Classical Mechanics: a Critical Introduction

Michael Cohen, Professor Emeritus
Department of Physics and Astronomy
University of Pennsylvania
Philadelphia, PA 19104-6396

with Solutions Manual by
Larry Gladney, Ph.D.
Edmund J. and Louise W. Kahn Professor for Faculty Excellence
Department of Physics and Astronomy
University of Pennsylvania

"Why, a four-year-old child could understand this...
Run out and find me a four-year-old child."

\- GROUCHO

REVISED PREFACE (Jan. 2013)

Anyone who has taught the "standard" Introductory Mechanics course more than a few times has most likely formed some fairly definite ideas regarding how the basic concepts should be presented, and will have identified (rightly or wrongly) the most common sources of difficulty for the student. An increasing number of people who think seriously about physics pedagogy have questioned the effectiveness of the traditional classroom with the Professor lecturing and the students listening (perhaps). I take no position regarding this question, but assume that a book can still have educational value.

The first draft of this book was composed many years ago and was intended to serve either as a stand-alone text or as a supplementary "tutor" for the student. My motivation was the belief that most courses hurry through the basic concepts too quickly, and that a more leisurely discussion would be helpful to many students. I let the project lapse when I found that publishers appeared to be interested mainly in massive textbooks covering all of first-year physics.

Now that it is possible to make this material available on the Internet to students at the University of Pennsylvania and elsewhere, I have revived and reworked the project and hope the resulting document may be useful to some readers. I owe special thanks to Professor Larry Gladney, who has translated the text from its antiquated format into modern digital form and is also preparing a manual of solutions to the end-of-chapter problems. Professor Gladney is the author of many of these problems. The manual will be on the Internet, but the serious student should construct his/her own solutions before reading Professor Gladney's discussion. Conversations with my colleague David Balamuth have been helpful, but I cannot find anyone except myself to blame for errors or defects. An enlightening discussion with Professor Paul Soven disabused me of the misconception that Newton's First Law is just a special case of the Second Law.

 Corrections, comments, criticisms, additional problems will be most welcome. Thanks.

Michael Cohen, Dept. of Physics and Astronomy, Univ. of Pa.,
Phila, PA 19104-6396
email: micohen@physics.upenn.edu

0.1 Introduction

Classical mechanics deals with the question of how an object moves when it is subjected to various forces, and also with the question of what forces act on an object which is not moving.

The word "classical" indicates that we are not discussing phenomena on the atomic scale and we are not discussing situations in which an object moves with a velocity which is an appreciable fraction of the velocity of light. The description of atomic phenomena requires quantum mechanics, and the description of phenomena at very high velocities requires Einstein's Theory of Relativity. Both quantum mechanics and relativity were invented in the twentieth century; the laws of classical mechanics were stated by Sir Isaac Newton in 1687[New02].

The laws of classical mechanics enable us to calculate the trajectories of baseballs and bullets, space vehicles (during the time when the rocket engines are burning, and subsequently), and planets as they move around the sun. Using these laws we can predict the position-versus-time relation for a cylinder rolling down an inclined plane or for an oscillating pendulum, and can calculate the tension in the wire when a picture is hanging on a wall.

The practical importance of the subject hardly requires demonstration in a world which contains automobiles, buildings, airplanes, bridges, and ballistic missiles. Even for the person who does not have any professional reason to be interested in any of these mundane things, there is a compelling intellectual reason to study classical mechanics: this is the example *par excellence* of a theory which explains an incredible multitude of phenomena on the basis of a minimal number of simple principles. Anyone who seriously studies mechanics, even at an elementary level, will find the experience a true intellectual adventure and will acquire a permanent respect for the subtleties involved in applying "simple" concepts to the analysis of "simple" systems.

I wish to distinguish very clearly between "subtlety" and "trickery". There is no trickery in this subject. The *subtlety* consists in the necessity of using concepts and terminology quite precisely. Vagueness in one's thinking and slight conceptual imprecisions which would be acceptable in everyday discourse will lead almost invariably to incorrect solutions in mechanics problems.

In most introductory physics courses approximately one semester (usually a bit less than one semester) is devoted to mechanics. The instructor and students usually labor under the pressure of being required to "cover" a certain amount of material. It is difficult, or even impossible, to "cover" the standard topics in mechanics in one semester without passing too hastily over a number of fundamental concepts which form the basis for everything which follows.

Perhaps the most common area of confusion has to do with the listing of the forces which act on a given object. Most people require a considerable amount of practice before they can make a correct list. One must learn to distinguish between the forces acting on a thing and the forces which it exerts on other things, and one must learn the difference between real forces (pushes and pulls caused by the action of one material object on another) and demons like "centrifugal force" (the tendency of an object moving in a circle to slip outwards) which must be expunged from the list of forces.

An impatient reader may be annoyed by amount of space devoted to discussion of "obvious" concepts such as "force", "tension", and "friction". The reader (unlike the student who is trapped in a boring lecture) is, of course, free to turn to the next page. I believe, however, that life is long enough to permit careful consideration of fundamental concepts and that time thus spent is not wasted.

With a few additions (some discussion of waves for example) this book can serve as a self-contained text, but I imagine that most readers would use it as a supplementary text or study guide in a course which uses another textbook. It can also serve as a text for an online course.

Each chapter includes a number of Examples, which are problems relating to the material in the chapter, together with solutions and relevant discussion. None of these Examples is a "trick" problem, but some contain features which will challenge at least some of the readers. I strongly recommend that the reader write out her/his own solution to the Example before reading the solution in the text.

Some introductory Mechanics courses are advertised as not requiring any knowledge of calculus, but calculus usually sneaks in even if anonymously (e.g. in the derivation of the acceleration of a particle moving in a circle or in the definition of work and the derivation of the relation between work and kinetic energy).

Since Mechanics provides good illustrations of the physical meaning of the "derivative" and the "integral", we introduce and explain these mathematical notions in the appropriate context. At no extra charge the reader who is not familiar with vector notation and vector algebra will find a discussion of those topics in Appendix A.

Contents

Chapter 1

KINEMATICS: THE MATHEMATICAL DESCRIPTION OF MOTION

Kinematics is simply the mathematical description of motion, and makes no reference to the forces which cause the motion. Thus, kinematics is not really part of physics but provides us with the mathematical framework within which the laws of physics can be stated in a precise way.

1.1 Motion in One Dimension

Let us think about a material object (a "particle") which is constrained to move along a given straight line (e.g. an automobile moving along a straight highway). If we take some point on the line as an origin, the position of the particle at any instant can be specified by a number x which gives the distance from the origin to the particle. Positive values of x are assigned to points on one side of the origin, and negative values of x are assigned to points on the other side of the origin, so that each value of x corresponds to a unique point. Which direction is taken as positive and which as negative is purely a matter of convention. The numerical value of x clearly depends on the unit of length we are using (e.g. feet, meters, or miles). Unless the particle is at rest, x will vary with time. The value of x at time t is denoted by $x(t)$.

The ***average velocity*** of a particle during the time interval from t to t' is defined as

$$v_{\text{avg}} = \frac{x(t') - x(t)}{t' - t} \tag{1.1}$$

i.e. the change in position divided by the change in time. If we draw a graph of x versus t (for example, Fig.1.1) we see that $[x(t') - x(t)]/[t' - t]$ is just the slope of the dashed straight line connecting the points which represent the positions of the particle at times t' and t.

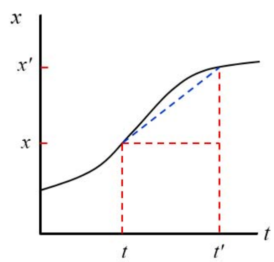

Figure 1.1: An example graph of position versus time.

A more important and more subtle notion is that of ***instantaneous velocity*** (which is what your car's speedometer shows). If we hold t fixed and let t' be closer and closer to t, the quotient $[x(t') - x(t)]/[t' - t]$ will approach a definite limiting value (provided that the graph of x versus t is sufficiently smooth) which is just the slope of the tangent to the x versus t curve at the point $(t, x(t))$. This limiting value, which may be thought of as the average velocity over an infinitesimal time interval which includes the time t, is called the " the instantaneous velocity at time t" or, more briefly, "the velocity at time t". We write

$$v(t) = \lim_{t' \to t} \frac{x(t') - x(t)}{t' - t}. \tag{1.2}$$

This equation is familiar to anyone who has studied differential calculus; the right side is called "the derivative of x with respect to t" and frequently denoted by dx/dt. Thus $v(t) = dx/dt$.

If $x(t)$ is given in the form of an explicit formula, we can calculate $v(t)$ either directly from equation 1.2 or by using the rules for calculating derivatives which are taught in all calculus courses (these rules, for example $d/dt\,(t^n) = n\ t^{n-1}$, merely summarize the results of evaluating the right side of (1.2) for various functional forms of $x(t)$). A useful exercise is to draw a qualitatively correct graph of $v(t)$ when $x(t)$ is given in the form of a graph, rather than as a formula. Suppose, for example, that the graph of $x(t)$ is Fig.1.2. We draw a graph of $v(t)$ by estimating the ***slope*** of the

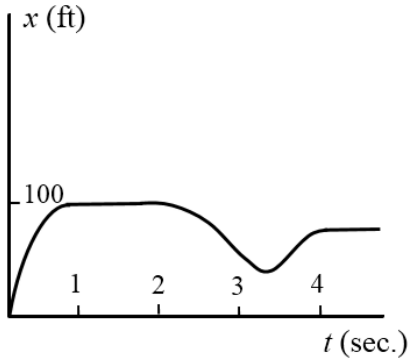

Figure 1.2: Another example of a position versus time graph.

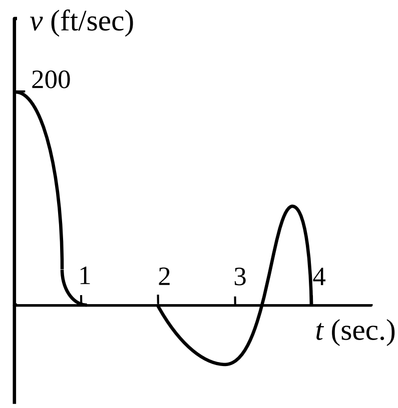

Figure 1.3: The corresponding graph of velocity versus time.

x-versus-t graph at each point. We see that the slope is positive at $t = 0$ (with a numerical value of about 200 ft/sec, though we are not interested in very accurate numbers here) and continues positive but with decreasing values until $t = 1$. The slope is zero between $t = 1$ and $t = 2$, and then becomes negative, etc. (If positive v means that the object is going forward, then negative v means that the object is going backward.) An approximate graph of $v(t)$ is given by Fig.1.3.

If we are given $v(t)$, either as a formula or a graph, we can calculate $x(t)$. The mathematical process of finding the function $x(t)$ when its slope $v(t)$ is given at all points is called “integration”. For example, if $v(t) = 9t^3$, then $x(t) = (9/4)t^4 + c$ where c is any constant (the proof is simply to calculate dx/dt and verify that we obtain the desired $v(t)$). The appearance of the arbitrary constant c in $x(t)$ is not surprising, since knowledge of the velocity at all times is not quite sufficient to fully determine the position at all times. We must also know where the particle started, i.e. the value of x when $t = 0$. If $x(t) = (9/4)t^4 + c$, then $x(0) = c$.

Suppose we are given the graph of $v(t)$, for example Fig.1.4. Let us consider the shaded rectangle whose height is $v(t)$ and whose wdith is Δ, where Δ is a very small time interval.

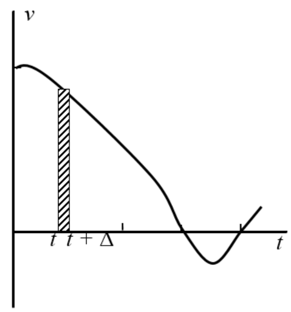

Figure 1.4: The shaded area is the displacement during $t \to t + \Delta$.

The area of this rectangle is $v(t)\Delta$, which is equal to the displacement (i.e. the change in x) of the particle during the time interval from t to $t + \Delta$. (Strictly speaking, the previous statement is not exactly true unless $v(t)$ is constant during the time interval from t to $t + \Delta$, but if Δ is small enough the variation of v during this interval may be neglected.) If t_1 and t_2 are any two times, and if we divide the interval between them into many small intervals, the displacement during any sub-interval is approximately equal to the area of the corresponding rectangle in Fig.1.5. Thus the net displacement $x(t_2) - x(t_1)$ is approximately equal to the sum of the areas of the rectangles. If the sub-intervals are made smaller and smaller, the error in this approximation becomes negligible, and thus we see that *the area under the portion of the v versus t curve between time t_1 and t_2 is equal to the displacement $x(t_2) - x(t_1)$ experienced by the particle during that time interval.*

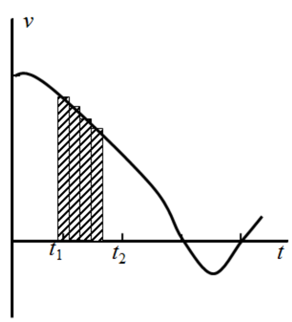

Figure 1.5: The shaded area is the displacement during $t_1 \to t_2$.

The above statement is true even if v becomes negative, provided we define the area as negative in regions where v is negative. In the notation of integral calculus we write

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t)\ dt \tag{1.3}$$

The right side of eqn.(1.3) is called the "integral of $v(t)$ with respect to t from t_1 to t_2" and is defined mathematically as the limit of the sum of the areas of the rectangles in Fig.1.5 as the width of the individual rectangles tends to zero.

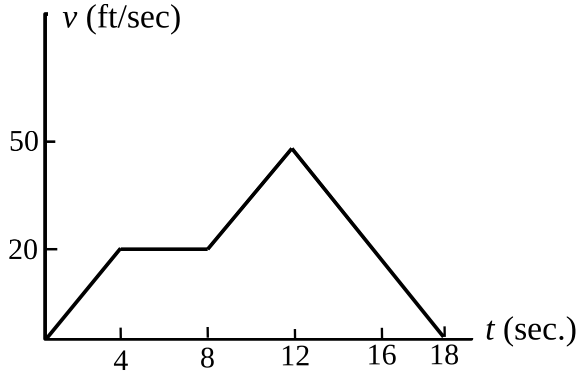

Figure 1.6: Plot of velocity versus time for an automobile.

Example 1.1 : Calculating distance and average velocity

Fig.1.6 shows the velocity of an auto as a function of time. Calculate the distance of the auto from its starting point at $t = 6,\ 12,\ 16$ and 18 sec. Calculate the average velocity during the period from $t = 4$ sec to $t = 15$ sec and during the period from $t = 0$ to $t = 18$ sec.

Solution: Calculating areas: $x(6) = 40 + 40 = 80'$; $x(12) = 40 + 80 + 140 = 260'$; $x(16) = 260 + 4(50 + 16.67)/2 = 393.3'$; $x(18) = 260 + 150 = 410'$. $x(15) - x(4) = 332.5'$; avg. vel. from $t = 4$ to $t = 15 = 30.23$ ft/sec; avg. vel. from $t = 0$ to $t = 18 = 22.78$ ft/sec [Note: After students have learned more formulas many will use formulas rather than simple calculation of areas and get this wrong.]

Example 1.2 :

A woman is driving between two toll booths 60 miles apart. She drives the first 30 miles at a speed of 40 mph. At what (constant) speed should she drive the remaining miles so that her average speed between the toll booths will be 50 mph?

Solution: If T is total time, $50 = 60/T$, so $T = 1.2$ hrs. Time for first 30 mi $= 30/40 = 0.75$ hr. Therefore, the time for the remaining 30 mi $= 1.2 - .75 = .45$ hr. The speed during the second 30 miles must be $30/.45 = 66.67$ mi/hr.

1.2 Acceleration

Acceleration is defined as the ***rate of change of velocity***. The ***average acceleration*** during the interval from t to t' is defined as

$$a_{\text{avg}} = \frac{v(t') - v(t)}{t' - t} \tag{1.4}$$

where $v(t')$ and $v(t)$ are the instantaneous values of the velocity at times t' and t. The instantaneous acceleration is defined as the average acceleration over an infinitesimal time interval, i.e.

$$a(t) = \lim_{t' \to t} \frac{v(t') - v(t)}{t' - t} \tag{1.5}$$

Since $v(t) = dx/dt$, we can write (in the notation of calculus) $a(t) = d^2x/dt^2$. We stress that this is simply shorthand for $a(t) = d/dt\ [dx/dt]$.

Comparing eqns.(1.5) and (1.2) we see that the relation between $a(t)$ and $v(t)$ is the same as the relation between $v(t)$ and $x(t)$. It follows that if $v(t)$ is given as a graph, the slope of the graph is $a(t)$. If $a(t)$ is given as a graph then we should also expect that the area under the portion of the graph between time t_1 and time t_2 is equal to the change in velocity $v(t_2) - v(t_1)$. The analogue of eqn.(1.3) is

$$v(t_2) - v(t_1) = \int_{t_1}^{t_2} a(t)\ dt \tag{1.6}$$

Example 1.3 : Instantaneous Acceleration

Draw a graph of the instantaneous acceleration $a(t)$ if $v(t)$ is given by Fig.1.6.

1.3 Motion With Constant Acceleration

All of the preceding discussion is entirely general and applies to any one-dimensional motion. An important special case is motion in which the acceleration is constant in time. We shall shortly see that this case occurs whenever the forces are the same at all times. The graph of acceleration versus time is simple (Fig.1.7). The area under the portion of this graph

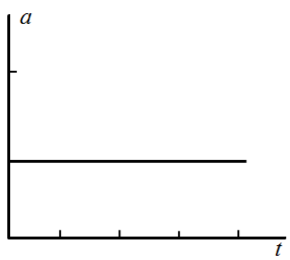

Figure 1.7: Plot of constant acceleration.

between time zero and time t is just $a \cdot t$. Therefore $v(t) - v(0) = a\ t$. To make contact with the notation commonly used we write v instead of $v(t)$ and v_0 instead of $v(0)$. Thus,

$$v = v_0 + at \tag{1.7}$$

The graph of v versus t (Fig.1.8) is a straight line with slope a. We can get an explicit formula for $x(t)$ by inserting this expression into eqn.(1.3) and performing the integration or, without calculus, by calculating the shaded area under the line of Fig.1.8 between $t = 0$ and t. Geometrically (Fig.1.9), the area under Fig.1.8 between $t = 0$ and t is the width t multiplied by the height at the midpoint which is $1/2\ (v_0 + v_0 + at)$. Thus we find $x(t) - x_0 = 1/2\ (2v_0 t + at^2)$. Finally,

$$x = x_0 + \frac{1}{2}(v + v_0)t \tag{1.8}$$

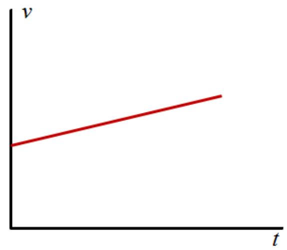

Figure 1.8: Plot of velocity versus time for constant acceleration.

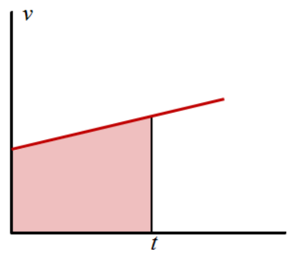

Figure 1.9: Area under the curve of v versus t.

If we want to use calculus (i.e. eqn.(1.3)) we write

$$x(t) - x(0) = \int_0^t (v_0 + at')\ dt' = v_0 t + \frac{1}{2}at^2 \tag{1.9}$$

(note that we have renamed the "dummy" integration variable t' in order to avoid confusion with t which is the upper limit of the integral).

Comparing eqn.(1.8) with the definition (eqn.(1.1)) of average velocity we see that the average velocity during any time interval is half the sum of the initial and final velocities. *Except for special cases, this is true only for uniformly accelerated motion.*

Sometimes we are interested in knowing the velocity as a function of the position x rather than as a function of the time t. If we solve eqn.(1.7) for t, i.e. $t = (v - v_0)/a$ and substitute into eqn.(1.8) we obtain

$$v^2 - v_0^2 = 2a(x - x_0) \tag{1.10}$$

We collect here the mathematical formulas derived above, all of which are applicable only to motion with ***constant acceleration***.

$$\begin{aligned} v &= v_0 + at && (1.11a) \\ x - x_0 &= \frac{1}{2}(v + v_0)t && (1.11b) \\ x &= x_0 + v_0 t + \frac{1}{2}at^2 && (1.11c) \\ v^2 &= v_0^2 + 2a(x - x_0) && (1.11d) \end{aligned}$$

There is often more than one way to solve a problem, but not all ways are equally efficient. Depending on what information is given and what question is asked, one of the above formulas usually leads *most* directly to the answer.

Example 1.4 : A constant acceleration problem

A car decelerates (with constant deceleration) from 60 mi/hr to rest in a distance of 500 ft. [Note: 60 mph = 88 ftsec]

1. Calculate the acceleration.
2. How long did it take?
3. How far did the car travel between the instant when the brake was first applied and the instant when the speed was 30 mph?
4. If the car were going at 90 mph when the brakes were applied, but the deceleration were the same as previously, how would the stopping distance and the stopping time change?

Solution: We will use the symbols 88 ft/s $= v_0$, 500 ft $= D$.

1. $0 = v_0^2 + 2aD \Rightarrow a = -7.74$ ft/s^2
2. $T =$ stopping time, $D = 1/2\ v_0 T \Rightarrow T = 11.36$ s (could also use eqn.(1.11a))
3. $D' =$ answer to (3), i.e.

$$\begin{aligned} (1/2\ v_0)^2 &= v_0^2 + 2aD' \left(\text{where } a = -\frac{v_0^2}{2D}\right) \\ \text{thus } \frac{D'}{D} &= \frac{3}{4} \Rightarrow D' = 375 \text{ ft} \end{aligned}$$

4. D'' = answer to (4). From (d) $D''/D = (90/60)^2 \Rightarrow D'' = 1125$ ft. From (a) the stopping time is $(3/2)11.36 = 17.04$ sec.

Example 1.5 : Another example of constant acceleration

A drag racer accelerates her car with constant acceleration on a straight drag strip. She passes a radar gun (#1) which measures her instantaneous speed at 60 ft/s and subsequently passes a second radar gun (#2) which measures her instantaneous speed as 150 ft/s.

1. What is her speed at the midpoint (in time) of the interval between the two measurements?
2. What is her speed when she is equidistant from the two radar guns?
3. If the distance between the two radar guns is 500 ft, how far from gun #1 is the starting point?

Solution: Symbols: $v_1 = 60$ $v_2 = 150$ T = time interval D = space interval.

1. $a = (v_2 - v_1)/T$. At $T/2$ $v = v_1 + aT/2 = (v_1 + v_2)/2 = 105$ ft/s.
2. v_3 = speed at $D/2$ $a = (v_2^2 - v_1^2)/2D$. $v_3^2 = v_1^2 + 2aD/2 = (v_1^2 + v_2^2)/2$ $v_3 = 114.24$ ft/s
3. D' = dist. from starting point to #1. $v_1^2 = 2aD' \Rightarrow D' = v_1^2 D/(v_2^2 - v_1^2) = 95.2$ ft.

1.4 Motion in Two and Three Dimensions

The motion of a particle is not necessarily confined to a straight line (consider, for example, a fly ball or a satellite in orbit around the earth), and in general three Cartesian coordinates, usually referred to as $x(t)$, $y(t)$, $z(t)$ are required to specify the position of the particle at time t. In almost all of the situations which we shall discuss, the motion is confined to a plane;

if we take two of our axes (e.g. the x and y axes) in the plane, then only two coordinates are required to specify the position.

Extensions of the notion of velocity and acceleration to three dimensions is straightforward. If the coordinates of the particle at time t are $(x(t), y(t), z(t))$ and at t' are $(x(t'), y(t'), z(t'))$, then we define the ***average x-velocity*** during the time interval $t \to t'$ by the equation $v_{x,\mathrm{av}} = [x(t') - x(t)]/[t' - t]$. Similar equations define $v_{y,\ \mathrm{av}}$ and $v_{z,\ \mathrm{av}}$. The instantaneous x-velocity, y-velocity, and z-velocity are defined exactly as in the case of one-dimensional motion, i.e.

$$v_x(t) \equiv \lim_{t' \to t} \frac{x(t') - x(t)}{t - t'} = \frac{dx}{dt} \tag{1.12}$$

and so on. Similarly, we define $a_{x,\mathrm{avg}} = [v_x(t') - v_x(t)]/[t' - t]$ with similar definitions for $a_{y,\ \mathrm{avg}}$ and $a_{z,\ \mathrm{avg}}$. The instantaneous x-acceleration is

$$a_x(t) \equiv \lim_{t' \to t} \frac{v_x(t') - v_x(t)}{t' - t} = \frac{d^2x}{dt^2} \tag{1.13}$$

with similar definitions of $a_y(t)$ and $a_z(t)$.

All of the above seems somewhat heavy-handed and it is almost obvious that by introducing more elegant notation we could replace three equations by a single equation. The more elegant notation, which is called ***vector notation***, has an even more important advantage: it enables us to state the laws of physics in a form which is obviously independent of the orientation of the particular axes which we have arbitrarily chosen. The reader who is not familiar with vector notation and/or with the addition and subtraction of vectors, should read the relevant part of Appendix A. The sections defining and explaining the dot product and cross product of two vectors are not relevant at this point and may be omitted.

We introduce the symbol $\vec{r}$ as shorthand for the number triplet (x, y, z) formed by the three coordinates of a particle. We call $\vec{r}$ the ***position vector*** of the particle and we call x, y and z the ***components of the position vector*** with respect to the chosen set of axes. In printed text a vector is usually represented by a boldfaced letter and in handwritten or typed text a vector is usually represented by a letter with a horizontal arrow over it.[1]

The velocity and acceleration vectors are defined as

$$\vec{v}(t) = \lim_{t' \to t} \frac{\vec{r}(t') - \vec{r}(t)}{t' - t} = \frac{d\vec{r}}{dt} \tag{1.14}$$

[1] *Since the first version of this text was in typed format, we find it convenient to use the arrow notation.*

and
$$\vec{a}(t) = \lim_{t' \to t} \frac{\vec{v}(t') - \vec{v}(t)}{t' - t} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{1.15}$$
[Again, we stress the importance of understanding what is meant by the difference of two vectors as explained in Appendix A.] In particular, even if the magnitude of the velocity vector (the "speed") remains constant, the particle is accelerating if the direction of the velocity vector is changing.

A very important kinematical problem first solved by Newton (in the year 1686) is to calculate the instantaneous acceleration $\vec{a}(t)$ of a particle moving in a circle with constant speed. We refer to this as ***uniform circular motion***. We shall solve the problem by two methods, the first being Newton's.

1.4.1 Circular Motion: Geometrical Method

The geometrical method explicitly constructs the vector $\Delta\vec{v} = \vec{v}(t') - \vec{v}(t)$ and calculates the limit required by eqn.(1.15). We let $t' = t + \Delta t$ and indicate in Fig.1.10 the position and velocity vector of the particle at time t and

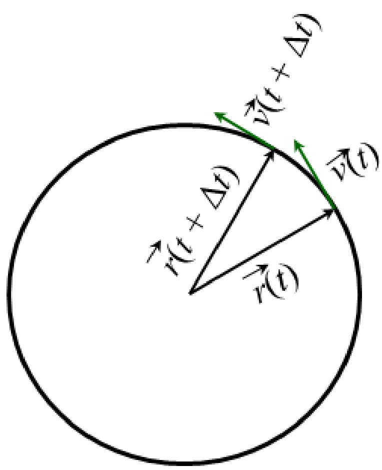

Figure 1.10: Geometric construction of the acceleration for constant speed circular motion.

at time $t+\Delta t$. The picture is drawn for a particle moving counter-clockwise, but we shall see that the same acceleration is obtained for clockwise motion. Note that $\vec{r}(t+\Delta t)$ and $\vec{r}(t)$ have the same length r and that $\vec{v}(t+\Delta t)$ and $\vec{v}(t)$ have the same length v since the speed is assumed constant. Furthermore, the angle between the two $\vec{r}$ vectors is the same as the angle between the two $\vec{v}$ vectors since at each instant $\vec{v}$ is perpendicular to $\vec{r}$. During time Δt the arc-length traveled by the particle is $v\Delta t$, and the radian measure of the angle between $\vec{r}(t+\Delta t)$ and $\vec{r}(t)$ is $(v\Delta t)/r$.

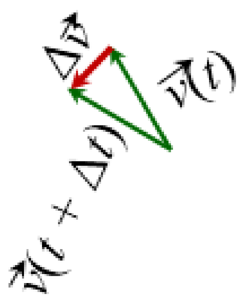

Figure 1.11: Geometric construction of the change in velocity for constant speed circular motion.

We are interested in the limit $\Delta\vec{v}/\Delta t$ as $\Delta t \to 0$. If we bring the tails of $\vec{v}(t)$ and $v(t+\Delta t)$ together by a parallel displacement of either vector, then $\Delta\vec{v}$ is the vector from tip of $\vec{v}(t)$ to the tip of $\vec{v}(t+\Delta t)$ (see Fig.1.11). The triangle in Fig.1.11 is isosceles, and as $\Delta t \to 0$ the base angles of the isosceles triangle become right angles. Thus we see that $\Delta\vec{v}$ becomes perpendicular to

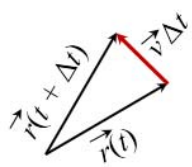

Figure 1.12: Geometric construction of the change in position for constant speed circular motion.

the instantaneous velocity vector $\vec{v}$ and is anti-parallel to $\vec{r}$ (this is also true for clockwise motion as one can see by drawing the picture). The magnitude of the acceleration vector is

$$|\vec{a}| = \lim_{\Delta t\to 0} |\Delta\vec{v}|/\Delta t. \tag{1.16}$$

Since the isosceles triangles of Figs.1.11 and 1.12 are similar, we have $|\Delta\vec{v}|/v = |\Delta\vec{r}|/r$. But since the angle between $\vec{r}(t)$ and $\vec{r}(t+\Delta t)$ is very small, the chord length $|\Delta\vec{r}|$ can be replaced by the arc-length $v\Delta t$. Thus $|\Delta\vec{v}| = v^2\Delta t/r$.

We have therefore shown that the acceleration vector has magnitude v^2/r and is directed from the instantaneous position of the particle toward the center of the circle, i.e.

$$\vec{a} = \left(-\frac{v^2}{r}\right)\left(\frac{\vec{r}}{r}\right) = -\frac{v^2}{r}\hat{r} \tag{1.17}$$

where $\hat{r}$ is a unit vector pointing from the center of the circle toward the particle. The acceleration which we have calculated is frequently called the ***centripetal acceleration***. The word "centripetal" means "directed toward the center" and merely serves to remind us of the direction of $\vec{a}$. If the speed v is not constant, the acceleration also has a tangential component of magnitude dv/dt.

1.4.2 Circular Motion: Analytic Method

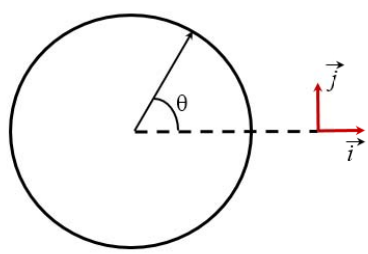

Figure 1.13: Geometric construction of the acceleration for constant speed circular motion.

If we introduce unit vectors $\hat{i}$ and $\hat{j}$ (Fig.1.13) then the vector from the center of the circle to the instantaneous position of the particle is $\vec{r} = r[\cos\theta\ \hat{i} + \sin\theta\ \hat{j}]$ where r and θ are the usual polar coordinates. If the particle is moving in a circle with constant speed, then $dr/dt = 0$ and $d\theta/dt = \text{constant}$. So,

$$\vec{v} = \frac{d\vec{r}}{dt} = r\left[-\sin\theta\frac{d\theta}{dt}\ \hat{i} + \cos\theta\frac{d\theta}{dt}\ \hat{j}\right]. \tag{1.18}$$

We have used the chain rule $d/dt\ (\cos\theta) = [d/d\theta\ (\cos\theta)][d\theta/dt]$ etc. Note that the standard differentiation formulas require that θ be expressed in radians. It should also be clear that $\vec{v}$ is tangent to the circle. Note that $v^2 = (r\ d\theta/dt)^2\ (\sin^2\theta + \cos^2\theta) = (r\ d\theta/dt)^2$. Therefore we have

$$\vec{a} = \frac{d\vec{v}}{dt} = r\left(\frac{d\theta}{dt}\right)^2 \left[-\cos\theta\ \hat{i} - \sin\theta\ \hat{j}\right] = -\frac{v^2}{r}\ \hat{r} \tag{1.19}$$

as derived above with the geometric method.

1.5 Motion Of A Freely Falling Body

It is an experimental fact that in the vicinity of a given point on the earth's surface, and in the absence of air resistance, all objects fall with the same constant acceleration. The magnitude of the acceleration is called g and is approximately equal to 32 ft/sec^2 or 9.8 meters/sec^2, and the direction of the acceleration is *down*, i.e. toward the center of the earth.

The magnitude of the acceleration is inversely proportional to the square of the distance from the center of the earth and the acceleration vector is directed toward the center of the earth. Accordingly, the magnitude and direction of the acceleration may be regarded as constant only within a region whose linear dimensions are very small compared with the radius of the earth. This is the meaning of "in the vicinity".

We stress that in the absence of air resistance the magnitude and direction of the acceleration do not depend on the velocity of the object (in particular, if you throw a ball upward the acceleration is directed downward while the ball is going up, while it is coming down, and also at the instant when it is at its highest point). At this stage of our discussion we cannot "derive" the fact that all objects fall with the same acceleration since we have said nothing about forces (and about gravitational forces in particular) nor about how a particle moves in response to a force. However, if we are willing to accept the given experimental facts, we can then use our kinematical tools to answer all possible questions about the motion of a particle under the influence of gravity.

One should orient the axes in the way which is mathematically most convenient. We let the positive y-axis point vertically up (i.e. away from the center of the earth). The x-axis must then lie in the horizontal plane. We choose the direction of the x-axis in such a way that the velocity $\vec{v}_0$ of the particle at time $t = 0$ lies in the x-y plane. The components of the

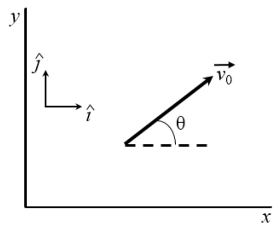

Figure 1.14: Initial velocity vector.

acceleration vector are $a_y = -g$, $a_x = a_z = 0$. Eqns. (1.11a- 1.11d) yield

$$\begin{aligned}
v_y &= v_{0,y} - gt && \text{(1.20a)}\\
v_y^2 &= v_{0,y}^2 - 2g(y - y_0) && \text{(1.20b)}\\
y &= y_0 + \tfrac{1}{2}(v_y + v_{0,y})t && \text{(1.20c)}\\
y &= y_0 + v_{0,y}t - \tfrac{1}{2}gt^2 && \text{(1.20d)}\\
v_x &= \text{constant} = v_{0,x} && \text{(1.20e)}\\
x &= x_0 + v_{0,x}t && \text{(1.20f)}\\
v_z &= \text{constant} = 0 \qquad z = \text{constant} = z_0 && \text{(1.20g)}
\end{aligned}$$

We shall always locate the origin in such a way that $z_0 = 0$ and thus the entire motion takes place in the x-y plane. Usually we locate the origin at the initial position of the particle so that $x_0 = y_0 = 0$, but the above formulas do not assume this.

We can obtain the equation of the ***trajectory*** (the relation between y and x) by solving eqn.(1.20f) for t and substituting the result into (1.20d). We find

$$y - y_0 = \frac{v_{0,y}}{v_{0,x}}(x - x_0) - \frac{1}{2}g\frac{(x - x_0)^2}{v_{0,x}^2} \tag{1.21}$$

This is, of course, the equation of a parabola. If we locate our origin at the initial position of the particle and if we specify the initial speed v_0 and the angle θ between the initial velocity and the x-axis (thus $v_{0x} = v_0 \cos\theta$ and $v_{0y} = v_0 \sin\theta$) then the equation of the trajectory is

$$y = x\tan\theta - 1/2\ gx^2/(v_0^2\cos^2\theta). \tag{1.22}$$

If a cannon is fired from a point on the ground, the horizontal range R is defined as the distance from the firing point to the place where the shell

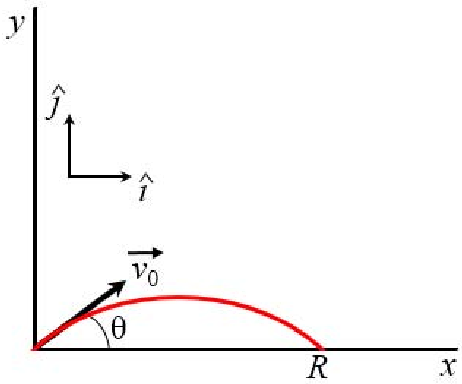

Figure 1.15: Path of a parabolic trajectory.

hits the ground. If we set $y = 0$ in eqn.(1.22) we find

$$0 = x\left[\tan\theta - \tfrac{1}{2}\frac{gx}{v_0^2\cos^2\theta}\right] \tag{1.23}$$

which has two roots, $x = 0$ and $x = (2v_0^2/g)\sin\theta\cos\theta = (v_0^2/g)\sin(2\theta)$. The first root is, of course, the firing point, and the second root tells where the shell lands, i.e.

$$R = \frac{v_0^2}{g}\sin(2\theta). \tag{1.24}$$

If we want to maximize the range for a given muzzle speed v_0, we should fire at the angle which maximizes $\sin(2\theta)$, i.e. $\theta = 45°$.

The simplest way to find the greatest height reached by a shell is to use eqn.(1.20b), setting $v_y = 0$. We find $y_{\max} - y_0 = (v_0^2/g)\sin^2\theta$. We could also set $dy/dx = 0$ and find $x = R/2$, which is obvious when you consider the symmetry of a parabola. We could then evaluate y when $x = R/2$.

Example 1.6 : Freely-falling motion after being thrown.

A stone is thrown with horizontal velocity 40 ft/sec and vertical (upward) velocity 20 ft/sec from a narrow bridge which is 200 ft above the water.

- How much time elapses before the stone hits the water?
- What is the vertical velocity of the stone just before it hits the water?

- At what horizontal distance from the bridge does the stone hit the water?

REMARK: It is NOT necessary to discuss the upward and downward part of the trajectory separately. The formulas of equations 1.20a - 1.20g apply to the entire trajectory.
Solution:

$$\begin{aligned}(h = 200'\ )\,(1.20d) \rightarrow -h &= v_{0,y}t - \tfrac{1}{2}gt^2 \Rightarrow \\ t &= \frac{v_{0,y} \pm \sqrt{v_{0,y}^2 + 2gh}}{g}\end{aligned}$$

The positive root (4.215 sec) is the relevant one. The negative root is the time at which the stone could have been projected upward from the river with a vertical velocity such that it would pass the bridge at $t = 0$ with an upward velocity of 20 ft/sec. Eqn.(1.20a) gives the answer to (b); $v_y = 20 - 32(4.215) = -114.88$ (i.e. 114.88 ft/sec downward). Part (b) can also be answered directly (without calculating t) by Eqn.(1.20b). Finally, for part (c) we have $x = 40(4.215) = 168.6$ ft.

Example 1.7 : Freely-falling motion of a batted ball.

The bat strikes a ball at a point 4 ft above the ground. The velocity vector of the batted ball initially is directed 20° above the horizontal.

1. What initial speed must the batted ball have in order to barely clear a 20 ft high wall located 350 ft from home plate?

2. There is a flat field on the other side of the wall. If the ball barely clears the wall, at what horizontal distance from home plate will the ball hit the ground?

Solution: We use the trajectory formula eqn.(1.22), taking the origin at the point where the bat strikes the ball. Note that $\tan 20^\circ = 0.3640$ and $\cos^2 20^\circ = 0.8830$. Therefore

$$\begin{aligned}16 &= 127.4 - 18.120(350/v_0)^2 \Rightarrow \\ v_0 &= 141.2 \text{ ft/sec}\end{aligned}$$

Using this value of v_0 in the trajectory formula eqn.(1.22), we can set $y = -4$ (ball on the ground). The positive root for x is 411.0 ft. An excellent simple approximation for x is to note that $y = 0$ when $x = (v_0^2/g)\sin 40° = 400.3$ ft (from the range formula) and then approximate the rest of the trajectory by a straight line. This would give an additional horizontal distance of $4/\tan 20° = 10.99$ ft. This gives $x = 411.3$ ft for the landing point, which is only about 3.5" long.

1.6 Kinematics Problems

1.6.1 One-Dimensional Motion

1.1. Fig.1.16 shows the velocity vs. time for a sports car on a level track. Calculate the following:

(a) the distance traveled by the car from $t = 0$ to $t = 40$ seconds

(b) the acceleration of the car from $t = 40$ seconds to $t = 60$ seconds

(c) the average velocity of the car from $t = 0$ to $t = 60$ seconds.

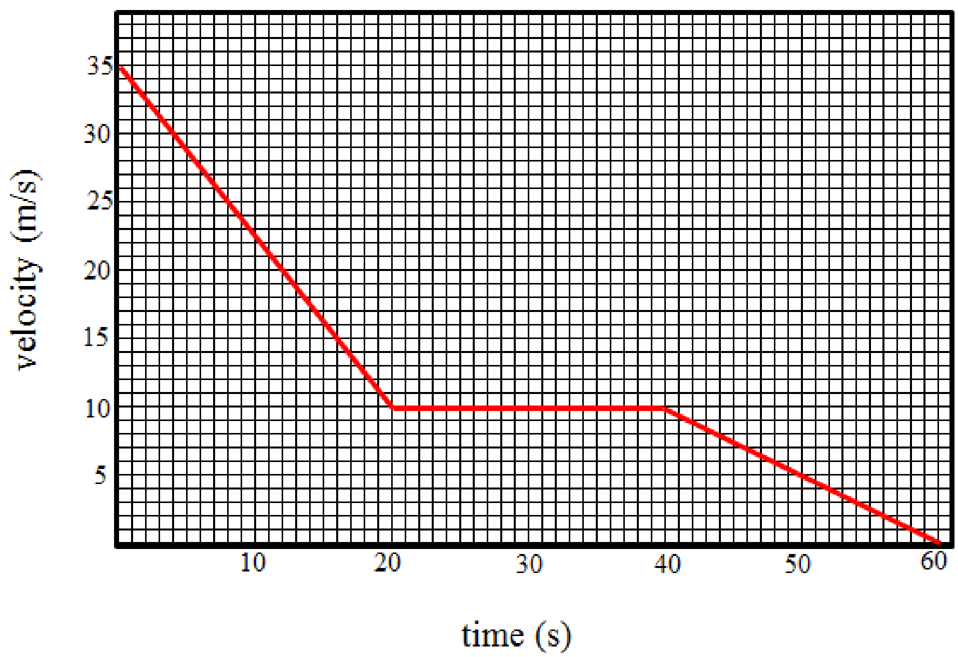

Figure 1.16: Graph for Problem **1.1**.

1.2. The fastest land animal is the cheetah. The cheetah can run at speeds of as much as 101 km/h. The second fastest land animal is the antelope, which runs at a speed of up to 88 km/h.

(a) Suppose that a cheetah begins to chase an antelope that has a head start of 50 m. How long does it take the cheetah to catch the antelope? How far will the cheetah have traveled by this time?

(b) The cheetah can maintain its top speed for about 20 seconds before needing to rest. The antelope can continue at its top speed for a considerably longer time. What is the maximum head start the cheetah can allow the antelope and still be able to catch it?

1.3. A window is 3.00 m high. A ball is thrown vertically from the street and, while going upward, passes the top of the window 0.400 s after it passes the bottom of the window. Calculate

(a) the maximum height above the top of the window which the ball will reach

(b) the time interval between the two instants when the ball passes the top of the window.

1.4. An elevator is accelerating upward with acceleration A. A compressed spring on the floor of the elevator projects a ball upward with velocity v_0 relative to the floor. Calculate the maximum height above the floor which the ball reaches.

1.5. A passageway in an air terminal is 200 meters long. Part of the passageway contains a moving walkway (whose velocity is 2 m/s), and passengers have a choice of using the moving walkway or walking next to it. The length of the walkway is less than 200m. Two girls, Alison and Miriam decide to have a race from the beginning to the end of the passageway. Alison can run at a speed of 7 m/s but is not allowed to use the moving walkway. Miriam can run at 6 m/s and can use the walkway (on which she runs, in violation of airport rules). The result of the race through the passageway is a tie.

(a) How long is the walkway?

(b) They race again, traversing the passageway in the opposite direction from the moving walkway. This time Miriam does not set foot on the walkway and Alison must use the walkway. Who wins?

1.6.2 Two and Three Dimensional Motion

1.6. The position of a particle as a function of time is given by

$$\vec{r} = [(2t^2 - 7t)\hat{\imath} - t^2\hat{\jmath}] \text{ m}.$$

Find:

(a) its velocity at $t = 2$ s.

(b) its acceleration at $t = 5$ s.

(c) its average velocity between $t = 1$ s and $t = 3$ s.

1.7. A ski jump is on a hill which makes a constant angle of 10.0° below the horizontal. The takeoff point is 6.00 meters vertically above the surface of the hill. At the takeoff point the ramp tilts upward at an angle of 15.0° above the horizontal. A jumper takes off with a speed of 30.0 m/s (and does not make any extra thrust with his knees). Calculate the horizontal distance from the takeoff point to his landing point.

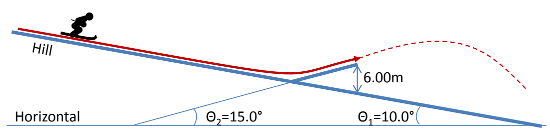

Figure 1.17: Figure for Problem **1.7**.

1.8. A doughnut-shaped space station has an outer rim of radius 1 kilometer. With what period should it rotate for a person at the rim to experience an acceleration of $g/5$?

Figure 1.18: Figure for Problem **1.8**.

1.9. A high-speed train through the Northeast Corridor (Boston to Washington D.C.) is to travel at a top speed of 300 km/h. If the passengers onboard are not to be subjected to more than $0.05g$, what must be the minimum radius of curvature for any turn in the track? [Will banking the track be useful?]

1.10. In a conical pendulum, a bob is suspended at the end of a string and describes a horizontal circle at a constant speed of 1.21 m/s (see Fig.1.19). If the length of the string is 1.20 m and it makes an angle of 20.0° with the vertical, find the acceleration of the bob.

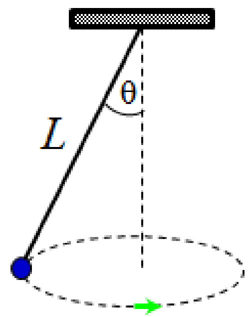

Figure 1.19: Figure for Problem **1.10**.

Chapter 2

NEWTON'S FIRST AND THIRD LAWS: STATICS OF PARTICLES

Perhaps the most appealing feature of classical mechanics is its *logical economy*. Everything is derived from Newton's three laws of motion. [Well, almost everything. One must also know something about the forces which are acting.] It is necessary of course, to understand quite clearly what the laws assert, and to acquire some experience in applying the laws to specific situations.

We are concerned here with the first and third laws. The second law will be discussed in the next chapter. Time spent in thinking about the meaning of these laws is (to put it mildly) not wasted.

2.1 Newton's First Law; Forces

The first law, in Newton's own words is: "Every body perseveres in its state of resting, or uniformly moving in a right line, unless it is compelled to change that state by forces impressed upon it."[New] In modern language, the first law states that the velocity of a body is constant if and only if there are no forces acting on the body or if the (vector) sum of the forces acting on the body is zero. Note that when we say the velocity is constant, we mean that both the magnitude and direction of the velocity vector are constant. In this statement we assume that all parts of the body have the same velocity. Otherwise, at this early point in the discussion we don't know what we mean by "the velocity of a body".

Two questions immediately arise:

(a). What do we mean by a force?

(b). With respect to what set of axes is the first law true? (Note that a body at rest or moving with constant velocity as measured with respect to one set of axes may be accelerating with respect to another set of axes.)

The answer to (a) and (b) are related. In fact, if we are willing to introduce a sufficiently complicated notion of force, the first law will be true with respect to every set of axes and says nothing. The "sufficiently complicated notion of force" would involve postulating that whenever we see the velocity of a particle changing a force is acting on the particle even if we cannot see the source of the force.

We shall insist on giving the word "force" a very restricted meaning which corresponds closely to the way the word is used in everyday language. We define a ***force*** as the *push or pull exerted by one piece of matter on another piece of matter*. This definition is not quantitative (a quantitative measure of force will be introduced shortly) but emphasizes the fact that we are entitled to speak of a "force" *only* when we can identify the piece of matter which is exerting the force and the piece of matter on which the force is being exerted.

Some simple examples will illustrate what we mean and do not mean when we use the word "force".

- As a stone falls toward the earth we observe that its velocity changes and we say that the earth is pulling on the stone. This pull (which we call the gravitational force exerted by the earth on the stone) is an acceptable use of the term "force" because we can see the piece of matter (the earth) which is exerting it. We have, of course, learned to live with the idea that one piece of matter can exert a force on another piece of matter without directly touching it.

- Consider a woman sitting in a moving railroad car. The earth is pulling down on her. The seat on which she is sitting is exerting an upward force on her; if there are coil springs in the seat, this upward force is exerted by the springs, which are compressed.[1] If the car accelerates

[1]Every seat has "springs" in it, but the springs may be very stiff. When you sit on a wooden bench, you sink slightly into the bench, compressing the wood until it exerts an upward force on you which is equal in magnitude but opposite in direction to the downward pull exerted by the earth on you.

in the forward direction, the seat exerts an additional force on the woman; this force is directed forward and is exerted by the back of the seat. Examination of the coil springs (or foam rubber) in the back of the seat will show that they are compressed during the period when the train is accelerating. While the train is accelerating, the woman has the feeling that something is pushing her back into her seat. Nevertheless, we do *not* acknowledge that there is any force pushing the woman toward the back of the car since we cannot point to any piece of matter which is exerting such a force on the woman. (If the car had a rear window and if we looked out that window and saw a huge piece of matter as large as a planet behind the car, then we could say that the gravitational force exerted by the planet is pulling the woman toward the rear. But of course we don't see this.)

If the floor of the car is very smooth, and if a box is resting on the floor, the box will start sliding toward the rear as the car accelerates. If we measure position and velocity in terms of axes which are attached to the car, we will say that the box is accelerating toward the rear of the car. Nevertheless we do *not* say that there is a force pushing the box toward the rear since we cannot point to the piece of matter which exerts the force. Thus, with our restricted notion of force, Newton's first law is not true if we use axes which are attached to the accelerating car. On the other hand, if we use axes attached to the ground, Newton's first law *is* true. With respect to the latter axes the velocity of the box does not change; this is consistent with the statement that there is no force on the box.

We shall see that the relatively simple notion of "force" which we have defined is quite sufficient for our purposes. Many forces are encountered in everyday life, but if we look closely enough they can all be explained in terms of the gravitational attraction exerted by one piece of matter (usually the earth) on another, and the electric and magnetic forces exerted by one charged body on another. We shall frequently refer to the "contact" forces exerted by one body on another when their surfaces are touching. These "contact" forces can, in general, have a component perpendicular to the surface and a component parallel to the surface; the two components are called, respectively, the ***normal*** force and the ***frictional*** force. If we examine the microscopic origin of these forces, we find that they are electric forces between the surface molecules (or atoms) of one body and the surface molecules of the other body. Even though the molecules have no net charge, each molecule contains both positive and negative charges; when two molecules are close enough, the forces among the various charges do not exactly cancel out and there is a net force. Fortunately, the application of Newton's laws does not require a detailed microscopic understanding of such things as contact forces; nevertheless we refuse to include any force on our list unless we are convinced that it can ultimately be explained in terms of gravitational, electric, and/or magnetic forces.[2]

2.2 Inertial Frames

Now that we know what we mean by a force, we can ask "With respect to which axes is it true that a particle subject to no forces moves with constant velocity?", i.e. with respect to which axes is Newton's first law true? A set of axes is frequently called a "frame of reference", and those axes with respect to which Newton's first law is true are called ***inertial frames***.

It is important to note that, as a consequence of Newton's first law, there is more than one inertial frame. If a set of axes XYZ are an inertial frame, and if another set of axes $X'Y'Z'$ are moving with constant velocity and not rotating with respect to XYZ, then $X'Y'Z'$ are also an inertial frame. This follows from the fact that a particle which has constant velocity with respect to the XYZ axes will also have a constant velocity with respect to the $X'Y'Z'$ axes.

We have already seen that axes attached to an accelerating railroad car are not an inertial frame, but axes attached to the earth *are* an inertial frame. Actually, this is not quite true. For most purposes, axes attached to the earth appear to be an inertial frame. However, due to the earth's rotation, these axes are rotating relative to the background of the distant stars. If you give a hockey puck a large velocity directed due south on a perfectly smooth ice rink in Philadelphia, it will not travel in a perfectly straight line relative to axes scratched on the ice, but will curve slightly to the west because of the earth's rotation. This effect is important in naval gunnery and illustrates that axes attached to the earth are not a perfect inertial frame. A better, but less convenient, set of axes are axes which are non-rotating with respect to the distant stars and whose origin moves with the center of the earth.

Another phenomenon which demonstrates that axes attached to the earth's surface are not a perfect inertial frame is the Foucault pendulum. A

[2] The fundamental particles which are the constituents of matter are subject to gravitational and electromagnetic forces, and also to two other forces, the *strong* and the *weak* force. The latter two forces do not come to play in everyday observations.

plumb bob attached by a string to the ceiling of a building at the North or South Pole will oscillate in a plane which is non-rotating with respect to the distant stars. The earth rotates relative to the plane of the pendulum.

Newton was principally interested in calculating the orbits of the planets. For this purpose he used axes whose origin is at rest with respect to the sun, and which are non-rotating with respect to the distant stars. Thse are the best inertial frame he could find and he seems to have regarded it as obvious that these axes are at rest in "absolute space" which "without relation to anything external, remains always similar and immovable". Kepler observed that, relative to these axes, the planets move in elliptical orbits and that the periodic times of the planets (i.e. the time required for the planet to make one circuit of the sun) are proportional to the 3/2 power of the semi-major axis of the ellipse. Newton used his laws of mechanics, plus his law of universal gravitation (which gave a quantitative formula for the force exerted by the sun on the planets) to explain Kepler's observations, *assuming that the axes in question are an inertial frame or a close approximation thereto*. Furthermore, he was able to calculate the orbit of Halley's comet with great accuracy.

Most (perhaps all) physicists today would say that the notion of "absolute space" is elusive or meaningless, and that the inertial frames are physically defined by the influence of distant matter. Ordinarily, when we enumerate the forces acting on a body we include only the forces exerted on it by other bodies which are fairly close to it; e.g. if the body is a planet we take account of the gravitational force which the sun exerts on the planet but we do not explicitly take account of the force which the distant stars exert on the planet (nor do we really know how to calculate that force). Nevertheless, the effect of the distant stars cannot be ignored since they determine which are the inertial frames, i.e. they single out the preferred axes with respect to which Newton's laws are true. The important characteristic of axes attached to the sun is not that they are at rest in absolute space, but that they are freely falling under the gravitational influence of all the matter outside the solar system.

In most of the homely examples which we shall discuss, axes attached to the surface of the earth can be treated as an inertial frame. In discussion of planetary motion we shall use axes attached to the sun which do not rotate relative to the distant stars.

2.3 Quantitative Definition of Force; Statics of Particles

Newton's second law, which we shall discuss in the next chapter, states that the acceleration of a body is proportional to the total force acting on the body. Some authors use this fact as the basis for a quantitative definition of force. We propose to define "force" quantitatively *before* discussing the second law. Thus it will be clear that the second law is a statement about the world and not just a definition of the word "force". We shall then gain familiarity with the analysis of forces by studying a number of examples of the static equilibrium of particles.

As the unit of force we choose some elementary, reproducible push or pull. This could, for example, be exerted by a standard spring stretched by a standard amount at a standard temperature. Two units of force would then be the force exerted by two such standard springs attached to the same object and pulling in the same direction (depending on the characteristics of the spring this might or might not be the same as the force exerted by a single spring stretched by twice the standard amount).

More whimsically, if we imagine that we have a supply of identical mice which always pull as hard as they can, then the "mouse" can be used as the unit of force. One mouse pulling in a given direction can be represented by an arrow of unit length pointing in that direction. Three mice pulling in the same direction can be represented by an arrow of length three pointing in that direction. The definition is easily extended to fractional numbers of mice; e.g. if it is found that seven squirrels pulling in a given direction produce exactly the same effect as nineteen mice pulling in that direction, then we represent the force exerted by one squirrel by an arrow of length 19/7 in the appropriate direction.[3] Thus any push or pull in a definite direction can be represented by an arrow in that direction, the length of the arrow being the number of mice required to perfectly mimic the given push or pull.

Since we have established a procedure for representing forces by arrows which have lengths and directions, it seems almost obvious that forces have all the properties of vectors. In particular, suppose two teams of mice are attached to the same point on the same body. Let one team consist of N_1 mice all pulling in the same direction (represented by a vector $\vec{N}_1$),

[3] Since any irrational number is the limit of a sequence of rational numbers, the definition is readily extended to forces which are equivalent to an irrational number of mice (which is not the same as a number of irrational mice!)

and let the other team consist of N_2 mice all pulling in another direction (represented by a vector $\vec{N}_2$). Is it obvious that the two teams of mice pulling simultaneously are in all respects equivalent to a single team of mice where the direction of the single team is the direction of the vector $\vec{N}_1 + \vec{N}_2$ and the number of mice in the single team is $|\vec{N}_1 + \vec{N}_2|$, i.e. the length of the vector $\vec{N}_1 + \vec{N}_2$? I think there is an important point here which needs proving and can be proved without recourse to experiment. Since many readers may regard the proof as an unnecessary digression, we present it as an Appendix (Appendix C.1).

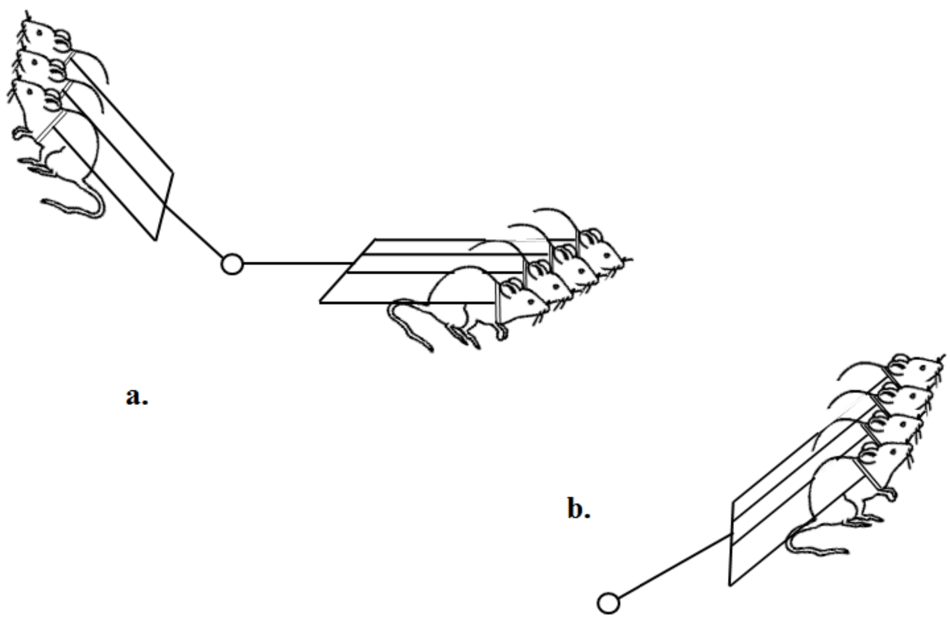

Figure 2.1: Two teams of mice are attached to the same point on the same body (figure **a.** above). One team consists of N_1 mice pulling in the same direction represented by the vector $\vec{N}_1$ and the other team consists of N_2 mice pulling in the same direction represented by the vector $\vec{N}_2$. Is it obvious that the two teams of mice are equivalent to a single team (figure **b.**), where the direction of the single team is the direction of the vector $\vec{N}_1+\vec{N}_2$ and the number of mice in the single team is the magnitude of the vector $\vec{N}_1 + \vec{N}_2$? See Appendix C for a proof.

At any rate, we should clearly understand that when we represent forces by vectors we are saying not only that a force has magnitude and direction, but that *two forces* (each represented by a vector) *acting simultaneously at* *the same point are equivalent to a single force, represented by the vector sum of the two force vectors.* It follows that when more than two forces act at a point, they are equivalent to a single force, represented by the vector sum of the vectors representing the individual forces.

Now we can discuss the *equilibrium* of point masses. A "point mass" is a body so small that we measure only its location and ignore the fact that different parts of the body may have different velocities. We shall see shortly that, as a consequence of Newton's third law, Newton's laws apply not only to point masses but to larger composite objects consisting of several or many point masses.

A body is said to be in ***equilibrium*** when it is at rest (not just for an instant, but permanently or at least for a finite period of time[4]) or moving with constant velocity. According to Newton's first law, *there is no force on a body in equilibrium.*

The simplest example of equilibrium would be a particle in the outer reaches of the solar system, sufficiently far from the sun and the planets so as to be subject to negligible gravitational forces. This example is not very interesting since no forces act on the particle. The more interesting examples of equilibrium, as encountered in everyday life, are situations in which the net (or "resultant") force on a body is zero, but several forces are acting on the body; thus *the equilibrium of the body results from the fact that the vector sum of all the forces acting on the body is zero.*

2.4 Examples of Static Equilibrium of Particles

Example 2.1 : Static equilibrium of a block on the floor.

As a first example of equilibrium, we consider a block at rest on the floor. We assume here that the block can be treated as a "point mass" which obeys Newton's first law. Even if the block is not very small we shall see shortly that Newton's third law justifies treating the block as a "point mass".

Two forces act on the block: the earth pulls downward on the block and the floor pushes upward on the block. Since the total force on the block

[4] If a ball is thrown vertically upward, it will be instantaneously at rest at the moment when it reaches its highest point. The ball is *not* in equilibrium at that instant since it does not remain at rest for a finite time and the net force on the ball is not zero. On the other hand, a ball lying on the ground *is* in equilibrium.

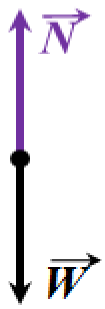

Figure 2.2: Free-body diagram for a block resting on the floor. $\vec{W}$ is the gravitational force exerted by the earth on the block. $\vec{N}$ is the contact force exerted by the floor on the block.

must vanish, the two forces must be equal in magnitude and opposite in direction. However they are entirely different in their origins. The downward pull exerted by the earth (frequently called "gravity" or the "weight" in colloquial terms) is simply the vector sum of the gravitational forces exerted on the block by every molecule of the earth. The contribution to this sum from nearby molecules (within a few miles from the block) is negligible; the gravitational force on the block is caused mainly by its attraction to distant molecules because there are so many distant molecules. On the other hand, the upward force exerted by the floor is a very short-range electrical force (the "contact" force which we have already mentioned) exerted by molecules located in the surface of the floor on molecules located in the bottom of the block.

The magnitude of the gravitational force exerted by the earth on a body is called the ***weight*** of the body and is usually denoted by the letter W. The numerical value of W depends on what we choose as the unit of force. In the so-called English system of units the unit of force is not the mouse but the ***pound***. The pound may be defined as the gravitational force exerted by the earth on a certain standard object placed at a certain location on the earth's surface. For example, the object could be 454 cubic centimeters (27.70 cubic inches) of water at a temperature of 4 degrees centigrade and atmospheric pressure, located at 33rd and Walnut Streets in Philadelphia. In the metric system the unit of force is the ***newton***, which is approximately .225 pounds.

[Note: the Google web site will do many kinds of common conversions for you. There's no need to memorize specific conversion factors. Nevertheless, it is useful to know (at least approximately) common conversion factors, e.g. meters $\leftrightarrow$ feet, centimeters $\leftrightarrow$ inches, kilowatts $\leftrightarrow$ horsepower, etc.] We often abbreviate the newton unit as N. It is often useful to draw a ***free-body diagram*** in which each of the forces acting on a body is represented by a vector. Usually all of the vectors are drawn originating from a common origin. If the body is in equilibrium, the vector sum of all the forces in the free-body diagram is zero. In the present example, the free-body diagram is very simple (Fig.2.2) and adds nothing to our verbal description. In the following more interesting examples, quantitative conclusions can be drawn from the free-body diagram.

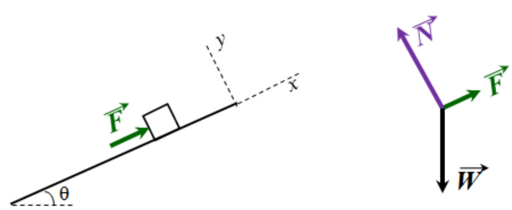

Figure 2.3: The block is held in place by a force applied parallel to the incline. The free-body diagram includes three forces.

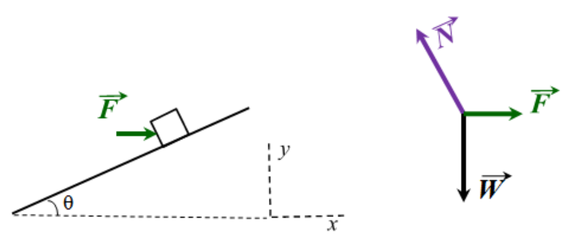

Figure 2.4: See example 2.2b. In this case the block is held in place by a horizontal applied force.

Example 2.2 : A block in equilibrium on a frictionless incline.

Consider a block in equilibrium on a smooth inclined plane. The earth pulls downward on the block and the plane exerts a contact force on the block in a direction normal (perpendicular) to the plane. Evidently, a third force must be applied to the block if it is to be in equilibrium. Experience tells us that the direction of this third force is not uniquely determined. For example, the block can be kept in equilibrium by an appropriate force applied parallel to the plane in the uphill direction (Fig.2.3) or by an appropriate horizontal force (Fig.2.4). In fact, there is a continuum of possible directions for the third force. If the block is to be in equilibrium, the third force must have the appropriate magnitude. This depends on the direction in which it is applied.

(a). Let us first look at the case where the third force is applied parallel to the incline. The free-body diagram for this case exhibits the forces acting on the block. $\vec{W}$ is the gravitational force exerted by the earth, $\vec{N}$ is the contact force exerted by the incline and $\vec{F}$ is the third force. We show it as indicating a push from below parallel to the incline but the same free-body diagram results from a *pull* directed parallel to the incline but from a point of contact on the uphill side of the body. In the latter case we might supply such a force with a string where $\vec{F}$ would then be the ***tension*** in the string. We wish to calculate $\vec{F}$ and $\vec{N}$.

Newton's first law says

$$\vec{F} + \vec{N} + \vec{W} = 0 \tag{2.1}$$

This is a vector equation and is true only if the sum of the x-components of the three vectors is zero and the sum of the y-components is also zero. We can choose the directions of the x- and y-axes to suit our convenience; the most convenient choice is to take the x-axis parallel to the incline and the y-axis perpendicular to the incline. Then the x- and y-components of eqn.(2.1) are

$$F - W \ \sin\theta \ = \ 0 \tag{2.2a}$$
$$N - W \ \cos\theta \ = \ 0 \tag{2.2b}$$

Thus $F = W\sin\theta$ and $N = W\cos\theta$. Note that if we had taken the x-axishorizontal and the y-axisvertical, then the x- and y-components of eqn.(2.1) would be $F\cos\theta - N\sin\theta = 0$ and $F\sin\theta + N\cos\theta - W = 0$, which again yield $F = W\sin\theta$ and $N = W\cos\theta$.

(b). If the third force is applied horizontally (we call it $\vec{F}$ again), the free-body diagram is shown in Fig.2.4. If we take the x-axis horizontal and the y-axis vertical, the components of the vector equation $\vec{F}+\vec{N}+\vec{W} = 0$ are $F - N\sin\theta = 0$ and $N\cos\theta - W = 0$. Thus $N = W/\cos\theta$ and $F = W\tan\theta$. Note the different expressions for N in this case and the previous one; note also that in this case F becomes infinite as θ approaches 90°. Does this agree with your "physical intuition"?

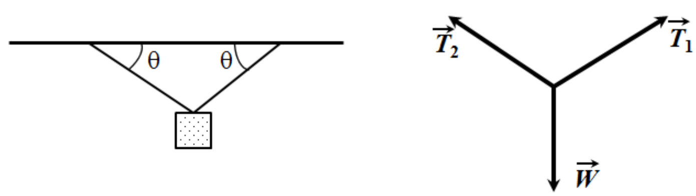

Figure 2.5: Illustration and free-body diagram for Example 2.3.

Example 2.3 : A block suspended from the ceiling by two strings.

Consider a block suspended by a pair of strings of equal length both of which are attached to the ceiling and make an angle θ with the horizontal (see Fig.2.5). The free-body diagram exhibits the forces acting on the block. $\vec{W}$ is the gravitational force exerted by the earth and $\vec{T}_1$ and $\vec{T}_2$ are the forces exerted by the two strings. From the symmetry of the problem it is evident that $\vec{T}_1$ and $\vec{T}_2$ have the same magnitude, which we call T. If we take the x-axis horizontal and the y-axis vertical, the components of the vector equation $\vec{T}_1 + \vec{T}_2 + \vec{W} = 0$ are

$$T\cos\theta - T\cos\theta = 0$$
$$2T\sin\theta - W = 0$$

The first of these equations tells us nothing but merely confirms our assumption that the tensions in both strings are equal. The second tells us that the required tension in the strings is $T = W/(2\sin\theta)$.

Note that T becomes very large when θ is very small (in fact $T \to \infty$ as $\theta \to 0$). Thus we see that a rather modest sideways force applied to a taut wire can break the wire. However, if we apply a sideways force to the center of a taut nylon rope, the rope will stretch; since θ does not remain small, the tension in the rope will not become very large.

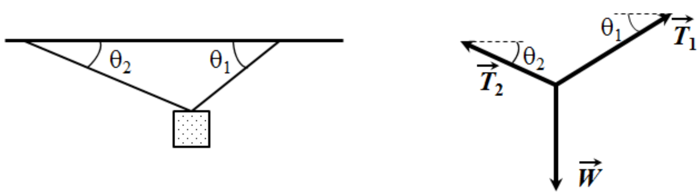

Figure 2.6: Illustration and free-body diagram for Example 2.4.

Example 2.4 : A block suspended from the ceiling by two strings of different lengths.

In the previous example the two strings might have been of different lengths, so that $\theta_1 \neq \theta_2$ (Fig.2.6). Note that the two strings must be seperately attached to the block. (If the block hangs from a ring which is free to slide along the string, the ring will slide until $\theta_1 = \theta_2$.) In the present case $T_1 \neq T_2$ and the components of the force equation are

$$\begin{aligned} T_1\cos\theta_1 - T_2\cos\theta_2 &= 0 \\ T_1\sin\theta_1 + T_2\sin\theta_2 - W &= 0 \end{aligned}$$

Solving this pair of simultaneous linear equations, we find

$$\begin{aligned} T_1 &= \frac{W}{\sin\theta_1 + \cos\theta_1\tan\theta_2} \\ T_2 &= \frac{W}{\sin\theta_2 + \cos\theta_2\tan\theta_1} \end{aligned}$$

Note that when $\theta_1 = \theta_2$ the result agrees with that of the previous example.

2.5 Newton's Third Law

We have not yet considered the static equilibrium of systems which consist of *several* bodies, which may exert forces on each other. In order to discuss such systems we must introduce a very important property of forces which has not yet been mentioned and cannot be deduced from anything which has been said up to this point. Newton's statement of this property is:

> "To every action there is always opposed an equal reaction; or, the mutual actions of two bodies upon each other are always equal, and directed to contrary points."

This is called ***Newton's Third Law of Motion***. Newton goes on to give some examples of the third law:

> "If you press a stone with your finger, the finger is also pressed by the stone. If a horse draws a stone tied to a rope, the horse (if I may say so) will be equally drawn back towards the stone ..."

In modern language, the third law can be stated as follows:

> "For every force which A exerts on B, B exerts an equal and oppositely directed force on A."

These forces are called an action-reaction pair.

The third law has very important consequences and it is essential to understand exactly what the third law asserts. Let us look again at Example 2.1 (the block on the floor). There are two forces acting on the block: $\vec{W}$ (the gravitational force exerted by the earth) and $\vec{N}$ (the contact force exerted by the floor). The reaction to $\vec{W}$ is the gravitational force exerted *on* the earch *by* the block. This force is represented by the vector $-\vec{W}$, i.e. the magnitude of the gravitational force exerted by the block on the earth is W but the direction of this force is upward. The reaction to $\vec{N}$ is the contact force exerted *by* the block *on* the floor. As we have previously noted, this force is a very short-range electrical force. This force has the same magnitude as $\vec{N}$ but opposite direction (downward), and is represented by the vector $-\vec{N}$.

A common misconception is that $\vec{W}$ and $\vec{N}$ are an action-reaction pair. Note that the two forces in an action-reaction pair do not act on the same body (one is exerted by A on B while the other is exerted by B on A). Since $\vec{W}$ and $\vec{N}$ act on the same body (the block), they cannot be an action-reaction pair. Furthermore, the two forces in an action-reaction pair are of the same physical origin, e.g. both are gravitational forces or both are contact forces. But $\vec{W}$ is a gravitational force and $\vec{N}$ is a contact force, so again we see that they cannot be an action-reaction pair.

The third law is true even when the bodies under consideration are *not* in equilibrium. For example, a body falling toward the earth exerts a gravitational force on the earth which is equal and opposite to the gravitational force exerted by the earth on the body. When a baseball bat strikes a ball, the ball exerts a force on the bat which is equal in magnitude and opposite in direction to the force which the bat exerts on the ball. A familiar paradox is posed by the question "If the force exerted by the ball on the bat is equal and opposite to the force exerted by the bat on the ball, then why does the ball accelerate?" The answer is provided by Newton's second law, which states that the acceleration of the ball is proportional to the *net* force acting <u>on the ball</u>. Thus, the force exerted by the ball on the bat is irrelevant to the question of whether the ball accelerates.

The temptation to (incorrectly) identify $\vec{W}$ and $\vec{N}$ as an action-reaction pair arises partially from the fact that in equilibrium $\vec{N} = -\vec{W}$. If we consider a non-equilibrium situation in which the block is on the floor of an elevator which is accelerating upward, then the net force acting on the block is not zero, i.e. $\vec{W} + \vec{N} \neq 0$. In this case the block accelerates upward because the upward contact force exerted by the floor on the block is *greater* that the downward gravitational force exerted on the block by the earth. Nevertheless, Newton's third law is valid: the gravitational force exerted on the earth by the block is equal and opposite to the gravitational force exerted on the block by the earth, and the contact force exerted on the floor by the block is equal in magnitude and opposite in direction to the contact force exerted on the block by the floor.

Strictly speaking, not all the forces in nature obey the third law exactly.[5] However, if we stay away (as we will in this book) from situations in which a large amount of electromagnetic radiation is being produced, the third law is true. A commonly (and erroneously) cited example of a force which violates the third law is the magnetic force between two short segments of wire, each of which carries an electric current. In fact, the force between two current-carrying segments is not measurable, and the only physically significant force is the force between two closed loops of wire. This obeys the third law.

[5]With appropriate modification of the statement of the third law, all forces obey the law. However, the modified statement is somewhat abstract and not useful for our purposes.

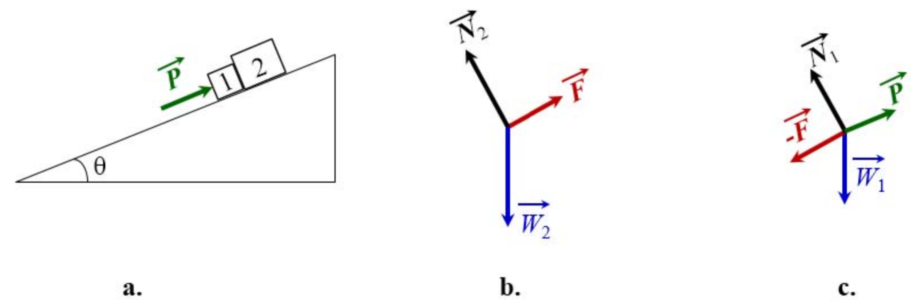

Figure 2.7: Illustration for Example 2.5. Two blocks are held in equilibrium by a force $\vec{P}$ applied parallel to the incline. Figure b. shows the free-body diagram for block #2 while figure c. shows the free-body diagram for block #1.

Example 2.5 : Static equilibrium of blocks on an incline.

As a first illustration of the use of Newton's third law we consider a slight generalization of Example 2.2. Suppose we have two blocks on a smooth inclined plane, the upper block (#2) being supported by a lower block (#1), which is supported by an external force $\vec{P}$ applied parallel to the incline.

The free-body diagram for block #2 (see Fig.2.7) includes the force $\vec{W}_2$ exerted by the earth, $\vec{N}_2$ exerted by the plane and the force $\vec{F}$ which block #1 exerts on block #2. We assume that the blocks are rectangular with smooth faces so that $\vec{F}$ is parallel to the incline. The free-body diagram for block #1 includes (in addition to $\vec{W}_1$ and $\vec{N}_1$) the force which block #2 exerts on block #1. By Newton's third law, this force is $-\vec{F}$. Furthermore the free-body diagram for block #1 must include the external force $\vec{P}$ which is applied to block #1. Note clearly that $\vec{P}$ is a force acting on block #1, *not* on block #2. The force exerted by #1 on #2 is $\vec{F}$.

The condition of equilibrium for block #2 is

$$\vec{F} + \vec{N}_2 + \vec{W}_2 = 0 \tag{2.3a}$$

and for block #1

$$\vec{P} - \vec{F} + \vec{N}_1 + \vec{W}_1 = 0 \tag{2.3b}$$

The components of eqn.(2.3a) and eqn.(2.3b) along axes parallel and perpendicular to the incline are

$$\begin{aligned} F - W_2 \sin\theta &= 0 \\ N_2 - W_2 \cos\theta &= 0 \\ P - F - W_1 \sin\theta &= 0 \\ N_1 - W_1 \cos\theta &= 0. \end{aligned}$$

These are four equations in the four unknowns $P,\ F,\ N_1,\ N_2$. Solving, we obtain

$$F = W_2 \sin\theta \tag{2.3c}$$
$$P = (W_1 + W_2) \sin\theta \tag{2.3d}$$
$$N_1 = W_1 \cos\theta \tag{2.3e}$$
$$N_2 = W_2 \cos\theta \tag{2.3f}$$

Note that without the third law we would have to introduce another unknown force (the force which block #2 exerts on block #1, which we might call $\vec{F}'$). We would then have four equations in five unknowns and the problem would not be mathematically determinate.

In the example above, the formulae for F and N_2 can be obtained directly from eqn.(2.2a) and eqn.(2.2b) since the problem of the equilibrium of the upper block is identical with the problem we have already solved in Example 2.2. More important is the observation that the formula (2.3d) for P can be obtained directly from eqn.(2.2a) if we think of the two blocks as one composite object of weight $W_1 + W_2$. Similarly, from eqn.(2.2b), the total normal force on this composite object is $N = (W_1 + W_2)\cos\theta$.

Is it always permissible to say that the total force on an object in equilibrium is zero, even when the object is a composite system consisting of several parts? With the aid of the third law we can prove that the answer is "yes". Were this not the case, Newton's first law would be applicable only to certain "elementary" objects (presumably of microscopic dimensions) and could not be applied to objects like balls and blocks which really consist of many molecules.

In short, the third law permits us to sidestep the delicate question of what are the "particles" which obey Newton's laws of motion. If sufficiently small objects obey Newton's first law, then the third law implies that larger objects will also obey the first law.[6]

We define a *system* as any collection of particles (a particle is an object sufficiently small that it obeys Newton's laws). The particles are enumerated by an index $i = 1,\ 2,\ ...,\ N$. We say that the system is in equilibrium when every particle of the system is in equilibrium (i.e. at rest or moving with constant velocity).

If $\vec{F}_i$ is the total force acting on the i^{th} particle, then in equilibrium $\vec{F}_i = 0$ for every i, and thus

$$\sum_{i=1}^{N} \vec{F}_i = 0$$

We can write $\vec{F}_i$ as the sum of two terms

$$\vec{F}_i = \vec{F}_{i,\ \text{ext}} + \sum_{j=1//j\neq i}^{N} \vec{f}_{ji} \tag{2.4}$$

where $\vec{F}_{i,\ \text{ext}}$ is the *external* force on the i^{th} particle (i.e. the force exerted on the i^{th} particle by particles which are not included in the system) and $\vec{f}_{ji}$ is the force exerted *by* the j^{th} particle *on* the i^{th} particle.[7] Thus the equation $\sum_i \vec{F}_i = 0$ becomes

$$\sum_i \vec{F}_{i,\ \text{ext}} + \sum_i \sum_j \vec{f}_{ji} = 0.$$

But the terms in the double sum cancel each other in pairs; for example, the sum includes both $\vec{f}_{21}$ and $\vec{f}_{12}$, which are an action-reaction pair and therefore add vectorially to zero. Thus the double sum vanishes (i.e. the total internal force vanishes; this is frequently summarized by the statement that "you can't lift yourself by your own bootstraps") and we have

$$\sum_i \vec{F}_{i,\ \text{ext}} = 0. \tag{2.5}$$

Therefore, if a system of particles is in equilibrium, the total **external** force on the system must vanish.

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[6]The third law also permits the second law to be applied to large "composite" objects as we shall shortly see.

[7]We assume that a particle cannot exert any force on itself, i.e. $\vec{f}_{ii} = 0$. Strictly speaking, this follows from Newton's third law which says $\vec{f}_{ii} = -\vec{f}_{ii}$ since the action and reaction would be identical in the case of a self-force.

The preceding theorem allows us to apply Newton's first law to either of the blocks in Example 2.5, or to the composite system consisting of both blocks. We are free, of course, to choose any convenient collection of particles as our "system". In subsequent examples we shall see that this freedom can frequently be used to simplify the solution of the problem.

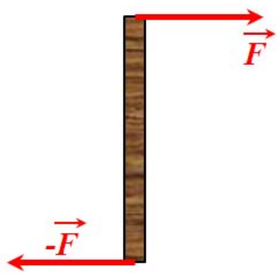


Figure 2.8: This illustrates that even though there is no total force on a system, the system may not be in equilibrium. Equal and opposite forces are applied to the two ends of the rod in this case.

We have proved that if a system is in equilibrium, then the total external force on the system must vanish. We have not proved that if the total force on a system vanishes, the system is in equilibrium (given that $\sum \vec{F}_i = 0$, we cannot infer that $\vec{F}_i = 0$ for every i). Even though the total external force on a system vanishes, the system may still not be in equilibrium. A simple example (Fig.2.8) is a rod acted on by a force $\vec{F}$ (perpendicular to the rod) applied at one end and a force $-\vec{F}$ applied at the other end. Clearly the rod will start to rotate even though the total external force vanishes. Discussion of the *necessary and sufficient* conditions for the equilibrium of rigid bodies is deferred until Chapter 8.

2.6 Ropes and Strings; the Meaning of "Tension"

Many simple and important devices (for example, a block and tackle) consist of several parts which are connected by strings or ropes. Before analyzing such devices we must understand what a string is and what is meant by the tension in a string. For conceptual purposes let us divide the string into two parts, A and B, by means of an imaginary plane perpendicular to the string at an arbitrary point (we stress that the plane is merely a mathematical construct and does not damage the string!). B exerts a force $\vec{T}$ on A (Fig.2.9) and by the third law A exerts a force $-\vec{T}$ on B.

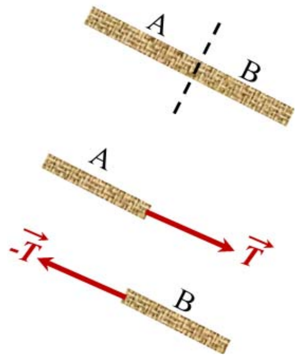


Figure 2.9: Conceptually, we divide the string into two pieces by means of an imaginary plane. B exerts a force $\vec{T}$ on A and A exerts a force $-\vec{T}$ on B. The common magnitude of these two forces is called the ***tension*** at this point in the string.

