wave. This is explored in one of the homework problems. Because fields penetrate a distance δ, the reflected wave appears to change direction behind the interface. (Goos-Hänchen Effect).

7.4 Action Principle for Maxwell's Equations

To formulate an action principle, it is best to introduce potentials so that two of the equations are automatically satisfied. Then we can show that

$$
S = \int d^3x dt \frac{1}{2} \left(\epsilon_0 \boldsymbol{E}^2 - \frac{\boldsymbol{B}^2}{\mu_0} \right) = \int d^3x dt \frac{1}{2} \left(\epsilon_0 \left(\nabla \phi + \frac{\partial \boldsymbol{A}}{\partial t} \right)^2 - \frac{(\nabla \times \boldsymbol{A})^2}{\mu_0} \right)
\tag{404}
$$

is stationary with respect to variations $\delta\phi, \delta\boldsymbol{A}$ on fields satisfying Maxwell's equations. Varying w.r.t. $\delta\phi$ leads to

$$
-\epsilon_0 \nabla \cdot \left(\nabla \phi + \frac{\partial \boldsymbol{A}}{\partial t} \right) = \nabla \cdot (\epsilon_0 \boldsymbol{E}) = 0
\tag{405}
$$

and varying w.r.t. $\delta\boldsymbol{A}$ leads to

$$\epsilon_0\frac{\partial}{\partial t}\left(-\nabla\phi-\frac{\partial\boldsymbol{A}}{\partial t}\right)-\frac{1}{\mu_0}\nabla\times(\nabla\times\boldsymbol{A})=\epsilon_0\frac{\partial\boldsymbol{E}}{\partial t}-\frac{1}{\mu_0}\nabla\times\boldsymbol{B}=0 \tag{406}$$

which are just the sourceless Maxwell equations.

The action is easily extended to include sources

$$S = \int d^3x dt\left(\frac{\epsilon_0}{2}\boldsymbol{E}^2-\frac{1}{2\mu_0}\boldsymbol{B}^2-\rho\phi+\boldsymbol{J}\cdot\boldsymbol{A}\right) \tag{407}$$

With sources, S is not manifestly gauge invariant. Under

$$\boldsymbol{A}\to\boldsymbol{A}+\nabla\Lambda,\qquad \phi\to\phi-\frac{\partial\Lambda}{\partial t} \tag{408}$$

$\boldsymbol{E}$ and $\boldsymbol{B}$ are invariant, but S changes by

$$\delta S = \int d^3x dt\left(\rho\frac{\partial\Lambda}{\partial t}+\boldsymbol{J}\cdot\nabla\Lambda\right)=-\int d^3x dt\Lambda\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J}\right)+\text{Surface Term} \tag{409}$$

where the surface terms would vanish for sufficiently localized Λ. Thus S will be locally gauge invariant provided charge is conserved:

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\boldsymbol{J} = 0. \tag{410}$$