Classical Mechanics and Electrodynamics

Exercises – FYS 3120

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Part I

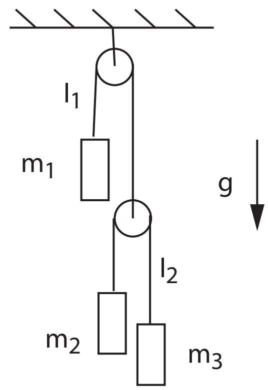
Exercises

Problem Set 1

Problem 1.1

An Atwood's machine consists of three parts, with masses $m_1 = 4m$, $m_2 = 2m$ and $m_3 = m$, that move vertically, and two rotating pulleys, which we treat as massless. The lengths of the ropes, which we also consider as massless, have fixed lengths l_1 and l_2.

Explain why the number of degrees of the system is 2 and and choose a corresponding set of generalized coordinates. Find the potential and kinetic energies of the system expressed as functions of the generalized coordinates and their time derivatives.



Problem 1.2

A particle with mass m moves in three-dimensional space under the influence of a constraint. The constraint is expressed by the equation

$$e^{-(x^2+y^2)} + z = 0 \tag{1}$$

for the Cartesian coordinates (x, y, z) of the particle.

a) Explain why the number of degrees of freedom of the particle is 2. Use x and y as generalized coordinates and find the expression for the position vector $\mathbf{r}$ of the particle in terms of x and y.

b) A virtual displacement is a change in the position of the particle $\mathbf{r} \to \mathbf{r} + \delta\mathbf{r}$ which is caused by an inifinitesimal change in the generalized coordinates, $x \to x + \delta x$ and $y \to y + \delta y$. Find $\delta\mathbf{r}$ expressed in terms of δx and δy.

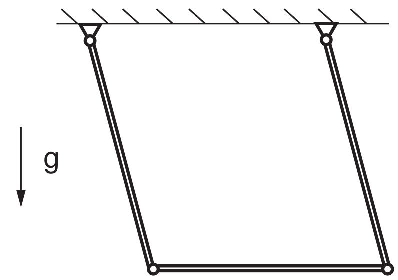
c) The constraint can be interpreted as a restriction for the particle to move on a two-dimensional surface in three-dimensional space. Any virtual displacement $\delta\mathbf{r}$ is a tangent vector to the surface while the constraint force $\mathbf{f}$ which acts on the particle is perpendicular to the surface. Use this to determine $\mathbf{f}$ as a function of x and y, up to an undetermined normalization factor (the length of the vector).

d) Make a drawing of a section through the surface for $y = 0$. Indicate in the drawing the direction of the two vectors $\mathbf{f}$ and $\delta\mathbf{r}$ for a chosen point on the surface.

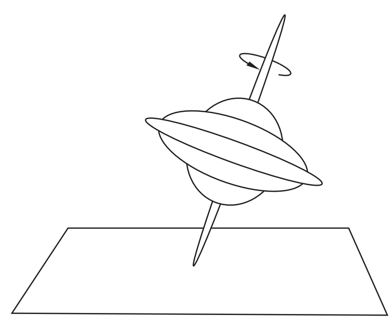
Problem 1.3

Three identicals rods of mass m and length l are connected by frictionless joints, as shown in the figure, with the distance between the points of suspension being equal to the length of the rods. The rods move in the plane. We remind about the expression for the moment of of inertia of one of the rods about its endpoint, $I = \frac{1}{3}ml^2$.

Choose a suitable generalized coordinate for the systems, and find the Lagrangian $L = T - V$ expressed as a function of the generalized coordinate and its time derivative.



Problem 1.4

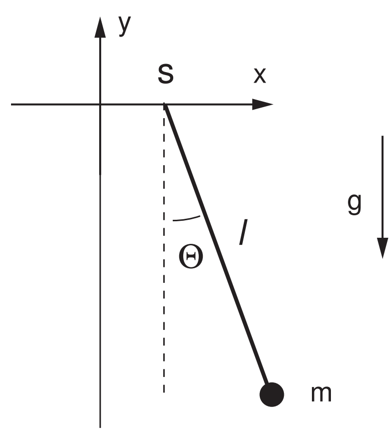


A rotating top is set into motion on a horizontal floor. Count the number of degrees of freedom of the top.

Problem Set 2

Problem 2.1

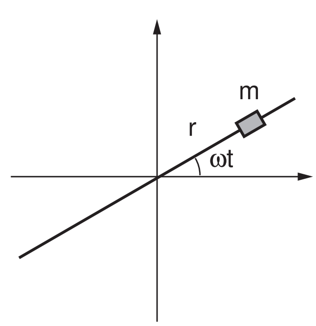
A pendulum consists of a rigid rod, which we consider as massless, and a pendulum bob of mass m. The point of suspension of the pendulum has horizontal coordinate $x = s$ and vertical coordinate $y = 0$.



a) Assume first that the point of suspension is kept fixed, with $s = 0$. Use the angle θ as generalized coordinate, find the Lagrangian of the system and determine the form of Lagrange's equation for the system. Check that it has the standard form of a pendulum equation.

b) The point of suspension is now released so it can move freely in the horizontal direction (x-direction). Use s and θ as generalized coordinates for the system and determine the corresponding set of Lagrange's equations. Show that the equations imply that the vertical motion of the pendulum bob is identical to free fall in the gravitational field (in reality restricted by the length l of the rod).

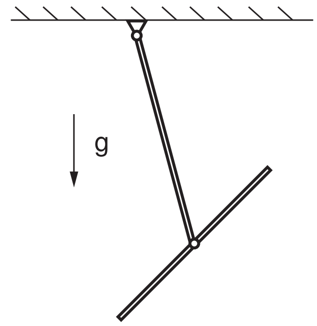
Problem 2.2



A small body of mass m moves without friction on a rod. The rod rotates in the horisontal plane about a fixed point with constant angular velocity ω. Find Lagrange's equation for the radial coordinate and solve the equation for the initial condition at $t = 0$, $\dot{r} = 0$ and $r = r_0$.

Problem 2.3

Two identical rods of mass m and length l are connected to each other with a frictionless joint. The first rod is connected to a joint in the ceiling and to a joint at the center of the second rod. Assume that the motion takes place in the vertical plane. Choose suitable generalized coordinates for the system, and find the corresponding Lagrangian. Formulate Lagrange's equations for the system.

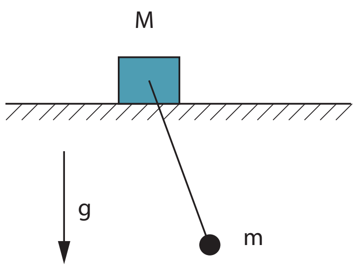


Problem Set 3

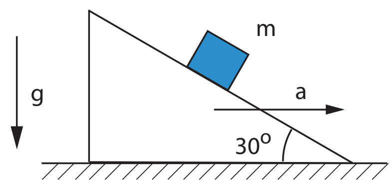
Problem 3.1

Choose suitable generalized coordinates for the systems specified below, and find the corresponding Lagrangians. Formulate in each case Lagrange's equations, and interpret the equations, when possible, in terms of other mechanical principles. Search for exact solutions, in the cases where they can be found, and look for possible constants of motion.

a) A pendulum is connected to a box that can slide without friction on a horizontal plane. Assume that the motion takes place in the vertical plane. The pendulum rod is considered as massless.

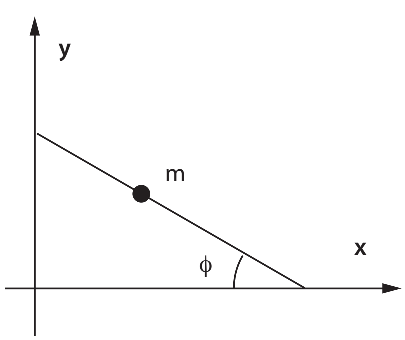


Problem 3.1 a)

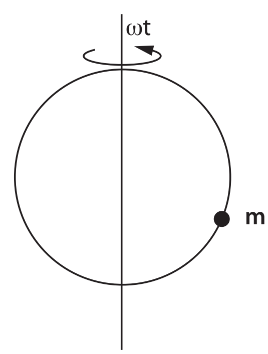


Problem 3.1 b)

b) A particle with mass m slides without friction on a tilted plane. The body that constitutes the the tilted plane is forced to to move horizontally with a constant acceleration a.



Problem 3.1 c)



Problem 3.1 d)

c) A rigid rod in the horizontal plane is forced to move in such a way that the end points are in contact with the coordinate axes. The angle φ increases linearly with time. A particle slides without friction along the rod.

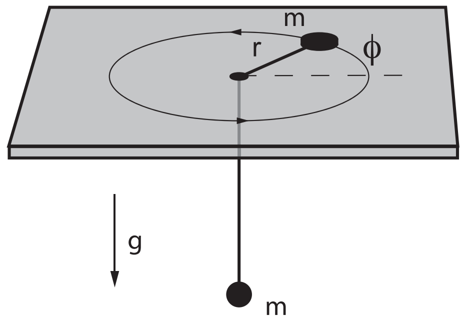
d) A rigid circular metal hoop rotates with constant angular velocity around an axis through the center. A particle slides without friction along the circle and there is no gravity.

Problem 3.2

Two bodies with the same mass, m, are connected with a massless rope through a small hole in a smooth horizontal plane. One body is moving on the plane, the other one is hanging at the end of the rope and can move vertically. At all instances the rope is tight. The acceleration due to gravity is g.

a) Find the Lagrange's equations of motion in polar coordinates (r, θ) and explain their physical meaning. Discuss special cases.

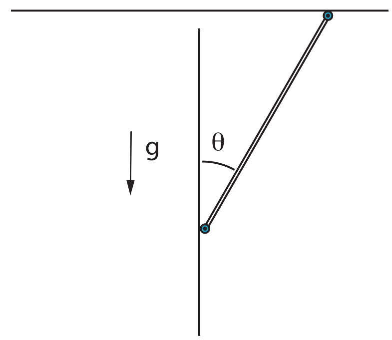
b) Reduce the equations of motion to a one dimensional problem in r with an effective potential and discuss the motion.



Problem Set 4

Problem 4.1

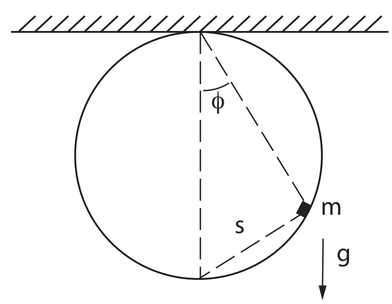
The figure shows a rod of length b and mass m. One endpoint of the rod is constrained to move along a horisontal line and the other endpoint along a vertical line. The two lines are in the same plane. There is no friction and the acceleration due to gravity is g.



a) Find Lagrange's equations with the angle θ as coordinate.
b) Find the period for small oscillations about the equilibrium position.
c) Find the period for oscillations with amplitude $\pi/2$.

Problem 4.2

A particle of mass m is attached to the circumference of a rigid circular hoop of radius r. The the hoop rolls on the underside of a horisontal line.



We assume the hoop to be massless and the motion to take place in a vertical plane. Find the Lagrangian, first with ϕ as generalized coordinate. Find the corresponding Lagrange's equation. Show next that the Lagrangian simplifies to that of a one-dimensional harmonic oscillator when s is used as generalized coordinate. What is the period of oscillations. Why is there a maximal allowed amplitude for the oscillations in s, and what happens when the total energy is larger than the energy corresponding to the maximum amplitude?

Problem 4.3

A particle of mass m and charge q is moving in a magnetic field given by the vectorpotential (in polar coordinates)

$$A_r = A_\theta = 0 \qquad\qquad A_\phi = \frac{k}{r}\tan(\theta/2), \tag{2}$$

where k is a constant. Throughout this problem we will use polar coordinates (r, ϕ, θ) and assume the motion to be non-relativistic. Assume also that there is no gravitational field.

a) Find the corresponding $\mathbf{B}$-field. Do you have a suggestion in what way such a magnetic field can be approximately realized.

b) Find the Lagrangian and Lagrange's equations for the charged particle.

c) Show that the kinetic energy is a constant of motion.

d) Explain the physical meaning of Lagrange's equation for r.

e) Show that there exists solutions of the form

$$r = (a^2t^2 + b^2)^{1/2} \qquad\qquad \theta = \theta_0, \tag{3}$$

where a, b,and θ_0 are constants.