Figure 2.10: A block of weight W is suspended by a string of weight w per unit length. To find the tension at point P it is useful to define our system as everything below P.

$\vec{T}$ must be parallel (tangent) to the string and must point in the direction from A to B. These two properties distinguish a string from a rod. One part of a rod can exert a sideways force (called a shear) on the adjacent part, as well as a force parallel to the rod. In the case of a rod, the parallel force exerted by B on A can point from A to B (in which case we say B is pulling on A) or from B to A (in which case we say B is pushing on A). One part of a string cannot exert a sideways force on the adjacent part and can only pull (not push) on the adjacent part.

The magnitude of $\vec{T}$ (which is, of course, equal to the magnitude of $-\vec{T}$) is called the ***tension*** in the string at the point in question. We shall see that **under certain circumstances** the tension is the same at all points in a string. However, this is not always the case (see Example 2.6). A question which is frequently asked, but really does not have a precise meaning, is "In what direction does the tension act?" The tension is the common magnitude of two forces; one force acts on A and points from A to B, the other force acts on B and points from B to A.

Example 2.6 : Tension in a string with weight.

Let a block of weight W be suspended from the ceiling by a vertical string of weight per unit length w (see Fig.2.10). We wish to find the tension at a point P which is at a distance x from the lower end of the string. We define our "system" as the block plus the portion of the string below P. Two external forces act on the system: the earth exerts a downward gravitational force of magnitude $W + wx$ and the portion of the string above P exerts an upward force whose magnitude is the tension T at point P. Since the total external force must be zero, we find $T = W + wx$. Thus the tension is not the same at all points in the string and has its maximum value at the highest point of the string. If the string is weightless ($w = 0$) the tension is the same at all points of the string.

More generally, even when a string passes over pulleys, we can show that **the tension is the same at all points of the string provided that the string is weightless and provided that the surface of contact between the string and the pulley is smooth**. This statement is true even in non-equilibrium sitations, but we prove it now only for the case where the string is in equilibrium. Consider a very short segment of the string. The rest of the string exerts forces on its two ends. In addition, if the segment is in contact with a smooth surface, the surface may exert a force perpendicular to the segment. If the string is weightless, there is no gravitational force on the segment. If the segment is in equilibrium, then the total force on it must vanish. In particular, the sum of all the forces along the direction parallel to the segment must vanish. This implies that the tensions at the two ends must be equal. It follows that the tension is the same at all points in the string.

In the following example and in all subsequent examples involving ropes and/or pulleys, it will be assumed (unless otherwise stated) that the weight of the ropes is negligible and the pulleys are smooth; consequently the tension is the same at all points in the rope. The pulleys will also be assumed weightless unless otherwise stated.

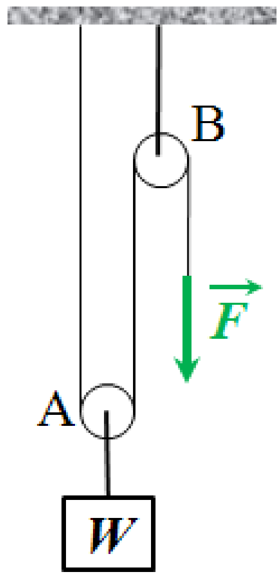


Figure 2.11a: A simple pulley system in Example 2.7.

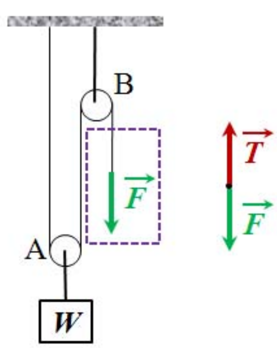


Figure 2.11b: Consideration of the free-body diagram for the system enclosed in the dashed box shows that the tension in the rope is equal to the force $\vec{F}$ applied to the end.

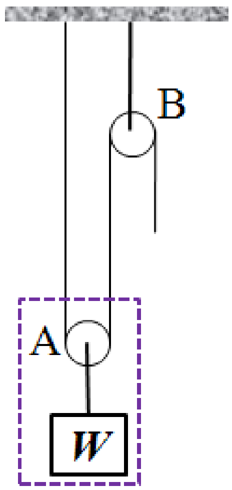


Figure 2.11c: To calculate F it is useful to define our system as the contents of the dashed box.

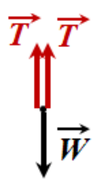


Figure 2.11d: Free-body diagram for the system defined in Fig. 2.11c.

Example 2.7 : Analysis of a simple pulley system.

A simple example of a pulley system is shown in Fig. 2.11a. Note that the pulley B is fixed in space, whereas pulley A moves up and down with the weight W. The obvious question is: what force F must be applied to the end of the rope in order to maintain equilibrium?

First, we must recognize that the tension in the rope is equal to F. To prove this, we take as our system the segment of rope contained within the dashed box in Fig. 2.11b. The only external forces acting on this system are the downward force F and an upward force of magnitude T exerted by the remainder of the rope on the upper end of the segment. Since the total force on the segment must vanish, we find $T = F$.

To calculate F we define our system as the contents of the dashed box in Fig. 2.11c. The only external forces on this system (see Fig. 2.11d) are the downward pull of gravity on the block (the pulleys are assumed weightless) and the upward forces exerted on the U-shaped segment of rope by the remainder of the rope. The condition for equilibrium is $2T - W = 0$. Since $T = F$, we find $F = W/2$. Thus, a 50 lb force applied to the end of the rope can hold a 100 lb block in equilibrium. This fact is summarized by saying that the pulley system provides a two-to-one ***mechanical advantage***. Note that if the pulley system is used to raise the weight W, the free end of the rope must be pulled down by a distance equal to twice the distance which W rises. The pulley system allows you to do the job with a smaller force, but you must exert that force through a larger distance; this illustrates a much more general principle called “conservation of energy”, which is the physicist’s version of the statement that “you can’t get something for nothing”.

With appropriate arrangements of ropes and pulleys we can design pulley systems with arbitrarily large mechanical advantage (see Fig. 2.12).

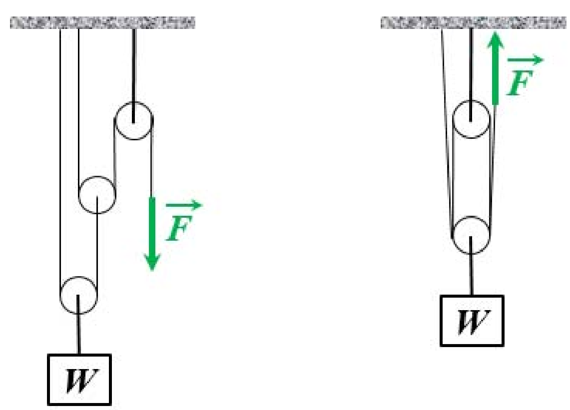


Figure 2.12: Both of these pulley arrangements have a mechanical advantage of four-to-one.

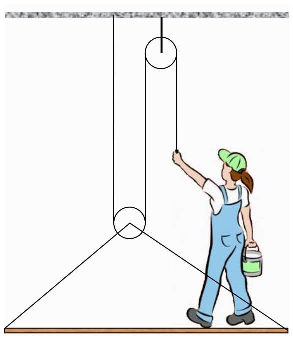


Figure 2.13a: Illustration for Example 2.8.

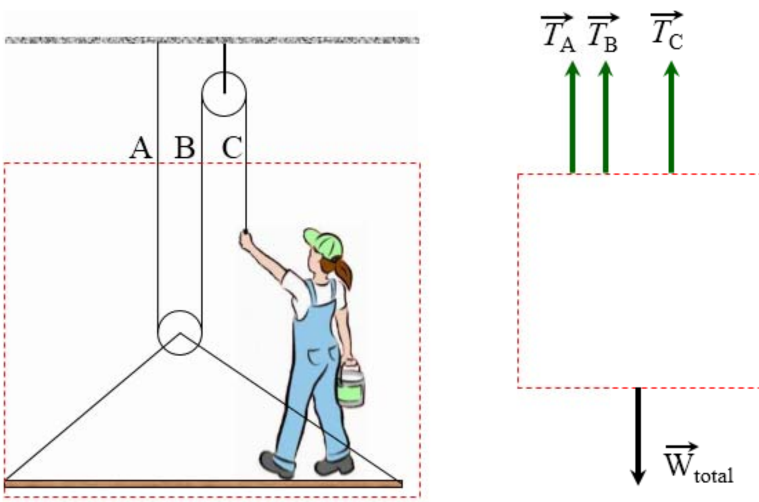


Figure 2.13b: To calculate the tension in the rope, it is useful to consider the system enclosed in the dashed box.

Example 2.8 : Analysis of a boswain pulley system.

Fig. 2.13a depicts a painter (whose weight is 160 lbs or 712 newtons) standing on a scaffold (which weighs 200 lbs or 890 newtons). With what force $\vec{F}$ must the painter pull on the rope to maintain equilibrium? Note that the painter and scaffold are in equilibrium not only when they are at rest, but also when they are ascending or descending with constant velocity.

We have already seen that the tension T must be the same everywhere in a massless rope and that it must be equal to F. The easiest way to calculate $T_A = T_B = T_C = T$ is to consider the system contained inside the dashed box in Fig. 2.13b. For bookkeeping purposes, we introduce a unit vector $\hat{k}$ which points vertically upward. The part of the rope outside the dashed line box exerts a force $\vec{T}_A + \vec{T}_B + \vec{T}_C = 3T\hat{k}$ since the outside part exerts an upward force $T\hat{k}$ on the inside part at each of the three points A, B, C. The force of gravity on the woman and the scaffold is $(-1602\ \mathrm{N})\hat{k}$. Thus the condition of equilibrium is

$$3T\hat{k} - (1602\ \mathrm{N}) = 0 \Rightarrow T = 534\ \mathrm{N}$$

This is about 120 pounds. Since $T = F$ we find the painter must pull with 120 pounds of force.

It is interesting to ask what force the scaffold exerts on the painter's feet. Let us call this force $N\ \hat{k}$. We can calculate N by defining our system as the painter alone or the scaffold alone (but the system of painter-plus-scaffold will not suffice since the force exerted by the scaffold on the painter is an *internal* force in this system and therefore will not appear in the equation of equilibrium). The forces on the painter (Fig. 2.13c) are a force $F\ \hat{k}$ exerted by the rope on the painter's hands, a force $N\ \hat{k}$ exerted by the scaffold on her feet, and a force $(-712\ \text{newtons})\ \hat{k}$ exerted by gravity. Since the painter is in equilibrium we have $F\ \hat{k} + N\ \hat{k} - (712\ \text{newton})\hat{k} = 0$. Since we already know $F = 534$ newtons, we find $N = 178$ newtons or about 40 pounds.

If the world is mathematically consistent, we ought to be able to obtain the same result by considering the equilibrium of the scaffold (to be precise we define our system as the scaffold plus a U-shaped bit of rope as in Fig. 2.13d). The forces on this system are $2T\ \hat{k}$ exerted by the rest of the rope, $(-890\ \text{newtons})\ \hat{k}$ exerted by gravity, and $-N\ \hat{k}$ exerted by the painter's feet. Thus we have $2T\ \hat{k} - (890\ \text{newtons})\ \hat{k} - N\ \hat{k} = 0$ which yields $N = 2T - 890\ \text{newtons} = 178$ newtons. This is about 40 pounds.

If the scaffold is too heavy, the painter will be lifted off it and will be unable to maintain equilibrium. Suppose the weight of the painter is W_{painter} and the weight of the scaffold is W_{scaffold}. Applying Newton's first law to the system painter-plus-scaffold, we obtain $3T\ \hat{k} - (W_{\text{painter}} + W_{\text{scaffold}})\ \hat{k} = 0$, or $T = (W_{\text{painter}} + W_{\text{scaffold}})/3$. Applying the first law to the painter alone we find $F\ \hat{k} + N\ \hat{k} - W_{\text{painter}}\hat{k} = 0$ or $N = W_{\text{painter}} - F$. Since we know $F = T$ we find $N = (2W_{\text{painter}} - W_{\text{scaffold}})/3$. The force which the floor exerts on the painter's feet was defined as $N\ \hat{k}$. As long as N is positive the floor is *pushing* up on the painter's feet. A negative value of N indicates that the floor must be pulling down on the painter's feet; this is not possible unless her feet are strapped down to the floor. Clearly the critical value of W_{scaffold} (beyond which equilibrium will not be possible unless the painter's feet are strapped down) is found by setting $N = 0$. This gives $W_{\text{scaffold}} = 2W_{\text{painter}}$. If the painter weighs 160 pounds, she will "lift off" if the scaffold weighs more than 320 pounds.

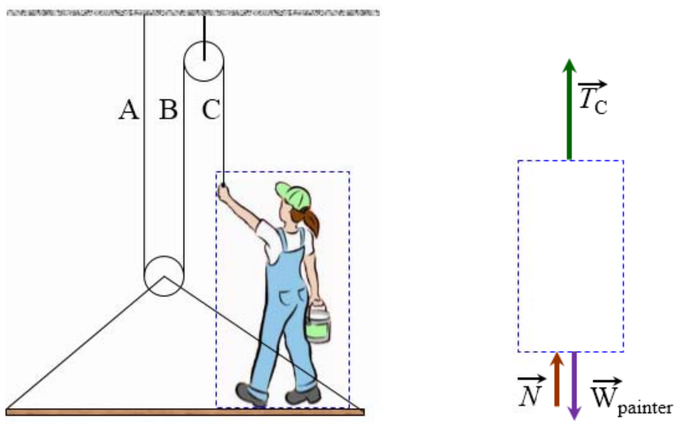


Figure 2.13c: The free-body diagram for the painter. This enables us to calculate the force exerted by the scaffold on the painter's feet.

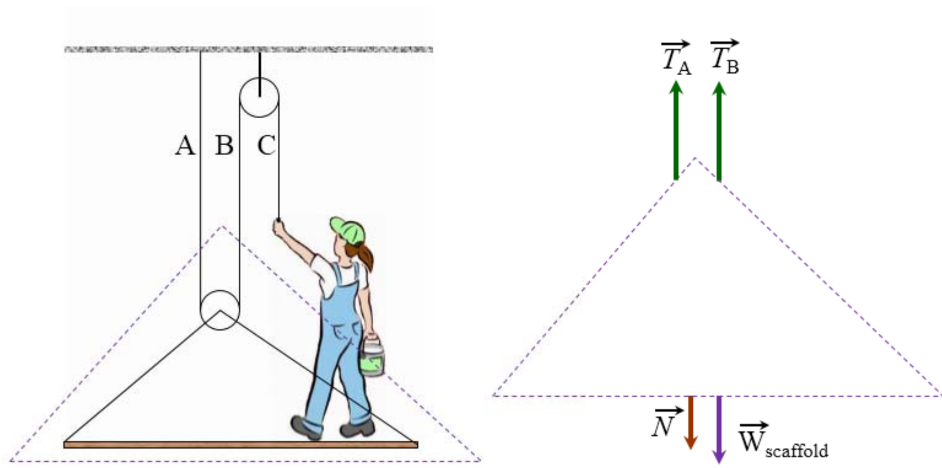


Figure 2.13d: We can also calculate the force exerted by the painter's feet on the scaffold by considering this free-body diagram.

2.7 Friction

When the surfaces of two bodies are in contact, it is frequently the case that the force which one body exerts on the other has not only a component perpendicular to the surfaces, but also a a component parallel to the surface. The latter force is called a ***frictional force*** and plays an essential role in many familiar phenomena. For example, an automobile could neither be driven up a hill nor parked on a hill if friction did not exist. Nor could the car negotiate a curve. In the absence of friction, a passenger standing in a subway car would slide to the rear of the car when the train accelerates. We are accustomed to think of friction as undesirable (and spend considerable money on lubrication to reduce the frictional force which exists when one surface slides along another), but the preceding examples show that friction is frequently desirable and essential.

It should hardly be necessary to emphasize that the validity of Newton's laws does not require the unrealistic assumption of a world without friction. Newton's laws describe the real world with real forces. Indeed, in some problems we assume that there is no friction, but this is not conceptually necessary. Most of Newton's predecessors did not realize that no force is required to keep a body moving with constant velocity; they observed that they had to push an object in order to keep it moving along a horizontal table. This does not imply that Newton's first law is false; the necessary push is simply equal and opposite to the frictional force exerted by the table. Today, with air tracks and air tables which actually suspend a puck or cart above a surface by means of a thin cushion of air, we can come very close to an experimental realization of a frictionless surface. No one who has played with an air track or air table believes that a force is necessary to keep a body moving.

The microscopic origin of frictional forces is not fully understood, nor is such an understanding necessary for our purposes. In some situations one can calculate the magnitude of the frictional forces without even knowning anything about the nature of the surfaces. In other situations it is necessary to know more about the surfaces (i.e. their composition and smoothness). Consideration of a few simple examples will probably prove more enlightening than an abstract discussion of this subject.

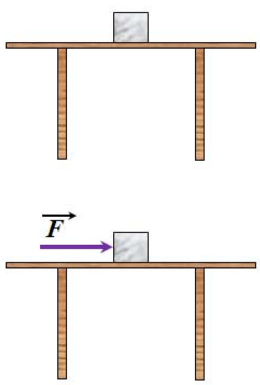


Figure 2.14d: Illustration for Example 2.9.

Example 2.9 : A block on a table with friction.

Consider a block at rest on a horizontal table (Fig. 2.14d). The only forces acting on the block are the gravitational force $\vec{W}$ (directed downward) and the normal force $\vec{N}$ (directed upward) which is exerted by the table. Newton's first law requires $\vec{N}+\vec{W}=0$. Now suppose that a horizontal force $\vec{F}$ is applied to the block. Our experience tells us that on a real table (i.e. a table which is not perfectly smooth) the block will remain at rest if the magnitude of $\vec{F}$ is not too large.

Newton's first law requires that another horizontal force, equal in magnitude and opposite in direction to $\vec{F}$, must be acting on the block. This force, which is exerted by the surface of the table on the bottom surface of the block, is called the frictional force and is denoted by the symbol $\vec{f}$. Newton's first law states[8] $\vec{F} + \vec{f} = 0$. For purposes of simple visualization, one can think of both the table and the block as being covered with minute roughnesses (ridges and valleys), so that their surfaces intermesh to some extent, like a set of gear teeth (Fig. 2.15d). The frictional force is simply the horizontal force exerted by the table's roughnesses on the block's roughnesses.

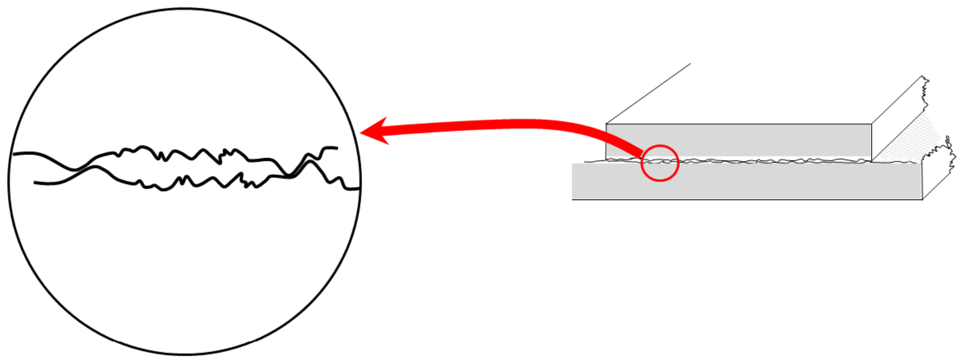


Figure 2.15d: Schematic representation of the origin of static friction. Actually, the roughnesses of surfaces are much smaller than represented here but they exist even for surfaces that are smooth to the touch.

As long as the block in Example 2.9 is at rest, the magnitude f of the frictional force is simply equal to the magnitude F of the applied force. In this case we have calculated f without knowing anything about the nature of the surfaces.

An obvious question is: how large a force F must be applied in order to cause the block to slip? The answer depends, of course, on the materials of which the block and the table are made and also on the smoothness of the surfaces. Even if we know the composition and smoothness of both surfaces, it is virtually impossible to answer this question from first principles since a detailed understanding of various interactions on a microscopic (molecular) scale is required. Fortunately, the question can be answered experimentally and a large amount of data can be summarized by a very simple "law". We stress that **this law is *not* fundamental (in contrast with Newton's**

[8]The full statement of the first law in this case is $\vec{N} + \vec{W} + \vec{F} + \vec{f} = 0$. But since $\vec{N}$ and $\vec{W}$ are vertical and $\vec{F}$ and $\vec{f}$ are horizontal, it follows that $\vec{N} + \vec{W} = 0$ and $\vec{F} + \vec{f} = 0$.

laws) but merely provides a useful summary of experimental data.

In general, when the surfaces of two bodies are in contact, body A exerts a normal force $\vec{N}$ (perpendicular to the surface) and a frictional force $\vec{f}$ (parallel to the surface) on body B, and Newton's third law requires that B exert forces $-\vec{N}$ and $-\vec{f}$ on A. We denote the magnitude of $\vec{N}$ (and $-\vec{N}$) by N and the magnitude of $\vec{f}$ (and $-\vec{f}$) by f. In Example 2.9, N is equal to the weight W of the block and f is equal to the magnitude F of the applied force. N can be varied by placing additional weights on top of the block. For each value of N we can measure $F_{\max}$, the largest value of F which can be applied without causing the block to slip. **It is found that the ratio $F_{\max}/N$ remains constant as N is varied**. This ratio is called the ***coefficient of static friction*** between the two surfaces and is denoted by the symbol μ_s.

The coefficient of static friction depends on the composition and smoothness of the surfaces, but does *not* depend on the area of contact. Thus, if the block were replaced by another block of the same material and smoothness but twice the bottom surface area, the ratio $F_{\max}/N$ would be found to be the same as for the original block.[9] Since $F = f$ as long as the block is not slipping, it follows that $F_{\max} = f_{\max}$, where $f_{\max}$ is the largest value of the frictional force which one surface can exert on the other (for a given value of the normal force).

Thus we arrive at a statement of the ***experimental "law of static friction"***: when two surfaces are in contact with no motion of one surface relative to the other (i.e. no slipping), the maximum value of the frictional force which one surface can exert on the other is proportional to the normal force, the coefficient of proportionality μ_s depending only on the composition and smoothness of the two surfaces, i.e

$$\frac{f}{N} \leq \mu_s \tag{2.6}$$

We stress that eqn.(2.6) is an *inequality.* Equality holds only when slipping is about to occur. Many students fall into the habit of automatically replacing f by $\mu_s N$ with disastrous consequences. For example, if no horizontal force $\vec{F}$ is applied to the block in Example 2.9, then there is **no** frictional force and the ratio f/N is zero even though μ_s is not zero.

[9]The fact that $F_{\max}/N$ does not vary when the area of contact A is varied can be deduced from the fact that $F_{\max}/N$ is independent of N for fixed A. The proof is left as a challenge to the interested reader.

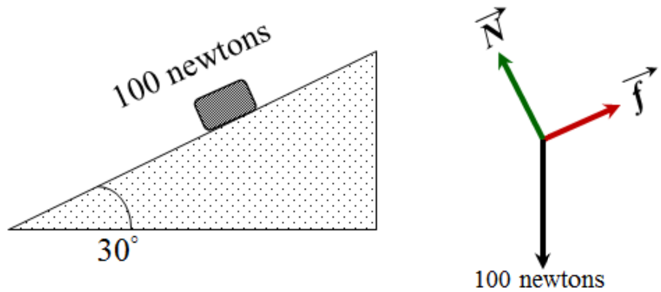


Figure 2.16d: Illustration and free-body diagram for Example 2.10.

Example 2.10 : A block on an incline with friction.

A block of weight 100 newton rests in equilibrium on a plane which is inclined at an angle of 30° with the horizontal. The coefficient of static friction between the plane and the block is $\mu_s = 0.6$. We wish to calculate the frictional and normal forces exerted by the plane on the block.

The free-body diagram for the block is shown in Fig. 2.16d. Taking components of the force equation along axes parallel and perpendicular to the plane, we obtain $f - (100 \text{ newtons}) \sin 30° = 0$ and $N - (100 \text{ newtons}) \cos 30° = 0$. Thus $f = 50$ newtons and $N = 86.6$ newtons.

Note that we have made no use of the given value of μ_s. The value of μ_s enters the discussion only if we ask "Is it really possible for the block to be in equilibrium on the plane?" Examining the ratio f/N, we find $f/N = \tan 30° = 0.577$. Therefore, we find from eqn.(2.6) that the equilibrium is possible, since $0.577 < .6$. More generally, if a block of weight W is in equilibrium on an inclined plane making angle θ with the horizontal, the frictional force is $f = W \sin\theta$ and the normal force is $N = W \cos\theta$. Thus we find $f/N = \tan\theta$. Therefore, static equilibrium is possible only if $\tan\theta \le \mu_s$.

In the above example, the maximum angle for which static equilibrium is possible is[10] $\theta_{\text{max}} = \tan^{-1}(0.6) = 31°$.

[10]We have implicitly assumed that as θ is increased the block eventually starts to slide down the plane. It is also possible that the block may tip over before it starts to slide. Analysis of this possibilty is deferred until Chapter 7 where the *necessary and sufficient* conditions for equilibrium of extended rigid bodies are discussed. Our experience tells us that if the width (dimension parallel to the plane) of the block is sufficiently large compared with the height (dimension perpendicular to the plane), slipping will occur before tipping.

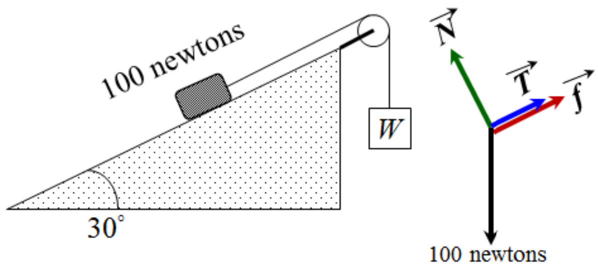


Figure 2.17a: Illustration and free-body diagram for the block when W has the minimum value consistent with equilibrium.

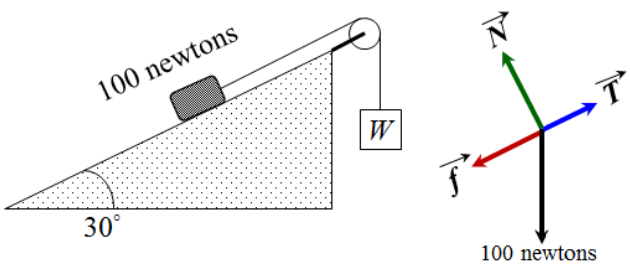


Figure 2.17b: Illustration and free-body diagram for the block when W has the maximum value consistent with equilibrium.

Example 2.11 : A block on an incline with friction connected to a weight.

A box with weight of 100 newtons rests on a 30° inclined plane. The coefficient of static friction between the box and the plane is $\mu_s = 0.4$. A weight W is connected to the box by means of a string passing over a smooth pulley (Fig. 2.17a). Find the maximum and minimum values of W for which static equilibrium is possible.

Note that if $W = 0$ equilibrium is not possible since $\tan 30° > 0.4$ and the block will therefore slide down the plane. When $W = W_{\text{min}}$ (the minimum value necessary for equilibrium) the frictional force f exerted by the plane on the block will be directed uphill and will have the maximum possible magnitude, i.e. $f = \mu_s N = 0.4(100 \text{ newtons}) \cos 30° = 34.6$ newtons. From the free-body diagram for the box we find $T + f - (100 \text{ newtons}) \sin 30^\circ = 0$ where T is the tension in the string. Thus $T = (100 \text{ newtons}) \sin 30^\circ - (34.6 \text{ newtons}) = 15.4$ newtons.

Consideration of the forces acting on the hanging weight yields $T = W$; therefore $W_{\text{min}} = 15.4$ newtons.

As W is increased beyond 15.4 newtons the block remains in equilibrium and the necessary frictional force f decreases. When $W = (100 \text{ newtons}) \sin 30^\circ = 50$ newtons, the frictional force f is zero. As W increases beyond 50 newtons the direction of the frictional force is downhill and f increases again. When $W = W_{\text{max}}$ the free-body diagram for the block is given by Fig. 2.17b, where f has its maximum possible magnitude $\mu_s N = 34.6$ newtons. The component of the force equation parallel to the plane gives $T - f - (100 \text{ newtons}) \sin 30^\circ = 0$. Therefore, $W_{\text{max}} = T = f + (100 \text{ newtons}) \sin 30^\circ = 84.6$ newtons.

Example 2.12 : A block pulled on a horizontal table with friction.

A block of weight 100 newtons rests on a horizontal table. The coefficient of static friction between the block and the table is $\mu_s = 0.8$.

a. If someone pulls on the block with a force F in the direction 37° above the horizontal, what is the minimum value of F which will cause the block to slip?

b. If someone pushes on the block with a force F in a direction 37° below the horizontal, what is the minimum value of F which will cause the block to slip?

c. Same question as (b), except the direction of the push is 53° below the horizontal.

There are some slightly subtle aspects to this problem, especially part (c). Therefore a careful discussion is worthwhile. (Note that 37° and 53° are popular angles in physics problems since they are the angles in the 3-4-5 right triangle; i.e. $\sin 37^\circ = \cos 53^\circ = 0.6$ and $\cos 37^\circ = \sin 53^\circ = 0.8$.)

Let us first consider part (a), assuming that the applied force $\vec{F}$ is small enough that the block is not slipping. From the free-body diagram for the block (Fig. 2.19) we obtain $F \cos 37^\circ - f = 0$ and $F \sin 37^\circ + N -$

(100 newtons) $= 0$. Therefore $f = F\cos 37° = .8F$ and $N = (100\text{ newtons}) - F\sin 37° = (100\text{ newtons}) - .6F$. We should note clearly that N, the normal force exerted by the table on the block is *not* equal to the weight of the block. Equilibrium is possible as long as $f/N < 0.8$. Drawing a graph of the ratio f/N as a function of the applied force magnitude F (as shown in Fig. 2.19), we see that f/N increases as F increases and that f/N tends to infinity as F approaches 166.67 newtons. This is the value of F which makes N vanish. For larger values of F the block is lifted off the plane. In order to find the value of F for which slipping occurs, we set $f/N = 0.8$. Thus we have $.8F/(100 - .6F) = 0.8$ which yields $F = 62.5$ newtons.

We analyze part (b) similarly. Assuming that the block is not slipping, we construct the free-body diagram for the block (Fig. 2.20). The horizontal and vertical components of the force equation yield $F\cos 37° - f = 0$ and $N - F\sin 37° - (100\text{ newtons}) = 0$. Solving for f and N we obtain $f/N = .8F/(100 + .6F)$. Again we draw a graph of f/N as a function of the applied force F (see Fig. 2.20). Note that f/N is an increasing function of F, but tends to a finite limit ($f/N \to 4/3$) as $F \to \infty$. Slipping occurs when $f/N = 0.8$, i.e. $.8F/(100 + .6F) = 0.8$. Solving for F, we find $F = 250$ newtons.

Carrying out a similar analysis for part (c), we obtain $f = F\cos 53° = .6F$ and $N = (100\text{ newtons}) + F\sin 53° = 100 + .8F$ and thus

$$\frac{f}{N} = \frac{.6F}{100 + .8F} \tag{2.7}$$

If we again draw a graph of f/N as a function of F (Fig. 2.21) we see that the limiting value of f/N, as $F \to \infty$, is 0.75. Thus it is clear that the ratio f/N will never attain the value 0.8 no matter how large F is. Therefore it is impossible to cause the block to slip by pushing on it in a direction 53° below the horizontal. As we push harder and harder, the normal force magnitude N increases fast enough so that the plane can always exert a large enough frictional force to counterbalance the horizontal component of the applied force $\vec{F}$.

If we set f/N (as given by eqn.(2.7)) equal to 0.8 and solve for F, we obtain $F = -2000$ newtons. Does this negative value of F have any physical significance? The obvious guess is that a negative push should be interpreted as a pull and that we have shown that a pull of 2000 newtons in a direction 53° above the horizontal will just suffice to make the block slip. This is incorrect! If we repeat the analysis of part (a) for the case when the applied force F is a pull in the direction 53° above the horizontal, we find that the critical value for slipping is $F = 64.5$ newtons. We conclude that the value $F = -2000$ newtons has no physical significance. This illustrates the usefulness of drawing graphs like Fig. 2.21 rather than just formally solving equations. Mathematically, the reason why a negative value of F does not correspond to a pull is that when we replace the symbol F by $-F$ the equations describing case (a) do not go over into those describing case (b).

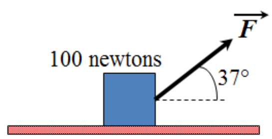


Figure 2.18a: Illustration for Example 2.12a.

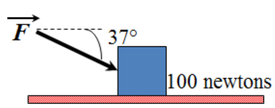


Figure 2.18b: Illustration for Example 2.12b.

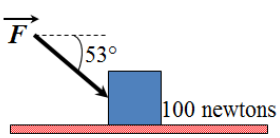


Figure 2.18c: Illustration for Example 2.12c.

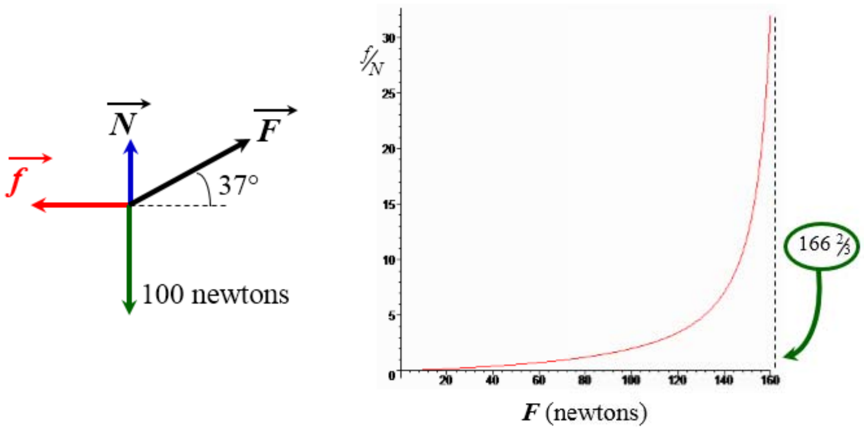


Figure 2.19: Free-body diagram for Example 2.12a.

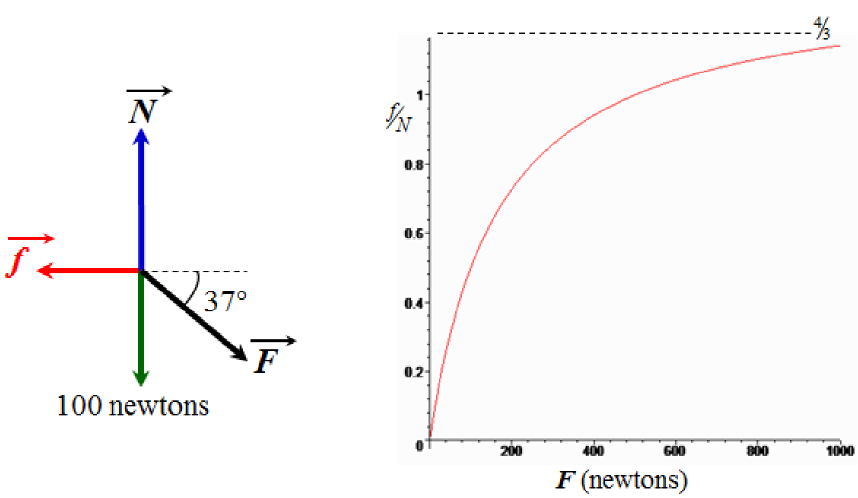


Figure 2.20: Free-body diagram for Example 2.12b.

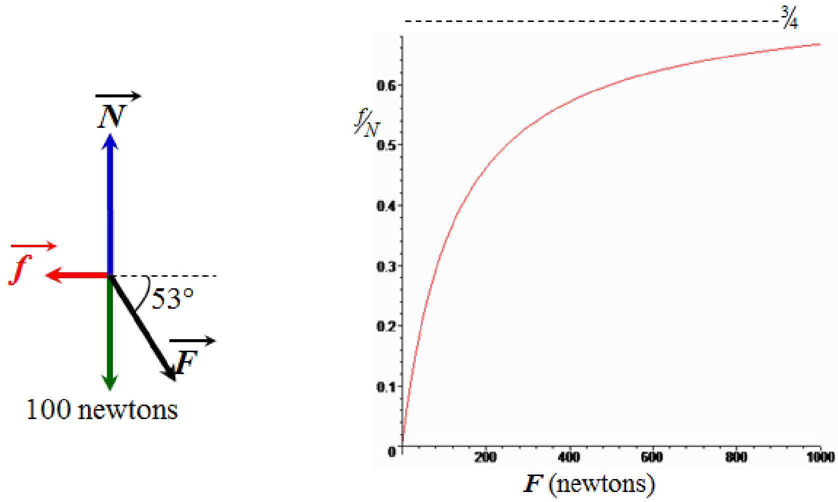


Figure 2.21: Graph of f/N versus F for Example 2.12c.

2.8 Kinetic Friction

We have not yet discussed the frictional force which one surface exerts on another when there is relative motion of the two surfaces. Our experience tells us that the frictional force *opposes* the relative motion. More specifically, if the surfaces of body A and body B are in contact and body A is moving with velocity $\vec{v}$ relative to body B (Fig. 2.22), then the frictional force exerted by B on A is anti-parallel to $\vec{v}$. We also note that to maintain the relative velocity there must be at least one other force acting. If that force is acting on A then it must have a component that is responsible for matching the frictional force in order to maintain the relative velocity of A with respect to B. By Newton's third law, the frictional force exerted by A on B is parallel to $\vec{v}$.

The magnitude f of the frictional force depends on the composition and smoothness of the surfaces, and also on the magnitude of the normal force which one surface exerts on the other. Furthermore, one might expect f to depend on the relative velocity of the two surfaces. Experimentally, it is found that, over a large range of velocities, f does not depend on the velocity and that f is proportional to the normal force N.

Thus, we can state the experimental "law of kinetic friction": If surface A is moving with velocity $\vec{v}$ relative to surface B and the two surfaces are in contact, then the frictional force exerted by B on A is directed anti-parallel to $\vec{v}$ and has magnitude

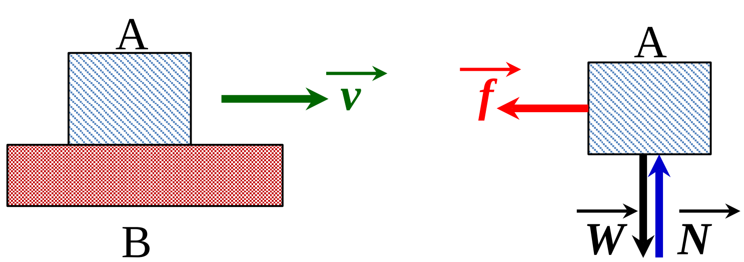


Figure 2.22: If A is moving to the right relative to B, then the frictional force exerted by B on A is directed to the left.

$$f = \mu_k N \tag{2.8}$$

where N is the magnitude of the normal force which one surface exerts on the other.

The coefficient μ_k is called the ***coefficient of kinetic friction*** and does not depend on the area of contact. One should note carefully that the law of kinetic friction is stated in the form of an *equality* $f = \mu_k N$, whereas the law of static friction is an *inequality* $f \leq \mu_s N$. When there is no relative motion of the two surfaces, the normal force does not uniquely determine the frictional force; where there is relative motion, N uniquely determines f. For a given pair of surfaces it is always found that $\mu_k \leq \mu_s$. This is an immediate consequence of the experimental procedure for measuring μ_s.[11]

Most examples involving kinetic friction are concerned with non-equilibrium situations (in which particles are accelerating) and are therefore not discussed until Chapter 3. However, if the velocities of the particles are constant, then the force on each particle must be zero and the example can be discussed in this chapter.

Example 2.13 : One block atop another connected through a pulley.

A block of weight 100 newtons is on top of a block of weight 200 newtons. The two blocks are connected by an inextensible string which passes over a smooth pulley which is attached to a wall (Fig. 2.23a). The coefficient of kinetic friction between the two blocks is 0.6 and the coefficient of kinetic friction between the bottom block and the floor is 0.5. What horizontal force F must be applied to the bottom block to keep it moving to the right with constant velocity?

Since the string is inextensible, the top block is moving to the left with constant velocity (equal in magnitude and opposite in direction to the velocity of the bottom block). By Newton's first law, the force on each block is zero. Force diagrams for the two blocks are shown in Fig. 2.23b and 2.23c. The forces acting on the top block are:

i. the gravitational force, of magnitude 100 newtons, directed downward

ii. the normal force (magnitude N_1) exerted by the bottom block on the top block, directed upward

[11] Suppose a block of weight W is at rest on a horizontal table. The block will slip if we apply a horizontal force $\vec{F}$ infinitesimally greater than $\mu_s W$ in magnitude. But as soon as the block has acquired a very small velocity, the table will exert a force $\mu_k W$ in the direction opposite to the motion. If $\mu_k > \mu_s$, this force will be greater than the applied force and the block will be very quickly decelerated to rest again. Therefore to "really" set the block in motion a force $\mu_k W$ would be necessary. From a macroscopic point of view it would appear in this case that the coefficient of static friction is the value μ_k rather than the value μ_s.

iii. the force exerted by the string (magnitude T), directed to the left

iv. the frictional force (magnitude f_1) exerted by the bottom block on the top block, directed to the right.

The vertical and horizontal components of Newton's first law give $N_1 = 100$ newtons and $T = f_1$. Furthermore, the law of kinetic friction gives $f_1 = 0.6(100 \text{ newtons}) = 60$ newtons. Therefore $T = 60$ newtons.

The forces acting on the bottom block are:

i. the gravitational force, of magnitude 200 newtons, directed downward

ii. the normal force (magnitude N_2) exerted by the floor on the bottom block, directed upward

iii. the force exerted by the string (magnitude T), directed to the left

iv. the frictional force (magnitude f_1) exerted by the top block on the bottom block, directed to the left

v. the frictional force (magnitude f_2) exerted by the floor on the bottom block, directed to the left

vi. the force F, directed to the right.

Newton's first law gives $N_2 - N_1 - (200 \text{ newtons}) = 0$ and $F - T - f_1 - f_2 = 0$. Since $N_1 = 100$ newtons, we find $N_2 = 300$ newtons. Since $f_2 = 0.5\ N_2$, we find $f_2 = 150$ newtons. Since $T = 60$ newtons and $f_1 = 60$ newtons, we find $F = 60 + 60 + 150 = 270$ newtons. (If F were applied to the top block rather than the bottom one, what value of F would be required? This answer is also 270! We can see this easily later when discussing energy considerations.)

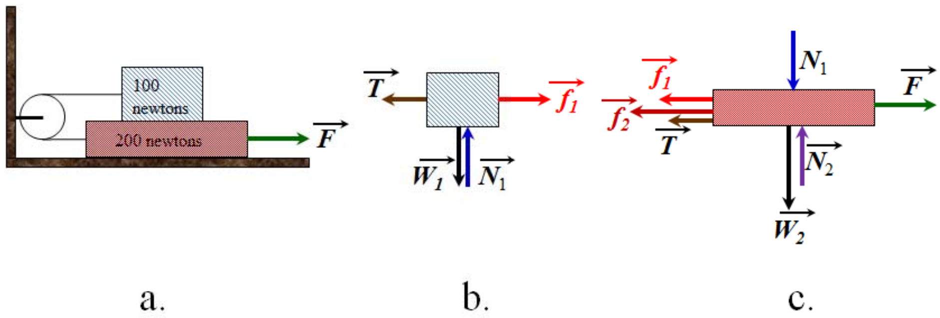


Figure 2.23: (a.) Illustration for Example 2.13. (b.) Force diagram for upper block. (c.) Force diagram for lower block.

Although the preceding example is quite simple, the analysis requires consideration of eleven distinct forces. The student is urged to acquire the habit of carefully listing (mentally if not in writing) all the forces which act in whatever situation is under study. Without such a listing, it is impossible to apply Newton's laws.

2.9 Newton's First Law of Motion Problems

2.1. A block with weight $W_1 = 100$ newtons is suspended vertically from a string which rounds a frictionless, massless pulley wheel to connect to a weight W_2 on an incline at angle $\theta = 30°$ as shown in Fig. 2.24. Answer the following.

(a) If the incline is frictionless, what must the weight W_2 be to maintain equilibrium?

(b) Suppose the incline has a coefficient of static friction of $\mu_s = 0.400$. What are the maximum and minimum values of W_2 consistent with equilibrium?

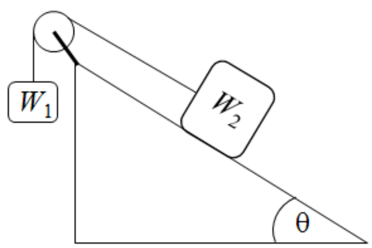


Figure 2.24: Problem **2.1**.

2.2. A weight W_1 is attached to the ceiling by a string. A second string connects W_1 to another weight W_2 which is also subject to a constant horizontal force of magnitude F. Calculate the angles between the two strings and the vertical direction, in equilibrium.

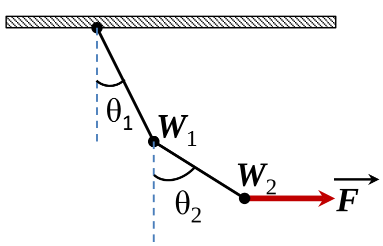


Figure 2.25: Problem **2.2**.

2.3. A skier weighs 620 newtons. She is going down a slope which makes a constant angle of 20.0° with the horizontal. The coefficient of kinetic friction between the skis and the snow is 0.150. When the skier is moving with velocity $\vec{v}$ (measured in m/sec), the air exerts a drag force 0.148 v^2 newton on her, antiparallel to her velocity. She accelerates until she reaches a constant speed v_f, which is called the "terminal velocity". Calculate her terminal velocity.

2.4. A cylindrical pipe (weight W) with smooth walls rests in a trough whose sides are planes making angles θ_1 and θ_2 with the horizontal. Calculate the force exerted by the pipe on each of the walls.

2.5. Three pipes with smooth walls rest in an open box (width $3D$) with a horizontal bottom and vertical walls. Two of the pipes have diameter D and weight W_1 and sit on the bottom of the box with centers separated by distance $2D$. The third pipe has diameter $2D$ and weight W_2 and rests on the other pipes. Calculate the force on each of the vertical walls. [Be careful with the geometry.]

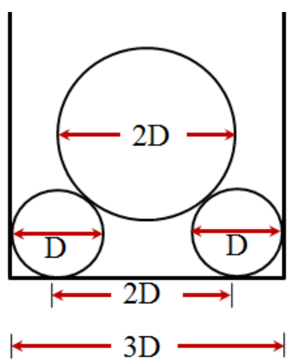


Figure 2.26: Problem **2.5**.

Chapter 3

NEWTON'S SECOND LAW; DYNAMICS OF PARTICLES

3.1 Dynamics Of Particles

Until now we have been discussing bodies in equilibrium, i.e. bodies subject to no net force. According to Newton's first law the velocity remains constant when no net force acts on the body. Newton's second law tells us in a quantitative way how a body modifies its motion when a force acts on the body.

In Newton's own words: "The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed."[New] Not all the words in this statement are familiar to today's physicist. Newton defines "motion" as the product of "the quantity of matter" and the velocity; Newton's "motion" is called "momentum" today. By "change of motion" he means "rate of change of motion".

In the concise language of vectors

$$\vec{F} = m\vec{a} \tag{3.1}$$

Two of the three symbols in eqn.(3.1) have already been defined; $\vec{a}$ is a purely kinematic quantity which was defined in Chapter 1, and $\vec{F}$ was defined in Chapter 2. The third symbol (m) requires some discussion. The definition of “mass” as the “quantity of matter” is somewhat vague. Eqn.(3.1) permits us to devise a precise operational procedure for measuring the mass of a body. The equation states that if we subject a particular body to various forces, the acceleration of the body is proportional to the force acting on the body, i.e. $\vec{a} = k\ \vec{F}$. The proportionality constant k is a property of the body; the larger the value of k, the easier it is to accelerate the body. The ***mass*** of the body is defined as the reciprocal of k, i.e. $m = 1/k$.

In the English system of units the fundamental quantities are length (foot), time (seconds), and force (pounds). The pound was defined as the force exerted by the earth on a certain standard body located at a certain place. We can use eqn.(3.1) to determine the mass m of a body. The mass is the force on the body (measured in pounds) divided by the acceleration (measured in ft/sec^2). The English unit of mass, called the ***slug***, is a derived unit having the dimensions of lb-sec^2/ft. A body whose mass is one slug will experience an acceleration of 1 ft/sec^2 when subjected to a force of 1 lb. A body whose mass is two slugs will experience an acceleration of 0.5 ft/sec^2 when subjected to a force of 1 lb, etc.

Let us apply eqn.(3.1) to a body of weight W freely falling at the “standard location”. It is found experimentally that all freely falling bodies have the same acceleration at a given place; at the standard location the numerical value of this acceleration (which is called g) is 32.174 ft/sec^2. Eqn.(3.1) says $W = mg$. Thus the mass m of a body is related to its weight W by the equation

$$m = \frac{W}{g} \tag{3.2}$$

Note that W and g depend on the place where the body is located, but Newton’s second law asserts that m is a property of the body, independent of the location of the body. A body whose mass is 1 slug weighs 32.174 lb at the standard location but can have a slightly different weight at another location.

In the metric system the fundamental units are length (meters), time (seconds), and mass (kilograms). The kilogram can be defined as the mass of 1000 cm^3 of water under standard conditions of temperature and pressure, but (for greater accuracy) may eventually be defined in terms of other fundamental constants. The metric unit of force, the ***newton***, is the force which must be applied to a body of mass 1 kilogram in order to give that body an acceleration of 1 meter/sec^2. One newton is equal to approximately 0.225 pounds and one pound is equal to approximately 4.45 newtons.

Once we have a body of unit mass, we can measure the mass of any other body by subjecting it and the standard body to the same force and measuring the ratio of the accelerations of the two bodies. If the same force is applied to body #1 and body #2, we have $M_1 a_1 = M_2 a_2$. Thus if $M_1 = 1$, we have $M_2 = a_1/a_2$. Note that this procedure for measuring the mass does not require weighing the object, nor does it require counting the number of protons, neutrons, and electrons in the object.[1]

If $\vec{F} = 0$, then eqn.(3.1) says that $\vec{a} = 0$ and therefore the velocity $\vec{v}$ is constant. Thus it is clear that the second law includes the first law as a special case. It should also be understood that if we include in our list of forces only the forces which have been discussed in Chapter 2, then Newton's second law is true only in an inertial frame. If, for example, we use axes attached to a rotating merry-go-round or to an accelerating freight car, then Newton's second law will not be true; a body will accelerate with respect to such axes even when no forces are acting on it.

We shall now use the second law to analyze a series of examples.

Example 3.1 : A block on a smooth incline.

A block of mass m is sliding down a smooth plane which is inclined at angle θ to the horizontal. Calculate the acceleration of the block and the force which the plane exerts on the block.

Two forces are acting on the block: the gravitational force of magnitude mg, directed vertically downward, and the normal force N exerted by the plane. If we introduce $x-$ and $y-$axes, then the vector equation $\vec{F} = m\vec{a}$

[1] It seems intuitively obvious that if an object is composed of constituent particles (protons, neutrons, and electrons) the mass of the object is equal to the sum of the masses of the constituents. However, if neutrons and protons are assembled to form a heavy nucleus, the mass of the nucleus is found to be about 1% less than the sum of the masses of the constituent particles. Einstein predicted (correctly) that this "mass defect" would be equal to the work necessary to disassemble the nucleus divided by the square of the velocity of light. Thus, Newton's idea of mass as the "quantity of matter" is untenable when the particles are held together by the very strong nuclear force. Similarly, the weight of the nucleus is significantly less than the sum of the weights of the constituent particles. Nevertheless, all objects (protons, neutrons, nuclei, baseballs) fall with the same acceleration g at a given location. This fact has been verified to fantastically high accuracy[HB01]. Thus, the proportionality of weight and mass, expressed by the equation $W = mg$, appears to be a much more sacred law of nature than the additivity of weight or mass. In everyday life, where nuclear forces are not involved, the "mass defect" is very small and, for all practical purposes, the additivity property holds.

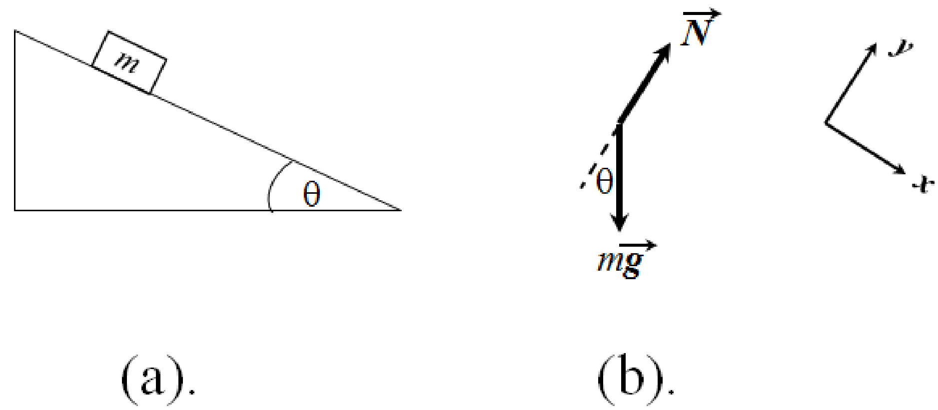


Figure 3.1: Illustration (a) and free-body diagram (b) for Example 3.1.

becomes a pair of equations

$$\sum F_x = ma_x \tag{3.3a}$$

$$\sum F_y = ma_y \tag{3.3b}$$

In this problem it is most convenient to take the x-axis parallel to the inclined plane and the y-axis perpendicular to the inclined plane (Fig. 3.1b). Then the acceleration is only in the $x-$direction; thus $a_x = a$, $a_y = 0$. The normal force has no component in the $x-$direction. The gravitational force has a component $mg\ \sin\theta$ in the $x-$direction and $mg\ \cos\theta$ in the $y-$direction. Thus, eqn.(3.3a) becomes

$$mg \sin\theta = ma$$

and eqn.(3.3b) becomes

$$N - mg\cos\theta = 0.$$

Therefore $a = g\sin\theta$ and $N = mg\cos\theta$.

Note that Newton's second law enabled us to calculate not only the acceleration, but also the magnitude of the normal force N (which was determined by the fact that the *direction* of the acceleration is known so that the sum of all forces perpendicular to that direction is zero). Many students acquire the habit of automatically writing $N = mg\cos\theta$ in all problems involving inclined planes. The following example is intended to show that N is not always equal to $mg\cos\theta$ and that there is no substitute for the orderly application of Newton's laws.

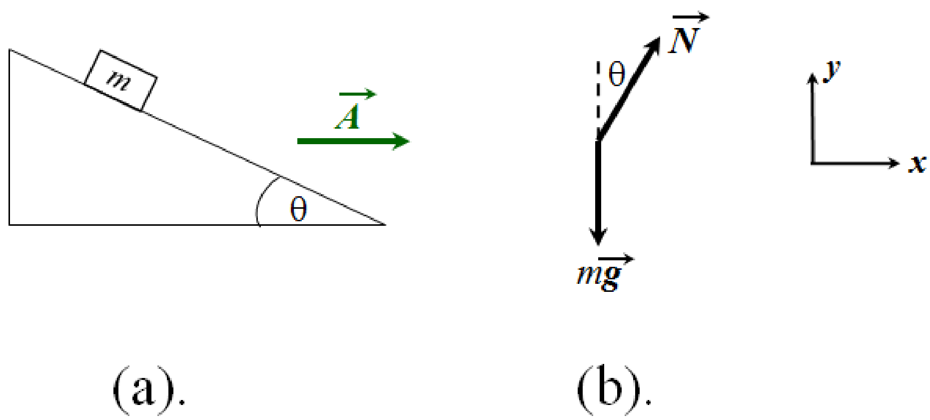


Figure 3.2: Illustration (a) and free-body diagram (b) for Example 3.2.

Example 3.2 : A block on an accelerated smooth incline.

Suppose that the inclined plane in Example 3.1 is one face of a wedge. Suppose that the wedge is accelerating horizontally to the right (for example, the wedge might be attached to an accelerating railroad car). If the acceleration A of the wedge is correctly chosen then the block will not slide up or down the wedge, but will stay at rest relative to the wedge. Calculate the correct value of A and calculate the force which the wedge exerts on the block.

As in Example 3.1, the only forces acting on the block are the gravitational force mg, directed downward, and the normal force $\vec{N}$ exerted by the wedge. If the block is at rest relative to the wedge, then the block has an acceleration $\vec{A}$ directed horizontally to the right (note that this is the acceleration of the block relative to an inertial frame; a frame accelerating with the wedge is "illegal" for use with Newton's second law as it is not an inertial frame).

In this case it is most convenient to take the x-axis horizontal and the

y-axis vertical (Fig. 3.2). The $x-$ and $y-$components of $\vec{F} = m\vec{a}$ yield $N \sin\theta = mA$ and $N \cos\theta - mg = 0$. Thus we find $N = mg/\cos\theta$ and $A = g \tan\theta$. Had we chosen the $x-$ and $y-$ axes parallel and perpendicular to the incline as we did in Example 3.1, then the acceleration would have an $x-$component $A \cos\theta$ and a $y-$component $A \sin\theta$. The $x-$ and $y-$components of $\vec{F} = m\vec{a}$ then become $mg \sin\theta = mA \cos\theta$ and $N - mg \cos\theta = mA \sin\theta$. Solving for A and N, we find $A = g \tan\theta$ and $N = mg/\cos\theta$, as expected.

Note carefully that $\vec{N}$ does not have the same magnitude as in Example 3.1. The difference arises from the fact that in Example 3.1 the acceleration of the block is parallel to the incline, while in this example the acceleration is horizontal.

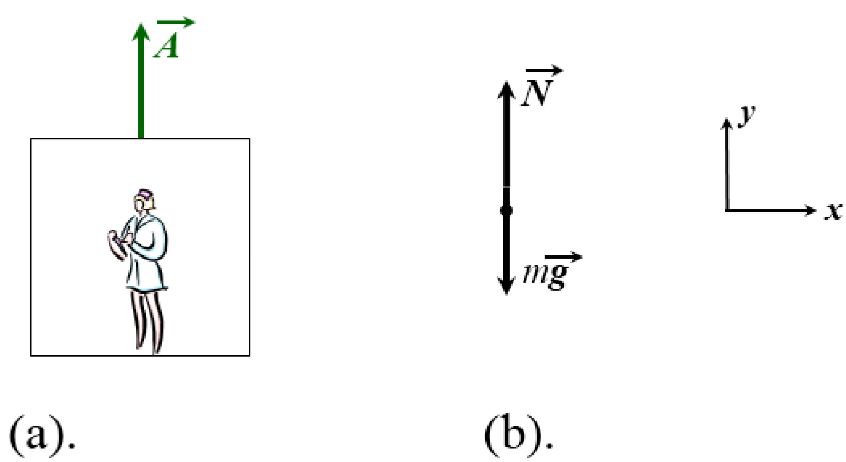


Figure 3.3: Illustration (a) and free-body diagram (b) for Example 3.3.

Example 3.3 : A woman in an elevator accelerating upward.

A woman of mass m is standing in an accelerating elevator. What force does the floor exert on her feet?

To avoid confusion associated with signs, we introduce a unit vector $\hat{k}$ pointing vertically upward. Let the acceleration of the elevator be $A\hat{k}$; thus positive A corresponds to upward acceleration and negative A corresponds to downward acceleration.

Two forces act on the woman (Fig. 3.3): the gravitational force $-mg\hat{k}$ (since $W = mg$) and the force $N\hat{k}$ exerted by the floor. Since the woman's acceleration in an inertial frame is $A\hat{k}$, Newton's second law says $-mg\hat{k} + N\hat{k} = mA\hat{k}$. Thus we find $N = m(A + g)$. If the woman is standing on a spring scale, the dial of the scale indicates the magnitude N. If A is positive (upward acceleration), then the scale reading is greater than mg. The woman "feels" heaver than usual. What she actually feels is the compression of the bones and cartilege in her legs, which enables her feet to exert the force N on the scale. If $A = -g$ (a freely falling elevator), then $N = 0$ and the woman feels weightless since her feet are exerting no force on the floor. In fact, the earth is still exerting a force $-mg\hat{k}$ on her, but she *feels* as though she were living in outer space subject to no gravitational forces.

More generally, we can prove that all phenomena inside the accelerating box are identical with the phenomena inside a box which is non-accelerating but which is situated on a planet where the acceleration of gravity is $-(g + A)\hat{k}$ rather than $-g\hat{k}$. If the box has no windows through which you can look out, there is no way you can tell whether the box is accelerating or is simply located on a different planet. The interested reader will find a proof of this theorem in Appendix D.

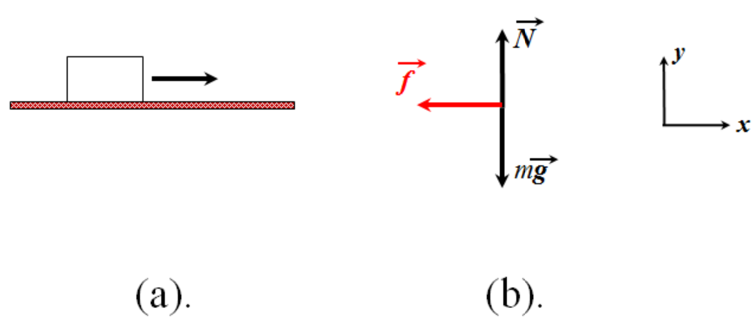


Figure 3.4: Illustration (a) and free-body diagram (b) for Example 3.4.

Example 3.4 : A block sliding horizontally with friction.

A block of mass m is sliding along a horizontal surface. The coefficient of kinetic friction between the block and the surface is μ_k. If the initial speed of the block is v_0, how far will the block travel before coming to rest and how much time will elapse before the block comes to rest?

This very simple problem involves both kinematics and dynamics. Professors like to put this type of problem on an exam since it tests the student's knowledge in two areas and also tests whether he/she has integrated the knowledge into usable form. Dynamics (i.e. $\vec{F} = m\vec{a}$) is needed to calculate the deceleration of the block; once the deceleration is known, the distance and time can be calculated from the kinematic formulae derived in Chapter 1.

We assume that the block is traveling to the right and introduce $x-$ and $y-$ axes as in Fig. 3.3b. The acceleration $\vec{a}$ is purely in the $x-$direction, i.e. $a_x = a,\ a_y = 0$ (we expect that a will be negative). The forces acting on the block (Fig. 3.3) are the gravitational force, the normal force $\vec{N}$ exerted by the surface, and the friction force $\vec{f}$ exerted by the surface. Writing $\sum F_x = ma_x$ and $\sum F_y = ma_y$ we find $-f = ma$ and $N - mg = 0$. Recalling that $f = \mu_k N$, we find $f = \mu_k mg$ and thus $a = -f/m = -\mu_k g$. The distance D traveled in stopping is most easily found from eqn.(1.11d), which yields $D = v_0^2/(2\mu_k g)$. The time T required to stop is most easily found from eqn.(1.11a), which yields $T = v_0/\mu_k g$.

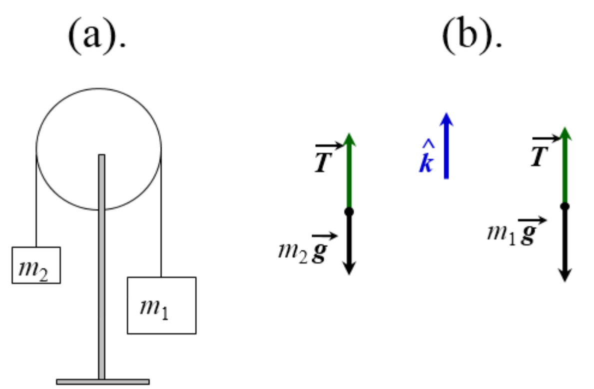


Figure 3.5: Illustration (a) and free-body diagram (b) for Example 3.4.

We now apply Newton's methodology to the analysis of Atwood's machine (Fig. 3.5a), which is simply a pair of masses (m_1 and m_2) connected by a string which passes over a pulley; the position of the center of the pulley is kept fixed by means of a support (e.g. the stand in Fig. 3.5a.). Calculate the acceleration of each mass and the tension[2] in the string.

Example 3.5 : Analysis of Atwood's machine.

In order to avoid sign confusion, we introduce a unit vector $\hat{k}$ pointing vertically up. We define the acceleration of m_2 as $a\hat{k}$ (i.e., if a is positive then the acceleration of m_2 is directed upward). Since the string is assumed inextensible, the acceleration of m_1 is $-a\hat{k}$. If the tension in the string is T the force equation (Newton's second law) for m_2 is

$$T\hat{k} - m_2\; g\hat{k} = m_2\; a\hat{k}$$

and the force equation for m_1 is

$$T\hat{k} - m_1\; g\hat{k} = -m_1\; a\hat{k}$$

Thus we obtain $T - m_2 g = m_2 a$ and $T - m_1 g = -m_1 a$. Eliminating T, we obtain

$$a = g\frac{m_1 - m_2}{m_1 + m_2} \tag{3.4}$$

Substituting this expression into either of the preceding equations we obtain

$$T = 2g\frac{m_1 m_2}{m_1 + m_2} \tag{3.5}$$

A common error is to make an incorrect list of the forces acting on m_1 and m_2 in the previous example. The forces acting on m_1 are gravity pulling down ($-m_1\; g\hat{k}$) and the string pulling up ($T\hat{k}$). The gravitational force on m_2 is not a force acting on m_1, and T should not be replaced by $m_2 g$ in the force equation for m_1. Similarly, T should not be replaced by $m_1 g$ in the force equation for m_2.

[2]We showed in Chapter 2 that if the string is weightless and the contact between string and pulley is smooth, then in equilibrium the tension is the same at all points in the string. The proof was based on the fact that the total force on each small element of the string is zero. Even in a non-equilibrium situation the force on each element of a weightless string must be zero; if the element is weightless it is massless and Newton's second law says that the total force acting on the element is equal to the mass of the element (zero) times its acceleration. Thus, we can show as before that the tension is the same at all points in the string. Even if the contact between string and pulley is rough, the tension will be the same at all points in the string if the pulley is massless and turns on a smooth axle; this will be shown in Chapter 8. Unless we state otherwise, it will be assumed that strings and pulleys are massless and axles are smooth (frictionless).

One should acquire the habit of examining an answer to see whether it is "reasonable". In particular, there are usually some special limiting cases in which we already know what the answer is and if our answer is correct it will reduce to the expected value in these special cases. This procedure will usually enable us to discover algebraic errors as well as errors in reasoning.

In the previous example we expect that $a = 0$ if $m_1 = m_2$, $a > 0$ if $m_1 > m_2$, and $a < 0$ if $m_1 < m_2$. Eqn.(3.4) agrees with these expectations. Furthermore, if $m_1 = m_2$ we have equilibrium and therefore expect $T = m_1 g$. Eqn.(3.5) agrees with this expectation if we set $m_1 = m_2$. We also know the answer, without calculation, in the special case when m_1 is much larger than m_2. In this case we expect m_1 to drop like a freely falling body, i.e. $a = g$. Indeed eqn.(3.4) yields this result when $m_1 \gg m_2$ (we can neglect m_2 in the numerator and denominator) and also yields $a = -g$ (as expected) when $m_2 \gg m_1$. Furthermore, since m_2 accelerates upward with acceleration g when $m_1 \gg m_2$, it follows that in this case the total force acting on m_2 must have magnitude $m_2 g$ and must be directed upward. Therefore the tension in the string must be $2m_2 g$ when $m_1 \gg m_2$. Eqn.(3.5) confirms this since $m_1/(m_1 + m_2)$ approaches 1 when $m_1 \gg m_2$.

Example 3.6 : Interaction of Atwood's machine with the floor.

A question frequently asked about Example 3.5 is the following: what upward force $\vec{N}$ does the ground exert on the stand? (Equivalently, if the entire apparatus is placed on a scale, what will the scale read?) If the entire apparatus were in equilibrium, we could immediately infer from Newton's first law that $\vec{N}$ is equal in magnitude to the weight of the apparatus. But m_1 and m_2 are accelerating and are therefore not in equilibrium.

At this point it is useful to extend Newton's second law to composite systems (consisting of more than one particle), just as we extended Newton's first law. We simply write the equation $\vec{F}_i = m_i \vec{a}_i$ for each particle in our system (the index i enumerates the particles) and sum the resulting equations to obtain $\sum_i \vec{F}_i = \sum_i m_i \vec{a}_i$. If we decompose $\vec{F}_i$ into an external part and an internal part, just as we did in Chapter 2, the internal forces cancel in pairs as the result of Newton's third law. Thus we obtain

$$\vec{F}_{\text{ext}} = \sum_i m_i \vec{a}_i \tag{3.6}$$

where $\vec{F}_{\text{ext}}$ is the total (i.e. the *net* or vector sum) external force on the system. **If all particles of the system have the same acceleration $\vec{\mathbf{a}}$,** eqn.(3.6) becomes

$$\vec{F}_{\text{ext}} = M\vec{a} \tag{3.7}$$

where M is the total mass of the system.

Let us apply eqn.(3.6) to Atwood's machine, taking the entire apparatus (including the stand, which is assumed weightless) as our system. We cannot use eqn.(3.7) since different parts of the system have different accelerations. So we must use eqn.(3.6). The only external forces on this system are gravity and the normal force exerted by the ground. Thus eqn.(3.6) states

$$-m_1 g\hat{k} - m_2 g\hat{k} + N\hat{k} = m_2 a\hat{k} - m_1 a\hat{k}$$

which yields $N = (m_2 - m_1)a + (m_1 + m_2)g$. Inserting our prior expression eqn.(3.4) for a and doing a little algebra, we find

$$N = 4g\frac{m_1 m_2}{m_1 + m_2} \tag{3.8}$$

If the stand is not weightless then the weight of the stand must be included in the external force and will simply be added to the right side of eqn.(3.8). Note that when $m_1 = m_2$ we find from eqn.(3.8) that $N = 2m_1 g$, which is what we expect in equilibrium. If $m_1 \neq m_2$, it follows from eqn.(3.8) that $N < m_1 g + m_2 g$. Thus, if the apparatus is placed on a scale, the scale will read less than the weight of the apparatus!

Although we calculated N by using the general theorem eqn.(3.6), we could also have found N by noting that the stand is in equilibrium. The only external forces on the stand are the two strings pulling downward, each with force T and the floor pushing up with force of magnitude N. Thus we find $N = 2T$ and using the expression eqn.(3.5) for T we obtain eqn.(3.8).

Example 3.7 : Analysis of a double pulley system.

Let us consider the pulley system shown in Fig. 3.6a. The pulleys are smooth and massless. The position of pulley B is fixed, but A can move. We have already seen (Example 2.7) that if $m_1 = \frac{1}{2}m_2$ the system will be in equilibrium. If m_1 and m_2 are arbitrarily chosen, we want to calculate the accelerations of both masses and the tension T in the string.

It is essential to recognize the relation between the acceleration of m_2 and the acceleration of m_1. Failure to understand this relation, which is an

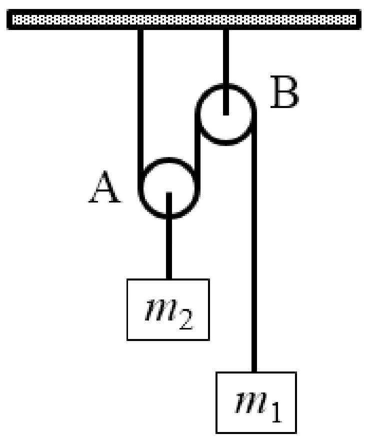


Figure 3.6a: Illustration for Example 3.7.

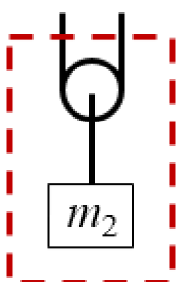


Figure 3.6b: It is useful to consider the subsystem in the dashed box.

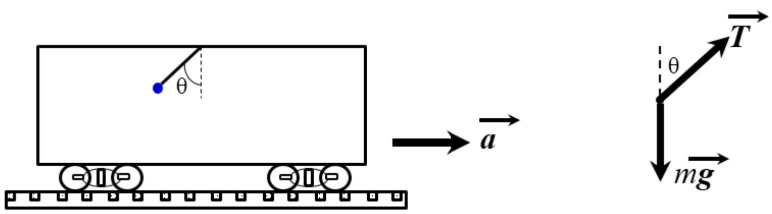
immediate consequence of the fact that the total length of string remains constant, is a very common source of error. In order that the string remain taut, m_1 must descend two inches for every inch which m_2 rises. If we introduce a unit vector $\hat{k}$ pointing vertically upward and define the acceleration of m_2 as $a\hat{k}$, then the acceleration of m_1 is $-2a\hat{k}$.

Applying Newton's second law to m_1 we find $T\hat{k} - m_1 g\hat{k} = m_1(-2a\hat{k})$. Applying Newton's second law to m_2 (or, more precisely, to the system contained within the dashed box in Fig. 3.6b) we find $2T\hat{k} - m_2 g\hat{k} = m_2 a\hat{k}$.

Solving these two simultaneous equations for a and T we find

$$\begin{aligned} a &= g\frac{2m_1 - m_2}{4m_1 + m_2} \\ T &= 3g\frac{m_1 m_2}{4m_1 + m_2}. \end{aligned}$$

When $m_1 = \frac{1}{2}m_2$ we recover the solution the statics problem. The reader should verify that the expressions for a and T reduce to the expected values when $m_1 \gg m_2$ and when $m_2 \gg m_1$.



(a.) (b.)

Figure 3.7: Illustration (a.) and free-body diagram (b.) for Example 3.8.

Example 3.8 : A simple accelerometer.

A mass m is suspended by a string from the ceiling of an accelerating railroad car (Fig. 3.7). The string makes a constant angle θ with the vertical. Calculate the acceleration of the railroad car.

Most students get the correct answer to this problem but many use dubious arguments which will surely lead to confusion when applied to more complicated situations. *This is not a statics problem*! The mass m is not in equilibrium; it has the same acceleration as the train. If we introduce axes attached to the ground (x-axis horizontal, y-axis vertical) then $a_x = a$ and $a_y = 0$. Fig. 3.7b shows the forces exerted on the mass m by the earth and by the string. The x and y components of $\vec{F} = m\vec{a}$ yield

$$T \sin\theta = ma \tag{3.9a}$$

and

$$T\cos\theta - mg = 0 \tag{3.9b}$$

Eliminating T we obtain $\tan\theta = a/g$ or $a = g\tan\theta$.

Many students use axes attached to the railroad car and try to use Newton's first law ($\vec{F} = 0$) since the mass m is not accelerating with respect to these axes. However, these axes are not an inertial frame and Newton's first law will not be true in this frame of reference unless we modify our definition of force to include a fictional force of precisely the sort which we excluded in Chapter 2. The third "force", which must be added to Fig. 3.7b to make the vector sum of the three forces vanish, points horizontally to the left and has magnitude ma (vectorially, the third "force" is $-m\vec{a}$). If one introduces this extra "force" and then uses Newton's first law, one obtains eqns.(3.9a) and 3.9b. However, this "force" cannot be explained as the push or pull exerted by some other piece of matter on m. Although at an advanced level it is occasionally useful to use axes which are not an inertial frame and to introduce appropriate fictional forces, we emphatically object to the use of such axes in an introductory course.

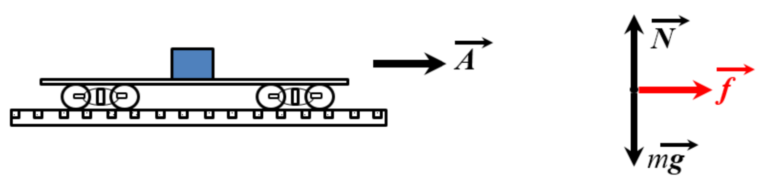


Figure 3.8: Illustration (a.) and free-body diagram (b.) for Example 3.9.

Example 3.9 : Example of friction in the direction of motion.

A box weighing 200 newtons rests on the floor of a freight car Fig. 3.8. The coefficients of static and kinetic friction between the box and the floor are $\mu_s = 0.2$ and $\mu_k = 0.1$. Suppose the freight car is initially at rest and then accelerates with constant acceleration $A = 0.65$ m/s^2. Calculate the acceleration of the box and the force exerted by the floor on the box. Answer the same questions when the acceleration of the freight car is $A = 2.5$ m/s^2.

It is important to have a qualitative understanding of this problem before writing the equations. We know from experience that if the acceleration $\vec{A}$

of the freight car has a small enough magnitude, the box will not slip and therefore the acceleration of the box will also be $\vec{A}$. The only horizontal force acting on the box is the frictional force exerted on it by the floor. This force must be directed forward (to the right in Fig. 3.8b) and (in order to satisfy Newton's second law) must have a magnitude equal to the mass of the box times the acceleration of the box. If A is larger than a certain critical value A_0, the required frictional force will be larger than the maximum possible force of static friction and the box will slip. As the box slips, the floor will still exert a forward force on the box (since the box is moving backward relative to the floor). This force, which may be calculated by the law of kinetic friction (eqn.(2.8)), will determine the acceleration of the box (which will be directed forward but will have a magnitude smaller than A).

First we want to know whether the box is accelerating with the freight car or slipping; therefore we should calculate the value of the critical acceleration A_0. If $A < A_0$ the acceleration of the box is A, and Newton's second law gives $f = mA$ where f is the frictional force exerted by the floor and m is the mass of the box. Since the box has no acceleration in the vertical direction we have $N - mg = 0$ and thus $f/N = A/g$. But the law of static friction states that $f/N \leq \mu_S$. Thus we find that slipping occurs if A is greater than the critical value $A_0 = \mu_s g$. Taking $g = 9.8$ m/s^2 we find $A_0 = .2(9.8) = 1.96$ m/s^2 (note that A_0 does not depend on the mass of the box). Therefore the box has the same acceleration as the freight car when $A = 0.65$ m/s^2 and the box is slipping when $A = 2.5$ m/s^2.

Thus, when $A = 0.65$ m/s^2, the acceleration of the box is 0.65 m/s^2 and the frictional force is

$$f = mA = \frac{200 \text{ newtons}}{9.8 \text{ m/s}^2} \cdot (0.65 \text{ m/s}^2) = 13 \text{ newtons}$$

When $A > A_0$ the frictional force is $f = \mu_k N = \mu_k mg$. The acceleration a of the box is found from Newton's second law $f = ma$, which gives $a = \mu_k g$. Thus, when $A = 2.5$ m/s^2 we find that the acceleration of the box is $a = (0.1)(9.8 \text{ m/s}^2) = 0.98$ m/s^2 (again independent of the mass of the box). The frictional force is

$$f = \mu_k mg = (0.1)(200 \text{ newtons}) = 20 \text{ newtons}$$

Note that when $A > A_0$ the acceleration of the box and the frictional force do not depend on A.

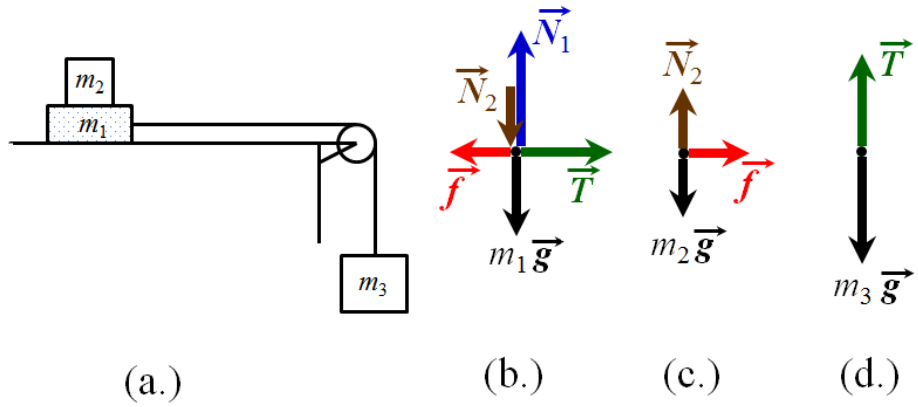


Figure 3.9: Illustration (a.) for Example 3.10 and free-body diagram for block m_1 (b.), block m_2 (c.) and block m_3 (d.).

Example 3.10 : Two blocks with mutual friction pulled by a third on a smooth surface.

Consider the apparatus shown in Fig. 3.9a. m_1 is on a smooth horizontal surface and the coefficients of friction between m_1 and m_2 are $\mu_s = 0.2$, $\mu_k = 0.1$. For all values of m_3 we wish to find the acceleration of m_1, the acceleration of m_2, the tension in the string, and the horizontal force which m_1 exerts on m_2.

This is a slightly more complicated versoin of Example 3.9. We expect that if m_3 is sufficiently small, m_1 and m_2 will have the same acceleration; but if m_3 is greater than a certain critical value then the upper block (m_2) will slip. By now we should hardly need to emphasize that the tension in the string is <u>not</u> equal to m_3g (a common error); if the tension in the string were m_3g then m_3 would be in equilibrium and would not accelerate.

We introduce a unit vector $\hat{i}$ pointing horizontally to the right and a unit vector $\hat{k}$ pointing vertically up. We define the acceleration of m_1 as $A_1\hat{i}$ and the acceleration of m_2 as $A_2\hat{i}$. Since the string is assumed inextensible, the acceleration of m_3 is $-A_1\hat{k}$. The unit vectors have been introduced only to avoid sign errors; by now the reader should be able to dispense with them. Free-body diagrams for the three blocks are shown in Fig. 3.9b, c, d. N_2 is the magnitude of the normal force which one block exerts on the other, f is the magnitude of the frictional force, and N_1 is the normal force which the table exerts on the lower block. Since neither m_1 nor m_2 has a vertical acceleration we find (from Fig. 3.9c) $N_2 = m_2 g$ and (from Fig. 3.9b) $N_1 = (m_1 + m_2)g$.

Application of Newton's second law to each of the blocks gives

$$\begin{aligned} f &= m_2 A_2 && (3.10a) \\ T - f &= m_1 A_1 && (3.10b) \\ T - m_3 g &= -m_3 A_1 && (3.10c) \end{aligned}$$

First we look at the case where m_1 and m_2 have the same acceleration ($A_1 = A_2$). Adding eqns.(3.10a) and (3.10b) we obtain $T = (m_1 + m_2)A_1$ (which also could have been obtained by applying Newton's second law to the composite object formed by the two blocks). Substituting this expression for T into eqn.(3.10c) we find $A_1 = gm_3/(m_1 + m_2 + m_3)$. The frictional force is given by

$$f = \frac{gm_2 m_3}{m_1 + m_2 + m_3}. \qquad (3.11)$$

Since $N_2 = m_2 g$ we have

$$\frac{f}{N_2} = \frac{m_3}{m_1 + m_2 + m_3} \qquad (3.12)$$

This solution is compatible with the law of static friction provided that $f/N_2 \leq \mu_s$. If $\mu_s = 0.2$ we can find the critical value of m_3 by setting $f/N_2 = 0.2$; thus we find $f/N_2 \leq 0.2$ if $m_3 \leq 0.25(m_1 + m_2)$.

Note that the right side of eqn.(3.12) is always less than 1. Therefore, if $\mu_s \geq 1$, m_2 never slips relative to m_1 no matter how large m_3 is. If $\mu_s < 1$ the critical value of m_3 (obtained by setting $f/N_2 = \mu_s$) is $m_3 = \mu_s(m_1 + m_2)/(1 - \mu_s)$, which agrees with the previous result when $\mu_s = 0.2$.

If $m_3 > 0.25(m_1 + m_2)$ then m_2 slips relative to m_1. The frictional force f is given by the law of kinetic friction $f = 0.1(m_2 g)$ and from eqn.(3.10a) we find $A_2 = 0.1g$. We can now solve eqn.(3.10b) and eqn.(3.10c) for the two unknowns A_1 and T, obtaining

$$\begin{aligned} A_1 &= g\frac{m_3 - .1m_2}{m_1 + m_3} \\ T &= m_3 g\frac{m_1 + .1m_2}{m_1 + m_3} \end{aligned}$$

The reader who has an insatiable appetite for this type of problem might amuse herself (himself) by attaching the string to m_2 rather than m_1, or by introducing friction at the contact between the table and m_1.