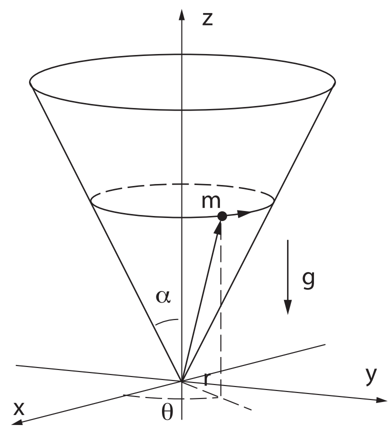
f) Give a physical interpretation of the constants a and b.

g) Make a sketch that shows the magnetic field and a trajectory of the type we have just found.

Problem Set 5

Problem 5.1

A particle moves on a circular cone of half angle α, which lies symmetrically about the positive z axis, as shown in the figure. The particle has mass m and moves without friction on the inner surface of the cone. The particle's position is given by the polar coordinates (r, ϕ) of the *projection* of the position vector into the x, y plane. The acceleration due to gravity is g in the negative z-direction.



a) Show that the Lagrangian for the particle is

$$L = \frac{1}{2}m(\dot{r}^2(1 + \cot^2\alpha) + (r\dot{\phi})^2) - mgr\cot\alpha. \tag{4}$$

b) Find Lagrange's equations for the particle.

c) Find two constants of motion and explain their physical meaning.

d) Which initial velocity $\mathbf{v}_0$ must be given to the particle in the point $(r = r_0, \phi = 0)$ to make it move in a horisontal trajectory? Show that the answer can be found from the equations of motion as well as from elementary ideas from Newtonian mechanics?

e) Show that the Hamiltonian is given by

$$H = \frac{p_r^2}{2m(1 + \cot^2\alpha)} + \frac{p_\phi^2}{2mr^2} + mgr\cot\alpha. \tag{5}$$

f) Show how the equations of motion that was found in part b) also can be found from Hamilton's equations.

g) Show how the two constants of motion in part c) can be found from the Hamiltonian formalism.

Problem 5.2 (Exam 2004 Problem 2)

A small sphere rolls on the inside of a hollow cylinder as shown in the figure. The motion is all the time taking place in a vertical plane (the x, y plane) under the influence of gravity. The inner radius of the cylinder is r an the mass of the sphere is m. The initial velocity of the small sphere is v_0 at the time when the sphere is at the bottom of the cylinder. We assume this velocity to be sufficiently large for the sphere to perform complete circulations inside the cylinder, so that at all times there is contact between the sphere and the cylinder

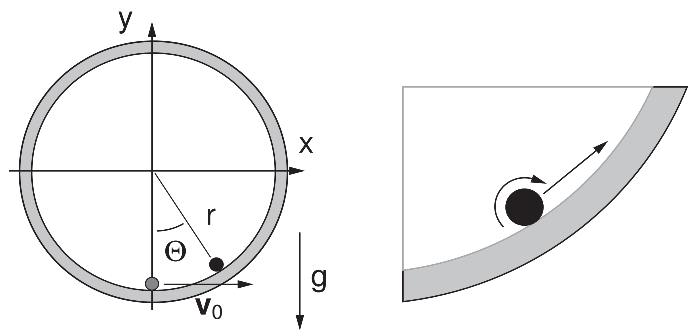


Fig 2a Fig 2b

In the first part of this problem we consider the radius of the sphere to be negligibly small and also disregard the moment of inertia of the sphere about its center of mass (Fig. 2a).

a) Choose a convenient generalized coordinate for the sphere and write the corresponding expression for the Lagrangian.

b) Formulate Lagrange's equation for the generalized coordinate and show that it has the form of a pendulum equation.

c) What is the minimum value for the initial velocity v_0 if the sphere should be in contact with the cylinder under a full revolution? Express this value in terms of r and m.

d) Take into account the effects of the sphere having a small, but finite radius a ($a << r$) and a non-vanishing moment of inertia $I = (2/5)ma^2$ about an axis through the center of mass (Fig. 2b).

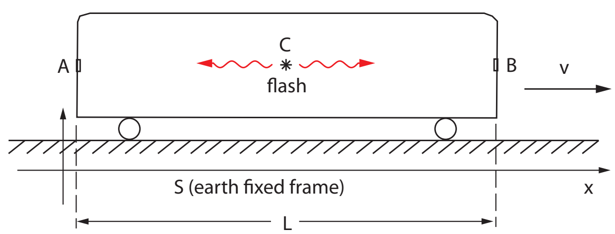
What is in that case the correct expression for the Lagrangian, and what is the smallest initial velocity v_0, if the sphere should stay in contact with the cylinder under a full revolution inside of the cylinder?

Problem Set 6

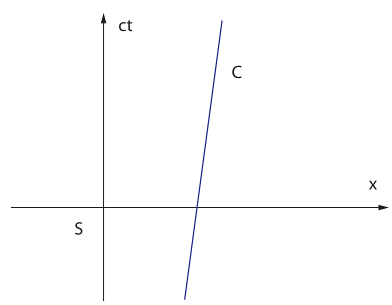
Problem 6.1

This problem is an exercise in using the postulates of special relativity in an elementary way.

A railway carriage is moving in a straight line with constant velocity v relative to the earth. The earth is considered as an inertial reference frame S, and in this reference frame the moving carriage has the length L. The situation is shown in Figure 1, where A and B indicate points on the rear wall and front wall of the carriage, respectively. C is a point in the middle of the carriage.



a) In Figure 2 we have drawn the world line (space-time trajectory) for the midpoint C in a two-dimensional Minkowski diagram of reference frame S. Draw the world lines for the points A and B in the same diagram and show that the angle α between these lines and the time axis is given by $\tan\alpha = v/c$. (Choose the origin of the coordinate system in S so that A has coordinate $x = 0$ at time $t = 0$.)



At a given time t_0 a flash tube is discharged at point C. We will call this event (space-time point) E_0. Some of the light will propagate backwards in the compartment and some will propagate forwards. Let E_1 and E_2 be the events where the light signals hit the rear wall and front wall, respectively. Let us assume that the light is reflected from A and B, and that the two reflected light signals meet at a space-time point E_3.

b) Draw the world lines of the light signals as well as the four events E_0, E_1, E_2 and E_3 in the Minkowski diagram of reference frame S.

c) We introduce the co-moving reference frame S' of the carriage. Explain why E_1 and E_2 are simultaneous in this reference frame and why E_0 and E_3 are at the same point in space in in S'. Is this consistent with the drawing of point b)?

d) Draw the straight line from from E_1 to E_2 in the Minkowski diagram of S and show that the angle between the x-axis and this line is α.

e) Show that if a signal should connect the two space-time points E_1 and E_2 it must have the velocity c^2/v (which is greater than c).

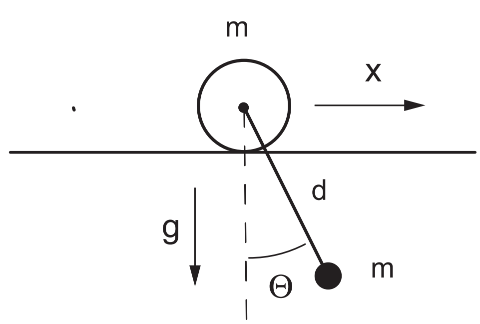
f) Let E be any event, *i.e.*, any space-time point, inside the carriage. Plot in the Minkowski diagram of S the events in the carriage system that are simultaneous with E in the co-moving frame S'. Plot in the same diagram the events that occur at the same place as E in reference frame S'.

Let the coordinates x' and t' of reference frame S' to be chosen such that $x' = 0$, $t' = 0$ corresponds to $x = 0$, $t = 0$.

g) In the Minkowski diagram of S, the coordinate axes of x and t appear as orthogonal lines. Draw in this diagram the coordinate axes of x' and t', corresponding to $t' = 0$ and $x' = 0$.

h) The lines plotted in g) define non-orthogonal axes for x' and ct'. The space-time position for any event E can in the diagram be read out either as x and ct in the orthogonal coordinate system of S or as x' and ct' in the non-orthognal coordinate system of S'. Explain how.

Problem 6.1. (Midterm Exam 2005)



Figur 1

A composite system is shown in Figure 1. A cylinder with mass m rolls without slipping on a horizontal table. To the axis of the cylinder is attached a pendulum which oscillates freely under the influence of gravity. The pendulum bob has the same mass m as the cylinder and the length of the pendulum rod is d. The rod is considered massless. Choose as generalized coordinates the horizontal displacement x of the cylinder and the angle θ of the pendulum rod relative to the vertical direction. The cylinder has radius R and has a homogeneous distribution of mass. Use the following initial conditions at time $t = 0$, $\dot{x} = 0$, $\dot{\theta} = 0$ and $\theta = \theta_0 \neq 0$.

a) Find the Lagrangian of the composite system.

b) Formulate Lagrange's equations for x og θ. What are the constants of motions that you can identify?

c) Show that by eliminating x that we obtain the following equation of motion for θ,

$$(1 - \frac{2}{5}\cos^2\theta)\ddot{\theta} + \frac{2}{5}\cos\theta\sin\theta\dot{\theta}^2 + \frac{g}{d}\sin\theta = 0 \tag{6}$$

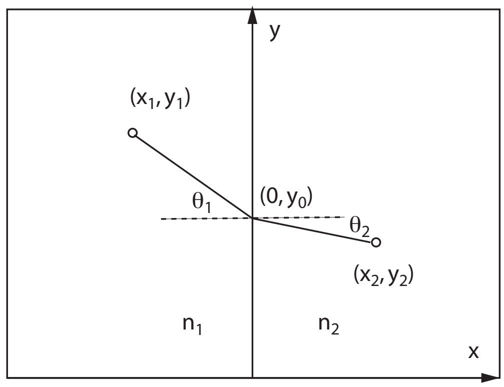
d) Assume $\theta_0 << 1$, so that the pendulum performs small oscillations around $\theta = 0$. Show that in this case the equation of motion reduces to an harmonic oscillator equation and determine the frequency of oscilations.

Problem Set 7

Problem 7.1 (Midterm Exam 2007)

According to Fermat's Principle, a light ray will follow the path between two points which makes the *optical path length* extremal. For simplicity we consider here paths constrained to a two dimensional plane (the x, y plane), in an optical medium with a position dependent index of refraction $n(x, y)$. The optical path length between two points (x_1, y_1) and (x_2, y_2) along $y(x)$ can be written as the integral

$$A[y(x)] = \int_{x_1}^{x_2} n(x, y)\sqrt{1 + y'^2}dx \;, \qquad y' = \frac{dy}{dx} \tag{7}$$



a) Find Lagrange's equation for the variational problem $\delta A = 0$, and express it as a differential equation for the function $y(x)$. Show that if the index of refraction is constant the equation has the straight line between the two points as solution.

b) Assume the medium to have two different, constant indices of refraction, $n = n_1$ for $x < 0$ og $n = n_2$ for $x > 0$ (ses Fig. 1). Explain why the variational problem can now be simplified to the problem of finding the coordinate $y = y_0$ for the point where the light ray crosses the boundary between the two media at $x = 0$. Find the equation for y_0 that gives the shortest optical path length. (Solving the equation is not needed.)

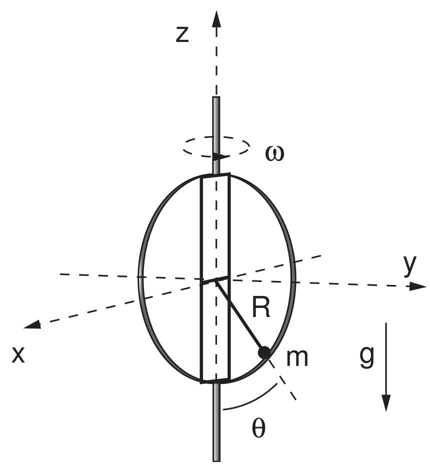
c) Show that the equation for y_0 at point b) implies that the path of the light ray satisfies Snell's law of refraction,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{8}$$

with θ_1 and θ_2 as the angle of the light ray relative to the normal on the two sides of the the boundary.

Problem 7.2 (Midterm Exam 2006)

A circular hoop is rotating with constant angular velocity ω around a symmetry axis with vertical orientation, as shown in Fig. 1. Inside the hoop a planar pendulum can perform free oscillations, while the plane of the pendulum rotates with the hoop. The mass of the pendulum bob is m, the length of the pendulum rod is R and the gravitational acceleration is g. The pendulum rod is considered as massless. As generalized coordinate we use the angle θ of the pendulum relative to the vertical axis.



a) Express the Cartesian coordinates of the pendulum bob as functions of θ and ω and find the Lagrangian of the pendulum.

b) Derive Lagrange's equation for the system. Find the oscillation frequency for small oscillations about the equilibrium point $\theta = 0$

c) Show that $\theta = 0$ is a *stable* equilibrium only for $\omega < \omega_{cr}$ and determine ω_{cr}. Show that for $\omega > \omega_{cr}$ there are two new equilibria $\theta_\pm \neq 0, \pi$ and determine the values of θ_+ and θ_- as functions of ω.

d) Study small deviations from equilibrium, $\theta = \theta_\pm + \chi$, with $\chi << 1$. Show that, for $\omega > \omega_{cr}$, the system will perform harmonic oscillations about the points θ_+ and θ_-. What are the corresponding oscillation frequencies?

The phenomenon where the original stable equilibrium $\theta = 0$ splits into two new equilibrium points θ_+ and θ_- is referred to as a *bifurcation*.

e) Find the Hamiltonian H of the system as function of θ and its conjugate momentum p_θ and derive the corresponding Hamilton's equations.

f) Consider the Hamiltonian $H(\theta, p_\theta)$ as a potential function of the two phase space variables θ and p_θ. Make a sketch of the equipotential lines $H(\theta, p_\theta) = const$ for the region around the equilibrium point $\theta = p_\theta = 0$, first in the case $\omega < \omega_{cr}$, and next in the case ω slightly larger than ω_{cr} (include in this case the new equilibrium points $(\theta_\pm, p_\theta = 0)$ in the drawing). Indicate in the drawing the direction of motion in the two-dimensional phase space. (A qualitative drawing is sufficient.)

Problem Set 8

Problem 8.1

We have below four equations that involve tensors of different ranks. Clearly the consistency rules for covariant equations are not satisfied in all places. Show where there are errors in each equation, and show how the equations can be modified to bring them to correct covariant form.

$$C^\mu = T^\mu{}_\nu A^\mu \,, \quad D_\nu = T^\mu{}_\nu A_\mu \,, \quad E_{\mu\nu\rho} = T_{\mu\nu} S^\nu{}_\rho \,, \quad G = S^{\mu\nu} T^\nu{}_\alpha A^\alpha \tag{9}$$

Problem 8.2

Assume A^μ to be a 4-vector and $T^{\mu\nu}$ a symmetric rank 2 tensor. Show that by making products and contracting indices with A^μ and $T^{\mu\nu}$, one can form at least four different scalars and at least one 4-vector (in addition to A^μ).

Problem 8.3

We have defined the following four tensor fields as functions of the space-time coordinates $x = (x^0, x^1, x^2, x^3)$,

$$f(x) = x_\mu x^\mu \,, \quad g^\mu(x) = \lambda x^\mu \,, \quad b^{\mu\nu}(x) = \alpha x^\mu x^\nu \,, \quad h^\mu(x) = \frac{x^\mu}{x_\nu x^\nu} \tag{10}$$

Calculate the following derivatives, and comment on what kind of tensor fields they represent

$$\partial_\mu f(x) \,, \quad \partial_\mu g^\mu(x) \,, \quad \partial_\mu b^{\mu\nu}(x) \,, \quad \partial_\mu h^\mu(x) \tag{11}$$

Problem 8.4

The *d'Alembertian* is the operator which generalizes the *Laplacian* $\boldsymbol{\nabla}^2$ to four space-time dimensions. It is defined as

$$\Box^2 \equiv \partial_\mu \partial^\mu = \frac{\partial^2}{\partial x^\mu \partial x_\mu} \tag{12}$$

Show that this operator is invariant under a change $x^\mu \to x'^\mu = L^\mu{}_\nu x^\nu$ between space-time coordinates of two different inertial reference frames. (Use the chain rule to express derivatives in one set of coordinates as derivatives in the other set.)

Problem 8.5

A thin rigid rod has rest length L_0 (length measured in the rest frame). It moves relative to an inertial frame S', so that the mid point of the rod has time dependent coordinates given by $x' = 0, y' = b - ut', z' = 0$, with b and u as positive constants. In this reference frame the rod is at all times parallel to the x' axis.

a) A point A on the rod has the distance a from the mid point, measured in S'. What are the time dependent coordinates of this point in the same reference frame?

b) The inertial frame S' moves with velocity v along the x axis of another inertial frame S. (The axes of the two frames are parallel.) Find the time dependent coordinates (x, y, z) of the point A in this reference frame.

c) What is the orientation of the rod relative to the coordinate axes of S, and what is the length of the rod measured in this frame?

Problem Set 9

Problem 9.1

An electron moving in a storage ring of radius $R = 10$m with a speed that gives $\gamma = 30$. Find the velocity of the particle. What is the lab time spent on one circulation in the ring, and what is the proper time. Find the acceleration a and the the proper acceleration a_0 (in the instantaneous rest frame).

Problem 9.2

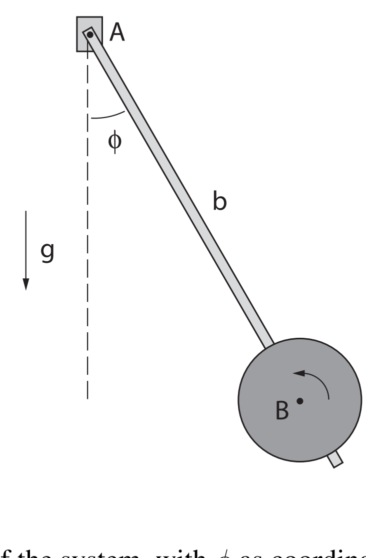
A particle moves with coordinates

$$x = ut, \qquad y = \frac{1}{2}gt^2, \qquad z = 0$$

(u and g are constants) in an inertial frame. What is the acceleration in this frame, and what is the proper acceleration a_0. What will happen with the proper acceleration as t increases?

Problem 9.3 (Midterm Exam 2008)

A pendulum can rotate freely about a horizontal axis A as shown in the figure. The pendulum consists of a rigid rod and attached to this a wheel which rotates about a point B on the rod. A motor (not included in the figure) affects the rotation of the wheel, so that the angular velocity measured *relative to the direction of the rod* changes linearly with time, $\omega = \alpha t$, where α is a constant over the period of time which we consider. For simplicity we assume that all other effects of the motor can be neglected and that friction can be disregarded. We also consider the mass of the pendulum rod to be negligible. The mass of the wheel is m and the moment of inertia about B is I. The distance between the points A and B is b. The gravitational acceleration is g and the angle of the pendulum rod relative to the vertical direction is denoted ϕ. We assume that the pendulum can perform full rotations about the axis A.



a) Show that the Lagrangian of the system, with ϕ as coordinate, is

$$L = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi \tag{13}$$

and find Lagrange's equation for the variable ϕ.

b) From general theory we know that we can modify the Lagrangian by adding a total time derivative

$$L(\phi, \dot{\phi}, t) \rightarrow L'(\phi, \dot{\phi}, t) = L(\phi, \dot{\phi}, t) + \frac{d}{dt} f(\phi, t) \tag{14}$$

without changing the equation of motion.

Show that if we in the present case choose

$$f(\phi, t) = -I\alpha\phi\, t - \frac{1}{6} I\alpha^2 t^3 \tag{15}$$

then the new Lagrangian L' will have no *explicit* time dependence. Show that Lagrange's equations for the new Lagrangian L' is the same as for L.

c) Introduce the dimensionless acceleration parameter

$$\lambda = \frac{I}{mgb}\alpha \tag{16}$$

and determine the angular position of *equilibrium points* of the pendulum, with ϕ given as function of λ. Explain why, depending on the value of λ, there are three different situations, so that for $|\lambda| < 1$ the pendulum has two equilibrium points, for $\lambda = 1$ it has one and for $|\lambda| > 1$ the pendulum has no equilibrium point. Draw a circle corresponding to different directions for the pendulum rod, and mark the positions of the equilibrium points for the parameter values $\lambda = 0,\ 0.5$ and 1.0. Indicate in the figure whether the equilibrium points are stable or unstable.

d) Find the canonical momentum p'_ϕ corresponding to ϕ with L' as Lagrangian and determine the corresponding Hamiltonian H'. Explain why H' is a constant of motion.

e) Make a two dimensional contour plot of the phase space potential function $H'(\phi, p'_\phi)$, for different values of λ, for example for $\lambda = 0,\ 0.5,\ 1.0$. Explain how the plots can be read as *flow diagrams* for the phase space motion and indicate in the plots the direction of motion. Give a qualitative description of the different types of motion that can be read out of the diagrams and comment on how the situation changes with increasing λ.

(The contour plot should show equidistant potential lines corresponding to constant values of H'. Choose length scales in the plots so that small oscillations for $\lambda = 0$ are described by circles in the phase space diagram – see corresponding figure in the lecture notes.)

Problem Set 10

Problem 10.1

A monocromatic light source is at rest in the laboratory and sends photons with frequency ν_0 towards a mirror which has its reflective surface perpendicular to the beam direction. The mirror moves away from the light source with velocity v. Use the transformation formula for 4-momentum and the Planck relation $E = h\nu$ to

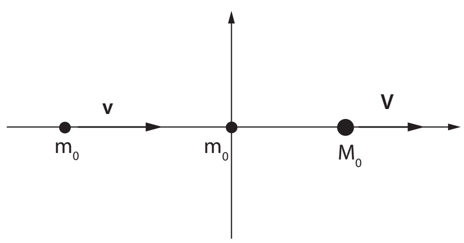
a) find the frequency of the emitted and reflected light in the rest frame of the mirror,

b) find the frequency of reflected light in the lab system.

Problem 10.2

Use conservation of relativistic energy and momentum to solve this problem.

Figure 1 shows a particle with rest mass m_0 and (relativistic) kinetic energy T in the laboratory frame S. The particle is moving towards another particle which is at rest in S, with the same rest mass m_0.

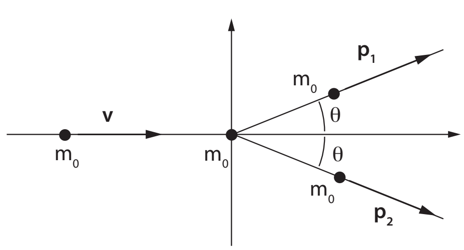


a) Find the velocity v of the first particle expressed in terms of the dimensionless quantity $\alpha = T/m_0c^2$ (and the speed of light).

First we will assume that the particles collide in such a way that they form one particle after the collision (totally inelastic collision.)

b) Determine the compound particle's energy E, momentum P, velocity V and rest mass M_0. Find the change in the total kinetic energy of the system due to the collision.

In the rest of the exercise we will assume that the situation before the collision is as described earlier, but that the particles now collide elastically, i.e. after the collision the two particles are the same as before the collision, with no change in their rest masses. The collision happens in such a way that the particles after the collision make the same angle, θ, with the x-axis in the lab frame S. See Figure 2.



c) Show that after the collision the particles have the same momentum ($|\mathbf{p}_1| = |\mathbf{p}_2|$) and energy ($E_1 = E_2$).

d) Determine $E \equiv E_1 = E_2$ and $p \equiv |\mathbf{p}_1| = |\mathbf{p}_2|$.

e) Determine the angle θ. Find θ in the limiting cases when $\alpha = T/m_0c^2$ goes to zero and to infinity. Show that $\theta < \pi/4$.

Problem 10.3 (Exam 2006)

An electron, with charge e, moves in a constant electric field $\mathbf{E}$. The motion is determined by the relativistic Newton's equation

$$\frac{d}{dt}\mathbf{p} = e\mathbf{E} \tag{17}$$

where $\mathbf{p}$ denotes the relativistic momentum $\mathbf{p} = m_e\gamma\mathbf{v}$, with m_e as the electron rest mass, $\mathbf{v}$ as the velocity and $\gamma = 1/\sqrt{1-(v/c)^2}$ as the relativistic gamma factor. We assume the electron to move along the field lines, that is, there is no velocity component orthogonal to $\mathbf{E}$.

a) Show that the electron has a constant proper acceleration $\mathbf{a}_0 = e\mathbf{E}/m_e$, which is the acceleration in an instantaneous rest frame of the electron.

b) Show that if $v = 0$ at time $t = 0$, then γ depends on time t as

$$\gamma = \sqrt{1+\kappa^2t^2} \tag{18}$$

and find κ expressed in terms of a_0.

c) Show that if we write $\gamma = \cosh\kappa\tau$ then τ is the proper time of the electron.