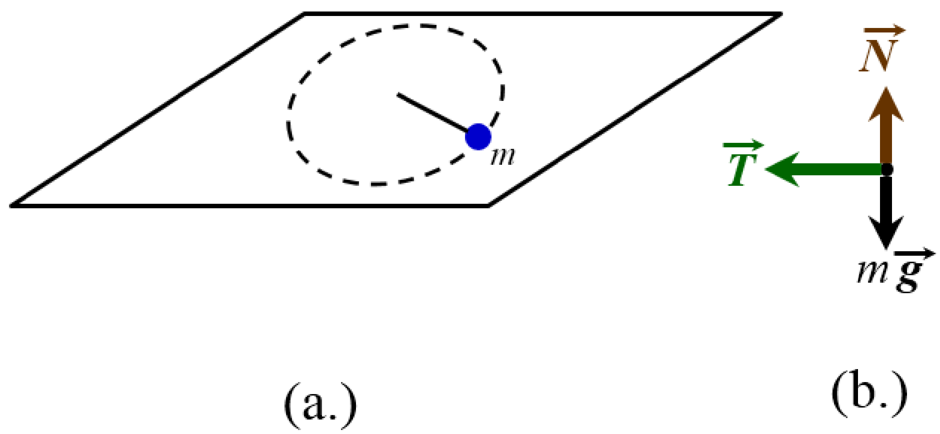


Figure 3.10: Illustration (a.) and free-body diagram (b.) for Example 3.11.

Example 3.11 : Point mass swung along a circular path.

This is perhaps the simplest dynamics problem involving circular motion. One end of a taut string is attached to a fixed point on a smooth horizontal surface. The other end is attached to a particle of mass m, which moves in a circle of radius r with constant speed v. Calculate the tension in the string.

The particle is not in equilibrium since the direction of the velocity vector is changing. According to eqn.(1.17) the particle has an acceleration of magnitude v^2/r directed toward the center of the circle. The only horizontal force acting on the particle is exerted by the string (Fig. 3.10b) and is also directed toward the center of the circle. Newton's second law requires that the tension in the string be $T = mv^2/r$.

The force exerted by the string is sometimes called a ***centripetal force***. The word "centripetal" simply means "directed toward the center". The term "centripetal force" is somewhat unfortunate since it conveys a certain aura which leads many students to think (erroneously) that such a force is somehow different in kind from other forces. If asked to enumerate the forces acting on m, some students would answer: “Gravity, the normal force exerted by the table, and the centripetal force.” A better answer is: “ Gravity, the normal force exerted by the table, and the force exerted by the string.” We shall generally avoid the term ”centripetal force”.

Even more vigorously to be avoided are both the term and the concept “centrifugal force” . This is a fictional force, directed radially outward and of magnitude mv^2/r, which would have to be added to Fig. 3.10b if one wanted to insist that the particle is in equilibrium. In an inertial frame the particle is not in equilibrium and one cannot point to any piece of matter which is exerting a radially outward force on m.

Note that if the string were suddenly cut there would then be no force acting on m and the magnitude and direction of the velocity vector would remain constant. Thus the particle would travel along a straight line tangent to its original circular path. Some people believe, on grounds of “intuition”, that if the string were cut the particle would fly out along a radial line; if this were true the particle would have to change the direction of its velocity vector suddenly at the instant when the string is cut. This would require an infinite acceleration at this instant, and thus an infinite force. Since the force exerted by the string on the particle is perfectly finite until the string is cut, and is zero thereafter, it follows that the velocity vector does not change its direction at the instant when the string is cut.

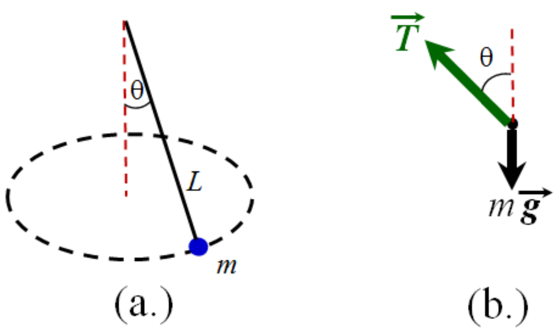


Figure 3.11: Illustration (a.) and free-body diagram (b.) for Example 3.12.

Example 3.12 : A Conical Pendulum.

A particle of mass m is attached to the ceiling by a string of length L. If the particle is started correctly it will travel in a horizontal circle with constant speed, the string making a constant angle θ with the vertical. Calculate the correct speed v with which the particle must be started.

The radius of the circle described by the particle is $L\ \sin\theta$, and thus the acceleration of the particle has magnitude $v^2/(L\ \sin\theta)$ and is directed horizontally toward the center of the circle. The only forces on the particle are exerted by the string and by gravity (Fig. 3.11b). The vertical and horizontal components of $\vec{F} = m\vec{a}$ yield $T\cos\theta - mg = 0$ and $T\sin\theta = mv^2/(L\ \sin\theta)$. Eliminating T from this pair of equations we obtain $v^2 = gL\ \sin^2\theta/\cos\theta$. The period of the pendulum (which we call t) is the time required for the particle to complete one cycle of the motion, i.e.

$$t = 2\pi\frac{L\ \sin\theta}{v} = 2\pi\sqrt{\frac{L\ \cos\theta}{g}} \tag{3.13}$$

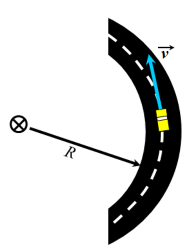


Figure 3.12a: Aerial view of a car driving along a curved section of a road (Example 3.13).

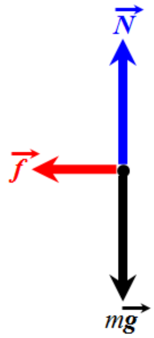


Figure 3.12b: (See Example 3.13) Free-body diagram for the car if the road is not banked.

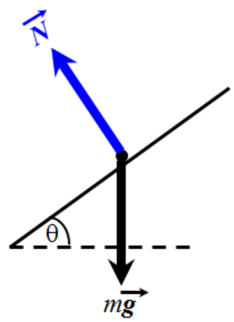


Figure 3.12c: Free-body diagram if there is no friction and the road is banked.

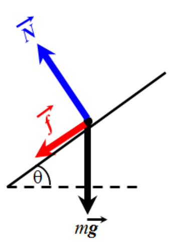


Figure 3.12d: Free-body diagram if there is friction and the road is banked and the car is driven faster than the "correct" speed v_0.

Example 3.13 : Highway Design

A car is driving along a road at constant speed v, approaching a curve. We want to determine whether the car will skid as it goes around the curve.

If the car negotiates the curve without skidding then its acceleration is directed toward the center of the curve and has magnitude v^2/R, where R is the radius of the curve (Fig. 3.12a). Therefore the net external force acting on the car must be directed toward the center of the curve and must have magnitude mv^2/R.

First let us suppose that the road is not banked. Then the only horizontal force acting on the car is the frictional force f exerted by the road on the tires (see Fig. 3.12b). This force must be directed toward the center of the curve and is therefore perpendicular to the direction of motion of the car. As long as the car is not skidding, the part of the tire which is touching the road is instantaneously at rest (the kinematics of rolling will be discussed more fully in Chapter 8); therefore the force f is produced by static, rather than kinetic, friction.

The radial component of $\vec{F} = m\vec{a}$ yields $f = mv^2/R$, and the vertical component yields $N - mg = 0$. The law of static requires $f/N \leq \mu_s$. Thus we see that the car can negotiate the curve without skidding if $v^2/gR \leq \mu_s$. For example, if $R = 100$ meters and $\mu_s = 1$ (a reasonable value for a rubber tire on a dry concrete road[Noo01]) we find that the critical value of v is 31 m/s or about 69 miles per hour. Many drivers have learned from unpleasant experiences that on a wet or icy day μ_s and the critical velocity are reduced.

If the road is banked, then even on a perfectly icy day a car can negotiate the curve without skidding, provided it is driven at the correct speed. To calculate this speed, which we call v_0, we consider the free-body diagram for the car (Fig. 3.12c). In drawing this diagram we have assumed that the road exerts **no frictional force** on the car. The road is banked at angle θ above the horizontal and the car is traveling perpendicular to the page. The acceleration of the car is directed horizontally to the left (if the car is not sliding uphill or downhill) and has magnitude v_0^2/R. Thus the horizontal and vertical components of $\vec{F} = m\vec{a}$ yield $N \sin\theta = mv_0^2/R$ and $N \cos\theta - mg = 0$. Eliminating N, we find $v_0^2 = gR\ \tan\theta$.

If the road is banked at an angle $\theta = \tan^{-1}(v_0^2/gR)$, where v_0 is the average speed at which people drive, then the road will exert no sideways force on a car negotiating the curve at speed v_0 and even on a slippery day the car will not skid if it is driven at this speed. { *Evidently, highway engineers don't use this formula, which yields $\theta = 36°$ if R=100 m and v_0*

= 60 mi/hr.}

What will happen if a driver drives through the curve at a speed other than the "correct" speed v_0? If the road is perfectly slippery ($\mu_s = 0$) the car will slip to the outside of the curve if $v > v_0$ and will slip to the inside of the curve if $v < v_0$.

A more interesting practical question is the following: for given values of the radius of curvature R, the banking angle θ, and the coefficient of static friction μ_s, what is the maximum speed v at which a car can drive through the curve without skidding? Fig. 3.12d is the free-body diagram for the car when it is being driven at a speed $v > v_0$. (v_0 is defined as $\sqrt{gR\ \tan\theta}$). In this case the frictional force $\vec{f}$ exerted by the road on the tires is directed downhill (if $v < v_0$ the frictional force is directed uphill). The horizontal and vertical components of $\vec{F} = m\vec{a}$ yield

$$\begin{aligned} N\sin\theta + f\cos\theta &= \frac{mv^2}{R} \\ N\cos\theta - f\sin\theta - mg &= 0. \end{aligned}$$

Solving for the two unknown quantities f and N, we obtain

$$f = \frac{mv^2}{R}\cos\theta - mg\sin\theta \tag{3.14a}$$

$$N = \frac{mv^2}{R}\sin\theta + mg\cos\theta \tag{3.14b}$$

Eqns.(3.14a) and (3.14.b) could have been obtained directly if we had taken our axes parallel and perpendicular to the incline, rather than horizontal and vertical. Note that the acceleration has a component $(v^2/R)\cos\theta$ along the incline and a component $(v^2/R)\sin\theta$ perpendicular to the incline.

Eqn.(3.14a) can be rewritten in the form

$$f = \frac{m}{R}\cos\theta\ \left(v^2 - v_0^2\right)$$

which makes it obvious that f is positive (i.e. the frictional force points downhill) for $v > v_0$. The negative value of f for $v < v_0$ means that the frictional force points uphill.

To determine whether the car skids we must examine the ratio f/N. From eqns.(3.14a) and (3.14.b) we find

$$\frac{f}{N} = \frac{(v^2/gR)\cot\theta - 1}{(v^2/gR) + \cot\theta} \tag{3.15}$$

A graph of the right side of eqn.(3.15) for $v > v_0$ is shown in Fig. 3.13. Note that f/N tends to the limiting value $\cot\theta$ as $v \to \infty$. If $\mu_s > \cot\theta$, the ratio f/N will never exceed μ_s and the car will not skid no matter how fast it is driven. If $\mu_s < \cot\theta$ then $f/N = \mu_s$ when

$$\frac{v^2}{gR} = \frac{\mu_s \cot\theta + 1}{\cot\theta - \mu_s} \tag{3.16}$$

and the speed given by eqn.(3.16) is the maximum speed with which the car can be driven through the curve without skidding.

If we want to investigate the possibility of downhill skidding for $v < v_0$, we should examine the ratio $|f|/N$ (since f is negative in this case). A graph of $|f|/N$ for $v < v_0$ is shown in Fig. 3.14. Note that the ratio $|f|/N$ is maximum when $v = 0$ and has the value $\tan\theta$. If $\mu_s > \tan\theta$ the car will not skid downhill no matter how slowly it is driven. If $\mu_s < \tan\theta$ we obtain the critical speed by setting $|f|/N = \mu_s$ and thus

$$\frac{v^2}{gR} = \frac{1 - \mu_s \cot\theta}{\mu_s + \cot\theta} \tag{3.17}$$

If the car is driven more slowly than the speed given by eqn.(3.17), it will skid downhill.

If μ_s is larger than both $\tan\theta$ and $\cot\theta$, the curve can be driven without skidding at *any speed*. This condition cannot be attained on an ordinary road since θ is quite small (and $\cot\theta$ is very large) but can be attained on a raceway. For example, we could have $\mu_s = 1.2$ and $\theta = 45°$. Note that if $R = 100$ meters and the banking angle is $45°$ then the "correct" speed at which to drive the curve is

$$\sqrt{gR\ \tan\theta} = 31 \text{ m/s} = 70 \text{ mi/hr}.$$

At this speed the road will exert no sideways force on the tires and the strain on mechanical components will be a minimum. A racing car will drive through the curve at a speed considerably higher than the "correct" speed.

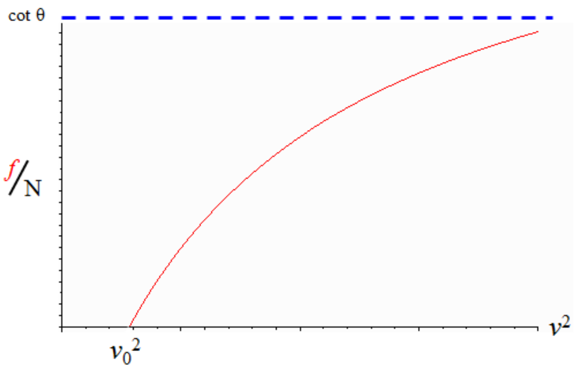


Figure 3.13: Graph of the ratio of frictional to normal force as a function of the square of the speed of the car when the speed is greater than the "correct" speed v_0.

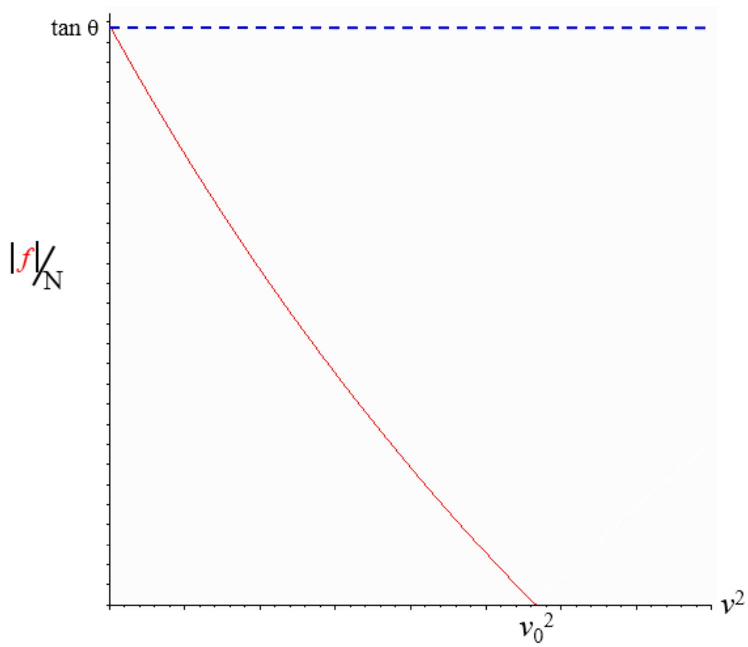


Figure 3.14: Graph of the same ratio when the speed is less than v_0.

3.2 Motion of Planets and Satellites; Newton's Law of Gravitation

Newton was primarily concerned with explaining the observed motions of the planets in the solar system and of their moons. A large quantity of observational data was available to Newton. The facts which were most important in his view were the following:

A. The moons of Jupiter move in essentially circular orbits around Jupiter, with periods[3] proportional to the 3/2 power of their distances from the center of Jupiter.

B. The same is true of the moons of Saturn.

C. The planets move in elliptical orbits having the sun as a focus.

D. The radius vector from the sun to any planet sweeps out equal areas in equal times, i.e. the rate at which area is swept out is constant.

E. The periods of the planets, referred to the background of fixed stars, are proportional to the 3/2 power of their mean[4] distances from the sun.

Johanes Kepler (1571 - 1630) inferred (C), (D), and (E) which are known respectively as Kepler's first, second, and third laws, from a large body of observational data.

From (D) Newton showed that the force acting on the planets is directed toward the sun, i.e. that it is a ***central force***. The mathematical derivation of this result will be presented in Chapter 8 (see section 8.1); at this point we emphasize that (D) tells us the *direction* of the force acting on the planets but tells us nothing about the magnitude of that force. Newton deduced from (C) and (E) that the magnitude of the force acting on a planet is inversely proportional to the square of the distance of the planet from the sun and directly proportional to the mass of the planet.

[3]The period is defined as the time required for the moon to make a complete circuit of Jupiter. If we take $t = 0$ at the instant when the vector from the center of Jupiter to the moon points in a particular direction relative to the background of the fixed stars, the period T is the elapsed time until that vector again points in the same direction. Note the important role of the fixed stars in providing the physical definition of a set of non-rotating axes.

[4]The "mean" distance referred to here is the average of the closest and furthest distances of the planet from the sun, and is thus equal to the semi-major axis of the ellipse.

Rather than deal with the somewhat difficult mathematics necessary to describe a planet moving in an elliptical orbit, let us focus on the moons of Jupiter (whose orbits are circles). A moon of mass m which moves in a circle of radius R with constant speed v has an acceleration v^2/R directed toward the center of the circle. Newton therefore concluded that a force of magnitude

$$F = \frac{mv^2}{R} \tag{3.18}$$

must be pulling the moon toward the center of Jupiter. Regarding the detailed "cause" of that force, Newton admitted to some puzzlement, but considered it obvious that the force is somehow exerted by Jupiter itself.

The period of the moon, i.e. the time required to travel around the circle once, is

$$T = \frac{2\pi R}{v} \tag{3.19}$$

Thus we can rewrite eqn.(3.18) as

$$F = \frac{4\pi^2 mR}{T^2} \tag{3.20}$$

Since (according to (A)) T is proportional to $R^{3/2}$, we can write

$$T^2 = k\ R^3 \tag{3.21}$$

The meaning of the observational result (A) is that the proportionality constant k has the same value for all the moons, i.e. k does not depend on the mass m of the moon.

Inserting eqn.(3.21) into eqn.(3.20) Newton found

$$F = 4\pi^2 \frac{m}{kR^2} \tag{3.22}$$

or, in words,

> The force which Jupiter exerts on a moon of mass m which is at distance R from Jupiter's center is proportional to m/R^2 and is directed toward Jupiter's center.

By Newton's third law, the moon exerts a force on Jupiter. This force is proportional to M/R^2, where M is the mass of Jupiter. Since the force which the moon exerts on Jupiter must be equal in magnitude (and opposite in direction) to the force which Jupiter exerts on the moon, we see that the force must be proportional to the product Mm (i.e. the constant $4\pi^2/k$

in eqn.(3.22) is proportional to M). Thus, the force law between Jupiter (mass M) and a moon (mass m) at a distance R from the center of Jupiter is

$$F = G\frac{mM}{R^2} \tag{3.23}$$

where G is a constant (called the ***gravitational constant***) which does not depend on m or M.

Evidently there is an attractive force, also of the form eqn.(3.23), between Saturn and each of its moons. Furthermore, Newton showed that a force law of this same form between the sun and each of its planets is necessary (and sufficient) to explain Kepler's laws. The earth presumably exerts a similar force on its own moon. Obvious as it may seem from our present vantage point, a considerable imaginative leap was required for Newton to recognize that this force is identical in kind with the force which the earth exerts on an apple falling from the branch of a tree. He also realized that this idea could be numerically tested. The genius of Newton in this regard is perhaps best stated by the French polymath Paul Valéry: "Il fallait être Newton pour apercevoir que la lune tombe, quand tout le monde voit bien qu'elle ne tombe pas."[Val45] [5]

If we denote the earth's mass by M_e, then the force exerted by the earth on a mass m at distance R from the center of the earth is GmM_e/R^2, and the acceleration of the mass m toward the center of the earth is GM_e/R^2 (note that the magnitude of the acceleration does not depend on m). Thus, if we compare the acceleration a_{moon} of the moon toward the earth with the acceleration g of a falling apple at the earth's surface, the two accelerations should be in the ratio $(R_e/R_m)^2$, where R_e is the distance of the apple from the center of the earth (i.e. R_e is the radius of the earth) and R_m is the distance of the moon from the center of the earth. Newton knew that to a good accuracy $R_e/R_m = 1/60$, and thus $(R_e/R_m)^2 = .0002777$. The moon's acceleration is v^2/R_m; the velocity v of the moon is $2\pi R_m/T$ where T is the period of the moon's motion around the earth. Inserting $T = 27.3$ days and $R_m = 3.844 \times 10^8$ meters[Sta], we obtain $v = 1023.97$ m/s and $v^2/R_m = 0.002728$ m/s which yields $a_{\text{moon}}/g = 0.002728/9.8 = 0.0002783$. Newton made this calculation and undoubtedly found it convincing when coupled with the other evidence for the inverse square force law.

Newton believed in the simplicity and universality of the laws of nature. Having concluded that an attractive force exists between Jupiter and its moons, between Saturn and its moons, between the earth and its moon, and

[5] Roughly translated: "One had to be a Newton to see that the moon falls when the whole world sees that it does not fall."

between the sun and the planets, he then postulated that a similar attractive force (which he called ***gravitation***) exists between any two bodies. He pointed out that in everyday experience we are unaware of the gravitational attraction between bodies because their mutual attractive force is so small compared with the gravitational force which the earth exerts on them. He postulated that any two particles (very small bodies) attract each other with a force of magnitude

$$F = G\frac{mM}{r^2} \tag{3.24}$$

directed from one particle to the other. In this equation m and M are the masses of the particles and r is the distance between them.

The question immediately arises: what force does a large spherical mass (like the earth) exert on a particle? If the distance of the particle from the center of the sphere is large compared with the radius of the sphere, then there will be at most a negligible error in using eqn.(3.24), with r taken as the distance from the center of the sphere to the particle. We have already seen that there is strong evidence that eqn.(3.24) is correct even when the distance of the particle from the center of the sphere is not large compared with the radius of the sphere (for example, take the particle as Newton's apple and the sphere as the earth), with r taken as the distance from the center of the sphere to the particle.

Newton recognized that if he assumed eqn.(3.24) as the fundamental force law between particles, then it should be possible to calculate the force between a particle and a spherical mass distribution by considering the distribution as composed of many small elements (each of which exerts a force on the particle given by eqn.(3.24)) and adding up the forces which these elements exert on the particle. He hoped that he would be able to show that if the particle is outside the spherical mass distribution the force is the same as if the entire mass distribution were replaced by a point mass (of the same total mass) at the center of the sphere. It is widely believed that he delayed the publication of the **Principia** for twenty years until he had a satisfactory proof of this point.

The proof in the **Principia** is geometrical. We omit a proof here, but a student who is familiar with integral calculus ought to be able to carry out the calculation. The idea is to decompose the mass distribution into many small elements ΔM. The force exerted by ΔM on a particle of mass m has magnitude $Gm\Delta M/r^2$ where r is the distance between ΔM and m; the direction of this force is along the line between ΔM and m. The total force on m is found by vectorially adding the forces exerted on m by all the mass elements ΔM (the sum of all the infinitesimal contributions is performed by integration). Newton (who invented integral calculus although the first complete published proofs came from Gottfried Leibniz) used a geometrical argument instead of integration in order not to overwhelm his readers. We stress that the proof (whether geometrical or by integration) depends heavily on the fact that the force between particles varies inversely as the *square* of the distance. If the exponent were other than two, it would not be the case that a spherical mass distribution produces the same gravitational effect as a point mass located at the center of the sphere.

A careful reader may have been disturbed by our discussion of the motion of the moon around the earth, and of Jupiter's and Saturn's moons around their respective planets. In each of these cases we employed Newton's second law in a frame of reference defined by axes which are non-rotating (with respect to the fixed stars) and whose origin moves with the center of the planet in question. Such axes are not an inertial frame since they are accelerating with respect to non-rotating axes whose origin is fixed in the sun.

The acceleration of these non-inertial axes is not negligible. For example, the earth's acceleration in its approximately circular motion around the sun is more than twice as large as the moon's acceleration relative to the earth (as the reader can verify using the known ratio of distances of the earth from the sun and the moon, and of the month to the year). Nevertheless the discussion is essentially correct because, when we applied Newton's second law ($\vec{F} = m\vec{a}$) to the moon, the only force which we considered was the gravitational force exerted by the earth on the moon. We omitted consideration of the gravitational force which the sun exerts on the moon. To a high degree of accuracy the sun exerts the same force per unit of mass on the moon and on the earth since the earth-moon distance is very small compared with the earth-sun distance. Therefore the sun produces the same acceleration of both the earth and the moon and their *relative* acceleration is due only to the mutual gravitational forces between the earth and the moon. Similar remarks apply to Jupiter and its moons, and to Saturn and its moons.

Direct measurement of the proportionality constant G in the force law eqn.(3.24) is difficult since the gravitational force is very small for experimentally accessible values of the masses and distance. The first accurate determination of G, obtained by direct measurement of the force between two known masses at a known separation, was performed by Cavendish in 1798[Ph.00]. The accepted value today is $G = 6.67384 \times 10^{-11}$ m^3 kg^{-1} s^{-2} = 6.67384×10^{-11} N $-$ m^2/kg^2[TN]. Since the acceleration g of a freely falling body at the earth's surface is given by

$$g = G\frac{M_e}{R_e^2} \tag{3.25}$$

we can calculate the mass of the earth M_e from the known values of g, G, and R_e. One finds $M_e = 5.97 \times 10^{24}$ kg.

A century before Cavendish, Newton was able to make a reasonable estimate of G by guessing the average density of the earth. He guessed that the average density is between five and six times the density of water ("it is probable that the quantity of the whole matter of the earth may be five or six times greater than it if consisted all of water"[New02]). From the assumed value of the density and the known value of R_e, Newton calculated M_e and thus G. In fact, Newton's guess was remarkably good. The true density of the earth is 5.5 times that of water!

Example 3.14 : An artificial satellite's orbit.

An artificial satellite is orbiting the earth in a circular orbit of radius R. Calculate the period T of the satellite.

(a). Evaluate T when $R = 1.05\ R_e$ (just large enough to eliminate air resistance).

(b). Find the radius R of a synchronous orbit. A synchronous orbit, which is used for a communications satellite, places the satellite in a circular orbit in a plane containing the equator, at a height such that the satellite is always directly above the same point on the ground.

The force on the satellite, directed toward the center of the earth, is GmM_e/R^2. The acceleration of the satellite is v^2/R, directed toward the center of the earth. Therefore $GmM_e/R^2 = mv^2/R$. Inserting $v = 2\pi R/T$ we find

$$T^2 = 4\pi^2\frac{R^3}{GM_e}. \tag{3.26}$$

Even if we did not know the values of G and M_e we could evaluate the right side of eqn.(3.26) using eqn.(3.25). We rewrite eqn.(3.26) as

$$T^2 = \left(\frac{4\pi^2 R_e}{g}\right)\cdot\left(\frac{R}{R_e}\right)^3 \tag{3.27}$$

Inserting $R_e = 6.3781 \times 10^6$ meters and $g = 9.81$ m/s^2 we obtain

$$T = (84.44 \text{ minutes})(R/R_e)^{3/2} \tag{3.28}$$

If $R/R_e = 1.05$, we find $T = 91$ minutes. For a synchronous orbit $T =$ 23.9 hours[MS82] (the period of the earth's rotation relative to the distant stars otherwise known as the ***sidereal day***), and thus we find $R/R_e)^{3/2} =$ 16.97 and $R/R_e = 6.60$. The student should calculate the speed (in meters per second) of a satellite in each of these orbits.

3.3 Newton's 2nd Law of Motion Problems

3.1. Two blocks with masses of $m_1 = 5.00$ kg and $m_2 = 2.00$ kg hang over a massless pulley as shown in Fig. 3.15. A force $F_0 = 100$ newtons acting at the axis of the pulley accelerates the system upward. Find

(a) The acceleration of each block measured by a stationary observer;

(b) The tension in the string.

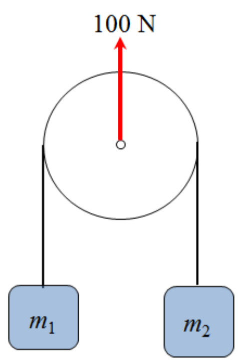


Figure 3.15: Problem 3.1.

3.2. An elevator is accelerating upward with acceleration $\vec{A}$. A compressed spring on the floor of the elevator projects a ball upward with velocity v_0 relative to the floor. Calculate the maximum height above the floor which the ball reaches.

3.3. A board of mass 30.0 kg is lying on an ice-covered pond. The coefficients of static and kinetic friction between the board and the ice are 0.200 and 0.100 respectively. Initially the board is at rest and a boy of mass 50.0 kg runs along the board with acceleration a relative to the board.

(a) What is the minimum value of a which will cause the board to slip?

(b) If the boy's acceleration relative to the board is 4.00 m/s^2, what is his acceleration relative to the ice?

3.4. A right-angled wedge (mass M) rests on a horizontal surface. The diagonal face of the wedge has length D and makes angle θ with the ground. All surfaces of the wedge are smooth. A block of mass m and negligible size is initially at the high end of the diagonal, and the wedge and block are initially at rest. The block then slides down the wedge, which recoils.

(a) How far from its initial position is the wedge when the block hits the ground?

(b) (harder)How much time did it take the block to slide down the wedge?

3.5. A closed rectangular box slides down a smooth inclined plane which makes an angle θ with the horizontal. A point mass m hangs by a string from the ceiling of the box.

(a) Calculate the angle α between the string and the normal to the ceiling. Does the string hang on the downhill or uphill side of the normal?

(b) Now suppose that there is a coefficient of kinetic friction μ between the box and the plane. Calculate α. (The string could oscillate but we are interested in the constant value of α when the box slides for a long time.)

3.6. A phonograph turntable rotates at 33 rpm (revolutions per minute). A coin placed on the turntable less than 15 cm from the center will not slip and will rotate with the turntable, but it will slip if it is more than 15 cm from the center. Calculate the coefficient of static friction between the coin and the turntable.

3.7. Kepler found that the squares of the periods of the planets are proportional to the cubes of the radii of their orbits around the sun (assume circular orbits). From this fact Newton deduced that the force between the sun and a planet is proportional to the inverse square of the mutual distance. Suppose Kepler had found that the squares of the periods are proportional to the n-th power of the radii of the orbits. What force law would Newton have inferred?

3.8. Example 5.4 concerns a sled going down a hill and continuing on a horizontal field, with a coefficient of kinetic friction μ between the sled and the snow. In Chapter 5 we use energy considerations to analyze this system, but you can (and should) also solve Example 5.4 by using Newton's second law and simple kinematics. In our description of the hill, we stated that the bottom of the hill is a short, smooth, frictionless section which changes the direction of the sled's velocity (from downhill to horizontal) continously, without a bump. One might think that it makes no difference (or very little difference) whether that short section is frictionless or has the same coefficient of kinetic friction μ as the slope and the field. This is not true. For simplicity, treat the sled as a point mass and treat the bottom of the hill as an arc of a circle of radius R. The angle of the arc must be equal to θ (θ is the angle between the hill and the horizontal) in order that the direction of the sled's velocity change continuously. We assume that the snow in the arc has a coefficient of kinetic friction μ. Note that if R is small, the length of the arc ($R\theta$) is small. If the sled is going fast and R is small, it is reasonable to assume $v^2/R \gg g$, and we can neglect the effect of gravity during the time when the sled is passing through the arc. If the speed of the sled when it enters the arc is v_0 and the speed when it exits the arc is v_f, then (independently of the radius R and neglecting gravity) $v_f = v_0 e^{-\mu\theta}$. PROVE THIS. [Note that when a particle is moving with varying speed in a circle of radius R, the acceleration has a radial component v^2/R and a tangential component dv/dt.]

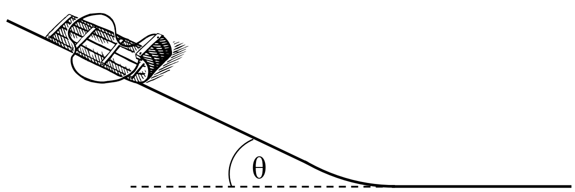


Figure 3.16: Problem 3.8.

3.9. A conveyor belt has width D and moves with speed V. The belt is in the same plane as the adjacent floor. A small rubber hockey puck approaches the belt with velocity $\vec{v}_0$ perpendicular to the edge of the belt. The puck slides onto the belt. The coefficient of kinetic friction between the belt and the puck is μ. Calculate the minimum value of v_0 which will permit the puck to reach the other edge of the belt. Evaluate your answer when $\mu = 0.200$, $D = 3.00$ m, and $V = 6.00$ m/s.
HINT: A problem which appears difficult in one inertial frame may be simpler in another inertial frame.

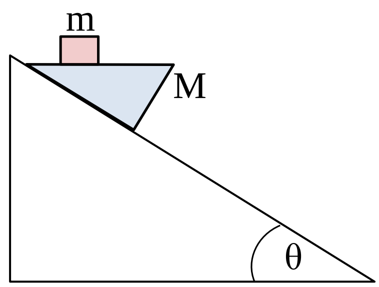


Figure 3.17: Problem 3.10.

3.10. A wedge of mass M slides down an inclined plane. The plane is inclined at angle θ to the horizontal and the upper face of the wedge is horizontal (i.e. the angle between that face and the inclined plane is θ). The surfaces of the wedge are perfectly smooth and a block of mass m is free to slide on the upper surface of the wedge. Find the acceleration of the wedge. (The plane is immovable).

3.11. A toy car of mass m can travel at a constant speed v. It moves in a circle on a horizontal table where the centripetal force is provided by a string *and* friction. The string is attached to a block of mass M that hangs as shown in Fig. 3.18. The coefficient of static friction is μ. Show that the ratio of the maximum radius to the minimum radius possible is

$$\frac{M + \mu m}{M - \mu m}$$

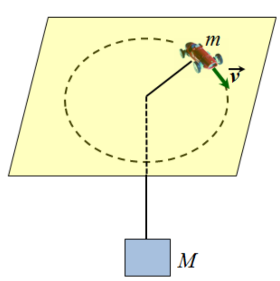


Figure 3.18: Problem 3.11.

Chapter 4

CONSERVATION AND NON-CONSERVATION OF MOMENTUM

4.1 PRINCIPLE OF CONSERVATION OF MOMENTUM

The only laws of classical mechanics are Newton's laws. All other "laws" or general principles are deduced from Newton's laws. The physicist is especially interested in statements about the behavior of systems which do not depend on the detailed nature of the forces which are involved. The best-known example of such a statement is the Principle of Conservation of Momentum:

> If a system is subject to *no external forces*, the total momentum of the system remains constant in time.

To understand this statement we must, of course, first define "momentum". If a particle of mass m has velocity $\vec{v}$, the momentum of the particle (usually denoted by the vector $\vec{p}$) is defined by

$$\vec{p} = m\vec{v}$$

The momentum of a system of particles is defined as the sum of the momenta of the individual particles

$$\vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i$$

In Chapter 3 we proved (eqn.(3.6)) that for any system

$$\vec{F}_{\text{ext}} = \sum_i m_i \vec{a}_i = \frac{d}{dt}\left(\sum_i m_i \vec{v}_i\right) = \frac{d\vec{P}}{dt} \tag{4.1}$$

where $\vec{F}_{\text{ext}}$ is the total external force acting on the system.

To derive eqn.(4.1) we added up the force equations for all the particles in the system; the internal forces canceled out in pairs as a consequence of Newton's third law.

If $\vec{F}_{\text{ext}} = 0$ we have

$$\frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} = \text{ constant} \tag{4.2}$$

which is the ***Principle of Conservation of Momentum***.

Example 4.1 : Analysis of a collision in which bodies stick together.

A body of mass 1 kg and a body of mass 2 kg collide on a smooth horizontal surface. Before the collision the 1 kg body has velocity 3 m/s directed due north and the 2 kg body has velocity 5 m/s directed northeast (i.e. 45° east of due north). The two bodies stick together, forming a body of mass 3 kg. Find the magnitude and direction of the velocity of the 3 kg body.

We orient our axes so that the x-axis points east, the y-axis points north, and the z-axis is perpendicular to the surface. We define our system as the two bodies. Because the surface is smooth there is no external force in the $x-$ or $y-$direction (note that there are internal forces in the system since the two bodies exert forces on each other during the time when they are colliding). Gravity exerts a force in the $z-$direction on each body, but the normal force exerted by the surface is equal in magnitude and opposite in direction to the gravitational force. Thus there is no net external force on the system and we can apply the Principle of Conservation of Momentum which states

$$\left(\sum m_i \vec{v}_i\right)_{\text{initial}} = \left(\sum m_i \vec{v}_i\right)_{\text{final}} \tag{4.3}$$

Note that eqn.(4.3) is a vector equation, equivalent to the three equations

$$\left(\sum m_i v_{i,\ x}\right)_{\text{initial}} = \left(\sum m_i v_{i,\ x}\right)_{\text{final}} \tag{4.4a}$$
$$\left(\sum m_i v_{i,\ y}\right)_{\text{initial}} = \left(\sum m_i v_{i,\ y}\right)_{\text{final}} \tag{4.4b}$$
$$\left(\sum m_i v_{i,\ z}\right)_{\text{initial}} = \left(\sum m_i v_{i,\ z}\right)_{\text{final}} \tag{4.4c}$$

A common *error* is to think that eqn.(4.3) implies

$$\left(\sum m_i v_i\right)_{\text{initial}} = \left(\sum m_i v_i\right)_{\text{final}}$$

where v_i is the magnitude of the velocity vector $\vec{v}_i$. This does not follow from eqn.(4.3) and is not, in general, true.

In the present example eqn.(4.4c) is uninteresting, merely stating that $0 = 0$. If we call the unknown final velocity vector $\vec{V}$, with components V_x and V_y, then eqn.(4.4a) and eqn.(4.4b) imply

$$\begin{aligned} 2(5)(.707) + 0 &= 3V_x \\ 2(5)(.707) + 1(3) &= 3V_y. \end{aligned}$$

Thus we find $V_x = 2.36$ m/s, $V_y = 3.36$ m/s. The speed of the 3 kg body is $V = \sqrt{V_x^2 + V_y^2} = 4.10$ m/s. The velocity vector V is directed 55° north of east ($\tan^{-1} V_y/V_x = 55°$).

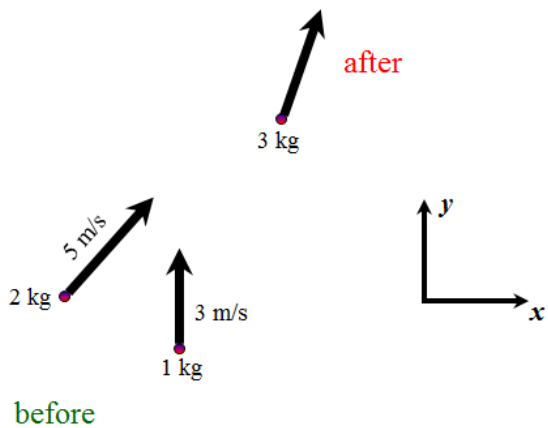


Figure 4.1: An inelastic collision.

Example 4.2 : Analysis of a collision in which the bodies bounce away.

Consider the same two bodies as in the previous example, colliding with the same initial velocities. However, they do not stick together. After the collision the 2 kg body has velocity 4 m/s directed 30° east of due north. Find the velocity of the 1 kg body.

Calling the unknown velocity $\vec{V}$, we find from eqns.(4.4a) and (4.4b)

$$\begin{aligned} 2(5)(.707) + 0 &= 2(4)(.500) + 1\ V_x \\ 2(5)(.707) + 1(3) &= 2(4)(.866) + 1\ V_y \end{aligned}$$

and thus $V_x = 3.07$ m/s, $V_y = 3.14$ m/s.

Note that when the two bodies stick together (this is called a ***completely inelastic collision***) the Principle of Conservation of Momentum uniquely determines the final velocity. When the two bodies do not stick together, the Principle of Conservation of Momentum does not uniquely determine the two final velocity vectors; if one of the final velocity vectors is given, as in the present example, then the other is determined by Conservation of Momentum. More generally, a vector in the $x - y$ plane is specified by two numbers (e.g. the two components of the vector, or the length of the vector and the angle it makes with the x-axis); thus, four numbers are required to specify the two final velocity vectors. Conservation of $x-$ and $y-$momentum imposes two conditions (eqns.(4.4a) and (4.4b)) which must be satisfied by these four numbers. Accordingly, the final state will be determined if any two of these numbers (e.g. the directions of the two final velocities) are specified. The multiplicity of possible final states corresponds to the fact that the bodies have shapes (and the contact may take place at various points on their surfaces) and various degrees of "squishiness" (e.g. two steel balls vs. two old tennis balls).

We now consider a skyrocket fired vertically from the ground. At the instant when it is rising with velocity 100 m/s it explodes into three fragments of equal mass. Just after the explosion one fragment has velocity 50 m/s directed vertically downward and another fragment has velocity 75 m/s directed horizontally. Find the velocity vector of the third fragment just after the explosion.

DIGRESSION (FAIRLY IMPORTANT): Most students will immediately solve this problem by equating the momentum of the skyrocket just before the explosion to the sum of the momenta of the fragments just after the explosion. This procedure is correct but requires some discussion since the system is not free of external forces; gravity acts on the skyrocket and also on the fragments. How do we justify neglecting the effect of gravity? If we integrate both sides of eqn.(4.1) with respect to time from t_1 to t_2, where t_1 and t_2 are arbitrary, we obtain

$$\int_{t_1}^{t_2} \vec{F}_{\text{ext}}\, dt = \int_{t_1}^{t_2} \frac{d\vec{P}}{dt}\, dt = \vec{P}(t_2) - \vec{P}(t_1) \tag{4.5}$$

Let us choose t_1 as a time just before the explosion and t_2 just after the explosion. In this problem $\vec{F}_{\text{ext}} = -Mg\hat{k}$ where M is the total mass of the system and $\hat{k}$ is a unit vector directed vertically upward. Then the left side of eqn.(4.5) becomes $-Mg\hat{k}(t_2 - t_1)$. An "ideal" explosion is one in which the skyrocket flies apart in an infinitesimal time, i.e. $t_2 - t_1 \to 0$. In this case the left side of eqn.(4.5) vanishes or is negligible, and thus the momentum just after the explosion is equal to the momentum just before the explosion.

Example 4.3 : An exploding rocket.

Consider the problem just stated of an exploding skyrocket. If we define $\hat{\imath}$ as a unit vector parallel to the velocity of the horizontally moving fragment, momentum conservation requires

$$M(100\ \hat{k}) = \frac{M}{3}(-50\ \hat{k}) + \frac{M}{3}(75\ \hat{\imath}) + \frac{M}{3}\vec{V}$$

where $\vec{V}$ is the velocity of the third fragment. Thus we find $\vec{V} = 350\ \hat{k} - 75\hat{\imath}$.

(SUGGESTION: Invent a problem in which you know the height at which the explosion occurred and you also know the locations (relative to the point directly under the explosion) of the points where the fragments landed and the times (relative to the time of the explosion) when the fragments landed. From this information you can calculate the velocity of the skyrocket just before the explosion. The solution will involve Conservation of Momentum as well as kinematic results from Chapter 1.)

Example 4.4 : Pulled together on a frictionless surface.

Two children, one with a mass of 30 kg and the other with a mass of 45 kg are on a frozen pond (the ice is assumed perfectly smooth). Initially they are both at rest, 30 meters apart, each holding an end of a 30 meter weightless rope. Both children then start pulling on the rope until they collide. Where will the collision occur? (The method of solution should make it clear that the location of the collision point does not depend on the details of how they pull on the rope.)

We define our system to consist of the two children plus the rope. Since there is no external force on the system, we have

$$m_1\vec{v}_1 + m_2\vec{v}_2 = \text{constant}$$

where $m_1,\ m_2,\ \vec{v}_1,\ \vec{v}_2$ are the masses and velocities of the two children. Since $\vec{v}_1$ and $\vec{v}_2$ are initially zero, the value of the constant is zero and therefore

$$\frac{d}{dt}(m_1\vec{r}_1 + m_2\vec{r}_2) = 0$$

where $\vec{r}_1$ and $\vec{r}_2$ are the positions of the children relative to a fixed origin. If the initial positions of the children are $\vec{r}_1(0)$ and $\vec{r}_2(0)$ and their position when they collide is $\vec{R}$, then

$$(m_1 + m_2)\vec{R} = m_1\vec{r}_1(0) + m_2\vec{r}_2(0).$$

It is convenient (but not necessary) to take our origin at the initial position of child #1 so that $\vec{r}_1(0) = 0$. Then

$$\vec{R} = \frac{m_2}{m_1 + m_2} \cdot \vec{r}_2(0)$$

which states that if the initial distance between the children is D, the collision occurs on the line between the initial positions at a distance $(m_2/m_1 + m_2)D$ from the initial position of #1. In the present example, the collision occurs 18 meters from the initial position of the less massive child.

4.2 Center of Mass

The discussion of the preceding example illustrates the usefulness of the concept of ***center of mass***. In general, if a system consists of particles enumerated by an index i, located at positions $\vec{r}_i$, the location of the center of mass $\vec{R}_{\rm cm}$ is defined by the equation

$$\vec{R}_{\rm cm} = \frac{\sum m_i \vec{r}_i}{\sum m_i} \tag{4.6}$$

In words, the position vector of the center of mass is a weighted average of the position vectors of the individual particles, each particle being weighted by the ratio of its mass to the total mass.

If $X_{\rm cm},\ Y_{\rm cm},\ Z_{\rm cm}$ are the Cartesian coordinates of the center of mass, then eqn.(4.6) is equivalent to the three equations

$$X_{\rm cm} = \frac{\sum m_i\ x_i}{\sum m_i} \tag{4.7a}$$

$$Y_{\rm cm} = \frac{\sum m_i\ y_i}{\sum m_i} \tag{4.7b}$$

$$Z_{\rm cm} = \frac{\sum m_i\ z_i}{\sum m_i} \tag{4.7c}$$

If we rewrite eqn.(4.6) as $M\vec{R}_{\text{cm}} = \sum m_i\ \vec{r}_i$ where $M = \sum m_i$ and differentiate both sides with respect to time, we obtain

$$M\ \vec{V}_{\text{cm}} = \sum m_i \vec{v}_i \tag{4.8}$$

where $\vec{V}_{\text{cm}} = d\vec{R}_{\text{cm}}/dt$. Differentiating both sides with respect to time again we obtain

$$\begin{aligned} M\vec{A}_{\text{cm}} &= \sum m_i \vec{a}_i \quad \text{where} \\ \vec{A}_{\text{cm}} &= \frac{d\vec{V}_{\text{cm}}}{dt} = \frac{d^2\vec{R}_{\text{cm}}}{dt^2} \end{aligned}$$

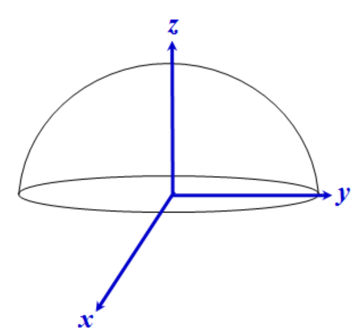
Combining this result with eqn.(3.6) we obtain the very important result

$$\vec{F}_{\text{ext}} = M\vec{A}_{\text{cm}} \tag{4.9}$$

which states that the motion of the center of mass of a system is identical with the motion of a particle of mass M (where M is the total mass of the system) subjected to a force $\vec{F}_{\text{ext}}$ (where $\vec{F}_{\text{ext}}$ is the total external force on the system). Therefore if you throw a chair into the air with an arbitrary amount of spin, the center of mass (generally abbreviated as CM) of the chair will move (we neglect air friction here) in a parabola.

In Example 4.4 the external force is zero and therefore $\vec{A}_{\text{cm}} = 0$ and $\vec{V}_{\text{cm}} =$ constant. Since initially $\vec{V}_{\text{cm}} = 0$ it follows that $\vec{V}_{\text{cm}} = 0$ at all times and $\vec{R}_{\text{cm}}$ is constant. Thus the center of mass never moves and the collision must take place at the center of mass of the two initial positions.

Frequently one is interested in the motion of a solid body of finite (i.e. not infinitesimal) size. The location of the center of mass is often obvious from symmetry considerations (e.g. the CM of a uniform rod is at its midpoint), but in other cases some calculation is necessary. Typically, we conceptually subdivide the body into infinitesimal pieces and perform the sums in eqns.(4.7a), (4.7b) and (4.7c) by means of integral calculus. As an example, let us calculate the location of the CM of a solid hemisphere of uniform density. For convenience we take our x and y axes in the flat face with the origin at the center of that face. From simple considerations of symmetry we see that the CM



is on the z-axis, i.e. $X_{\rm cm} = Y_{\rm cm} = 0$. To calculate $Z_{\rm cm}$ we must convert the sums in eqn.(4.7c) into integrals. This can be done fairly simply in either of two ways. One way is to subdivide the body into thin slices by means of planes perpendicular to the z-axis. A plane of constant z intersects the hemisphere in a circle of radius $\sqrt{a^2 - z^2}$ where a is the radius of the hemisphere. Thus the volume of the slice contained between the plane at height z and the plane at height $z + dz$ is $\pi(a^2 - z^2)\ dz$ and the mass of this slice is $\rho\pi(a^2 - z^2)\ dz$ where ρ is the mass density (mass per unit volume). Converting the sums in eqn.(4.7c) into integrals, we find

$$z_{\rm cm} = \frac{\rho\pi \int_0^a dz\ z(a^2 - z^2)}{\rho\pi \int_0^a dz\ (a^2 - z^2)} = \frac{3}{8}a$$

Alternatively, we can divide the body into volume elements defined by natural coordinate surfaces in spherical coordinates. The volume element is $dV = r^2\ \sin\theta\ dr\ d\theta\ d\phi$. Using $z = r\cos\theta$ we find

$$z_{\rm cm} = \frac{\rho \int_0^a dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\ r^3\ \cos\theta \sin\theta}{\rho \int_0^a dr \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\ r^2\ \sin\theta} = \frac{3}{8}a$$

in agreement with the previous calculation. Note that the answer is *reasonable*; we expect $z_{\rm cm} < a/2$ since more than half the mass is located below the plane $z = a/2$.

> REMARK: You can, if you are interested, calculate the location of the CM of various bodies, e.g. a pyramid. This is more an exercise in calculus than in physics.

If the solid hemisphere is moved to a different position or is tilted, the center of mass continues to be the same physical point of the body. More generally, the definition (eqn.(4.6)) of the center of mass implies (we omit the proof, which is left as an exercise, Problem 4.1, for the interested reader) that the center of mass of a rigid body continues to be the same physical point of the body even when the location and orientation of the body are changed. For some bodies, such as a hollow sphere, there is no matter located at the center of mass. Nevertheless, one can imagine the center of mass as attached to the body by rigid weightless rods.

Suppose we divide the particles in a system into two groups (subsystems) which we call 1 and 2. Suppose the total masses of the two subsystems are M_1 and M_2 and the CM's of the two subsystems are located at $\vec{R}_1$ and $\vec{R}_2$. Then it follows from the definition, eqn.(4.6), of the CM that the CM of the entire system is just the CM of two point masses: M_1 located at $\vec{R}_1$ and M_2 located at $\vec{R}_2$. Thus if two rods are welded together, the CM of the composite is just the CM of two masses located at the midpoints of the rods.

Eqn.(4.9) is of important practical interest when applied to a piece of rotating machinery such as a flywheel. If the flywheel rotates around a fixed axis, any physical point of the flywheel moves in a circle. Suppose the CM of the flywheel is not on the axis; then the CM moves in a circle and has an acceleration of magnitude v^2/R directed toward the center, where v is the speed of the CM and R is the distance of the CM from the center. If the flywheel makes n revolutions per second (n is called the ***frequency***), then $v = 2\pi\ Rn$ and $v^2/R = 4\pi^2 n^2 R$. If the mass of the flywheel is M, eqn.(4.9) states that an external force of magnitude $4\pi^2 n^2 RM$ is being applied to the flywheel. This force is exerted by the bearings and is directed radially inward, along the instantaneous direction from the CM to the axis. The flywheel exerts an equal and opposite force on the bearings, which wears out the bearings or causes the bearings to vibrate, or both. For example, if the flywheel has a mass of 145 kg and is rotating 6000 times per minute ($n = 100$) and if the CM is 3.175 millimeters (1/8 of an inch) from the axis, then the magnitude of this rapidly rotating force is 182,000 newtons or 40,800 pounds or more than 20 tons. In order to make $R = 0$ one attaches a point mass to the wheel, choosing the magnitude and location of the point mass in such a way as to put the CM on the axis. This procedure is called ***statically balancing*** the wheel.

ADDENDUM FOR CAR OWNERS

Even after an object has been statically balanced, it may still exert fluctuating forces on the bearings. Consider, for example, the system (see Fig. 4.2)

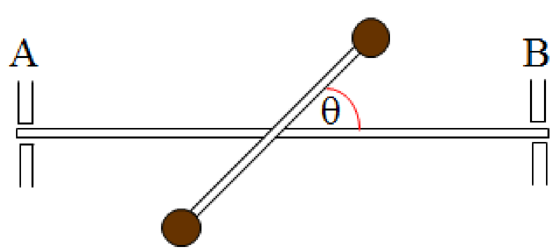


Figure 4.2: A dumbbell on an axle.

consisting of an axle AB and a dumbbell welded to the axle. The weld is at the CM of the dumbbell, but the angle θ between the dumbbell and the axle is not 90°. The dumbbell and axle rotate about the direction AB, the axle being supported by bearings at A and B. Since the CM is on the axis it is not accelerating and therefore the bearings exert no net force (except for a constant upward force equal to the weight of the dumbbell and axle). However, at any instant the bearings at A and B exert equal and opposite fluctuating forces on the axle, canceling the "tendency" of the dumbbell to align itself prependicular to the axis of rotation. These fluctuating forces are eliminated by ***dynamic balancing***, the theory of which is discussed in most intermediate texts on mechanics.

4.3 Time-Averaged Force

In many situations the force on an object varies rapidly with time and the quantity of physical interest is the time-averaged force (for brevity we call it the ***average force***). For example, as the molecules of a gas collide with a wall, each molecule exerts a force on the wall for a very short time; any macroscopic apparatus which we use to measure the force exerted by the molecules on the wall will have a response time which is long compared to the duration of a collision or the time between collisions. Thus the apparatus will measure only the time-averaged force.

Suppose $f(t)$ is some function of the time t. We define the time average of $f(t)$ by the equation

$$f_{\mathrm{avg}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} f(t)\ dt. \tag{4.10}$$

The average thus defined depends on t_1 and t_2, but in most cases of interest f_{avg} does not depend on t_1 and t_2 provided that the length of the averaging interval $t_2 - t_1$ is not too small. For example, Fig. 4.3 is a graph of the force exerted on the wall of a container by molecules colliding with the wall. If the averaging interval includes many collisions then f_{avg} does not depend on the length of the averaging interval. Note that the definition of f_{avg} implies that the area under the horizontal line is equal to the area under the actual fluctuating graph of force-versus-time.

If $\vec{F}(t)$ is a vector which varies with time, the time average of $\vec{F}$ is similarly defined, i.e.

$$\vec{F}_{\mathrm{avg}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \vec{F}(t)\ dt. \tag{4.11}$$

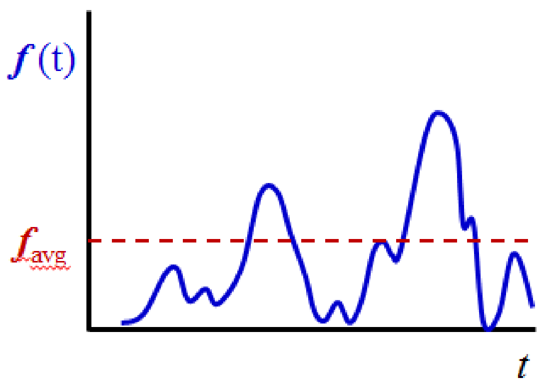


Figure 4.3: Graph of a rapidly time-varying force vs. time.

Thus the $x-$component of $\vec{F}_{\text{avg}}$ is the time average of $F_x(t)$ and similarly for the $y-$ and $z-$components.

We are now able to calculate the time-averaged force exerted by particles colliding with a wall. Rather than deal with gas molecules, which have a statistical distribution of velocities, we shall assume that the particles are bullets fired by a machine gun; thus all the particles approach the wall with the same velocity. We define our system to consist of all the bullets which hit the wall during a time interval of length T where T is large compared with the time between bullets. From eqn.(4.1) we have

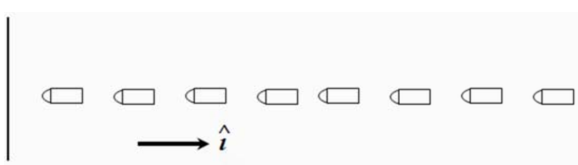


Figure 4.4: A rapid sequence of bullets impacting the wall at the left.

$$\int_0^T \vec{F}_{\text{ext}}\ dt = \int_0^T \frac{d\vec{P}}{dt} = \vec{P}_{\text{final}} - \vec{P}_{\text{initial}} \tag{4.12}$$

where $\vec{P}_{\text{final}}$ is the momentum of the system at time T and $\vec{P}_{\text{initial}}$ is the momentum at time 0.

If the bullets come to rest in the wall, then $\vec{P}_{\text{final}} = 0$. Initially all the bullets are traveling to the left with speed v (the velocity of a bullet is $-v\hat{\imath}$ where $\hat{\imath}$ is a unit vector pointing to the right). The number of bullets in our system is nT where n is the number of bullets hitting the wall per unit time. Thus we find $\vec{P}_{\text{initial}} = -nTmv\hat{i}$, where m is the mass of a bullet. The only external force on our system is exerted by the wall. Using eqn.(4.11) we find $\vec{F}_{\text{avg}} = nmv\hat{i}$. This is the average force exerted by the wall on the bullets; the average force exerted by the bullets on the wall is $-nmv\hat{i}$. If, instead of coming to rest in the wall, the bullets were to bounce off the wall with velocity $v\hat{i}$, then we would have $\vec{P}_{\text{final}} = nTmv\hat{i}$ and the force on the wall would be double the previous value.

The example below is slightly advanced but will impress or alienate your friends if you discuss this at a party.

Example 4.5 : Sand in an hourglass.

An hourglass is placed on a scale. When all the sand is in the bottom of the hourglass, the scale reading is W (i.e. the weight of the hourglass plus all the sand is W). What will the scale read while sand is falling through the hourglass? For definiteness we assume that a constant mass of sand per unit time (which we call ρ) falls through the hourglass and that all grains of sand fall the same distance d (i.e. we ignore the piling-up of sand).

This question can be answered either by applying the general theorem eqn.(4.9) to the system consisting of the hourglass and the sand, or by examining in detail what is happening in the hourglass. Both methods are instructive.

Let the z-axis point vertically upward and let us take $z = 0$ at the bottom of the hourglass. The height of the CM of the system (hourglass + sand) is determined by the equation

$$M\ Z_{\text{cm}} = \sum m_i\ z_i$$

where M is the total mass of the system. At any instant of time the sum on the right can be decomposed into four pieces:

(I). the contribution from the sand in the upper chamber

(II). the contribution from the sand in the lower chamber

(III). the contribution from the sand which is falling

(IV). the contribution from the hourglass itself.

I is equal to $m(t)d$ where $m(t)$ is the mass of sand in the upper chamber at time t. II vanishes because $z = 0$ at the bottom. III is constant in time because a photograph of the falling stream of sand looks the same at all times. And IV is obviously constant in time. Thus we have $MZ_{\rm cm} = m(t)d +$constant. Differentiating both sides with respect to time and noting that $dm/dt = -\rho$ we obtain $M\ dZ_{\rm cm}/dt = -\rho d$ and $M\ d^2Z_{\rm cm}/dt^2 = 0$ since ρ is assumed to be constant. It follows from eqn.(4.9) that the net external force on the system is zero. But the external force is $-Mg\ \hat{k} + F\ \hat{k}$, where $-Mg\ \hat{k}$ is the gravitational force and $F\ \hat{k}$ is the force exerted by the scale. Thus we find $F = Mg = W$; the scale reading is the same whether or not sand is falling. (Actually there is a brief transient effect at the beginning and the end since the picture of the falling sand is changing.)

Some people will argue that F should be less than Mg because the scale does not feel the weight of the sand which is freely falling. However, there is another effect: the impact of the falling sand on the bottom, which increases F. The preceding analysis of the center of mass motion, which completely bypasses the necessity of discussing these effects, also implies that the two effects must exactly cancel. We can see this in detail. If t is the time required for a grain of sand to fall a distance d, then the weight of the sand in free fall is $\rho\ gt$. The machine-gun calculation preceding this example tells us that the impact of the falling sand creates an extra force $\rho\ v$ on the scale, where v is the speed of a sand grain just before it hits the bottom (and ρ is indentical with nm in the machine-gun calculation). Since $v = gt$, the impact force exactly cancels the weight reduction, as expected. (Indeed, there is a transient decrease in the scale reading before the first grain of sand hits the bottom and a transient increase while the last grains are falling.)

Example 4.6 : Transporting pigeons in a truck.

A large tractor-trailer truck stops at an intersection; a pedestrian observes that the driver jumps out of the cab, bangs furiously on the side of the trailer with a two-by-four, and jumps back into the cab. In response to a query from the pedestrian, the driver shouts "The safe load limit for the tires is 60,000 lbs. The rig weighs 40,000 lbs empty and I've got 40,000 lbs of live pigeons inside. So I've got to keep half the pigeons in the air."

Will this scheme work?

Let our system be the truck plus all its contents. If $\vec{F}_{\rm avg}$ is the time-averaged force on the system, averaged over the time interval from $t = 0$

to $t = T$, then $\vec{F}_{\text{avg}} = [\vec{P}(T) - \vec{P}(0)]/T$ where $\vec{P}(T)$ and $\vec{P}(0)$ are the total momentum of the system at the two times. If we assume (not really necessary, but simplifies the discussion) that there is an upper limit to the speed with which a pigeon can fly, then the magnitudes of $\vec{P}(T)$ and $\vec{P}(0)$ are bounded. Thus, if we let the averaging interval T be long enough, we obtain $\vec{F}_{\text{avg}} = 0$. In particular, the average magnitude N of the upward force exerted by the road on the tires must be equal to W, the gravitational force on the truck plus all its contents.

As in the previous example, our conclusion is not based on a detailed analysis of the *internal* forces in the system. However, if one wants to know why the tires are not relieved of the weight of the pigeons which are in the air, the answer is that the flapping of the pigeons' wings increases the pressure which the air exerts on the floor of the truck. Note that our analysis assumes that the truck is closed so that all aerodynamic forces are internal to our system. If the trailer is enclosed only with chicken wire, then some of the aerodynamic forces are transmitted to nearby parts of the road. It should be clear that, in this case, if the walls of the trailer are high enough, the driver's strategy may work.

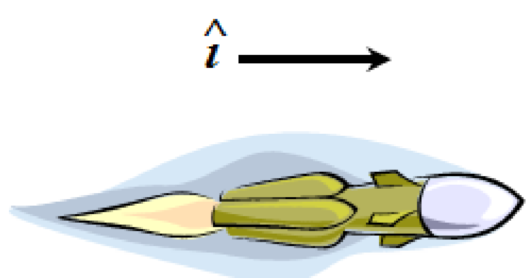


Figure 4.5: A rocket flying in deep space.

Example 4.7 : Rocket science.

Consider a rocket in outer space (no gravity). Initially the rocket is at rest and the mass of the rocket plus all its fuel is M_0. As fuel is burned, it is ejected to the rear with constant speed u relative to the rocket. A perfect analogy is that of a woman, armed with a machine gun, sitting on a sled on smooth ice. As she fires the gun to the rear the recoil accelerates the sled. What is the velocity of the rocket at the instant when the mass of the rocket plus the remaining fuel is M? (This relation does not depend on any assumptions about the rate of burning, which is not necessarily constant. If gravity is included, then the burning program is important.)

We take as our system the rocket plus all the fuel on board at any given instant. Let the mass of the system be M, and let the velocity of the rocket at this instant be $v\ \hat{\imath}$, where $\hat{\imath}$ is a unit vector pointing in the direction of motion of the rocket. We look at the same system (i.e. the same collection of particles) at a slightly later instant. At this time the mass of the rocket plus on-board fuel is $M + dM$ (note that dM is a negative quantity) and the velocity of the rocket is $(v + dv)\hat{\imath}$. Some of the particles of our system are no longer aboard the rocket; in fact, a mass $-dM$ of fuel has been ejected from the rocket and has velocity $(v - u)\hat{\imath}$ relative to an inertial observer. The forces exerted on the combustion chamber by the burning fuel, and the reactions to those forces, are all *internal* forces in our system. Thus there are no external forces on the system and the total momentum of the system at the initial instant must be equal to the total momentum at the slightly later instant. Hence we find

$$Mv\ \hat{\imath} = (M + dM)(v + dv)\ \hat{\imath} - dM(v - u)\ \hat{\imath}. \tag{4.13}$$

Since the two instants can be as close in time as we wish, the second order infinitesimal term $dM\ dv$ can be neglected in comparison with the terms proportional to dM or dv. (In fact, we already neglected second order infinitesimals in the second term of the above equation since the ejected particles can have a range of velocities from $(v - u)\ \hat{\imath}$ to $(v + dv - u)\ \hat{\imath}$.) Canceling out the unit vector $\hat{\imath}$, we obtain $M\ dv + u\ dM = 0$. Rewriting this as

$$\frac{dM}{M} + \frac{dv}{u} = 0$$

we find $d(\ln M + v/u) = 0$ which implies $\ln M + v/u = \text{constant}$.

If the initial velocity (at the instant when $M = M_0$) is zero, then the constant has the value $\ln M_0$ and we find $v = u\ \ln(M_0/M)$. If the initial velocity is v_0 we get what is called the ***ideal rocket equation***

$$v = v_0 + u\ \ln\left(\frac{M_0}{M}\right). \tag{4.14}$$

From an engineering point of view it is instructive to write this equation in the form $M_0/M = \exp[(v - v_0)/u]$. Usually one wants to put a certain mass M_1 (called the payload) into an orbit which requires that $v - v_0$ have a certain value w. In this case one must start with a mass M_0, where $M_0/M_1 = \exp(w/u)$. Suppose two fuels exist, the second having a value of u twice as large as the first. Then if $M_0/M_1 = 100$ for the first fuel, we will have $M_0/M_1 = 10$ for the second fuel.

4.4 Momentum Problems

4.1. Show that the center of mass of a rigid body continues to be the same physical point of the body even if the location and orientation of the body are changed.

4.2. A rope is hung vertically so that the bottom of the rope is just touching the floor. The rope has length L and mass M. The rope is released. Find the following:

(a) the force on the floor as a function of the distance fallen by the top end of the rope;

(b) the maximum force on the floor and the moment in time after the rope release when this occurs.

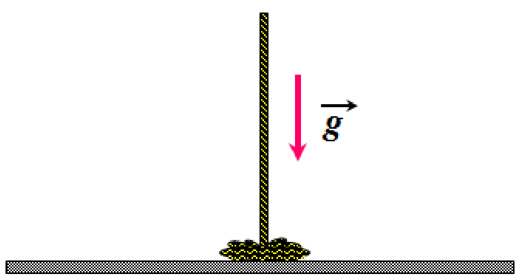


Figure 4.6: Problem 4.2.

4.3. A skyrocket is fired vertically and, when it reaches its high point, explodes into two fragments. One fragment lands 10 seconds after the explosion at a point 120 meters away from the launch point. The other fragment lands 4 seconds after the explosion at a point 24 meters from the launch point. Calculate

(a) the height at which the explosion occurred

(b) the maximum height reached by a fragment.

4.4. (For this problem take all velocities as horizontal. The positive x-axis is east and positive y-axis is north.)
A shell of mass 3.00 kg is traveling due east with velocity 350 m/s. It explodes into two fragments. Fragment #1 has velocity 900 m/s

directed 20.0° south of due east. Fragment #2 has a velocity v_2 directed 40.0° north of due east. Calculate v_2 (Dont make unwarranted assumptions about the masses of the fragments.)

4.5. The Saturn V rocket (total mass 2,800,000 kg [1]) launched astronauts to the moon. The first stage (section) of the rocket lifted it to 67 kilometers and was then discarded. The gross mass of the first stage (case + fuel) on the launch pad was 2,300,000 kg, and the mass of the case and the rest of the rocket was 131,000 kg. The burnout time of the first stage was 150 sec, and the thrust was 34,020,000 newtons. Calculate the velocity (relative to the rocket) with which the first-stage fuel was ejected and the final velocity of the rocket at 67 kilometers altitude.

4.6. Fig. 4.7 shows a cannon mounted on a flatcar and aimed at an angle θ above the horizontal. The flatcar and cannon are locked together and are initially at rest. Together they have mass M. A cannonball of mass m is fired at speed V *relative to the cannon*. Find the recoil speed of the flatcar and cannon and show that the angle α to the horizontal at which the cannonball emerges from the cannon is given by

$$\tan\alpha = \frac{M+m}{M}\tan\theta$$

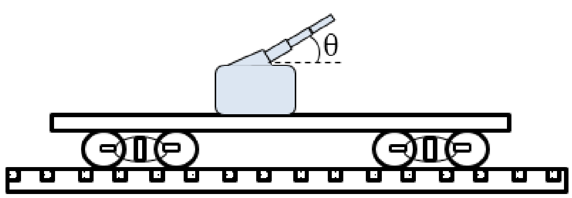


Figure 4.7: Problem 4.6.

[1]See `http://en.wikipedia.org/wiki/Saturn_V` for the numbers.

Chapter 5

WORK AND ENERGY

An important general consequence of Newton's laws is the ***Work-Energy Theorem***. This theorem enables us, in many situations, to give an explicit relation between the speed of a particle and its location in space. We have already seen an example of such a relation in the description of a freely falling body, but the reasoning in the present chapter is applicable to a much wider class of examples.

5.1 Definition of Work

Suppose a force $\vec{F}$ is acting on a particle which undergoes a very small displacement $\Delta\vec{r}$. The ***work*** done by the force on the particle is defined as[1]

$$\text{Work} = \vec{F} \cdot \Delta\vec{r} = |\vec{F}||\Delta\vec{r}| \ \cos\theta \qquad (5.1)$$

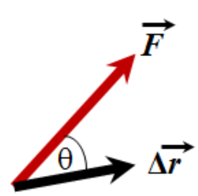


Figure 5.1: Calculating the work.

where θ is the angle between $\vec{F}$ and $\Delta\vec{r}$. Note that it makes no difference whether we take θ as the interior or exterior angle between $\vec{F}$ and $\Delta\vec{r}$ since $\cos\theta = \cos(360° - \theta)$. The work can be positive or negative, depending on whether θ is between 0° and 90° or between 90° and 180°. Thus, if you are pushing a box uphill on an inclined plane, you are doing positive work on the box and gravity is doing negative work. If you are restraining the box as it slides downhill you are doing negative work and gravity is doing positive

[1] If the reader is not familiar with the dot product (scalar product) of two vectors, he/she should read Appendix A before continuing.

work. Note that if $\theta = 90°$ (force is perpendicular to displacement) the force does no work. Therefore, if a particle moves on a smooth surface, the normal force exerted by the surface does no work on the particle.

The definition (eqn.(5.1)) of work can be used even if the displacement is not very small, provided that the force $\vec{F}$ does not vary during the displacement. If $\vec{F}$ varies, the definition (eqn.(5.1)) would be ambiguous (what value of $\vec{F}$ should we use?) and the only "natural" and useful definition of work is the one which follows.

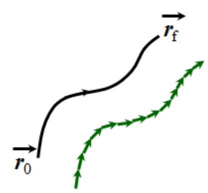


Figure 5.2: Subdividing a path.

Suppose that a particle is subjected to a displacement, not necessarily small, from an initial position $\vec{r}_o$ to a final position $\vec{r}_f$. We also specify the path taken by the particle, which is not necessarily a straight line. Conceptually, we can subdivide the path into a sequence of very small displacements $\Delta\vec{r}_n$, each of which is straight (see Fig. 5.2). Let $\vec{F}_n$ be the force acting on the particle when it undergoes the displacement $\Delta\vec{r}_n$. The work done on the particle during this short step is $\vec{F}_n \cdot \Delta\vec{r}_n$, and the total work done on the particle as it goes from $\vec{r}_o$ to $\vec{r}_f$ is defined as

$$W = \lim_{n\to\infty} \sum_n \vec{F}_n \cdot \Delta\vec{r}_n \tag{5.2}$$

where "lim" means that we are interested in the limiting value of the sum as the length of the steps becomes shorter and shorter and the number of terms in the sum consequently becomes larger and larger.

The limit which we have defined is clearly a generalization of the notion of an integral, and is generally represented by the symbol

$$\int_{\vec{r}_o}^{\vec{r}_f} \vec{F} \cdot d\vec{r}$$

which is usually referred to as the "***line integral*** of $\vec{F}$ from $\vec{r}_o$ to $\vec{r}_f$." Thus we can write

$$W = \int_{\vec{r}_o}^{\vec{r}_f} \vec{F} \cdot d\vec{r} \tag{5.3}$$

The student may find it useful to remember eqn.(5.2) rather than eqn.(5.3), since the former is easily visualized. Depending on the nature of the force $\vec{F}$, the right side of eqn.(5.3) may or may not have the same value for all paths between a given pair of endpoints $\vec{r}_o$ and $\vec{r}_f$. In the special case where $\vec{F}$ has the same value at all points on the path we have (using the distributive property of the dot product)

$$W = \lim_{n\to\infty} \sum_n \vec{F}_n \cdot \Delta\vec{r}_n = \vec{F} \cdot (\vec{r}_f - \vec{r}_o) \tag{5.4}$$

Example 5.1 : Work done by gravity.

Calculate the work done by gravity on a particle which moves from an initial position $\vec{r}_o$ to a final position $\vec{r}_f$. It is assumed that $\vec{r}_o$ and $\vec{r}_f$ are close enough to the earth's surface, and to each other, so that the gravitational force is constant, i.e. $\vec{F}_{\text{grav}} = -mg\ \hat{k}$.

We can write $\vec{r}_0 = x_0\ \hat{i} + y_0\ \hat{j} + z_0\ \hat{k}$ and a similar expression for $\vec{r}_f$. Using eqn.(5.4) and $\hat{k}\cdot\hat{i} = 0$, $\hat{k}\cdot\hat{j} = 0$, $\hat{k}\cdot\hat{k} = 1$ we find

$$W_{\text{grav}} = mg\ (z_0 - z_f). \tag{5.5}$$

Note that the work done by gravity depends only on the initial and final positions and does not depend on what path the particle traversed between those positions. This is true even when we take account of the variation of the magnitude and the direction of the gravitational force when the particle moves through large distances. (The work done by gravity is the same for all paths between two given points, but is not in general given by eqn.(5.5)). The signs with which z_0 and z_f enter eqn.(5.5) are easily remembered by noting that gravity does positive work on a particle which moves downward (force is parallel to displacement) and negative work on a particle which moves upward.

5.2 The Work-Energy Theorem

Let us consider a particle of mass m which at one instant of time t_0 is at a position $\vec{r}_0$ and has velocity $\vec{v}_0$, and at a later instant t_f is at a position $\vec{r}_f$ and has velocity $\vec{v}_f$. Let W be the total work done on the particle as it goes from $\vec{r}_0$ to $\vec{r}_f$. The ***Work-Energy Theorem*** asserts:

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 \tag{5.6}$$

The quantity $\frac{1}{2}mv^2$ is called the ***kinetic energy*** of the particle; thus the work-energy theorem can be stated in words as:

> The work done on a particle during any period of time is equal to the change[2] in its kinetic energy.

To prove this theorem we start with Newton's second law $\vec{F} = m\vec{a}$ and take the dot product of both sides with the instantaneous velocity vector $\vec{v}$, obtaining

$$\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v} \tag{5.7}$$

Using $d/dt(\vec{A} \cdot \vec{B}) = \vec{A} \cdot d\vec{B}/dt + \vec{B} \cdot d\vec{A}/dt$ (see Appendix A) we find

$$\frac{d}{dt}(\vec{v} \cdot \vec{v}) = 2\vec{v} \cdot \frac{d\vec{v}}{dt} = 2\vec{v} \cdot \vec{a}$$

Thus

$$m\vec{a} \cdot \vec{v} = \frac{d}{dt}\left(\tfrac{1}{2}m\vec{v} \cdot \vec{v}\right) = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right).$$

If we multiply both sides of eqn.(5.7) by a very short time interval Δt, we obtain

$$\vec{F} \cdot \vec{v}\Delta t = \frac{d}{dt}\left(\tfrac{1}{2}mv^2\right)\Delta t. \tag{5.8}$$

Since $\vec{v}\Delta t = \Delta\vec{r}$ where $\Delta\vec{r}$ is the displacement which the particle undergoes during the time interval Δt, the left side of eqn.(5.8) is the work ΔW done on the particle during the time interval Δt. The right side of eqn.(5.8) is just the change in the quantity $\frac{1}{2}mv^2$ during the time interval Δt. Thus, the work done on the particle during any short time interval is equal to the change in its kinetic energy during this time interval. Subdividing the period from t_0 to t_f into many short time intervals, we see that the total work done on the particle during this period is equal to the final kinetic energy minus the initial kinetic energy, which proves eqn.(5.6).

Application of the work-energy theorem to a freely falling particle yields

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = mg(z_0 - z_f). \tag{5.9}$$

This result follows immediately from our work in Chapter 1 (recall that v_x and v_y are constant during free fall, so only v_z is changing), but now we can show that eqn.(5.9) is also true in many situations where the particle is <u>not</u> freely falling. Consider, for example, a particle moving under the influence of gravity on a smooth surface of arbitrary shape. In this case, both the magnitude and direction of the particle's acceleration will usually vary with time; thus the analysis of motion with constant acceleration in Chapter 1 is not applicable.

[2] The change in a quantity is defined as the final value of the quantity minus the initial value of the quantity.

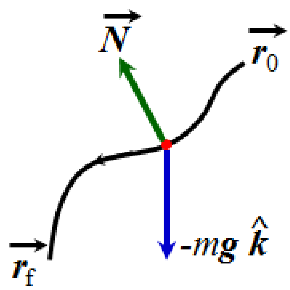


Figure 5.3: Work done by gravity.

However, the work-energy theorem is always applicable, provided we take care to calculate the total work done on the particle by *all* the forces acting on it. We note that a smooth surface exerts no force parallel to the surface. In this case there are only two forces acting on the particle: the gravitational force $-mg\ \hat{k}$ and the normal force $\vec{N}$ which is exerted by the surface. As we have already noted, any force perpendicular to the surface can do no work and hence $\vec{N}$ does no work since $\vec{N} \cdot \Delta\vec{r} = 0$ if $\Delta\vec{r}$ is a small displacement in the surface. The only force which does work is the gravitational force, and eqn.(5.9) is therefore true.

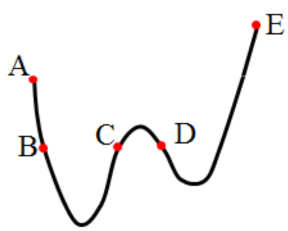


Figure 5.4: A particle starting from rest at A will have the same speed at B, C, and D and will never reach E.

If we let the "final" state in eqn.(5.9) be an arbitrary point in the motion of the particle, we can omit the subscript "f" and write

$$v^2 = v_0^2 + 2g(z_0 - z) \tag{5.10}$$

Eqn.(5.10) makes it clear that the speed of the particle depends only on its altitude z (and on the initial values v_0 and z_0), and not on the shape of the surface. If the particle starts from rest it can never reach an altitude greater than its initial altitude, since eqn.(5.10) would yield a negative value of v^2 if $z > z_0$.

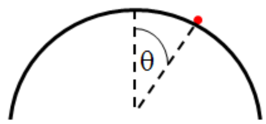


Figure 5.5: A particle starting from rest at the top of the hemisphere is given an infinitesimal push in Example 5.2.

Example 5.2 : Particle sliding on a hemisphere.

A particle (mass m) slides on a surface of a smooth inverted hemisphere (radius R), starting from rest at the top (an infinitesimal push starts the motion). At the instant when it has descended through an angle θ, what is its speed and what is the magnitude of the force exerted by the hemisphere on the particle? At what value of θ does the particle fly off the surface?

If we take our origin at the center of the hemisphere, then $z = R\cos\theta$ and eqn.(5.10) yields $v^2 = 2g(R - R\cos\theta)$. To calculate the force exerted by the hemisphere on the particle, we must use Newton's second law ($\vec{F} = m\vec{a}$). Two forces act on the particle:

1. the gravitational force, of magnitude mg directed vertically downward;

2. the normal force $\vec{N}$ exerted by the surface, directed radially outward.

There are *no other forces* acting on the particle.

The acceleration vector of the particle has a component v^2/R directed radially inward and a component dv/dt directed tangentially (downhill). We are interested only in the radial component of $\vec{F} = m\vec{a}$, which yields

$$mg\cos\theta - N = \frac{mv^2}{R} \tag{5.11a}$$

and thus

$$N = mg\cos\theta - \frac{mv^2}{R}. \tag{5.11b}$$

Inserting the expression for v^2 which we obtained from the work-energy theorem, we find

$$N = mg\cos\theta - 2mg(1-\cos\theta) = mg(3\cos\theta - 2). \tag{5.12}$$

The formula eqn.(5.12) for N makes sense in two respects: when $\theta = 0$ it yields $N = mg$ and as θ increases N decreases. [WARNING: Some students will write eqn.(5.11b) immediately, without writing eqn.(5.11a). The person who does this is almost certainly thinking of mv^2/R as a third force on the particle and will eventually become confused. One should always start by putting all the forces on one side of the equal sign and $m\vec{a}$ on the other side.]

At what value of θ does the particle fly off? Many (or even most) students have difficulty in stating the criterion which determines the point where the particle leaves the surface. It is important to recognize that the surface can only *push* on the particle and cannot pull on it. Examing eqn.(5.12) we see that $N = 0$ when $\theta = \cos^{-1}(2/3) = 48°$, $N > 0$ for $\theta < 48°$ and $N < 0$ for $\theta > 48°$. A negative value of N means that the surface is required to pull radially inward on the particle. Since the surface cannot do this, the particle will fly off when $N = 0$ ($\theta = 48°$). Note that if we were discussing a bead sliding on a smooth wire bent into the shape of an inverted semi-circle, the wire could (and would) supply the necessary inward force when $\theta > 48°$.

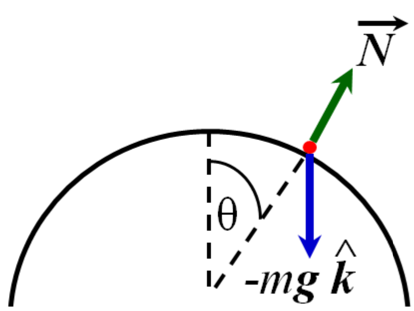


Figure 5.6: Force diagram for a particle sliding on a hemispherical surface in Example 5.2.

Oscillations will be discussed at greater length in Chapter 6. This topic is usually approached via a differential equation, but the work-energy theorem is sufficent for a complete analysis of the pendulum as we show in the next example.

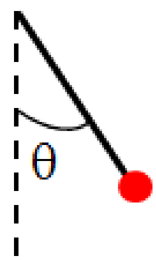


Figure 5.7: A simple pendulum.

Example 5.3 : Motion of a pendulum.

A pendulum consists of a point mass m attached to the ceiling by a massless string of length L. The pendulum swings back and forth, remaining always in the same vertical plane. The angular amplitude of the oscillation is $\theta_{\max}$ (i.e. when the pendulum is at one extreme of its motion, the angle between the string and the vertical is $\theta_{\max}$).

a. Find the speed of the pendulum and the tension in the string at the instant when the pendulum makes an angle θ with the vertical.

b. (More difficult) Assuming that $\theta_{\max}$ is small ($<$ 0.1 radian), use the result of (a) to calculate the period of the pendulum, i.e. the time required for the pendulum to execute one complete oscillation.