As a reminder we give the following functional relations:

$$\cosh^2 x - \sinh^2 x = 1\,, \quad \frac{d}{dx}\cosh x = \sinh x\,, \quad \frac{d}{dx}\sinh x = \cosh x \tag{19}$$

Problem Set 11

Problem 11.1

Two photons in the laboratory system have frequencies ν_1 and ν_2. The angle between the propagation directions is θ.

a) Write down the expressions for the total energy and momentum of the photons in the laboratory system.
b) Find the photons' frequency in the center of mass system.
c) Is it allways possible to find a center of mass system for the photons?

Problem 11.2

We send a photon towards an electron at rest.

a) What is the minimum energy of the photon required for the following process to take place

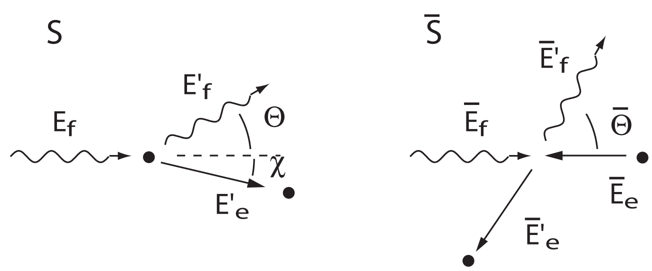
$$\gamma + e^- \rightarrow e^- + e^- + e^+ \tag{20}$$

The particles e^- and e^+ have the same rest mass m_0.
b) Show that the process

$$\gamma \rightarrow e^- + e^+ \tag{21}$$

is impossible.

Problem 11.3 (Exam 2005)



Figur 2a Figur 2b

A photon with energy $E_f = 100\,\text{keV}$ is scattered on a free electron which is, before the scattering, at rest in the laboratory frame. After the scattering the energy of the photon is E_f', and the direction of propagation makes an angle θ relative to the direction of the incoming photon. The rest energy of the electron is $E_e = m_e c^2 = 0.51\text{MeV}$, and after the scattering it has an energy which we denote by E_e'. The electron is scattered in a direction which makes an angle χ relative to the direction of the incoming photon.

We examine this process both in the lab frame S (Figur 2a) and in the center of mass system $\bar{S}$ (Figur 2b). In the center of mass system all variables are marked with a "bar", for example with $\bar{E}_f$ as the energy of the incoming photon.

a) What is meant by the center of mass system? Use the transformation formulas for energy and momentum to determine the relative velocity between the lab system and the center of mass system.

b) Explain why the energy of the incoming and outgoing photon is the same in the center of mass system and find this energy.

c) If $\theta = 90^\circ$ what is the energy of the outgoing photon in the lab system? What is the corresponding energy of the outgoing electron?

Problem Set 12

Problem 12.1

A thin straight conducting cable, oriented along the z axis in an inertial reference frame S, carries a constant current I. The cable is charge neutral.

a) Show, by use of Ampere's law, that the current produces a rotating magnetic field $\mathbf{B} = B(r)\mathbf{e}_\phi$, where (r, ϕ) are polar coordinates in the x, y plane and $\mathbf{e}_\phi$ is a unit vector in the direction of increasing ϕ. Determine the function $B(r)$.

Consider next the same situation in a reference frame S' that moves with velocity v along the z axis.

b) Use the fact that charge and current densities transform under Lorentz transformation as components of a current 4-vector to show that in S' the conducting cable will be charged. Determine the charge per unit length, λ' and the current I' in this reference frame.

c) Use Gauss' and Ampere's laws to determine the electric and magnetic fields, $\mathbf{E}'$ and $\mathbf{B}'$, as functions of the polar coordinates (r', ϕ') in reference frame S'.

d) Show that if the fields in S' are derived from the fields in S by use of the relativistic transformation formulas for $\mathbf{E}$ and $\mathbf{B}$, that gives the same results as found in c).

Problem 12.2

We consider a monochromatic plane wave that propagates in the z direction in a Cartesian coordinate system. For a given position $\mathbf{r}$ in space the electric field component $\mathbf{E}$ will describe a time dependent, periodic orbit in the x, y plane. The orbit will depend on the form of polarization of the electromagnetic wave.

The electromagnetic wave can generally be viewed as a superposition of two *linearly polarized* waves that propagate in the same direction, and which are polarized in orthogonal directions. We first choose these directions to be defined by the coordinate axes x and y. The amplitudes and the phases of the two partial waves may be different, and the general form of the electric field is therefore

$$\mathbf{E}(\mathbf{r}, t) = a\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_1)\mathbf{i} + b\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \phi_2)\mathbf{j} \tag{22}$$

where in general $a \neq b$ and $\phi_1 \neq \phi_2$.

In the following we consider the case where the two partial waves are 90^o out of phase and write this as

$$\mathbf{E}(\mathbf{r}, t) = a\cos(\mathbf{k}\cdot\mathbf{r} - \omega t)\mathbf{i} + b\sin(\mathbf{k}\cdot\mathbf{r} - \omega t)\mathbf{j} \tag{23}$$

a) Show that the orbit described by the time dependent electric field (23) is an ellipse with symmetry axes along the coordinate axes in the x, y plane. What determines the *eccentricity* of the ellipse?

We consider now a different decomposition of the same wave, in linearly polarized components along the rotated directions

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})\,, \quad \mathbf{e}_2 = \frac{1}{\sqrt{2}}(\mathbf{i} - \mathbf{j}) \tag{24}$$

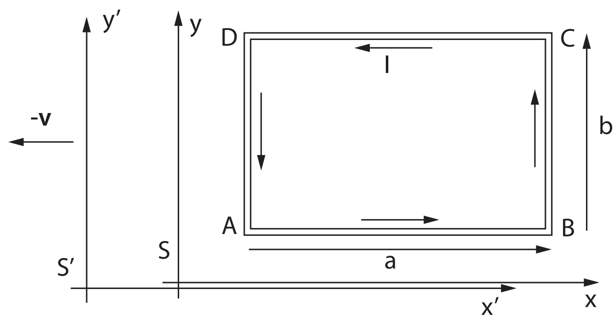
b) Show that in this new decomposition, the amplitudes of the two linearly polarized components are equal, $a' = b'$, but the relative phase $\Delta\phi = \phi'_1 - \phi'_2$ is different from 90^o (or $\pi/2$ in radians). Show that the relative phase $\Delta\phi$ is determined by the ration a/b in the first decomposition.

c) Assume the amplitude $|\mathbf{E}| = \sqrt{a^2 + b^2}$ to be fixed. Plot the orbit of $\mathbf{E}$ with $\mathbf{e}_1$ and $\mathbf{e}_2$ defining the horizontal and vertical axes for a set of different values of the relative phase $\Delta\phi$. Include the cases that correspond to linear and circular polarization and give the values of $\Delta\phi$ for these cases.

Problem Set 13

Problem 13.1

The figure shows a rectangular current loop ABCD. In the loop's rest frame, S, the loop has length a in the x direction and width b in the y direction, the current is I and the charge density is zero.



a) Show that in the rest frame the loop's electric dipole moment is zero and the magnetic moment is $\mathbf{m} = I\mathbf{a} \times \mathbf{b}$, where $I = j\Delta$ with j as the current density and Δ as the cross section area of the current wire.

In the following we will observe the loop from the system S' where the loop is moving with velocity $\mathbf{v}$ to the right ($\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$). We will now examine the system in S'.

b) What is the length and width of the loop in S'?

c) Show that the parts AB and CD of the loop have charge $\pm \frac{aIv}{c^2}$ in S'.

d) Show that the loop's electrical dipole moment is $\mathbf{p}' = -\frac{1}{c^2}\mathbf{m} \times \mathbf{v}$.

e) Find the current density in the four parts of the loop. Show that the current is $I\gamma$ in the AB and CD and I/γ in BC and DA.

f) Show that the magnetic moment in S' is $\mathbf{m}' = (1 - \beta^2/2)\mathbf{m}$.

Problem 13.2 (Exam 2007)

In a circular loop of radius a an oscillating current of the form $I = I_0 \cos\omega t$ is running. The current loop lies in the x, y plane. We use the notation $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ for the Cartesian unit vectors in the directions x, y and z, in order to reserve the symbol $\mathbf{j}$ for the current density. The current loop is at all times charge neutral.

a) Explain why the electric dipole moment $\mathbf{p}$ of the current loop vanishes, and show that the magnetic dipole moment has the following time dependence, $\mathbf{m}(t) = m_0 \cos\omega t\, \mathbf{e}_z$, with m_0 as a constant. Find m_0 expressed in terms of a and I_0.

As a reminder, the general expressions for the radiation fields of a magnetic dipole are

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi cr}\ddot{\mathbf{m}}_{ret} \times \mathbf{n}\,; , \quad \mathbf{B}(\mathbf{r}, t) = -\frac{1}{c}\mathbf{E}(\mathbf{r}, t) \times \mathbf{n} \tag{25}$$

with $\mathbf{m}_{ret} = \mathbf{m}(t - r/c)$ and $\mathbf{n} = \mathbf{r}/r$. In the following we assume that we study the fields far from the current loop (in the radiation zone) where the expressions (25) are valid.

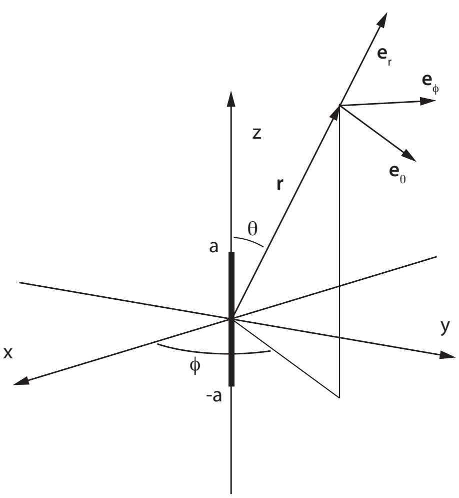
b) Write down the expressions for the radiation fields for points on the x axis far from the current loop and show that that they have the form of electromagnetic waves that propagate in the direction away from the loop. What is the polarization of the waves?

c) Use the general expression for Poynting's vector $\mathbf{S}$ to find the radiated power per unit solid angle $\frac{dP}{d\Omega}$, in the x direction. What is the corresponding radiated power in the direction of the z axis?

Problem Set 14

Problem 14.1

The figure shows a straight antenna of length $2a$ lying along the z-axis with its center at the origin. We assume that the charge of the antenna is at all times located at the endpoints. The current in the antenna (between the charged end points) is given by $I = I_0 \sin \omega t$ where ω and I_0 are constants. The antenna is electrical neutral at time $t = 0$. The field point is given by the position vector $\mathbf{r}$ and in spherical coordinates (r, θ, ϕ).



a) Show that the antenna's electrical dipole moment at time t is given by $\mathbf{p}(t) = \frac{2aI_0}{\omega}(1-\cos \omega t)\mathbf{k}$, where $\mathbf{k}$ is the unit vector in the z-direction.

We will now assume that the fields can be treated as electrical dipole radiation.

b) Find the components of the $\mathbf{B}$ and $\mathbf{E}$ fields in the directions $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ in the field point (r, θ, ϕ) at time t.

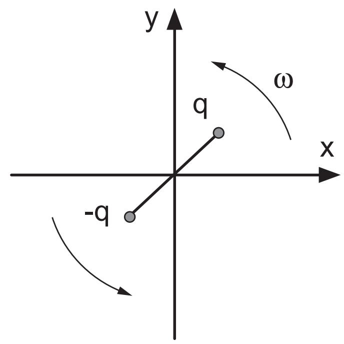
c) Show that the time average of the total radiated power in all directions kan be written as $\langle P \rangle = \frac{RI_0^2}{2}$ and find R (radiation resistance). What is the time average of the total power consumed by the antenna if it has an 'ordinary' resistance R_0 as well?

d) Find R for an antenna of length $2a = 5$ cm which is conducting a current with frequency $f = 150$ MHz. What is the time average of the total radiated power when $I_0 = 30$ A?