The force exerted by the string is directed along the string and is perpendicular to the velocity of the point mass. Thus, in any small time interval Δt the displacment $\Delta\vec{r}$ of the point mass is perpendicular to the force exerted by the string, and the string therefore does no work. Only gravity does work on the mass so we can use eqn.(5.10). If we choose "0" as the instant when the string makes the maximum angle $\theta_{\max}$ with the vertical (and therefore $v_0 = 0$) and "f" as the instant when the string makes angle θ with the vertical and has speed v, then eqn.(5.10) yields

$$v^2 = 2g\left(-L\cos\theta_{\max} + L\cos\theta\right) \tag{5.13}$$

In writing eqn.(5.13) we took our origin at the ceiling and used the relation $z = -L\cos\theta$. Thus we find

$$v = \sqrt{(2gL)(\cos\theta - \cos\theta_{\max}} \tag{5.14}$$

Eqn.(5.14) gives the speed at any point in the pendulum's motion. During a complete oscillation the pendulum passes through each point twice, once going to the right and once going to the left.

The radial component of $\vec{F} = m\vec{a}$ yields

$$T - mg\cos\theta = \frac{mv^2}{L} \tag{5.15}$$

where T is the tension in the string. Solving for T and using eqn.(5.13) we find

$$T = mg(3\cos\theta - 2\cos\theta_{\text{max}})$$

This says that T is largest when $\theta = 0$ and smallest when $\theta = \theta_{\text{max}}$. If θ_{max} is very small, then both cosines in the formula for T are close to 1 and $T \simeq mg$ as expected.

We can use eqn.(5.14) to calculate the period of the pendulum. As the string's angle with the vertical changes from θ to $\theta + d\theta$ the mass travels a distance $L\,d\theta$. The time required for the mass to travel this distance is

$$dt = L\frac{d\theta}{v} = \sqrt{\frac{L}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{\text{max}}}} \tag{5.16}$$

The time required for the pendulum to go from its low point to one extreme of its oscillation is found by integrating the right side of eqn.(5.16) with respect to θ from $\theta = 0$ to $\theta = \theta_{\text{max}}$. This time is equal to one-fourth of the period τ. Thus we find

$$\frac{\tau}{4} = \sqrt{\frac{L}{2g}}\int_0^{\theta_{\text{max}}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_{\text{max}}}}. \tag{5.17}$$

The integral is not elementary (it is called an elliptic integral), but is easily evaluated when θ_{max} is sufficiently small. Using the Maclaurin series for cosine[oST10] (with θ measured in radians), $\cos\theta = 1 - \theta^2/2! + \theta^4/4! - \ldots$ and omitting all terms after the first two, we obtain $\cos\theta - \cos\theta_{\text{max}} = \frac{1}{2}(\theta_{\text{max}}^2 - \theta^2)$ (with an error of less than one part in a thousand if $\theta_{\text{max}} < 0.1$ radian). Eqn.(5.17) then becomes

$$\frac{\tau}{4} = \sqrt{\frac{L}{g}}\int_0^{\theta_{\text{max}}}\frac{d\theta}{\sqrt{\theta_{\text{max}}^2 - \theta^2}}. \tag{5.18}$$

If we make the change of variable $x = \theta/\theta_{\text{max}}$ the integral becomes

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}}$$

At this stage, even before we evaluate the final integral, it is evident that the period does not depend on the angular amplitude θ_{max} (provided $\theta_{\text{max}} \ll 1$). This means that if there is some slight damping in the system (due to air resistance or friction in the suspension) which causes θ_{max} to decrease slowly, the period of the pendulum does not change as the angular amplitude decreases. It is this property which allows the pendulum to be used as a reliable clock.

To evaluate the final integral we make a further substitution $x = \sin\phi$. Using $dx = \cos\phi \; d\phi$ and $\sqrt{1-x^2} = \cos\phi$, we find

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} d\phi = \frac{\pi}{2}$$

and thus

$$\tau = 2\pi\sqrt{\frac{L}{g}} \tag{5.19}$$

If the angular amplitude of the oscillation is not small, the period depends slightly on the amplitude, increasing with increasing amplitude.

The preceding reasoning can be extended to give a complete description of the motion of the pendulum, i.e. to give an explicit formular for the angle θ as a function of the time t. If we take $t = 0$ at the instant when the pendulum passes through its low point, moving to the right, then the time t required for the pendulum to go from its low point out to an angle θ is

$$t = L\int_0^\theta \frac{d\theta'}{v(\theta')}$$

We have called the variable of integration θ' in order to distinguish it from θ, which is the upper limit of the integral.

Assuming again that θ_{max} is very small, we obtain

$$t = \sqrt{\frac{L}{g}}\int_0^\theta \frac{d\theta'}{\sqrt{\theta_{\text{max}}^2 - \theta'^{\,2}}}.$$

With the change of variable $\theta' = \theta_{\max} \sin\phi$ we find

$$t = \sqrt{\frac{L}{g}} \int_0^{\sin^{-1}\theta/\theta_{\max}} d\phi = \sqrt{\frac{L}{g}} \sin^{-1}\left(\frac{\theta}{\theta_{\max}}\right).$$

Solving for θ we obtain

$$\theta = \theta_{\max} \sin\left[\sqrt{\frac{g}{L}} t\right] \tag{5.20a}$$

which obviously describes an oscillatory motion. If we had taken $t = 0$ at the instant when $\theta = \theta_{\max}$, then eqn.(5.20a) would be replaced by

$$\theta = \theta_{\max} \cos\left[\sqrt{\frac{g}{L}} t\right] \tag{5.20b}$$

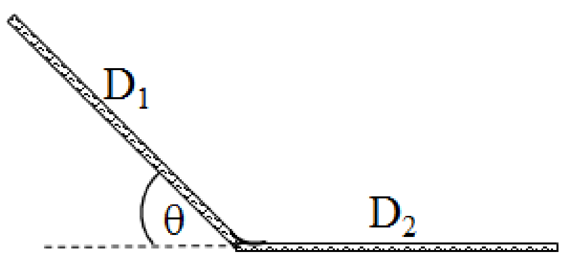


Figure 5.8: Sliding downhill and onto a flat field.

Example 5.4 : Sliding downhill.

A sled starts from rest, goes down a hill of length D_1 and angle θ, and continues along a flat field, coming to rest at distance D_2 from the bottom of the hill. By means of the work-energy theorem, derive an expression for the coefficient of kinetic friction μ_k between the sled and the snow, in terms of D_1, D_2, and θ. [The corner is a short frictionless curve.] The hill and and the field are covered with snow, but we assume that the effect of many sleds hitting the corner has turned the corner into smooth ice. If the corner has a non-zero coefficient of kinetic friction it is not correct to neglect the effect of the corner even if the corner is very short (see Problem 3.8).

One way to solve this problem is to use $\vec{F} = m\vec{a}$ and the kinematic equations of Chapter 1 (you should do this).

The work-energy theorem permits a brief and direct solution. If we choose "0" as the instant when the sled is at rest at the top of the hill and "f" as the instant when it comes finally to rest at the bottom, then $v_0 = v_f = 0$ and therefore $W_{\rm grav} + W_{\rm fric} = 0$ (where $W_{\rm grav}$ and $W_{\rm fric}$ are the work done by gravity and friction, respectively). From eqn.(5.5) we have $W_{\rm grav} = mg\mathrm{D}_1 \sin\theta$. On the the incline, the frictional force has magnitude $\mu_k mg \cos\theta$ and is directed opposite to the motion of the sled; thus the work done by friction as the sled descends the incline is $-\mu_k \mathrm{D}_1 mg \cos\theta$. Similarly, the work done by friction as the sled travels along the flat field is $-\mu_k mg\mathrm{D}_2$. Thus we find

$$\begin{aligned} mg\mathrm{D}_1 \sin\theta - \mu_k mg\mathrm{D}_1 \cos\theta - \mu_k mg\mathrm{D}_2 &= 0 \rightarrow \\ \mu_k &= \frac{\mathrm{D}_1 \sin\theta}{\mathrm{D}_1 \cos\theta + \mathrm{D}_2}. \end{aligned}$$

5.3 Potential Energy

Whenever we find a situation in which the only force doing work on a particle is the force of gravity (which, for the moment, is assumed constant in magnitude and direction), eqn.(5.9) is applicable. This equation can be rewritten in the form

$$\frac{1}{2}mv_0^2 + mgz_0 = \frac{1}{2}mv_f^2 + mgz_f.$$

Since 0 and f are arbitrary instants, we see that the quantity $\frac{1}{2}mv^2 + mg\ z$ has the same value at all times, i.e.

$$\frac{1}{2}mv^2 + mg\ z = \text{constant}. \tag{5.21}$$

Eqn.(5.21) is identical in content with eqn.(5.9), but we think about it in a somewhat different way. We have already given a name to the quantity $\frac{1}{2}mv^2$, calling it the kinetic energy. Eqn.(5.21) states that the sum of the kinetic energy and the quantity mgz remains constant during the motion;

thus an increase (decrease) in the kinetic energy must be accompanied by a corresponding decrease (increase) in mgz. The quantity mgz is called the ***potential energy***, and eqn.(5.21) states that

$$\text{Kinetic energy} + \text{Potential energy} = \text{constant} \tag{5.22}$$

The constant sum of the kinetic energy and the potential energy is called the ***total mechanical energy***, and eqn.(5.22) is called the ***Principle of Conservation of Energy***[3]. It must be borne in mind that we have proved this Principle only for the special case in which eqn.(5.9) is valid (in Example 5.4 the work done by kinetic friction invalidates the Principle). Extension of the Principle to other situations is not always possible; we shall now discuss the circumstances under which such an extension is possible.

The Work-Energy Theorem, which is always true (since it follows from $\vec{F} = m\vec{a}$ with no additional assumptions), asserts

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = \int_{\vec{r}_0}^{\vec{r}_f} \vec{F} \cdot d\vec{r} \tag{5.23}$$

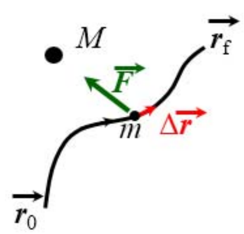


Figure 5.9: Work done by gravity.

where the right side is the total work done on the particle as it goes from $\vec{r}_0$ to $\vec{r}_f$. Let us consider, in particular, the case of a particle of mass m moving under the gravitational influence of a point mass M (whose position is held fixed). Then

$$\vec{F} = -G\frac{Mm}{r^2}\hat{r}$$

where r is the distance between M and m, and $\hat{r}$ is a unit vector pointing from M to m. We are specifically interested in situations where $\vec{r}_0$ and $\vec{r}_f$ are sufficiently different so that $\vec{F}$ cannot be treated as a constant along the path from $\vec{r}_0$ to $\vec{r}_f$. The path traveled by m is some curve which can be broken up into many small steps, a typical step being represented by the vector $\Delta\vec{r}$. We can decompose $\Delta\vec{r}$ into two pieces, one parallel to $\hat{r}$ and the other perpendicular to $\hat{r}$. When we calculate the work $\vec{F} \cdot \Delta\vec{r}$, only the piece of $\Delta\vec{r}$ parallel to $\hat{r}$ contributes to the dot product. If we introduce polar coordinates (taking the origin at M), then the vector $\Delta\vec{r}$ goes from the point whose polar coordinates are

[3] We refer here to “mechanical energy” since there are other “types” of energy. Mechanical energy refers specifically to energy associated with position and speed of constituents within a system.

$(r,\ \theta,\ \phi)$ to the nearby point $(r+\Delta r,\ \theta+\Delta\theta,\ \phi+\Delta\phi)$. The piece of $\Delta\vec{r}$ along the radial direction is $(\Delta r)\hat{r}$, and thus we find

$$\vec{F}\cdot\Delta\vec{r} = \left[-\frac{GMm}{r^2}\ \hat{r}\right]\cdot[(\Delta r)\hat{r}] = -\frac{GMm}{r^2}\ \Delta r$$

since $\hat{r}\cdot\hat{r}=1$.

Letting $\Delta r\to 0$ and adding up the contributions from all the steps, we find

$$\int_{\vec{r}_0}^{\vec{r}_f}\vec{F}\cdot d\vec{r} = -\int_{r_0}^{r_f}\frac{GMm}{r^2}\ dr = \frac{GMm}{r_f}-\frac{GMm}{r_0}. \tag{5.24}$$

This calculation shows that the work done by gravity depends only on the end points of the path, and is therefore the same for all paths between those end points (previously we showed this under the assumption that the gravitational force is constant). Inserting eqn.(5.24) into eqn.(5.23) we find

$$\frac{1}{2}mv_f^2-\frac{1}{2}mv_0^2 = \frac{GMm}{r_f}-\frac{GMm}{r_0} \tag{5.25}$$

or equivalently

$$\frac{1}{2}mv^2-\frac{GMm}{r} = \text{constant}. \tag{5.26}$$

In this case we call $-GMm/r$ the ***gravitational potential energy*** since the sum of this quantity and the kinetic energy remains constant.

In Chapter 3 we stated that (assuming the earth is spherical) the gravitational force exerted by the earth on an object outside the earth is identical with the gravitational force exerted by a point mass M_e (M_e = mass of earth) located at the center of the earth. (We sketched how to prove this by integral calculus, but did not give the proof.) Since g is the gravitational force per unit mass acting on an object near the earth's surface, it follows that

$$g = \frac{GM_e}{R^2} \tag{5.27}$$

where R is the radius of the earth. Futhermore, if we are discussing the motion of a particle near the surface of the earth, we ought to be able to show that the potential energy $-GM_em/r$ (where r is the distance of m from the center of the earth) is equivalent to the previously defined potential energy mgz. To see this we write $r=z+R$, where z is the altitude of the particle above the earth's surface. IF $z\ll R$ we can use the binomial theorem[oST10] to write

$$-\frac{GM_em}{z+R}\simeq -GM_em\left(\frac{1}{R}-\frac{z}{R^2}\right) = -\frac{GM_em}{R}+mgz.$$

Thus we see that the two expressions for the potential energy differ only by an additive constant which does not depend on z. Since **in all calculations only the difference of potential energy between two points enters**, the two expressions are indeed equivalent.

The property of the gravitational force which enabled us to define a potential energy (i.e. to find a quantity depending only on the position of the particle such that the sum of that quantity and the kinetic energy remains constant) is the following: the work done by gravity on a particle traveling between two points does not depend on the path. We shall now show that whenever the force $\vec{F}$ on a particle has the property that the work does not depend on the path, a potential energy can be defined.

In order to save words, we introduce a definition: a force $\vec{F}$ is called ***conservative*** if the work done by $\vec{F}$ on a particle traveling between any two points is the same for all paths between these two points.

We have already seen that gravity is a conservative force. We only proved this for the case when the gravitational force is due to a single particle, but if the gravitational force is due to several point masses at different locations, the force is vectorially additive (i.e. the net force is the vector sum of the forces exerted by the individual masses) and thus the work is additive, and therefore the total work is the same for all paths between any two particular points.

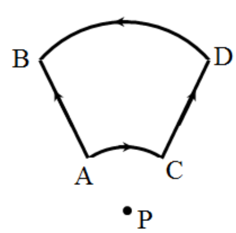


Figure 5.10: A force directed toward or away from P may or may not be conservative.

The same reasoning which showed that the gravitational force due to a single point mass is conservative also shows that any force which is directed toward (or away from) a fixed point in space, and whose magnitude depends only on the distance from the point, is conservative. However, if the force is directed toward a fixed point but its magnitude depends on both the distance and *direction* from the fixed point, the force is not conservative. In Fig. 5.10, in which the force is always directed toward point P, let us compare the work done on path AB with the work done on path ACDB (AC and DB are arcs of circles centered at P). No work is done on AC or DB, since the force is perpendicular to the displacement at every little step. If the magnitude of the force depends only on the distance from P, then the work done on path AB is the same as the work done on path CD. However, if the magnitude of the force depends on the direction from P, then the work done on path AB is not in general equal to the work done on path CD, and therefore the work done on path AB is not equal to the work done on path ACDB.

An important example of a non-conservative force is friction. If space is filled with a medium (e.g. air or water) which exerts a drag on a moving body, then the work done by the drag force (friction) depends on the length of the path traveled, and also on the speed with which the particle traverses the path.

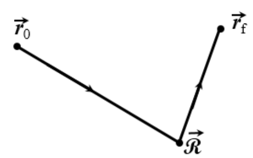


Figure 5.11: Diagram to prove eqn.(5.29).

If a force $\vec{F}$ is conservative, then the work done by $\vec{F}$ on a particle which goes from $\vec{r}_0$ to $\vec{r}_f$ is a function only of $\vec{r}_0$ and $\vec{r}_f$ and does not depend on the path. We call the work $W(\vec{r}_0,\ \vec{r}_f)$. The function $W(\vec{r}_0,\ \vec{r}_f)$ has a special form; it is the *difference* of a function of $\vec{r}_0$ and the same function of $\vec{r}_f$. To see this, we arbitrarily pick some fixed point $\vec{\mathcal{R}}$ (called the ***reference point***) and define

$$V(\vec{r}) = W(\vec{r},\ \vec{\mathcal{R}}) \tag{5.28}$$

i.e. $V(\vec{r})$ is the work done by $\vec{F}$ on a particle traveling from $\vec{r}$ to the reference point. Since $W(\vec{r}_0,\ \vec{r}_f)$ does not depend on the path, we can choose a path which goes from $\vec{r}_0$ to $\vec{\mathcal{R}}$ and then from $\vec{\mathcal{R}}$ to $\vec{r}_f$, and thus we find $W(\vec{r}_0,\ \vec{r}_f) = W(\vec{r}_0,\ \vec{\mathcal{R}}) + W(\vec{\mathcal{R}},\ \vec{r}_f)$. But $W(\vec{\mathcal{R}},\ \vec{r}_f) + W(\vec{r}_f,\ \vec{\mathcal{R}}) = W(\vec{\mathcal{R}},\ \vec{\mathcal{R}}) = 0$ since one of the paths from $\vec{\mathcal{R}}$ to $\vec{\mathcal{R}}$ is the path of zero length. Therefore

$$W(\vec{r}_0,\ \vec{r}_f) = V(\vec{r}_0) - V(\vec{r}_f) \tag{5.29}$$

Consequently, **if all the forces which do work are conservative**, the work-energy theorem yields

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = V(\vec{r}_0) - V(\vec{r}_f) \tag{5.30}$$

which implies

$$\frac{1}{2}mv^2 + V(\vec{r}) = \text{constant} \tag{5.31}$$

The function $V(\vec{r})$ is called the ***potential energy*** and eqn.(5.31) is the statement of the ***Principle of Conservation of Energy***.

If a particle moves under the influence of a conservative force (or forces) with associated potential V and also under the influence of a non-conservative force (such as friction), then the work-energy theorem yields (using K.E. and P.E. to denote kinetic and potential energies)

$$\text{K.E.}_f + \text{P.E.}_f = \text{K.E.}_0 + \text{P.E.}_0 + W' \tag{5.32}$$

where W' is the work done by the non-conservative forces as the particle moves from $\vec{r}_0$ to $\vec{r}_f$. If the non-conservative force is a frictional drag directed opposite to the motion, then $W' < 0$.

The reference point $\vec{\mathcal{R}}$ is arbitrary. A change in the reference point causes the potential energy at all points to change by an additive constant (prove this!). Since eqn.(5.30) involves the *difference* of the potential energies at two points, an additive constant in V will not change anything. The definition eqn.(5.28) implies $V(\vec{\mathcal{R}}) = 0$. Thus, if we define the gravitational potential energy as $V(\vec{r}) = -GMm/|\vec{r}|$, we have chosen the reference point $\vec{\mathcal{R}}$ as a point infinitely far away from the mass M. If we choose the reference point as a point on the earth's surface, then $V(\vec{r}) = -GMm/|\vec{r}| + GMm/R$. In this case the potential at points near the earth's surface is mgz, in agreement with our earlier work.

Example 5.5 : The earth's escape velocity.

With what speed must a projectile be fired from the earth's surface in order to escape to infinity? (Ignore air resistance, the effect of the earth's rotation, and the effects of the sun and the moon.)

If v_e is the speed of the projectile when it leaves the earth, and v_∞ is the speed of the projectile when it is infinitely far (i.e. its distance from the earth is a large multiple of the earth's radius R), then eqn.(5.31) implies

$$\frac{1}{2}mv_{\mathrm{e}}^2 - GM_e\frac{m}{R} = \frac{1}{2}mv_\infty^2 \tag{5.33}$$

We are interested in the minimum value of v_{e} which will permit the projectile to reach infinity. Setting $v_\infty = 0$, we find $v_{\mathrm{e}} = \sqrt{2GM_e/R} = \sqrt{2gR}$. This speed is called the ***escape velocity*** from the earth. Note that $v_e = \sqrt{2}v_o$, where v_o is the velocity of a satellite in a circular orbit with radius equal to the earth's radius (see Chapter 3). Numerically, $v_o = 18,000$ mi/hr and $v_e = 25,000$ mi/hr. Space scientists and science fiction writers sometimes describe low earth orbit as being "halfway to anywhere" or "halfway to infinity", since the kinetic energy of a projectile in low earth orbit is half the kinetic energy required to escape entirely from the earth's gravitational influence.

5.4 More General Significance of Energy (Qualitative Discussion)

The reader will probably have recognized that, as far as solving problems is concerned, the Work-Energy Theorem is sufficient for our purposes in beginning physics. The introduction of the concept of potential energy was not really necessary. Nevertheless, every text on mechanics introduces the concept "potential energy" and encourages students to use the Principle of Conservation of Energy whenever possible; the student must keep in mind, however, that when non-conservative forces are present (the most common being friction) mechanical energy is not conserved.

What is the reason for this emphasis on energy and Conservation of Energy, when the latter principle appears not always to be true? The answer is: *if we look at things in sufficient detail* (this may require a very powerful microscope) *we will find that energy is always conserved.*

Consider, for example, a block which slides on a horizontal table and eventually comes to rest because of friction. Initially the block has kinetic energy, and finally it has none. The gravitational potential energy of the block is the same in the initial and final states. Thus *mechanical* energy (as we have defined it) is not conserved, and the final mechanical energy minus the initial mechanical energy is equal to W', the work done by friction. However, if we could examine the block with a sufficiently powerful microscope we could see that the individual molecules in the block are not at rest, but jiggle around their equilibrium positions in a rather random fashion. Furthermore, we would see that the amount of molecular jiggling in the final state of the block is somewhat greater than the amount of molecular jiggling in the initial state. Macroscopically, we would find that the block in its final state is slightly warmer than it was in the initial state. Similarly, the molecules of the table increase their jiggling when the block slides over them, and the surface of the table becomes slightly warmer. Each jiggling molecule has kinetic energy, and there is a significant amount of "invisible" kinetic energy in the form of molecular jiggling.

When we speak of "kinetic energy" in classical mechanics, we refer only to the visible kinetic energy of the block (i.e. $\frac{1}{2}MV^2$ where M and V are the mass and velocity of the block) and do not keep track of the "invisible" kinetic energy. If we were to include the invisible kinetic energy in our account books we would find that the total mechanical energy of the system (block + table) in the final state is equal to the total mechanical energy of the system (block + table) in the initial state. In short, the macroscopic kinetic energy which the block loses is converted into increased kinetic energy of jiggling of the molecules in the block and in the table. From the macroscopic point of view of classical mechanics energy can be lost, but from a microscopic point of view some energy is merely being transformed into a less visible form.

For completeness we should note that "invisible" energy need not necessarily be kinetic energy, but can also be potential energy. For example, when a bullet comes to rest in a block of wood, not all the kinetic energy of the bullet is converted into kinetic energy of molecular jiggling. The wood has an internal potential energy which depends on the arrangement of the molecules, which changes when the bullet enters the block. The kinetic energy of the bullet is converted partly into energy of molecular jiggling and partly into internal potential energy of the wood.

The proof of the fact that energy (appropriately defined) is always conserved falls outside the scope of Newtonian mechanics. Remarkably, Newtonian mechanics is able to make many predictions which do not depend on what is happening at the microscopic level; the accurate description of many of these phenomena requires ***quantum mechanics***, which is radically different from Newtonian mechanics. We stress again that within the framework of our (narrow, but useful) macroscopic definition of energy, the energy of a system may or may not be conserved.

5.5 Elastic and Inelastic Collisions

We have already discussed a collision in which the two colliding objects stick together. In this case the final velocity is determined by the Principle of Conservation of Momentum. In particular, let us consider a particle of mass m_1 and velocity $\vec{v}_1$ which collides with a particle of mass m_2 which is initially at rest. If the two particles stick together, the final velocity is $\vec{V} = (m_1/m_1 + m_2)\vec{v}_1$. The final kinetic energy is

$$\frac{1}{2}(m_1 + m_2)V^2 = \frac{1}{2}\frac{m_1^2 v_1^2}{(m_1 + m_2)}.$$

Since the initial kinetic energy was $\frac{1}{2}m_1 v_1^2$, we have

$$\frac{\text{K.E.}_{\text{final}}}{\text{K.E.}_{\text{initial}}} = \frac{m_1}{m_1 + m_2}.$$

Thus if $m_1 = m_2$ we find that half of the original kinetic energy is lost in the collision. According to the discussion in the preceding section, the missing energy has been converted into molecular kinetic energy and internal potential energy of the colliding objects. The fact that these objects stick together guarantees that there is some mechanism for converting macroscopic kinetic energy into "invisible" microscopic kinetic and potential energy. If no such mechanism exists, then the two colliding objects cannot stick together.

A collision in which the final (macroscopic) kinetic energy is less than the initial (macroscopic) kinetic energy is called ***inelastic***. If the final kinetic energy is equal to the initial kinetic energy, the collision is called ***elastic***. The collision between two smooth steel balls is almost perfectly elastic. There are, of course, varying degrees of inelasticity; for example, even though the two colliding objects do not stick together, the final kinetic energy may be somewhat less than the initial kinetic energy.

When two particles collide, there are (as we have previously discussed) many possible final states consistent with Conservation of Momentum. It is easy to show that, among all these states, the one of lowest kinetic energy is the state in which the two particles have the same final velocity, i.e. they stick together. A collision in which the two particles stick together is called "***completely inelastic***". More precisely, we should call such a collision "***maximally inelastic***" since in general there is still some macroscopic kinetic energy in the final state. Momentum conservation places a limit on the amount of kinetic energy that can be converted into microscopic form. When two colliding particles stick together the final kinetic energy is $P^2/2M$ where P is the total momentum and M is the total mass.

If the system is free from external forces, the total momentum is conserved. Unlike energy, momentum cannot be converted into "invisible" form. Random molecular jiggling will not carry any net momentum since as many molecules are going in one direction as in another. If there is a non-random "drift velocity" superimposed on the jiggling, the body as a whole will move with this drift velocity. Therefore all momentum is visible and we can expect momentum to be conserved even when energy (apparently) is not, provided that no external force acts on the system.

Figure 5.12: Before and after a one-dimensional elastic collision.

We have seen that momentum conservation fully determines the final state in a completely inelastic collision. Another interesting case in which the final state is fully determined is the **elastic collision in one dimension**. By "one dimension" we mean that all particles live on a line, e.g. the x-axis. There are two unknown velocities in the final state (we call them v_{1f} and v_{2f}). Momentum conservation imposes one condition on the two unknowns; if we also require energy conservation, we have a second equation and the final state is therefore determined.

For simplicity we assume that m_2 is initially at rest and that m_1 initially has velocity v_{1i}. Momentum conservation requires

$$m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f} \tag{5.34}$$

and energy conservation requires

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{5.35}$$

If we solve eqn.(5.34) for v_{1f}

$$v_{1f} = v_{1i} - \frac{m_2}{m_1} v_{2f} \tag{5.36}$$

and substitute eqn.(5.36) into eqn.(5.35) we obtain the quadratic equation

$$\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 \left[v_{1i}^2 + \frac{m_2}{m_1} v_{2f}^2 - 2 \frac{m_2}{m_1} v_{1i} v_{2f} \right] + \frac{1}{2} m_2 v_{2f}^2$$

for the unknown v_{2f}. With a little algebra this can be rewritten as

$$0 = v_{2f} \left[\left(1 + \frac{m_2}{m_1} \right) v_{2f} - 2 v_{1i} \right]$$

which has roots $v_{2f} = 0$ and $v_{2f} = 2v_{1i}/(1 + m_2/m_1)$. What is the physical significance of the two roots? If $v_{2f} = 0$ then eqn.(5.36) implies $v_{1f} = v_{1i}$. This solution, in which the final velocities of both particles are identical with their initial velocities, obviously satisfies energy and momentum conservation but is physically meaningful only if there is no interaction between the particles (so that m_1 can pass through m_2 without exerting a force on it). The other solution

$$v_{2f} = \frac{2 v_{1i}}{1 + m_2/m_1} \tag{5.37}$$

which implies (by eqn.(5.36))

$$v_{1f} = v_{1i} \frac{1 - m_2/m_1}{1 + m_2/m_1} \tag{5.38}$$

is the physically interesting one.

It is instructive to examine eqns.(5.37) and (5.38) in various limiting cases, in order to see whether the formulae conform with our expectations. There are three limiting cases which can be understood simply:

1. Suppose $m_2/m_1 \gg 1$ (a ping-pong ball striking a stationary bowling ball head-on). We expect the bowling ball to remain essentially at rest and the ping-pong ball to rebound (as from a rigid wall) with velocity equal in magnitude and opposite in direction to its initial velocity. Thus we expect $v_{2f} = 0$ and $v_{1f} = -v_{1i}$, which agrees with eqns.(5.37) and (5.38) when $m_2/m_1 \gg 1$.

2. Suppose $m_1 = m_2$. Then eqns.(5.37) and (5.38) yield $v_{2f} = v_{1i}$ and $v_{1f} = 0$, i.e. the two particles exchange velocities. This solution is familiar to shuffleboard and billiards players, and can be readily understood by looking at the collision from the point of view of another inertial frame, the ***center-of-mass system***. The center of mass has velocity $\frac{1}{2}v_{1i}$, and in this frame the collision is *symmetric*, i.e. particle #1 initially has velocity $\frac{1}{2}v_{1i}$ and #2 initially has velocity $-\frac{1}{2}v_{1i}$. In a symmetric elastic collision the final velocity of each particle is just the negative of its initial velocity. Therefore, in the center-of-mass system, the final velocity of #1 is $-\frac{1}{2}v_{1i}$ and the final velocity of #2 is $\frac{1}{2}v_{1i}$. To translate back into the language of a stationary observer we add $\frac{1}{2}v_{1i}$ (the velocity of the center of mass) to both of these velocities and thus find that $v_{1f} = 0$ and $v_{2f} = v_{1i}$.

3. Suppose $m_2/m_1 \ll 1$ (a bowling ball striking a stationary ping-pong ball head-on). The collision causes negligible change in the velocity of the heavy object. Therefore $v_{1f} = v_{1i}$, in agreement with eqn.(5.38). In this case the center of mass has essentially the same velocity as the heavy object, i.e. $v_{\rm cm} = v_{1i}$. In the CM system the light object approaches the heavy object with velocity $-v_{1i}$ and will rebound (as from a rigid wall) with velocity v_{1i}. Thus, from the point of view of a stationary observer, $v_{2f} = 2v_{1i}$, in agreement with eqn.(5.37).

The preceding discussion illustrates an important aspect of good problem-solving technique. If there is some variable parameter (like m_2/m_1) in the problem, simple reasoning will frequently suffice to predict the answer for certain special values of that parameter. If your solution does not agree with your expectations in these special cases, then either your solution is wrong (algebraic error or something more serious?) or your expectations are incorrect.

5.5.1 Relative Velocity in One-Dimensional Elastic Collisions

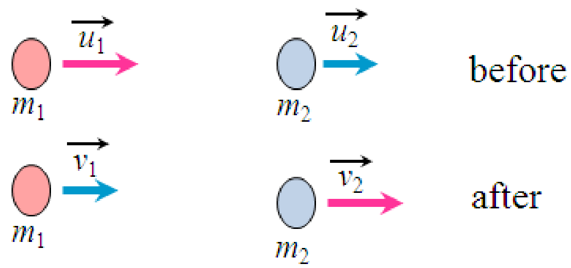


Figure 5.13: An elastic collision in one dimension where, to simplify the analysis, all velocities have been drawn in the same direction.

Suppose we want to analyze a one-dimensional elastic collision in which the target particle (#2) is not initially at rest. We can do the algebra or, if we are lazy, go to a frame A′ in which the particle is initially at rest. In this frame we can use the results of Section 5.5. Translation of velocities from the frame A′ back to the original frame (call it A) is trivial because velocities in A′ differ from the corresponding velocities in A by an additive constant (the relative velocity of the two frames).

If we denote initial and final velocities in frame A′ by v' and u', then eqns.(5.37) and 5.38 yield (noting that $v_2' = 0$)

$$u_2' - u_1' = \frac{v_1'}{1 + m_2/m_1} \cdot 1 + m_2/m_1 = v_1' - v_2'. \tag{5.39}$$

But since $u_2' - u_1' = u_2 - u_1$ and $v_2' - v_1' = v_2 - v_1$ (where v and u denote initial and final velocities in frame A), we have

$$u_1 - u_2 = (v_2 - v_1). \tag{5.40}$$

Thus, in a one-dimensional elastic collision, the relative velocity of the particles is *reversed* in direction but *unchanged* in magnitude. This is a useful result for many problems involving collisions in one dimension if kinetic energy is conserved.

5.5.2 Two Dimensional Elastic Collisions

In contrast with the one-dimensional elastic collision, the final state in a two-dimensional elastic collision is <u>not</u> uniquely determined by momentum and energy conservation. There are four unknown quantities in the final

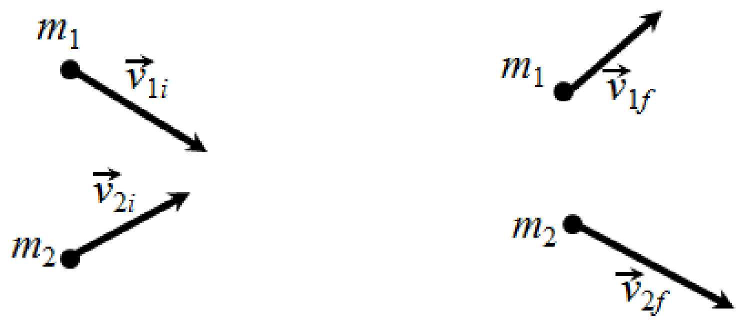


Figure 5.14: An elastic collision in two dimensions.

state (the $x-$ and $y-$components of $\vec{v}_{1f}$ and $\vec{v}_{2f}$, or the magnitudes and directions of the two velocities). Conservation of the $x-$ and $y-$components of momentum imposes two conditions, and conservation of energy imposes a third condition. Thus, there is one free parameter in the final state. If, for example, we specify the direction of $\vec{v}_{1f}$, then the magnitude of $\vec{v}_{1f}$ and the magnitude and direction of $\vec{v}_{2f}$ are determined. What additional information

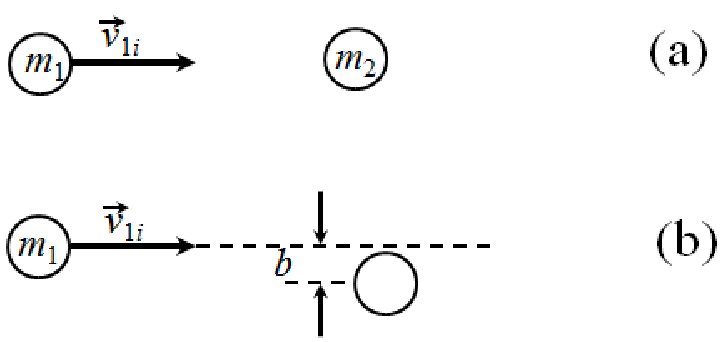


Figure 5.15: Collision of two objects with an impact parameter.

(in addition to the two initial velocity vectors) about the initial state must we have in order to completely predict the final state? For simplicity, let us assume that m_2 is initially at rest and that the colliding objects are hockey pucks. In Fig. 5.15 (a) and (b) puck #1 has the same velocity, but the final states will be different in the two cases. In (a) the collision is head-on, whereas in (b) puck #1 has a glancing collision with #2. To fully describe the initial state we must specify how well #1 is aimed at #2. This is usually done by the giving the value of the ***impact parameter*** b (see Fig. 5.15), the distance by which the center of #1 would miss the center of #2 if #1 were not deflected. The interested reader may find it instructive to calculate $\vec{v}_{1f}$ and $\vec{v}_{2f}$ in terms of $\vec{v}_{1i}$ and the impact parameter, but we omit the calculation here. The key point is that, if the surfaces of the pucks are smooth, the direction of $\vec{v}_{2f}$ (i.e. the direction of the impulse given to #2) is along the line from the center of #1 to the center of #2 at the instant when the pucks touch. In one dimension everyone is a perfect marksman, i.e. the impact parameter is always zero.

5.6 Power and Units of Work

Work has the units of (force)$\times$ (distance). The unit of work in the English system is the ***foot-pound*** (ft-lb). In the mks system the unit of work is the newton-meter, usually called the ***joule***. The reader should verify that 1 joule = .738 ft-lb.

In many situations one is interested in the *rate* at which work is done. For example, the horsepower rating of a motor measures the rate at which it can do work; any motor can do an arbitrarily large amount of work if it is allowed to run for a sufficiently long time. Similarly, the ***wattage*** rating of an electric bulb measures the amount of energy **per unit time** which must be supplied to the bulb to keep it operating. If the bulb is operated by a man-powered or steam-powered generator, the generator must do work at a rate equal to the wattage.

The rate of work is called ***power***; power has the units of (work)/ (time) or (energy)/(time), since work and energy have the same units. In the English system the unit of power is 1 ft-lb/sec. This unit has no other name, but 550 ft-lb/sec is called 1 ***horsepower***. Thus a 1/2 horsepower motor can do work at the rate of 275 foot-pounds per seconed. By attaching appropriate gears to the output shaft we can use the motor to lift a 25 lb weight at the rate of 11 ft/sec or a 50 lb weight at the rate of 5.5 ft/sec, etc. In the mks system the unit of power is 1 joule/sec, which is called 1 ***watt***. Since 1 watt = .738 ft-lb/sec it follows that 1 horsepower = 746 watts = .746 kilowatts. Since the watt and kilowatt have the units of (energy)/(time), the kilowatt-hour (kwh) can be used as a unit of energy; 1 kwh = (1000 joules/sec)$\times$ (3600 sec) = 3.6×10^6 joules = 2.66×10^6 ft-lb.[4]

[4] A kilowatt-hour, for which a consumer in Philadelphia pays 15 cents, is the energy required to lift an average automobile from the ground to the observation deck of the Empire State Building.

The energy content of food is measured in ***food calories***. 1 food calorie = 4184 joules = 3085.96003 ft-lbs. The food calorie is equal to 1000 of the scientific unit ***calorie***, the little calorie being the amount of energy required to raise the temperature of one gram of water by one degree centigrade. The average American adult male takes in about 3000 calories per day (estimates vary); his rate of energy consumption is (3000× 4184) joules/(24× 3600 sec) = 145 watts. Most of this energy is used to support biological processes and eventually appears as heat given off by the body. Thus it is feasible for universities, which are beset by high energy costs, to use students as the major source of heat for auditoriums.

A mountaineer in good physical condition can climb (at low altitudes) at the rate of 300 vertical meters per hour for a few hours. If she/he has a mass of 60 kg, the rate at which she/he does useful work (i.e. work which appears as increased gravitational potential energy rather than as heat given off by the body) is (60 kg)× (9.8 m/s^2)× (300 m) / 3600 sec = 49 watts = 0.066 hp. Thus, a 60-watt bulb is equivalent to the full-time effort of a healthy human being. Some athletes can perform useful work at the rate of one horsepower for a very short time (for example, a football player who weighs 184 lb might run up a flight of stairs at the rate of 3 vertical feet per second). An excellent method of inducing a heart attack is to push hard on the rear of a moving car. If you push with a force of 55 lb on a car moving with a velocity of 10 ft/sec you are working at the rate of one horsepower.

If a force $\vec{F}$ is acting on an object which undergoes a small displacment $\Delta\vec{r}$ during a small time interval Δt, the work done by the force is $\vec{F}\cdot\Delta\vec{r}$ and the rate of work is $\vec{F}\cdot\Delta\vec{r}/\Delta t = \vec{F}\cdot\vec{v}$. If the object to which the force is applied is a particle of mass m and if $\vec{F}$ is the total force acting on the particle, then $\vec{F} = m\ d\vec{v}/dt$ and the instantaneous power delivered by $\vec{F}$ is $m(d\vec{v}/dt)\cdot\vec{v} = d/dt(\frac{1}{2}mv^2)$.

Example 5.6 : Power to the particle.

A particle of mass 2 kg moves along the x-axis. Its position at time t is given by $x = 4t + 4t^2 - t^3$ where x is in meters and t is in seconds. Calculate the instantaneous power being delivered to the particle at $t = 1$ and $t = 2$.

The velocity is $\vec{v} = (dx/dt)\ \hat{i} = \hat{i}(4 + 8t - 3t^2)$ and the acceleration is $d\vec{v}/dt = (8 - 6t)\ \hat{i}$. Evaluating $m(d\vec{v}/dt)\cdot\vec{v}$ at $t = 1$ and $t = 2$ we find: instantaneous power = 36 watts when $t = 1$ and -64 watts when $t = 2$. The negative value at $t = 2$ results from the fact that at that instant the force and the velocity have opposite directions.

5.7 Work and Conservation of Energy Problems

5.1. A tennis racket (held firmly in a hand) is moving at a speed u_{racket}. It hits a ball moving at speed u_{ball}. What is the maximum possible speed of the ball after it is hit?

5.2. A block of mass m is released on a wedge of mass M at a height h above the floor as shown in Fig. 5.16. All surfaces are frictionless. Show that the speed of the wedge when the block hits the floor is given by

$$\sqrt{\frac{2m^2gh\cos^2\theta}{(M+m)(M+m\sin^2\theta)}}$$

[Solve by energy and momentum conservation.]

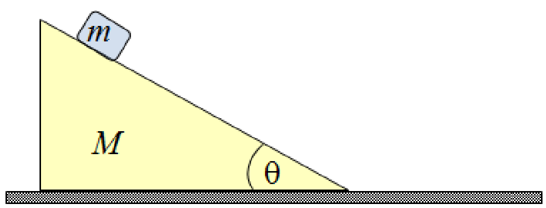


Figure 5.16: Problem 5.2.

5.3. A bungee jumper (mass = 80.0 kg) jumps from a high bridge. The bungee cord is an elastic climbing rope of negligible mass. One end of the rope is tied to the bridge, and the other end is tied to the jumper's harness. The unstretched length of the rope is 50.0 meters, and when the rope is stretched to a length $50.0 + x$ it exerts a restoring force of magnitude kx (this is called Hooke's Law), where $k = 200$ N/m.

(a) At what distance below the bridge does the jumper have maximum velocity?

(b) Calculate the maximum velocity.

(c) At what distance below the bridge does the jumper have maximum acceleration?

(d) Calculate the maximum acceleration.

(e) Calculate the jumper's maximum distance from the bridge.

5.4. If you double the length of a rope which obeys Hooke's law, the Hooke's law constant (k) decreases by a factor of 2. More generally (show this), if K is the Hooke's law constant of a rope of unit length, then the Hooke's law constant of a rope whose unstretched length is L is $k = K/L$. You are a bungee jumper using a Hooke's law rope whose unstretched length is L (one end is tied to the bridge and the other end is tied to your harness). Show that the maximum subsequent tension in the rope does not depend on L. (Assume that the mass of the rope is negligible compared with your mass and can therefore be neglected.)

5.5. Prove that if a collision conserves momentum and kinetic energy in an inertial frame A, it will also conserve energy and momentum in any other inertial frame A′. (Note that if the origin of the frame A′ has velocity $\vec{V}$ as measured in frame A, then if observers in the two frames measure the velocity of the same particle, the relation between their measurements is $\vec{v}' = \vec{v} - \vec{V}$.)

5.6. Prove that if a particle collides elastically with another particle of the same mass, and if the target particle is initially at rest, then the subsequent velocities of the two particles are perpendicular to each other.

5.7. In the figure below a 500 gram block is dropped from a height 60.0 cm above the top of the platform of mass 1.00 kg. The platform sits above a vertically-aligned spring with a spring constant $k = 120$ N/m. Assuming the block makes a completely inelastic collision with the platform, find the maximum compression of the spring.

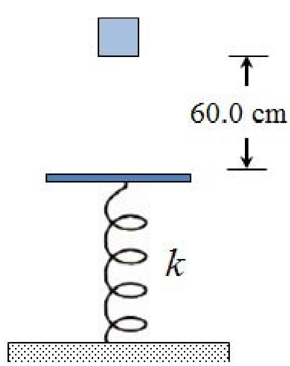


Figure 5.17: Problem 5.7.

5.8. Two planets, both of mass M, are separated by a distance D. Their relative velocity is negligible, and there is an inertial frame in which both planets are essentially at rest. The gravitational potential $u(\vec{r})$ is defined as the gravitational potential energy of a particle of unit mass when it is at the position $\vec{r}$. In the presence of the two planets, located at $\vec{R}_1$ and $\vec{R}_2$, the potential is

$$u(\vec{r}) = -\frac{GM}{\left|\vec{R}_1 - \vec{r}\right|} - \frac{GM}{\left|\vec{R}_2 - \vec{r}\right|} \ .$$

This problem takes place far out in space. There are no other massive objects in the vicinity of the two planets.

(a) Draw a qualitatively correct graph of u as a function of position along the line between the two planets.

(b) There are space stations Alpha and Beta located on the line between the planets. Both space stations are at rest with respect to the planets. Alpha is at distance $D/4$ from planet #1 and Beta is at distance $D/3$ from planet #2. A projectile of mass m is fired from station Alpha, with its velocity $\vec{v}$ pointing directly at planet #2. What is the minimum speed v which will permit the projectile to reach station Beta?

Chapter 6

SIMPLE HARMONIC MOTION

Oscillations are a common phenomenon in our experience. We have already discussed the motion of a pendulum (Chapter 5, Example 5.3). Another example, which (we shall see) is mathematically identical with the small oscillations of a pendulum, is the vertical oscillation of a mass suspended from the ceiling by a spring. A slightly more complicated example is the motion of a rocking chair, and a considerably more complicated example is the motion of a violin string. The common feature of these examples is that initially a system is somehow pushed away from its equilibrium configuration; however, the forces which act on the system drive it *back toward* the equilibrium configuration. When the system reaches the equilibrium configuration it has a finite velocity and therefore overshoots; once it is on the other side, the forces are again directed toward the equilibrium configuration to which it eventually returns, overshooting in the opposite direction and so on.

6.1 Hooke's Law and the Differential Equation for Simple Harmonic Motion

The motion of a particle under the influence of a force whose magnitude is proportional to the distance of the particle from its ***equilibrium position*** and whose direction is always *toward* the equilibrium position is called ***simple harmonic motion*** (SHM). Let us consider, for example, a particle moving in one dimension on a smooth horizontal surface. A spring is attached to the particle and the other end of the spring is attached to a wall

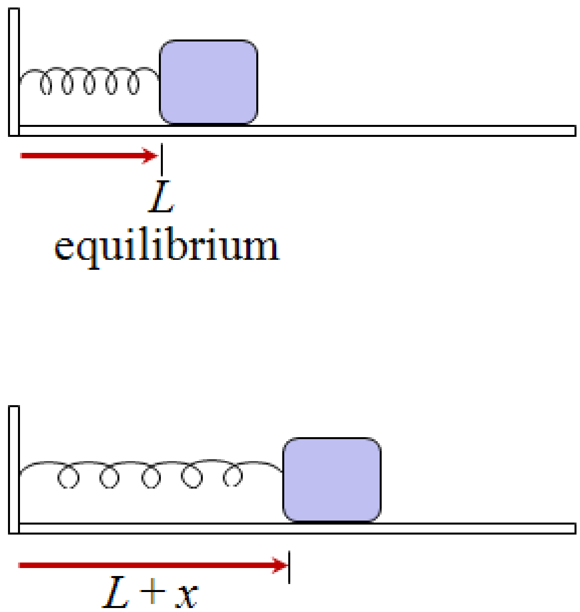


Figure 6.1: Defining equilibrium for simple harmonic motion.

as in Fig. 6.1.

Let the equilibrium length of the spring be L and let the actual length of the spring at a given instant be $L+x$. It is assumed that in equilibrium the coils of the spring are partially open so that x can be positive or negative. If x is positive the spring exerts a force to the left and, if x is negative, the spring exerts a force to the right. An "ideal" spring is one which obeys ***Hooke's Law***, which says that the magnitude of the force is proportional to the magnitude of x. If we introduce a unit vector $\hat{i}$ pointing away from the wall and let the force exerted on the particle by the spring be $F\hat{i}$, then the quantitative statement of Hooke's Law is

$$F = -kx \tag{6.1}$$

where k is a constant called the ***spring constant***. The minus sign in eqn.(6.1) ensures that if x is positive (negative) the force is directed to the left (right).

Hooke's Law (unlike Newton's Laws) is not a fundamental law of nature, but most springs obey Hooke's Law if x is small enough. Every spring will deviate from Hooke's Law if it is stretched or compressed too far. We shall assume that the magnitude of x is small enough so that Hooke's Law is valid. Since the acceleration of the particle is $d^2x/dt^2\ \hat{\imath}$, Newton's second law yields

$$m\frac{d^2x}{dt^2} = -kx \tag{6.2}$$

Eqn.(6.2), plus the initial conditions (the initial position and velocity, i.e. the values of x and dx/dt at $t = 0$), fully determines the subsequent motion. Mathematically, our problem is to find the function $x(t)$ which satisfies eqn.(6.2) (which is called a "differential equation") and takes on prescribed values for x and dx/dt at time $t = 0$.

Example 6.1 : Mathematical Digression

To see that eqn.(6.2) plus specified initial values for x and dx/dt uniquely determine $x(t)$, we can imagine solving eqn.(6.2) numerically. Let ε be a very small time increment, and (for notational convenience) denoted dx/dt by v and denote d^2x/dt^2 by a. Since $x(0)$ and $v(0)$ are known, we can compute $x(\varepsilon)$ and $v(\varepsilon)$ by using $x(\varepsilon) = x(0)+v(0)\varepsilon$ and $v(\varepsilon) = v(0)+a(0)\varepsilon$, where eqn.(6.2) gives us the value of $a(0)$ since we know $x(0)$. Now we know $x(\varepsilon)$, $v(\varepsilon)$, and (using eqn.(6.2)) $a(\varepsilon)$. Now we can compute $x(2\varepsilon)$, $v(2\varepsilon)$, and $a(2\varepsilon)$ by using $x(2\varepsilon) = x(\varepsilon) + v(\varepsilon)\varepsilon$ and $v(2\varepsilon) = v(\varepsilon) + a(\varepsilon)\varepsilon$. Thus we can advance by small time increments. This procedure can be used to solve a second-order differential equation (an equation whose highest order derivative occuring is a second derivative) numerically even when we can't get a solution in terms of familiar functions.

6.2 Solution by Calculus

The mathematical problem can be solved by several different methods, which we shall now discuss. We rewrite eqn.(6.2) as

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{6.3}$$

where $\omega = \sqrt{k/m}$ (ω is the lower-case Greek letter "omega"). A perfectly legitimate (though not very systematic) way to solve eqn.(6.3) is to guess at the solution and then to verify that the guess satisfies eqn.(6.3). In the preceding discussion we have already indicated that we expect x to be an oscillating function of the time t. The simplest oscillating function with which we are familiar is sin t. Since $d/dt(\sin\ t) = \cos\ t$ and $d/dt(\cos\ t) = -\sin\ t$, we have $d^2/dt^2(\sin\ t) = -\sin\ t$. Thus we see that sin t *almost* satisfies eqn.(6.3), up to the factor ω^2. This is easily fixed by trying the function $\sin\omega t$. Since $d/dt(\sin\omega t) = \omega\cos\omega t$ and $d/dt(\cos\omega t) = -\omega\sin(\omega t)$ we have $d^2/dt^2(\sin\omega t) = -\omega^2\sin\omega t$. Thus we see that the function $x(t) = \sin\omega t$ satisfies eqn.(6.3). Similarly we see that $x(t) = \cos\omega t$ also satisfies eqn.(6.3). This is not surprising since the graph of $\cos\omega t$ is exactly the same as the graph of $\sin\omega t$ provided that the time origin is shifted, i.e. $\cos\omega t = \sin\omega(t + \pi/2\omega)$. Finally,

$$x(t) = A\sin\omega t + B\cos\omega t \tag{6.4}$$

satisfies eqn.(6.3) and is, in fact, its most general solution.

By choosing A and B appropriately we can arrange for x and dx/dt to take on any prescribed values at $t = 0$. Suppose that we want x to have the value x_0 and dx/dt to have the value v_0 at $t = 0$ (physically we can let x_0 and v_0 have any desired values if we start the motion "by hand" and then let go). Letting $t = 0$ in eqn.(6.4) we find $B = x_0$. Differentiating eqn.(6.4) with respect to time, we obtain

$$\frac{dx}{dt} = \omega A\cos\omega t - \omega B\sin\omega t \tag{6.5}$$

Thus, the values of A and B which fit the initial conditions are $B = x_0$ and $A = v_0/\omega$, and we obtain

$$x(t) = \frac{v_0}{\omega}\sin\omega t + x_0\cos\omega t \tag{6.6}$$

Note that if $x_0 = 0$ the displacement $x(t)$ is proportional to $\sin\omega t$, and if $v_0 = 0$ the displacement is proportional to $\cos\omega t$ (we have already seen this in our discussion of the small oscillations of a pendulum (Example 5.3)). Even when $x_0 \neq 0$ and $v_0 \neq 0$ we can, by an appropriate shift of the time origin, exhibit $x(t)$ as a pure sine function or a pure cosine function. To do this we write eqn.(6.4) as

$$x(t) = \sqrt{A^2 + B^2}\left[\frac{A}{\sqrt{A^2 + B^2}}\sin\omega t + \frac{B}{\sqrt{A^2 + B^2}}\cos\omega t\right] \tag{6.7}$$

where it is understood that we always choose the positive square root. For given values of A and B there is a unique angle δ in the range $-\pi < \delta \leq \pi$ such that

$$\cos\delta = \frac{B}{\sqrt{A^2 + B^2}} \tag{6.8}$$

and

$$\sin\delta = \frac{A}{\sqrt{A^2+B^2}}. \tag{6.9}$$

Note that eqn.(6.8) and eqn.(6.9) are consistent with the identity $\cos^2\delta + \sin^2\delta = 1$. From the graphs of $\cos\delta$ and $\sin\delta$ we see that

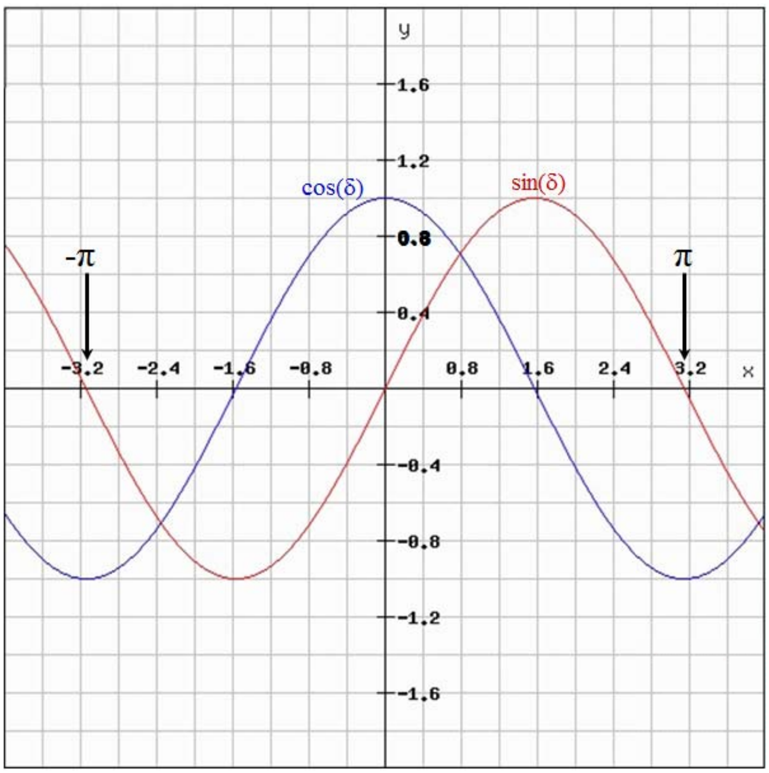


Figure 6.2: Cosine and sine of the phase angle for simple harmonic motion.

if $A > 0,\ B > 0$ then $0 < \delta < \pi/2$

if $A > 0,\ B < 0$ then $\pi/2 < \delta < \pi$

if $A < 0,\ B > 0$ then $-\pi/2 < \delta < 0$

if $A < 0,\ B < 0$ then $-\pi < \delta < -\pi/2$

Using eqn.(6.8) and eqn.(6.9) we can rewrite eqn.(6.7) as

$$\begin{aligned} x(t) &= \sqrt{A^2+B^2}\,[\sin\omega t\sin\delta+\cos\omega t\cos\delta] \\ &= \sqrt{A^2+B^2}\cos(\omega t-\delta) \\ x(t) &= \sqrt{A^2+B^2}\cos\omega\left(t-\frac{\delta}{\omega}\right). \end{aligned} \tag{6.10}$$

If we let $t' = t-\delta/\omega$ then x is proportional to $\cos\omega t'$. The change of variable from t to t' corresponds to moving our time origin by an amount δ/ω, i.e. $t' = 0$ when $t = \delta/\omega$. The significance of δ/ω is that it is the time when $x(t)$ has its maximum positive value (since the cosine in eqn.(6.10) has the value $+1$ when $t = \delta/\omega$). Thus we see that if we measure time from the instant when x has its maximum positive value, the displacement is a pure cosine function. It is obvious from eqn.(6.7) or eqn.(6.10) that $x(t)$ is periodic with period $T = 2\pi/\omega$, i.e. $x(t+2\pi/\omega) = x(t)$. Therefore x attains its maximum positive value not only at time δ/ω, but also at times $\delta/\omega \pm 2\pi/\omega$, $\delta/\omega \pm 4\pi/\omega$, etc. The number of oscillations per second (the ***frequency***) is $f = 1/T = \omega/2\pi$. We also note that ω is called the ***angular frequency***.

By now it should be clear that if we measure time from the instant when $x = 0$ and dx/dt is positive (i.e. the instant when the particle passes through its equilibrium position, traveling to the right) the displacement is a pure sine function. To prove this we note that $\cos x = \sin(x+\pi/2)$ and rewrite eqn.(6.10) as

$$x(t) = \sqrt{A^2+B^2}\sin\omega\left(t-\frac{\delta-\pi/2}{\omega}\right). \tag{6.11}$$

Thus we see that $x = 0$ and $dx/dt > 0$ when

$$t = \frac{\delta-\pi/2}{\omega},\ \frac{\delta-\pi/2}{\omega}\pm\frac{2\pi}{\omega},\ \frac{\delta-\pi/2}{\omega}\pm\frac{4\pi}{\omega},\ \text{ etc.} \tag{6.12}$$

The maximum value of x is called the ***amplitude*** of the oscillation. Inserting the calculated values of A and B we find

$$x_{\max} = \sqrt{(v_0/\omega)^2+x_0^2} \tag{6.13}$$

Example 6.2 : See Fig. 6.1.

The equilibrium length of the spring is 1.00 meter and the mass is 0.100 kg. The spring constant is $k = 4.00$ N/m. At $t = 0$ the mass is 0.800 m from the wall and is given a velocity of 0.700 m/s toward the left. Calculate (a) the angular frequency ω of the oscillation; (b) the period of the oscillation; (c) the amplitude of the oscillation; (d) the maximum and minimum distance of the mass from the wall; (e) the time when the mass is closest to the wall (calculate the smallest such time); (f) the time when the mass is furthest from the wall (calculate the smallest such time); (g) the time when the mass first is at a distance 1.10 m from the wall, traveling to the right; (h) the time when the mass first is at a distance 1.10 m from the wall, traveling to the left.

Solution: The angular frequency is $\omega = \sqrt{k/m} = 6.33\ \mathrm{s}^{-1}$. The period is $T = 2\pi/\omega = 0.993$ s. The amplitude is given by eqn.(6.13). Note that $x_0 = -0.200$ m since initially the mass is 0.200 m to the left of its equilibrium position. Thus $x_{\max} = \sqrt{(.700/6.33)^2 + (.200)^2} = .229$ m. The maximum distance of the mass from the wall is $1.00 + .229 = 1.23$ m. The minimum distance is $1.0 - .229 = 0.771$ m.

From eqn.(6.6) we have $A = -.1107$ m and $B = -.200$ m, and thus, from eqn.(6.9), $\sin\delta = -.483$. Since $A < 0$ and $B < 0$ the angle δ is in the range $-\pi < \delta < -\pi/2$. Thus $\delta = -2.638$ radians and $\delta/\omega = -0.4168$ s (the radian is a dimensionless quantity since it is the ratio of two lengths). Finally, from eqn.(6.10) we have

$$x(t) = .229\cos[6.325(t + .4167)]. \tag{6.14}$$

The mass is closest to the wall when x has its largest negative value, i.e. $6.325(t + .4167) = \pi,\ 3\pi,\ 4\pi$, etc. The smallest value of t is obtained by letting the argument of the cosine equal π. Thus we find $t = \pi/6.325 - .4167 = 0.080$ s. The mass is furthest from the wall a half-period later, i.e. when $t = 0.080 + .993/2 = .577$ s. A rough graph of eqn.(6.14) is useful when answering (g) and (h). As shown in Fig. 6.3, we graph $x = .229\cos\theta$ where $\theta = 6.325(t + .4167)$. Note that $\theta = 2.6356$ radians when $t = 0$. Although this graph can easily be made by a computer, there is some educational value in graphing eqn.(6.14) by hand using a calculator as it forces understanding of which sections of the cosine curve are relevant. For example, point **a** on the graph represents the position of the mass at $t = 0$. The mass is closest to the wall at point **b** and furthest from the wall at point **c**. At point **d** the mass is 1.10 m from the wall (x = 0.1 m) and traveling to the right. Thus, at

d, $\cos\theta = 0.1/.229 = .4367$. A calculator shows that $\cos^{-1}(0.4367) = 1.119$ rad. Since $\cos(x) = \cos(-x)$ we have $\cos(1.119) = \cos(-1.119) = .4367$. Also, of course, $\cos(x + 2\pi) = \cos(x)$. The value of θ at point **d** must be between 1.5π and 2π. Thus, at point **d**, $\theta = -1.119 + 2\pi = 5.1642$. Setting $6.325(t + .4167) = 5.1642$ we find $t = .400$ s (which is the answer to (g)). Similarly, at point **e** the mass is 1.10 m from the wall and traveling to the left. From the graph we see that θ must be between 2π and 2.5π. Thus $\theta = 1.119 + 2\pi = 7.4022$ rad, and $t = .754$ s (which is the answer to (h)). Note that $t(\mathbf{c})$, the time at which the mass is furthest from the wall, is midway between $t(\mathbf{d})$ and $t(\mathbf{e})$, as expected.

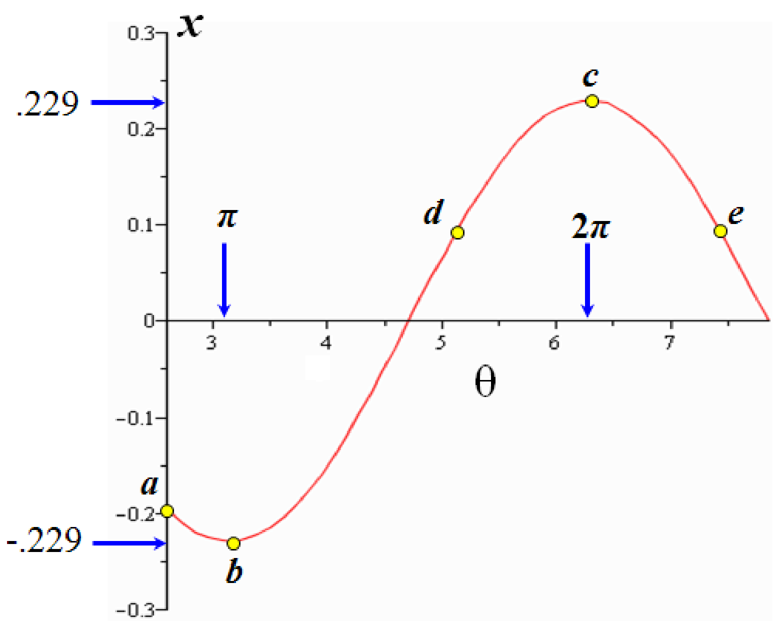


Figure 6.3: Graph of the horizontal displacment of the mass for Fig. 6.1 given the conditions set in Example 6.2.

6.3 Geometrical Solution of the Differential Equation of Simple Harmonic Motion; the Circle of Reference

The differential equation (eqn.(6.2)) can be solved without calculus by means of a simple geometrical construction. Since our objective in this section is to refrain from using calculus, we replace the symbol d^2x/dt^2 by a_x (the subscript x may seem unnecessary but will help avoid subsequent confusion). Our problem is to find the motion of a particle along the x-axis, given that the acceleration is proportional to the negative of the displacement, i.e.

$$a_x = -\omega^2 x \quad \text{where } \omega^2 = \frac{k}{m}. \tag{6.15}$$

We recall from Chapter 1 that a particle moving in a circle of radius r with constant speed v has an acceleration

$$\vec{a} = -\frac{v^2}{r}\,\hat{r} = -\frac{v^2}{r^2}\,\vec{r} \tag{6.16}$$

where $\vec{r}$ is the vector from the center of the circle to the instantaneous position of the particle and $\hat{r} = \vec{r}/r$ is the unit vector having the same direction as $\vec{r}$. If the circle is in the $x-y$ plane, then $\vec{r} = x\hat{i} + y\hat{j}$ and it follows from eqn.(6.16) that

$$a_x = -\left(\frac{v}{r}\right)^2 x. \tag{6.17}$$

Let us choose v and r such that v/r is equal to the ω in eqn.(6.15), so that eqn.(6.17) becomes identical with eqn.(6.15). Since v/r is the angular velocity (the angular displacement per unit time, measured in radians per second) of the particle traveling around the circle, we see that the angular velocity is equal to ω.

Let P be the instantaneous position of our particle on the circle and let us drop a perpendicular from P to the x-axis, intersecting the x-axis at point Q. Then, since the acceleration of Q along the x-axis is a_x, we see that the motion of Q satisfies eqn.(6.15). In summary: if uniform circular motion is projected onto a line, the resulting linear motion is Simple Harmonic Motion.[1] The circle, a purely mathematical construct, is called the ***circle of reference***.

[1] The claim that this discussion does not rely on calculus is not quite true since the derivation of eqn.(6.16), which involves constructing the difference of the velocity vectors at two nearby times and dividing by the time difference IS calculus.

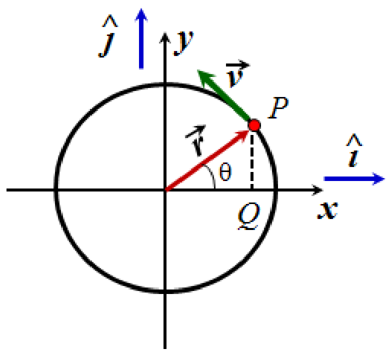


Figure 6.4: Formulating the graphical solution to the simple harmonic motion equation.

By appropriately choosing the radius r of the circle of reference and the initial position P of the particle on the circle of reference, we can arrange for Q to have any desired initial position and velocity. The $x-$coordinate of P is $r\cos\theta$ where θ is the angle between $\vec{r}$ and the x-axis(see Fig. 6.4). If the value of θ at $t = 0$ is $-\delta$ (this is the definition of δ) and if P travels around the circle in a counterclockwise sense with angular velocity ω, then $\theta = \omega t - \delta$ and thus

$$x = r\cos(\omega t - \delta) \tag{6.18}$$

The velocity of the point P has the direction of the tangent to the circle and has magnitude $v = r\omega$. The $x-$component of the velocity of P, which is the velocity of Q, is (see Fig. 6.4)

$$v_x = -v\sin\theta = -r\omega\sin(\omega t - \delta). \tag{6.19}$$

Note that we could have obtained eqn.(6.19) by differentiating eqn.(6.18) with respect to time, but in this section we are using geometry and not calculus.

Setting $t = 0$ in eqn.(6.18) and eqn.(6.19) we obtain $x_0 = r\cos\delta$ and $v_0 = r\omega\sin\delta$ (we have used $\cos(-\delta) = \cos\delta$ and $\sin(-\delta) = -\sin(\delta)$ where x_0 and v_0 are the initial displacement and velocity. Thus

$$x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = r^2 \tag{6.20}$$

which is identical with eqn.(6.13) since obviously $x_{\max} = r$. The angle δ is determined by

$$\begin{aligned}\cos\delta &= \frac{x_0}{r} = \frac{x_0}{\sqrt{x_0^2 + (v_0/\omega)^2}} \\ \sin\delta &= \frac{(v_0/\omega)}{\sqrt{x_0^2 + (v_0/\omega)^2}}\end{aligned}$$

which are identical with eqn.(6.8) and eqn.(6.9). Thus we have obtained all the equations of the previous section.

It is instructive to draw qualitatively correct graphs of x, v_x, and a_x versus t by visualizing the motion of the point P on the circle of reference. You should compare your graphs with eqn.(6.18) and eqn.(6.19). Take $t = 0$ at the instant when $x = x_{\max}$, i.e. $\theta = 0$. It should be clear that the speed is zero when $x = \pm x_{\max}$ and the speed is greatest when $x = 0$.

6.4 Energy Considerations in Simple Harmonic Motion

If the force on a particle is $\vec{F} = -kx\ \hat{i}$, then the work done on the particle as it goes from x_0 to x_f is (think of the path as a sequence of small steps $dx\ \hat{i}$)

$$W = -k\int_{x_0}^{x_f} x\ dx = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_0^2$$

(note that the force is conservative since the work depends only on the endpoints and does not depend on whether the particle went directly from x_0 to x_f or backtracked during its path).

The work-energy theorem states

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = -\frac{1}{2}kx_f^2 + \frac{1}{2}kx_0^2$$

i.e.

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \text{constant} = E \tag{6.21}$$

Eqn.(6.21) is already contained in our previous solution. Recall $x = x_{\max}\cos(\omega t - \delta)$, $v = -\omega x_{\max}\sin(\omega t - \delta)$. Using $\omega^2 = k/m$ and $\sin^2\delta + \cos^2\delta = 1$ we obtain

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_{\max}^2.$$

Since $v_{\max} = \omega x_{\max}$ we can write the energy as $\frac{1}{2}mv_{\max}^2$ or $\frac{1}{2}kx_{\max}^2$. We recognize $\frac{1}{2}kx_{\max}^2$ as the potential energy with the reference point taken as the equilibrium position ($x = 0$). The two expressions for the energy correspond to looking at the particle at $x = 0$, when it has maximum kinetic energy and no potential energy, or at $x = \pm x_{\rm max}$ when it has maximum potential energy and no kinetic energy.

A person who has never heard of energy but has a good mathematical eye might recognize that if you multiply both sides of eqn.(6.2) by dx/dt you obtain $d/dt(\frac{1}{2}mv^2 + \frac{1}{2}kx^2) = 0$, which is identical with eqn.(6.21). Furthermore, if x_0 and v_0 are specified then eqn.(6.21) gives v as a function of x. Since $dt = dx/v(x)$ you can integrate both sides to get $t(x)$ and thus $x(t)$. The details of this calculation are identical (with changes in notation) with our discussion (Example 5.3) of the small oscillations of a pendulum.

6.5 Small Oscillations of a Pendulum

We have already discussed the small oscillations of a pendulum by means of energy considerations (Example 5.3) and obtained an explicit formula for the period and for the angular displacement θ as a function of the time t. A brief re-examination of this problem will show that (if $\theta_{\rm max}$ is small) it is an example of simple harmonic motion.

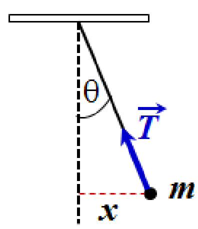


Figure 6.5: A pendulum with small amplitude oscillations undergoes simple harmonic motion.

If the pendulum is a point mass attached to the ceiling by a string of length L, the only forces acting on m are the tension $\vec{T}$ of the string and the gravitational force $m\vec{g}$. If the amplitude of oscillation is small, then $T \simeq mg$. The horizontal component of $\vec{F} = m\vec{a}$ is $m\ d^2x/dt^2 = -T\sin\theta$. Using the geometrical relation $\sin\theta = x/L$ and $T \simeq mg$ we find

$$m\frac{d^2x}{dt^2} = -\left(\frac{mg}{L}\right)x \tag{6.22}$$

Eqn.(6.22) is of the same form as eqn.(6.2), with the spring constant k replaced by mg/L. Thus the horizontal motion of the small-amplitude pendulum is simple harmonic motion with $\omega = \sqrt{g/L}$ and period $T = 2\pi/\omega = 2\pi\sqrt{L/g}$ independent of the amplitude (provided the amplitude is small). The angular frequency ω is independent of the mass because the "spring constant" is proportional to m.

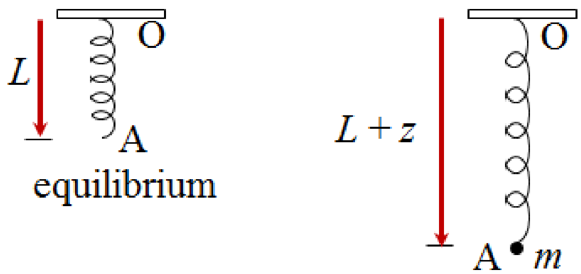


Figure 6.6: A particle of mass m hanging from a vertical spring in Example 6.3.

Example 6.3 : A spring oscillating in the vertical direction.

A massless spring OA has equilibrium length L and spring constant k. The spring hangs vertically with O attached to the ceiling and a mass m attached to A. Find the frequency with which the mass will oscillate if it is displaced from its equilibrium position.

Solution: Let the instantaneous length of the spring be $L + z$ and let $\hat{e}$ be a unit vector pointing *downward*. The the spring exerts a force $-kz\ \hat{e}$ on the mass and the gravitational force on the mass is $mg\ \hat{e}$. The acceleration of the mass is $d^2z/dt^2\ \hat{e}$ and thus

$$m\frac{d^2z}{dt^2} = -kz + mg \qquad (6.23)$$

If we define $u = z - mg/k$ (note that u is the distance of the mass from its equilibrium position with positive u corresponding to points below the equilibrium position) then

$$m\frac{d^2u}{dt^2} = -ku \qquad (6.24)$$

Eqn.(6.24) describes simple harmonic motion with angular frequency $\omega = \sqrt{k/m}$ and frequency $f = (1/2\pi)\sqrt{k/m}$. Thus the effect of gravity is to change the equilibrium position but to leave the frequency of oscillations unchanged.

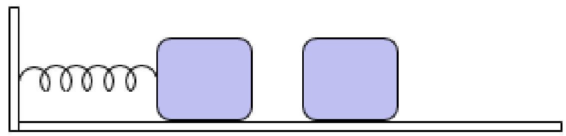


Figure 6.7: A mass M connected to a spring collides with an identical mass in Example 6.4.

Example 6.4 : A mass connected to a spring collides with another mass.

A mass (M = 0.100 kg), which moves on a smooth horizontal table, is attached to the right end of a spring (equilibrium length = 1.00 m, $k = 4.00$ N/m). The left end of the spring is attached to a wall. Initially the spring is compressed to a length of 0.700 m and then released. Another mass (also 0.100 kg) is sitting on the table at a distance of 1.00 m from the wall. When the moving mass hits the stationary mass, the two masses stick together.

(a). Find the smallest distance of the 0.200 kg mass from the wall in the subsequent motion.

(b). If the stationary mass had been sitting at a distance 1.200 m from the wall and the two masses stick together in the collision, find the smallest subsequent distance from the wall of the 0.200 kg mass.

(c). Does the answer to (a) depend on the numerical value of the masses (assuming that the two masses are equal) and on the numerical value of the spring constant k?

Solution:

(a). The energy E of the 0.100 kg mass is purely potential energy, $(1/2)kx_{\rm max}^2$, at the instant of release. Since the collision takes place at $x = 0$ the energy is purely kinetic at this point. We could calculate the speed v of the 0.100 kg mass just before the collision, but this is unnecessary. The speed of the 0.200 kg mass just after the collision is $v/2$ (by momentum conservation; energy is *not* conserved in the collision). The kinetic energy of the 0.200 kg mass just after the collision is $(1/2)(2M)(v/2)^2 = (1/4)Mv^2 = E/2$. Since the total energy after the collision is one half of the total energy before the collision, the value of $x_{\rm max}$ after the collision is $1/\sqrt{2}$ times the value of $x_{\rm max}$ before the collision. Therefore, after the collision $x_{\max} = 0.3/1.414 = 0.212$ m and the smallest distance from the wall is 1.0 - .212 = .788 m. This result clearly did not depend on the numerical value $M = 0.100$ kg or on the value of k.

(b). Frequently it is more instructive to solve a problem in terms of symbols (representing the input quantities, like k and M) than to insert numerical values prematurely. If we do this we will see that that the answer to (b) also does not depend on the value of k or M. Let x_0 denote the value of x at the point where M $(= 0.100$ kg) was initially released $(x_0 = -0.300$ m). Let x_1 denote the value of x at the point where the collision occurs $(x_1 = 0.200$ m). Before the collision the mass M has total energy $(1/2)kx_0^2$. Just before the collision the mass M has the kinetic energy $(1/2)kx_0^2 - (1/2)kx_1^2$. Let $X_{\max}$ denote the maximum value of x in the motion after the collision. Then the total energy of the mass $2M$ after the collision is $(1/2)kX_{\max}^2$ and the kinetic energy of the mass $2M$ just after the collision is $(1/2)kX_{\max}^2 - (1/2)kx_1^2$. We have shown above that, when a moving particle hits a stationary particle of the same mass and the two masses stick together, the kinetic energy just after the collision is one-half of the kinetic energy just before the collision (*As we noted in Chapter 4, a finite external force (the spring) acting during the infinitesimal time of the collision contributes no momentum to the system, so it is fine to use Conservation of Momentum.*). Thus we find

$$\frac{1}{2}kX_{\max}^2 - \frac{1}{2}kx_1^2 = \frac{1}{2}\left[\tfrac{1}{2}kx_0^2 - \tfrac{1}{2}kx_1^2\right].$$

Simple algebra yields

$$\left(\frac{X_{\max}}{x_0}\right)^2 = \frac{1}{2}\left[1 + \left(\frac{x_1}{x_0}\right)^2\right]. \tag{6.25}$$

Inserting numbers, we find $X_{\max} = .255$ m and the smallest distance from the wall is 1.0 - .255 = .745 m. This solution made no use of the numerical values of M or k.

(c). There are of course other (equivalent) ways to solve parts (a) and (b) of this problem, all of which will lead to eqn.(6.25). *A VERY IMPORTANT* lesson can be learned here: even before solving the problem, we can *know* that the answer will not involve the symbols k or M. We see this by considering the *dimensions* of the input quantities. In the mks system, the fundamental units are length, mass, and time. The unit of force (the newton) is a derived quantity and has dimensions (mass) $\times$ (length)/(time)^2. The spring constant k has dimensions force/length, i.e. (mass)/(time)^2. The quantity $X_{\max}/x_0$ is dimensionless, being the ratio of two lengths. The solution will give $X_{\max}/x_0$ in terms of the input quantities x_0, x_1, k, and M. The only dimensionless quantity which you can form from the four inputs is x_0/x_1. There is no way that k or M can enter into a dimensionless quantity (the situation would be different if the problem involved two spring constants or two different masses, in which case there would be additional dimensionless ratios). This kind of reasoning, which is called ***dimensional analysis***, can be very useful.

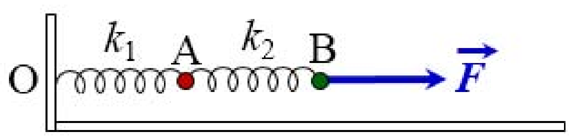


Figure 6.8: Two springs are connected and subjected to a force in Example 6.5.

Example 6.5 : Two springs are connected.

As shown in Fig. 6.8, OA is a spring with equilibrium length L_1 and spring constant k_1. O is attached to a wall and A is attached to a second spring AB which has equilibrium length L_2 and spring constant k_2. What force is required to hold B at distance $L_1 + L_2 + x$ from the wall?

Solution: Let $\hat{i}$ be a unit vector directed to the right. If we exert a force $F\hat{i}$ on B, then B exerts a force $-F\hat{i}$ on us and the length of AB must therefore be $L_2 + F/k_2$. In equilibrium the net force on AB must vanish. Therefore OA must exert a force $-F\hat{i}$ on AB and the length of OA is therefore $L_1 + F/k_1$. Therefore

$$x = F\left(\frac{1}{k_1} + \frac{1}{k_2}\right).$$

If we write $F = k_{\text{eq}}x$ (where k_{eq} is the spring constant of a single spring equivalent to the combination), then

$$\begin{aligned} \frac{1}{k_{\text{eq}}} &= \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow \\ k_{\text{eq}} &= \frac{k_1 k_2}{k_1 + k_2}. \end{aligned}$$

If $k_1 = k_2$, then $k = \frac{1}{2}k_1$.

6.6 Simple Harmonic Oscillation Problems

6.1. (See Example 3.8)
We saw that if a mass is attached to a string which is attached to the ceiling of a railroad car which has constant acceleration $\vec{a}$, the string can hang at a constant angle (θ) from the vertical, where $\tan\theta = a/g$. If the mass is given a slight displacement from that position, what is the period of the subsequent oscillations? (There is a very simple way to solve this problem.)

6.2. A particle slides frictionlessly inside a spherical surface of radius R as shown in Fig. 6.9. Show that the motion is simple harmonic for small displacements and find the period of this motion.

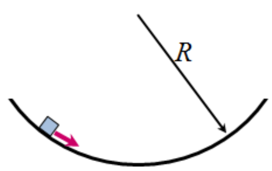


Figure 6.9: Problem 6.2.

6.3. Suppose a tunnel is dug along a chord through the earth as shown in Fig. 6.10. Under the assumption that the earth has uniform mass density, total mass M and radius R, show that a particle of mass m that is dropped into one end of the tunnel executes simple harmonic motion and find the time taken for the particle to just reach the other end of the tunnel. Assume the particle moves without friction in the tunnel. Note that, for a particle inside a uniform spherically symmetric mass distribution, the force of gravity on the particle is directed toward the center of the mass distribution. If the particle is at distance r from the center, the matter further from the center exerts no gravitational force on the particle, and the matter closer to the center has the same gravitational effect as though it were all concentrated at the center.

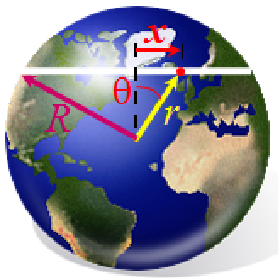


Figure 6.10: Problem 6.3.

6.4. Fig. 6.11 shows a block of mass $m = 1.00$ kg atop another block of mass $M = 5.00$ kg that is connected to a spring with spring constant $k = 20.0$ N/m. Block M slides frictionlessly on a horizontal surface but there is a coefficient of static friction μ between the two blocks. If the amplitude of oscillation is $A = 0.400$ m, what is the minimum value of μ such that the upper block does not slip relative to the lower block?

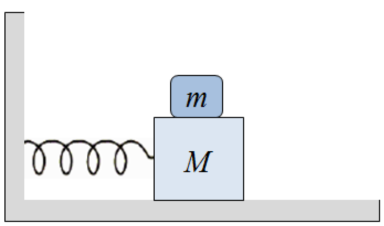


Figure 6.11: Problem 6.4.

Chapter 7

Static Equilibrium of Simple Rigid Bodies

[If the reader is not familiar with the vector cross-product, he/she should read the discussion of that subject in Appendix A.]

Until now we have concerned ourselves mainly with the statics and dynamics of point masses. The only significant statement we have made about the motion of more complicated systems was eqn.(4.9), which states that (as a consequence of Newton's third law) the total external force acting on a system is equal to the total mass times the acceleration of the center of mass. In particular, if the system is in equilibrium (i.e. every particle is at rest or moving with constant velocity) the total external force must be zero. However, as we can see from the example illustrated in Fig. 7.1, the vanishing of the total external force is not sufficient to ensure that the system will be in equilibrium. Under the infludence of the pair of equal and opposite forces applied to its ends, the rod in Fig. 7.1 will start to spin. In this chapter we shall be concerned with what conditions must be satisfied in order that a body may remain in equilibrium. In the following chapter we shall discuss how the body moves and spins when subjected to arbitrary forces.

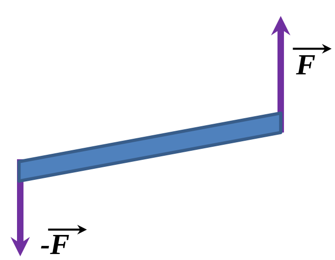


Figure 7.1: Net force is zero but the motion of the bar changes.

The discussion of equilibrium relies heavily on the concept of *torque*, and the discussion of non-equilibrium situations also requires the concept of *angular momentum* which will be subsequently defined.

7.1 Definition of Torque

Our experience tells us that when a force acts on an extended body (e.g. a seesaw), the effect of the force depends not only on the magnitude and direction of the force, but also on the location of the point where the force acts. If we introduce an origin O and draw a vector $\vec{r}$ from O to the point where the force $\vec{F}$ acts, then the vector cross-product $\vec{r} \times \vec{F}$ is called "the ***torque*** produced by the force $\vec{F}$ about the origin O". The torque depends on the location of the origin since the vector $\vec{r}$ changes if we change the origin.

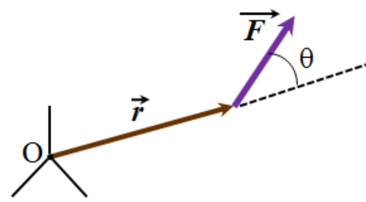


Figure 7.2: Definition of Torque

We denote torque by the vector $\vec{\tau}$ (Greek letter "tau"), i.e.

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{7.1}$$

The direction of $\vec{\tau}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ and (according to the right-hand rule which is carefully discussed in Appendix A) would point out of the page if $\vec{r}$ and $\vec{F}$ are the vectors shown in Fig. 7.3. The magnitude of $\vec{\tau}$ is $F\ r \sin\theta$, which can also be written $F\ r_\perp$ or $r\ F_\perp$, where $r_\perp$ is the magnitude of the component of $\vec{r}$ which is perpendicular to $\vec{F}$, and $F_\perp$ is the magnitude of the component of $\vec{F}$ which is perpendicular to $\vec{r}$ (see Fig. 7.3).

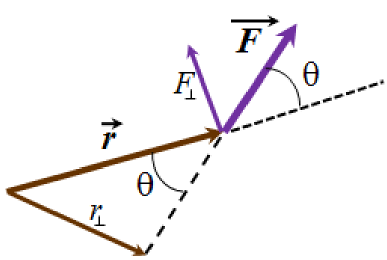
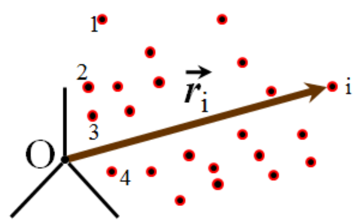


Figure 7.3: Definitions of $F_\perp$ and $r_\perp$.

7.2 Static Equilibrium of Extended Bodies



Let us consider a system (any collection of particles) which is in equilibrium (i.e. each particle is in equilibrium). We enumerate the particles by an index i. The total force on each particle is zero.

$$\vec{F}_i = 0 \tag{7.2}$$

Furthermore, if we choose an origin O and take the cross-product of both sides of eqn.(7.2) with $\vec{r}_i$ (the vector from O to the position of the i-th particle), we find

$$\vec{r}_i \times \vec{F}_i = 0 \tag{7.3}$$

We recall that if we add up the equations eqn.(7.2) for all i, the internal forces cancel out as a result of Newton's third law and we obtain

$$\vec{F}_{\text{ext}} = 0 \tag{7.4}$$

where $\vec{F}_{\text{ext}}$ is the total external force on the system. Can we also say that the internal forces produce no net torque on the system?

To examine this question we decompose $\vec{F}_i$ into an external and internal part (as we did in Chapter 2)

$$\vec{F}_i = \vec{F}_{i,\text{ ext}} + \sum_j \vec{f}_{ji} \tag{7.5}$$

where $\vec{f}_{ji}$ is the force exerted on i by j. If we now add up the torque equations (eqn.(7.3)) for all the particles we obtain

$$\sum_i \vec{r}_i \times \vec{F}_{i,\text{ ext}} + \sum_{ij} \vec{r}_i \times \vec{f}_{ji} = 0. \tag{7.6}$$

The terms in the double sum do not obviously cancel in pairs. For example, the terms $(i = 1,\ j = 2)$ and $(i = 2,\ j = 1)$ give

$$\vec{r}_1 \times \vec{f}_{21} + \vec{r}_2 \times \vec{f}_{12}$$

which can be combined (using Newton's third law $\vec{f}_{21} = -\vec{f}_{12}$) to give $(\vec{r}_1 - \vec{r}_2) \times \vec{f}_{21}$. This cross-product will vanish if $\vec{f}_{21}$ is parallel or antiparallel to $\vec{r}_1 - \vec{r}_2$. But $\vec{r}_1 - \vec{r}_2$ is the vector from particle #2 to #1. Thus if the force between particles is parallel or antiparallel to the line between the particles, the cross-product will vanish and the internal forces will not contribute to the total torque. Forces which act along the line between the particles are called ***central forces***; familiar examples are the gravitational force and the electrostatic force (which are mathematically of the same form). There *are* forces in nature which are not central forces, the most familiar example being magnetic forces. Even in this case, however, it can be shown by a more elaborate argument that the internal forces make no contribution to the total torque. Were this not true then an isolated system would, under some circumstances, begin to rotate faster and faster and could continuously perform work without any energy input.

Accordingly, we assert (though a completely general proof is beyond the scope of this discussion) that, if a system is in equilibrium, the internal forces make no contribution to the total torque. Thus,

$$\sum_i \vec{r}_i \times \vec{F}_{i,\ \mathrm{ext}} = \vec{\tau}_{\mathrm{ext}} = 0 \tag{7.7}$$

where $\vec{\tau}_{\mathrm{ext}}$ is the torque produced by the *external* forces which act on the system.

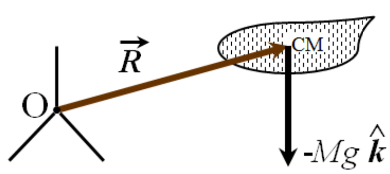


Figure 7.4: Torque produced by gravity on a body about the origin O.

In most of the examples which we shall consider here, one of the external forces acting on a system is the force of gravity. This force acts on every particle of the system, and different particles are at different distances $\vec{r}$ from the origin O. However, there is an important theorem which makes it easy to calculate the torque produced by gravity[1].

THEOREM: For purposes of computing torques, the entire gravitational force on a system may be considered to act at the center of mass. (As illustrated in Fig. 7.4, the gravitational torque on the system is $\vec{\tau}_{\mathrm{grav}}$ =

[1] We assume here that the region of interest is small enough so that the magnitude and direction of the gravitational force per unit mass is the same for all particles in our system.

$\vec{R} \times (-Mg\ \hat{k})$ where $\vec{R}$ is the vector from the origin to the CM of the system, M is the total mass, and $\hat{k}$ is a unit vector pointing vertically up. The proof follows immediately from the definition of the center of mass,

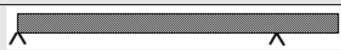
$$M\vec{R} = \sum_i m_i \vec{r}_i. \tag{7.8}$$

The gravitational torque is

$$\vec{\tau}_{\text{grav}} = \sum_i \vec{r}_i \times (-m_i g \hat{k}) = \left(\sum_i m_i \vec{r}_i\right) \times (-g\ \hat{k}). \tag{7.9}$$

Inserting eqn.(7.8) into eqn.(7.9) we obtain $\vec{\tau}_{\text{grav}} = \vec{R} \times (-Mg\ \hat{k})$ which is the desired result. With the aid of this theorem we can now work out some examples.

Example 7.1 : Static equilibrium of a balanced beam.



A uniform beam of weight W and length L rests on two supports, one at the left end and the other at distance 3/4 L from the left end. Find the force on the beam exerted by each support.

Solution: We introduce unit vectors: $\hat{k}$ pointing vertically up, $\hat{\imath}$ pointing to the right, and $\hat{\jmath}$ into the paper as shown in Fig. 7.5. The forces acting on the beam are $F_1\ \hat{k}$ acting on the left end, $F_2\ \hat{k}$ acting at distance 3/4 L from the left end, and the force of gravity $-W\ \hat{k}$ which may be considered to act at the midpoint. Since the total force on the beam is zero, we have

$$F_1 + F_2 - W = 0. \tag{7.10}$$

The total torque around any origin must be zero. If we take the origin at the left end of the beam, the force $F_1\ \hat{k}$ contributes no torque and we obtain

$$\vec{\tau} = (L/2\ \hat{\imath}) \times (-W\ \hat{k}) + (3L/4\ \hat{\imath}) \times (F_2\ \hat{k}) = 0. \tag{7.11}$$

Thus

$$W \cdot \frac{L}{2}\ \hat{\jmath} - \frac{3}{4} F_2 L\ \hat{\jmath} = 0 \tag{7.12}$$

and therefore $F_2 = 2/3W$. Using eqn.(7.10) we find $F_1 = 1/3W$. We could equally well have taken torques about some other origin, e.g. the other support. In this case we find

$$(-3L/4\ \hat{\imath}) \times (F_1\ \hat{k}) + (-L/4\ \hat{\imath}) \times (-W\ \hat{k}) = 0 \tag{7.13}$$

which yields $F_1 = W/3$. Thus we see that the problem can be solved by writing one force equation and one torque equation, or by writing torque equations around two origins.

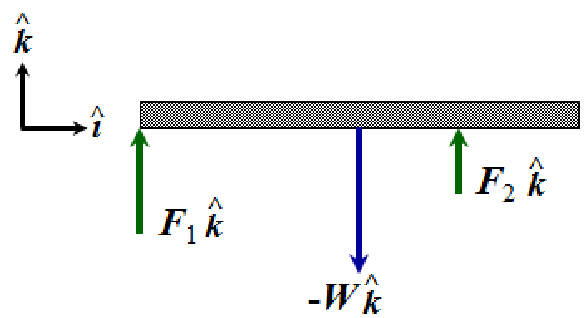


Figure 7.5: The forces acting on a beam balanced at two points.

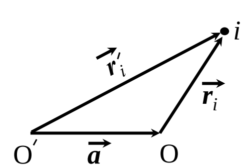


Figure 7.6:

If the total force on a system is zero, and if the torque around some particular origin is zero, then it follows that the torque around any other origin is zero. To see this, let O and O′ be two origins, and let $\vec{a}$ be the vector from O′ to O. Suppose the total external force is zero: $\vec{F}_{\text{ext}} = \sum_i \vec{F}_{i,\text{ ext}} = 0$ and the external torque around O is zero: $\vec{\tau}_{\text{O, ext}} = \sum_i \vec{r}_i \times \vec{F}_{i,\text{ ext}} = 0$ where $\vec{r}_i$ is the vector from O to the i-th particle. The torque around O′ is $\vec{\tau}_{\text{O}',\text{ ext}} = \sum_i \vec{r}_i{}' \times \vec{F}_{i,\text{ ext}}$ where $\vec{r}_i{}'$ is the vector from O′ to the i-th particle. From Fig. 7.6 we see that $\vec{r}_i{}' = \vec{a} + \vec{r}_i$ and thus $\vec{\tau}_{\text{O}',\text{ ext}} = \sum_i (\vec{a} + \vec{r}_i) \times \vec{F}_{i,\text{ ext}} = \vec{a} \times \vec{F}_{\text{ext}} + \vec{\tau}_{\text{O, ext}}$. Therefore, if $\vec{F}_{\text{ext}}$ and $\vec{\tau}_{\text{O, ext}}$ vanish, then $\vec{\tau}_{\text{O}',\text{ ext}}$ also vanishes.

This means that all the information about a given system is contained in the force equation plus the torque equation about one origin. Additional equations obtained by taking torques about other origins will be algebraic consequences of the force equation and the first torque equation.

Figure 7.7:

We have shown that the vanishing of the total external force and torque are necessary conditions for the equilibrium of a system. Are these conditions also sufficient for equilibrium? If the system is not a rigid body the answer is evidently "no", as can be seen by consideration of many simple examples, such as that illustrated in Fig. 7.7. The total force and torque are zero, and the two particles will accelerate toward each other. If the system is a rigid body, and if at one instant all the particles of the system are at rest, it can be shown that all particles of the system will remain at rest if the total external force and torque vanish. The proof, which is given in the next chapter, involves an analysis of the possible motions of a rigid body (which are greatly restricted compared with the motions of an arbitrary collection of particles).

All the statics examples which we shall discuss are two-dimensional, i.e. all the particles and all the forces are in a plane (which we take as the plane of the paper). Accordingly, all torques are perpendicular to this plane and (in the notation of Example 7.1) are proportional to $\hat{j}$ or $-\hat{j}$. A torque proportional to $\hat{j}$ is directed *into* the page and is frequently called a clockwise torque; a torque proportional to $-\hat{j}$ is directed out of the page and is frequently called a counterclockwise torque. Thus we can omit all vectors from the torque equation, provided that we remember to put opposite signs in front of clockwise and counterclockwise torques.

In Fig. 7.5 the force $F_2\hat{k}$ produces a counterclockwise torque around the left end of the beam. The force $-W\ \hat{k}$ produces a clockwise torque. Note that in Fig. 7.5 the force $F_2\ \hat{k}$ is "trying" to turn the beam in a counterclockwise sense around the left end, whereas the force $-W\ \hat{k}$ is "trying" to turn the beam in a clockwise sense around the left end. If we take the right end of the beam as our origin, then $F_1\ \hat{k}$ and $F_2\ \hat{k}$ produce clockwise torques and $-W\ \hat{k}$ produces a counterclockwise torque. Students who have trouble with the signs should simply calculate the cross-products.

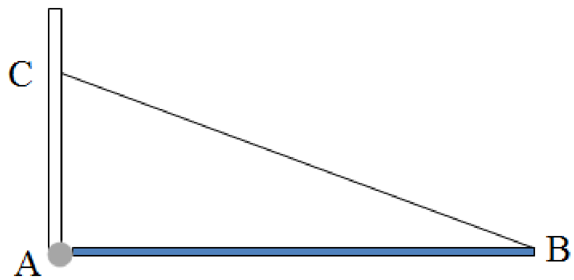


Figure 7.8: A rod hinged at a wall and held at the other end by a string.

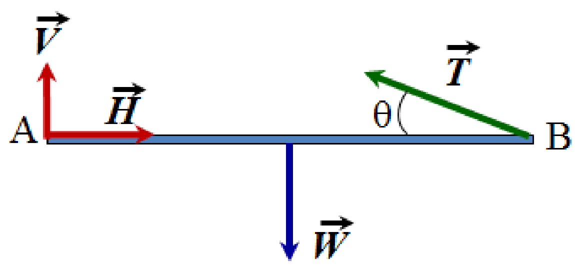


Figure 7.9: Force diagram for Example 7.2.

Example 7.2 : Static equilibrium for a hinged rod.

A uniform rod AB, of weight W, is attached to a wall by a smooth hinge at A and is kept horizontal by a weightless wire CB which makes angle θ with the horizontal. Calculate the tension in the wire and the horizontal and vertical force components exerted by the wall on the rod.

Solution: In Fig. 7.9 we exhibit all the forces acting *on* the rod. V and H are the magnitudes of the vertical and horizontal forces exerted by the wall at A; these forces are assumed to have the directions indicated by the arrows. If V turns out to be negative, the equations would be telling us that the vertical force exerted by the wall is directed down rather than up; similarly, a negative value for H would mean that the wall exerts a horizontal force directed to the left. Note that there are three unknowns (V, H, T) and three equations which state the equilibrium conditions for the rod (two components of the force equation and one torque equation). Thus the problem is mathematically determinate.

The horizontal and vertical force equations are

$$H - T\cos\theta = 0 \tag{7.14}$$

and

$$V + T\sin\theta - W = 0 \tag{7.15}$$

If we take torques around A, we find

$$WL/2 - TL\sin\theta \ = 0 \tag{7.16}$$

where L is the length of the rod, and thus $T = W/(2\sin\theta)$. Substituting into eqn.(7.14) and eqn.(7.15) we find $H = (W/2)\cot\theta$ and $V = W/2$. The value of V could have been found directly by taking torques around B.

Example 7.3 : Static equilibrium for a hinged, attached rod.

Most introductory courses devote very little time to statics and consequently few students acquire adequate technique to solve problems like this one, and still fewer can solve it efficiently. Nevertheless it is recommended that the student study this example which illustrates a number of important points.

The geometry of this example is similar to that of Example 7.2; AB is a rod of weight W, CB is a rod of weight W' and the joints at A, B, and C are smooth hinges. Calculate the horizontal and vertical forces exerted by the wall at A and at C, and the horizontal and vertical forces exerted by each rod on the other at B.

Solution: In this problem there are many unknown forces. The algebra will be greatly simplified if, from the outset, we eliminate some of the unknowns by making use of the force equations and Newton's third law. It is also important to realize that there are three different systems (the rod AB, the rod CB, and the system ABC consisting of both rods) for which one can write force and torque equations. [However, not all equations are algebraically independent. If the force and torque on any two of these systems vanish, the force and torque on the third system will also vanish.]

In Fig. 7.10 we have exhibited all the external forces which act on ABC (the forces at the joint are internal forces in the system ABC). We denote the horizontal and vertical forces at A by $\vec{H}$ and $\vec{V}$, with assumed directions as indicated by the arrows. The horizontal and vertical forces at C are then determined by the requirement that the total horizontal and total vertical force on ABC must vanish. Similarly, Fig. 7.11 shows the forces acting on AB (note that the forces at the right end are the forces exerted *by* rod CB *on* rod AB). Fig. 7.12 shows the forces acting on rod CB. Thus, by using the force equations we have reduced the number of unknowns to two.

We can determine $\vec{V}$ most easily by taking torques on AB around the origin B, obtaining $VL - WL/2 = 0$ (L = length of AB) and thus $V = W/2$. An easy way to calculate $\vec{H}$ is to take torques on ABC around the origin A (Fig. 7.10), obtaining $WL/2 + W'L/2 - HL\tan\theta = 0$ and thus $H = \frac{1}{2}(W + W')\cot\theta$. One can, of course, write other torque equations which lead to the same values of H and V. Inserting the values of H and V into Figs. 7.10- 7.12 we obtain all the forces.

A common error is to assume that the force which the rod CB exerts on the wall is parallel to the rod CB. If this were so then the force which the wall exerts on CB would also be parallel (or antiparallel) to CB and would produce no torque about B; thus the only torque on CB around the origin B would be the torque produced by $\vec{W}\,'$ and the rod could not be in equilibrium (unless $W' = 0$ in which case the force exerted by CB on the wall *is* parallel to CB).

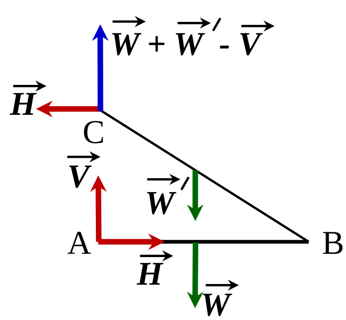


Figure 7.10: Force diagram for the composite system ABC of Example 7.3.

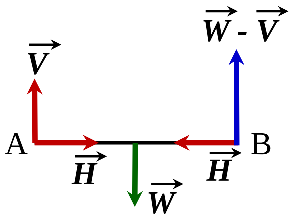


Figure 7.11: Force diagram for the rod AB of Example 7.3.

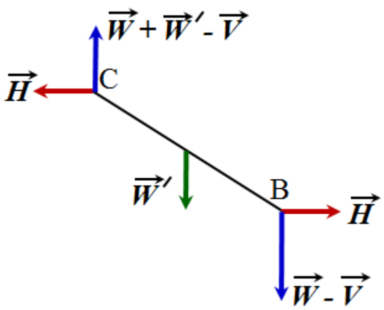


Figure 7.12: Force diagram for the rod CB in Example 7.3.

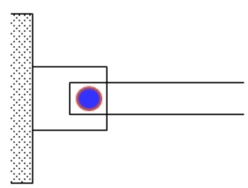


Figure 7.13: A smooth hinge.

It is important also to understand what is meant by a "smooth hinge", and why we usually assume that a joint is smoothly hinged. We assume that the rods are connected to each other, and to brackets on the wall, by means of pins which are perpendicular to the plane of the paper and pass through circular holes in the rods (see Figs. 7.13- 7.15). It is assumed that the surface of contact between the pin and the hole is well lubricated so that the only forces which act at that surface are perpendicular to the surface. Thus, if we take the center of the hole as origin, we see that the pin exerts no net torque on the rod (but usually does exert a net force). If a rod is attached to a wall bracket by means of a sufficiently rusty hinge (Fig. 7.15), the rod can remain in a horizontal position without additional support. If we take our origin at the center of the hole, the weight W produces a clockwise torque on the rod; however, the tangential component of the force which the pin exerts on the surface of the hole produces a counterclockwise torque which (if the hinge is sufficiently rusty) has the same magnitude as the gravitational torque, and thus the rod is in equilibrium.

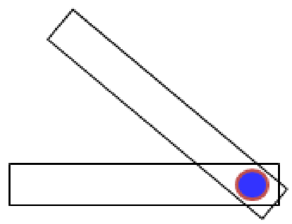


Figure 7.14: A hinge connecting two rods at an angle to one another.

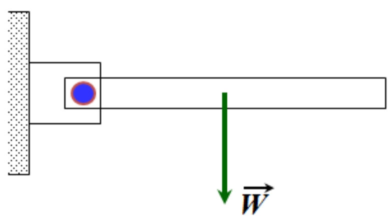


Figure 7.15: A rod with weight W attached to a wall with a rusty hinge.

This analysis illustrates the importance of not over-idealizing the situation. If we think of the rod as a purely one-dimensional object, so that the pin and hole have zero radius, then we cannot understand how the rusty pin can produce a torque about the center of the hole.

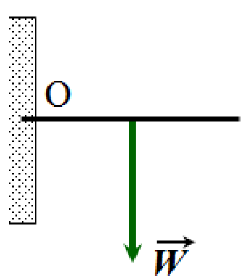


Figure 7.16: A rod in a hole in a wall.

A closely related situation, of importance in architecture and engineering, is that of a horizontal rod, one end of which is inserted into a hole in a wall.

If we define our system as the portion of the rod outside the wall, and if we take our origin O at the point where the rod enters the wall (Fig. 7.16), then the only torque on the system appears to be the gravitational torque. Again, we must recognize that the rod has a finite thickness. Actually, the protruding portion of the rod droops a little so that the upper portion of the rod is slightly stretched ("in tension") and the lower portion is slightly compressed. In Fig. 7.17 we have divided the rod into a portion I outside the wall and a portion II inside the all by means of an imaginary plane. Fig. 7.17 illustrates schematically the forces which II exerts on I across the dividing plane. In the upper portion of the rod, II pulls to the left on I, and in the lower portion of the rod II pushes to the right on I. If we take our origin O at the midpoint of the dividing plane, it is clear that the system of forces illustrated in Fig. 7.17 produces a counterclockwise torque which cancels the clockwise gravitational torque. Furthermore, II exerts a vertical force ("shear") on I which enables the force equation to be satisfied. Analysis of the internal forces in beams, and of the small deformations associated with those forces, is beyond the scope of this discussion.

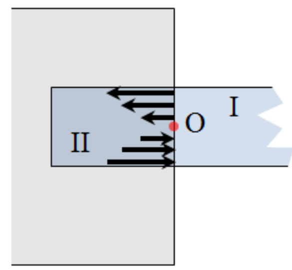


Figure 7.17:

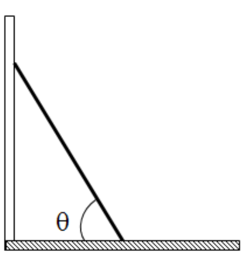


Figure 7.18: A ladder leaning against a smooth wall in Example 7.4.

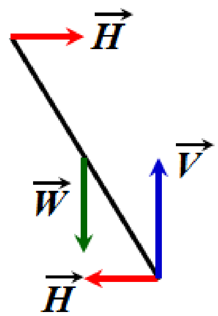


Figure 7.19: Force diagram for a ladder leaning against a smooth wall.

Example 7.4 : Ladder leaning against a smooth wall.

A uniform ladder stands with its upper end against a smooth wall and its lower end on a rough floor (coefficient of static friction μ_s). The ladder is inclined at angle θ with the horizontal. Calculate the horizontal and vertical forces exerted by the floor and calculate the minimum angle θ at which the ladder can stand without slipping.

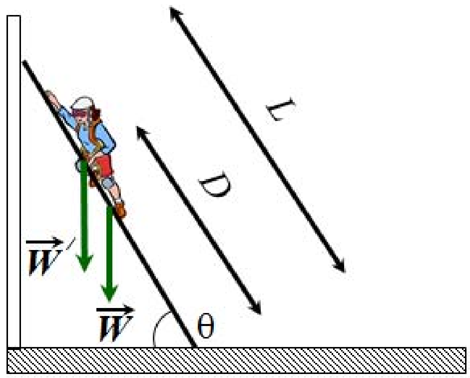
Solution: Fig. 7.19 shows the forces acting on the ladder. The wall, being smooth, can only exert a horizontal force $\vec{H}$ directed to the right. In order that the total force on the ladder vanish, the floor must exert a force of magnitude H directed to the left and a vertical force equal to the weight W of the ladder. Taking torques around the bottom end of the ladder we find

$$HL\sin\theta - W\frac{L}{2}\cos\theta = 0$$

where L is the length of the ladder. Therefore $H = (W/2)\cot\theta$. In order that the ladder not slip we must have $H/W \leq \mu_s$, i.e. $\frac{1}{2}\cot\theta \leq \mu_s$. Therefore $\theta_{\min} = \cot^{-1}(2\mu_s)$.

Example 7.5 : Climbing a ladder leaning against a smooth wall.

In Fig. 7.20 we show a woman of weight W' at a distance D from the bottom of a ladder of length L and weight W inclined at an angle of 60.0° above the horizontal. Let W = 222 newtons and L = 6.10 meters. The coefficient of static friction between the ladder and the ground is μ_s.



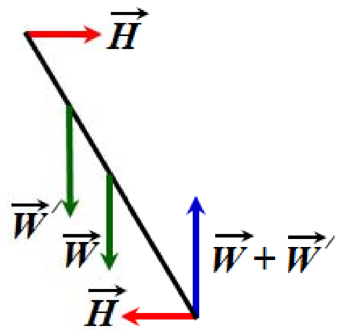


Figure 7.20: Climbing a ladder leaning against a smooth wall in Example 7.5.

(a). If $\mu_s = 0.600$, calculate the weight of the heaviest person who can climb to the top of the ladder without causing the ladder to slip. Answer the same question if $\mu_s = 0.500$.

(b). If $\mu_s = 0.500$, how far up the ladder can a person with a weight of 1110 newtons ($5W'$) ascend before the ladder slips?

Solution: The force diagram in Fig. 7.20 shows the forces acting on the ladder (it is assumed that all the woman's weight is balanced on her knee as she leans on the ladder and that the knee is a distance D from the bottom of the ladder). Taking torques on the ladder around the bottom end we find $H\,L\sin\theta - W\,L/2\cos\theta - W'\,D\cos\theta = 0$ and thus $H = (W/2 + W'D/L)\cot\theta$. In order that the ladder not slip, we must have $H/(W+W') \le \mu_s$, i.e.

$$\frac{\left(\frac{1}{2} + \frac{W'D}{WL}\right)\cot\theta}{(1 + W'/W)} \le \mu_s. \tag{7.17}$$

Note that the left side of eqn.(7.17) increases as D increases. Therefore, if eqn.(7.17) is satisfied when $D = L$, it is also satisfied for $D < L$. Thus the woman can reach the top of the ladder if

$$\frac{(\frac{1}{2} + W'/W)\cot\theta}{(1 + W'/W)} \le \mu_s. \tag{7.18}$$

Fig. 7.21 is a graph of $(1/2 + W'/W)/(1 + W'/W)$ as a function of the ratio W'/W. The graph is an increasing function of W'/W and approaches the asymptotic value 1 as $W'/W \to \infty$. Accordingly if $\cot\theta \le \mu_s$ any person (no matter how heavy) can climb to the top without causing the ladder to slip. In this problem $\theta = 60°$ and $\cot\theta = .577$. Thus if $\mu_s = 0.6$ any person can get to the top. If $\mu_s = 0.5$ we find the weight of the heaviest person who can get to the top by setting the lhs of eqn.(7.18) equal to 0.5. This yields $W'/W = (\mu_s - 1/2\ \cot\theta)/(\cot\theta - \mu_s) = 2.73$ and thus $W' = 606$ newtons or about 136 pounds. If $W' = 1110$ newtons, the ladder will slip before the person reaches the top. Setting the lhs of eqn.(7.17) equal to 0.5 and inserting $W'/W = 5$ we find $(1/2 + 5D/L)(.577)/6 = .5$ which yields $D/L = .940$ and thus $D = 5.73$ meters or about 18.8 feet for a ladder about 20 feet long.

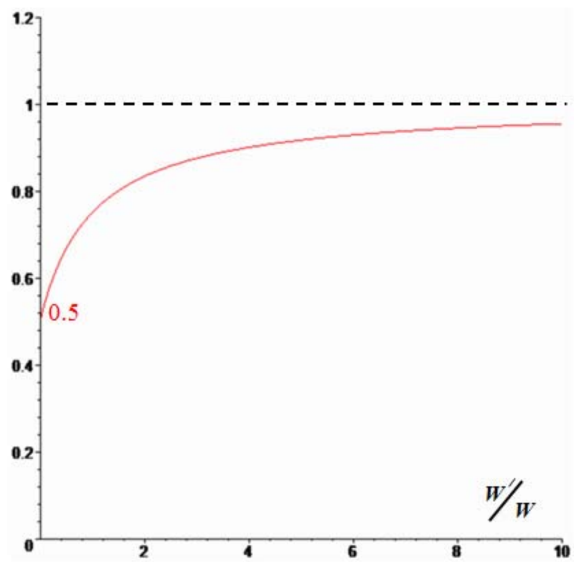


Figure 7.21: Plot of $(1/2 + W'/W)/(1 + W'/W)$ as a function of W'/W for Example 7.5.

Before leaving the subject of statics we should recognize that we have confined our attention to situations which are mathematically ***determinate***, i.e. situations in which all the forces can be determined by means of the force and torque equations without further detailed information about the system. The following examples, which are ***statically indeterminate***, illustrate the fact that the force and torque equations are not always sufficient to answer all questions.

(a). A ladder is leaning against a rough wall with its base on a rough floor. There are four unknown forces (a vertical force and a horizontal force at each end of the ladder) and only three equations (one torque equation, a horizontal force equation and a vertical force equation).

(b). A horizontal beam supported at three points. There are three unknown forces and two equations (one force equation and one torque equation).

(c). A sign attached to a wall by smooth hinges at two points (see Fig. 7.22). There are four unknown forces (two at each hinge) and three equations.

Figure 7.22: A sign hanging by smooth hinges from a wall.

Nature, of course, is not indeterminate. In each of these examples the apparent indeterminacy can be resolved only by knowing something about the elastic properties of the bodies involved; all bodies experience small deformations when subjected to forces. In these cases we need to know the relation between the deformations and the forces. With this information, plus the force and torque equations, all the forces can be determined.

7.3 Static Equilibrium Problems

7.1. An "A frame" consists of two rods of equal length connected by a smooth hinge where they meet and connected by a wire at their midpoints. The weights of the rods are W_1 and W_2, the floor is smooth and both rods make angle θ with the horizontal. The floor is smooth. Calculate the tension in the wire and the force exerted by the floor on each leg of the frame.

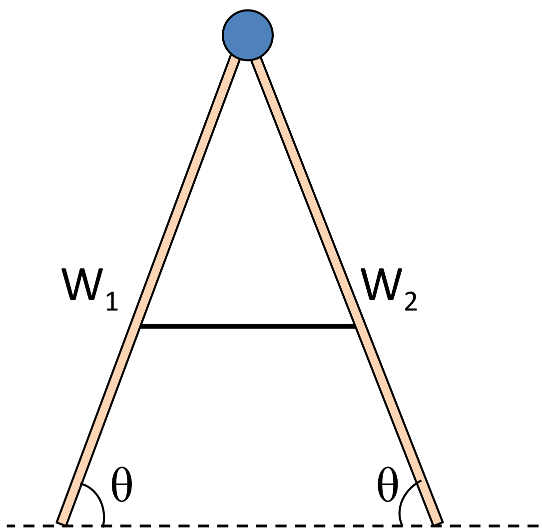


Figure 7.23: Problem **7.1**.

7.2. [*] A board is inclined at an angle θ (which can be varied) above the horizontal. A block rests on the board. The coefficient of static friction between the block and the board is μ. The height of the block (i.e. the dimension perpendicular to the incline) is 10.0 cm and the width (dimension parallel to the incline) is 6.00 cm. Suppose we slowly increase θ, starting with $\theta = 0$. It is clear that eventually, depending on the value of μ, the block will either slide downhill or tip over.

(a) Calculate the critical value μ_c which separates the sliding and tipping regimes.

(b) If the block has height h and width w, calculate μ_c.

7.3. (Compare this problem with problem **2.2**) A uniform rod AB (weight W, length L) is connected to the ceiling by a smooth hinge at A. A second uniform rod BC (weight W', length L') is connected to rod AB by a smooth hinge at B. A horizontal force $\vec{F}$ is applied to rod BC at the bottom end (C). Calculate the angle between each rod and the vertical in equilibrium.

7.4. A horizontal beam (#1) which has mass M_1 (uniformly distributed) and length L_1 is attached to a vertical wall by a smooth hinge at its left end (A). A mass M_3 hangs from the right end, attached by a weightless vertical cord. Beam #1 is supported from below by a diagonal beam (#2) of mass M_2 (uniformly distributed). The left (lower) end of #2 is attached to the wall by a smooth hinge at a point (B) below A, and the right end of #2 is attached to #1 by a smooth hinge at point C, the midpoint of #1. The angle between #2 and the horizontal is θ. Calculate the horizontal and vertical forces exerted by the wall at A and B, and the horizontal and vertical forces exerted by #2 on #1 at C. [Your solution should be as economical as possible and should not involve a lot of algebra.]

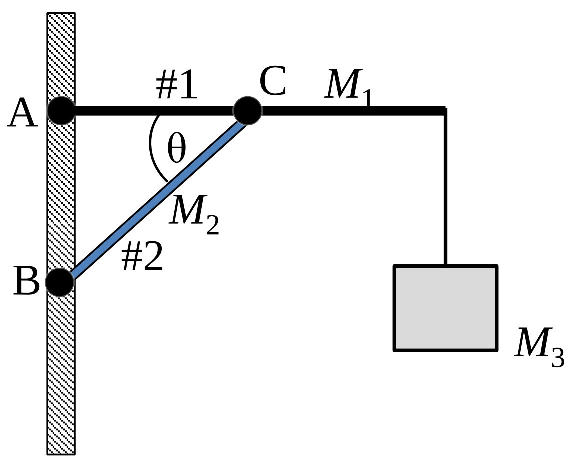


Figure 7.24: Problem **7.4**.

7.5. A hoop of mass M has a point mass (also M) attached to a point on the circumference. The hoop is in static equilibrium on an inclined plane, held in place by static friction. The plane makes an angle of 20° with the horizontal. Find the angle α between the lines AB and

BC, where A is the point where the hoop touches the plane, B is the center of the hoop, and C is the position of the point mass.

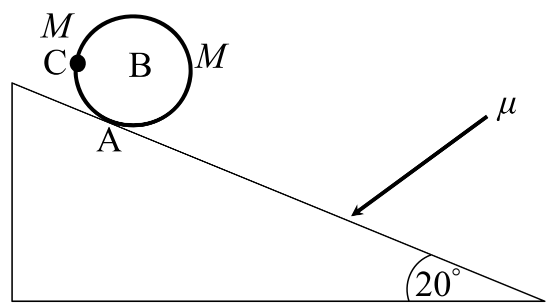


Figure 7.25: Problem **7.5**.

Chapter 8

Rotational Motion, Angular Momentum and Dynamics of Rigid Bodies

Recall that an inertial frame is a set of axes such that, if you measure positions and velocities with respect to those axes, Newton's first law is true, i.e. a particle subject to no forces will move with constant velocity. In particular, the axes of an inertial frame must be non-rotating with respect to the background of the distant stars. Regarding the motion of the origin of an inertial frame, there is some arbitrariness due to imprecision in the notion of "no force". We shall assume here that we understand the meaning of "inertial frame" well enough to solve elementary problems.

Consider a particle of mass m whose position vector with respect to the origin O of an inertial frame is $\vec{r}$ and whose velocity and acceleration are $\vec{v} = d\vec{r}/dt$ and $\vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2$. Taking the cross-product of both sides of the equation of motion $\vec{F} = m\vec{a}$ with $\vec{r}$ we obtain

$$\vec{r} \times \vec{F} = m\vec{r} \times \vec{a} \tag{8.1}$$

where $\vec{F}$ is the total force acting on the particle. The left side of eqn.(8.1) is, of course, the torque $\vec{\tau}$ (about the origin O) acting on the particle. We also define the ***angular momentum*** $\vec{L}$ of the particle about the origin O by the equation

$$\vec{L} = m\ \vec{r} \times \vec{v} \tag{8.2}$$

Using the rule for differentiating a cross-product (see Appendix A) we find

$$\frac{d\vec{L}}{dt} = m\ \vec{v} \times \vec{v} + m\ \vec{r} \times \vec{a}. \tag{8.3}$$

Since $\vec{v} \times \vec{v} = 0$ we can combine eqn.(8.1) and eqn.(8.3) to obtain

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{8.4}$$

In words: the torque is equal to the rate of change of angular momentum (similar to the statement that the force is equal to the rate of change of linear momentum).

8.1 Angular Momentum and Central Forces

Eqn.(8.4) has important consequences when applied to the central force problem, i.e. a particle moving under the influence of a force which is always directed toward a fixed point. If we take our origin O at the fixed point, then the torque vanishes because $\vec{r}$ and $\vec{F}$ are parallel (or antiparallel). Therefore $d\vec{L}/dt = 0$ and the angular momentum $\vec{L}$ is constant. The constancy of $\vec{L}$ implies

(a). the motion of the particle takes place in a fixed plane, namely the plane containing the force center, the initial position of the particle, and the initial velocity vector of the particle;

(b). as the particle moves in this plane, the vector from the force center to the particle sweeps out area at a constant rate (this is ***Kepler's Second Law*** and is a property of all central forces, not just the inverse-square law).

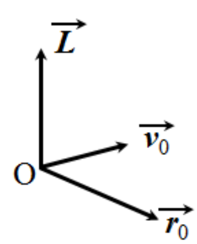


Figure 8.1: Direction of $\vec{L}$.

To prove (a), we pass a plane through the force center O, perpendicular to the constant vector $\vec{L}$. Eqn.(8.2) implies that $\vec{r}$ is perpendicular to $\vec{L}$; therefore $\vec{r}$ lies in this plane. But since $\vec{L} = m\vec{r}_0 \times \vec{v}_0$ (where $\vec{r}_0$ and $\vec{v}_0$ are the initial position and velocity vectors), the plane perpendicular to $\vec{L}$ is the plane containing $\vec{r}_0$ and $\vec{v}_0$.

In proving (a) we made use only of the constancy of the *direction* of $\vec{L}$. The magnitude of $\vec{L}$ is also constant. Using the definition of cross-product, we see that the magnitude of $\vec{L}$ is

$$|\vec{L}| = m\ r\ v \sin\theta = mrv_{\tan} \tag{8.5}$$

where θ is the angle between $\vec{r}$ and $\vec{v}$, and $v\ \sin\theta = v_{\tan}$ is the tangential

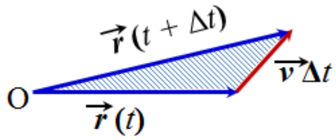


Figure 8.2: Area swept out by the radial vector.

velocity (i.e. the component of velocity perpendicular to $\vec{r}$). The shaded area in Fig. 8.2 is the area swept out by the vector $\vec{r}$ in the small time interval Δt. Through first order in Δt the area is $\Delta A = \frac{1}{2}\, r\, v_{\tan}\Delta t$. Thus, the rate at which area is swept out is $dA/dt = \frac{1}{2}\, r\, v_{\tan} = L/2m$. Since L is constant, dA/dt is constant.

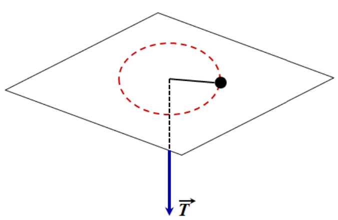


Figure 8.3: A particle moves on a horizontal table in a circular path maintained by tension in the string attached to the particle in Example 8.1.

Example 8.1 : Particle moving on a horizontal plane in a circular path.

A particle of mass m moves on the surface of a smooth horizontal table, constrained by a string which passes through a hole in the table (Fig. 8.3). Initially the particle is moving with speed v_1 in a circle of radius r_1. The string is pulled in slowly until the particle is moving in a smaller circle of radius r_2. Calculate

(a). the new speed v_2 of the particle;

(b). the ratio T_2/T_1 (where T_1 and T_2 are initial and final tensions in the string;

(c). the work done on the particle by the string.

Solution: The force exerted by the string on the particle is always directed toward the hole, and therefore angular momentum is conserved, i.e. $v_1 r_1 = v_2 r_2$. Therefore $v_2 = v_1(r_1/r_2)$. Since $T = mv^2/r$ we have

$$\frac{T_2}{T_1} = \left(\frac{v_2}{v_1}\right)^2 \left(\frac{r_1}{r_2}\right) = \left(\frac{r_1}{r_2}\right)^3 . \tag{8.6}$$

The easiest way to calculate the work W done by the string is to use the work-energy theorem, i.e.

$$W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \frac{1}{2}mv_1^2 \left[\left(\frac{r_1}{r_2}\right)^2 - 1\right] . \tag{8.7}$$

It is also instructive to calculate the work directly from the definition $W = \int \vec{F} \cdot d\vec{r}$ (note that the tension in the string changes as the string is pulled in so we cannot treat the force as constant). At the instant when the length (from the hole to the particle) of the string is r, the tension is (by eqn.(8.6)) $T = (mv_1^2/r_1)(r_1/r)^3$ and the force on the particle is $\vec{F} = -T\ \hat{r}$, where $\hat{r}$ is a until vector pointing radially outward. When the length of the string is changed from r to $r + dr$ (note that dr is negative if the string is being shortened), the displacement of the particle is $\hat{r}\ dr$ plus a tangential component which does not contribute to the work. Thus

$$\begin{aligned} W &= \int_{r_1}^{r_2} \left[-\left(\frac{mv_1^2}{r_1}\right)\left(\frac{r_1}{r}\right)^3 \hat{r}\right] \cdot [\hat{r}\ dr] = -m\ v_1^2 r_1^2 \int_{r_1}^{r_2} \frac{dr}{r^3} \\ &= (-m\ v_1^2 r_1^2)(-\tfrac{1}{2}) \left[\frac{1}{r_2^2} - \frac{1}{r_1^2}\right] \\ &= \tfrac{1}{2}mv_1^2 \left[\left(\frac{r_1}{r_2}\right)^2 - 1\right] \end{aligned} \tag{8.8}$$

in agreement with eqn.(8.7).

Moreover, if we apply the work-energy theorem to the infinitesimal process in which the length of the string is changed from r to $r + dr$ and the speed of the particle changes from v to $v + dv$ we find $-(mv^2/r)\ dr = \frac{1}{2}m(v + dv)^2 - \frac{1}{2}mv^2$ which yields $dv/v = -dr/r$. Thus $d(\ln v + \ln r) = 0$ which implies $\ln v + \ln r = \text{const.}$, i.e. $v\ r = \text{constant}$, which is the statement of conservation of angular momentum. Mechanics is an elegant and consistent logical structure.

8.2 Systems Of More Than One Particle

We now turn our attention to systems consisting of more than one particle (the index i enumerates the particles). Each particle obeys eqn.(8.4), i.e.

$$\vec{\tau}_i = \frac{d\vec{L}_i}{dt} \quad [\vec{\tau} = \vec{r}_i \times \vec{F}_i, \quad \vec{L}_i = m_i \vec{r}_i \times \vec{v}_i]. \tag{8.9}$$

If we add up the torque equations eqn.(8.9) for all the particles in the system, the torques due to internal forces cancel out for reasons discussed in Chapter 7. Defining the total angular momentum of the system as the sum of the angular momenta of the individual particles

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i \tag{8.10}$$

we obtain

$$\vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt} \tag{8.11}$$

where $\vec{\tau}_{\text{ext}}$ is the total external torque acting on the system.

In deriving eqn.(8.11) it was assumed that $\vec{r}_i$ and $\vec{v}_i$ are the position and velocity vectors of the i-th particle in an ***inertial frame***. In fact eqn.(8.11) is also true if we use axes which are non-rotating (with respect to the distant stars) and whose origin is the *center of mass* of the system[1]. Such axes are not an inertial frame if the center of mass is accelerating, but are frequently the most convenient axes.

The force equation eqn.(4.9) [$\vec{F}_{\text{ext}} = M\vec{A}_{\text{CM}}$] and the torque equation eqn.(8.11) completely determine the motion if the system is a rigid body. Our object here is to develop the techniques for solving simple problems, keeping the mathematics as simple as possible. Therefore we shall confine our attention mainly to the *two-dimensional* rigid body which moves always in the plane of the page and has negligible thickness in the direction perpendicular to this plane. The analysis is also applicable to rigid bodies which are not of negligible thickness, provided that all the motions of the body are parallel to a fixed plane and the body possesses sufficient symmetry[2].

If we paint a line on a two-dimensional rigid body, the line will, in general, have a different position and direction at time $t + \Delta t$ than it did at time t. In Fig. 8.4 the solid and dotted curves represent the configuration of the body at time t and $t + \Delta t$, respectively. Let the angle between the

[1] The proof is given in Appendix B

[2] A full discussion of this point would take us too far afield. See Appendix B.

Figure 8.4: A two-dimensional body rotated through an angle.

initial (time t) and final (time $t + \Delta t$) orientations of the line be $\Delta\theta$; we measure $\Delta\theta$ in radians and call $\Delta\theta$ positive if a clockwise rotation carries the line from its initial to its final orientation, and negative if the rotation is counterclockwise. The angular velocity ω of the body is defined as

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t}. \tag{8.12}$$

The value obtained for ω does not depend on what line we painted on the body since all lines rotate through the same angle as a consequence of the fact that the body is rigid.

Suppose some point O of the body is kept stationary (the obvious way to do this is to pass an axle, perpendicular to the page, through the body at O). We choose as our axes an inertial frame with origin at O. What is the angular momentum $\vec{L}$ of the body around the origin O? All mass points move in circles around O (Fig. 8.5) since their distance from O cannot change. Thus a mass point whose vector distance from O is $\vec{r}_i$ has a velocity vector $\vec{v}_i$ of magnitude $v_i = |\omega r_i|$ and direction perpendicular to $\vec{r}_i$. Fig. 8.5 illustrates the situation for positive ω (clockwise rotation); if ω is negative, $\vec{v}_i$ points in the opposite direction. In either case

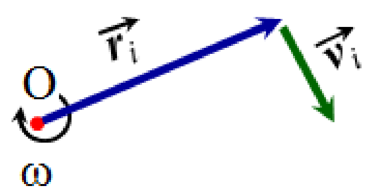


Figure 8.5: Velocity of a mass point in a rotating rigid body.

$$\vec{r}_i \times \vec{v}_i = \omega |\vec{r}_i|^2 \ \hat{j} \tag{8.13}$$

where $\hat{j}$ is a unit vector directed into the page.

The angular momentum $\vec{L}$ of the body about the origin O is

$$\vec{L} = I\omega \ \hat{j} \tag{8.14}$$

where

$$I = \sum_i m_i \; r_i^2. \tag{8.15}$$

For three-dimensional bodies (including a sphere with center at O) possessing sufficient symmetry about the origin O, Eqns.(8.14) and 8.15 are still valid provided $\hat{j}$ is the axis of rotation and (in eqn.(8.15)) r_i is replaced by $r_{i\perp}$, the perpendicular distance from the axis of rotation to m_i.

I is usually called the ***moment of inertia*** of the body about the axis $\hat{j}$ through the origin O. I is sometimes called the "***rotational inertia***" of the body. This is excellent terminology since I is indeed the measure of how hard it is to change the angular velocity of a body just as M is the measure of how hard it is to change the linear velocity.

In a two-dimensional problem the torque is perpendicular to the page $[\vec{\tau}_{\text{ext}} = \tau_{\text{ext}} \; \hat{j}]$ and thus eqn.(8.11) becomes $\tau_{\text{ext}} = I \; d\omega/dt$. Defining the ***angular acceleration*** $\alpha = d\omega/dt$ we obtain

$$\tau_{\text{ext}} = I\alpha \tag{8.16}$$

Eqn.(8.16) is the "recipe" we were aiming toward; it relates the angular acceleration of a rigid body to the torque acting on the body, and is obviously analogous to Newton's second law (with force replaced by torque, linear acceleration replaced by angular acceleration, and mass replaced by rotational inertia).

In order to use eqn.(8.16) we need to know the moments of inertia of some simple rigid bodies:

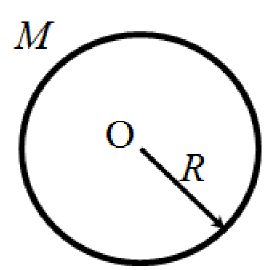


Figure 8.6: A hoop of mass M and radius R.

(a). Moment of inertia of a ***hoop*** (mass M, radius R) about its center (Fig. 8.6). In this case all the mass is at the same distance from the origin O and thus

$$I = \sum_i m_i \; r_i^2 = \left(\sum_i m_i\right) \; R^2 = MR^2. \tag{8.17}$$

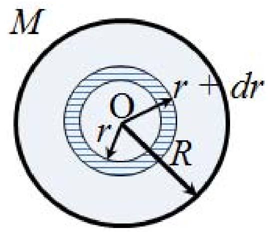


Figure 8.7: A flat disc of radius R and mass M.

(b). Moment of inertia of a uniform disc (mass M, radius R) about its center. In this case, different mass elements are at different distances from the origin. If we divide the body into many rings (Fig. 8.7), the area of the ring bounded by circles of radius r and $r + dr$ is $2\pi r\ dr$, and the mass of this ring is $(2\pi\ r\ dr)\sigma$ where σ is the mass per unit area. The moment of inertia is

$$I = \sum_i (2\pi\ r\ dr)\sigma\ r^2 = \int_0^R 2\pi\sigma\ r^3\ dr = \pi\sigma\frac{R^4}{2}. \tag{8.18}$$

The mass of the disc is $M = \pi R^2\sigma$ and thus $I = \frac{1}{2}MR^2$. Eqn.(8.18) makes sense when compared with eqn.(8.17) since, in the case of a uniform disc, the "average" distance of the mass elements from the center is less than R.

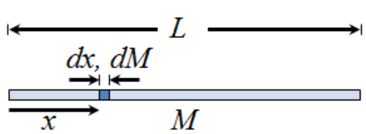


Figure 8.8: A rod of length L and mass M.

(c). Moment of inertia of a ***uniform rod*** (mass M, length L) about an <u>end</u>. A small portion of the rod, of length dx, has mass $(M/L)\ dx$ (see Fig. 8.8). If x is measured from the end of the rod, we find

$$I = \frac{M}{L}\int_0^L x^2\ dx = \frac{1}{3}ML^2. \tag{8.19}$$

Similarly, the moment of inertia of the rod about its midpoint is

$$I = \frac{M}{L}\int_{-L/2}^{L/2} x^2\, dx = \frac{1}{12}ML^2. \tag{8.20}$$

(d). Moment of inertia of a hoop or uniform disc about a point on the rim (this is needed if we wish to apply eqn.(8.16) to an object which is rolling without slipping on an incline using the point of contact as an origin). In these cases it is difficult to do the integration. However, a simple theorem enables us to write down the answer instantly in terms of the results of (a) and (b).

8.3 Simple Rotational Motion Examples

If I_0 is the moment of inertia of a two-dimensional body about a point O, and if $I_{\rm CM}$ is the moment of inertia of the same body about its center of mass, then $I_0 = I_{\rm CM} + Ma^2$ where M is the total mass of the body and a is the distance between O and the center of mass. The above theorem, and the three-dimensional version of the theorem, are proved in Appendix B. Using the theorem, we see that the moments of inertia of a hoop and a uniform disc about a point on the rim are $2MR^2$ and $3/2MR^2$, respectively.

We have now developed sufficient apparatus to solve some problems.

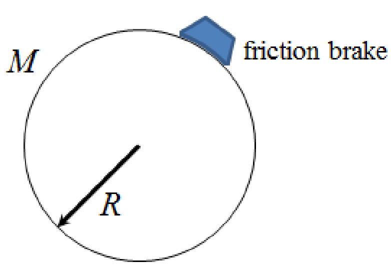


Figure 8.9: A flywheel and a friction brake.

Example 8.2 : A flywheel with a friction brake.

A flywheel is a uniform disc of mass 100 kg and radius 0.50 m and is initially rotating at the rate of 20.0 revolutions per second. A friction brake is applied to the rim and exerts a tangential braking force of 20.0 newtons. Calculate

(a). the time required for the wheel to stop, and

(b). the number of revolutions it makes from the instant the brake is applied until it stops.

Solution: First we shall solve the problem with symbols. For convenience we assume that the wheel is spinning clockwise. The tangential component F of the braking force is directed counterclockwise (note that the radial component of the force exerted by the brake produces no torque about the center of the wheel). The brake produces a counterclockwise torque of magnitude $F\ R$ ($R =$ radius). From eqn.(8.16) we have $-F\ R = \frac{1}{2}MR^2\alpha$ and thus $\alpha = -2\ F/MR$. If the number of revolutions per second is initially n_0, the initial angular velocity is $\omega_0 = 2\pi n_0$.

Now we note that *The kinetic formulas derived in Chapter 1 to describe one-dimensional motion with constant acceleration apply equally well to rotational motion with constant angular acceleration*, with appropriate change of symbols ($x \to \theta,\ v \to \omega,\ a \to \alpha$). The derivations are identical with those of Chapter 1. Thus we have

$$\theta = \theta_0 + \omega_0 t + \tfrac{1}{2}\alpha t^2 \tag{8.21}$$
$$\omega = \omega_0 + \alpha t \tag{8.22}$$
$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{8.23}$$
$$\theta - \theta_0 = \tfrac{1}{2}(\omega_0 + \omega)t \tag{8.24}$$

We define θ as positive in the clockwise sense, consistently with our convention that ω is positive for clockwise rotation.

The time T required to stop the wheel is found from eqn.(8.22); setting $\omega = 0$, we find $T = -\omega_0/\alpha = MR\omega_0/(2F)$. The angle θ through which the wheel turns while stopping is found most easily from eqn.(8.24), which yields $\theta = \frac{1}{2}\omega_0 T = (1/4)MR\omega_0^2/F$. The number of revolutions the wheel makes while stopping is $\theta/2\pi$. Inserting numbers, we find $\alpha = -0.800$ rad/s^2, $\omega_0 = 40.0\pi$ rad/s, $T = 157$ s, number of revs = 1570.

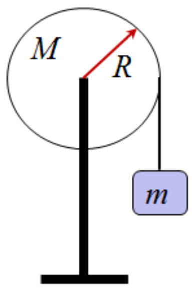


Figure 8.10: A pulley wheel of mass M and a weight with mass m.

Example 8.3 : A massive pulley wheel connected to a weight.

In Fig. 8.10 the pulley wheel is a uniform disc of mass M, radius R. The string (massless) does not slip relative to the pulley wheel. Calculate the acceleration of the block, the angular acceleration of the wheel, and the force exerted by the axle on the wheel.

Solution: Let us call the angular acceleration of the wheel α and the downward acceleration of the block a (i.e. the acceleration of the block is $a\ \hat{e}$ where $\hat{e}$ is a unit vector pointing vertically downward). In this problem we expect both α and a to be positive. We shall write the torque equation for the wheel and the force equation for the block. The torque equation for the wheel states

$$T\ R = \frac{1}{2} M R^2 \alpha \tag{8.25}$$

where T is the tension in the string. The force equation for the block states

$$mg - T = ma \tag{8.26}$$

(A common error is to assume that $T = mg$ when writing the torque equation. If this were so, then the block would not accelerate.) Eqns.(8.25) and 8.26 are two equations in the three unknown quantities α, a, and T. The missing information is the kinematic relation between a and α, which results from the fact that the total length of string remains constant. If the wheel rotates through a small angle $\Delta\theta$ ($\Delta\theta$ is positive for clockwise rotation) a length of string $R\Delta\theta$ is unwound from the wheel. Therefore the block must descend by a distance Δx, where

$$\Delta x = R\Delta\theta \tag{8.27}$$

Dividing both sides of eqn.(8.27) by Δt and letting $\Delta t \to 0$, we find

$$\frac{dx}{dt} = R\frac{d\theta}{dt} = R\omega \tag{8.28}$$

and differentiating eqn.(8.28) with respect to t we find

$$a = \frac{d^2x}{dt^2} = R\alpha. \tag{8.29}$$

Inserting eqn.(8.29) into eqn.(8.25) we obtain $T = \frac{1}{2}Ma$ and thus, from eqn.(8.26),

$$a = \frac{g}{1 + M/2m} = R\alpha.$$

To find the force exerted by the axle on the wheel, we take as our system the wheel plus a slightly greater length of string than is wound around the wheel (see Fig. 8.11). The center of mass of this system is permanently at rest and therefore the force on the system vanishes. The forces on the system are $\vec{T}$ (acting downward), $M\vec{g}$ (acting downward), and an upward force $\vec{F}$ exerted by the axle. Since $F - T - Mg = 0$ we find

$$F = T + Mg = Mg\frac{3 + M/m}{2 + M/m}.$$

We can also find F by applying eqn.(3.6) $[\vec{F}_{\text{ext}} = \sum_i m_i\vec{a}_i]$ to the system (wheel + block + string), obtaining $F - (M + m)g = -ma$. This yields the same value of F as the preceding method.

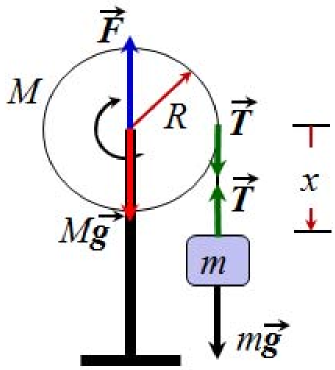


Figure 8.11: Forces acting on the block and the pulley wheel of Example 8.3.

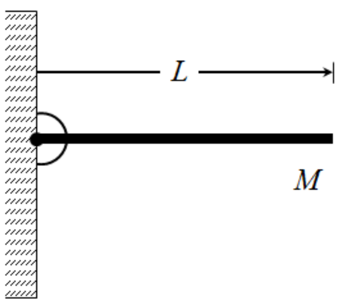


Figure 8.12: A rod attached to a wall by a hinge in Example 8.4.

Example 8.4 : A rod connected to a wall by a hinge.

A uniform rod of length L and mass M is attached to a wall by a smooth hinge at its left end. Initially the right end of the rod is supported so that the rod is in equilibrium in a horizontal position. The support is then suddenly removed. Calculate

(a). the force $\vec{F}$ exerted by the hinge on the rod before the support is removed;

(b). the angular acceleration of the rod just after the support is removed;

(c). the force $\vec{F'}$ exerted by the hinge on the rod just after the support is removed.

Solution: Part (a) is a problem in static equilibrium. Taking torques around the right end of the rod, we have $FL - MgL/2 = 0 \to F = Mg/2$. In equilibrium, a force of $Mg/2$ must also be exerted on the right end. After the rod is released, we will take torques about an origin at the hinge (thus, the force exerted by the hinge will not appear in the torque equation). The only torque is produced by gravity (acting at the CM), and we obtain just after the support is removed $Mg\ L/2 = 1/3ML^2\alpha$ and thus $\alpha = 3/2g/L$. Since the right end moves in a circle of radius L, its tangential (vertical) acceleration is $L\alpha$, i.e. $3/2g$. The fact that the right end has a vertical acceleration larger than g can be verifed by a simple demonstration (Fig. 8.13). If a meter stick is arranged so that one end is supported by a table or hinged to a wall so that the meter stick can swing freely in the vertical direction, then a line of pennies placed along the top of the meter stick while it is held horizontal will show the effect of the end of the meter stick falling with acceleration greater than gravitational acceleration. The image clearly shows the line of pennies "leaving" the stick as it swings downward. Similarly, if several persons are sitting on a toboggan which goes over a bump, the sled will fall way from the front person unless he/she holds on to a handle.

To find the force exerted by the hinge just after the support is removed, we apply the force equation $[\vec{F}_{\text{ext}} = M\vec{A}_{\text{CM}}]$ to the rod at this moment. The acceleration of the CM is $\frac{1}{2}L\alpha$, or $3/4\ g$ vertically downward. The forces acting on the rod are Mg (downward) and the upward force F' exerted by the hinge. Thus $F' - Mg = M(-3g/4)$ and hence $F' = Mg/4$. Note that the force exerted by the hinge changes its value (from $Mg/2$ to $Mg/4$) suddenly at the instant of release.

The force $\vec{F'}$ can also be obtained by writing the torque equation, using the center of mass as the origin (recall that eqn.(8.11) is also valid in this case). Eqn.(8.2) can be used to calculate the angular momentum of a rigid body about its CM, provided we use I_{CM} as the moment of inertia. Note that when we write the torque equation about the CM, gravity produces no torque since it may be considered to act at the CM; however, the vertical force F' exerted by the hinge produces a clockwise torque $F'\ L/2$. Thus the torque equation about the CM is

$$F'\frac{L}{2} = \frac{1}{12}ML^2\alpha.$$

Inserting $\alpha = 3/2\ g/L$ we obtain $F' = 1/4\ Mg$ in agreement with the previous calculation.

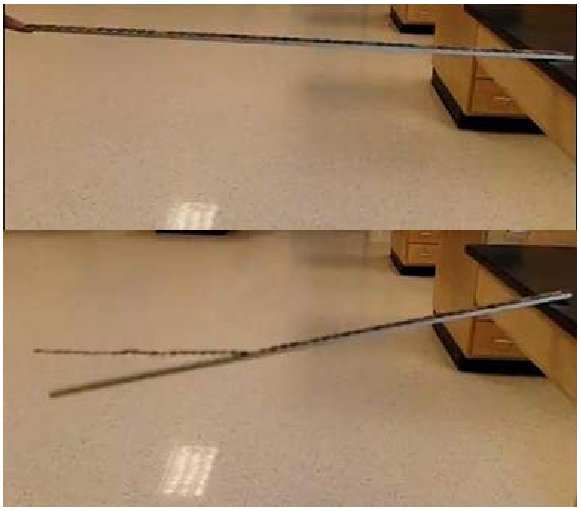


Figure 8.13: A meter stick with a line of pennies arranged on top of it is released at one end.

Figure 8.14: A toboggan going over a bump. The front end falls faster than g. See Example 8.4.

It is now easy to show that the vanishing of the external force and torque are not only necessary but also sufficent to ensure the equilibrium of a rigid body, provided that at some instant all points of the body are at rest or in uniform motion. If the external force vanishes, then the CM remains at rest or moves with constant velocity. We have already seen that if the external force vanishes and the external torque about some origin vanishes, then the external torque about any origin (including the CM) vanishes. Since we have

$$\tau_{CM} = I_{\rm CM}\frac{d\omega}{dt}$$

it follows that $\omega = $ const., and thus $\omega = 0$ since at some instant the body is not rotating [without this assumption the body could be rotating with constant angular velocity]. If $\omega = 0$ the relative velocity of any two points on the body is zero, so all points have the same velocity as the CM. The proof is easily extended to three dimensions.

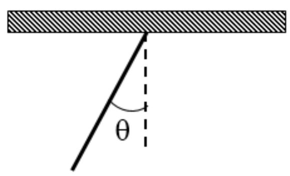


Figure 8.15: A rod attached to the ceiling by a smooth hinge.

Example 8.5 : A swinging rod connected to the ceiling by a hinge.

A uniform rod (mass M, length L) is attached to the ceiling by a smooth hinge and executes oscillations of small angular amplitude in a vertical plane. Calculate the period of the oscillation.

Solution: Let θ be the angle between the rod and the vertical. To maintain consistency with the sign convention we have been using, we call θ positive when the rod is to the left of the vertical (thus θ increases as the rod rotates in the clockwise sense). The only torque around the hinge is due to gravity $\tau = (-MgL/2)\sin\theta$ (the minus sign occurs because the torque is counterclockwise for positive θ). Thus, the torque equation is

$$-Mg\frac{L}{2}\sin\theta = \frac{1}{3}ML^2\frac{d^2\theta}{dt^2}.$$

This is the exact equation describing the oscillations of the rod. If θ is small, we can replace $\sin\theta$ by θ, obtaining

$$\frac{1}{3}ML^2\frac{d^2\theta}{dt^2} = -\frac{1}{2}MgL\theta.$$

This differential equation is of the form which we studied in Chapter 6. The rod executes simple harmonic motion, i.e.

$$\theta(t) = \theta_{\rm max}\cos(\omega t + \delta)$$

where $\omega^2 = (MgL/2)/(ML^2/3) = 3/2g/L$. The period is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2L}{3g}}.$$

More generally, if a body oscillates about an axis passing through a point O, the torque equation is

$$-Mgh\sin\theta = I\frac{d^2\theta}{dt^2}$$

where h is the distance from O to the CM and I is the moment of inertia about O. For small oscillations we replace $\sin\theta$ by θ and have SHM with $\omega^2 = Mgh/I$ and period $T = 2\pi\sqrt{I/Mgh}$.

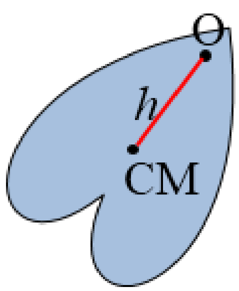


Figure 8.16: An arbitrarily shaped body free to oscillate about point O under the influence of gravity. See Example 8.5.

8.4 Rolling Motion

Before discussing an example which involves wheels which roll without slipping on a surface, it is useful to consider what is meant by ***rolling without slipping***. A wheel is said to be rolling without slippling if, at all times, the molecule of the wheel which is touching the ground has zero *instantaneous* velocity (i.e. that molecule is at rest relative to the ground). It is, of course, not necessary to talk about individual molecules; if a mark is painted on the rim of a wheel which is rolling without slipping, the mark has zero velocity at the instant when it touches the ground. By way of contrast, we note that if a driver brakes and skids (without swerving), the bottom molecule of the tire has a finite forward velocity relative to the ground (this is obvious in the extreme case when the wheels are locked). If a driver "peels rubber" by stepping hard on the accelerator, the bottom molecule on each of the drive wheels has a backward velocity relative to the road (even if the car has a forward velocity).

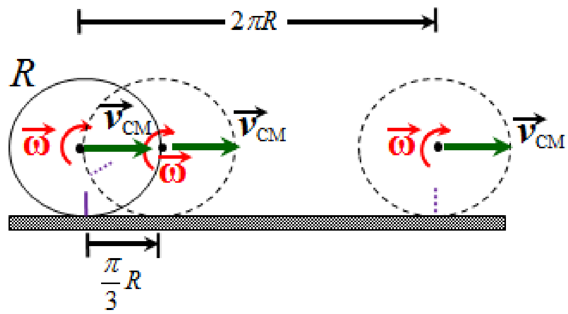


Figure 8.17: A circular object of radius R rolling without slipping on a horizontal surface. The angular velocity is $\vec{\omega}$ and the speed of the CM is $v_{\rm CM} = \omega R$.

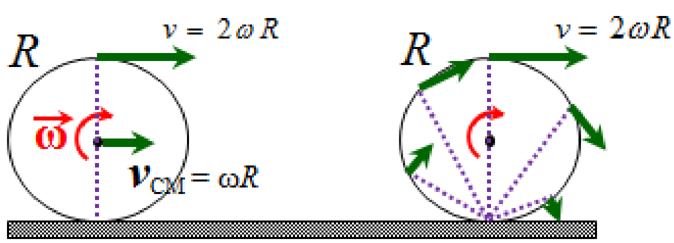


Figure 8.18: A round object of radius R rolling without slipping on a horizontal surface always has a point of contact with the surface that is instantaneously at rest. The speed at any other point on the body can be calculated by assuming the point of contact is the instantaneous axis of rotation.

If the bottom molecule of the wheel is at rest, then the velocities of all other molecules can be calculated by thinking of the wheel as rotating about an axis which passes through the bottom molecule (the "point of contact") and is instantaneously at rest. Thus a molecule at distance d from the point of contact has a velocity of magnitude ωd (where ω is the angular velocity) directed perpendicular to the line from the point of contact to the molecule.

In Fig. 8.18 we exhibit the velocity vectors of various molecules on a wheel which is rolling (clockwise) without slipping. The velocity of the center of the wheel is found by setting d equal to the radius R of the wheel; the speed is $v = \omega R$ and the direction of the velocity is parallel to the road. Note that the highest molecule on the wheel has a speed of $2\omega R$, directed forward. Thus we see that if a car is moving at 60 miles per hour, a mark painted on the tread has a velocity of 120 mph when it is at the top of the tire and 0 mph when it is at the bottom of the tire. The path of the mark is a cycloid (Fig. 8.19) [Derive the equation!]. We can differentiate $v = \omega R$ with respect to time to obtain $a = \alpha R$ [the acceleration of center of wheel rolling without slipping].

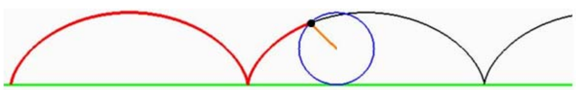


Figure 8.19: The cycloid is the shape drawn out by a point on a wheel rolling without slipping on a flat surface.

Rather than think of the wheel as rotating about an axis which is instantaneously at rest and passes through the point of contact, we can equally well think of the wheel as rotating about a *moving axis through its center*.[3] The moving axis has a velocity of magnitude ωR, directed to the right. The velocities of various molecules relative to an axis through the center are shown in Fig. 8.20. If we vectorially add the velocity of the center to all the velocity vectors in Fig. 8.20, the lowest molecule has a velocity ωR (directed to the left) relative to the center; adding this to the velocity of the center, we find a net velocity of zero. For the top molecule, the two velocities have the same sign and we find $v = 2\omega R$. This way of thinking about the motion has the advantage that it enables us to understand the velocities of individual molecules on a wheel which is slipping as it rolls. In this case the speed v of the center is not equal to ωR, but we can still obtain the velocities of all molecules by adding $\vec{v}$ vectorially to the velocities in Fig. 8.20. We shall make use of these remarks in discussing the motion of a skidding billiard ball (Example 8.8).

The preceding discussion illustrates the fact that a given motion of a rigid body can be achieved by a combination of translation and rotation in a variety of ways. The location in space of the instantaneous axis of rotation

[3]The angular velocity $\vec{\omega}$ is the same whichever description we use. If we paint a line segment on the sidewall, $\vec{\omega}$ is defined by the rate of change of the direction in which that segment points.

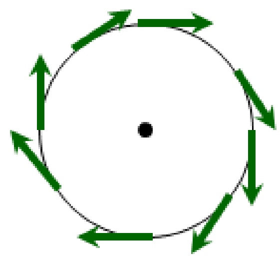


Figure 8.20: A wheel rotating about an axis through its center or a wheel rolling along a flat surface as seen from a reference frame moving with the center of the wheel has molecule velocities as shown.

is not uniquely defined; but the angular velocity and the direction of the axis of rotation are uniquely defined.

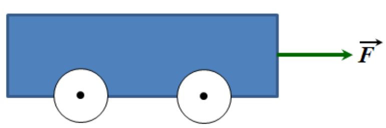


Figure 8.21: A cart is pulled so that its wheels roll without slipping on a horizontal road.

Example 8.6 : Pulling a cart so that its wheels roll without slipping.

A cart consists of a body of mass M plus four wheels, each a uniform solid disc of mass m and radius R. The wheels are attached to the axles by smooth bearings and roll without slipping on a horizontal road. A horse exerts a horizontal force $\vec{F}$ on the cart. Calculate the acceleration of the cart.

Solution: An important point should be emphasized here. The equation $\vec{F}_{\text{ext}} = (\text{total mass}) \cdot \vec{A}_{\text{CM}}$ is applicable to any system without exception (some students appear to think that the equation does not apply to a system which contains rotating objects). Thus, if we take the entire cart (body + wheels) as our system, can we conclude that the acceleration is equal to F divided by the total mass $M + 4m$? NO, because F is not the total external force on the system. The ground exerts horizontal forces on the wheels and these must be included in the net external force (*the ground also* *exerts vertical forces, which are cancelled out by the gravitational force on the system*).

If we focus our attention on any particular wheel (as shown in Fig. 8.22), we can write the torque equation for the wheel about an origin at the center of the wheel (this is not an inertial origin but is allowed because it is the CM of the wheel). The only torque about this origin is produced by the horizontal force which the ground exerts on the wheel. All other forces have no lever arm as they either act at the axle or are directed along the line between the point of application of the force and the radial vector from the axle to that point. Since the angular velocity and angular acceleration of the wheel are clockwise (positive, according to our sign conventions), the horizontal force $\vec{f}$ exerted by the ground must be directed to the left (Fig. 8.22). The torque equation for the wheel is

$$fR = \frac{1}{2}mR^2\alpha = \frac{1}{2}mR^2\frac{a}{R} \tag{8.30}$$

where α is the angular acceleration and a is the linear acceleration magnitude for the center of the wheel (which is the same as the magnitude for acceleration of the CM of the entire cart). Since all wheels have the same a, the horizontal force f is the same for all wheels. The force equation for the entire cart (body + wheels) is

$$F - 4f = (M + 4m)a \tag{8.31}$$

We have have two equations in the two unknowns f and a. Solving them, we find $a = F/(M + 6m)$ and $f = (mF/2)/(M + 6m)$. The net horizontal force on a wheel must be directed forward since the CM of the wheel is accelerating foreward. If F' is the magnitude of the horizontal force which the axle exerts on a wheel, then the force equation for a wheel is $F' - f = ma$. Inserting the values of f and a we find $F' = (3/2)(mF)/M + 6m)$. The horizontal forces acting on the body of the cart are $\vec{F}$ forward and $4F'$ backward. Thus, the force equation for the body is

$$F - 4F' = Ma. \tag{8.32}$$

Inserting the calculated values of F' and a, we see that eqn.(8.32) is indeed satisfied.

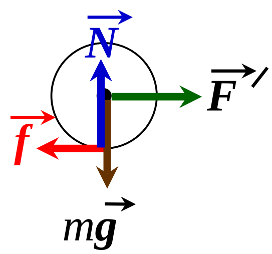


Figure 8.22: Forces acting on one of the wheels of the cart. $\vec{F}'$ is the horizontal force exerted by the axle, and $\vec{N}$ is the sum of the vertical forces exerted by the ground and the axle.

We shall discuss some simple examples involving the motion of objects which are not strictly two-dimensional. The discussion of the motion of a three-dimensional solid body is, in the most general cases, rather complicated because the angular momentum and angular velocity vectors are not necessarily parallel. Certain simple cases, however, are easily dealt with. We shall deal only with spheres (solid or hollow) and right circular cylinders (solid or hollow). Furthermore, we shall consider only motions in which the velocities of all particles are always parallel to the plane of the page; in the case of the cylinder we insist that the axis of the cylinder be perpendicular to the page. The angular velocity $\vec{\omega}$ is defined as in the two dimensional case; if we consider a thin slice of the body between two planes parallel to the page, $\vec{\omega}$ is the angular velocity of that slice (i.e. $\vec{\omega} = \omega\hat{j}$).

In order to use eqn.(8.4) we need to express $\vec{L}$ in terms of $\vec{\omega}$. We assume that the origin O, about which $\vec{L}$ is to be calculated, is a body-fixed point (i.e. a point fixed in the body) in the midplane of the body. The midplane of the sphere is the plane parallel to the page and containing the center of the sphere. The midplane of the right cylinder is the plane parallel to the page and equidistant from the ends of the cylinder. Once the midplane is established, it is easily shown (Appendix B) that the angular momentum $\vec{L}$ about the origin O is

$$\vec{L} = I\omega\ \hat{j} \quad \text{where} \quad I = \sum_i m_i\ r_{i\perp}^2 \tag{8.33}$$

and $r_{i\perp}$ is the distance of particle i from an axis which is perpendicular to the page and passes through O. If we take the x and z axes in the plane of the page, then $r_{i\perp}^2 = x_i^2 + z_i^2$ and thus $I = \sum_i m_i(x_i^2 + z_i^2)$. If O is either the CM of the body or the origin of an inertial frame (e.g. if a stationary axle passes through O) then we can use the torque equation (eqn.(8.4)). Since $\vec{L}$ has the direction $\hat{j}$, the torque must have this direction ($\vec{\tau}_{\text{ext}} = \tau_{\text{ext}}\ \hat{j}$) and we obtain

$$\vec{\tau}_{\text{ext}} = I\frac{d\vec{\omega}}{dt} = I\vec{\alpha} \tag{8.34}$$

which is identical with eqn.(8.16) except for the redefinition of I.

From eqn.(8.33) it is clear that the moment of inertia has the same value for all points O on a given axis, so one can speak of the moment of inertia about a given axis. We can think of a right cylinder as composed of many identical slices. It follows that the two-dimensional formulae also are valid for the right cylinder. If the axis is parallel to the axis of the cylinder:

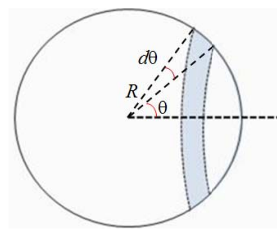


Figure 8.23: We subdvide the hollow sphere by planes perpendicular to an axis.

. I of hollow cylinder about axis through center $= MR^2$

. I of hollow cylinder about axis through point on rim $= 2MR^2$

. I of solid cylinder about axis through center $= \frac{1}{2}MR^2$

. I of solid cylinder about axis through point on rim $= (3/2)MR^2$.

To calculate the moment of inertia of a hollow sphere about an axis through its center, we subdivide the surface of the sphere into many rings (Fig. 8.23). This is easily done in polar coordinates. If θ is the polar angle with respect to the axis, then the distance from the axis is $R\ \sin\theta$ and the area of the ring whose angular width is $d\theta$ is $(2\pi R\ \sin\theta)(R\ d\theta)$. The mass of the ring is equal to its area multiplied by the mass per unit area (which we call σ). Thus the moment of inertia is

$$I = \int_0^\pi \sigma\ (2\pi R\ \sin\theta)(R\ d\theta)(R\ \sin\theta)^2 = \frac{8}{3}\pi\sigma\ R^4.$$

Since the mass is $M = 4\pi R^2\sigma$ we have

$$I = \frac{2}{3}MR^2 \quad \text{(hollow sphere about axis through center).} \tag{8.35}$$

[A quick derivation of eqn.(8.35) is obtained by noting that if the mass is distributed uniformly on the surface of a sphere, we have, by symmetry,

$\sum_i m_i x_i^2 = \sum_i m_i y_i^2 = \sum_i m_i z_i^2$. But $\sum_i m_i(x_i^2+y_i^2+z_i^2) = MR^2$. Therefore $\sum_i m_i(x_i^2 + z_i^2) = (2/3)MR^2$.]

The moment of inertia of the hollow sphere about an axis tangent to the sphere is obtained by means of the parallel-axis theorem (Appendix B), i.e.

$$I = \frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2. \tag{8.36}$$

To find the moment of inertia of a solid sphere about an axis through its center, we think of the sphere as an "onion" composed of many spherical shells. We know the moment of inertia of a shell. The rest of the calculation is an exercise for the reader. Finally, we find $I = (2/5)MR^2$ (solid sphere about an axis through the center). Note that, as expected, this is smaller than the I (about the same axis) of a hollow sphere of the same mass and radius. The I of a solid sphere about an axis tangent to the sphere is

$$\frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2.$$

Now we can discuss an example which involves spheres and cylinders.

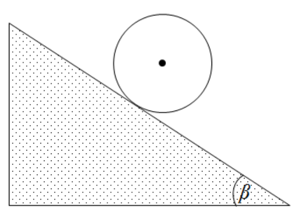


Figure 8.24: A solid sphere rolls without slipping down an incline.

Example 8.7 : A solid sphere rolls without slipping down an incline.

A solid sphere (mass M, radius R) rolls without slipping down an inclined plane (angle β to the horizontal). Calculate the acceleration of the center of the sphere. Do the same for a hollow sphere, a solid cylinder, and a hollow cylinder. If we conduct a downhill race with these four objects, which will win?

Solution: The forces acting on the sphere are shown in Fig. 8.25. The tangential force $\vec{f}$ is produced by static friction if the sphere is not slipping. The torque equation for the sphere about an origin at its center is $fR = I_{\rm CM}\alpha$. The force $\vec{f}$ must have the indicated direction (uphill) in order that the angular acceleration α may be positive (clockwise). The force equation for the sphere (i.e. the component of $\vec{F} = m\vec{a}$ parallel to the incline) is $Mg\ \sin\beta - f = Ma$ where a is the acceleration (positive in the downhill sense. If the sphere is not slipping, then $a = \alpha R$. We have two equations in the two unknowns f and a. Solving, we find

$$a = \frac{g\ \sin\beta}{1 + I_{\rm CM}/MR^2}, \qquad f = \frac{Mg\ \sin\beta}{1 + MR^2/I_{\rm CM}}.$$

The ratio $I_{\rm CM}/MR^2$ has the values 2/5, 2/3, 1/2, and 1 for the solid sphere, hollow sphere, solid cylinder, and hollow cylinder respectively. Thus, the sold sphere wins the race, followed by the solid cylinder, hollow sphere, and hollow cylinder (in that order).

Note that the acceleration a does not depend on the mass or radius of the object, so it is not necessary that the contestants have equal masses or radii. Even without solving the problem in detail one can predict simply from dimensional analysis that the acceleration of each of these objects will be independent of its mass and radius. We must calculate the acceleration and the only input quantities with dimensions are g, M, and R. There is no way we can use M or R in the answer and still obtain a quantity with the dimensions of acceleration.

We also note that there is a deceptively "simple" method of calculating a, namely, by writing the torque equation about the instantaneous point of contact. Let us paint a mark (an inertial origin) on the incline. At the instant when the rolling object is touching the mark, its angular momentum around the mark is $I\omega$, where $I = I_{\rm CM} + MR^2$ (parallel-axis theorem) and the external torque around the mark is $Mg\ R\ \sin\beta$. If we equate the time derivative of $I\omega$ to the torque, we obtain the correct values of α and a. The trouble is that the angular momentum L about the mark is equal to $I\omega$ only at the instant when the object is touching the mark, and there is an additional term in L at nearby instants (since the mark is not a point fixed in the body). To justify this "solution" one must show that the time derivative of this additional term vanishes at the instant in question.

As an additional challenge question, the reader can determine how steep the hill must be to cause the sphere to slip. (Assume you know the value of μ_s.)

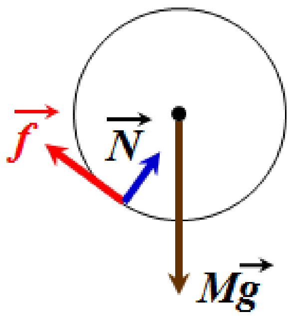


Figure 8.25: Forces acting on the solid sphere as it rolls down the incline.