Problem 14.2 (Exam 2006)

A thin rigid rod of length ℓ rotates in a horizontal plane (the x,y-plane) as shown in Fig. 2. At the two end points there are fixed charges of opposite sign, $+q$ and $-q$. The rod is rotating with constant angular frequency ω. This gives rise to a time dependent electric dipole moment

$$\mathbf{p}(t) = q\ell(\cos \omega t\, \mathbf{i} + \sin \omega t\, \mathbf{j}) \tag{26}$$



a) Use the general expression for the radiation fields of an electric dipole (see the formula collection of the course) to show that the magnetic field in the present case can be written as

$$\mathbf{B}(\mathbf{r},t) = B_0(r)(\cos\theta\sin(\omega(t-\frac{r}{c}))\,\mathbf{i} - \cos\theta\cos(\omega(t-\frac{r}{c}))\mathbf{j} - \sin\theta\sin(\omega(t-\frac{r}{c})-\phi)\,\mathbf{k} \tag{27}$$

with (r,θ,ϕ) as the polar coordinates of $\mathbf{r}$. Find the expression for $B_0(r)$.

What is the general relation between the electric field $\mathbf{E}(\mathbf{r},t)$ and the magnetic field $\mathbf{B}(\mathbf{r},t)$ in the radiation zone? (A detailed expression for $\mathbf{E}(\mathbf{r},t)$ is not needed.)

b) Show that radiation in the x-direction is linearly polarized. What is the polarization of the radiation in the z-direction?

c) Find the time-averaged expression for the energy density of the radiation. In what direction has the radiated energy its maximum?

Part II

Midterm Exams 2004-2009

FYS 3120/4120: Klassisk mekanikk og elektromagnetisme

Midtterminevaluering våren 2004

Obligatorisk sett innleveringsoppgaver

Godkjent besvarelse
er nødvendig for å kunne avlegge eksamen. Besvarelsen teller med ved fastsettelse av eksamenskarakter.

Utlevering av oppgaver
foregår fredag 19. mars, oppgavene legges også ut på kursets web-side.

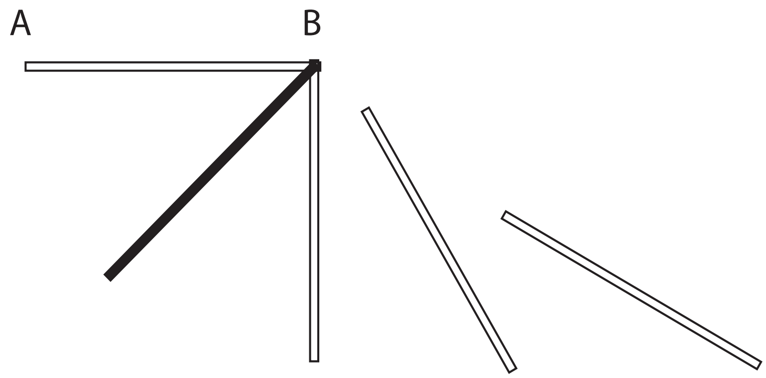
Frist for innlevering
er satt til fredag 26. mars.

Innlevering av besvarelser
Besvarelser kan leveres enten i skriftlig form eller som vedlegg i e-post.
Skriftlig innlevering kan gjøres på ekspedisjonskontoret eller til foreleser.
Lever en ekstra kopi som retter kan beholde til vurdering ved eksamen.
Innlevering pr. e-post: Lever besvarelsen som én fil, fortrinnsvis i pdf-format.
Sendes til **j.m.leinaas@fys.uio.no** med cc til **matsho@fys.uio.no**.

Spørsmål om oppgavene
kan rettes til Mats Horsdal, rom Ø466 eller Jon Magne Leinaas rom Ø471 (bortreist fredag 19 og mandag 22).

Husk
å skrive tilstrekkelig (men ikke unødvendig mye) med tekst til å gjøre forstått hva som foregår i besvarelsen.

Oppgavesettet
består av 3 oppgaver på de 4 neste sidene.



OPPGAVE 1
Stav i tyngdefelt.

En stiv jevntykk stav med lengde L og masse m beveger seg i tyngdefeltet. Bevegelsen foregår hele tiden i et vertikalt plan (x, y-planet). I utgangpunktet holdes staven fast i de to endepunktene (A og B) slik at den ligger horisontalt. Staven slippes så i det ene endepunktet (A) slik at den under den første del av bevegelsen roterer fritt om det andre endepunktet (B). Ved det tidspunkt ($t = 0$) som A befinner seg rett under B slippes staven helt og under den andre del av bevegelsen faller den fritt i tyngdefeltet.

a) Bestem vinkelhastigheten til staven og hastigheten til massesenteret ved tidspunktet $t = 0$.

b) Innfør (generaliserte) koordinater for å beskrive bevegelsen til staven under det frie fallet ($t > 0$) og sett opp den tilhørende Lagrangefunksjon.

c) Sett opp Lagranges ligninger og finn løsningene med de gitt initialbetingelser.

d) Formuler fallproblemet ($t > 0$) ved hjelp av Hamiltons formalisme. (Finn Hamiltonfunksjonen uttrykt ved generaliserte koordinater og impulser og sett opp Hamiltons ligninger.)

OPPGAVE 2
Betatron.

I denne oppgaven skal vi studere akselerasjon av en partikkel ved bruk av *elektromagnetisk induksjon*. Dette er et prinsipp som benyttes i partikkelakseleratorer kalt *betatroner*. Vi studerer en partikkel med masse m og ladning q som beveger seg i et horisontalplan (x, y-planet). I første del av oppgaven regner vi bevegelsen som ikke-relativistisk.

Partikkelen er utsatt for et elektromagnetisk felt som er beskrevet ved et vektorpotensial som i sylinderkoordinater har komponenter

$$A_r = 0,\ A_z = 0,\ A_\phi = \frac{1}{2} r \mathcal{B}(t) \tag{1}$$

for $r < r_{\mathrm{max}}$. Vi regner med at partikkelen hele tiden beveger seg i området $r < r_{\mathrm{max}}$. $\mathcal{B}(t)$ er konstant i dette området, men varierer med tiden.

a) Finn magnetfeltet $\vec{B}$ og det induserte elektriske felt $\vec{E}$ begge uttrykt i sylinderkoordinater. Hvordan varierer feltstyrkene med r?

b) Finn den ikke-relativistiske Lagrangfunksjonen for den ladete partikkelen uttrykt ved polarkoordinater i planet og sett opp Lagranges ligninger for de to variablene. Vis at disse ligningene er i overensstemmelse for bevegelsesligningen på vektorform for en ladet partikkel i et elektromagnetisk felt

$$\frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B}) \tag{2}$$

c) Anta først (for $t < 0$) at $\mathcal{B}$ er tidsuavhengig, $\mathcal{B} = \mathcal{B}_0$. Lagrangefunksjonens symmetrier tilsier at det finnes to bevegelseskonstanter, hvilke? Vis at bevegelsesligningene har løsninger som svarer til bevegelse i sirkel, $r = r_0$. Finn de to bevegelseskonstantene uttrykt ved $\mathcal{B}_0$ og r_0. Finn også sirkelfrekvensen (vinkelhastigheten) ω_0.

d) Anta nå at $\mathcal{B}$ har $\mathcal{B}_0$ som begynnelsesverdi ved $t = 0$ og øker langsomt med tiden t, slik at den ved tidspunktet $t = t_1$ har verdien $\mathcal{B}_1$. Partikkelen går i sirkelbane med radius r_0 ved $t = 0$. Denne forandres til en bane med radius r_1 ved $t = t_1$. Hvilke av størrelsene under punkt c) er det som nå er bevart når feltet økes? Bestem radius r_1 og vinkelhastighet ω_1 som partikkelen har ved $t = t_1$ uttrykt ved r_0 og ved feltstørrelsene $\mathcal{B}_0$ og $\mathcal{B}_1$. Hva blir energien til partikkelen $\mathcal{E}_1$

ved $t = t_1$ uttrykt ved energien $\mathcal{E}_0$ ved $t = 0$?

I det neste punktet behandler vi partikkelen relativistisk. Bevegelsesligningen på vektorform (2) er fortsatt gyldig forutsatt, $\vec{p}$ betegner den relativistiske impuls

$$\vec{p} = m\vec{v} = \gamma m_0 \vec{v}, \quad \gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \tag{3}$$

I dette uttrykket har vi innført m_0 som partikkelens hvilemasse, mens m nå er dens relativistise masse, som øker med v.

e) Anta igjen at feltet er tidskonstant, $\mathcal{B} = \mathcal{B}_0$. Vis at partikkelen også i den relativistiske beskrivelse kan bevege seg i en sirkulær bane med (vilkårlig) radius r_0. Hva blir uttrykkene for vinkelhastighet og energi? Sammenlign med de ikke-relativistiske uttrykk.

OPPGAVE 3
Tidsdilatasjon.

Et romskip forlater jordbane ved tiden $t = 0$ og setter kursen mot nærmeste stjerne, *Proxima Centauri*, som ligger i en avstand på 4.2 lysår. Romskipet har en utgangshastighet $v = 0$, og under første del av turen har det en konstant akselerasjon $a = g$ i retning mot Proxima Centauri. ($g = 9.8 m/s^2$ er tyngdeakselerasjonen ved jordoverflaten.)

Ved tidspunktet τ_0 målt i romskipets egentid forandres akselerasjonen til $a = -g$ slik at hastigheten bremses ned og romskipet når Proxima Centauri med sluthastighet $v = 0$. Vi betegner den første del av turen med I ($a = g$), og den andre del med II ($a = -g$). Romskipet tilbringer kun kort tid ved Proxima Centauri (vi ser bort fra denne tiden) før det returnerer mot jorden. Reisen tilbake foregår på samme måte som turen til Proxima Centauri. (Vi kaller de to delene av denne reisen III og IV.)

Den første del av turen (I) beskrives som en hyperbolsk tid-romsbane i den jordfaste referanseramme S. Posisjon og tid er under denne del av ferden gitt ved ligningene

$$x - x_I = \frac{c^2}{a} \cosh(\frac{a}{c}(\tau - \tau_I)), \quad t - t_I = \frac{c}{a} \sinh(\frac{a}{c}(\tau - \tau_I)) \tag{4}$$

hvor x_I, t_I og τ_I er konstanter. Tilsvarende uttrykk gjelder for de andre delene av turen, men med andre konstanter. Vi setter jortidskoordinater $x = 0, t = 0$ ved avreise av romskipet fra jorda. Vi setter også $\tau = 0$ ved dette tidspunkt.

a) Vis at tidsparameteren τ i ligning (4) svarer til egentiden til romskipet. Bestem romskipets 4-hastighet og 4-akselerasjonen (for del I av reisen) som funksjon av τ og vis at romskipet har en konstant egenakselerasjon lik a (akselerasjon i det instantane, inertiale hvilesystem). Forklar hvorfor banen kalles hyperbolsk.

b) Bestem konstantene i ligningene (4) for første del (I) av reisen og skriv opp ligningene på den form de da får, uttrykt ved tyngdeakselerasjonen g. Tegn et to-dimensjonalt tidrom-diagram (Minkowski-diagram) som viser hele reisen tur/retur Proxima Centauri.

c) Bestem egentidsparameteren τ_0. Finn hvor lang tid hele turen tar målt med romskipets egentid og målt med jordtid.

d) Hva er den største hastigheten romskipet oppnår under turen, målt i jordas referanseramme.

FYS 3120/4120: Klassisk mekanikk og elektromagnetisme

Midttermineksamen våren 2005

Obligatorisk sett innleveringsoppgaver

Godkjent besvarelse
er nødvendig for å kunne avlegge eksamen. Besvarelsen teller med ved fastsettelse av eksamenskarakter.

Utlevering av oppgaver
foregår fredag 1. april. Oppgavene legges ut på kursets web-side.

Frist for innlevering
er satt til fredag 8. april.