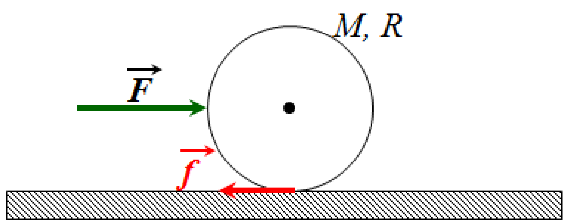


Figure 8.26: A billiard ball is struck with an impulsive force large enough that it begins rolling *with* slipping in Example 8.8.

Example 8.8 : A billiard ball that slips then rolls without slipping.

A billiard cue strikes a cue ball horizontally along a line directed through the center of the ball. The speed of the center of the ball just after it is struck is v_1. The coefficient of kinetic friction between the ball and the table is μ_k. Calculate the distance the ball travels before it stops skidding, the elapsed time before its stops skidding, and the speed of the ball after it stops skidding.

Solution: One should first understand qualitatively what is happening. We assume that the cue exerts an ***impulsive*** force on the ball, i.e. the cue exerts a large force F for a very short time t_1 such that the ***impulse***

$$\Im = \int F\ dt$$

has a finite value (which is the momentum given to the ball by the cue). The torque equation for the ball (around the CM) is $f\ R = I_{\rm cm} d\omega/dt$. Since the frictional force f is proportional to the weight of the ball, it does not become large during the time when the cue is hitting the ball. Therefore, if t_1 is sufficiently small, the angular speed ω is essentially zero just after the cue strikes the ball. Thus, initially the ball has velocity $\vec{v}_1$ directed to the right and zero angular velocity. The bottom is skidding and the frictional force $\mu_k Mg$ creates a clockwise angular acceleration $\alpha = fR/I_{\rm CM} = (5/2)\mu_k g/R$. The frictional force is directed to the left and causes a linear acceleration $-\mu_k g$ of the CM.

As the speed of the center decreases and the clockwise angular velocity increases, the (rightward) velocity of the bottom molecule relative to the table decreases. This continues until the instant when the velocity of the bottom molecule is zero. At this instant rolling without slipping commences (the minute roughnesses on the ball engage the roughnesses on the table, like gear teeth), and subsequently the velocity and angular velocity remain constant.

During the skidding phase, a is constant and the speed of the center at time t is $v = v_1 - \mu_k gt$. Since the initial angular velocity is zero, we have (*Note that we cannot write $\alpha = a/R$ since this is obtained by taking $d/dt(v = \omega R)$, but $v = \omega R$ only holds when the bottom molecule has zero velocity, i.e. no skidding.*) $\omega = \alpha t = 5/2(\mu_k gt/R)$. The speed of the bottom molecule is $v_{\rm bot} = v - \omega R = v_1 - (7/2)\mu_k gt$. The time T when skidding stops is found by setting $v_{\rm bot} = 0$, i.e. $T = (2/7)(v_1/\mu_k g)$. The distance the ball travels while skidding is $D = v_1 T - (1/2)\mu_k g T^2 = (12/49)v_1^2/u_k g$. The velocity of the center at time T is $v(T) = v_1 - \mu_k gT = 5/7v_1$. Note that the velocity of the ball when it stops skidding does not depend on μ_k or g!

If we model the contact between the ball and a floor as taking place at only one point, then a ball rolling without slipping on a horizontal floor will never stop rolling. The proof, in words, is: if the frictional force $\vec{f}$ is antiparallel to the velocity of the center, then the velocity will decrease, but the torque produced by $\vec{f}$ will increase the angular velocity (and, if no slipping, the linear velocity must increase); a similar paradox occurs if $\vec{f}$ is parallel to the velocity; therefore $f = 0$ and $a = 0$. You can easily write the equations. In order to understand how the ball can stop, we must allow the surface to exert a force on the ball at more than one point, e.g. by making the ball fuzzy or soft, or by letting it sink into a sandy or grassy surface.

8.5 Work-Energy for Rigid Body Dynamics

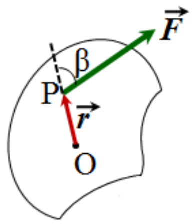


Figure 8.27: A force $\vec{F}$ is applied to a 2-dimensional rigid body which can rotate about an axis which passes through O and is perpendicular to the page.

The Work-Energy Theorem, which we proved for a point mass, can be extended to systems of particles. To see what the theorem says about rigid bodies, let us first examine a simple situation (Fig. 8.27): a two-dimensional rigid body which can rotate about a fixed axis through a point O, and subject to a force $\vec{F}$ which acts at a point P on the body. The work done by the force when the body rotates through a small angle $\Delta\theta$ is $\vec{F} \cdot \Delta\vec{r}$, where $\Delta\vec{r}$ is the vector displacement of P. Since P moves in a circle around O, the magnitude of $\Delta\vec{r}$ is $r\Delta\theta$ (where r is the distance from O to P) and its direction is perpendicular to OP (clockwise if $\Delta\theta > 0$). Thus, the work done by $\vec{F}$ is $\Delta W = F\ r\Delta\theta \sin\beta = \tau\Delta\theta$ where β is the angle between $\vec{F}$ and OP and $Fr\ \sin\beta$ is the torque acting on the body. The *rate of work* is

$$\frac{dW}{dt} = \lim_{\Delta t \to 0} \tau \frac{\Delta\theta}{\Delta t} = \tau\omega. \tag{8.37}$$

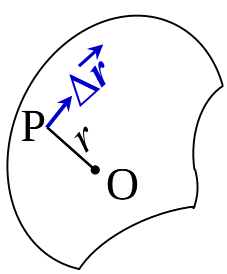


Figure 8.28: If the body rotates through angle $\Delta\theta$, the point P moves through a distance $r\Delta\theta$ perpendicular to OP.

The speed of a particle of mass m_i (Fig. 8.28) whose distance from O is r_i, is $\omega\, r_i$. Therefore the kinetic energy (KE)[4] of the body is $\frac{1}{2}\sum_i m_i r_i^2 \omega^2 = \frac{1}{2} I\omega^2$. If we write the torque equation $\tau = I\ d\omega/dt$ and multiply both sides by ω, we obtain

$$\frac{dW}{dt} = \frac{d}{dt}(KE). \tag{8.38}$$

Integrating both sides of eqn.(8.38) with respect to t from an arbitrary time t_0 to another arbitrary time t_f, we obtain

$$W = (KE)_f - (KE)_0 = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_0^2 \tag{8.39}$$

where W is the work done on the body between t_0 and t_f and *the axis of rotation is fixed*.

We should note that a critical reader might say that the previous derivation, and extensions thereof, are unnecessary: we have proved that, for each particle in a system, the work done on the particle is equal to the change in its kinetic energy, so the total work must be equal to the change in the total KE. This argument is alright but involves a hidden assumption, i.e. that the internal forces make no contribution to the total work. In Appendix B we prove that this assumption is correct if the system is a ***rigid body***.

The Work-Energy Theorem enables us to calculate the acceleration of the block in Example 8.3 without calculating the tension in the string. In eqn.(8.39) let t_f be an arbitrary time t, and let t_0 be the time when the block was released from rest. We let x be the distance the block has fallen from its initial position (x is positive and increases as the block falls). Applying eqn.(8.39) to the pulley wheel, we obtain

$$W' = \frac{1}{2}\left(\tfrac{1}{2} MR^2\right)\omega^2 \tag{8.40}$$

where W' is the work done by the string on the wheel between times t_0 and t, and ω is the angular velocity magnitude at time t. Note that $W' \neq mgx$ since the tension in the stirng is not equal to mg. We do not need to know the value of W'. The Work-Energy Theorem for the block yields

$$W_{\text{grav}} + W'' = \frac{1}{2} mv^2 \tag{8.41}$$

where W_{grav} and W'' are the work done on the block by gravity and the string (between t_0 and t), and v is the speed of the block at time t.

[4]This is also true in three dimensions, with r_i replaced by the perpendicular distance of m_i from the axis of rotation. Eqn.(8.38) is readily extended to the case when the axis is not fixed, so the K.E. includes contributions from both translational and rotational motion.

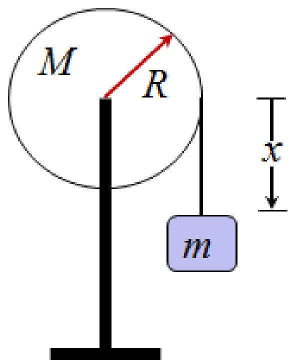


Figure 8.29: A pulley wheel of mass M and a weight with mass m with displacement in the vertical direction of distance x.

The important point to recognize is that $W'' = -W'$. To see this we note that when the block drops a small distance Δx, the string does work $T\Delta x$ on the wheel and work $-T\Delta x$ on the block. Thus, if we add eqns.(8.40) and 8.41, W and W'' cancel out. Note that this argument depends on the inextensibility of string; otherwise the block would not move the same distance as the point where the string touches the wheel. Inserting $W_{\text{grav}} = mgx$ and $\omega = v/r$ (which also depends on the inextensibility of the string), we find

$$mgx = \left(\tfrac{1}{2}m + \frac{1}{4}M\right)v^2 \to v^2 = \frac{2gx}{1 + M/2m}. \tag{8.42}$$

From the kinematic formulae of Chapter 1, we know that when there is a linear relation between v^2 and x, the acceleration is constant and is equal to one-half the coefficient of proportionality. If the reader wants to see a proof of this statement, differentiate both sides of eqn.(8.42) with respect to t. Since $d(v^2)/dt = 2v\ dv/dt = 2va$ and $dx/dt = v$, we obtain

$$a = \frac{g}{1 + M/2m},$$

in agreement with our solution of Example 8.3.

The significance of eqn.(8.42) should be obvious by now. If we consider the system (pulley + block + string) the kinetic energy is $(1/2m + 1/4M)v^2$ and the potential energy is $-mgx$ [the gravitational potential energy (PE) of the pulley wheel remains constant and can be taken as zero], and eqn.(8.42) simply states that the total energy (PE + KE) at time t is equal to the total energy (zero) at time t_0.

The Work-Energy Theorem enables us to calculate the angular speed of the rod which we considered in Example 8.4 (see Fig. 8.30) after it has fallen through an angle θ.

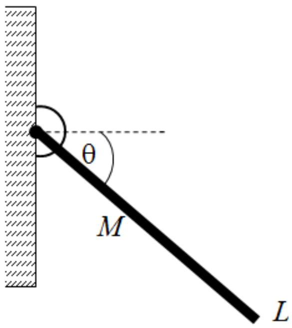


Figure 8.30: A rod of mass M and length L falls through an angular displacement θ while attached to a wall via a smooth hinge.

Example 8.9 : A rod connected to a wall by a hinge falls through an angle θ.

The rod in Fig. 8.30 is attached to the wall by a smooth hinge and is released from a horizontal position, initially with zero angular velocity. Calculate (a) the angular velocity when it has fallen through angle θ (b) the horizontal and vertical forces exerted by the hinge at that instant.

Solution: From the definition of the CM of a system of particles, it follows immediately that the work done by gravity as the system goes from an initial configuration (0) to a final configuration (f) is $Mg\,(z_0 - z_f)$, where M is the total mass of the system and z_0 and z_f are the initial and final heights of the CM above an arbitrary reference level. Thus, by eqn.(8.39)

$$\left(Mg\frac{L}{2}\right)\sin\theta = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 \tag{8.43}$$

In particular, when $\theta = \pi/2$ (just before the rod strikes the wall) we have $\omega^2 = 3g/L$. [WARNING: A common error is to use formulae which are applicable only when the angular acceleration is constant, which is not the case here.]

The torque equation when the rod makes angle θ with the horizontal is

$$\frac{1}{3}ML^2\alpha = \left(\frac{MgL}{2}\right)\cos\theta. \tag{8.44}$$

Eqn.(8.43) is derivable as a mathematical consequence of eqn.(8.44) even if we never mention work or kinetic energy. Noting that $\alpha = d^2\theta/dt^2$ and multiplying both sides of eqn.(8.44) by $d\theta/dt$ we obtain

$$\left(\frac{1}{3}ML^2\right)\left(\frac{d\theta}{dt}\right)\left(\frac{d^2\theta}{dt^2}\right) = \left(\frac{MgL}{2}\right)\cos\theta\frac{d\theta}{dt}. \qquad (8.45)$$

Since $d/dt[(d\theta/dt)^2] = 2(d\theta/dt)(d^2\theta/dt^2)$ and $d/dt(\sin\theta) = \cos\theta \; d\theta/dt$ we can rewrite eqn.(8.45) as

$$\frac{d}{dt}\left[\frac{1}{6}ML^2\left(\frac{d\theta}{dt}\right)^2 - \frac{MgL}{2}\sin\theta\right] = 0.$$

Therefore the quantity in square brackets is constant, but the value of the constant is zero since at time zero both θ and $d\theta/dt$ vanish. Noting that $d\theta/dt = \omega$, we obtain eqn.(8.43).

We can calculate the horizontal and vertical forces exerted by the hinge at the instant when the rod has fallen through angle θ by using the theorem (eqn.(4.9)) $\vec{F}_{\text{ext}} = M\vec{A}_{\text{CM}}$. The external forces acting on the rod are gravity and $\vec{H}$ and $\vec{V}$, the horizontal and vertical forces exerted by the hinge. Thus $H = M \; d^2X/dt^2$ and $V - Mg = d^2Z/dt^2$ where X $(= L/2 \; \cos\theta)$ and $Z(= -L/2 \; \sin\theta)$ are the coordinates of the CM of the rod. Differentiating, we find $dX/dt = -L/2 \; \sin\theta \; d\theta/dt$ and $d^2X/dt^2 = -L/2 \; \sin\theta \; d^2\theta/dt^2 - L/2 \; \cos\theta \; (d\theta/dt)^2$. From eqn.(8.43) and eqn.(8.44) we have

$$\begin{aligned} \frac{d^2\theta}{dt^2} &= \frac{3}{2}\frac{g}{L}\cos\theta \\ \left(\frac{d\theta}{dt}\right)^2 &= \frac{3g}{L}\sin\theta. \end{aligned}$$

Thus we find $H = -9/4 \; Mg \; \sin\theta\cos\theta = -9/8 \; Mg \; \sin 2\theta$. Note that the horizontal force is always directed to the left. A similar calculation (do it!) yields $V = Mg/4 \; (1 + 9\sin^2\theta)$.

The Work-Energy Theorem (the change in KE of a system is equal to the work done on the system) is true even when some or all of the forces acting on the system are not conservative. If a force is conservative, we can define a potential energy (PE) for each configuration of the system. The PE is the amount of work done by the force as the system moves from that configuration to some standard configuration (which, by definition, has zero PE). Thus, if we take $\theta = 0$ (i.e. the rod is horizontal) as the standard configuration of the rod in Example 8.9, then the PE when the rod is at angle θ below the horizontal is $-MgL/2\ \sin\theta$. If all the forces acting on a system are conservative, then KE + PE = constant, and the value of the constant is determined by the initial conditions. Thus, in eqn.(8.43), we can carry the term on the left across the equal sign and obtain KE + PE = 0.

Many problems in rigid-body dynamics can be solved simply by energy considerations, without introducting forces and torques. Returning to Example 8.7, we can take the standard configuration as one in which the point of contact between the rolling object and the inclined plane is at the lowest point on the inclined plane (see Fig. 8.24). Then the gravitational PE is $Mgx\ \sin\beta$, where x is the distance from the lowest point on the inclined plane to the point of contact. When applying the Work-Energy Theorem to this problem, it is important to realize that the only force which does work on the rolling object is gravity. Although the plane exerts a force on the rolling object, this force does no work. The rate of work done by the inclined plane on the rolling object is $\vec{F}\cdot\vec{v}$, where $\vec{F}$ is the force exerted by the inclined plane and $\vec{v}$ is the velocity of the matter in the very small surface element (ideally a line or a point) of the rolling object which is touching the incline. in the absence of slipping, $\vec{v} = 0$ and thus $\vec{F}\cdot\vec{v} = 0$.

The KE of the rolling object is $\frac{1}{2}I_p\omega^2$, where I_p is the moment of inertia about an axis through the point of contact and perpendicular to the screen (or page). As previously discussed, the parallel-axis theorem yields $I_p = I_{CM} + MR^2$, where $I_{\rm CM} = (2/5,\ 2/3,\ 1/2,\ 1)MR^2$ and the four numerical prefactors refer to a solid sphere, a hollow sphere, a solid cylinder, and a hollow cylinder respectively. If we write KE + PE = constant and replace ω by V/R, where V is the sped of the center of the rolling object, we obtain

$$\frac{1}{2}\left(1 + \frac{I_{\rm CM}}{MR^2}\right)MV^2 + Mgx\ \sin\beta = \text{ const.} \tag{8.46}$$

Differentiating eqn.(8.46) with respect to t and noting $dx/dt = -V$ and $dV/dt = a$, we obtain the same value for the acceleration which we previously found.

Before discussing the next Example, we shall state two simple theorems, which are proved in Appendix B and are frequently useful. First, some definitions: Let S be any system (a collection of particles, not necessarily a rigid body); let $\hat{\imath},\ \hat{\jmath},\ \hat{k}$ be mutually perpendicular axes attached to an arbitrary origin O; let O′ be the CM of the system S, and let $\hat{\imath}',\ \hat{\jmath}',\ \hat{k}'$ be mutually perpendicular axes which are attached to O′ and are non-rotating with respect to $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$. [Usually O and the axes $\hat{\imath}\hat{\jmath}\hat{k}$ are an inertial frame, but this assumption is not necessary.] The vector from O to O′ is $\vec{R}_{\rm CM} = X\hat{\imath} + Y\hat{\jmath} + Z\hat{k}$ and the velocity of the CM in the frame O$\hat{\imath}\hat{\jmath}\hat{k}$ is

$$\vec{V}_{\rm CM} = \hat{\imath}\frac{dX}{dt} + \hat{\jmath}\frac{dY}{dt} + \hat{k}\frac{dZ}{dt}.$$

The position and velocity vectors (in the frame O$\hat{\imath}\hat{\jmath}\hat{k}$) of the particles are similarly defined. i.e. $\vec{r}_n = \hat{\imath}x_n + \hat{\jmath}y_n + \hat{k}z_n$ and $\vec{v}_n = \hat{\imath}dx_n/dt + \hat{\jmath}dy_n/dt + \hat{k}dz_n/dt$ with similar definitions in the CM frame O′$\hat{\imath}'\hat{\jmath}'\hat{k}'$. The angular momentum of the system S in the frame O$\hat{\imath}\hat{\jmath}\hat{k}$ is called $\vec{L}_{\rm O}$, and the angular momentum of the same system in the frame O′$\hat{\imath}'\hat{\jmath}'\hat{k}'$ is called $\vec{L}_{\rm CM}$. Similarly, the kinetic energy of S in the frame $O\hat{\imath}\hat{\jmath}\hat{k}$ is called $KE_{\rm O}$, and the kinetic energy in the frame $O'\hat{\imath}'\hat{\jmath}'\hat{k}'$ is called $KE_{\rm CM}$. Let M = the total mass of the particles in system S.

$$\textbf{THEOREM}: \vec{L}_{\rm O} = M\vec{R}_{\rm CM}\times\vec{V}_{\rm CM} + \vec{L}_{\rm CM} \quad (8.47)$$
$$\textbf{THEOREM}: KE_{\rm O} = \tfrac{1}{2}MV_{\rm CM}^2 + KE_{\rm CM} \quad (8.48)$$

The following example requires the use of the above theorems.

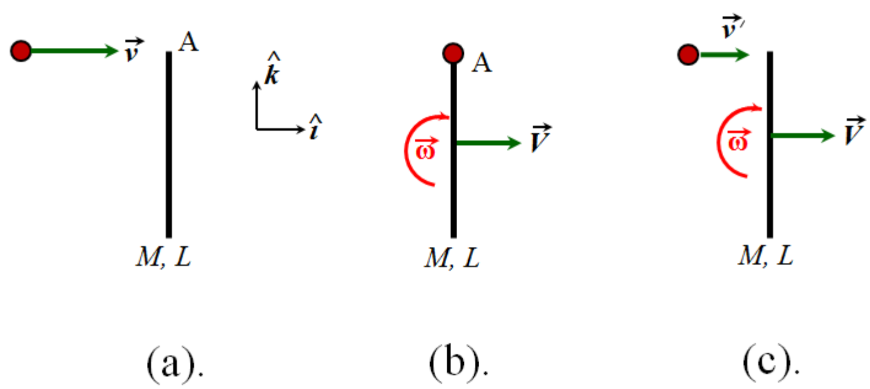


Figure 8.31: A rod of mass M and length L is struck by a particle of mass m in Example 8.10.

Example 8.10 : A rod struck on one end by a particle.

A rod (mass M, length L) of uniform density lies on a smooth table (see Fig. 8.31). A particle of mass m slides on the table with velocity $\vec{v}$ perpendicular to the rod and collides with the rod at one end (A). Defining unit vectors $\hat{k}$, $\hat{\imath}$, and $\hat{\jmath} = \hat{k} \times \hat{\imath}$ as in Fig. 8.31(a), we let $V\hat{\imath}$, $v'\hat{\imath}$, and $\omega\hat{\jmath}$ be the velocity of the center of the rod, the velocity of the particle, and the angular velocity of the rod immediately after the collision.

(a). (Fig. 8.31(b)) Calculate V, v', and ω if the particle sticks to the rod at A (a maximally inelastic collision).

(b). (Fig. 8.31(c)) Calculate V, v', and ω if the collision is elastic (total KE after collision = total KE before collision).

Solution: Since there are no external forces on the system (rod + particle), the total momentum and angular momentum (about an appropriate origin) are conserved. Momentum conservation yields

$$mv = mv' + MV \tag{8.49}$$

The most convenient origin about which to conserve angular momentum is a mark (which we call O) painted on the table directly under the point A. The axes $\hat{\imath}\hat{\jmath}\hat{k}$ are attached to the table with this mark as origin. Before the collision, the angular momentum of the system about this mark is zero. Just after the collision, the angular momentum of the particle about this mark is zero. By Eqn.(8.47), the angular momentum of the rod about O just after the collision is

$$\left(-\frac{L}{2}\,\hat{k}\right) \times \left(MV\,\hat{\imath}\right) + \left(\frac{1}{12}\,ML^2\right)\omega\hat{\jmath}.$$

Therefore

$$0 = -\frac{MVL}{2} + \frac{1}{12}ML^2\omega. \tag{8.50}$$

We have two equations in three unknowns (v', V, ω). In part (a), the additional equation is the kinematic relation

$$v' = V + \frac{1}{2}\omega L. \tag{8.51}$$

In part (b) the additional equation is the statement of conservation of energy, which (by eqn.(8.48)) is

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^{\,2} + \frac{1}{2}MV^2 + \frac{1}{24}ML^2\omega^2. \tag{8.52}$$

From eqn.(8.50) we find $V = \omega L/6$. Using eqn.(8.51) we obtain $v' = 4V$ and (using eqn.(8.49)), for part (a):

$$\begin{aligned} V &= \frac{v}{4 + M/m} \\ v' &= \frac{v}{1 + M/4m} \\ \omega &= \frac{(6v/L)}{4 + M/m}. \end{aligned}$$

From eqn.(8.49) we have $m(v-v') = MV$, and eqn.(8.50) yields $\omega = 6V/L$. In eqn.(8.52) we rewrite $v^2 - v'^{\,2}$ as $(v-v')(v+v')$. Thus we find, in part (b), $MV(v+v') = 4MV^2$. One solution is $V = 0$, which implies $v = v'$ and $\omega = 0$. This is the "nothing happens" solution which we previously encountered when discussing elastic scattering of two particles in one dimension, and is physically relevant only when where there is no interaction between the projectile and the target. If $V \neq 0$ we can divide by V, obtaining $v+v' = 4V$. Combining this with $v - v' = (M/m)V$ we obtain, finally, for part (b),

$$\begin{aligned} V &= \frac{v}{2 + M/2m} \\ v' &= \frac{v(1 - M/4m)}{1 + M/4m} \\ \omega &= \frac{3v/L}{1 + M/4m}. \end{aligned}$$

Note that when $M \gg m$ we obtain $v' = -v$, and when $M \ll m$ we obtain $v' = v$, as expected.

8.6 Rotational Motion Problems

8.1. Several turns of string are wrapped around a solid cylinder. One end of the string is attached to the cylinder, and a child holds the other end in her hand.

(a) The cylinder falls vertically and spins simultaneously as the string unwinds. Calculate the acceleration of the center of the cylinder.

(b) (Prof. J. Kikkawa) If the child accelerates her hand upward at the proper acceleration, the center of the cylinder will stay fixed in space. Calculate the proper acceleration.

8.2. The wheels of a bicycle have radius R; the mass of each wheel is m and may be assumed to be concentrated entirely at the rim. The mass of the frame plus cyclist is M. By pedaling, the cyclist applies (via the chain) a torque τ to the rear wheel. The tires roll without slipping on the road. Calculate the acceleration a of the bike.

8.3. The upper end of a uniform rod of mass m is attached to the ceiling by a vertical string and the lower end rests on a smooth floor, making angle θ with the floor.

(a) Calculate the force which the floor exerts on the rod.

(b) The string is suddenly cut. Calculate the force exerted by the floor and the vertical acceleration of the midpoint of the rod just after the string is cut.

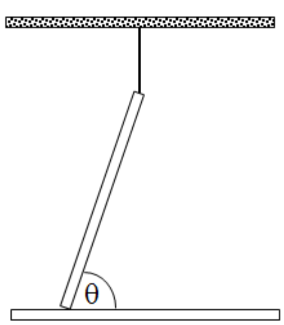


Figure 8.32: Problem **8.3**.

8.4. The coefficient of static friction between a solid cylinder and the surface of a hill is μ. If the hill is not too steep, the cylinder can roll down the hill without skidding. Calculate the steepest angle of inclination of the hill such that the cylinder will not skid.

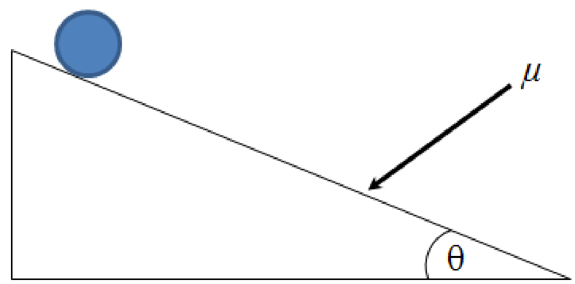


Figure 8.33: Problem **8.4**.

8.5. A car has wheels of radius R and hubcaps of radius r. A hubcap drops off while the car is traveling on a horizontal road at speed v. The hubcap hits the road and (after a very brief transient period) rolls parallel to the car without skidding. Calculate the speed of the hubcap. (Treat the hubcap as a solid disk.)

8.6. The top surface of a cube is horizontal and the adjacent planes are vertical. A solid sphere of radius R rolls along the top surface, approaching an edge with a velocity $\vec{v}_0$ which is perpendicular to the edge. [Think of the edge as being rounded with a very small radius of curvature and a very large coefficient of static friction.] If v_0 is greater than a certain critical value v_c, the sphere will leave the cube immediately when it reaches the edge, i.e. the velocity of the center of the sphere will be horizontal just after it loses contact with the cube. If $v_0 < v_c$ the sphere will maintain contact (without slipping) with the edge until the line from the edge to the center of the sphere has rotated away from the vertical by an amount (angle) θ_c. At the instant when the sphere loses contact with the cube, the velocity of the center of the sphere will be directed at angle θ_c below the horizontal.

(a) Calculate v_c.

(b) If $v_0 < v_c$ calculate θ_c.

(c) If $v_0 = 0$ (we really mean v_0 is infinitesimal), evaluate θ_c numerically.

(d) Prove that if $v_0 = 0$ the sphere (which is in free fall after it loses contact with the edge) will not bump into the edge as it falls [This is DIFFICULT!]

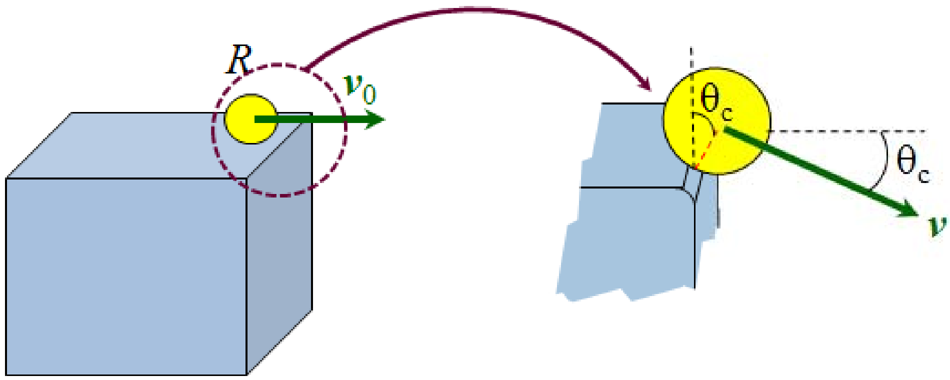


Figure 8.34: Problem **8.6**.

8.7. [Theory of the "sweet spot"] Consider a uniform rod of length L (call the ends A and B). An impulse P, perpendicular to the rod, is delivered to the rod at a point p whose distance from A is $(L/2) + s$.

(a) Calculate the velocity, just after the impulse, of a point p' on the rod whose distance from A is $(L/2) - y$. [Note: $0 < s,\ y < L/2$]

(b) For each value of y there is a "magic" value of s such that the velocity of the point p' will be zero. If your hand is holding the rod at the point p' and the impulse is delivered at the "magic" value of s, your hand will feel no shock. Considering the value of y as given, calculate the magic value of s. If $y = 0.400L$, what is the magic value of s? [5]

(c) (Another, equivalent, way to locate the sweet spot). Consider a baseball bat, which does not have a uniform mass distribution, whose moment of inertia about the center of mass is Ma^2 ($M =$ mass of bat). The bat is aligned along the z-axis. A hole parallel to the x-axis is drilled through the handle of the bat at distance d from the CM and the bat is mounted on an axle passing through the hole (the axle is fixed in space, but the bat can rotate around the axle). An impulse $\vec{P}$ in the y-direction is delivered to the bat

[5] A simple generalization of this calculation, taking account of the non-uniformity of the mass distribution, will locate the sweet spot of a baseball bat or tennis racket.

at a distance d' from the CM (on the fat part of the bat, i.e. the impulse and the axle are on opposite sides of the CM). Calculate the impulse communicated to the axle and show that this impulse will be zero if $d \cdot d' = a^2$.

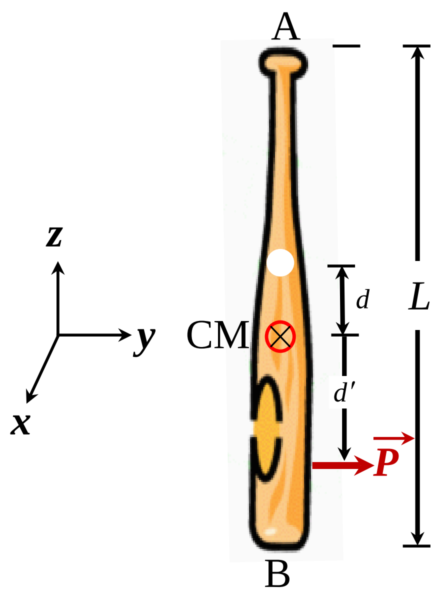


Figure 8.35: Problem **8.7**.

8.8. A rod (mass M uniformly distributed, length L) hangs from the ceiling, attached by a smooth universal hinge. A lump of clay (mass m) approaches the rod with velocity $\vec{v}$ perpendicular to the rod and sticks to the rod at its midpoint

(a) Calculate θ_{max}, the largest angle between the rod and the vertical in the subsequent motion.

(b) Calculate the impulse given by the rod to the hinge. Give clear justification of any conservation theorem(s) which you use.

Chapter 9

REMARKS ON NEWTON'S LAW OF UNIVERSAL GRAVITATION - contributed by Larry Gladney

For centuries humankind had understood that there are repeating, predictable patterns in the motion of the planets in night sky, in the position and phases of the Moon, and in the ocean tides. Not until Newton were all of these patterns to be understood through the application of a single law of physics which, with the accompanying advances in mathematics also made by Newton, literally brought the universe into the realm of human knowledge. This ***Law of Universal Gravitation*** is so complete that even today it is still used as the basis for understanding the orbits of the hundreds of artificial satellites that circle the earth and for engineering the Mars landing missions. It was not until two centuries after Newton that Einstein revolutionized our understanding of gravity yet again.

We can only touch on the ramifications that come from Newton's Law of Universal Gravitation here. This law, when combined with his three laws of motion, is the basis for innumerable applications in astronomy, geology, and physics. Remarkably, these connections to other fields are all accessible to the beginning student of physics. A testament to the genius of Newton is that his Law of Universal Gravity still has not been supplanted after more than three centuries of research and thinking on the subject. Although

Einstein's General Theory of Relativity offers a completely original idea for how gravity works, it does not supplant Newton's for almost any application on size scales from an earth-bound laboratory to the interactions of galaxies. This is also another testament to the simplicity and depth of the basic laws of nature.

9.1 Determination of g

We mentioned in section 3.2 that the straightforward use of Newton's Law of Universal Gravitation was not strictly valid because typically we are looking at application of the law in non-inertial frames. This applies to the determination of "g", the local acceleration near the surface of the earth as the earth rotates around its axis. Hence the surface is not, strictly speaking, an inertial frame. Yet we typically ignore this in introductory physics. Let's evaluate whether this is justified.

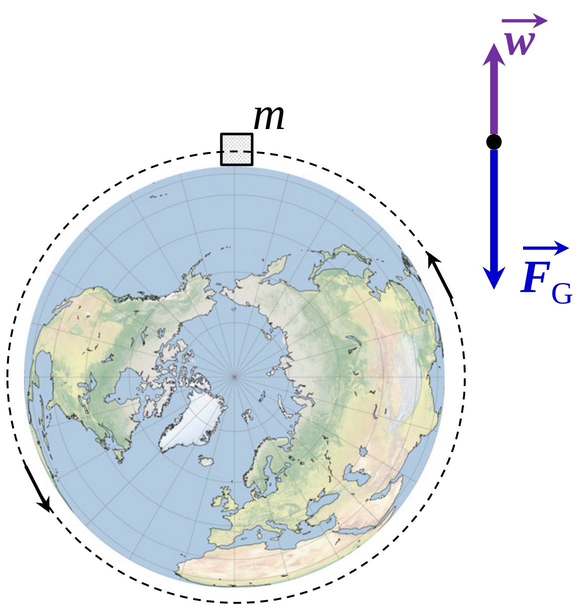


Figure 9.1: An orthographic projection of earth shown from directly above the North Pole. The mass m rotates with the surface of the earth. We wish to determine the effect of earth's rotation on the weight $\vec{w}$.

Example 9.1 : Earth's rotational effect on g.

Figure 9.1 is a schematic view of a mass m sitting on the surface of the earth as viewed from above the North Pole. The size of m is highly exaggerated. We wish to know what weight w would be measured, say by a spring balance, given the gravitational force with magnitude $F_G = GmM_{\mathrm{E}}/R_{\mathrm{E}}^2$.

Solution: We first note that the body is *not* in equilibrium with just the forces shown since the surface of the earth is not an inertial frame. The body m has a net acceleration a_R toward the center of the earth, hence what we measure as the weight is less than F_G since

$$F_G - w = ma_R \tag{9.1}$$

Plugging in Newton's formula for F_G and using the usual definition of the weight as $w = mg$, we get, with one step of algebra,

$$g = \frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}^2} - a_R. \tag{9.2}$$

So only at the poles is $g = GM_{\mathrm{E}}/R_{\mathrm{E}}^2$. In actuality the direction of $\vec{a_R}$ is not toward the center of the earth unless you are at the equator. Its actual direction at other locations is perpendicular to the earth's axis of rotation at your given latitude. The largest the effect of earth's rotation can be is at the equator where, ignoring the slight bulging of the earth, we have

$$a_R = \omega^2 R_{\mathrm{E}} = \frac{4\pi^2 R_{\mathrm{E}}}{T^2} \tag{9.3}$$

where T is the period of the rotation - 1 day or 8.64×10^4 s, so given $R_{\mathrm{E}} = 6.37 \times 10^6$ m, we have $a_R = 0.0337$ m/s^2 or about a third of a percent of the typical value of $g = 9.8$ m/s^2. Thus the difference between the typical value of g at the equator and at the South Pole is not negligible, but sufficiently small that we are usually justified in ignoring it for intermediate latitudes.

Example 9.2 : Effect of altitude on g.

We also take the acceleration g due to gravity as a constant. It is obvious from the gravitation law that this is not true as the altitude of a particular location on the earth's surface can place it farther away or closer to the earth's center. About how large is this altitude effect for "typical" altitudes you are likely to encounter?

Solution: We want to see how g changes as a function of altitude or change in distance from the earth's center. For such problems it is usually acceptable to calculate the fractional change. For the case in point, a mass m a distance r from the earth's center experiences force F due to gravity where

$$\begin{aligned} F &= \frac{GmM_{\mathrm{E}}}{r^2} \Rightarrow \\ dF &= -2\frac{GmM_{\mathrm{E}}}{r^3}\, dr \Rightarrow \\ \frac{dF}{F} &= -2\frac{dr}{r}. \end{aligned} \tag{9.4}$$

The result shows that the fractional change in the force is just twice the *negative* fractional change in the radius where we note that the negative sign is crucial since it indicates that the force magnitude decreases if the change in distance is positive. Let's assess the fractional change for someone taking off in an airplane from ground level and cruising at an altitude of 10 km (about 33,000 feet).

$$\begin{aligned} F &= mg \Rightarrow dF = m\ dg \Rightarrow \\ \frac{dF}{F} = \frac{dg}{g} &= -2\frac{dr}{r} \qquad (9.5) \\ &= -2\frac{10^4\ \mathrm{m}}{6.37 \times 10^6\ \mathrm{m}} \\ \frac{dg}{g} &= -0.31\% \qquad (9.6) \end{aligned}$$

No wonder you never feel any lighter on an airplane ride! A change of 3 parts in a thousand is comparable to the effect due to the earth's rotation and is not noticeable for people. However, instruments can easily measure smaller fractions than this and hence use this method to make a crude estimate of the shape of the earth, the so-called **geoid**.

9.2 Kepler's First Law of Planetary Motion

A comet (e.g. Halley's comet) moves in a long thin elliptical orbit with the sun at one focus of the ellipse. From our previous discussion of angular momentum conservation in section 8.1, it follows that $v_{\tan}$ is maximum when the comet is closest to the sun and minimum when the comet is farthest away. In section 3.2 we enumerated Kepler's Laws of Planetary Motion and showed how the second and third of these could be derived from Newton's Law of Universal Gravitation. The fact that the orbits of planets or of Halley's comet must be ellipses was not shown as it requires a little more math. For completeness though, it is reasonable to take what we have learned in previous chapters to show that elliptical orbits are the exact prediction of Newton's prescription of a $1/r^2$ attractive force law for gravity.

First we note that the problem of considering the trajectory for a point particle moving in the gravitational field of a much more massive body is more easily considered in polar coordinates rather than Cartesian ones. The argument for why this should be so is rather simple and due to Newton himself. Experience, and the mathematics of constant acceleration tell us that all trajectories of freely moving objects near the surface of the earth are parabolic. Given that we know that objects further from the surface experience a different acceleration, how does that affect the trajectory? Newton began his thinking on the subject by first assuming constant gravitational acceleration and doing a thought experiment for the question: what if you fired a projectile horizontally from a cannon atop a very high mountain? Assume the mountain is so high that the projectile moves mostly above the atmosphere so that we may ignore air resistance[1] (see Fig. 9.2). The projectile will still initially fall at whatever the local gravitational acceleration determines although its horizontal motion is unchanged. The parabolic path therefore is still expected. Now however, given the height of the mountain, we can expect that the range of the projectile is extended beyond what would happen for the same initial speed of a projectile fired from the ground. Given the curvature of the earth, we can expect that the projectile must fall *further* than just the vertical height of the mountain. Let's say that the local gravitational acceleration is still about 10 meters/s^2. Then, for a sufficiently powerful cannon we expect that the initial speed of the projectile can be high enough that the range is such that the earth's surface has "curved away" from the bottom of the mountain so that the projectile must travel further

[1] Clearly this is all imaginary! There are no such mountains on earth.

horizontally (all the while falling vertically). Is there an initial horizontal velocity for which the projectile never hits the earth? We can estimate the value for such an initial horizontal speed by noting that the projectile falls about 5 meters in the first second of flight. If the horizontal speed is such that the earth has curved away from the horizontal line of the bottom of the mountain by a vertical distance of 5 meters, then the projectile is no closer to the ground after one second than when it was fired since we define "height" as the distance above the earth's surface immediately below (i.e. along a radius of the earth) the projectile. "Vertically down" means perpendicular to the earth's surface along the radius of the earth so "horizontal" is the direction parallel to the surface, i.e. perpendicular to the radius, so the projectile continues to move horizontally while "falling" even though it gets no closer to the earth's surface!

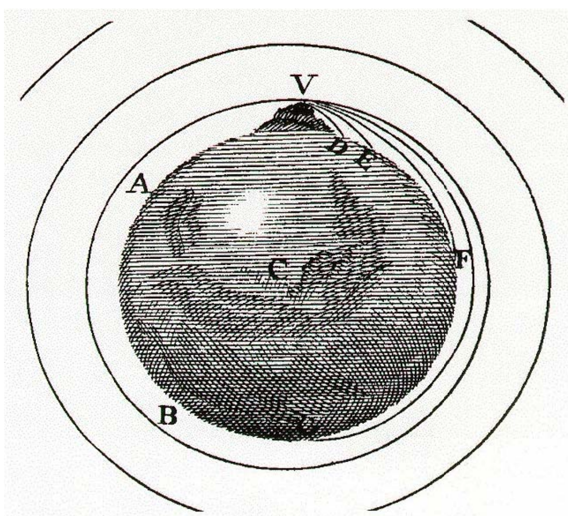


Figure 9.2: Illustration from Newton for why we expect orbits to occur for sufficiently fast projectiles near a gravitating body.

So as Fig. 9.2 shows, successive shots with faster and faster initial speeds eventually get to the point where the projectile never lands. It is in orbit around the earth. The slower initial speeds lead to parabolic paths, the orbit speed leads to a circular path. Circles and parabolas belong to a general mathematical class of shapes called ***conic sections*** since they can all be produced by a plane slicing through a cone at various angles (Fig. 9.3). The circle we see from the figure is a special case of an ellipse. Hence, if the orbit is closed, that is to say, the projectile repeats its motion around the earth,

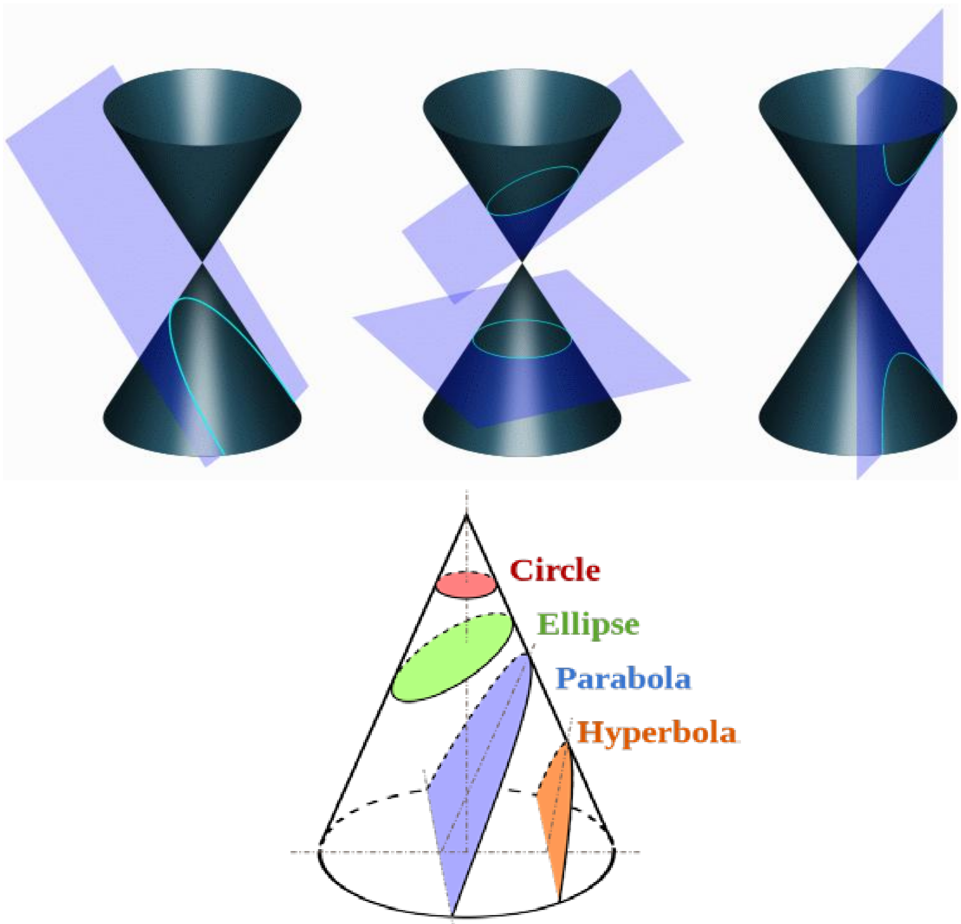


Figure 9.3: Illustration of conic sections.

then the general solution should be an ellipse.

Of course the gravitational acceleration is not constant but changes as a function of distance from the center of the gravitating body - the earth in this case. So, how do we do the general problem? As shown above, it's easiest to visualize the solution in terms of radius and angular position. As shown in section 8.1, a central force always produces motion confined to a plane so only 2 not 3 variables are needed. If we make these r and θ rather than, say x and y, the resulting solution of the equation of motion for the case of a circle must turn out to be rather trivial since r would be a constant. Since the gravitational force is directed only along r we know that the acceleration of a particle m around a much more massive planet of mass M is

$$a = \frac{d^2r}{dt^2} - r\omega^2 = \frac{F_{\text{grav}}}{m} = -\frac{GM}{r^2} \tag{9.7}$$

The right-hand side of the equation results from Newton's gravitation law while the left-hand side is the radial acceleration which, in the general case, is $d^2r/dt^2 - \omega^2 r$. Note that in eqn.(9.7) ω is also a function of time. We can eliminate it though by noting that the angular momentum, L, is constant and $L = mr^2\omega$, so

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + r\left(\frac{L}{mr^2}\right)^2 = -\frac{GM}{r^2} + \frac{L^2}{m^2r^3} \tag{9.8}$$

This is tricky to solve but we can do it if we make use of a change of variables that's handy for all central force equations of motion: $u \equiv 1/r$ and make use of the constancy of the angular momentum to convert from time to angular position θ as the independent variable. So, using the first of these we have

$$L = mr^2\omega = \frac{m}{u^2}\frac{d\theta}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{Lu^2}{m} \tag{9.9}$$

and use of the chain rule gives

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta} = \frac{Lu^2}{m}\frac{d}{d\theta} \tag{9.10}$$

We can therefore write the equation of motion as

$$\begin{aligned} \frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) &= -\frac{1}{u^2}\frac{du}{dt} = -\frac{L}{m}\frac{du}{d\theta} \Rightarrow \\ \frac{d^2r}{dt^2} &= -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2} \end{aligned} \tag{9.11}$$

Our equation of motion eqn.(9.8) becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{GMm^2}{L^2} \tag{9.12}$$

The right-hand side is a constant so it's easy to verify that the solution is

$$u = \frac{1}{r} = \frac{GMm^2}{L^2} + C\cos\theta \tag{9.13}$$

where C is a constant of integration to be determined by the initial conditions of the orbit[2]. To give this a more familiar feel for the mathematically-inclined, this equation is generally written in the form

$$\frac{\alpha}{r} = 1 + \varepsilon \cos\theta \tag{9.14}$$

where ε is termed the ***eccentricity*** of the orbit and 2α is the orbit's ***latus rectum***. The form of eqn.(9.14) is easy to relate to that of an ellipse with ***semi-major axis*** a and eccentricity ε if the origin of the coordinate system ($r = 0$) is set at one focus of the ellipse. Such a formula would be

$$\frac{a\left(1-\varepsilon^2\right)}{r} = 1 + \varepsilon \cos\theta \tag{9.15}$$

You can easily prove to yourself that eqns.(9.14) and 9.15 represent ellipses for fixed values of α, ε, and a with any computer graphing application. It's clear that the values of the constants must be connected to the physical characteristics of the orbit. An obvious choice is

$$\alpha = \frac{L^2}{GMm^2} \Rightarrow a = \frac{\alpha}{1-\varepsilon^2} \tag{9.16}$$

We need to do more work to show it, but the eccentricity can be written as

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}} \tag{9.17}$$

with E being the total mechanical energy of the system.

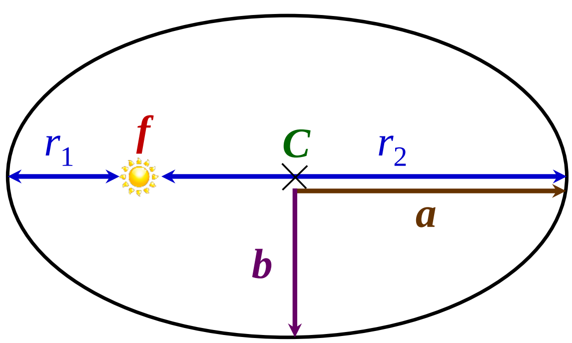


Figure 9.4: Parameters of an elliptical orbit with parameters measured from the origin at the ellipse center, C: the semi-major axis a, the semi-minor axis b, the sun at focal point f. The minimum and maximum distances r_1 and r_2, respectively, are measured from f.

[2]Note that r returns to the same value when θ increases by 2π. This would not be true if the exponent in the force law were other than 2 except for the case $F = -kr$.

Example 9.3 : Orbits and Halley's comet.

Sir Edmund Halley determined the orbit of the famous comet that bears his name. To date it is still the only short-period (namely you can see it more than once in a human lifetime) comet that is clearly visible to the human eye from earth. The orbit has a period of about 76 years and an eccentricity of 0.967. Find the maximum and minimum distances of Halley's comet from the sun.

Solution: First we should note that the properties of ellipses make it easy to find the minimum and maximum radii (which we denote by r_1 and r_2, respectively) from the focus (the location of the sun) once we know the mean radius and the eccentricity. You should remember from math class that

$$\begin{aligned} r_1 &= a(1-\varepsilon) \\ r_2 &= a(1+\varepsilon) \end{aligned}$$

where a is the mean radius or semi-major axis, and that

$$\begin{aligned} a &= \frac{r_1+r_2}{2} \\ b &= \sqrt{r_1 r_2} \end{aligned}$$

with b being the semi-minor axis. To get the mean radius in this case we need to extend the proof of Kepler's Third Law of Planetary Motion from section 3.2 to handle elliptical rather than just circular orbits. To do so it is convenient to make use of the proof of Kepler's Second Law from section 8.1, namely that the area swept out for one orbit is the area of the ellipse and this is a constant due to angular momentum conservation. That is to say, the area A swept out by a line from the focus to the orbiting comet per unit time is

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{L}{2m} \tag{9.18}$$

where L is the angular momentum for the orbit. For one complete period T the area swept out is just

$$T\frac{dA}{dt} = T\frac{L}{2m} = \pi ab \tag{9.19}$$

since πab is the area of an ellipse. From eqn.(9.13) we see that the extrema of r occur when $\theta = 0$ and π, and thus

$$\frac{1}{r_1} + \frac{1}{r_2} = \frac{r_1+r_2}{r_1 r_2} = \frac{2GMm^2}{L^2} \tag{9.20}$$

which can be rewritten as

$$\frac{2a}{b^2} = \frac{1}{2}\frac{GM}{(dA/dt)^2} = \frac{1}{2}\frac{GM}{(\pi ab/T)^2} \tag{9.21}$$

and finally

$$T^2 = \frac{4\pi^2 a^3}{GM} \tag{9.22}$$

This is the general form of Kepler's Third Law appropriate for any closed orbit. For a circle a is just the constant radius. Finally, with this result in hand, we can answer the original question. Note that there are 3.15×10^7 seconds in a year (it's actually a useful shorthand to remember that this is about $\pi \times 10^7$ s/y).

$$\begin{aligned}
T^2 &= \frac{4\pi^2 a^3}{GM} \Rightarrow \\
a &= \left[\frac{GMT^2}{4\pi^2}\right]^{1/3} \\
&= \left[\frac{\left(6.67 \times 10^{-11}\ \frac{\mathrm{N \cdot m^2}}{\mathrm{kg^2}}\right)\left(1.99 \times 10^{30}\ \mathrm{kg}\right)\left(76\ \mathrm{y} \cdot \pi \times 10^7\ \frac{\mathrm{s}}{\mathrm{y}}\right)^2}{4\pi^2}\right]^{1/3} \\
a &= 2.68 \times 10^{12}\ \mathrm{m} \Rightarrow \\
r_{\min} &= a(1-\varepsilon) = 2.68 \times 10^{12}\ \mathrm{m}(1-0.967) = 8.84 \times 10^{10}\ \mathrm{m} \\
r_{\max} &= a(1+\varepsilon) = 2.68 \times 10^{12}\ \mathrm{m}(1+0.967) = 5.27 \times 10^{12}\ \mathrm{m}
\end{aligned}$$

9.3 Gravitational Orbit Problems

9.1. A space vehicle is initially stationary (with respect to the earth) 300 kilometers above the earth's surface. What velocity (parallel to the earth's surface) must the vehicle be projected at to achieve circular orbit at that altitude? Also find the period of the orbit.

9.2. The most efficient (in terms of energy expended) way to send a spacecraft from the Earth to another planet is to use the Hohmann transfer orbit in which the spacecraft is put into an elliptical orbit with a perihelion at the (nearly) circular orbit of the Earth around the Sun and the aphelion at the (nearly) circular orbit of the planet to be visited/invaded. Ignore Earth and Mars gravity. You can find relevant solar system parameters (mass and distance of Mars from the sun, etc. from any number of textbooks or Internet sources).

   (a) Determine the direction the rockets are to be fired for going from the Earth to Mars and for the return trip of Mars to Earth.

   (b) With what speed must the spacecraft be launched from low Earth orbit and how long does the journey to Mars take?

   (c) Where must Mars be (in relation to Earth) when the spacecraft is launched from Earth?

Chapter 10

APPENDICES

Appendix A

Appendix A

A.1 Vectors

Many of the entities with which physics deals (including forces, velocities and accelerations) have both a *magnitude* and a *direction*. Such entities are called ***vectors***. Notationally, vectors are represented by boldface letters or (as in this text) by ordinary letters with arrows above them (for example, $\vec{v}$ = velocity).

Newton's Second Law (in vector notation $\vec{F} = m\vec{a}$) states that the magnitude of the acceleration of a particle of mass m is proportional (with proportionality factor $1/m$) to the magnitude of the force acting on the particle and the direction of the acceleration is the same as the direction of the force. Vector notation provides a compact way of stating the relation between the acceleration and the force. If we introduce a set of mutually perpendicular x, y, and z axes, the vector equation $\vec{F} = m\vec{a}$ is equivalent to three numerical equations ($F_x = ma_x$, $F_y = ma_y$, $F_z = ma_z$), where F_x is the force in the x-direction and a_x is the acceleration in the x-direction, etc. In addition to the virtue of providing a compact way of writing the mathematical relation between acceleration and force, the vector notation makes it clear that this relation does not depend on the orientation of our axes (i.e. the directions in which the axes point).

More generally, when we use vector notation in writing an equation, we are stating in compact form a relation (or relations) among the magnitudes and directions of the physical entities represented by the vectors. This relation is true, independently of the orientation of our axes, and we are free to choose the orientation of the axes in whatever way we find convenient. Sometimes (e.g. as in the derivation of the Work-Energy Theorem) it is not necessary to introduce any axes. Furthermore, there are various mathematical operations involving vectors (we shall introduce only the operations which are relevant in elementary Mechanics) which, in many cases, provide insights and reduce labor, compared with what we would do if we had chosen a particular set of axes and had to manipulate a set of simultaneous equations involving the components (= perpendicular projections) of the vectors along those axes.

The preceding remarks are an advertisement for the usefulness of familiarizing oneself with vector notation and simple operations involving vectors.

A.1.1 Definitions and Proofs

Let P_1 and P_2 be any two points in space, and draw a line ("arrow") from

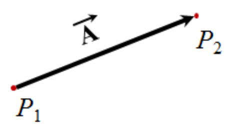


Figure A.1: A vector $\vec{A}$ between two points P_1 and P_2.

P_1 to P_2. We call P_2 the "head" of the arrow and indicate the head by the symbol $>$. Occasionally we call P_1 the "tail" of the arrow. The arrow (which we call $\vec{A}$ here) represents a ***displacement***, i.e. the change in position of a particle which moves (or is moved) from P_1 to P_2. If we think of P_1 and P_2 as two points on the path of a particle, we realize that there are many possible paths from P_1 to P_2. For example (Fig. A.2), if we guide

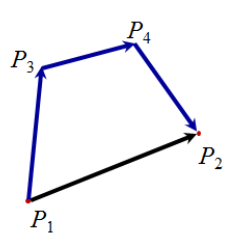


Figure A.2: There are many paths from P_1 to P_2.

the particle with our hand, we could move it on a straight line from P_1 to P_3, then on a line from P_3 to P_4, and then on a line from P_4 to P_2. The vector $\vec{A}$ represents the *net* effect of those three successive displacements. The length (or "magnitude") of $\vec{A}$ is denoted by $|\vec{A}|$ or A, and is the distance (always a positive number) between P_1 and P_2. We prefer the word "magnitude" to "length" because vectors may represent entities (e.g. velocities or accelerations) which dimensionally are not lengths. The direction of $\vec{A}$ can be described mathematically (for example by two polar coordinates on a sphere), but at this point we are deliberately refraining from introducing any particular set of axes.

If α is a positive real number, we define the vector $\alpha\vec{A}$ as a vector having the same direction as $\vec{A}$ and having magnitude $\alpha|\vec{A}|$. If α is a negative real number, $\alpha\vec{A}$ is defined as a vector pointing in the direction opposite (antiparallel) to $\vec{A}$ and having magnitude $-\alpha|\vec{A}|$. Thus $-1.7\vec{A}$ points in the direction opposite to $\vec{A}$ and has magnitude equal to $1.7\times$(magnitude of $\vec{A}$).

When we represent a displacement by a vector, the actual location of the vector in space may or may not be relevant. In general we consider the vector to be fully defined by its magnitude and direction; these properties are unchanged by moving the vector to another location while preserving the direction of the vector (this movement is called a ***parallel translation***). If the location of the vector is important we would specify not only the magnitude and direction of the vector but also the location of the head or tail.

We have already defined the operation of *multiplying* a vector by a real number; the result of this operation is also a vector. We now define ***vector addition***, i.e. the operation of adding two or more vectors. One may think of the vectors as representing displacements, but the same definition is used when the vectors represent velocities, forces, or anything else.

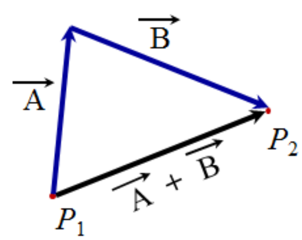


Figure A.3: The resultant vector from adding $\vec{A}$ and $\vec{B}$.

If we think of $\vec{A}$ and $\vec{B}$ as displacements, the vector $\vec{A}+\vec{B}$ is defined as the total displacement which results when an object (e.g. a point mass) is subjected to the displacement $\vec{A}$ followed by the displacement $\vec{B}$. Geometrically (Fig. A.3), if we draw the vector $\vec{A}$ and then draw the vector $\vec{B}$, placing the tail of $\vec{B}$ at the head of $\vec{A}$, then $\vec{A}+\vec{B}$ is the vector from the tail of $\vec{A}$ to the head of $\vec{B}$. Similarly, $\vec{B}+\vec{A}$ is the vector representing the total displacement which results when an object is subjected to the displacement $\vec{B}$ followed by the displacement $\vec{A}$.

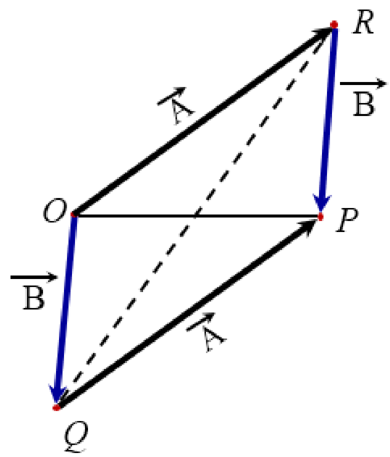


Figure A.4: A parallelogram formed from vectors $\vec{A}$ and $\vec{B}$.

If we draw the parallelogram $ORPQ$ (Fig. A.4) with $\vec{A}$ and $\vec{B}$ as edges, we see that the vector from O to P represents the result of making the displacement $\vec{A}$ followed by $\vec{B}$, and also represents the result of making the displacement $\vec{B}$ followed by $\vec{A}$. For any two vectors (not necessarily representing displacements) the vector sum $\vec{A}+\vec{B}$ is defined by the geometric construction shown in Fig. A.3. Thus we see that $\vec{A}+\vec{B}=\vec{B}+\vec{A}$, i.e. vector addition is ***commutative***. We define $\vec{A}-\vec{B}$ as the sum of the vector $\vec{A}$ and the vector $-\vec{B}$, i.e. $\vec{A}-\vec{B}=\vec{A}+(-\vec{B})$. In Fig. A.4, suppose we replace the vector from R to P by a vector from P to R. Since the vector from Q to P is $\vec{A}$ and the vector from P to R is $-\vec{B}$, we see that the vector from Q to R is $\vec{A}+(-\vec{B})=\vec{A}-\vec{B}$.

In Fig. A.2, let $\vec{B}$, $\vec{C}$, and $\vec{D}$, respectively, be the vectors from P_1 to P_3, from P_3 to P_4, and from P_4 to P_2. Thus $\vec{B}+\vec{C}$ is the vector from P_1 to P_4, and $(\vec{B}+\vec{C})+\vec{D}$ is the vector from P_1 to P_2. $\vec{C}+\vec{D}$ is the vector from P_3 to P_2 and $\vec{B}+(\vec{C}+\vec{D})$ is the vector from P_1 to P_2. Thus $(\vec{B}+\vec{C})+\vec{D}=\vec{B}+(\vec{C}+\vec{D})$, i.e. vector addition is ***associative*** and the parentheses may be omitted. Note that the proof does not assume that P_1, P_2, P_3 and P_4 are in the same plane. Also, the proof is easily extended to the addition of any number of vectors.

Finally, the definitions of addition of vectors and of multiplication of a vector by a real number α imply the ***distributive*** property $\alpha(\vec{A}+\vec{B}) = \alpha\vec{A}+\alpha\vec{B}$.

We have deliberately presented the preceding definitions and proofs without introducing any particular set of axes (a set of axes is a coordinate system) in order to emphasize that vectors enable us to state relations among physical quantities without committing ourselves to any particular choice of axes. In actual calculations, it is often convenient to introduce axes.

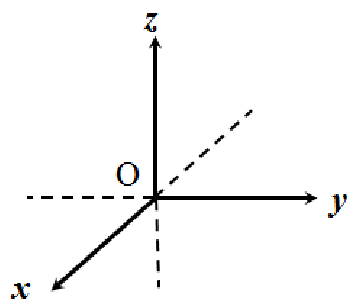


Figure A.5: Three mutually perpendicular axes ($x,\ y,\ z$).

We now introduce three mutually perpendicular axes ($x,\ y,\ z$) passing through a common origin O. If we are interested in vectors which represent lengths, then each point on the x-axis has a number associated with it, the magnitude of the number being equal to its distance from the origin (measured in whatever units of length we use). The sign of the number is positive on one side of the origin and negative on the other side. Similarly, numbers are assigned to each point on the y- and z-axis. In Fig. A.5 the positive part of each axis is a solid line and the negative part is a dashed line. If (for example) we are interested in vectors which represent velocities, then the numbers on the x-axis would be velocities (measured in whatever velocity units we use), with the positive x-axis representing velocities in the direction of increasing x and the negative x-axis representing velocities in the direction of decreasing x.

If we place the tail of a vector $\vec{A}$ at the origin, then we call the Cartesian components of the head of the vector ($A_x,\ A_y,\ A_z$). Reminder: if we pass a plane, perpendicular to the x-axis, through the head of the vector $\vec{A}$, the x-coordinate of the point where the plane intersects the x-axis is A_x. Furthermore, $A_x = |\vec{A}|\cos\alpha$, where α is the angle between $\vec{A}$ and the positive x-axis. Similar remarks apply to A_y and A_z. Frequently, A_x is called "the

x-component of the vector A".

From the definition of the sum of two vectors, we see that the x-component, y-component and z-component of $\vec{A}+\vec{B}$ are A_x+B_x, A_y+B_y and A_z+B_z. Similarly, the x-component of $\vec{A}+\vec{B}+\vec{C}$ is $A_x+B_x+C_x$, etc.

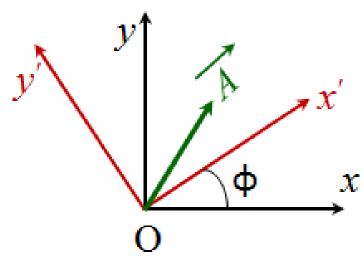


Figure A.6: The components of the vector $\vec{A}$ depend on which axes we use.

The magnitude $|\vec{A}|$ of the vector $\vec{A}$ is (according to Pythagoras)

$$\sqrt{A_x^2+A_y^2+A_z^2}.$$

If my axes $(x,\ y,\ z)$ are oriented differently from your axes $(x',\ y',\ z')$, nevertheless we both agree on the length of the vector $\vec{A}$, i.e. $A_x^2+A_y^2+A_z^2 = A_{x'}^2+A_{y'}^2+A_{z'}^2$. In general, three angles are required to specify the orientation of your axes relative to mine. For the simple case in which the z and z' axes coincide and the angle between the unprimed and primed x-axes is ϕ, the relation between the unprimed and primed components of $\vec{A}$ is

$$\begin{aligned} A_{z'} &= A_z \\ A_{x'} &= A_x\cos\phi + A_y\sin\phi \\ A_{y'} &= A_y\cos\phi - A_x\sin\phi \end{aligned}$$

and we immediately see that, as expected $A_x^2+A_y^2+A_z^2 = A_{x'}^2+A_{y'}^2+A_{z'}^2$. In the more general case the algebra is a bit messier but, of course, the result still holds.

Another quantity which obviously does not depend on the orientation of the axes is the *angle* between two vectors. If we call this angle θ, the relevant formula (using components) is

$$\cos\theta = \frac{A_xB_x+A_yB_y+A_zB_z}{|\vec{A}||\vec{B}|} \tag{A.1}$$

To prove this, recall the formula from high school trigonometry (see Fig.

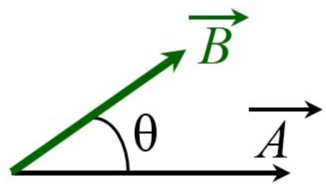


Figure A.7: Two vectors at different angles.

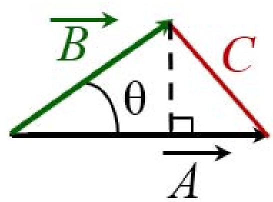


Figure A.8: The law of cosines applied to vector lengths.

A.8)

$$C^2 = A^2 + B^2 - 2AB\cos\theta \tag{A.2}$$

(which is proved by constructing the dashed line in Fig. A.8 and applying Pythagoras' Theorem to the right triangle whose hypotenuse is C and whose other sides are $B\sin\theta$ and $A - B\cos\theta$). But C is the length of the vector $\vec{A} - \vec{B}$, which has components $A_x - B_x$, etc. Thus $C^2 = (A_x - B_x)^2 + ... = A^2 + B^2 - 2(A_xB_x + A_yB_y + A_zB_z)$ which, combined with eqn.(A.2), yields eqn.(A.1). Since $\cos\theta = \cos(360 - \theta)$ it makes no difference whether we define θ as the interior or exterior angle in Fig. A.7.

The ***dot product*** (also called the ***scalar product***) of two vectors $\vec{A}$ and $\vec{B}$ is defined as $|\vec{A}||\vec{B}|\cos\theta$ where θ is the angle between $\vec{A}$ and $\vec{B}$ and is denoted by $\vec{A}\cdot\vec{B}$. From eqn.(A.1) we have

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = A_xB_x + A_yB_y + A_zB_z. \tag{A.3}$$

The dot product is especially useful in the derivation of the Work-Energy Theorem in Section 5.2. We emphasize that the dot product is a number (which may have dimensions), not a vector, and that this number does not depend on the orientation of our axes. From A.3 it is clear that $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$

and $(\alpha\vec{A})\cdot\vec{B} = \alpha(\vec{A}\cdot\vec{B})$. Furthermore, since the components of $\vec{B}+\vec{C}$ are $(B_x + C_x, ...)$, we see that $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C} = (\vec{B}+\vec{C})\cdot\vec{A}$.

If the vector $\vec{A}$ represents a displacement, then $A_x,\ A_y,\ A_z$ and $|\vec{A}|$ all have the dimensions of length. If we construct the vector $\vec{A}/|\vec{A}|$, the components of this vector are *dimensionless* numbers and the magnitude of this vector is the dimensionless number "1". A dimensionless vector whose magnitude is "1" is called a ***unit vector***. We denote a unit vector by a letter with a carat (i.e. $\hat{\ }$) rather than an arrow over it. Thus, if we denote $\vec{A}/|\vec{A}$ by the symbol $\hat{e}$, then $\hat{e}$ is a unit vector pointing in the same direction as $\vec{A}$.

We define unit vectors $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ pointing (in the positive direction) along our x, y, and z axes respectively. Then $\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and $\hat{\imath}\cdot\hat{\jmath} = \hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0$. If the components of the vector $\vec{A}$ are $(A_x,\ A_y,\ A_z)$ then $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$.

The dot product constructs a number from two vectors. There is another type of product, called the ***cross product*** or ***vector product***, which constructs a *vector* from two vectors. The definition of the cross product may seem slightly weird, but it is actually a mathematically "natural" object. It is the only way to combine two vectors ($\vec{A}$ and $\vec{B}$) to form a third vector $\vec{C}$ subject to the requirements:

1. the magnitude and direction of $\vec{C}$ do not depend on the orientation of our axes and no preferred directions in space enter into the rules for constructing $\vec{C}$;

2. $\vec{C}$, considered as a function of $\vec{A}$ and $\vec{B}$, has the distributive property in both variables.

The cross-product is useful in discussing planetary motion and in discussing objects or collections of objects which can have both rotational and translational motion.

The cross-product (denoted by $\vec{A}\times\vec{B}$) is defined as follows:

1. Bring the tails of $\vec{A}$ and $\vec{B}$ together (by a parallel translation) and let the page be the plane containing $\vec{A}$ and $\vec{B}$. Define $\hat{e}$ as a unit vector perpendicular to the page and pointing *into* the page; define $\hat{e}'$ as a unit vector perpendicular to the page and pointing *out* of the page. Clearly $\hat{e}' = -\hat{e}$.

2. Now imagine rotating $\vec{A}$ until it points in the same direction as $\vec{B}$. This can be done by a clockwise rotation through an angle θ or by a counter-clockwise rotation through angle θ' $(\theta + \theta' = 360°)$.

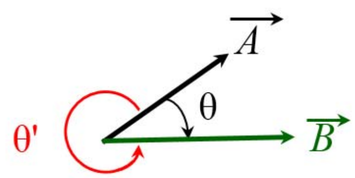


Figure A.9: Rotating from the direction of one vector into the direction of another.

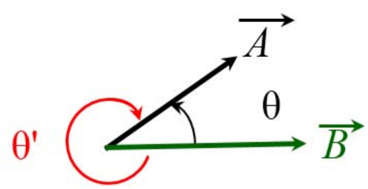


Figure A.10: Reversing the rotation of one vector direction into another.

3. We define

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\theta \ \hat{e} = |\vec{A}||\vec{B}| \sin\theta' \ \hat{e}' \tag{A.4}$$

REMARKS: $\hat{e} \ \sin\theta = \hat{e}' \ \sin\theta'$ since $\hat{e}' = -\hat{e}$ and $\sin(360° - \theta) = -\sin\theta$. Furthermore, the definition of $\vec{A} \times \vec{B}$ does not depend on which side of the page you are on. If you are on this side and another observer is on the other side, then your "clockwise rotation" is what the other observer calls a "counter-clockwise rotation". According to the other observer, your $\hat{e}'$ is a unit vector pointing into the page (i.e. away from the observer) and your $\hat{e}$ points out of the page (i.e. toward the observer). Thus, the other observer will also write eqn.(A.4).

Equations A.4 are frequently summarized by reference to a right-handed screw, which is the only kind you can buy in most hardware stores. Bring the tails of $\vec{A}$ and $\vec{B}$ together by a parallel translation and align the axis of a right-handed screw along the line L which passes through the common tail and is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$. It does not matter in which direction the screw points. Then imagine rotating $\vec{A}$ around the line L as axis until the directions of $\vec{A}$ and $\vec{B}$ coincide. Then

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin\alpha \ \hat{u} \tag{A.5}$$

where α is the angle through which $\vec{A}$ was rotated and $\hat{u}$ is a unit vector pointing along the line L in the direction in which the screw would have moved during the rotation. Since the rotation could have been performed in either sense, eqn.(A.5) covers both terms in A.4.

It is important to note that in the definition of $\vec{A} \times \vec{B}$, we rotate $\vec{A}$ (the first vector in the cross-product) until its direction coincides with that of $\vec{B}$. If a clockwise rotation of $\vec{A}$ through angle θ will make the directions of $\vec{A}$ and $\vec{B}$ coincide, then a counter-clockwise rotation of $\vec{B}$ through angle θ will make the directions coincide. It follows from eqn.(A.4) that

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B} \tag{A.6}$$

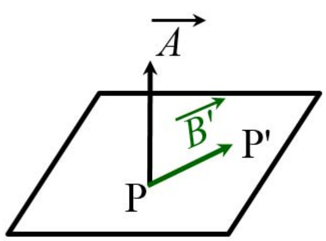


Figure A.11: The plane perpendicular to vector $\vec{A}$.

We note that the cross-product of two vectors which are parallel or anti-parallel is zero since $\sin(0°) = \sin(180°) = 0$.

From equations A.4 it is easy to show that for any real number a (positive or negative) $(a\vec{A}) \times \vec{B} = a(\vec{A} \times \vec{B}) = \vec{A} \times (a\vec{B})$.

Proof of the distributive property

$$\vec{A} \times (\vec{B}_1 + \vec{B}_2) = \vec{A} \times \vec{B}_1 + \vec{A} \times \vec{B}_2 \tag{A.7}$$

is more challenging and may, of course, be omitted by the reader who is interested only in the result. To prove eqn.(A.7) it is useful to visualize $\vec{A} \times \vec{B}$ in a slightly different way. We draw the vectors $\vec{A}$ and $\vec{B}$ with their tails touching at a point which we call P, and construct a plane which is perpendicular to $\vec{A}$ and contains P. Let $\vec{B}'$ be the projection of $\vec{B}$ onto this plane. [A line perpendicular to the plane and passing through the head of $\vec{B}$ will intersect the plane at a point P' and $\vec{B}'$ is the vector from P to P'.] If θ is the smaller of the two angles between $\vec{A}$ and $\vec{B}$, then $0 \leq \theta \leq 180°$ and $\sin\theta \geq 0$; thus $|\vec{B}'| = |\vec{B}|\sin\theta$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{B}' = |\vec{A}|\vec{B}''$ where the vector $\vec{B}''$ is in the plane and (viewed from a vantage point on the head of $\vec{A}$) is obtained by rotating $\vec{B}'$ counter-clockwise through an angle of 90°.

Let $\vec{B}_1$ and $\vec{B}_2$ be arbitrary vectors. Then $\vec{B}_1''$ is constructed by projecting $\vec{B}_1$ onto the plane perpendicular to $\vec{A}$ and rotating the projection counter-clockwise through 90°. Similarly, we construct B_2''. We recall that $\vec{B}_1+\vec{B}_2$ is constructed by placing the tail of $\vec{B}_1$ at P and placing the tail of $\vec{B}_2$ at the head of $\vec{B}_1$, and then drawing an arrow from P to the head of $\vec{B}_2$. Thus, if the projections of $\vec{B}_1$ and $\vec{B}_2$ onto the plane perpendicular to $\vec{A}$ are $\vec{B}_1'$ and $\vec{B}_2'$, then the projection of $(\vec{B}_1+\vec{B}_2)$ onto that plane is $\vec{B}_1'+\vec{B}_2'$. And if we rotate the vector $\vec{B}_1'+\vec{B}_2'$ counter-clockwise through 90°, the resulting vector, which we call $(\vec{B}_1+\vec{B}_2)''$ is the same as we would get by first rotating $\vec{B}_1'$ and $\vec{B}_2'$ through 90° (to obtain $\vec{B}_1''$ and $\vec{B}_2''$) and then forming the vector sum $\vec{B}_1''+\vec{B}_2''$. Thus we have shown $(\vec{B}_1+\vec{B}_2)''=\vec{B}_1''+\vec{B}_2''$; and, since $\vec{A}\times\vec{B}=|\vec{A}|\vec{B}''$, we have proved eqn.(A.7). Finally, since $\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$, the distributive property also applies to the first factor

$$(\vec{B}_1+\vec{B}_2)\times\vec{A}=\vec{B}_1\times\vec{A}+\vec{B}_2\times\vec{A} \tag{A.8}$$

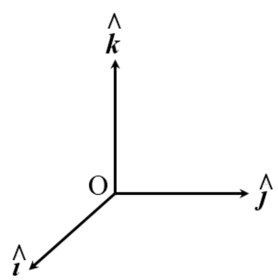


Figure A.12: These unit vectors form a set of right-handed axes, i.e. a right-handed coordinate system.

If we introduce a particular set of axes with unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ pointing in the positive direction along the x, y, and z axes, and write $\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k}$ and $\vec{B}=B_x\hat{i}+...$ then the distributive property of the cross-product enables us to calculate the components of $\vec{A}\times\vec{B}$. We know that $\hat{i}\times\hat{i}=\hat{j}\times\hat{j}=\hat{k}\times\hat{k}=0$. Conventionally, we use *right-handed axes* (Fig. A.12) which have the property $\hat{i}\times\hat{j}=\hat{k}$ (from which it follows that $\hat{j}\times\hat{k}=\hat{i}$ and $\hat{k}\times\hat{i}=\hat{j}$). [If you extend the thumb ($\hat{i}$), index finger ($\hat{j}$) and middle finger ($\hat{k}$) of your right hand so that they are all perpendicular to each other, they form a right-handed set of axes.] We find

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{\imath} + (A_z B_x - A_x B_z)\hat{\jmath} + (A_x B_y - A_y B_x)\hat{k}. \quad \text{(A.9)}$$

Frequently we deal with vectors which vary with time. For example, if $\vec{r}$ is the vector from a fixed origin to the instantaneous position of a moving particle, then $d\vec{r}/dt$ is the velocity vector of the particle. The derivative of a vector $\vec{A}(t)$ is defined in exactly the same way as the derivative of a function $f(t)$, i.e.

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t + \Delta t) - \vec{A}(t)}{\Delta t} \quad \text{(A.10)}$$

If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and the unit vectors $\hat{\imath},\ \hat{\jmath},\ \hat{k}$ are constant in time, then

$$\frac{d\vec{A}}{dt} = \hat{\imath}\ \frac{dA_x}{dt} + \hat{\jmath}\ \frac{dA_y}{dt} + \hat{k}\ \frac{dA_z}{dt}.$$

The derivative of the dot product can be obtained directly from eqn.(A.3) or (without introducing unit vectors) from eqn.(A.10). For example

$$\frac{d}{dt}\left(\vec{A} \cdot \vec{B}\right) = \lim_{\Delta t \to 0} \frac{\left[\vec{A}(t + \Delta t) \cdot \vec{B}(t + \Delta t) - \vec{A}(t) \cdot \vec{B}(t)\right]}{\Delta t}. \quad \text{(A.11)}$$

If we add zero (in the form $\vec{A}(t+\Delta t)\cdot\vec{B}(t) - \vec{A}(t+\Delta t)\cdot\vec{B}(t)$) to the numerator of eqn.(A.11), then the right side of that equation becomes

$$\frac{\left[\vec{A}(t + \Delta t) \cdot \left(\vec{B}(t + \Delta t) - \vec{B}(t)\right) + \left(\vec{A}(t + \Delta t) - \vec{A}(t)\right) \cdot \vec{B}(t)\right]}{\Delta t}$$

Letting $\Delta t \to 0$ we obtain

$$\frac{d}{dt}\left(\vec{A} \cdot \vec{B}\right) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{B} \quad \text{(A.12)}$$

and note that the order of the factors in the two terms is irrelevant since $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. We can carry out the same manipulations for the cross-product (substituting $\times$ for $\cdot$) but must be careful to preserve the order of the factors since $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. Thus we obtain

$$\frac{d}{dt}\left(\vec{A} \times \vec{B}\right) = \vec{A} \times \left(\frac{d\vec{B}}{dt}\right) + \left(\frac{d\vec{A}}{dt}\right) \times \vec{B}. \quad \text{(A.13)}$$

Appendix B

Appendix B

B.1 Useful Theorems about Energy, Angular Momentum, & Moment of Inertia

Recall the definition of the center of mass (CM) of a collection of particles:

$$M\ \vec{R}_{\mathrm{CM}} = \sum_i m_i\ \vec{r}_i \qquad (M = \sum_i m_i). \tag{B.1}$$

Equivalently

$$\sum_i m_i\ \vec{r}_i{}' = 0 \qquad \text{where } \vec{r}_i{}' = \vec{r}_i - \vec{R}_{\mathrm{CM}} \tag{B.2}$$

Differentiating eqn.(B.2) with respect to time we obtain

$$\sum_i m_i \vec{v}_i{}' = 0 \quad \text{where } \vec{v}_i{}' = \frac{d\vec{r}_i{}'}{dt} = \vec{v}_i - \vec{V}_{\mathrm{CM}} \tag{B.3}$$

Suppose we have an inertial frame $O\hat{\imath}\hat{\jmath}\hat{k}$ (an appropriate origin O and a set of axes non-rotating with respect to distant stars) and another frame $O'\hat{\imath}'\hat{\jmath}'\hat{k}'$ (where O' is the CM of our system and $\hat{\imath}'\hat{\jmath}'\hat{k}'$ are also non-rotating). The kinetic energy (KE) of our system, as measured in the frame $O\hat{\imath}\hat{\jmath}\hat{k}$, is

$$KE_{\mathrm{O}} = \frac{1}{2}\sum_i m_i\ v_i^2.$$

The KE as measured in the frame $O'\hat{\imath}'\hat{\jmath}'\hat{k}'$ is

$$KE_{\mathrm{CM}} = \frac{1}{2}\sum_i m_i\ v_i'^2. \tag{B.4}$$

THEOREM: $KE_{\rm O} = KE_{\rm CM} + \frac{1}{2}MV_{\rm CM}^2$.
The proof is simply to write $v_i^2 = \left(\vec{v}_i{}' + \vec{V}_{\rm CM}\right) \cdot \left(\vec{v}_i{}' + \vec{V}_{\rm CM}\right)$ and to note that the cross-term in the square vanishes because $\sum m_i\ \vec{v}_i{}' = 0$.

The angular momentum, as measured in the frame $O\hat{i}\hat{j}\hat{k}$, is

$$\vec{L}_O = \sum_i m_i \vec{r}_i \times \vec{v}_i.$$

We replace $\vec{r}_i$ by $\vec{R}_{\rm CM} + \vec{r}_i{}'$ and $\vec{v}_i$ by $\vec{V}_{\rm CM} + \vec{v}_i{}'$. Two of the four terms in the expansion of $\vec{L}_O$ vanish [$\vec{R}_{\rm CM} \times \sum m_i \vec{v}_i{}' = 0$ and $\sum m_i \vec{r}_i{}' \times \vec{V}_{\rm CM} = 0$]. Defining

$$\vec{L}_{\rm CM} = \sum_i m_i \vec{r}_i{}' \times \vec{v}_i{}'$$

we have the

$$\text{THEOREM}: \ \vec{L}_O = \vec{L}_{\rm CM} + M\vec{R}_{\rm CM} \times \vec{V}_{\rm CM}. \tag{B.5}$$

We have already discussed the fact that the internal forces in a system make no contribution to the total force and total torque acting on the system. We derived the result $\vec{\tau}_{0,\ \rm ext} = d\vec{L}_0/dt$ where

$$\vec{\tau}_{0,\ \rm ext} = \sum_i \vec{r}_i \times \vec{F}_{i,\ \rm ext}$$

and $\vec{F}_{\rm ext}$ is the external force acting on the $i^{\rm th}$ particle. Writing $\vec{r}_i = \vec{r}_i{}' + \vec{R}_{\rm CM}$ we find

$$\vec{\tau}_{0,\ \rm ext} = \vec{\tau}_{\rm CM,\ ext} + \vec{R}_{\rm CM} \times \vec{F}_{\rm ext}$$

where $\vec{\tau}_{\rm CM,\ ext}$ is the torque (due to external forces) measured in the CM system and $\vec{F}_{\rm ext}$ is the total external force. We note from eqn.(B.5)

$$\frac{d\vec{L}_0}{dt} = \frac{d\vec{L}_{\rm CM}}{dt} + M\vec{V}_{\rm CM} \times \vec{V}_{\rm CM} + \vec{R}_{\rm CM} \times M\vec{A}_{\rm CM}. \tag{B.6}$$

The second term on the right of eqn.(B.6) is zero and the third term is equal to $\vec{R}_{\rm CM} \times \vec{F}_{\rm ext}$. Thus we find

$$\text{THEOREM}: \ \vec{\tau}_{\rm CM,\ ext} = \frac{d\vec{L}_{\rm CM}}{dt}. \tag{B.7}$$

In particular, this theorem implies that when an object (or collection of objects) is falling under the influence of a uniform gravitational field, the angular momentum of the system around its CM remains constant (since gravity exerts no torque around the CM).

Recall that the moment of inertia of a body, around a line L, is the sum over the mass elements of the body of the mass of the element multiplied by the square of the distance of the element from the line L. A useful theorem, most easily stated in words, is the ***parallel axis theorem***: If I is the moment of inertia of a body about a line L and I' is the moment of inertia of the body about a line L' which is parallel to L and passes through the CM, then $I = I' + Ma^2$ where a is the perpendicular distance between L and L' and M is the mass of the body.

Proof : Let O be a point on the line L and let $\hat{e}$ be a unit vector pointing (in either sense) along L. Let $\vec{r}_i$ be a vector from O to the mass element m_i. Then

$$I = \sum_i m_i \left[\vec{r}_i \cdot \vec{r}_i - (\hat{e} \cdot \vec{r}_i)^2\right].$$

If O' is the CM and $\vec{r}_i{}'$ is the vector from O' to mass element m_i, then $I' = \sum m_i \left[\vec{r}_i{}' \cdot \vec{r}_i{}' - (\hat{e} \cdot \vec{r}_i{}')^2\right]$. If $\vec{R}$ is the vector from O to O' then $\vec{r}_i = \vec{R} + \vec{r}_i{}'$. Inserting this into the expression for I we see that the terms $2\vec{R} \cdot \sum m_i \vec{r}_i{}'$ and $2\hat{e} \cdot \sum m_i \vec{r}_i{}'$ vanish since $\sum m_i \vec{r}_i{}' = 0$. Thus

$$I = I' + M\left[\vec{R} \cdot \vec{R} - \left(\hat{e} \cdot \vec{R}\right)^2\right] = I' + Ma^2.$$

In Chapter 8 we discussed some simple examples of rigid body motion in which the angular velocity $\omega \hat{\jmath}$ is perpendicular to the page and the body is a figure of revolution about the axis of rotation $\hat{\jmath}$. We are interested in the angular momentum $\vec{L}_O = \sum \vec{r}_i \times m_i \vec{v}_i$, where O is a point fixed in the body and $\vec{r}_i$ is the vector from O to m_i and $\vec{v}_i = d\vec{r}_i/dt$. We write $\vec{r}_i = x_i\ \hat{\imath} + y_i\ \hat{\jmath} + z_i\ \hat{k}$ where $\hat{\imath},\ \hat{\jmath},\ \hat{k}$ are vectors fixed in the body. The coordinates $x_i,\ y_i,\ z_i$ do not change in time, but the unit vectors $\hat{\imath}$ and $\hat{k}$ rotate with the body. One easily sees $d\hat{\imath}/dt = \omega\ \hat{\jmath} \times \hat{\imath} = -\omega\ \hat{k}$ and $d\hat{k}/dt = \omega\ \hat{\jmath} \times \hat{k} = \omega\ \hat{\imath}$. Thus $\vec{v}_i = -\omega x_i\ \hat{k} + \omega z_i\ \hat{\imath}$ and

$$\vec{L}_O = \omega \sum_i m_i \left[\left(x_i^2 + z_i^2\right)\hat{\jmath} - x_i y_i\ \hat{\imath} - y_i z_i\ \hat{k}\right]. \tag{B.8}$$

The first term on the right side of eqn.(B.8) is just the moment of inertia about the axis $\hat{\jmath}$ through the point O. The other terms vanish if the object has sufficient symmetry: if the body is a figure of revolution about the y-axis, then if there is a mass element at $(x,\ y)$ there is an equal mass element at $(-x,\ y)$ and the second sum on the right vanishes (as does the third sum); similarly, if the body is symmetric about the plane $y = 0$ so that there are equal mass elements at $(x,\ y)$ and $(x,\ -y)$, then the second and third sums on the right vanish. In these cases we can write $\vec{L} = I\omega\ \hat{j}$, as in the two-dimensional case.

In Chapter 8 we proved the Work-Energy Theorem for a rigid body rotating about a fixed axis (see eqns.(8.37) and (8.38)). Extension of the proof of Chapter 8, by using the force and torque equations describing an arbitrary motion of a rigid body, involves some slightly complicated vector manipulations which are not appropriate for inclusion here. However, as we mentioned in Chapter 8, since we have proved that, for each particle in a system, the work done on the particle is equal to the change in its K.E., the Work-Energy Theorem will apply to the system as a whole provided we can show that the internal forces make no contribution to the total work. It is essential to assume that the system is a rigid body since the internal forces *will* usually do work if the distances between the particles change.

Consider a pair of particles whose positions (with respect to some axes) at a certain instant are $\vec{r}_i$ and $\vec{r}_j$, and slightly later the positions are $\vec{r}_i + \Delta\vec{r}_i$ and $\vec{r}_j + \Delta\vec{r}_j$. The vector from i to j is $\vec{r}_{ij} = \vec{r}_j - \vec{r}_i$ and the square of the distance between the two particles is $\vec{r}_{ij} \cdot \vec{r}_{ij}$. If the body is rigid, the interparticle distances don't change; thus (through first order in the small quantities) $\vec{r}_{ij} \cdot \Delta\vec{r}_{ij} = 0$, where $\Delta\vec{r}_{ij} = \Delta\vec{r}_j - \Delta\vec{r}_i$. If the force exerted by i on j is $\vec{f}_{ij}$ (and the force exerted by j on i is $-\vec{f}_{ij}$) then the work done by i on j is $\vec{f}_{ij} \cdot \Delta\vec{r}_j$ and the work done by j on i is $-\vec{f}_{ij} \cdot \Delta\vec{r}_i$. The sum of the two works is $\vec{f}_{ij} \cdot \Delta\vec{r}_{ij}$. This dot product will be zero if $\vec{f}_{ij}$ is parallel or anti-parallel to $\vec{r}_{ij}$, i.e. if the forces are central forces. The only non-central forces are those between steady currents (i.e. moving charges) in matter. The magnetic force on a moving charge is always perpendicular to the velocity of the charge and therefore does no work [this part of the argument does not require that the system be a rigid body].

CONCLUSION: the Work-Energy Theorem applies to a rigid body with only external forces contributing to the work.

Appendix C

Appendix C

C.1 Proof That Force Is A Vector

It is frequently asserted that it is an experimental fact that force is a vector. Indeed this is true since if $\vec{F} = m\vec{a}$, then $\vec{F}$ must be a vector since $\vec{a}$ is definitely a vector. Recall that in chapter 2 we defined $\vec{F}$ independently of motion, the unit of force being the push or pull exerted by some standard object ("the mouse"). A team of n mice pulling in a given direction was represented by an arrow of length n pointing in that direction (see Fig.2.1 which we reproduce here). If we introduce x and y axes making angles of θ and $90° - \theta$ with the direction in which the n mice are pulling, do we need to do an experiment to demonstrate that the team of n mice is equivalent to a pair of teams: $n \cos\theta$ mice pulling along the x-axis and $n \sin\theta$ mice pulling along the y-axis?

I claim we can prove this by pure thought.

Suppose $0 \leq \theta \leq 90°$, i.e. the team are pulling in the first quadrant. I regard it as clear that if appropriate numbers (n_1, n_2) of mice pull in the negative directions along the x- and y-axes, the tug-of-war will be a tie. If one requires a "proof" of this, it is clear that as the ratio n_1/n_2 is varied the direction of the resultant pull due to the teams on the axes will vary, and will be in the opposite direction from the pull of the n-team when the ratio has the "right" value. Then, by multiplying n_1 and n_2 by the same factor, one can adjust the magnitude of the resultant pull so as to cancel that of the n-team.

If we now let the teams of n_1 and n_2 pull in the positive directions along the x and y-axes, we can say that the pair of teams is equivalent to the

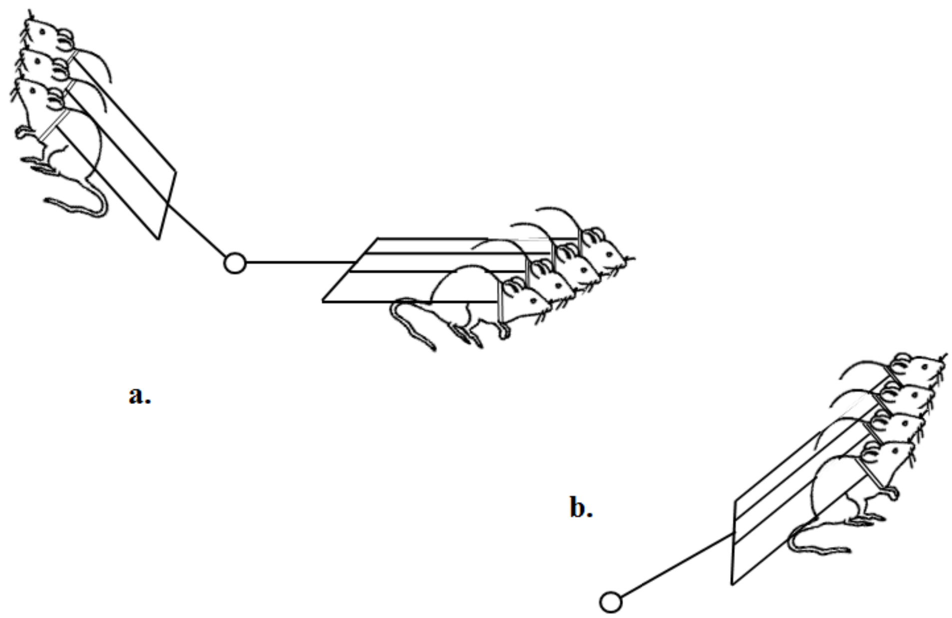


Figure C.1: Two teams of mice are attached to the same point on the same body (figure **a.** above). One team consists of N_1 mice pulling in the same direction represented by the vector $\vec{N}_1$ and the other team consists of N_2 mice pulling in the same direction represented by the vector $\vec{N}_2$. Is it obvious that the two teams of mice are equivalent to a single team (figure **b.**), where the direction of the single team is the direction of the vector $\vec{N}_1 + \vec{N}_2$ and the number of mice in the single team is the magnitude of the vector $\vec{N}_1 + \vec{N}_2$?

original team of n mice. Furthermore,

$$n_1 = n\ f(\theta) \quad \text{and} \quad n_2 = n\ f(\pi/2 - \theta) \tag{C.1}$$

The fact that n_1 and n_2 are proportional to n follows from the fact that if a three-way tug-of-war is a tie and you multiply all the teams by the same factor, it is still a tie. Since all directions in space are equivalent, the same function of the angle between the n-team and an axis must enter into the formulas for n_1 and n_2. Note that we use radian measure for later convenience.

The important constraint on the function $f(\theta)$ is consistency. Suppose we use a set of axes (x and y) and you use another set of axes (x' and y') with the same origin as ours, but your axes are rotated clockwise (through

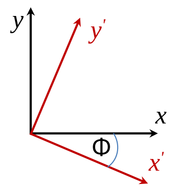


Figure C.2: Two sets of axes, one rotated with the respect to the other.

an angle ϕ) with respect to our axes (see Fig. C.2). The n-team could be replaced by $n_1 = n\ f(\theta)$ mice on our x-axis and $n_2 = n\ f(\pi/2 - \theta)$ mice on our y-axis, or equally well by $n'_1 = n\ f(\theta + \phi)$ mice on your x' axis and $n'_2 = n\ f(\pi/2 - \theta - \phi)$ mice on your y' axis. But the mice on our axes could be replaced by appropriate numbers of mice on your axes (or vice versa). The $n\ f(\theta)$ mice on our x-axis can be replaced by $n\ f(\theta)f(\phi)$ mice on your x'-axis and $n\ f(\theta)f(\pi/2 - \phi)$ on your y' axis. Similarly, the $n\ f(\pi/2 - \theta)$ mice on our y-axis can be replaced by $n\ f(\pi/2 - \theta)f(\pi/2 + \phi)$ mice on your x'-axis and $n\ f(\pi/2 - \theta)f(-\phi)$ mice on your y'-axis. The number of mice on your x'-axis should be the same whether the replacement is made in one stage or two stages, i.e.

$$f(\theta + \phi) = f(\theta)f(\phi) + f(\pi/2 - \theta)f(\pi/2 + \phi) \tag{C.2}$$

The corresponding statement for the y'-axis adds no new information.

We know some additional facts about $f(\theta)$: $f(0) = 1$, $f(\pi/2) = 0$. Furthermore, by comparing the situations when the n-team is in the upper right quadrant of the clock, and when the n-team is in the lower right quadrant, we see that $f(\theta) = f(-\theta)$ and $f(\pi/2 - \theta) = -f(\pi/2 + \theta)$.

We differentiate eqn.(C.2) twice with respect to ϕ and then set ϕ equal to zero. We use the symbol $''$ to denote the second derivative. Since $f(\pi/2 - \theta) = -f(\pi/2 + \theta)$ it follows that $f''(\pi/2) = 0$. We obtain

$$f''(\theta) = f''(0)f(\theta) \tag{C.3}$$

This differential equation is familiar to us in the context of our discussion of harmonic oscillations. If $f''(0) > 0$ (in that case let $f''(0) = a^2$), then the most general solution of eqn.(C.3) is $f(\theta) = A\ \exp(a\theta) + B\ \exp(-a\theta)$. In order that $f(\theta) = f(-\theta)$ we must have $A = B$, and in order that $f(0) = 1$ we must have $A = B = 1/2$. But then it will be impossible to satisfy $f(\pi/2) = 0$.

Therefore we conclude that $f''(0) < 0$ and let $f''(0) = -b^2$. The most general solution of eqn.(C.3) is then $f(\theta) = A\ \cos b\theta + B\ \sin b\theta$. In order that $f(\theta) = -f(\theta)$ and $f(0) = 1$ we must have $A = 1$ and $B = 0$. In order that $f(\pi/2) = 0$ we must have $b = 1, 3, 5, ...$ The odd integers greater than 1 may be eliminated by the obvious (surely!) requirement that $f(\theta) \geq 0$ for $0 \leq \theta \leq \pi/2$. Thus $f(\theta) = \cos\theta$, which proves that force has all the characteristics of a vector. In particular, if two teams (each represented by an arrow) are pulling on the same point, they are equivalent to a single team represented by the arrow which is the vector sum of the two arrows.

Appendix D

Appendix D

D.1 Equivalence of Acceleration of Axes and a Fictional Gravitational Force

We provide here a proof for the addition of a fictional gravitational force to explain the motion of a particle in a non-inertial frame when the acceleration of the frame can be determined relative to an inertial frame.

PROOF: If the axes of the inertial frame are $\hat{\mathbf{i}}\hat{\mathbf{j}}\hat{\mathbf{k}}$ and axes attached to the (accelerating) box are $\hat{\mathbf{i}}'\hat{\mathbf{j}}'\hat{\mathbf{k}}'$, and the primed axes are non-rotating with respect to the unprimed axes and have acceleration $\vec{A}$ relative to the unprimed axes, then a particle which has acceleration $\vec{a}'$ relative to the primed axes has acceleration $\vec{a}' + \vec{A}$ relative to the inertial frame. The equation of motion of the particle is $\vec{F} = m(\vec{a}' + \vec{A})$, where $\vec{F}$ is the total force acting on the particle. We can rewrite this as $\vec{F}' = m\vec{a}'$, where $\vec{F}'$ is the sum of the real force $\vec{F}$ and the fictional force $-m\vec{A}$. Q.E.D.

Appendix E

Appendix E

E.1 Developing Your Problem-Solving Skills: Helpful(?) Suggestions

A Dialogue: S = student   P = professor or instructor

S: I understand the principles but have trouble with many of the problems. Is there some systematic method, something like a computer program or a set of rules, for approaching a problem and solving it without going up a bunch of blind alleys? When you solve a problem on the blackboard, you proceed in a straight line from the problem to the solution. At each stage you seem to know which principle is relevant, and how to turn that principle into a useful equation. Do I have to solve a thousand problems and make several thousand mistakes before I become pretty good at this game?

P: That would definitely be a good idea if you have enough time and patience, but you still won't become proficient unless you *learn from your mistakes.* Roughly speaking (and omitting discussion of embarrassments like arithmetic errors and not knowing the difference between the sine and the cosine), there are two kinds of errors: errors of *commission* and errors of *omission.* A catalogue, even if edited to include only interesting (i.e. instructive) errors, would be quite voluminous.

Some examples of errors of commission would be to write $\vec{F} = m\vec{a}$ in a non-inertial (illegal) frame (e.g. to measure $\vec{a}$ with respect to axes attached to a rotating merry-go-round), or, when describing the motion of a pendulum, to use a kinematic formula which is applicable only to the motion of an object moving with constant acceleration.

An error of omission would be to omit a useful force or torque equation, or to omit important kinematic information, such as the relation between the velocities of various parts of a pulley system. In either case the number of unknown quantities will exceed the number of equations.

S: Well, you haven't really told me how to improve my proficiency, except by watching your elegant solutions on the blackboard.

P: Be that as it may, you have taught me something about the quality of my teaching. Maybe I should give back my teaching prize, which was probably based on my avuncular charm plus my skill in quickly solving freshman mechanics problems. Evidently, I haven't adequately shown you how I organize my thinking about a problem before I write down any equations. Of course, experience with similar problems is helpful, but I think there is something here which can be taught.

Before you write down any equations, you should understand how many unknown quantities there are in a problem, and should have a clear program for constructing a set of equations, based on Newton's Laws and/or kinematic constraints, which are sufficient to determine the values of those quantities. It may be useful to write down a concise list of the steps of the program, or, with experience , you can hold the program in your head. Of course, you may have found several distinct programs, and, if time permits, you should execute each of them. If they don't all yield the same answers, it is important to identify the error(s). Learning will surely occur.

S: It seems you've done most of the talking.

P: That's my specialty. But you've got to do most of the thinking.

Appendix

PREFACE TO SOLUTIONS MANUAL

This instructor's manual has complete solutions to problems for the open-source textbook "Classical Mechanics: A Critical Introduction" by Professor Michael Cohen. Students are strongly urged to solve the problems before looking at the solutions.

The responsibility for errors rests with me and I would appreciate being notified of any errors found in the solutions manual.

Sincerely,

Larry Gladney
Department of Physics and Astronomy
University of Pennsylvania
Philadelphia, PA 19104-6396
email: gladney@physics.upenn.edu

Appendix 1

KINEMATICS

1.1 Kinematics Problems Solutions

1.1.1 One-Dimensional Motion

1.1. (a) We find the area under the velocity vs. time graph to get the distance traveled (i.e. the displacement). From Fig.Soln:1.1 we can easily divide the area under the velocity vs. time plot into 3 areas. Adding the areas gives the answer.

$$\begin{aligned}
\text{area 1} &= \tfrac{1}{2}(35 \text{ m/s} - 10 \text{ m/s})(10 \text{ s} - 0) = 125 \text{ m} \\
\text{area 2} &= (10 \text{ m/s} - 0)(20 \text{ s} - 0) = 200 \text{ m} \\
\text{area 3} &= (10 \text{ m/s} - 0)(40 \text{ s} - 20 \text{ s}) = 200 \text{ m}
\end{aligned}$$

$$\text{total distance from } t = 0 \text{ to } t = 40 = 525 \text{ m}$$

(b) The acceleration is just the slope of the velocity vs. time graph, so

$$a = \frac{v(t = 60) - v(t = 40)}{60 \text{ s} - 40 \text{ s}} = -0.5 \text{ m/s}^2$$

(c) We need the total displacement of the car at $t = 60$ seconds.

$$\text{area t} = 40 \rightarrow 60 = \frac{1}{2}(60 \text{ s} - 40 \text{ s})(10 \text{ m/s} - 0) = 100 \text{ m}$$

The total displacement is therefore 525 + 100 = 625 m and the average velocity is

$$v_{\text{avg}} = \frac{x(t=60) - x(t=0)}{60\text{ s} - 0} = 10.4\text{ m/s}$$

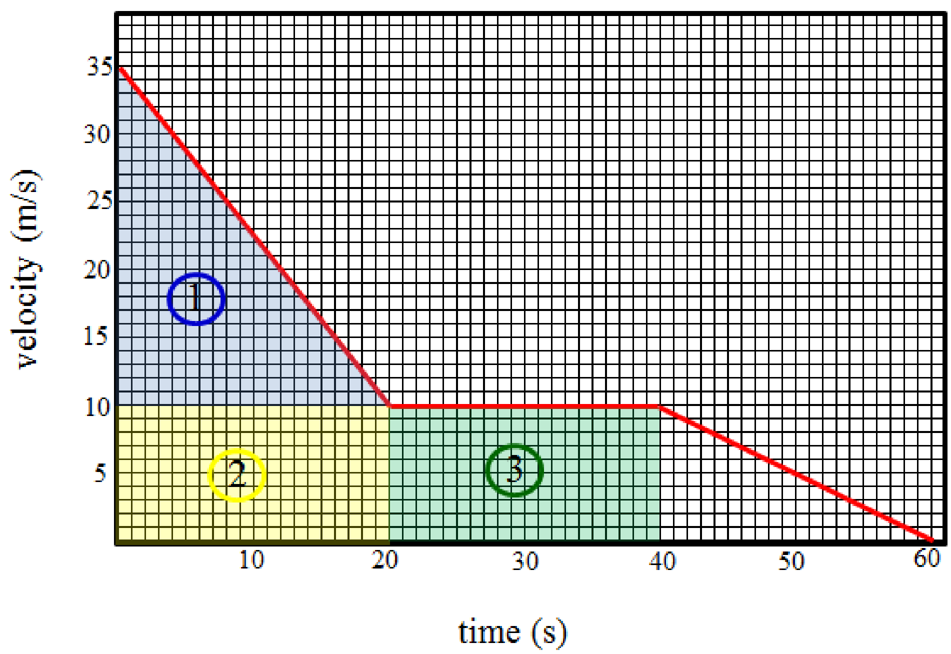


Figure Soln:1.1: Measuring the displacement of a car from the velocity vs. time graph.

1.2. (a) We start by describing the positions of the antelope and cheetah for the initial time and the time when the cheetah catches the antelope (the "final" time). When first beginning problem-solving in physics it can help to have a table listing the known and unknown information. All units are meters, m/s, or m/s^2 as appropriate. Since we want these units, we need to do some conversions.

$$\begin{aligned} 101\text{ km/h} &= \left(1.01\times10^5\text{ m/h}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 28.1\text{ m/s} \\ 88\text{ km/h} &= \left(8.80\times10^4\text{ m/h}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 24.4\text{ m/s} \end{aligned}$$

	cheetah	antelope
initial position	$x_{0\mathrm{c}} = 0.$	$x_{0\mathrm{a}} = 50.0$
final position	$x_{\mathrm{c}} = ?$	$x_{\mathrm{a}} = x_{\mathrm{c}}$
initial velocity	$v_{0\mathrm{c}} = 28.1$	$v_{0\mathrm{a}} = 24.4$
final velocity	$v_{\mathrm{c}} = 28.1$	$v_{\mathrm{a}} = 24.4$
acceleration	$a_{\mathrm{c}} = 0.$	$a_{\mathrm{a}} = a_{\mathrm{c}} = 0$

The problem specifies that we get the time at which the positions of the cheetah and antelope are the same. Using eqn.(1.11c), we have

$$\begin{aligned} x_{\mathrm{c}} &= x_{\mathrm{a}} \\ x_{0\mathrm{c}} + v_{0\mathrm{c}}t &= x_{0\mathrm{a}} + v_{0\mathrm{a}}t \Rightarrow \\ t &= \frac{x_{0\mathrm{a}}}{v_{0\mathrm{c}} - v_{0\mathrm{a}}} \\ &= \frac{50.0\ \mathrm{m}}{(28.1 - 24.4)\ \mathrm{m/s}} \\ t &= 13.5\ \mathrm{s} \end{aligned}$$

The distance traveled by the cheetah is

$$x_c = v_{0\mathrm{c}}t = (28.1\ \mathrm{m/s})(13.5\ \mathrm{s}) = 379\ \mathrm{m}.$$

(b) The situation is the same as for part a. except that now the starting position of the antelope is unknown and the time of capture is $t = 20$ s. Thus,

$$\begin{aligned} x_c &= x_a \\ x_{0\mathrm{c}} + v_{0\mathrm{c}}t = v_{0\mathrm{c}}t &= x_{0\mathrm{a}} + v_{0\mathrm{a}}t \Rightarrow \\ x_{0\mathrm{a}} &= (v_{0\mathrm{c}} - v_{0\mathrm{a}})t \\ &= (28.1\ \mathrm{m/s} - 24.4\ \mathrm{m/s})(20.0\ \mathrm{s}) = 74.0\ \mathrm{m} \end{aligned}$$

1.3. (a) We do not know the height of the bottom of the window above the release point of the throw but this does not matter as the acceleration is constant and known and the time of travel across a known distance is also known. Therefore, if v_0 is the speed of the ball as it passes the bottom of the window, t is the time for the ball to cross from the bottom to the top of the window, y_0 is the position of the bottom of the window, and y_1 is the position of the top of the window, then

$$\begin{aligned}
y_1 &= y_0 + v_0 t - \tfrac{1}{2} g t^2 \Rightarrow \\
v_0 &= \frac{(y_1 - y_0) + (1/2) g t^2}{t} \\
&= \frac{(3.00\ \mathrm{m}) + (1/2)(9.80\ \mathrm{m/s^2})(0.400\ \mathrm{s})^2}{0.400\ \mathrm{s}} \\
v_0 &= 9.46\ \mathrm{m/s}
\end{aligned}$$

This gives us the position of the top of the ball's flight since the final speed is zero. So,

$$\begin{aligned}
v_f^2 = 0 &= v_0^2 - 2g(y_f - y_0) \Rightarrow \\
y_f - y_0 &= \frac{v_0^2}{2g} = \frac{(9.46\ \mathrm{m/s})^2}{2(9.80\ \mathrm{m/s^2})} = 4.57\ \mathrm{m}
\end{aligned}$$

Therefore the maximum height above the top of the window is

$$y_f - y_1 = (4.57 - 3.00)\ \mathrm{m} = 1.57\ \mathrm{m}$$

(b) The speed of the ball when it first reaches the top of the window is

$$v_1 = v_0 - gt = 9.46\ \mathrm{m/s} - (9.80\ \mathrm{m/s^2})(0.400\ \mathrm{s}) = 5.54\ \mathrm{m/s}$$

The time interval, Δt, between passing the top of the window on the way up (v_1) and on the way down (we call this v_{down}) is

$$\begin{aligned}
v_{\mathrm{down}} &= v_{\mathrm{up}} - g\Delta t \Rightarrow \\
\Delta t &= \frac{-v_1 - v_1}{-g} \\
&= \frac{-2(5.54\ \mathrm{m/s^2})}{-9.80\ \mathrm{m/s^2}} \\
\Delta t &= 1.13\ \mathrm{s}
\end{aligned}$$

1.4. We solve the problem in a reference frame which has the velocity which the elevator had when the ball was released (we assume that the ball does not hit the ceiling of the elevator). If we take t=0 at the instant when the ball was released and denote the vertical coordinate by y, then $y_{\mathrm{floor}} = \frac{1}{2} A\ t^2$ and $y_{\mathrm{ball}} = v_0 t - \frac{1}{2} g t^2$. The height of the ball above the floor is $y_{\mathrm{b}} - y_f = v_0 t - \frac{1}{2}(g + A)t^2$. This is maximum when $t = v_0/(g + A)$ and has the value $h_{\max} = v_0^2/2(g + A)$.

We see that the maximum height above the floor which the ball reaches is the same as the maximum height it would reach if it were projected upward with velocity v_0 in a non-accelerating box on a planet where the acceleration of gravity is $g + A$ (rather than g). We shall see (Example 3.2) that the upward force which the floor exerts on a person of mass m in an accelerating elevator (this is the force which a scale measures) is $m(g + A)$, which is what the scale would read if the elevator were not accelerating but were on a planet where the acceleration of gravity is $(g + A)$.

Quite generally (see Appendix D), we can show that if a box has acceleration A (and is not rotating) relative to an inertial frame (i.e. a frame in which Newton's laws hold), we can treat axes attached to the box as though they were an inertial frame, provided that when we list the forces acting on an object in the box we add to the list a fictional force $-mA$. This extra "force" has the same form as the gravitational force mg. We call it "fictional" because it is not due to the action of any identifiable piece of matter.

1.5. (a) The walkway is less than 200 m in length. If we call the length of the walkway x, then for the race to end in a tie, Miriam must run the distance $200 - x$ m at a rate of 6 m/s so that sum of time on the walkway (determined by her speed of $6 + 2 = 8$ m/s) and off the walkway equals Alison's 200 m divided by her constant speed of 7 m/s. So,

$$\frac{200\ \mathrm{m}}{7.00\ \mathrm{m/s}} = \frac{x}{8.00\ \mathrm{m/s}} + \frac{200\ \mathrm{m} - x}{6.00\ \mathrm{m/s}} \Rightarrow$$

A common factor for the numbers 6, 7, and 8 is 168 $(7 \cdot 8 \cdot 3)$ so multiply through by this factor (i.e. 168 m/s) and solve for x.

$$\begin{aligned} 24(200\ \mathrm{m}) &= 21x + 28(200\ \mathrm{m} - x) \Rightarrow \\ 7x &= (28 - 24)(200\ \mathrm{m}) \Rightarrow \\ x &= 114\ \mathrm{m} \end{aligned}$$

(b) Now we must look at the times separately for Alison and Miriam.

$$\begin{aligned} t_{\text{Miriam}} &= \frac{200 \text{ m}}{6.00 \text{ m/s}} = 33.3 \text{ s} \\ t_{\text{Alison}} &= \frac{114 \text{ m}}{(7.00 - 2.00) \text{ m/s}} + \frac{(200 - 114) \text{ m}}{7.00 \text{ m/s}} = 35.1 \text{ s} \end{aligned}$$

Miriam wins the race.

1.1.2 Two and Three Dimensional Motion

1.6. We make use of equations from the chapter.

(a) We make use of eqn.(1.15).

$$\begin{aligned} \vec{v}(t) &= \frac{d\vec{r}}{dt} = [(4t - 7)\hat{\imath} - 2t\hat{\jmath}] \text{ m/s} \Rightarrow \\ \vec{v}(t = 2) &= [\hat{\imath} - 4\hat{\jmath}] \text{ m/s} \end{aligned}$$

(b)

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = [4\hat{\imath} - 2\hat{\jmath}] \text{ m/s}^2 = \vec{a}(t = 5 \text{ s})$$

(c) The average velocity is given by eqn.(1.1).

$$\begin{aligned} v_{\text{av}} &= \frac{\vec{r}(t = 3) - \vec{r}(t = 1)}{3 - 1} \\ &= \frac{[(18 - 21) - (2 - 7)]\hat{\imath} - (9 - 1)\hat{\jmath}}{2} \text{ m/s} \\ v_{\text{av}} &= (\hat{\imath} - 4\hat{\jmath}) \text{ m/s} \end{aligned}$$

1.7. Note that the slope of the hill can be described in terms of

$$-\tan\theta_1 = \frac{y_h}{x_h}$$

with $\theta_1 = 10.0°$ and (x_h, y_h) being coordinates of the hill surface. We adopt the usual convention of up being the $+y$ direction so we need to have the angle θ_1 of the hill as directed *below* the horizontal. The position of the skier as a function of time, assuming the takeoff point is set as coordinate $(0, +6.00 \text{ m})$, is

$$\begin{aligned} x_s &= v_0 \cos\theta_2 t \\ y_s &= 6.00 \text{ m} + v_0 \sin\theta_2 t - \tfrac{1}{2} g t^2 \end{aligned}$$

Eliminating t from the two equations gives

$$y_s = 6.00 \text{ m} + x_s \tan\theta_2 - \frac{g}{2}\frac{x_s^2}{(v_0\cos\theta_2)^2}$$

The definition of "landing" is to have $x_s = x_h$ simultaneously with $y_s = y_h$. Let's just drop the subscripts then and define x as horizontal distance from the takeoff point. Then

$$\begin{aligned}
y = -x\tan\theta_1 &= 6.00 \text{ m} + x\tan\theta_2 - \frac{gx^2}{2(v_0\cos\theta_2)^2} \Rightarrow \\
0 &= 6.00 \text{ m} + x\left(\tan 15.0^\circ + \tan 10.0^\circ\right) - \frac{\left(9.80 \text{ m/s}^2\right)x^2}{2\left(30.0 \text{ m/s}\cdot\cos 15.0^\circ\right)^2} \\
&= 6.00 \text{ m} + x(0.44428) - \left(5.83534\times 10^{-3} \text{ m}^{-1}\right)x^2 \Rightarrow \\
x &= \frac{-0.44428 \pm \sqrt{0.197381 + 4(0.00583534)(6.00)}}{2\left(-0.00583534 \text{ m}^{-1}\right)} \\
&= 87.8 \text{ m}
\end{aligned}$$

Note that the other root from the square root gives the answer corresponding to where the trajectory would *start* on the incline if the ramp were not present.

1.8. The period is derived from the speed of a point on the outer rim and this is in turn determined by the centripetal acceleration value. If r is the radius to the outer rim and the period is τ, then

$$\begin{aligned}
a = \frac{g}{5} &= \frac{v^2}{r} \Rightarrow \\
v &= \sqrt{\frac{rg}{5}} \\
\tau &= \frac{2\pi r}{v} = 2\pi\sqrt{\frac{5r}{g}} \\
&= 2(3.141593)\sqrt{\frac{5(1000 \text{ m})}{9.81 \text{ m/s}^2}} \\
\tau &= 142 \text{ s}
\end{aligned}$$

1.9. The radius can be derived from the definition of centripetal acceleration as

$$
\begin{aligned}
|\vec{a}| &= \frac{v^2}{r} \Rightarrow \\
r &= \frac{v^2}{a} \\
&= \frac{[(3.00 \times 10^5 \text{ m/h})(1 \text{ h}/3600 \text{ s/h})]^2}{0.05(9.81 \text{ m/s}^2)} \\
r &= 1.42 \times 10^4 \text{ m} = 14.2 \text{ km}
\end{aligned}
$$

The answer looks anomously large. In reality, engineers must bank roads (just as an airplane banks in flight to make a turn) to avoid such a large radius of curvature. Banking changes the direction of g so that coffee doesn't spill, i.e. the perceived "pull" of gravity always appears to be vertically down.

1.10. We need the radius of the circular path to calculate the acceleration. Once we have this, since we know the velocity, we can use eqn.(1.17).

$$
\begin{aligned}
r &= L \sin\theta \Rightarrow \\
|\vec{a}| &= \frac{v^2}{r} \\
&= \frac{(1.21 \text{ m/s})^2}{(1.20 \text{ m}) \sin 20.0^\circ)} \\
|\vec{a}| &= 3.57 \text{ m/s}^2
\end{aligned}
$$

The direction of the acceleration is always toward the center of the circular path.

Appendix 2

NEWTON'S FIRST AND THIRD LAWS

2.1 Newton's First and Third Laws of Motion Solutions

2.1. (a) The free-body diagrams for the two blocks for this case are as follows:

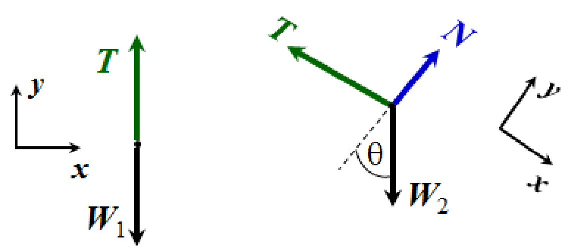


Figure Soln:2.1: Free-body diagram for Problem **2.1**a.

The first law equations for the weights are:

$$\begin{aligned}
T - W_1 &= 0 \Rightarrow T = W_1 = 100 \text{ newtons} \\
T - W_2 \sin\theta &= 0 \Rightarrow \\
W_2 &= \frac{T}{\sin\theta} = \frac{W_1}{\sin\theta} = \frac{100 \text{ newtons}}{\sin 30^\circ} = 200 \text{ newtons}
\end{aligned}$$

(b) Now, depending on the weight of W_2 it may tend to be pulled up the plane or slide down the plane. Let $W_{2,\text{min}}$ correspond to the lowest weight before W_2 slides up the incline and $W_{2,\text{ max}}$ to the maximum weight before W_2 slides down the incline. The free-body diagrams for these two cases are shown above. Note that only for these two special cases is the friction force at its maximum magnitude, $f = \mu_s N$. For the two cases the first law

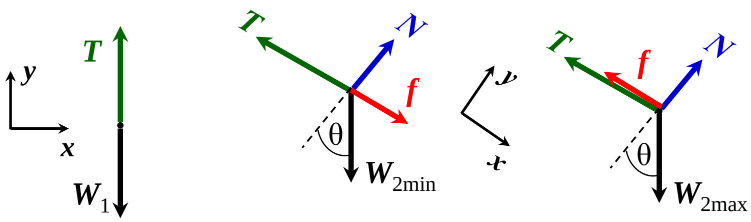


Figure Soln:2.2: Free-body diagram for Problem **2.1**b.

equation for W_1 remains the same as for part (a) and hence we continue to have $T = W_1$. For the minimum W_2 value, the first law equations for W_2 are

$$
\begin{aligned}
y: \; N - W_2 \cos\theta &= 0 \Rightarrow N = W_2 \cos\theta \\
x: \; -T + f + W_2 \sin\theta &= 0 \Rightarrow \\
W_1 - \mu_s N - W_{2,\text{ min}} \sin\theta &= 0 \Rightarrow \\
W_1 &= W_{2,\text{ min}}(\sin\theta + \mu_s \cos\theta) \Rightarrow \\
W_{2,\text{ min}} &= \frac{W_1}{\sin\theta + \mu_s \cos\theta} \\
&= \frac{100 \text{ newtons}}{\sin 30^\circ + 0.4 \cos 30^\circ} \\
W_{2,\text{ min}} &= 118 \text{ newtons}
\end{aligned}
$$

For the maximum W_2, the normal force expression is unchanged, but now the net force along the incline is defined by

$$
\begin{aligned}
W_{2,\text{ max}} \sin\theta - f - T &= 0 \rightarrow \\
W_{2,\text{ max}} \sin\theta - \mu_s W_2 \cos\theta - W_1 &= 0 \rightarrow \\
W_{2,\text{ max}} &= \frac{W_1}{\sin\theta - \mu_s \cos\theta} \\
&= \frac{100 \text{ newtons}}{\sin 30^\circ - 0.4 \cos 30^\circ} \\
W_{2,\text{ max}} &= 651 \text{ newtons}
\end{aligned}
$$

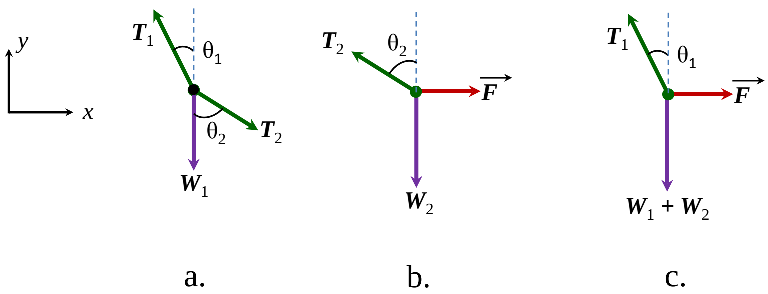


Figure Soln:2.3: Free-body diagrams for Problem **2.2**.

2.2. We can use free-body diagrams (a and b) shown in the figure above to look at the horizontal and vertical forces acting. This will lead to 4 equations (one in x and one in y for each of the two weights) but 4 unknowns (the tensions T_1 and T_2 and the two angles). Since we are only interested in the angles, it is actually easier to consider free-body diagram c. This has a system including both weights and hence the gravitational force acting is $W_1 + W_2$ in magnitude, $\vec{T_2}$ is an internal force and hence does not get shown, and the only external forces besides the weights are $\vec{T_1}$ and $\vec{F}$. Hence our force equations are

$$\begin{aligned} x:\ F - T_1 \sin\theta_1 &= 0 \Rightarrow T_1 \sin\theta_1 = F \\ y:\ T_1 \cos\theta_1 - W_1 - W_2 &= 0 \Rightarrow T_1 \cos\theta_1 = W_1 + W_2 \end{aligned}$$

Divide the first equation by the second to get

$$\begin{aligned} \frac{T_1 \sin\theta_1}{T_1 \cos\theta_1} = \tan\theta_1 &= \frac{F}{W_1 + W_2} \Rightarrow \\ \theta_1 &= \tan^{-1}\frac{F}{W_1 + W_2} \end{aligned}$$

To find θ_2 consider free-body diagram b.

$$\begin{aligned} x:\ F - T_2 \sin\theta_2 &= 0 \Rightarrow T_2 \sin\theta_2 = F \\ y:\ T_2 \cos\theta_2 - W_2 &= 0 \Rightarrow T_2 \cos\theta_2 = W_2 \end{aligned}$$

Again divide the first equation by the first to get

$$\begin{aligned} \frac{T_2 \sin\theta_2}{T_2 \cos\theta_2} = \tan\theta_2 &= \frac{F}{W_2} \Rightarrow \\ \theta_2 &= \tan^{-1}\frac{F}{W_2} \end{aligned}$$

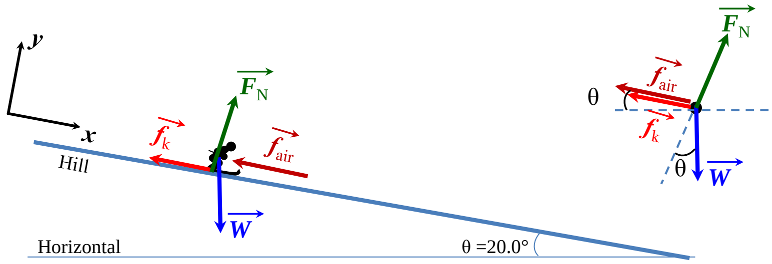


Figure Soln:2.4: Free-body diagram for Problem **2.3**.

2.3. The free-body diagram is shown in the diagram above. The force equations are below. Note that since the velocity is constant, the net force acting on the skier is zero.

$$\begin{aligned}
x: \ W\sin\theta - f_{\text{air}} - f_{\text{k}} &= 0 \\
f_{\text{air}} &= 0.148v^2 = W\sin\theta - \mu_k W\cos\theta \\
y: \qquad F_N - W\cos\theta &= 0 \Rightarrow \\
F_N &= W\cos\theta = (620\text{ N})\cos 20.0^\circ = 583\text{ N}
\end{aligned}$$

where $\vec{f}_{\text{air}}$ is the friction due to air drag corresponding to the terminal velocity. Now we can go back to the x equation to get

$$\begin{aligned}
0.148v^2 &= (620\text{ N})\left[\sin 20.0^\circ - (0.150)(0.940)\right] \Rightarrow \\
v &= \sqrt{\frac{125\text{ N}}{0.148\text{ N}/(\text{m/s})^2}} = 29.0\text{ m/s}
\end{aligned}$$

2.4. The force diagram is shown below for this situation. The x and y force equations are:

$$\begin{aligned}
x: \ F_1\sin\theta_1 - F_2\sin\theta_2 &= 0 \Rightarrow F_2 = F_1\frac{\sin\theta_1}{\sin\theta_2} \\
y: \ F_1\cos\theta_1 + F_2\cos\theta_2 - W &= 0 \Rightarrow \\
F_1\left[\cos\theta_1 + \sin\theta_1\cot\theta_2\right] - W &= 0
\end{aligned}$$

Replacing F_2 in the y equation with the expression from the x equation yields

$$\begin{aligned}
F_1 &= \frac{W}{\cos\theta_1 + \sin\theta_1\cot\theta_2} \Rightarrow \\
F_2 &= \frac{F_1}{\sin\theta_2/\sin\theta_1} = \frac{W}{\cos\theta_2 + \cot\theta_1\sin\theta_2}
\end{aligned}$$

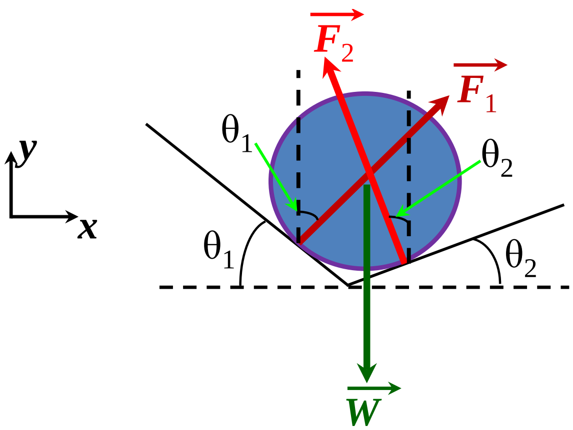


Figure Soln:2.5: Force diagram for Problem **2.4**.

Note that you get a check on the answer by seeing that you get F_2 by exchanging subscripts in the formula for F_1.

2.5. The force diagram below lays out all the angles we need. The line between the center of the $2D$ pipe and the center of either of the smaller pipes makes an angle θ with respect to the vertical where

$$\sin\theta = \frac{D}{3D/2} = \frac{2}{3} \Rightarrow \cos\theta = \sqrt{1-\sin^2\theta} = \frac{\sqrt{5}}{3}$$

From the force diagram we see in the x direction that equilibrium requires the horizontal components of $\vec{F}_1$ and $\vec{F}_2$ (the normal forces of the bottom pipes on the $2D$ pipe) to be equal, so

$$F_1 \sin\theta = F_2 \sin\theta \Rightarrow F_1 = F_2$$

Note that this also clear by symmetry. By Newton's Third Law these are also the magnitudes of the normal forces of the $2D$ pipe on either of the smaller pipes. The y components must then give

$$\begin{aligned} F_1\cos\theta + F_2\cos\theta - W_2 &= 0 \\ 2F_1\cos\theta &= W_2 \Rightarrow \\ F_1 = F_2 &= \frac{W_2}{2\cos\theta} = \frac{3W_2}{2\sqrt{5}} \end{aligned}$$

In equilibrium the vertical and horizontal net force on each of the pipes is zero. In particular, the net horizontal force on each of the

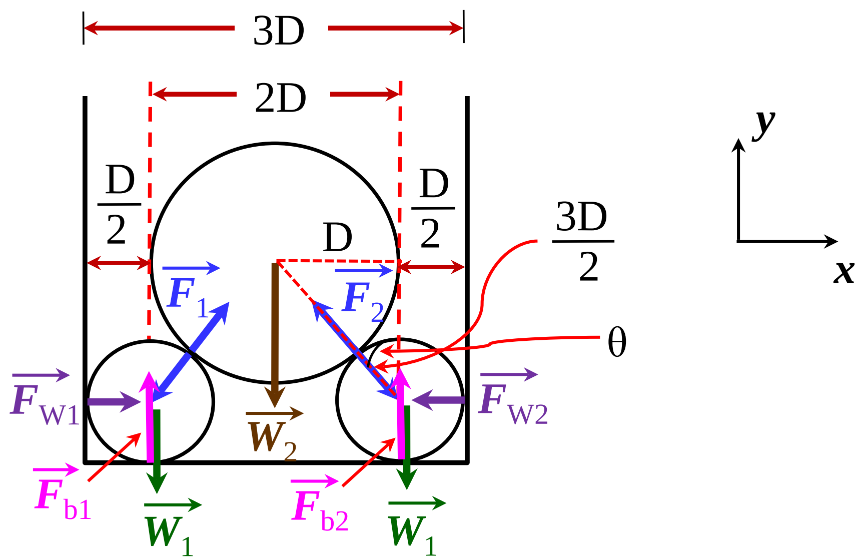


Figure Soln:2.6: Free-body diagrams for Problem **2.5**.

small pipes can be written as

$$\begin{aligned} F_{\mathrm{W1}} - F_1 \sin\theta &= 0 \Rightarrow \\ F_{\mathrm{W1}} = F_{\mathrm{W2}} &= \frac{1}{2} W_2 \tan\theta = \frac{W_2}{\sqrt{5}} \end{aligned}$$

Appendix 3

NEWTON'S SECOND LAW

3.1 Newton's Second Law of Motion Problem Solutions

3.1. (a) The free-body diagram for this case is as follows: The pulley is

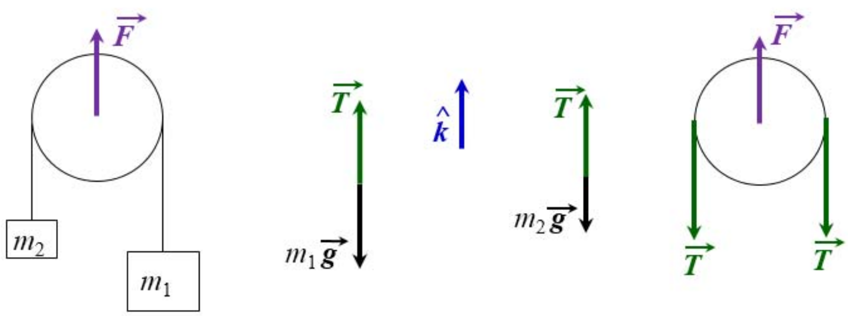


Figure Soln:3.1: Free-body diagram for Problem **3.1**a.

massless and hence the net force acting on it must be zero, so, if up is the positive direction (the direction of the unit vector $\hat{k}$ shown in the figure), then

$$F - 2T = 0 \Rightarrow T = \frac{1}{2}F = 50 \text{ newtons}$$

The acceleration of each block in an inertial frame is the vector sum of the acceleration of the center of the disk and the acceleration of that block relative to the center of the disk. Call the latter $\vec{a}_1$ for m_1 and $\vec{a}_2$ for m_2. Since the string is unstretched we must insist that $\vec{a}_1 = -\vec{a}_2$. If the acceleration of the center of the disk is $a\hat{k}$, then, writing $\vec{a}_2 = a_2\hat{k}$, we have $\vec{a}_1 = -a_2\hat{k}$. Now we can write the second law equations for the blocks.

$$\begin{aligned} T\hat{k} - m_1 g\hat{k} &= m_1(a - a_2)\hat{k} \\ T\hat{k} - m_2 g\hat{k} &= m_2(a + a_2)\hat{k} \end{aligned}$$

We can rewrite these equations as

$$\begin{aligned} \frac{T}{m_1} - g &= a - a_2 \\ \frac{T}{m_2} - g &= a + a_2 \end{aligned}$$

Summing the two equations we can solve for a and then for a_2 as follows:

$$\begin{aligned} T\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - 2g &= 2a \Rightarrow \\ a &= \frac{F_0}{4}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) - g \\ &= (25.0 \text{ newtons})\left[\frac{1}{5.00 \text{ kg}} + \frac{1}{2.00 \text{ kg}}\right] - 9.80 \text{ m/s}^2 \\ a &= 7.70 \text{ m/s}^2 \end{aligned}$$

If we instead subtract the first equation from the second, then

$$\begin{aligned} 2a_2 &= \frac{F_0}{2}\left[\frac{1}{m_2} - \frac{1}{m_1}\right] \Rightarrow \\ a_2 &= \frac{F_0}{4}\left[\frac{m_1 - m_2}{m_1 m_2}\right] \\ &= (25.0 \text{ newtons})\left[\frac{3.00 \text{ kg}}{10.0 \text{ kg}^2}\right] \\ a_2 &= 7.50 \text{ m/s} \end{aligned}$$

Hence the acceleration of m_1 is $(a - a_2)\hat{k} = (0.20\hat{k})$ m/s^2 and the acceleration of m_2 is $(a + a_2)\hat{k} = (15.2\hat{k})$ m/s^2.

(b) The tension is already derived as 50.0 newtons.

3.2. We solve the problem in an inertial frame which has the velocity which the elevator had when the ball was released (we assume that the ball does not hit the ceiling of the elevator). If we take $t = 0$ at the instant when the ball was released and denote the vertical coordinate by y, then $y_{\text{floor}} = \frac{1}{2}At^2$ and $y_{\text{ball}} = v_0 t - \frac{1}{2}gt^2$. The height of the ball above the floor is $y_{\text{ball}} - y_{\text{final}} = v_0 t - \frac{1}{2}(g + A)t^2$. This is maximum when $t = v_0/(g + A)$ and has the value

$$h_{\max} = \frac{v_0^2}{2(g + A)}.$$

We see that the maximum height above the floor which the ball reaches is the same as the maximum height it would reach if it were projected upward with velocity v_0 in a non-accelerating box on a planet where the acceleration of gravity is $\vec{g} + \vec{A}$ (rather than $\vec{g}$ where $\vec{g} = -g\hat{k}$ where $\hat{k}$ points vertically upward from the earth). We have already seen (Example 3.2) that the upward force which the floor exerts on a person of mass m in an accelerating elevator (this is the force which a scale measures) is $m(g + A)$, which is what the scale would read if the elevator were not accelerating but were on a planet where the acceleration of gravity is $(\vec{g} + \vec{A})$.

Quite generally, we can show that if a box has acceleration $\vec{A}$ (and is not rotating) relative to an inertial frame, we can treat axes attached to the box as though they were an inertial frame, provided that when we list the forces acting on an object in the box we add to the list a fictional force $-m\vec{A}$. This extra "force" has the same form as the gravitational force $m\vec{g}$. We call it "fictional" because it is not due to the action of any identifiable piece of matter.

PROOF: If the axes of the inertial frame are $\hat{\imath}\hat{\jmath}\hat{k}$ and axes attached to the box are $\hat{\imath}'\hat{\jmath}'\hat{k}'$, then a particle which has acceleration $\vec{a}'$ relative to the primed axes has acceleration $\vec{a}' + \vec{A}$ relative to the inertial frame. The equation of motion of the particle is $\vec{F} = m(\vec{a}' + \vec{A})$, where $\vec{F}$ is the total force acting on the particle. We can rewrite this as $\vec{F'} = m\vec{a}'$, where $\vec{F'}$ is the sum of the real force $\vec{F}$ and the fictional force $-m\vec{A}$. Q.E.D.

3.3. (a) If the board is not slipping then the acceleration of the boy in an inertial frame is also $\vec{a}$. The boy must therefore have force $\vec{F} = m_{\text{boy}}\vec{a}$ acting on him and he must exert a force of the same magnitude on the board. The minimum acceleration of the boy which will cause slipping is given by

$$\begin{aligned} f_{\text{s, max}} = \mu_s(m_{\text{board}} + m_{\text{boy}})g &= m_{\text{boy}} a_{\text{min}} \Rightarrow \\ a_{\text{min}} &= \mu_s g \left[1 + \frac{m_{\text{board}}}{m_{\text{boy}}}\right] \\ &= (0.200)\left(9.80 \text{ m/s}^2\right)\left[1 + \frac{30.0 \text{ kg}}{50.0 \text{ kg}}\right] \\ a_{\text{min}} &= 3.14 \text{ m/s}^2 \end{aligned}$$

(b) The boy's acceleration exceeds a_{min}. Let $\hat{\imath}$ be the direction of the boy's acceleration. The board is slipping on the ice so call its acceleration $\vec{a}_{\text{bd}} = -a_{\text{bd}}\hat{\imath}$. The acceleration of the boy is therefore $(4.00 - a_{\text{bd}})\hat{\imath}$ (in units of meters/s^2) and the force exerted by the board on the boy must be

$$F_{\text{boy},x} = m_{\text{boy}}(4.00 - a_{\text{bd}})\hat{\imath}$$

By Newton's Third Law, the net horizontal force on the board is thus

$$\begin{aligned} \vec{F}_{\text{bd}} = -m_{\text{bd}} a_{\text{bd}}\hat{\imath} &= \mu_k(m_{\text{boy}} + m_{\text{bd}})g\hat{\imath} - m_{\text{boy}}(4.00 \text{ m/s}^2 - a_{\text{bd}})\hat{\imath} \Rightarrow \\ -(m_{\text{bd}} + m_{\text{boy}})a_{\text{bd}} &= \mu_k(m_{\text{boy}} + m_{\text{bd}})g - m_{\text{boy}}(4.00 \text{ m/s}^2) \Rightarrow \\ a_{\text{bd}} &= -\frac{(0.100)(80.0 \text{ kg})(9.80 \text{ m/s}^2) - (50.0 \text{ kg})(4.00 \text{ m/s}^2)}{80.0 \text{ kg}} \\ &= 1.52 \text{ m/s}^2 \end{aligned}$$

The acceleration of the boy relative to the ice is

$$a_{\text{boy}} = (4.00 - 1.52) \text{ m/s}^2 = 2.48 \text{ m/s}^2$$

3.4. The situation initially and finally are shown in Fig.Soln:3.2. The free-body diagrams for the wedge and the block are shown in Fig.Soln:3.3. We choose the inertial frame of the earth with the x-axis along the horizontal (to the right) and the y-axis vertically upward. The 2nd law equations are therefore

$$\begin{aligned} x:\ F_{Mm}\sin\theta &= ma_x \quad \text{and} \quad -F_{Mm}\sin\theta = MA \\ y:\ F_{Mm}\cos\theta - mg &= ma_y \end{aligned}$$

where a_x and a_y are the x and y components of the acceleration of the block and A is the horizontal acceleration of the wedge. We know

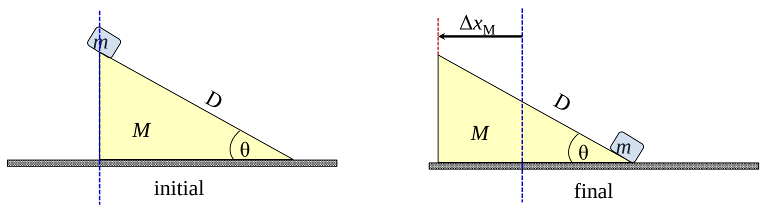


Figure Soln:3.2: Initial and final positions for the sliding wedge and block for Problem 3.4.

Figure Soln:3.3: Free-body diagrams for the sliding wedge and block for Problem 3.4.

that the table and gravity maintain no vertical acceleration for the wedge. With F_{Mm}, a_x, a_y and A all being unknown, we need one more equation to solve for everything. The remaining equation is the constraint that the block maintain contact with the wedge during the whole journey (otherwise the horizontal force on the wedge would cease). If the wedge were stationary, then the block moving down the wedge a distance s would yield $\Delta x = s\cos\theta$ and $\Delta y = s\sin\theta$. In the moving frame of the wedge free to slide, (we use primes for the coordinates in this frame), the ratio of the $\Delta x'$ and $\Delta y'$ distances must always be $\Delta y'/\Delta x' = -\tan\theta$ for the block to maintain contact with the surface of the wedge. Hence, in the inertial frame of the earth we have,

$$\begin{aligned}\Delta x = \Delta x' + \Delta x_M &= -\frac{\Delta y}{\tan\theta} + \Delta x_M \Rightarrow \\ a_x &= -\frac{a_y}{\tan\theta} + A\end{aligned}$$

where we just took the second derivative with respect to time to get the accelerations. Now we have, from our force equations

$$\begin{aligned}
a_x &= \frac{F_{Mm}\sin\theta}{m} \\
A &= -\frac{F_{Mm}\sin\theta}{M} \\
a_y &= \frac{F_{Mm}\cos\theta}{m} - g
\end{aligned}$$

Substituting into our equation coupling a_x and a_y yields

$$\begin{aligned}
a_x &= -\frac{a_y}{\tan\theta} + A \\
\frac{F_{Mm}\sin\theta}{m} &= -\frac{F_{Mm}\cos\theta - mg}{m\tan\theta} - \frac{F_{Mm}\sin\theta}{M} \Rightarrow \\
F_{Mm} &= \frac{mM\cos\theta}{M + m\sin^2\theta} g \Rightarrow \\
A &= -\frac{m\sin\theta\cos\theta}{M + m\sin^2\theta} g \\
a_x &= \frac{M\sin\theta\cos\theta}{M + m\sin^2\theta} g \\
a_y &= -\frac{(M+m)\sin^2\theta}{M + m\sin^2\theta} g
\end{aligned}$$

Call the distance traveled by the wedge in the time Δt needed for the block to reach the bottom of the wedge x_{Mf}. We determine them from the vertical distance $-D\sin\theta$ traveled by the block

$$\begin{aligned}
-D\sin\theta &= \frac{1}{2}a_y\Delta t^2 \Rightarrow \Delta t = \sqrt{\frac{-2D\sin\theta}{a_y}} \Rightarrow \\
\Delta t &= \sqrt{\frac{2D(M + m\sin^2\theta)}{(m+M)g\sin\theta}} \\
x_{Mf} &= \frac{1}{2}A\Delta t^2 = \frac{A}{2}\cdot\frac{2D\sin\theta}{|a_y|} = -\frac{m}{M+m}D\cos\theta
\end{aligned}$$

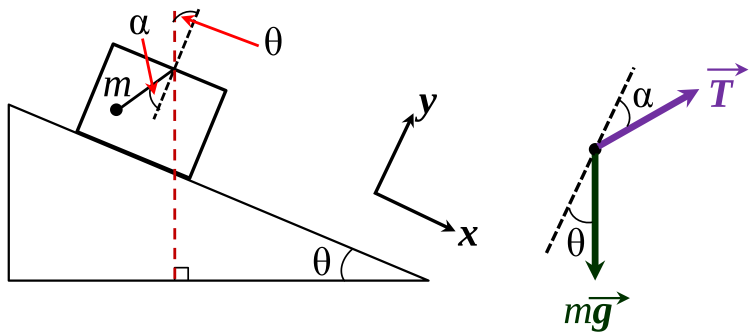


Figure Soln:3.4: Free-body diagram for Problem **3.5** part a.

3.5. (a) The free-body diagram is shown in Fig.Soln:3.4. The x and y components of the forces are parallel and perpendicular to the incline, respectively. The string keeps the acceleration along x of the pendulum bob the same as the acceleration of the rectangular box down the incline, i.e. $g \sin\theta$ since the entire system of box plus m can be considered as subject only to gravity and the normal force of the incline. For a system consisting only of m we have, along x

$$T \sin\alpha + mg \sin\theta = mg \sin\theta$$

The only solution for non-zero tension is $\alpha = 0$. Once a steady state is achieved the tension points along the normal to the ceiling so that there is zero acceleration relative to the ceiling and $mg \sin\theta$ provides the acceleration parallel to the incline so as to maintain the same acceleration for m as for the system of the box plus m.

(b) With friction acting the system of box plus m will fall with an acceleration less than $g \sin\theta$. If the system has total mass M, then

$$\begin{aligned}
y: \quad F_{\text{normal}} - Mg\cos\theta &= 0 \Rightarrow F_{\text{normal}} = Mg\cos\theta \\
x: \quad Mg\sin\theta - \mu F_{\text{normal}} &= Ma_x \\
Mg\sin\theta - \mu Mg\cos\theta &= Ma_x \Rightarrow \\
a_x &= g(\sin\theta - \mu\cos\theta)
\end{aligned}$$

The mass m as a system is now subjected to only the forces considered in part a, but the acceleration along x has to be the new value just derived. We see from this that the tension must pull *up* along x to keep the $mg\sin\theta$ component from accelerating m down along the incline faster than the rectangular box. Hence Fig.Soln:3.4 shows the string in the wrong place. It has to be *downhill* of the normal to the ceiling. Let's confirm by solving for α.

$$
\begin{aligned}
y: \quad T\cos\alpha - mg\cos\theta &= 0 \Rightarrow T = mg\frac{\cos\theta}{\cos\alpha} \\
x: \quad T\sin\alpha + mg\sin\theta &= ma_x = mg(\sin\theta - \mu\cos\theta) \\
mg\tan\alpha\cos\theta + mg\sin\theta &= mg\sin\theta - \mu mg\cos\theta \Rightarrow \\
\tan\alpha\cos\theta &= -\mu\cos\theta \Rightarrow \\
\alpha &= \tan^{-1}(-\mu) = -\tan^{-1}\mu
\end{aligned}
$$

The minus sign reflects that the string must hang downhill of the normal (which should be obvious if μ is very large and, consequently, the acceleration of the box is very small).

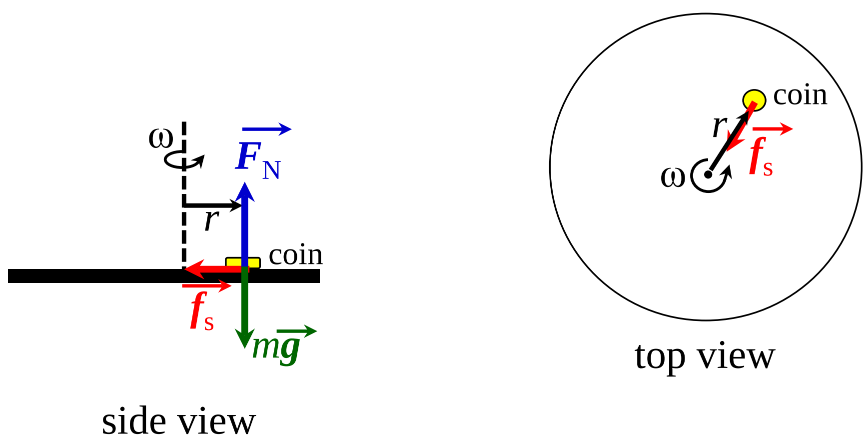


Figure Soln:3.5: Free-body diagram for Problem **3.6**.

3.6. Forces in the vertical direction (i.e. perpendicular to the turntable) must balance and hence we see immediately that $F_N = mg$ where F_N is the normal force of the turntable on the coin and mg is the magnitude of the gravitational force acting on the coin. In the horizontal (plane of the turntable surface) direction the only force acting is static friction (static since the coin maintains position with respect to the turntable). Hence static friction must satisfy the centripetal condition to keep the coin traveling in a circle at a constant speed of $v = \omega r$. The maximum radius the coin can be located from the center of rotation is therefore determined by the maximum static frictional force, hence

$$\begin{aligned} f_{\text{s, max}} = \mu_s F_N = \mu_s mg &= \frac{mv^2}{r} \Rightarrow \\ \mu_s g &= \omega^2 r \Rightarrow \\ \mu_s &= \frac{\omega^2 r}{g} = \frac{(33 \cdot 2\pi/60 \text{ s})^2 (0.15 \text{ m})}{9.8 \text{ m/s}^2} = 0.18 \end{aligned}$$

3.7. If we assume circular orbits are possible with any n-th power of the radius force law, then we can run the derivation of Kepler's Third Law in reverse. Let the period of the circular orbit be T. Then if the proportionality constant in Kepler's Third Law is k, we find

$$T^2 = kr^n \Rightarrow \frac{4\pi^2 r^2}{v^2} = kr^n$$

Since the constant k can be defined as we want, we absorb the factor of $4\pi^2$ and see that

$$\begin{aligned} kv^2 &= r^{2-n} \Rightarrow \\ k\frac{v^2}{r} &= r^{1-n} \\ F_{\text{grav}} &= k' r^{1-n} \end{aligned}$$

So the force law is proportional to the $1 - n$th power of the radius. We see that if $1 - n = -2$ then $n = 3$ as we expect.

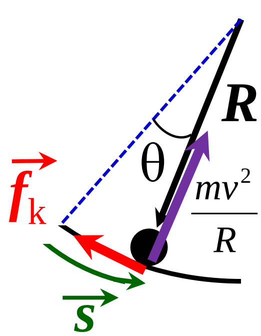


Figure Soln:3.6: Diagram for Problem **3.8**.

3.8. Fig.Soln:3.6 shows just the curve. We have idealized the sled to just a point at position θ from the start of the curve. We can also characterize the position in terms of the arc-length s from the start of the curved path. We are ignoring gravitational force but must consider the kinetic friction (f_k) and the normal force of the track on the sled that satisfies the condition

$$F_N = \frac{mv^2}{R}$$

to keep the sled following the curved track. The tangential equation of motion is therefore

$$ma = m\frac{dv}{dt} = -f_s = -\frac{\mu m v^2}{R}.$$

We want the velocity as a function of angular displacement, not time, so we make use of the chain rule of differentiation.

$$\frac{dv}{dt} = \frac{dv}{ds}\cdot\frac{ds}{dt} = \frac{dv}{ds}\cdot v$$

where v is the tangential velocity of the sled. The differential equation is easy to solve if we set the total distance of the circular track as just $R\theta$, i.e. we now let θ be the value of the inclination of the ramp leading into the curved track.

$$\begin{aligned}
\frac{dv}{dt} = v\frac{dv}{ds} &= -\frac{\mu v^2}{R} \Rightarrow \\
\int_{v_0}^{v_f}\frac{dv}{v} &= \int_0^{R\theta} -\frac{\mu}{R}\,ds \\
\ln v\Big|_{v_0}^{v_f} &= -\frac{\mu}{R}(R\theta) \Rightarrow \\
\ln\frac{v_f}{v_0} &= -\mu\theta \Rightarrow \\
v_f &= v_0 e^{-\mu\theta}
\end{aligned}$$

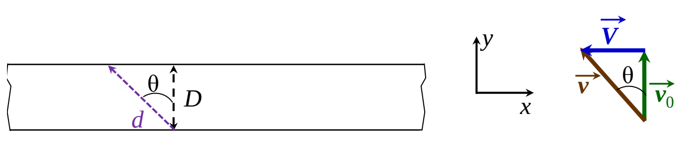


Figure Soln:3.7: Diagram for Problem 3.9.

3.9. The hint steers us to consider the inertial frame to be that where the conveyor belt is at rest, i.e. a frame moving with respect to the floor at velocity $\vec{V}$. In the floor's frame, the puck would execute a curved path as it slows its velocity relative to the conveyor belt. This makes calculating the effect of friction tricky. In the rest frame of the conveyor belt, the puck moves along a straight line since there are no forces acting perpendicular to the path of the puck in this frame (except for gravity which points downward and hence has no direct effect on the puck's motion relative to the conveyor belt). The puck goes diagonally backward in this frame and, since the frictional force is bounded, the velocity of the puck (relative to the floor) is still $\vec{v}_0$ just after the puck is on the belt.

The kinetic friction force is μmg with m being the mass of the puck so the acceleration along the path has magnitude μg. Looking at the y component of the puck's motion, the acceleration due to friction is

$$a_y = -\mu g \cos\theta = \mu g \frac{v_0}{\sqrt{v_0^2 + V^2}}$$

The minimum value of v_0 has the puck just reaching the other edge of the belt as its y velocity goes to zero, so

$$\begin{aligned} v_y^2 &= v_0^2 + 2a_y y \\ 0 &= v_0^2 - \frac{2\mu g v_0 D}{\sqrt{v_0^2 + V^2}} \Rightarrow \\ v_0^4 &= \frac{(2\mu g D)^2 v_0^2}{v_0^2 + V^2} \end{aligned}$$

We could also get to this equation by considering motion along the diagonal. In that case we have

$$\begin{aligned} 0 &= v_0^2 + V^2 - 2\mu g \frac{D}{\cos\theta} \Rightarrow \\ &= \sqrt{v_0^2 + V^2} - \frac{2\mu g D}{v_0} \Rightarrow \\ v_0^4 &= \frac{(2\mu g D)^2 v_0^2}{v_0^2 + V^2} \end{aligned}$$

Change variables to let $u = v_0^2$ and note that

$$\begin{aligned}
u^2 &= \frac{(2\mu g D)^2 u}{u+V^2} \Rightarrow \\
u(u+V^2) &= (2\mu g D)^2 \Rightarrow \\
u &= \frac{-V^2 + \sqrt{V^4 + 16(\mu g D)^2}}{2} \\
&= \frac{-(6.00 \text{ m/s})^2 + \sqrt{(1296 \text{ m}^4/\text{s}^4) + 16[(0.2)(9.8)(3)]^2 \text{ m}^4/\text{s}^2}}{2} \\
u &= 3.50 \text{ m}^2/\text{s}^2 \Rightarrow \\
v_0 &= 1.87 \text{ m/s}
\end{aligned}$$

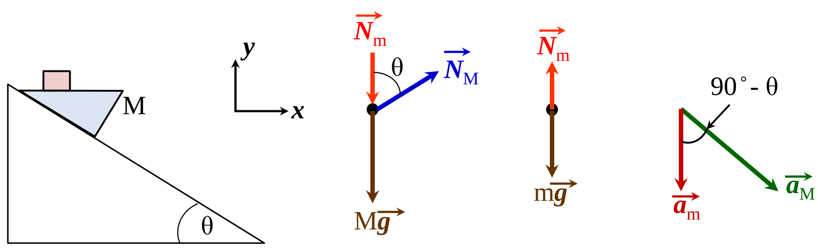


Figure Soln:3.8: Diagram for Problem 3.10.

3.10. The free-body diagrams for the wedge and the block on top of the wedge are shown in Fig.Soln:3.8. We know by looking at the forces acting that there is no horizontal motion of the block in the inertial frame of the fixed incline since there are no forces with a horizontal component acting on the block. Hence the block maintains contact with the wedge only if their vertical accelerations match while the wedge slides also horizontally relative to the incline or the block. Let the acceleration of the wedge along the incline have magnitude a_M. The direction of $\vec{a}_M$ is along the incline at angle θ below the horizontal. Let $\vec{N}_M$ be the normal force of the incline on the wedge and $\vec{N}_m$ be the normal force of the wedge on the block. The equations of motion are:

$$\begin{aligned}
m: \quad N_m - mg &= -m a_M \sin\theta \\
M,\ y: \quad N_M \cos\theta - N_m - Mg &= -M a_M \sin\theta \\
M,\ x: \quad N_M \sin\theta &= M a_M \cos\theta
\end{aligned}$$

The simultaneous solution for these three equations in the three unknowns N_M, N_m, and a_M is

$$\begin{aligned} N_M &= \frac{(m+M)Mg\cos\theta}{M+m\sin^2\theta} \\ N_m &= \frac{mMg\cos^2\theta}{M+m\sin^2\theta} \\ a_M &= \frac{(m+M)g\sin\theta}{M+m\sin^2\theta} \end{aligned}$$

The acceleration $\vec{a}_M$ parallel to the incline is the solution requested.

We can use less algebra to get the solution if we note that the acceleration of the block along the vertical direction must be

$$a_y = a_M \sin\theta$$

and its acceleration along the direction of the incline is

$$a_y \cos(90^\circ - \theta) = a_M \sin^2\theta .$$

Then looking only at the components of the force on the wedge and block which are parallel to the incline, we have

$$\begin{aligned} mg\sin\theta - N_m \sin\theta &= m a_M \sin^2\theta \\ Mg\sin\theta + N_m\sin\theta &= M a_M \end{aligned}$$

Adding the two equations gives the wedge acceleration immediately.

$$\begin{aligned} (m+M)g\sin\theta &= a_M(M+m\sin^2\theta) \Rightarrow \\ a_M &= \frac{(m+M)g\sin\theta}{M+m\sin^2\theta} \end{aligned}$$

3.11. For M to be in equilibrium we should have the tension in the string equal to the weight Mg. The conditions for minimum and maximum radius of the toy car are therefore set when the maximum static frictional force points inward and outward, respectively, with respect to the center of the circular path (see Fig.Soln:3.9). We see this because the centripetal condition requires, according to Fig.Soln:3.9,

$$\vec{T} + \vec{f}_{\rm s} = -\frac{mv^2}{r}\ \hat{\imath} \Rightarrow -\left(T \pm f_{\rm s}\right)\ \hat{\imath} = -\frac{mv^2}{r}\ \hat{\imath}$$

where $-\hat{\imath}$ is the direction toward the center of the circular path and therefore the direction of the tension. The direction of the tension is fixed along $-\hat{\imath}$, so the direction (i.e. the sign) needed for $\vec{f}_s$ is determined by the frictional force magnitude such that the vector equation above is true. In scalar form, the two conditions for the friction are determined by

$$
\begin{aligned}
T + \mu m g &= \frac{m v^2}{r_{\text{min}}} \\
T - \mu m g &= \frac{m v^2}{r_{\text{max}}} \Rightarrow
\end{aligned}
$$

Since $T = Mg$, dividing the first equation by the second yields

$$\frac{r_{\text{max}}}{r_{\text{min}}} = \frac{(M + \mu m)g}{(M - \mu m)g} = \frac{M + \mu m}{M - \mu m}$$

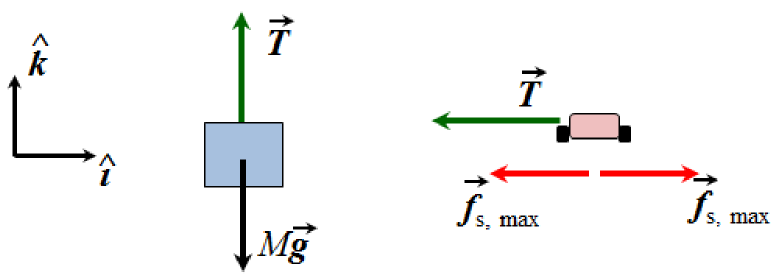


Figure Soln:3.9: Free-body diagram for Problem **3.11**.

Appendix 4

MOMENTUM

4.1 Momentum Problem Solutions

4.1. Note that the definition of center of mass location with respect to the origin is

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

For each particle n in the system

$$\begin{aligned}\left|\vec{R} - \vec{r}_n\right|^2 &= \left(\frac{\sum_i m_i \vec{r}_i}{M} - \vec{r}_n\right) \cdot \left(\frac{\sum_j m_j \vec{r}_j}{M} - \vec{r}_n\right) \\ &= \frac{1}{M^2}\left[\sum_i m_i \left(\vec{r}_i - \vec{r}_n\right)\right] \cdot \left[\sum_j m_j \left(\vec{r}_j - \vec{r}_n\right)\right]\end{aligned}$$

Now $(\vec{r}_i - \vec{r}_n) \cdot (\vec{r}_j - \vec{r}_n)$ is unchanged under all possible motions of the rigid body of which m_i, m_j, and m_n are part. Therefore the distance of the CM from all particles (index n) in the body is unchanged under all motions of the body. Hence CM is a point of the body.

4.2. The total force on the floor at any given amount of rope drop y, i.e. whenever the top of the rope has fallen a distance y below its initial release point and therefore there is a total length of rope y already on the floor, is

$$F(y) = \frac{dp}{dt} + mg$$

where dp/dt is the magnitude of the force necessary to bring an infinitesimal length of rope with mass dm to a halt if it has acquired speed v in its fall and hence, by Newton's Third Law, the rope contributes this magnitude of force to the table. m is the mass of rope already on the table and is therefore equal to λy with $\lambda = M/L$. The instantaneous velocity for any infinitesimal length of rope dm can be called v where $v^2 = 2gy$ since we consider dm to be in free-fall. Therefore, $dp = dm\ v$ and $dm = \lambda \cdot dy$ with dy being the infinitesimal length corresponding to mass dm. Thus,

$$\begin{aligned}
F(y) &= v\,\frac{dm}{dt} + \lambda y g \\
&= v \cdot \left(\frac{\lambda\, dy}{dt}\right) + \lambda y g \\
&= v^2\lambda + \lambda y g \\
F(y) &= 2gy \cdot \lambda + \lambda y g = 3\lambda g y
\end{aligned}$$

To answer part b. we just note that the maximum force occurs when y is maximum, i.e. $y = L$ so the maximum force is $3Mg$ and occurs when the last bit of rope hits the table. This makes sense because this last dm hits with the maximum speed (since it falls from the greatest height, L) and the (nearly) maximum weight of rope is already on the table.

4.3. (a) Linear momentum conservation demands

$$\vec{p}_1 = -\vec{p}_2$$

where $\vec{p}_1$ and $\vec{p}_2$ are the momenta of fragments 1 and 2, respectively. The fragments must fly off in directions that are collinear. So, the horizontal velocities can be determined from the distances traveled and the times given.

$$\begin{aligned}
v_{1x} &= \frac{120 \text{ m}}{10.0 \text{ s}} = 12.0 \text{ m/s} \\
v_{2x} &= \frac{24.0 \text{ m}}{4.00 \text{ s}} = 6.00 \text{ m/s}
\end{aligned}$$

Given that the sum of the x momenta for the fragments must add to zero, we have

$$p_{1x} = m_1 v_{1x} = p_{2x} = m_2 v_{2x} \Rightarrow \tag{E.1}$$

$$m_2 = m_1 \frac{12.0}{6.00} = 2m_1 \tag{E.2}$$

When the skyrocket reaches its highest point its velocity is zero so the vertical momenta of the fragments must also be equal in magnitude and opposite signs.

$$m_1 v_{1y} = -m_2 v_{2y} \Rightarrow v_{1y} = -2v_{2y}$$

If we use the explosion site as the origin of our coordinate system then the vertical drop of the fragments is described by

$$\begin{aligned} -h &= v_{1y}(10 \text{ s}) - \frac{9.8 \text{ m/s}^2}{2}(100 \text{ s}^2) \\ -h &= (10v_{1y} - 490) \text{ m} \end{aligned}$$

and

$$\begin{aligned} -h &= v_{2y}(4 \text{ s}) - \frac{9.8 \text{ m/s}^2}{2}(16 \text{ s}^2) \\ &= (-2v_{1y} - 78.4) \text{ m} \Rightarrow \\ 10v_{1y} - 490 &= -2v_{1y} - 78.4 \Rightarrow \\ v_{1y} &= 34.3 \text{ m/s} \Rightarrow v_{2y} = -17.2 \text{ m/s} \Rightarrow \\ h &= (4 \text{ s})(17.15 \text{ m/s}) + 78.4 \text{ m} \\ &= 147 \text{ m} \end{aligned}$$

(b) Given that fragment 1 goes up, we get the maximum height from that fragment's travel. With the ground as the origin, we have

$$\begin{aligned} 0 &= v_{1y}^2 - 2(9.8 \text{ m/s}^2)(y_{\max} - 147 \text{ m}) \Rightarrow \\ y_{\max} &= 147 \text{ m} + \frac{(34.3 \text{ m/s})^2}{19.6 \text{ m/s}^2} = 207 \text{ m} \end{aligned}$$

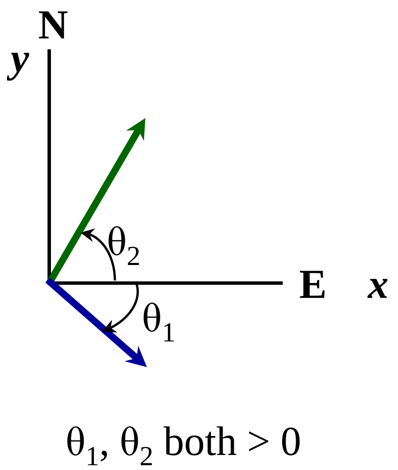


Figure Soln:4.1: Diagram for Problem 4.4.