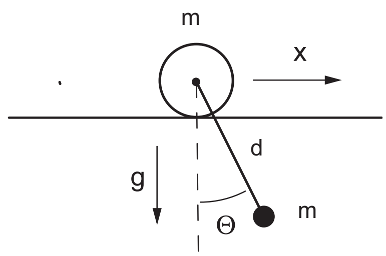
Innlevering av besvarelser
Besvarelser kan leveres enten i skriftlig form eller som vedlegg i e-post.
Skriftlig innlevering kan gjøres på ekspedisjonskontoret eller til foreleser.
Lever en ekstra kopi som retter kan beholde til vurdering ved eksamen.
Innlevering pr. e-post: Lever besvarelsen som én fil, fortrinnsvis i pdf-format. Sendes til **j.m.leinaas@fys.uio.no** med cc til **matsho@fys.uio.no**.

Spørsmål om oppgavene
kan rettes til Mats Horsdal, rom Ø466 eller Jon Magne Leinaas rom Ø471.

Husk
å skrive tilstrekkelig (men ikke unødvendig mye) med tekst til å gjøre forstått hva som foregår i besvarelsen.

Oppgavesettet
består av 3 oppgaver på de 3 neste sidene.

OPPGAVE 1
Sylinder og pendel.
Et sammensatt system er vist på figur 1. En sylinder med masse m ruller uten å gli på et horisontalt



Figur 1

bord. Til sylinderaksen er det festet en pendel som kan svinge fritt under påvirkning av tyngden. Pendelkula har samme masse m som sylinderen og lengden av pendelstanga er d. Pendelstanga regnes som masseløs. Som generaliserte koordinater velges forskyvningen x av sylinderen i horisontal retning og vinkelen θ for pendelutslaget. Sylinderen har radius R og har en homogen (konstant) massefordeling. Som initialbetingelse ved $t = 0$ setter vi $\dot{x} = 0$ og $\dot{\theta} = 0$, mens $\theta = \theta_0 \neq 0$.

a) Finn systemets Lagrangefunksjon.

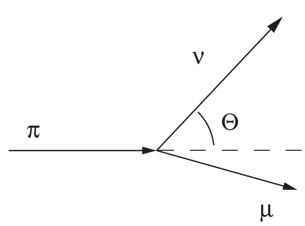
b) Sett opp Lagranges ligninger for variablene x og θ. Hvilke bevegelseskonstanter kan du identifisere?

c) Vis at ved å eliminere x får vi følgende bevegelsesligning for θ,

$$(1 - \frac{2}{5}\cos^2\theta)\ddot{\theta} + \frac{2}{5}\cos\theta\sin\theta\dot{\theta}^2 + \frac{g}{d}\sin\theta = 0 \qquad (1)$$

d) Anta små utslag om $\theta = 0$, dvs. $\theta_0 << 1$. Vis at ligningen i det tilfelle reduseres til en harmonisk oscillator-ligning og bestem svingefrekvensen.

OPPGAVE 2
Et akseleratorproblem.
Ved hjelp av en partikkelakselerator kalt en *protonsynkrotron* produseres en skarp stråle av ener-



Figur 2

girike π-mesoner. Vi regner at energien til π-mesonene er $E_\pi = 5300 MeV$, målt i laboratoriesystemet. π-mesonene er ustabile og disintegrerer i en μ-partikkel (myon) og et nøytrino (se figur 2, hvor nøytrinoet betegnes med ν). Levetiden (halveringstiden) til π-mesonene er $\tau_0 = 2.5 \cdot 10^{-8} s$, målt i partiklenes hvilesystem. Nøytrinoets hvilemasse regner vi lik 0, de andre massene er, uttrykt ved elektronmassen m_e,

$$m_\pi = 273 m_e \quad m_\mu = 207 m_e \tag{2}$$

Hvileenergien til elektronet er

$$m_e c^2 = 0.51 MeV \tag{3}$$

Energienheten MeV er relatert til SI-enheten Joule ved, $1MeV = 1.60 \cdot 10^{-13} J$.

a) Finn levetiden τ_L til π-mesonene i laboratoriesystemet. Hvor langt beveger de seg (i gjennomsnitt) før de disintegrerer.

b) Hvilken energi E_ν, målt i π-mesonets hvilesystem, har nøytrinoet som produseres ved disintegrasjonen av π-mesonet?

I det følgende betegner vi vinkelen mellom retningen til nøytrinoet og stråleretningen for π-mesonene med θ når den måles i π-mesonets hvilesystem og med θ' når den måles i laboratoriesystemet. Tilsvarende betegnelser for nøytrinoets energi er E_ν og E'_ν.

c) Anta et nøytrino sendes ut i en retning $\theta = \pi/2$, som i π-mesonets hvilesystem står vinkelrett på stråleretningen. Benytt transformasjonsformlene for impuls og energi til å finne hvilken retning θ' og hvilken energi E'_ν nøytrinoet vil ha i laboratoriesystemet? Begrunn ut fra det hvorfor halvparten av de produserte nøytrinoene vil ha energi større enn ca. $1125 MeV$ og retning innenfor en vinkel på $1.5°$ i forhold stråleretningen, begge målt i laboratoriesystemet.

d) Vis at generelt kan sammenhengen mellom θ og θ' uttrykkes ved

$$\tan\theta' = \frac{1}{\gamma}\frac{\sin\theta}{\beta + \cos\theta} \approx \frac{1}{\gamma}\tan\frac{\theta}{2} \tag{4}$$

hvor $\beta = v/c$ og $\gamma = 1/\sqrt{1-\beta^2}$, med v som π-mesonenes hastighet målt i laboratoriesystemet.

e) Vinkelfordelingen til nøytrinoene i hvilesystemet til π-mesonene kan uttrykkes ved funksjonen $n_0(\theta) = dN/d\theta$, hvor $N = N(\theta)$ er antallet nøytrinoer som kommer innenfor vinkelen θ fra strålingsretningen. Tilsvarende gir $n_L(\theta') = dN/d\theta'$ vinkelfordelingen i labsystemet. Ved uniform fordeling i hvilesystemet, vis at fordelingen $n_0(\theta)$ er gitt ved

$$n_0(\theta) = \frac{N_{tot}}{2}\sin\theta \tag{5}$$

hvor $N_{tot} = N(\pi)$ er det totale antall nøytrinoer, spredd i alle retninger. Finn den tilsvarende funksjon $n_L(\theta')$ i laboratoriesystemet.

OPPGAVE 3
Ladet partikkel i magnetfelt.

Ladete partikler som beveger seg i et magnetfelt vil tendere til å spiralere rundt de magnetiske flukslinjene. Hvis flukslinjene konvergerer, vil partiklenes bevegelse i retning langs magnetfeltet stoppe opp og de sendes tilbake i motsatt retning. Dette er et velkjent fenomen i sammenheng med det jordmagnetiske felt hvor ladete partikler fra sola kan fanges inn i en bevegelse langs feltlinjene med refleksjon mellom polene. Et resultat av dette er oppbyggingen av strålingsbeltet (Størmer- van Allen-beltet) rundt jorda

I denne oppgaven studerer vi fenomenet i en forenklet utgave, hvor magnetfeltet i hovedsak er homogent (konstant i rommet), med retning langs z-aksen, men med at lite inhomogent tillegg som gjør at feltlinjene har en langsom konvergens mot z-aksen.

a) Anta først at magnetfeltet $\mathbf{B} = \mathbf{B}_0 = B_0\mathbf{k}$ er tidsuavhengig og homogent. En ladet partikkel med ladning q og masse m beveger seg i feltet. Den har (ved $t = 0$) en utgangshastighet $\mathbf{v}_0$, hvor u_0 er hastighetskomponenten i z-retningen og $\mathbf{w}_0$ er hastighetskomponent i x, y-planet. Gjør rede for størrelsen og retningen til kraften som virker på partikkelen. Vis at bevegelsen generelt er en skruelinje om en akse i z-retningen, og at vinkelhastigheten rundt aksen er

$$\omega_0 = -\frac{q}{m}B_0 \tag{6}$$

Finn radius ρ_0 i sirkelbevegelsen rundt aksen.

I det følgende innfører vi et inhomogent tillegg til magnetfeltet. Uttrykt i sylinderkoordinater er vektorpotensialet

$$A_\rho = 0 \quad A_\phi = \frac{1}{2}\rho B_0 f(z) \quad A_z = 0\,. \tag{7}$$

med

$$f(z) = 1 + z^2/d^2 \tag{8}$$

hvor d er en typisk avstand som har å gjøre med forandringen i styrken på magnetfeltet.

b) Vis at magnetfeltet har følgende komponenter i sylinderkoordinater,

$$B_\rho = -\frac{1}{2}\rho B_0 \frac{df}{dz} \quad B_\phi = 0 \quad B_z = B_0 f(z)\,. \tag{9}$$

c) Sett opp uttrykket for Lagrangefunksjonen og Lagranges ligninger i sylinderkoordinater.

d) Vi antar at følgende relasjoner er (tilnærmet) oppfylt

$$\dot{\rho} = 0\,, \quad \dot{\phi} = -\frac{q}{m}B_z \tag{10}$$

Gi en begrunnelse for når dette kan være en god tilnærmelse?

d) Vis at med tilnærmelsenene ovenfor har vi følgende ligninger

$$\begin{aligned} qB_z\rho^2 &= -L_z \\ \frac{1}{2}m\dot{z}^2 - \frac{1}{2m}qB_zL_z &= T \end{aligned} \tag{11}$$

hvor L_z og T er bevegelseskonstanter. Kan du gi en fysisk fortolkning av disse størrelsene?

e)Vi antar følgende initialbetingelser,

$$z = 0 \quad \rho = \rho_0 \quad \dot{z} = u_0 \tag{12}$$

Vis at bevegelsen til partikkelen i z-retningen er begrenset av ytterpunkter $\pm a$ og bestem a. Skisser bevegelsen i en figur som viser feltlinjene og partikkelbanen

FYS 3120/4120: Classical mechanics and electrodynamics
Midtterm Exam, Spring semester 2006

Compulsory Homework Set

Return of a set of written solutions is necessary to be accepted for the final exam.

Return of solutions
The problem set is available on the course page from Friday, March 24.
Deadline for return of solutions is Friday, March 31.
Return of solutions to *Ekspedisjonskontoret* in the Physics building.

Language
The solutions can be written in Norwegian or English.

Questions
concerning the text can be posed to Mats Horsdal (office 469) or Jon Magne Leinaas (office 471).

Remember
to write sufficient comments to make clear what is the reasoning behind the derivations.

The problem set
consists of 3 problems printed on 3 pages.

PROBLEM 1
Rotating pendulum.

A circular hoop is rotating with constant angular velocity ω around a symmetry axis with vertical

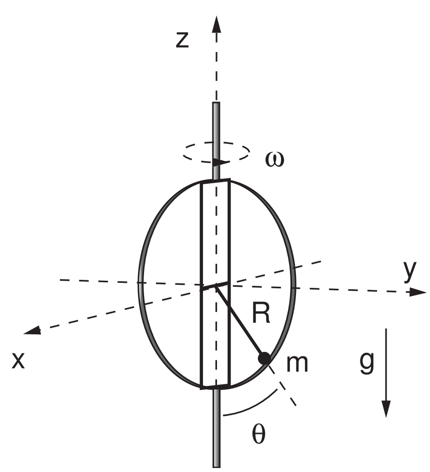


Figure 1:

orientation, as shown in Fig. 1. Inside the hoop a planar pendulum can perform free oscillations, while the plane of the pendulum rotates with the hoop. The mass of the pendulum bob is m, the length of the pendulum rod is R and the gravitational acceleration is g. The pendulum rod is considered as massless. As generalized coordinate we use the angle θ of the pendulum relative to the vertical axis.

a) Express the Cartesian coordinates of the pendulum bob as functions of θ and ω and find the Lagrangian of the pendulum.

b) Derive Lagrange's equation for the system. Find the oscillation frequency for small oscillations about the equilibrium point $\theta = 0$

c) Show that $\theta = 0$ is a *stable* equilibrium only for $\omega < \omega_{cr}$ and determine ω_{cr}. Show that for $\omega > \omega_{cr}$ there are two new equilibria $\theta_\pm \neq 0, \pi$ and determine the values of θ_+ and θ_- as functions of ω.

d) Study small deviations from equilibrium, $\theta = \theta_\pm + \chi$, with $\chi << 1$. Show that, for $\omega > \omega_{cr}$, the system will perform harmonic oscillations about the points θ_+ and θ_-. What are the corresponding oscillation frequencies?