4.4. We must have momentum conservation along the directions parallel and perpendicular to the original shell direction (i.e. east). In addition, mass is conserved. The three equations that result from these conditions are as follows.

$$\begin{aligned}
\text{mass}: \; m_1 + m_2 &= M = 3.00 \text{ kg} \\
x: \; Mv_0 &= m_1 v_1 \cos\theta_1 + m_2 v_2 \cos\theta_2 \\
y: \; 0 &= m_1 v_1 \sin\theta_1 - m_2 v_2 \sin\theta_2
\end{aligned}$$

From the y equation and mass conservation equation we get

$$v_2 = \frac{m_1}{m_2} v_1 \frac{\sin\theta_1}{\sin\theta_2}$$

Plugging into the x equation gives

$$\begin{aligned}
Mv_0 &= m_1 v_1 \cos\theta_1 + m_1 v_1 \sin\theta_1 \cot\theta_2 \Rightarrow \\
m_1 &= \frac{Mv_0}{v_1 \left[\cos\theta_1 + \sin\theta_1 \cot\theta_2\right]} \\
&= \frac{(3.00 \text{ kg})(350 \text{ m/s})}{(900 \text{ m/s}) \left[\cos 20.0^\circ + (\sin 20.0^\circ)(\cot 40.0^\circ)\right]} \\
m_1 &= 0.866 \text{ kg} \Rightarrow m_2 = 3.00 - 0.866 = 2.134 \text{ kg}
\end{aligned}$$

Therefore

$$\begin{aligned} v_2 &= \frac{m_1}{m_2} v_1 \frac{\sin\theta_1}{\sin\theta_2} \\ &= \frac{0.866 \text{ kg}}{2.134 \text{ kg}} (900 \text{ m/s}) \frac{\sin 20.0^\circ}{\sin 40.0^\circ} \\ v_2 &= 194 \text{ m/s} \end{aligned}$$

Note: Many students miss this problem because they have two equations (x and y momenta) in 3 unknowns and did not realize the constraint on the sum of the masses of the fragments constitutes the third equation!

4.5. The thrust is a force with magnitude equal to the time rate of momentum change of the exhaust gas. The time rate of change of the mass of the rocket is the time rate of mass of fuel expended as exhaust gas, so

$$\left|\frac{dM}{dt}\right| = \frac{\text{fuel burned}}{\text{burn time}} = \frac{2.300 \times 10^6 \text{ kg} - 1.310 \times 10^5 \text{ kg}}{150 \text{ s}} = 1.446{\times}10^4 \text{ kg/s}$$

The thrust magnitude, T, is therefore

$$\begin{aligned} T &= u \left|\frac{dM}{dt}\right| \Rightarrow \\ u &= \frac{T}{|dM/dt|} \\ &= \frac{3.402 \times 10^7 \text{ N}}{1.446 \times 10^4 \text{ kg/s}} \\ u &= 2.35 \times 10^3 \text{ m/s} \end{aligned}$$

We should note that the ideal rocket equation (Eqn.(4.14)) is not so accurate here since we have to consider the effect of gravity. The modification comes in Eqn.(4.13) where, instead of having momentum be conserved, we must look at the change in momentum due to an external gravitational force, i.e.

$$\begin{aligned} (Mv - Mg\ dt)\ \hat{i} &= (M + dM)(v + dv)\ \hat{i} - dM(v - u)\ \hat{i} \Rightarrow \\ Mv - Mg\ dt &= Mv + M\ dv + v\ dM + dM\ dv - v\ dM + u\ dM \Rightarrow \\ M\ dv &= -u\ dM - Mg\ dt \end{aligned}$$

where we drop the product of differentials and we note that dM is negative. Dividing through by M as in Chapter 4 and eliminating dt by noting that

$$dt = \frac{dM}{dM/dt}$$

yields

$$dv = -u\,\frac{dM}{M} - g\frac{dM}{dM/dt}$$

Integrating then gives (remember that $dM < 0$)

$$\begin{aligned}
\int_{v_0}^{v} dv' &= -u\int_{M_0}^{M}\frac{dM'}{M'} - \frac{g}{dM/dt}\int_{M_0}^{M} dM' \Rightarrow \\
v - v_0 &= -u\ln\frac{M}{M_0} - g\frac{M-M_0}{dM/dt}.
\end{aligned}$$

The ideal rocket equation modified by gravity is

$$v = v_0 - u\ \ln\left(\frac{M}{M_0}\right) - g\frac{M-M_0}{dM/dt}$$

There are two assumptions made here. The first is that 67 kilometers is a small enough altitude relative to the earth's radius that we can continue to use $g = 9.80\ \mathrm{m/s^2}$ as the gravitational acceleration. We have also assumed that at time $t = 0$

$$M_0\frac{dv}{dt} = -u\frac{dM}{dt} - M_0 g > 0 \Rightarrow -u\frac{dM}{dt} > M_0 g$$

otherwise the rocket does not initially lift off but just sits on the launch pad burning fuel until the total mass is sufficiently small enough for the thrust to lift the rocket.

The relevant numbers for this problem require us to get the mass at the end of the first stage firing. These numbers are

$$\begin{aligned}
M_0 &= 2.80\times 10^6\ \mathrm{kg} \\
M &= \text{launch mass} - \text{mass of 1st stage} \\
&= (2.8\times 10^6 - 2.3\times 10^6)\ \mathrm{kg} \\
u &= \text{exhaust velocity} = 2.35\times 10^3\ \mathrm{m/s} \\
t &= \text{time for first stage} = 150\ \mathrm{s}
\end{aligned}$$

Therefore the velocity of the rocket at 67 kilometers altitude would be

$$\begin{aligned} v &= 0 - \left(2.35 \times 10^3 \text{ m/s}\right) \ln\left(\frac{5.00 \times 10^5 \text{ kg}}{2.80 \times 10^6 \text{ kg}}\right) - (9.80 \text{ m/s}^2)(150 \text{ s}) \\ &= 2.578 \times 10^3 \text{ m/s} \end{aligned}$$

4.6. We know the velocity of the cannonball, but in a non-inertial frame. As the cannonball accelerates along the barrel of the cannon, the fact that the cannonball is accelerating means that a force is being exerted on it by the cannon and hence the cannon must have a force exerted on it by the cannonball according to Newton's third law. As the cannonball exits the cannon, it is moving at angle α with respect to the horizontal in the inertial frame (fixed with respect to the ground). The speed of the flatcar is calculable since the momentum of the system along the horizontal (x) is conserved as there are no external forces acting along x. Call the speed of the flatcar along the x-axis u just as the cannonball leaves the barrel. The appropriate vector diagram for finding the velocity of the cannonball is Fig.Soln:4.2 where $\vec{V}'$ is the final velocity of the cannonball with respect to the ground. Momentum conservation along the x-axis yields (assuming to the right is the positive x direction)

$$\begin{aligned} mV' \cos\alpha &= Mu \\ mV \cos\theta - mu &= Mu \Rightarrow \\ u &= \frac{mV \cos\theta}{m + M} \end{aligned}$$

The angle α is derived from the vector diagram by noting that $V'_y = V_y$, i.e. the vertical components are equal. The horizontal component of $\vec{V}'$ we have already calculated above. Thus,

$$\begin{aligned} V'_y &= V' \sin\alpha = V \sin\theta \\ V'_x &= V' \cos\alpha = \frac{Mu}{m} \Rightarrow \\ \frac{V'_y}{V'_x} = \tan\alpha &= \frac{m}{Mu} \cdot V \sin\theta \Rightarrow \\ &= \frac{mV \sin\theta}{M} \cdot \frac{m + M}{mV \cos\theta} \\ \tan\alpha &= \frac{m + M}{M} \tan\theta \end{aligned}$$

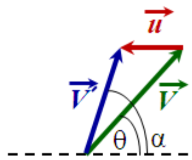


Figure Soln:4.2: Velocity vectors for Problem 4.6.

Appendix 5

WORK AND ENERGY

5.1 Work and Conservation of Energy Problem Solutions

5.1. Assume that the racket and ball meet in a one-dimensional elastic collision as conserving kinetic energy in the collision leaves more energy for the ball and hence corresponds to the highest velocity it can have. In such a case we can use the velocity exchange formula, i.e. $v_{\text{ball}} - v_{\text{racket}} = -(u_{\text{ball}} - u_{\text{racket}})$ where u indicates initial velocity and v indicates final velocity just after the collision. Using this, we see that the relative velocity of racket and ball becomes

$$v_{\text{ball}} - v_{\text{racket}} = -(-u_{\text{ball}} - u_{\text{racket}})$$

where the $-u_0$ occurs since the ball is moving opposite to the direction of the racket (which we assume is positive x direction). However the racket is held by the player and its velocity does not change (or at least not by an appreciable amount). So $v_{\text{racket}} = u_1$. Therefore

$$v_{\text{ball}} - u_{\text{racket}} = u_{\text{ball}} + u_{\text{racket}} \Rightarrow v_{\text{ball, max}} = u_{\text{ball}} + 2u_{\text{racket}}$$

Even if you didn't know the velocity exchange formula, you can visualize the collision in an inertial frame moving with the same constant velocity as the racket. In this frame the racket looks like a stationary wall and the ball approaches the wall with speed $u_{\text{ball}} + u_{\text{racket}}$ and rebounds with equal speed in the opposite direction. Relative to the ground, the velocity of the rebounding ball is therefore $u_{\text{ball}} + 2u_{\text{racket}}$.

5.2. Let u be the velocity of the block *relative* to the wedge. We use this velocity since its direction is always known relative to the wedge. For the system consisting of the block and wedge but not the table on which the wedge rests, we have (assuming the x-axisis along the horizontal with $+x$ being to the right)

$$\begin{aligned}
\sum P_x = 0 &= -MV + m(u\cos\theta - V) \Rightarrow \\
u\cos\theta &= \frac{M+m}{m}V \Rightarrow \\
u &= \frac{M+m}{m}\frac{V}{\cos\theta} \Rightarrow \\
u\sin\theta &= \frac{M+m}{m}V\tan\theta \ .
\end{aligned}$$

where V is the speed of the wedge. By use of energy conservation and the substitution of $u\cos\theta$ from the momentum equation we have

$$\begin{aligned}
KE_0 + U_0 &= KE_{\text{final}} + U_{\text{final}} \\
mgh &= \frac{1}{2}MV^2 + \frac{1}{2}m\left[(u\cos\theta - V)^2 + (u\sin\theta)^2\right] \Rightarrow \\
2mgh &= MV^2 + m\left[\left(\frac{M}{m}V + V - V\right)^2 + \left(\frac{M+m}{m}V\tan\theta\right)^2\right] \\
&= V^2\left[M + \frac{M^2}{m} + \frac{(M+m)^2}{m}\tan^2\theta\right] \Rightarrow \\
2m^2gh &= V^2\left[M(M+m) + (M+m)^2\tan^2\theta\right] \Rightarrow \\
\frac{2m^2gh}{M+m} &= V^2\left[M + (M+m)\tan^2\theta\right] \\
&= V^2\left[\frac{M\cos^2\theta + M\sin^2\theta + m\sin^2\theta}{\cos^2\theta}\right] \Rightarrow \\
V &= \sqrt{\frac{2m^2gh\cos^2\theta}{(M+m)(M+m\sin^2\theta)}}
\end{aligned}$$

5.3. (a) The jumper's maximum velocity occurs at the point where the bungee cord exerts a force which just counteracts gravity (at which point the jumper's acceleration is zero). After this point the acceleration is upward and the jumper decelerates until coming to rest momentarily and then moves upward.

$$\begin{aligned}
F_{\text{jumper}} &= mg - kx = 0 \Rightarrow \\
x &= \frac{mg}{k} \\
&= \frac{(80.0 \text{ kg})(9.80 \text{ m/s}^2)}{200. \text{ N/m}} \\
x &= 3.92 \text{ m}
\end{aligned}$$

So the maximum velocity occurs at 53.9 meters below the bridge.

(b) The maximum velocity occurs when the bungee cord is stretched by 3.92 meters, so, setting the coordinate system origin at 50.0 meters below the bridge,

$$\begin{aligned}
KE_0 + PE_0 &= KE_f + PE_f \\
0 + mg(50.0) &= \frac{1}{2}mv_{\text{max}}^2 + mg(-3.92 \text{ m}) + \frac{1}{2}k(3.92 \text{ m})^2 \Rightarrow \\
v_{\text{max}} &= \sqrt{2(9.80 \text{ m/s}^2)(53.92 \text{ m}) - \frac{(200 \text{ N/m})(3.92 \text{ m})^2}{80.0 \text{ kg}}} \\
&= 32.4 \text{ m/s}
\end{aligned}$$

(c) The maximum acceleration occurs for the maximum net force acting on the jumper. Until the 50.0 meters is reached only gravity acts and the jumper's acceleration is 9.80 m/s^2. Once the bungee cord is stretched by more than 3.92 meters, the net force is upward. The question is then whether the net force of $kx - mg > |mg|$ for any point before the velocity goes to zero since at the stop point the cord is at its maximum length for this jumper and exerting the maximum upward force. Hence we want to find the maximum stretch of the rope, x_{max}. Let $mg/k = \delta$ as a shorthand.

$$\begin{aligned}
mg(50.0 + x) &= \frac{1}{2}kx_{\text{max}}^2 \Rightarrow \\
100\delta + 2\delta x_{\text{max}} &= x_{\text{max}}^2 \Rightarrow \\
x_{\text{max}}^2 - 2\delta x_{\text{max}} - 100\delta &= 0 \Rightarrow \\
x_{\text{max}} &= \delta\left[1 + \sqrt{1 + \frac{100}{\delta}}\right]
\end{aligned}$$

The last step is just the usual quadratic equation solution simplified. The value of δ is

$$\delta = \frac{(80.0\text{ kg})(9.80\text{ m/s}^2)}{200\text{ N/m}} = 3.92\text{ m}$$

So

$$x_{\text{max}} = (3.92\text{ m})\left[1+\sqrt{1+\frac{100\text{ m}}{3.92\text{ m}}}\right] = 24.1\text{ m}$$

This indeed leads to an acceleration upward which is greater than 9.80 m/s^2 in magnitude.

$$kx_{\text{max}}-mg = (200\text{ N/m})(24.1\text{ m})-(80.0\text{ kg})(9.80\text{ m/s}^2) = 4036\text{ N} > m|g|.$$

kx is monotonically increasing so the maximum displacement always corresponds to the maximum upward force and hence the largest acceleration.

(d) The value of the maximum acceleration is derived from

$$\begin{aligned} F &= kx_{\text{max}} - mg \\ &= mg\left[1+\sqrt{1+\frac{100}{\delta}}-1\right] \Rightarrow \\ a_{\text{max}} &= \left(9.8\text{ m/s}^2\right)\left[\sqrt{1+\frac{100\text{ m}}{3.92\text{ m}}}\right] = 50.4\text{ m/s}^2 \end{aligned}$$

Being subjected to $5g$'s is an exhilarating experience.

(e) The maximum distance from the bridge is 50.0 m plus x_{max}, so 74.1 m. You need a tall bridge!

5.4. The Hooke's Law constant of a rope whose unstretched length is 1.0 (in arbitrary length units) is K. This means that if equal and opposite forces of magnitude F are applied to both ends, the rope will stretch to length $1 + F/K$, i.e. $F = K \cdot x$ (change in length). Now suppose we have a piece of the same kind of rope with unstretched length L, where L is an integer number of unit lengths. We can put marks on the rope, conceptually dividing it into L (note that L is an integer with no units) equal length segments. If equal and opposite forces of magnitude F are applied to the two ends, every segment will be in mechanical equilibrium and will have equal and opposite forces F

pulling on its two ends. Thus the increase in length of each segment is F/K and the increase in length of the rope is LF/K. The Hooke's Law constant k of the rope is defined by $F = k \cdot x$ (with x being the change in length). Thus $k = F/(LF/K) = K/$L.

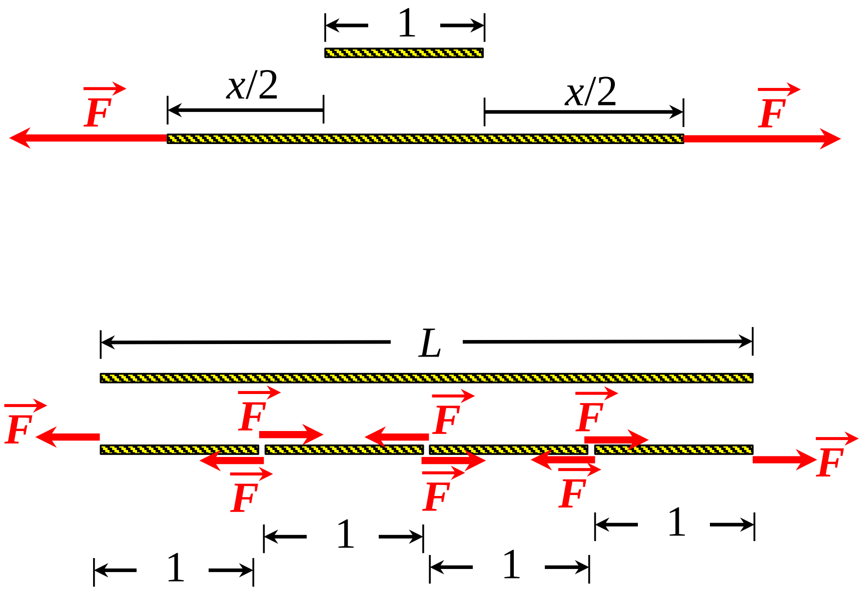


Figure Soln:5.1: Forces acting on sections of a rope under tension.

If you are a quibbler and want to see how to extend the argument to non-integer values of L, divide the unstretched rope conceptually into very many very short pieces, each of length Δ. With negligible error you can assume that $1/\Delta$ and L/Δ are integers. The number of segments in a piece of rope of unit length is $1/\Delta$. If the Hooke's Law constant for each little segment is α, then the preceding argument shows that the Hooke's Law constant for a piece of rope of (unstretched) unit length is $\alpha/(1/\Delta) = \alpha\Delta$. Thus $K = \alpha\Delta$. The number of segments in a rope of length L is L$/\Delta$ and thus the Hooke's Law constant for the rope is $k = \alpha/(\mathrm{L}/\Delta) = \alpha\Delta/\mathrm{L} = K/\mathrm{L}$ - Q.E.D.

A bungee jumper of mass M has a number of different ropes, all cut from the same big spool but of different lengths L. She chooses a rope, ties one end to a rail on the bridge and the other end to her harness, and jumps. We want to show that the maximum subsequent tension is the same for all of the ropes, i.e. that $T_{\max}$ does not depend on L.

Clearly the maximum tension occurs when the rope is maximally stretched, i.e. when the jumper is at her low point and has zero velocity. Let the length of the rope at this point be $L + x$. We take the jumper's gravitational potential energy as zero at the bridge and thus it equals $-Mg(L+x)$ at the low point. Because the rope's mass is much less than the jumper's, we neglect its gravitational P.E. The P.E. of the unstretched rope is zero and the P.E. of the stretched rope is $\frac{1}{2}kx^2$. The jumper's K.E. is zero on the bridge and zero at the low point.

Thus $0 = -Mg(L+x) + \frac{1}{2}kx^2$. We could solve this quadratic equation for its positive root and then calculate the maximum tension $T_{\max} = kx = (K/\mathrm{L})x$. If the statement of the problem is correct, then $T_{\max}$ does not depend on L, i.e. x/L does not depend on L. We can see this even without solving the quadratic equation. Let $y = x/\mathrm{L}$. Then $x = y\mathrm{L}$ and our equation is

$$\begin{aligned} 0 &= -Mg(L + y\mathrm{L}) + \tfrac{1}{2}(K/\mathrm{L})(y\mathrm{L})^2 \Rightarrow \\ &= -Mg(1+y) + \tfrac{1}{2}Ky^2 \end{aligned}$$

Note that L has disappeared, so its clear that y and $T_{\max}$ do not depend on L. This fact is known to many jumpers and rock climbers.

5.5. We seek to prove that if linear momentum is conserved in one frame, i.e. for a system of n particles

$$\sum_{j=1}^{n} m_j \vec{v}_{ji} = \sum_{j=1}^{n} m_j \vec{v}_{jf}$$

where the i and f subscripts refer to the velocities before and after the collision, then it is true in all other inertial frames. We can write down the initial and final momenta in any other inertial frame moving with velocity $\vec{V}$ with respect to the first inertial frame as follows:

$$\begin{aligned} \sum_{j=1}^{n} m_j \vec{v}_{ji} - \vec{V}\sum_{j=1}^{n} m_j &= \sum_{j=1}^{n} m_j \vec{v}_{jf} - \vec{V}\sum_{j=1}^{n} m_j \\ \sum_{j=1}^{n} m_j \left(\vec{v}_{ji} - \vec{V}\right) &= \sum_{j=1}^{n} m_j \left(\vec{v}_{jf} - \vec{V}\right) \\ \sum_{j=1}^{n} m_j \vec{v}\,'_{ji} &= \sum_{j=1}^{n} m_j \vec{v}\,'_{jf} \end{aligned}$$

Hence momentum is also conserved in the new inertial frame. We now turn to the argument for kinetic energy associated with motion. Kinetic energy conservation is

$$\sum_{j=1}^{n} \frac{1}{2} m_j v_{ji}^2 = \sum_{j=1}^{n} \frac{1}{2} m_j v_{jf}^2$$

For a different inertial frame A′, as stated above,

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{2} m_j {v'_{ji}}^2 &= \sum_{j=1}^{n} \frac{1}{2} m_j \left(\vec{v}_{ji} - \vec{V}\right)^2 \\
&= \sum_{j=1}^{n} \frac{1}{2} m_j \left[v_{ji}^2 - 2\vec{V} \cdot \vec{v}_{ji} + V^2\right] \Rightarrow \\
&= \sum_{j=1}^{n} \frac{1}{2} m_j \left[v_{jf}^2 - 2V \cdot \vec{v}_{ji} + V^2\right]
\end{aligned}$$

where the last line is from our previous expression for kinetic energy conservation in the original inertial frame. Momentum conservation in the original inertial frame means

$$\begin{aligned}
\sum_{j=1}^{n} m_j \vec{v}_{ji} &= \sum_{j=1}^{n} m_j \vec{v}_{jf} \Rightarrow \\
\vec{V} \cdot \sum_{j=1}^{n} m_j \vec{v}_{ji} &= \vec{V} \cdot \sum_{j=1}^{n} m_j \vec{v}_{jf} \Rightarrow \\
2\vec{V} \cdot \sum_{j=1}^{n} \frac{1}{2} m_j \vec{v}_{ji} &= 2\vec{V} \cdot \sum_{j=1}^{n} \frac{1}{2} m_j \vec{v}_{jf}
\end{aligned}$$

Therefore, we can rewrite our previous initial kinetic energy for the new inertial frame as

$$\begin{aligned}
\sum_{j=1}^{n} \frac{1}{2} m_j {v'_{ji}}^2 &= \sum_{j=1}^{n} \frac{1}{2} m_j \left[v_{jf}^2 - 2\vec{V} \cdot \vec{v}_{jf} + V^2\right] \\
&= \sum_{j=1}^{n} \frac{1}{2} m_j \left(\vec{v}_{jf} - \vec{V}\right)^2 \\
\sum_{j=1}^{n} \frac{1}{2} m_j {v'_{ji}}^2 &= \sum_{j=1}^{n} \frac{1}{2} m_j {v'_{jf}}^2
\end{aligned}$$

so kinetic energy is conserved in this frame.

5.6. The collision of two particles produces two outgoing momentum vectors. Those two vectors define a plane hence we have a two-dimensional problem. Momentum conservation for identical mass particles with one at rest requires that

$$\begin{aligned} m\vec{v}_{1i} &= m\vec{v}_{1f} + m\vec{v}_{2f} \Rightarrow \\ \vec{v}_{1i}^2 &= \vec{v}_{1f}^2 + \vec{v}_{2f}^2 + 2\vec{v}_{1f} \cdot \vec{v}_{2f} \end{aligned}$$

but kinetic energy conservation requires that

$$\frac{1}{2}mv_{1i}^2 = \frac{1}{2}mv_{1f}^2 + \frac{1}{2}mv_{2f}^2 \Rightarrow v_{1i}^2 = v_{1f}^2 + v_{2f}^2.$$

To make both momentum and kinetic energy conservation equations true requires

$$2\vec{v}_{1f} \cdot \vec{v}_{2f} = 0$$

which in turn requires the final momentum vectors to be orthogonal, i.e. particles moving perpendicular to one another.

5.7. The collision exerts an impulsive force on the platform which, due to inertia, brings the block to the same velocity as the platform before the spring can act to exert any more force than it exerts at equilibrium with just the platform alone. We measure the maximum compression of the spring from the equilibrium position of the platform and spring before the collision. The speed of the block just before the collision comes from mechanical energy conservation, so, if $m = 0.500$ kg and $y_0 = 0.600$ m, then

$$\begin{aligned} \tfrac{1}{2}mv^2 &= mgy_0 \Rightarrow \\ v &= \sqrt{2gy_0} = \sqrt{2(9.80\text{ m/s}^2)(0.600\text{ m})} \\ &= 3.43\text{ m/s} \end{aligned}$$

We use linear momentum conservation to find the speed of the platform and block immediately after the collision. Let M be the mass of the platform, v the initial speed of the block just before the collision and V the speed of the block and platform just after the collision. Then,

$$\begin{aligned} mv &= (m+M)V \Rightarrow \\ V &= \frac{mv}{m+M} = \frac{(0.500\text{ kg})(3.43\text{ m/s})}{1.50\text{ kg}} \\ &= 1.14\text{ m/s} \end{aligned}$$

After the collision we can use energy conservation. Let y be the distance the spring compresses from the equilibrium position (which is to say the equilibrium position at which the platform on top of the spring is at rest before the collision with the block). The spring is already compressed by a distance Mg/k before the block hits the platform, so

$$\begin{aligned}
K_f + U_f &= K_0 + U_0 \\
\tfrac{1}{2}k\left(\frac{Mg}{k}+y\right)^2 &= \tfrac{1}{2}(m+M)V^2 + (m+M)gy + \frac{1}{2}\left(\frac{Mg}{k}\right)^2 \Rightarrow \\
0 &= ky^2 - 2gmy - (m+M)V^2 \Rightarrow \\
y &= \frac{mg + \sqrt{m^2g^2 + kV^2(m+M)}}{k} \\
&= \frac{(4.90\text{ N}) + \sqrt{(4.90\text{ N})^2 + (120\text{ N/m})(1.14\text{ m/s})^2(1.50\text{ kg})}}{120\text{ N/m}} \\
y &= 0.175\text{ m} = 17.5\text{ cm}
\end{aligned}$$

5.8. (a) The qualitative figure is below where we represent x as the distance from one of the planets along the line between the planets. We are free to place the origin of the coordinate system anywhere we like. An easy choice is on the line between the planets. For the figure below it is obvious that

$$|\vec{R}_1 - \vec{r}| = D - x \quad \text{and} \quad |\vec{R}_2 - \vec{r}| = x$$

where x is a positive distance. The potential energy function is

$$\begin{aligned}
u(x) &= -GM\left[\frac{1}{x} + \frac{1}{D-x}\right] \\
&= -GM\left[\frac{D}{x(D-x)}\right] \\
u(x) &= -\frac{GM}{D(x/D)(1-x/D)}
\end{aligned}$$

We plot $u(x)$ in arbitrary units of energy.

(b) This problem is somewhat unphysical as the stations cannot stably be at rest with respect to the planets at their positions. Nevertheless we take the positions and note that the initial total mechanical energy of the projectile must exceed the maximum

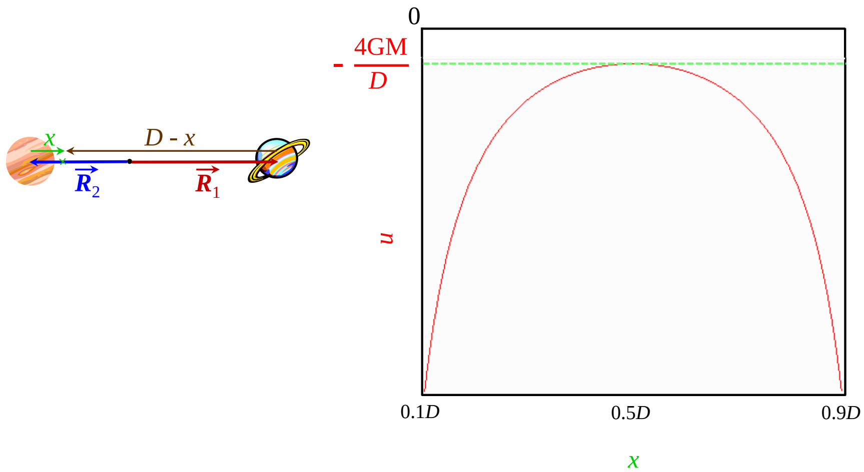


Figure Soln:5.2: Gravitational potential for Problem 5.8.

potential energy at $D/2$ to make it to the other station. We know the position of the maximum since

$$\frac{d}{dx}\left[\frac{1}{(x/D)(1-x/D)}\right] = 0 \Rightarrow x = \frac{D}{2}$$

and you can easily verify that this extremum is a maximum. Along the line between the planets, if the initial kinetic energy is such that the kinetic energy is non-zero at $D/2$, then the projectile "falls" to the other station.

$$\begin{aligned}
u_{\text{Alpha}} &= -GmM\left[\frac{1}{D/4} + \frac{1}{3D/4}\right] = -\frac{16GmM}{3D} \Rightarrow \\
K_0 + u_{\text{Alpha}} &= 0 + u(x = D/2) \\
\frac{1}{2}mv^2 - \frac{16GMm}{3D} &= -\frac{4GMm}{D} \Rightarrow \\
v_0 &= \sqrt{\frac{8GM}{3D}}
\end{aligned}$$

Appendix 6

Simple Harmonic Motion

6.1 Simple Harmonic Motion Problem Solutions

6.1. We showed that the equation of motion is the same as that for a frame where the gravitational acceleration is modified to $\vec{g}\,'$ where $\vec{g}\,' = \vec{g} - \vec{a}$. The period of the pendulum must therefore be

$$\text{period} = 2\pi\sqrt{\frac{\ell}{g'}} = 2\pi\sqrt{\frac{\ell}{\sqrt{g^2 + a^2}}}$$

6.2. The motion of the particle is identical to that of the bob of a simple pendulum. The tension in this case is replaced by the normal force of the bowl surface on the particle. We could use the forces acting on the particle to derive the equation of motion. Let's instead take the approach of section 6.4 and look at the mechanical energy. From Fig.Soln:6.1 we see that the height of the particle above the lowest point of the spherical surface is

$$y = R - R\cos\theta = R(1 - \cos\theta)$$

There is no friction so all the forces are conservative and we can write

$$\text{KE} + \text{PE} = \text{E} \Rightarrow \frac{1}{2}mv^2 + mgy = \text{constant}$$

where m is the mass of the particle. We note that the angular speed of the particle at any instant is $d\theta/dt = v/R$. We take the derivative of the energy equation with respect to time to find

$$\begin{aligned}
\frac{d}{dt}E = 0 &= \frac{1}{2}m\left(2v\frac{dv}{dt}\right) + mgR\left(0 - \sin\theta \cdot \frac{d\theta}{dt}\right) \\
&= mR\frac{d\theta}{dt} \cdot \frac{d}{dt}\left(R\frac{d\theta}{dt}\right) + mgR\sin\theta \cdot \frac{d\theta}{dt} \\
0 &= mR^2\frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + m\frac{d\theta}{dt}gR\sin\theta \Rightarrow \\
\frac{d^2\theta}{dt^2} &= -\frac{g}{R}\sin\theta
\end{aligned}$$

where $d^2\theta/dt^2$ of course is the angular acceleration. For small displacements from the lowest point we need $\theta \sim 0$ and hence $\sin\theta \sim \theta$. Therefore

$$\frac{d^2\theta}{dt^2} \simeq -\frac{g}{R}\theta$$

This has the same form as eqn.(6.3) if we replace ω^2 by (g/R) hence we see immediately that the equation of motion predicts simple harmonic oscillation with a period of

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g/R}} = 2\pi\sqrt{\frac{R}{g}}$$

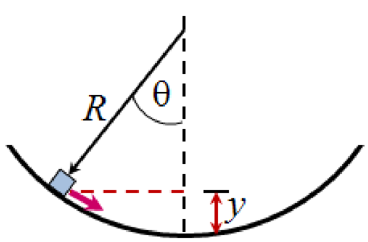


Figure Soln:6.1: Solution 6.2

6.3. The position of the particle can be determined as x as shown in Fig. 6.10. This position is

$$x = r\sin\theta$$

The force on the particle at this position has magnitude

$$F = \frac{GmM(r)}{r^2} = \frac{mgr}{R}$$

where $M(r) = Mr^3/R^3$ since the mass density is uniform (and hence equals $M/[(4/3)\pi R^3]$) and $g = GM/R^2$. The vector component of $\vec{F}$ directed perpendicular to the tunnel is offset by the normal forces holding the particle in the tunnel. The component parallel to the tunnel surface is

$$F_x = -\frac{mgr}{R} \cdot \sin\theta$$

where the minus sign indicates that the direction of the force always points towards the center of the tunnel. At this point $\theta = 0$ and the parallel-directed force is zero hence this point represents an equilibrium position. Hence the equation describing the particle's motion is

$$\begin{aligned} ma_x = F_x &= -\frac{mgr}{R}\sin\theta = -\frac{mgx}{R} \Rightarrow \\ a_x = \frac{d^2x}{dt^2} &= -\frac{g}{R}x \end{aligned}$$

This has the same form as eqn.(6.3) so we determine this motion to be simple harmonic oscillation along the tunnel with angular frequency

$$\omega = \sqrt{\frac{g}{R}}.$$

The time to cross from one end of the tunnel to the other is half a period of oscillation

$$\begin{aligned} \tfrac{1}{2}T &= \tfrac{1}{2} \cdot 2\pi \cdot \sqrt{\frac{R}{g}} \\ &= (3.14159)\sqrt{\frac{6.37 \times 10^6 \text{ m}}{9.81 \text{ m/s}^2}} \\ t &= 2530 \text{ s} = 42.2 \text{ minutes} \end{aligned}$$

Note that the result is independent of the mass of the particle or length of the tunnel! Another interesting note is that the period is equal to that of a satellite in a circular orbit skimming the rooftops on earth.

6.4. For the top block the normal force due to the bottom block is mg. The maximum horizontal force that can be exerted on the top block is given by the maximum static frictional force of μmg. Therefore the maximum acceleration possible for the top block before slipping occurs is μg. To get the maximum acceleration for the amplitude given we refer to eqn.(6.13). In this case we are free to set the reference point $x_0 = 0$. The angular frequency is given by

$$\omega^2 = \frac{k}{m+M}$$

From the equation of motion, eqn.(6.13) we see that the maximum acceleration magnitude occurs at the amplitude positions (the maximum magnitude for x), so

$$\begin{aligned} a_{\max} &= -\omega^2 \cdot x_{\max} = -\omega^2 \cdot A \\ \mu_{\min} g &= \frac{kA}{m+M} \Rightarrow \\ \mu_{\min} &= 0.136 \end{aligned}$$

Of course it is simpler to just note that

$$a_{\max} = \frac{F_{\max}}{M+m} = \frac{kA}{M+m} = \mu_{\min} g$$

Appendix 7

Static Equilibrium of Simple Rigid Bodies

7.1 Static Equilibrium Problem Solutions

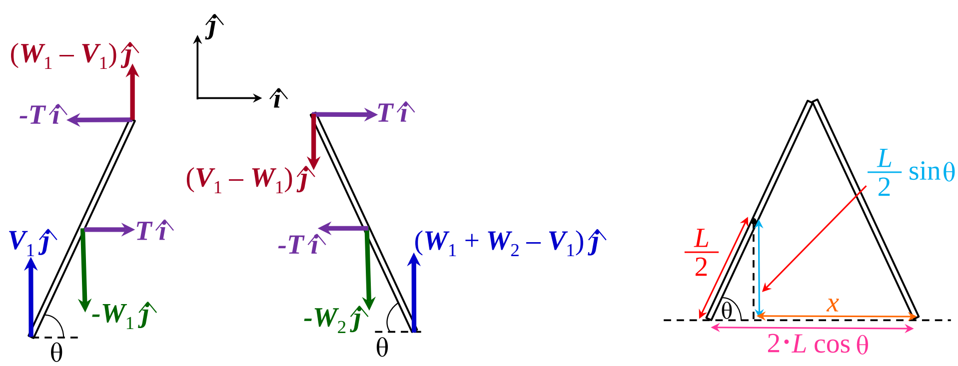


Figure Soln:7.1: Force diagrams on rods of the ladder in Prob. **7.1**.

7.1. If we take the ladder as a whole as the system, then the only external forces are the vertical forces from the floor (which cannot exert horizontal forces as there is no friction) and the weights. Looking at the torque about the bottom of the right leg, we have (note that in the diagram above $x = 2L\cos\theta - (L/2)\cos\theta = (3L/2)\cos\theta)$

$$\begin{aligned}
0 &= (2L\cos\theta)V_1 - (r\sin\phi)W_1 - \left(\frac{L}{2}\cos\theta\right)W_2 \\
&= V_1\cdot 2L\cos\theta - W_1\left(\frac{3L}{2}\cos\theta\right) - W_2\left(\frac{L}{2}\cos\theta\right) \Rightarrow \\
V_1 &= \frac{3}{4}W_1 + \frac{1}{4}W_2
\end{aligned}$$

The force of the floor on the right leg is therefore

$$\begin{aligned}
V_2 &= -(V_1 - W_1) - (-W_2) \Rightarrow \\
&= \frac{1}{4}W_1 + \frac{3}{4}W_2
\end{aligned}$$

Now to get the tension in the cross-support we can look at the torque on the left leg about its midpoint.

$$\begin{aligned}
0 &= \left(\frac{L}{2}\cos\theta\right)V_1 - \left(\frac{L}{2}\sin\theta\right)T - \left(\frac{L}{2}\cos\theta\right)(W_1 - V_1) \Rightarrow \\
T &= 2V_1\frac{\cos\theta}{\sin\theta} - W_1\frac{\cos\theta}{\sin\theta} \\
&= \left[\frac{3}{2}W_1 + \frac{1}{2}W_2 - W_1\right]\cot\theta \\
T &= \frac{1}{2}(W_1 + W_2)\cot\theta
\end{aligned}$$

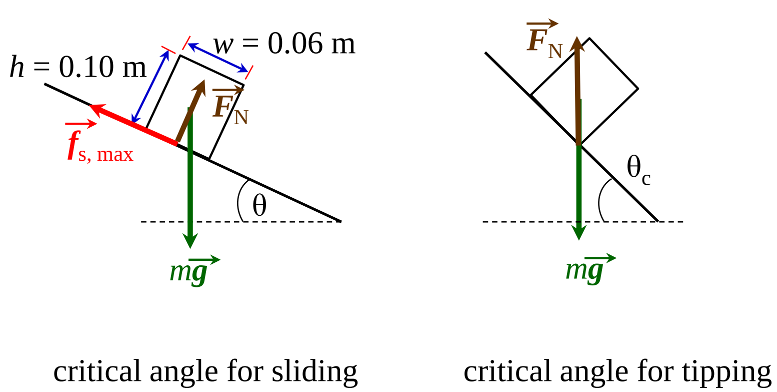


Figure Soln:7.2: A block on an inclined board at the critical angle for sliding or tipping in Prob. **7.2**.

7.2. The situation shown in Fig.Soln:7.2 depicts the block at an angle where the maximum static frictional force is just sufficient to prevent sliding and the angle where the block is just at unstable equilibrium for tipping. The problem asks us to determine the condition whereby these angles are the same, i.e. the value of $\mu = \mu_c$ that determines the angle for these two conditions (tipping and sliding) be the same. The tipping condition is straightforward as the equilibrium cannot be maintained if the box has its center of gravity unsupported by the bottom of the box. So the unstable equilibrium occurs for an angle at which the right bottom corner is direcly underneath the center of gravity, hence

$$\tan\theta_c = \frac{w/2}{h/2} = \frac{w}{h}$$

with w being the width and h the height of the box and θ_c is the critical angle at which tipping occurs. The critical angle for sliding is determined by the maximum static friction force and the component of the gravitational force pointing down the incline. The maximum static friction is determined from the normal force F_N.

$$\begin{aligned} F_N &= mg\cos\theta \Rightarrow f_{s,\ \mathrm{max}} = \mu F_N = \mu mg\cos\theta \\ \text{critical angle : } \quad \mu mg\cos\theta_c &= mg\sin\theta_c \Rightarrow \\ \tan\theta_c &= \mu \end{aligned}$$

For the tipping and sliding critical angles to be the same, we need

$$\mu_c = \tan\theta_c = \frac{w}{h} = 0.600$$

So the general condition is that if the static friction coefficient is greater than θ_c for tipping, then increasing θ causes tipping otherwise the box begins to slide before tipping can occur. That is to say, if $\mu > w/h$ it tips and if $\mu < w/h$ it slides.

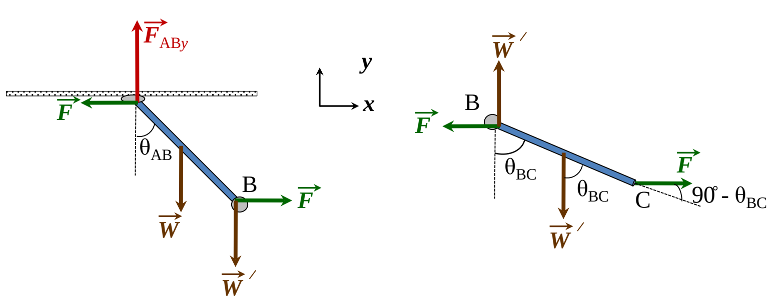


Figure Soln:7.3: Two rods hung from a ceiling in Prob. **7.3**.

7.3. For rod BC we have the torque about an axis through point B as

$$\begin{aligned}\tau_B = 0 &= FL'\cos\theta_{BC} - W'\frac{L'}{2}\sin\theta_{BC} \Rightarrow \\ \tan\theta_{BC} &= \frac{2F}{W'} \Rightarrow \\ \theta_{BC} &= \tan^{-1}\frac{2F}{W'}\end{aligned}$$

For rod AB we have the torque about the hinge at the ceiling as

$$\begin{aligned}0 &= FL\cos\theta_{AB} - W'L\sin\theta_{AB} - W\frac{L}{2}\sin\theta_{AB} \Rightarrow \\ \theta_{AB} &= \tan^{-1}\frac{2F}{W + 2W'}\end{aligned}$$

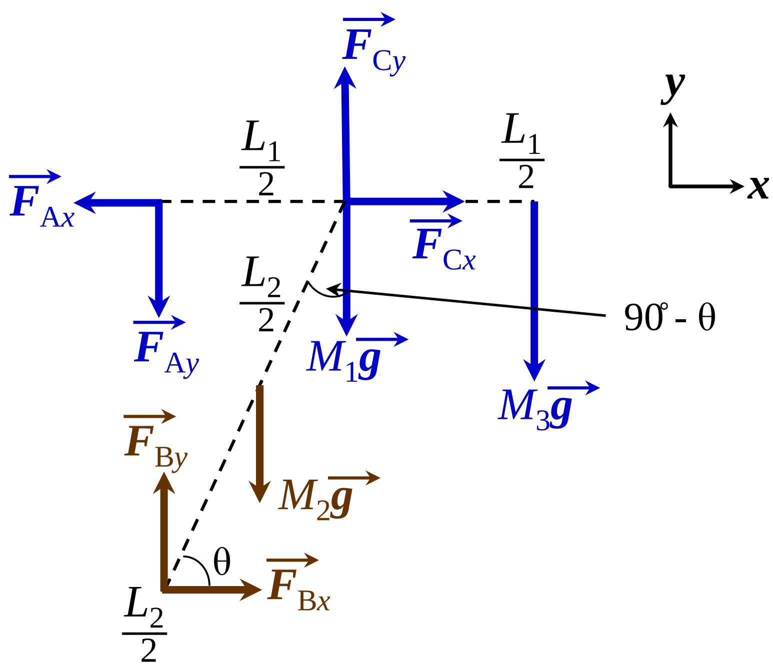


Figure Soln:7.4: Force diagram for Prob. **7.4**.

7.4. The force diagram is shown in Fig.Soln:7.4. By choosing point C for the axis of rotation, we can quickly determine the vertical force acting on beam #1 at point A as the hinge at C and the weight of M_1 have no lever arm at C. The tension in the string attached at the right end of beam #1 must be equal to the weight M_3g since that mass is in equilibrium.

$$F_{Ay}\frac{L_1}{2} - M_3g = 0 \Rightarrow F_{Ay} = M_3g$$

Now we use point A as the axis and look at the torque equation for beam #1 with F_{Cx} and F_{Cy} being the force exerted by the hinge at point C on beam #1.

$$\begin{aligned} 0 &= F_{Cy}\frac{L_1}{2} - M_1g\frac{L_1}{2} - M_3gL_1 \Rightarrow \\ F_{Cy} &= (2M_3 + M_1)g \end{aligned}$$

We could also have gotten this from the vertical force equilibrium equation for beam #1. F_{Cy} isn't a force component we wanted, but all other axis points have contributions from two unknown force components. Looking at the y force equation for beam #2 and noting that, by Newton's Second Law, $\vec{F}_{Cy}$ points opposite to the direction of $\vec{F}_{Cy}$ on beam #1,

$$-F_{Cy} - M_2g + F_{By} = 0 \Rightarrow F_{By} = F_{Cy} + M_2g = (2M_3 + M_1 + M_2)g$$

Now calculating the torque about point C for beam #2 gives us

$$\begin{aligned} 0 &= F_{Bx}L_2\sin\theta + M_2g\frac{L_2}{2}\cos\theta - F_{By}L_2\cos\theta \Rightarrow \\ F_{Bx} &= \left(F_{By} - \frac{1}{2}M_2g\right)\cot\theta = \left(2M_3 + M_1 + \frac{1}{2}M_2\right)g\cot\theta \end{aligned}$$

Now we see that the only horizontal force components acting on beam #2 are F_{Bx} and F_{Cx} so these must be equal in magnitude (remember that $\vec{F}_{Cx}$ points to the left on beam #2). This is also true for $\vec{F}_{Ax}$ and $\vec{F}_{Cx}$ so these must also be equal in magnitude, hence

$$F_{Ax} = F_{Cx} = F_{Bx} = \left(2M_3 + M_1 + \frac{1}{2}M_2\right)g\cot\theta$$

If we follow the rule stated of minimizing the algebra, then the shorter path to the answers is to treat #1 and #2 and M_3 as one rigid body. The torque about point A is then

$$\begin{aligned} 0 &= M_1g\frac{L_1}{2} + M_3gL_1 + M_2g\frac{L_1}{4} - F_{Bx}\frac{L_1}{2}\tan\theta \Rightarrow \\ F_{Bx} &= \left(M_1 + 2M_3 + \frac{1}{2}M_2\right)g\cot\theta \end{aligned}$$

For this system, there are no external forces acting at C so the only horizontal forces are at A and B hence

$$F_{Ax} = F_{Bx}$$

Now we can consider beam #1 on its own, where

$$F_{Cx} = F_{Ax}$$

As before, calculating the torque about C for beam #1 yields $F_{Ay} = M_3 g$ while the torque about A for that beam gives

$$F_{Cy} = (2M_3 + M_1)g$$

and the force equations for beam #1 give

$$F_{By} = F_{Cy} + M_2 g = (2M_3 + M_1 + M_2)g.$$

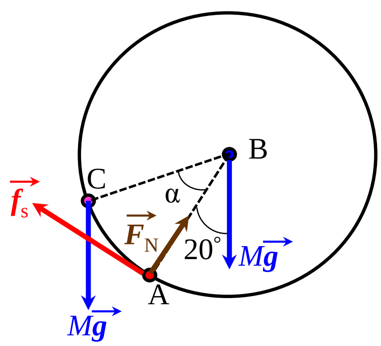


Figure Soln:7.5: Free-body diagram for Prob. **7.5**.

7.5. In the text it was proven that the net result of summing the gravitional torque on each piece of the hoop is that the force acts as if it were acting at the center of mass, i.e. the geometric center of the hoop. We can note that that the net force on the hoop in the direction parallel to the incline must be

$$Mg\sin 20^\circ + Mg\sin 20^\circ - f_s = 0 \Rightarrow f_s = 2Mg\sin 20^\circ$$

since the net force on the hoop must be zero. If we look at the torque about an axis through the center of the hoop, we see that the weight of the hoop has no lever arm and the normal force points along the radial vector to point A where the normal force acts. Hence the torque due to the point mass and the torque due to the static friction force must cancel about this axis. We call the radius of the hoop R, so

$$\begin{aligned} MgR\sin(\alpha + 20^\circ) - f_s R &= 0 \Rightarrow \\ MgR\sin(20^\circ + \alpha) &= 2MgR\sin 20^\circ \Rightarrow \\ (20^\circ + \alpha) &= \sin^{-1}\left[2\sin 20^\circ\right] \Rightarrow \\ \alpha &= 23.2^\circ \end{aligned}$$

Appendix 8

Rotational Motion, Angular Momentum and Dynamics of Rigid Bodies

8.1 Rotational Motion Problem Solutions

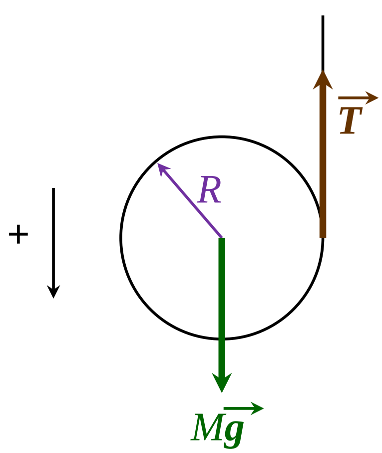


Figure Soln:8.1: Forces acting on a falling cylinder wrapped with a string in Prob. **8.1**.

8.1. (a) The forces acting are shown in Fig.Soln:8.1. The torque about an axis through the center of mass is

$$\begin{aligned}
\tau = TR &= I_{\mathrm{CM}}\alpha = \frac{1}{2}MR^2\frac{a_{\mathrm{CM}}}{R} \Rightarrow \\
T &= \frac{1}{2}Ma_{\mathrm{CM}}
\end{aligned}$$

where T is the tension in the string and we have used the rolling without slipping condition that the rotational acceleration equals the linear acceleration of the center of mass divided by the radius of the cylinder. Now we can look at the force equation for the center of mass.

$$\begin{aligned}
Mg - T &= Ma_{\mathrm{CM}} \\
Mg - \frac{1}{2}Ma_{\mathrm{CM}} &= Ma_{\mathrm{CM}} \Rightarrow \\
a_{\mathrm{CM}} &= \frac{2}{3}g
\end{aligned}$$

(b) The force diagram is unchanged but now the acceleration of the center of mass of the cylinder is zero with respect to a stationary observer. Therefore the tension must equal the weight, i.e. $T = Mg$. The torque about an axis through the center of mass is then

$$\begin{aligned}
TR &= I\alpha \\
MgR &= \frac{1}{2}MR^2\alpha \Rightarrow \\
\alpha &= \frac{2g}{R}
\end{aligned}$$

The rolling without slipping condition is assumed to still apply, that is to say, the cylinder does not slip on the string, so the hand needs to accelerate upward at value

$$a_{\mathrm{hand}} = R\alpha = 2g$$

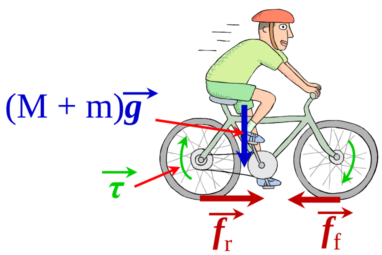


Figure Soln:8.2: Force and torque diagram for Prob. **8.2**.

8.2. The torque applied by the bicycle chain is considered known. We do not need to know the radius at which it is applied. Since the angular velocity of both wheels is increasing and the only torque on the front wheel (around its center) is due to the static frictional force, that force must be directed backward. Since the net force on the bicycle and rider must be forward, the static frictional force exerted by the ground on the rear wheel must be forward. Thus, for the rear wheel,

$$\tau - f_{\mathrm{r}}R = I\alpha = mR^2\frac{a}{R}$$

while for the front wheel we have

$$f_{\mathrm{f}}R = I\alpha = mR^2\frac{a}{R}.$$

The static frictional forces are the external forces from the ground acting on the system of bicycle plus rider, so

$$\begin{aligned} f_{\mathrm{r}} - f_{\mathrm{f}} &= (M+2m)a \\ \left[\frac{\tau}{R} - ma\right] - ma &= (M+2m)a \Rightarrow \\ a &= \frac{\tau}{(M+4m)R} \end{aligned}$$

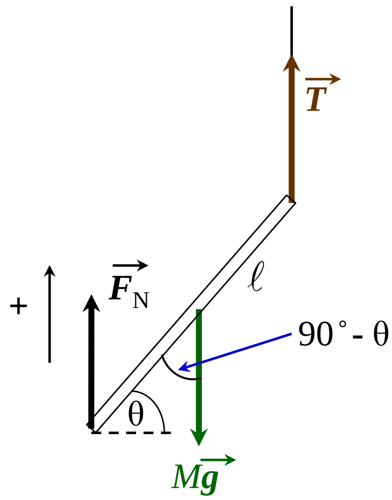


Figure Soln:8.3: Forces acting on a rod suspended from the ceiling and in contact with the floor in Prob. **8.3**.

8.3. (a) The net torque about the end of the rod attached to the string is zero, so

$$\begin{aligned}\tau_{\text{net}} &= 0 = mg\frac{\ell}{2}\cos\theta - F_N\ell\cos\theta \Rightarrow \\ F_N &= \frac{1}{2}mg\cos\theta\end{aligned}$$

where we note that the force exerted by the floor is only along the vertical as the smooth floor cannot exert any horizontal force component on the rod.

(b) When the string is cut there are still no horizontal components to either the contact force from the floor or gravity so the x-component of the center of mass position stays constant. This means the contact point with the floor is subject to acceleration to the left. We cannot use this point for calculations of torque. For an axis through the center of mass though, we can calculate the torque.

$$\tau_{\text{CM}} = -F_N\frac{\ell}{2}\cos\theta = -I_{\text{CM}}\alpha = -\frac{1}{12}m\ell^2\alpha$$

The vertical acceleration of the center of mass is the result of the normal force and gravity, so

$$F_N - mg = -ma_{\rm CM} = -m\left|\frac{d^2y(t)}{dt^2}\right|$$

assuming y is the vertical (up) direction and $a_{\rm CM}$ is the magnitude of the acceleration of the CM. To relate the angular acceleration to the vertical acceleration we note that the vertical position of the floor as an origin determines that

$$\begin{aligned}
y(t) &= \frac{\ell}{2}\sin\theta(t) \Rightarrow \\
\frac{dy(t)}{dt} &= \frac{\ell}{2}\cos\theta(t)\frac{d\theta(t)}{dt} \Rightarrow \\
\frac{d^2y(t)}{dt^2} &= \frac{\ell}{2}\left[-\sin\theta(t)\left(\frac{d\theta(t)}{dt}\right)^2 + \cos\theta(t)\frac{d^2\theta(t)}{dt^2}\right]
\end{aligned}$$

For the time $t = 0$, we have the initial linear and angular velocity of the CM as zero, so $d\theta(t)/dt\Big|_{t=0} = 0$. Hence, the vertical acceleration of the CM of the rod at $t = 0$ is

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=0} = -a_{\rm CM} = -\frac{\alpha\ell}{2}\cos\theta.$$

We can use the force and torque equations to solve for $a_{\rm CM}$, i.e. the vertical acceleration.

$$\begin{aligned}
F_N &= \frac{m\ell\alpha}{6\cos\theta} \\
&= \frac{m\ell}{6\cos\theta}\cdot\frac{2a_{\rm CM}}{\ell\cos\theta} \\
m(g - a_{\rm CM}) &= \frac{m}{3\cos^2\theta}a_{\rm CM} \Rightarrow \\
g &= a_{\rm CM}\left[\frac{1}{3\cos^2\theta} + 1\right] \Rightarrow \\
a_{\rm CM} &= \frac{3g\cos^2\theta}{1+3\cos^2\theta} \Rightarrow \vec{a}_{\rm CM} = -\frac{3g\cos^2\theta}{1+3\cos^2\theta}\hat{y}
\end{aligned}$$

and the normal force is

$$\vec{F}_N = \frac{ma_{\rm CM}}{3\cos^2\theta}\hat{y} = \frac{mg}{1+3\cos^2\theta}\hat{y}$$

8.4. We know that the static frictional force acts up along the incline on the cylinder. The normal force of the incline on the cylinder is $mg\cos\theta$ with m being the mass of the cylinder. The maximum static frictional force for any given angle θ is therefore

$$f_{\text{s, max}} = \mu mg\cos\theta$$

which decreases as θ increases. The maximum torque that could be exerted about an axis through the center of the cylinder also decreases. This torque must provide the angular acceleration that matches the condition for rolling without slipping of the linear acceleration of the center of mass of the cylinder. Hence, if R is the radius of the cylinder,

$$\begin{aligned}
\tau_{\text{max}} = f_{\text{s, max}}R &= I\alpha \\
\mu mgR\cos\theta &= \frac{1}{2}mR^2\frac{a_{\text{CM, max}}}{R} \Rightarrow \\
a_{\text{CM, max}} &= 2\mu g\cos\theta
\end{aligned}$$

But the acceleration of the center of mass must also satisfy Newton's Second Law. Along the direction of down the incline

$$\begin{aligned}
a_{\text{CM, max}} &= \frac{mg\sin\theta - f_{\text{s, max}}}{m} \\
2\mu g\cos\theta_{\text{max}} &= g\sin\theta_{\text{max}} - \mu g\cos\theta_{\text{max}} \Rightarrow \\
\theta_{\text{max}} &= \tan^{-1}(3\mu)
\end{aligned}$$

Contrast this result with the fact that the steepest incline on which a block can rest without slipping has $\theta_{\text{max}} = \tan^{-1}\mu$.

8.5. The hubcap's CM is originally moving along with the wheel with linear velocity v_0. The rotational velocity is originally therefore

$$\omega_0 = \frac{v_0}{R}$$

The impact with the road subjects the hubcap to an impulse which changes the linear velocity of the center of mass of the hubcap from v_0 to v_f and the angular velocity from v_0/R to $\omega_f = v_f/r$. Note that if the direction of the impulse slows (or quickens) the linear velocity, it must speed up (or reduce) the angular velocity given the direction of the torque about the center of mass.

$$\begin{aligned}
\Delta p &= m(v_f - v_0) \Rightarrow \\
\Delta \mathcal{L} &= -\Delta p \cdot r \\
\frac{1}{2}mr^2(\omega_f - \omega_0) &= -mr(v_f - v_0) \Rightarrow \\
\frac{r}{2}\left(\frac{v_f}{r} - \frac{v_0}{R}\right) &= v_0 - v_f \Rightarrow \\
3v_f &= v_0\left[2 + \frac{r}{R}\right] \Rightarrow \\
v_f &= v_0\left[\frac{2}{3} + \frac{r}{3R}\right] \Rightarrow \\
\omega_f &= \frac{v_f}{r} = \frac{v_0}{R}\left[\frac{2}{3}\frac{R}{r} + \frac{1}{3}\right]
\end{aligned}$$

Note that $v_f < v_0$ while $\omega_f > \omega_0$. We should expect that the linear velocity would decrease since the lowest point of the hubcap has forward velocity $v_0 - \omega_0 r = v_0(1 - r/R) > 0$ before it hits the road and therefore the frictional force is directed backward. This direction of frictional force also contributes torque about the CM which *increases* the rotational velocity.

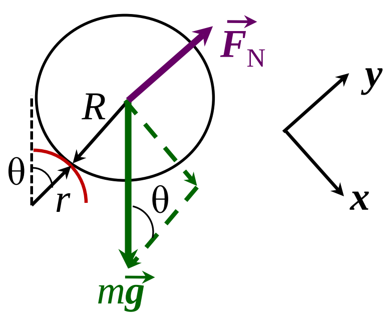


Figure Soln:8.4: Forces acting on a solid sphere rolling over the edge of a cube in Prob. **8.6**.

8.6. (a) The force diagram above shows that if $v_0 < v_c$ then the condition that maintains the sphere on the small radius r of the curved edge of the cube (in red) is

$$F_N - mg\cos\theta = -\frac{mv_0^2}{R+r}$$

This is the centripetal condition required for a circular path. Assuming that $r \ll R$ we can ignore r and just assume the radius of the curved path is just R the radius of the sphere. The definition of "maintain contact" is that the normal force not be zero, so if the initial velocity of the center of mass of the sphere is too high, F_N would have to be less than zero (non-physical!) to satisfy the equation. Hence the sphere leaves the cube without any deviation in the direction of $\vec{v}_0$. Therefore we want to evaluate v_c for $\theta = 0$.

$$mg\cos\theta - \frac{mv_c^2}{R} = 0 \Rightarrow v_c = \sqrt{Rg\cos\theta} = \sqrt{Rg}$$

(b) Now we allow for $\theta > 0$. The linear velocity of the center of mass and the angular velocity about the center of mass increase as the sphere's center drops as it rolls over the edge. The initial kinetic energy can be calculated as

$$\begin{aligned} K_0 &= \frac{1}{2}mv_0^2 + \frac{1}{2}I\omega_{\mathrm{CM}}^2 \\ &= \frac{1}{2}mv_0^2 + \frac{1}{2}\cdot\frac{2}{5}mR^2\cdot\frac{v_0^2}{R^2} \\ K_0 &= \frac{7}{10}mv_0^2 \end{aligned}$$

It is more convenient to consider the motion as pure rotation about the stationary point (assumes no slipping!) of contact of the sphere with the cube edge. Then, using the parallel-axis theorem and the rolling without slipping condition that angular speed around the center of mass is the same as angular speed of the center of mass around the axis through the point of rolling contact, we have

$$\begin{aligned} K_0 &= \frac{1}{2}I_{\mathrm{edge}}\omega_{\mathrm{edge}}^2 \\ &= \frac{1}{2}\left(\frac{2}{5}mR^2 + mR^2\right)\frac{v_0^2}{R^2} \\ K_0 &= \frac{7}{10}mv_0^2 \end{aligned}$$

The speed increases as the sphere rolls over the edge as the center of mass goes down. Considering the cube's top surface as the zero point of the vertical position, we calculate this increase in speed as

$$\begin{aligned} -\Delta U = mgR - mgR\cos\theta &= \Delta K = \frac{7}{10}mv_f^2 - \frac{7}{10}mv_0^2 \Rightarrow \\ gR(1-\cos\theta) &= \frac{7}{10}v_f^2 - \frac{7}{10}v_0^2 \end{aligned}$$

From our analysis in part a of this problem, we know that there is a maximum speed corresponding to maintaining contact, i.e. when the normal force goes to zero.

$$\begin{aligned} F_N - mg\cos\theta &= -\frac{mv^2}{R} \Rightarrow \\ 0 - mg\cos\theta_c &= -\frac{mv_f^2}{R} \Rightarrow \\ v_f^2 &= gR\cos\theta \end{aligned}$$

Plugging this into our previous energy equation, we see

$$\begin{aligned} gR(1-\cos\theta_c) &= \frac{7}{10}gR\cos\theta_c - \frac{7}{10}v_0^2 \Rightarrow \\ \frac{17}{10}gR\cos\theta_c &= gR + \frac{7}{10}v_0^2 \Rightarrow \\ \theta_c &= \cos^{-1}\left[\frac{7}{17}\frac{v_0^2}{gR} + \frac{10}{17}\right] \end{aligned}$$

(c) As before, the normal force goes to zero at the critical angle θ_c, so if $v_0 = 0$, the formula from part b gives us

$$\theta_c = \cos^{-1}\frac{10}{17} = 54.0^\circ$$

(d) Consider Fig.Soln:8.5 where we show the sphere just leaving the edge of the cube. We know the critical angle θ_c for which this happens if $v_0 = 0$. From the figure we see that if, in the time it takes for the sphere's center of mass to travel horizontally by a distance Δx, the sphere has traveled down vertically *less* than a distance $\Delta y = R\cos\theta$, then the left edge of the sphere misses the last point of contact of sphere and cube and therefore there is no bump. The free-fall formulae for the center of mass are

$$\begin{aligned} \Delta x &= v_{\text{CM}}\cos\theta_c(\Delta t) \\ \Delta y &= v_{\text{CM}}\sin\theta_c(\Delta t) + \frac{1}{2}g(\Delta t)^2 \end{aligned}$$

The first equation can be solved for Δt and this can be substituted into the Δy equation.

$$\begin{aligned}
\Delta t &= \frac{\Delta x}{v_{\mathrm{CM}} \cos\theta_c} \Rightarrow \\
\Delta y &= \Delta x \tan\theta_c + \frac{g}{2}\frac{(\Delta x)^2}{v_{\mathrm{CM}}^2 \cos^2\theta_c} \\
&= \Delta x \tan\theta_c + \frac{7(\Delta x)^2}{20R\cos^2\theta_c(1-\cos\theta_c)}
\end{aligned}$$

where in the last equation we used the result for v_{CM} from part c. Since we have calculated $\cos\theta_c = 10/17$ and we find from Fig.Soln:8.5 that we want $\Delta x = R(1-\sin\theta_c)$ to calculate the distance fallen, Δy, then

$$\begin{aligned}
\Delta y &= \Delta x \tan\theta_c + \frac{7(\Delta x)R(1-\sin\theta_c)}{20R\cos^2\theta_c(1-\cos\theta_c)} \\
&= \Delta x \left[\tan\theta_c + \frac{7(1-\sin\theta_c)}{20\cos^2\theta(1-\cos\theta_c)}\right] \\
&= \Delta x \left[1.3748 + \frac{7(0.19131)}{20(0.3402)(0.41175)}\right] \\
\Delta y &= 1.8447\Delta x \\
&= 1.8447R(1-\sin\theta_c) \\
\Delta y &= 0.353R
\end{aligned}$$

This is the distance the leftmost edge point of the sphere falls (since the sphere is a solid body) in the time it takes for this same point to just pass the edge of the cube. To bump the edge the sphere would have to fall vertically a distance

$$\Delta y = R\cos\theta = 0.588R$$

in this time. The sphere does not fall this far before the leftmost point clears the edge so no bump occurs.

Alternatively, we could just as well look at the position of the center of the sphere as a function of time after the sphere loses contact with the cube edge. The velocity of the CM at this position we calculated in part b so call this $\vec{V}_0$.

$$V_0^2 = gR\cos\theta_c$$

where again we know $\cos\theta_c = 10/17$. The initial x and y velocities of the CM just after the sphere loses contact are

$$\begin{aligned} V_{0x} &= V_0\cos\theta_c = \sqrt{gR}(\cos\theta_c)^{3/2} \\ V_{0y} &= -V_0\sin\theta_c = -\sqrt{gR}(\sin\theta_c)(\cos\theta_c)^{1/2} \end{aligned}$$

The position of the CM as a function of time is

$$\begin{aligned} x(t) &= x_0 + V_{0x}t = R\sin\theta_c + \sqrt{gR}(\cos\theta_c)^{3/2}t \\ y(t) &= y_0 + V_{0y}t - \frac{1}{2}gt^2 \\ &= R\cos\theta_c - \sqrt{gR\cos\theta_c}(\sin\theta_c)t - \frac{1}{2}gt^2 \end{aligned}$$

Therefore, the square of the distance of the CM from the last point of contact, as a function of time, is

$$\begin{aligned} R(t)^2 &= x^2(t) + y^2(t) \\ &= R^2\sin^2\theta_c + 2\sqrt{gR^3}\sin\theta_c(\cos\theta_c)^{3/2}t + gR\cos^3\theta_c t^2 + R^2\cos^2\theta_c \\ &\qquad - 2\sqrt{gR^3}\sin\theta_c(\cos\theta_c)^{3/2}t - gR\cos\theta_c t^2 + \\ &\qquad gR\cos\theta_c\sin^2\theta_c t^2 + \sqrt{g^3R\cos\theta_c}\sin\theta_c t^3 + \frac{g^2t^4}{4} \\ R(t)^2 &= R^2 + \sqrt{g^3R\cos\theta_c}\sin\theta_c t^3 + \frac{g^2t^4}{4} \end{aligned}$$

since $R(t)^2 > R^2$ for all $t > 0$, we have shown that the sphere always misses bumping the edge.

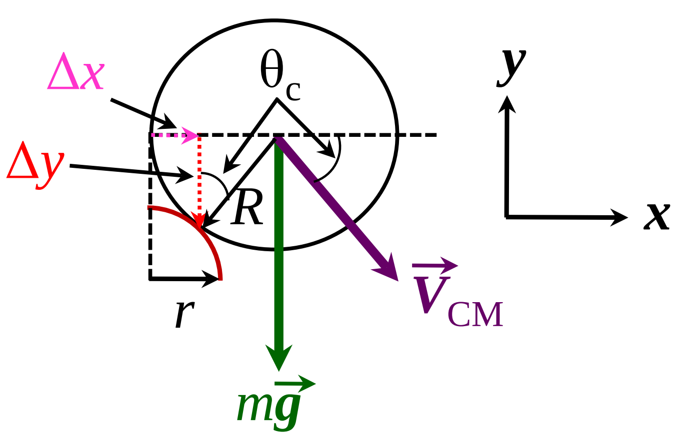


Figure Soln:8.5: Figure of a sphere just leaving the edge of a cube.

8.7. (a) The velocity of this point is due to a combination of rotational and translational motion. The translational motion is due to the motion of the center of mass (CM) in response to the momentum transfer of the impulse and the fact that the rod is a rigid body. Thus the translational velocity of the point p' is

$$Mv_{\text{CM}} = -P \Rightarrow v_{\text{trans}} = v_{\text{CM}} = -\frac{P}{M}$$

where M is the mass of the rod and we assume the impulse is in the $-x$ direction. The rotation of p' about the center of mass comes from the angular momentum imparted by the impulse, so we calculate ω_{CM}, the angular velocity about the center of mass, as follows.

$$\begin{aligned} \mathcal{L} = I_{\text{CM}}\omega_{\text{CM}} &= P \cdot s \\ \frac{ML^2}{12}\omega_{\text{CM}} &= P \cdot s \Rightarrow \\ \omega_{\text{CM}} &= \frac{12Ps}{ML^2} \end{aligned}$$

The velocity of p' due to rotation is thus in the $+x$ direction (since it is on the opposite side of the CM from the impulse) and hence

$$v_{\text{rot}} = \omega_{\text{CM}} \cdot y = \frac{12Ps}{ML^2}y$$

Thus the net velocity of p' is

$$v_{p'} = v_{\text{trans}} + v_{\text{rot}} = \frac{12Ps}{ML^2}y - \frac{P}{M}$$

(b) The magic value of s makes $v_{p'} = 0$, so

$$\begin{aligned} v_{p'} &= 0 \Rightarrow \\ \frac{12Ps}{ML^2}y &= \frac{P}{M} \Rightarrow \\ s &= \frac{L^2}{12y} \end{aligned}$$

If $y = 0.400L$ then

$$s = \frac{L^2}{12(0.400L)} = 0.208L$$

Note that p and p' are interchangeable ($s \cdot y = L^2/12$). If the impulse is delivered at distance $0.4L$ from the CM, then a point $0.208L$ from the CM on the opposite side will not recoil.

(c) Calculate the angular momentum about the axle after the impulse. Use the parallel-axis theorem to calculate the rotational inertia, I_{axle}, about the axle.

$$\begin{aligned}
\mathcal{L}_{\text{axle}} &= P(d+d') \\
I_{\text{axle}}\omega_{\text{axle}} &= P(d+d') \\
(Ma^2 + Md^2)\omega_{\text{axle}} &= P(d+d') \Rightarrow \\
\omega_{\text{axle}} &= \frac{P(d+d')}{M(a^2+d^2)}
\end{aligned}$$

The velocity of the CM is derived from the fact that it executes purely rotational motion about the fixed axle.

$$v_{\text{CM}} = \omega_{\text{axle}} d = \frac{Pd(d+d')}{M(a^2+d^2)}$$

The linear momentum of the CM comes from the combination of the impulses $\vec{P}$ and from the axle. We know these are in opposite directions as otherwise the bat would fly off in the direction of $\vec{P}$.

$$\begin{aligned}
P - P_{\text{axle}} &= Mv_{\text{CM}} \Rightarrow \\
P_{\text{axle}} &= P\left[1 - \frac{d^2 + d\cdot d'}{a^2+d^2}\right] \\
&= P\left[\frac{a^2 - d\cdot d'}{a^2+d^2}\right]
\end{aligned}$$

For no impulse at the axle we need

$$P_{\text{axle}} = 0 \Rightarrow a^2 = d \cdot d'$$

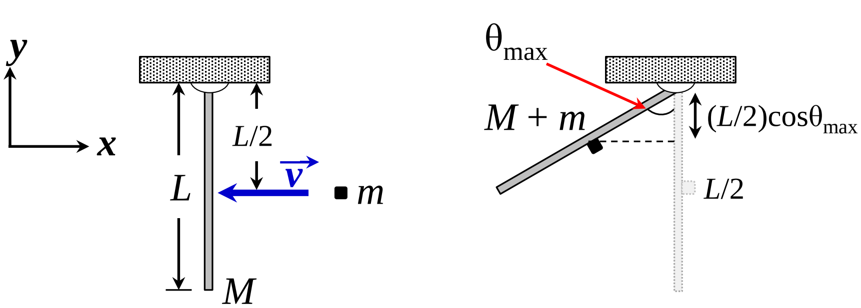


Figure Soln:8.6: Problem **8.8**.

8.8. (a) The collision is inelastic so mechanical energy is not conserved. The hinge exerts external forces on the rod so the linear momentum of the rod and clay is not conserved in the collision either. Angular momentum of the rod and clay about an axis through the hinge and perpendicular to the $x-y$ plane is conserved since the hinge can apply no torque about that axis. The angular velocity just after the collision is therefore calculable as follows.

$$\begin{aligned}
\mathcal{L}_{\text{initial}} &= \mathcal{L}_{\text{final}} = I_{\text{total}}\omega \\
mv \cdot \frac{L}{2} &= \left[I_{\text{rod about end}} + m \cdot \left(\frac{L}{2}\right)^2 \right] \omega \\
\frac{mvL}{2} &= \left[\frac{1}{3}ML^2 + \frac{1}{4}mL^2 \right] \omega \Rightarrow \\
\omega &= \frac{6mv}{L[4M+3m]}
\end{aligned}$$

After the collision, rotational mechanical energy is conserved as again the hinge provides no external torque for an axis through the hinge and gravity acting on the center of the mass of the system of rod and clay is conservative. So we can find the maximum angle of swing from equilibrium as follows:

$$\begin{aligned}
\Delta K &= -\Delta U_{\text{grav}} \\
0 - \frac{1}{2} I_{\text{total}} \omega^2 &= -(m+M)\frac{gL}{2}(1-\cos\theta_{\text{max}}) \Rightarrow \\
\frac{3m^2v^2}{2(4M+3m)} &= \frac{1}{2}(m+M)gL\,(1-\cos\theta_{\text{max}}) \Rightarrow \\
\theta_{\text{max}} &= \cos^{-1}\left[1 - \frac{3m^2v^2}{gL(m+M)(3m+4M)} \right]
\end{aligned}$$

Note: what happens if the term in the square brackets is less than -1? In that case the rod would rotate full circle forever (if the ceiling weren't there).

(b) Gravity plays no role in the collision since it is a finite force (and can therefore produce no impulse) and even if it could produce an impulse that impulse would be orthogonal to the momentum change in this problem.

We take our system to consist of the rod and clay. The linear momentum of the system just before the collision is

$$\vec{p}_{\text{initial}} = mv\hat{\imath}$$

and just after the collision

$$\begin{aligned}
\vec{p}_{\text{after}} &= (M+m)\vec{V}_{\text{CM}} \\
&= (M+m)\frac{L}{2}\omega\hat{\imath} \\
\vec{p}_{\text{after}} &= \frac{M+m}{4M+3m}(3mv)\hat{\imath}
\end{aligned}$$

The only external impulse on the system is exerted by the hinge. Call this impulse $\Delta\vec{p}_{\text{hinge}}$. Then

$$\begin{aligned}
\Delta\vec{p}_{\text{hinge}} &= \vec{p}_{\text{after}} - \vec{p}_{\text{initial}} \\
&= mv\left[\frac{3(M+m)}{3m+4M} - 1\right]\hat{\imath} \\
&= -\frac{mMv}{3m+4M}\hat{\imath}
\end{aligned}$$

The impulse given by the rod to the hinge is

$$\Delta\vec{p}_{\text{hinge}} = \frac{mMv}{3m+4M}\hat{\imath}$$

We should note that the total impulse given to the hinge by the rod *and* ceiling is zero.

Appendix 9

Remarks on Gravitation

9.1 Remarks on Gravity Problem Solutions

9.1. The kinetic energy of a circular orbit is half the gravitational potential energy of the orbit. So

$$\begin{aligned}
K + U &= \frac{1}{2}U \Rightarrow \\
K &= -\frac{1}{2}U = \frac{GM_E m}{2(R_E + 3.00 \times 10^5\ \text{m})} \Rightarrow \\
v &= \sqrt{\frac{GM_E}{R_E + 3.00 \times 10^5}} \\
&= \sqrt{\frac{(6.67 \times 10^{-11}\ \text{m}^3/(\text{kg} \cdot \text{s}^2))\,(5.97 \times 10^{24}\ \text{kg})}{6.67 \times 10^6\ \text{m}}} \\
v &= 7.73 \times 10^3\ \text{m/s}
\end{aligned}$$

The period of such an orbit is easy to get from Kepler's Third Law.

$$\begin{aligned}
\text{period} &= \left[\frac{4\pi^2 r^3}{GM_E}\right]^{1/2} \\
&= \left[\frac{4(3.14159)^2\left(6.67 \times 10^6\right)^3}{(6.67 \times 10^{-11}\ \text{m}^3/(\text{kg} \cdot \text{s}^2))\,(5.97 \times 10^{24}\ \text{kg})}\right]^{1/2} \\
\text{period} &= 5420\ \text{s} = 1.51\ \text{hours}
\end{aligned}$$

9.2. (a) The orbit diagram for this case is as follows: The Hohmann or-

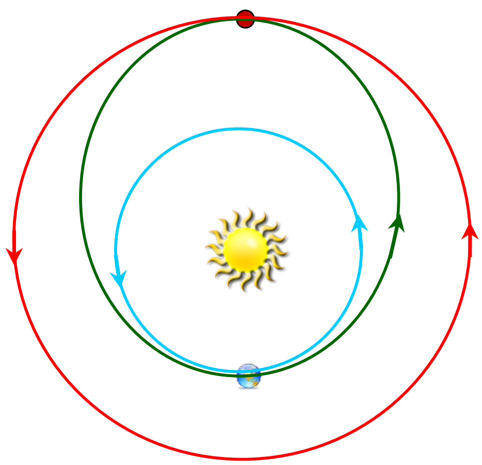


Figure Soln:9.1: Orbit diagram for Problem **9.2**a.

bit goes further away from the sun than earth so we need to *increase* the total energy at the perihelion of the transfer orbit. That means increasing the kinetic energy and therefore we fire the rockets so as to speed up in the direction of earth in its orbit.

(b) To determine the speed necessary for the launch from low earth orbit we can use energy conservation. Since the earth's orbit is nearly circular we can approximate (the spacecraft mass is m, the earth's speed in its orbit is v_0, and the radius of earth's orbit from the sun is R_{earth}).

$$\begin{aligned}
K &= \frac{1}{2}U \\
\frac{1}{2}mv_0^2 &= \frac{GM_{\text{sun}}m}{2R_{\text{earth}}} \Rightarrow \\
v_0 &= \sqrt{GM_{\text{sun}}}R_{\text{earth}} \\
&= \sqrt{\frac{(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}{1.50\times10^{11}\ \text{m}}} \\
v_0 &= 2.97\times10^4\ \text{m/s}
\end{aligned}$$

Of course the ship is in orbit so the actual speed is higher than this but the correction is about 25%. Let's use the v_0 above.

For the elliptical orbit (the transfer orbit) the mechanical energy is still conserved as is the angular momentum. Now we use v_p as the speed of the probe relative to the sun at the perihelion point (the launch from earth).

$$
\begin{aligned}
E_{\text{perihelion}} &= E_{\text{aphelion}} \Rightarrow \\
\frac{1}{2}mv_{\text{peri}}^2 - \frac{GM_{\text{sun}}}{R_{\text{earth}}} &= \frac{1}{2}mv_{\text{aph}}^2 - \frac{GM_{\text{sun}}m}{R_{\text{Mars}}}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\text{peri}} &= \mathcal{L}_{\text{aph}} \\
mv_{\text{peri}}R_{\text{earth}} &= mv_{\text{aph}}R_{\text{Mars}} \Rightarrow \\
v_{\text{aph}} &= v_{\text{peri}}\frac{R_{\text{earth}}}{R_{\text{Mars}}}
\end{aligned}
$$

Combining the angular momentum result with the energy conservation result, we see that

$$
\begin{aligned}
\frac{1}{2}v_{\text{peri}}^2 - \frac{GM_{\text{sun}}}{R_{\text{earth}}} &= \frac{1}{2}v_{\text{peri}}^2\frac{R_{\text{earth}}^2}{R_{\text{Mars}}^2} - \frac{GM_{\text{sun}}}{R_{\text{Mars}}} \Rightarrow \\
v_{\text{peri}}^2\left[\frac{R_{\text{Mars}}^2 - R_{\text{Mars}}^2}{2R_{\text{Mars}}^2}\right] &= GM_{\text{sun}}\left[\frac{R_{\text{Mars}} - R_{\text{earth}}}{R_{\text{earth}}R_{\text{Mars}}}\right] \Rightarrow \\
v_{\text{peri}} &= \sqrt{GM_{\text{sun}}\frac{2R_{\text{Mars}}}{R_{\text{earth}}(R_{\text{Mars}} + R_{\text{earth}})}} \\
&= \sqrt{\frac{2(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})(2.28\times10^{11}\ \text{m})}{(1.50\times10^{11}\ \text{m})(2.28+1.50)\times10^{11}\ \text{m}}} \\
v_{\text{peri}} &= 3.27\times10^4\ \text{m/s}
\end{aligned}
$$

So the launch speed must be

$$
v_{\text{launch}} = (3.27 - 2.97)\times10^4\ \text{m/s} = 2.93\times10^3\ \text{m/s} \simeq 3\ \text{km/s}
$$

To get the time needed we can just take advantage of our derivation of Kepler's Third Law for elliptical orbits.

$$
\begin{aligned}
T &= \frac{1}{2}\sqrt{\frac{4\pi^2}{GM_{\text{sun}}}\left[\frac{r_{\text{earth}} + r_{\text{Mars}}}{2}\right]^2} \\
&= \sqrt{\frac{(3.141593)^2(3.78\times10^{11}\ \text{m})^3}{8(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99\times10^{30}\ \text{kg})}} \\
T &= 2.24\times10^7\ \text{s} \simeq 8.5\ \text{months}
\end{aligned}
$$

(c) Since Mars needs to be at the apogee point when the spacecraft arrives there we need Mars to be at angle θ as shown in the figure below when the probe is launched.

The angle we want to calculate is the difference between the

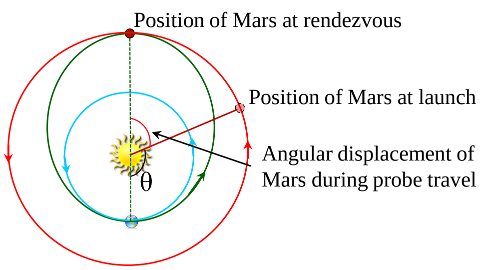


Figure Soln:9.2: Positions of Mars for Hohmann transfer orbit.

launch position of Mars and its position at the apogee of the transfer orbit.

$$\begin{aligned}\theta &= \pi - \frac{(2.24 \times 10^7 \text{ s})(2\pi/\text{Martian year})}{687 \text{ days/Martian year})(24 \cdot 3600 \text{ s/day})} \\ \theta &= 0.770 \text{ radians} = 44.1^\circ\end{aligned}$$

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