The phenomenon where the original stable equilibrium $\theta = 0$ splits into two new equilibrium points θ_+ and θ_- is referred to as a *bifurcation*.

e) Find the Hamiltonian H of the system as function of θ and its conjugate momentum p_θ and derive the corresponding Hamilton's equations.

f) Consider the Hamiltonian $H(\theta, p_\theta)$ as a potential function of the two phase space variables θ and p_θ. Make a sketch of the equipotential lines $H(\theta, p_\theta) = const$ for the region around the equilibrium point $\theta = p_\theta = 0$, first in the case $\omega < \omega_{cr}$, and next in the case ω slightly larger than ω_{cr} (include in this case the new equilibrium points $(\theta_\pm, p_\theta = 0)$ in the drawing). Indicate in the drawing the direction of motion in the two-dimensional phase space. (A qualitative drawing is sufficient.)

PROBLEM 2

π disintegration

A charged pion ($\pi^\pm$) is an unstable particle with life time (measured in the rest frame) $\tau_\pi = 2.6 \cdot$

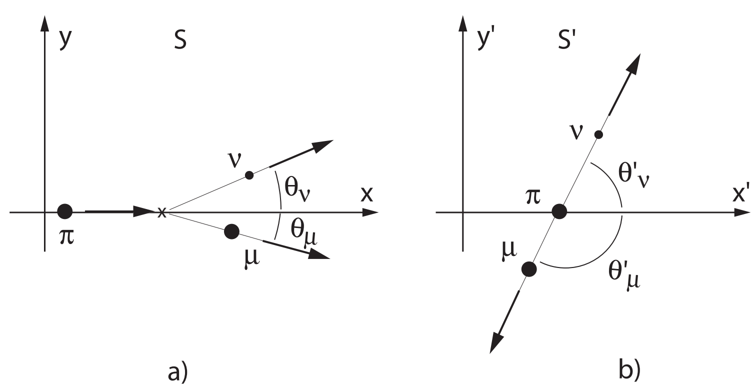


Figure 2:

$10^{-8} s$. It disintegrates into a muon ($\mu^\pm$) and a neutrino (ν_μ). The neutrino we consider as massless, while the pion rest energy is $m_\pi c^2 = 139\ MeV$ and the muon rest energy is is $m_\mu c^2 = 106\ MeV$.

The kinetic energy of a particle is defined as the total relativistic energy minus the rest mass energy. For energy the unit eV is used, while the momentum of the particle is measured in eV/c (energy divided by the speed of light). The energy unit MeV is related to the SI unit $Joule$ by $1 MeV = 1.60 \cdot 10^{-13} J$.

We consider a pion which in an inertial frame S moves with velocity $v = 0.8c$ in the x-direction before it disintegrates (see Fig. 2a). Use conservation of total 4-momentum, as well as the fact that the 4-momenta of the three particles transform as 4-vectors under Lorentz transformations to solve the problems below.

a) In the center-of-mass system S', where the pion is at rest (see Fig. 2b), explain why the neutrino is emitted with equal probability in all directions and why the muon will always be emitted in the direction opposite of the neutrino. Find the (relativistic) energy and momentum of the muon and of the neutrino, as measured in S'.

b) In the center-of-mass system S' what is the total kinetic energy released in the disintegration, and what is the velocity of the muon? (Since the neutrino is massless we consider all its energy to be kinetic.)

c) What is the life time of the pion in the inertial reference frame S, and how far does it move before it disintegrates?

d) Asume the neutrino is emitted at an angle $\theta'_\nu = \pi/2$ relative to the x-axis in S'. What is the corresponding angle θ_ν of the neutrino in S and what is the angle θ_μ of the muon? Find the energies and the momenta of the muon and the neutrino measured in S.

PROBLEM 3
A time dilatation problem

A space ship leaves earth orbit at local time $t = 0$ and head for the closest star, *Proxima Centauri*, at a distance of $d = 4.2$ light years. The space ship starts with a velocity $v = 0$ relative to earth, and during the first part of the journey it has a constant proper acceleration $a = g$ in the direction of Proxima Centauri, with $g = 9.8\ m/s^2$ as the gravitational acceleration at the earth surface.

At time τ_0 measured in the proper time of the space ship, the acceleration is reversed, so that $a = -g$ and the velocity decreases until it reaches Proxima Centauri with final velocity $v = 0$. The first part of the trip, with acceleration $a = g$, we refer to as part I, the second part, with $a = -g$, as part II. The space ship visits Proxima Centauri only for a short time, and we neglect this time interval in our description of the journey. The return travel to earth is carried out in the same way as the travel towards Proxima Centauri, and we refer to these parts of the journey as parts III and IV.

During the first part of the journey (part I), the coordinates of the space ship in the earth fixed reference frame S, define a hyperbolic space-time orbit, given by

$$x - x_I = \frac{c^2}{a}\cosh(\frac{a}{c}(\tau - \tau_I)), \quad t - t_I = \frac{c}{a}\sinh(\frac{a}{c}(\tau - \tau_I)) \tag{1}$$

with x_I, t_I og τ_I as constants. Similar expressions are valid for the other parts of the journey, but with other constants. The coordinates of the earth fixed reference system are set to $x = 0, t = 0$ at departure of the space ship. Also the time parameter τ is set to 0 at this event.

a) Show that the time parameter τ of Eq.(1) can be identified as the proper time of the space ship. Find the 4-velocity and 4-acceleration for Part I of the journey as a function of τ and check that the proper acceleration (the acceleration measured in the instantaneous inertial rest frame) defined by the path (1) is a. Explain why the space-time path is called hyperbolic.

b) Determine the constants of Eq. (1) for Part I of the journey, and write the form of the equations with the correct constants and with $a = g$. Draw a two-dimensional space-time diagram (Minkowski diagram) which shows the full journey to Proxima Centauri and back.

c) Determine the proper time value τ_0. Find the total time of the journey as measured on earth and on the space ship.

d) What is the maximum speed reached by the space ship on the journey, as measured on earth?

FYS 3120/4120: Klassisk mekanikk og elektrodynamikk

Midttermineksamen våren 2007
Obligatorisk sett innleveringsoppgaver

Innlevering av oppgaver
Oppgavene er tilgjengelige fra fredag 16. mars og kan lastes ned fra kursets webside. Frist for innlevering er satt til fredag 23. mars (i løpet av dagen). Besvarelsene leveres på ekspedisjonskontoret i 1. etg. i Fysikkbygget.

Spørsmål om oppgavene
kan rettes til Mats Horsdal, rom Ø469 eller Jon Magne Leinaas rom Ø471.
Merk: vi er begge tilgjengelige for spørsmål fra fredag t.o.m. tirsdag, men er begge bortreist f.o.m. onsdag 21. mars.

Tegning av figurer
Det blir noen steder bedt om at det tegnes figurer basert på matematiske uttrykk. Du kan gjerne benytte et matematikkprogram med plottefunksjoner til å lage disse hvis du har det tilgjengelig, men det er også helt i orden å lage en håndtegnet skisse som gir et kvalitativt riktig bilde.

Oppgavesettet
består av 3 oppgaver trykket på 4 sider.

OPPGAVE 1
Et oscillatorproblem
En partikkel med masse m beveger seg i én dimensjon (langs x-aksen), i et potensial

$$V(x) = \frac{1}{2}kx^2 + \frac{1}{4}ax^4 \tag{1}$$

der k og a er positive konstanter. Et slikt potensial omtales gjerne som et *anharmonisk* oscillatorpotensial.

a) Sett opp Lagrangefunksjonen for partikkelen og finn bevegelsesligningen fra Lagranges ligning.

b) Vis at hvis partikkelens utslag om likevektspunktet $x = 0$ er lite vil ligningen få form av en harmonisk oscillatorligning. Hva er sirkelfrekvensen ω? Hva betyr "lite utslag" i denne sammenhengen? Sett det opp som en betingelse på svingeamplituden A uttrykt ved konstantene i Lagrangefunksjonen.

c) Finn Hamiltonfunksjonen $H(x, p)$ og sett opp Hamiltons ligninger.

d) Lag et faseromsdiagram for tilfellet $a = 0$ i form av et to-dimensjonalt diagram med x og p som akser og med ekvipotensialkurver (høydekurver) for funksjonen $H(x, p)$. Bruk gjerne dimensjonsløse koordinater, hvor vi setter $m = k = 1$. Tegn inn et sett med ekvidistante høydekurver, der $H(x, p) = E$ er konstant.

Forklar hvorfor partikkelens bevegelse foregår langs høydekurvene i faseromsdiagrammet. Hva bestemmer bevegelsesretningen? Angi retningen på kurvene i diagrammet. Hva sier Hamiltons ligninger om sammenhengen det mellom farten til partikkelen i x,p-planet og hvor tett kurvene ligger, dvs. hvor stor gradienten til $H(x, p)$ er?

e) Tegn et tilsvarende faseromsdiagram med $a \neq 0$, f.eks. med $a = 1$. Angi også her bevegelsesretningen og gi en kvalitativ beskrivelse av forandringen fra diagrammet for den harmoniske oscillatoren

($a = 0$).

f) For den harmoniske oscillatoren er svingeperioden uavhengig av amplituden. Basert på en kvalitativ vurdering av faseromsdiagrammet, vil du anta at det også gjelder den anharmoniske oscillatoren der $a \neq 0$, eller vil perioden øke evt. minske med amplituden? Begrunn svaret.

Anta nå at x^2-leddet i potensialet skifter fortegn. Vi skriver potensialet som

$$V(x) = -\frac{1}{2}kx^2 + \frac{1}{4}ax^4 \tag{2}$$

der k og a fortsatt er positive konstanter.

g) Vis at punktet $x = 0$ nå er et ustabilt likevektspunkt, men at to nye stabile likevektspunkter har dukket opp ved $x = \pm b$. Bestem b.

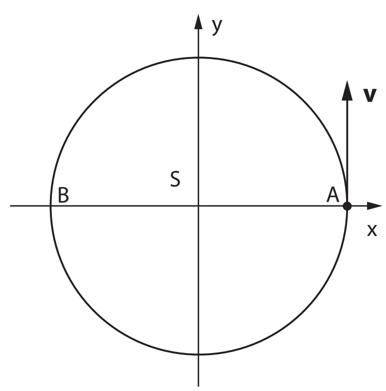
h) Finn svingefrekvensen for små svingninger om et av de nye stabile likevektspunktene.

i) Tegn et faseromsdiagram også i dette tilfellet og gi en kvalitativ beskrivelse av de forskjellige typer bevegelser partikkelen kan ha, basert på å studere diagrammet. Angi spesielt posisjonen til likevektspunktene i diagrammet.

OPPGAVE 2

Partikkel i akseleratorring.

En ladet partikkel beveger seg i sirkelbane i en partikkelakselerator. Den sirkelformete akseleratoren



Figur 1:

har radius $R = 10$ m og partikkelens hastighet svarer til en relativistisk gammafaktor $\gamma = 10$. Vi studerer partikkelens bevegelse, først i et inertialsystem S som er i ro i forhold til akseleratoren, med origo i sentrum av ringen (se fig. 1). Bevegelsen foregår i x,y-planet. Det betyr at $z = 0$ hele tiden, og vi kan derfor betrakte bevegelsen i et redusert 3-dimensjonalt tidrom med koordinater x, y og t. Vi antar at partikkelen befinner seg i punktet A på figuren (dvs. med $x = R$ og $y = 0$) ved $t = 0$.

a) Bestem farten v til partikkelen, vinkelhastigheten ω og omløpstiden T, alle størrelser bestemt i referansesystem S. Hva er akselerasjonen a i S?

b) Hva menes med partikkelens egentid τ? Gi sammenhengen mellom τ og koordinattiden t. Hva er omløpstiden T_τ målt i egentid?

c) Finn partikkelens tidrom-bane, beskrevet ved koordinatene som funksjon av egentid, $x^\mu(\tau)$, for $\mu = 0, 1, 2$. Hva slags kurve beskriver dette i det 3-dimensjonaldimensjonale tidrommet? Finn også de tre tidrom-komponentene til 4-hastigheten $U^\mu(\tau)$ og 4-akselerasjonen $A^\mu(\tau)$ i referansesystem S.

d) Akselerasjonen målt i partikkelens momentane hvilesystem kaller vi dens *egenakselerasjon* a_0? Sammenlign størrelsen av denne med størrelsen av akselerasjonen a målt i S.

Vi innfører et nytt inertialsystem S' som beveger seg relativt til S slik at partikkelen ved tidspunkt $t = 0$ ($\tau = 0$) er i ro i S' (punkt A i figuren). Vi velger koordinater i S' slik at partikkelen ved dette tidspunktet befinner seg i origo i S'. Koordinataksene i S og S' velger vi slik at x'-aksen er parallell med x-aksen og y'-aksen er parallell med y-aksen.

e) Skriv opp transformasjonsligningene som forbinder koordinatene i de to inertialsystemene S og S'.

f) Ved tidspunkt $t' = 0$ beskriver akseleratorringen en deformert sirkel i S'. Vis at det er en ellipse og bestem største og minste halvakse.

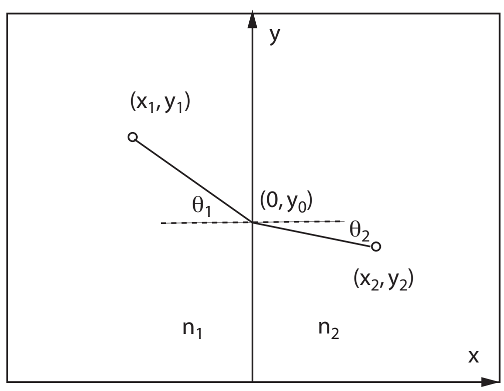
g) Finn partikkelbanen $x^{\mu\prime}(\tau)$, $\mu = 0, 1, 2$ i referansesystem S'. Tegn et 2-dimensjonalt plot som viser banen i x',y'-planet.

h) I inertialsystemet S' vil akselerasjonen a' til partikkelen variere med posisjonen i akseleratorringen. Hvor på akseleratorringen har partikkelen størst akselerasjon a'? Gi et begrunnet svar (beregning er ikke nødvendig).

OPPGAVE 3
Fermats prinsipp

Fermats prinsipp sier at en lysstråle vil følge den banen mellom to gitte punkter som minimaliserer den *optiske veilengden* mellom punktene. For enkelhets skyld antar vi at lysstrålen er begrenset til et to-dimensjonalt plan (x,y-planet), i et optisk medium med en posisjonsavhengig brytningsindeks $n(x, y)$. Den optiske veilengde mellom to punkter (x_1, y_1) og (x_2, y_2) langs banen $y(x)$ er da gitt ved virkningsintegralet

$$A[y(x)] = \int_{x_1}^{x_2} n(x, y)\sqrt{1 + y'^2}dx \,, \qquad y' = \frac{dy}{dx} \tag{3}$$



Figur 2:

a) Skriv opp Lagranges ligning, som svarer til variasjonsproblemet $\delta A = 0$. Uttrykk den som en differensialligning for funksjonen $y(x)$. Vis at hvis brytningsindeksen er konstant gir ligningen den rette linjen mellom endepunktene som løsning.

b) Anta at mediet har to forskjellige konstante brytningsindekser, $n = n_1$ for $x < 0$ og $n = n_2$ for $x > 0$ (se fig. 2). Forklar hvorfor minimaliseringsproblemet nå kan reduseres til å finne koordinaten

$y = y_0$ for punktet hvor lysstrålen krysser grensen $x = 0$. Finn ligningen for y_0, som gir den minste optiske veilengden. (Det er ikke nødvendig å løse ligningen.)

c) Vis at ligningen for y_0 i punkt b) impliserer at lysbanen tilfredsstiller Snells brytningslov,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \tag{4}$$

hvor θ_1 og θ_2 er lysstrålens vinkel med normalretningen på de to sidene av grenseflaten.

FYS 3120 Classical Mechanics and Electrodynamics
Midterm Exam, Spring semester 2008

Return of solutions
The problem set is available on the course page from Friday, April 4.
Deadline for return of solutions is Friday, April 11.
Return of solutions to *Ekspedisjonskontoret* in the Physics building.

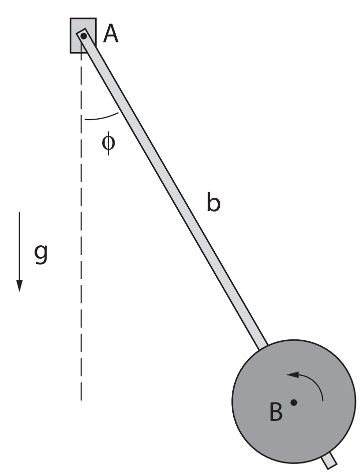
Language
The solutions can be written in Norwegian or English.

Questions
concerning the text can be posed to Per Øyvind Sollid (office 469) or Jon Magne Leinaas (office 471).

The problem set
consists of 3 problems printed on 5 pages.

PROBLEM 1
Pendulum with rotating wheel.

A pendulum can rotate freely about a horizontal axis A as shown in the figure. The pendulum consists of a rigid rod and attached to this a wheel which rotates about a point B on the rod. A motor (not included in the figure) affects the rotation of the wheel, so that the angular velocity measured *relative to the direction of the rod* changes linearly with time, $\omega = \alpha t$, where α is a constant over the period of time which we consider. For simplicity we assume that all other effects of the motor can be neglected and that friction can be disregarded. We also consider the mass of the pendulum rod to be negligible. The mass of the wheel is m and the moment of inertia about B is I. The distance between the points A and B is b. The gravitational acceleration is g and the angle of the pendulum rod relative to the vertical direction is denoted ϕ. We assume that the pendulum can perform full rotations about the axis A.



a) Show that the Lagrangian of the system, with ϕ as coordinate, is

$$L = \frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}I(\dot{\phi} + \alpha t)^2 + mgb\cos\phi \tag{1}$$

and find Lagrange's equation for the variable ϕ.

b) From general theory we know that we can modify the Lagrangian by adding a total time derivative

$$L(\phi, \dot{\phi}, t) \to L'(\phi, \dot{\phi}, t) = L(\phi, \dot{\phi}, t) + \frac{d}{dt}f(\phi, t) \tag{2}$$

without changing the equation of motion.

Show that if we in the present case choose

$$f(\phi, t) = -I\alpha\phi\, t - \frac{1}{6}I\alpha^2 t^3 \tag{3}$$

then the new Lagrangian L' will have no *explicit* time dependence. Show that Lagrange's equations for the new Lagrangian L' is the same as for L.

c) Introduce the dimensionless acceleration parameter

$$\lambda = \frac{I}{mgb}\alpha \tag{4}$$

and determine the angular position of *equilibrium points* of the pendulum, with ϕ given as function of λ. Explain why, depending on the value of λ, there are three different situations, so that for $|\lambda| < 1$ the pendulum has two equilibrium points, for $\lambda = 1$ it has one and for $|\lambda| > 1$ the pendulum has no equilibrium point. Draw a circle corresponding to different directions for the pendulum rod, and mark the positions of the equilibrium points for the parameter values $\lambda = 0,\ 0.5$ and 1.0. Indicate in the figure whether the equilibrium points are stable or unstable.

d) Find the canonical momentum p'_ϕ corresponding to ϕ with L' as Lagrangian and determine the corresponding Hamiltonian H'. Explain why H' is a constant of motion.

e) Make a two dimensional contour plot of the phase space potential function $H'(\phi, p'_\phi)$, for different values of λ, for example for $\lambda = 0,\ 0.5,\ 1.0$. Explain how the plots can be read as *flow diagrams* for the phase space motion and indicate in the plots the direction of motion. Give a qualitative description of the different types of motion that can be read out of the diagrams and comment on how the situation changes with increasing λ.

(The contour plot should show equidistant potential lines corresponding to constant values of H'. Choose length scales in the plots so that small oscillations for $\lambda = 0$ are described by circles in the phase space diagram – see corresponding figure in the lecture notes.)

PROBLEM 2
The brachistochrone challenge.

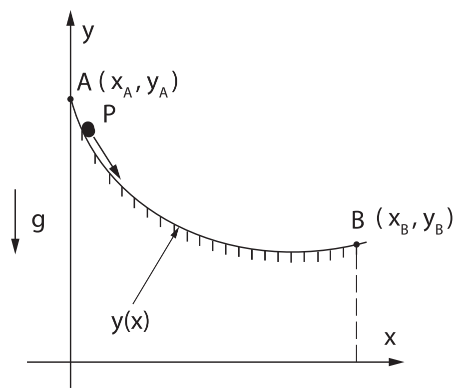
This is a classical problem in analytical mechanics. It was discussed by Galileo Galilei, who suggested a solution (but not the correct one), and studied the problem experimentally. In 1696 the problem was formulated as a challenge to the mathematicians at the time by Johann Bernoulli. He wrote in the journal *Acta Eruditorum*:

I, Johann Bernoulli, address the most brilliant mathematicians in the world. Nothing is more attractive to intelligent people than an honest, challenging problem, whose possible solution will bestow *fame and remain as a lasting monument. Following the example set by Pascal, Fermat, etc., I hope to gain the gratitude of the whole scientific community by placing before the finest mathematicians of our time a problem which will test their methods and the strength of their intellect. If someone communicates to me the solution of the proposed problem, I shall publicly declare him worthy of praise.*

The problem he formulated was the following:

Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time.

Five solutions were obtained from scientist and mathematicians we are all acquainted with, Newton, Jacob Bernoulli (the older brother of Johann), Leibniz and de L'Hôpital, in addition to Johann himself. Johann Bernoulli gave a formulation of the problem where he could use an anology to Snell's law of refraction in optics to solve the problem.



Let us now rephrase the problem with a few more words:
Assume a small body (P in the figure) moves in a vertical plane under the influence of gravity. It leaves a point A with zero velocity and follows (without friction) a given path in the plane which passes through a second point B, as shown in the figure. Assume the path between the two points A and B can be changed, while the points themselves stay fixed. For which path between the two points does the body spend the least time on the transit from point A to point B?

The challenge for you is the following:
Find the solution to the brachistochrone problem by using the correspondence between the variational problem (finding the "path of shortest time") and the Lagrange equation, in the way discussed in the lectures.

The body P is to be treated as a point particle of mass m and the path is represented by a function $y(x)$ with x as the horizontal and y as the vertical coordinate. The boundary conditions, which fix the positions of point A and B, are specified as, $y(x_A) = y_A\,,\ y(x_B) = y_B$. To simplify the equations assume in the following the initial coordinates are $x_A = y_A = 0$.

a) Show that the time T spent by the body on the way between A and B can be expressed as an integral of the form

$$T[y(x)] = \int_{x_A}^{x_B} L(y, y')dx \tag{5}$$

with $y' = \frac{dy}{dx}$ and with

$$L(y, y') = \sqrt{\frac{1 + y'^2}{-2gy}} \tag{6}$$

For the derivation it is convenient to make use of energy conservation, $\frac{1}{2}mv^2 + mgy = 0$ $(y < 0)$.

b) With $L(y, y')$ interpreted as a Lagrangian (x then plays the role of t in the usual formulation) the canonically conjugate momentum is $p = \frac{\partial L}{\partial y'}$ and the Hamiltonian is $H = py' - L$. Explain why H is a constant of motion and use this fact to show that $y(x)$ satisfies a differential equation of the form

$$(1 + y'^2)y = -k^2 \tag{7}$$

with k as a constant.

c) The equation has a solution which can be written on *parametric form* as

$$\begin{aligned} x &= \frac{1}{2}k^2(\theta - \sin\theta) \\ y &= \frac{1}{2}k^2(\cos\theta - 1) \end{aligned} \tag{8}$$

where θ has been introduced as a curve parameter. Show that (8) is a solution of the differential equation (7) by changing from x to θ as a variable in the equation and by using the above expression for $y(\theta)$. In what way are the boundary conditions taken care of by this solution?

d) The curve $y(x)$ defined by the solution of the brachistochrone problem is a section of a *cycloid*, known for example as the curve traced out by a point on the periphery of a rolling wheel. Make a plot which shows the form of the curve.

e) Assume that point B lies at the lowest point of the cycloid. Show that in this case the following relation has to be satisfied, $y_B = -\frac{2}{\pi}x_B$. Calculate for that situation the time used by the body to reach point B from A and compare with the time used when the body instead follows a straight line between the two points.

PROBLEM 3
Hyperbolic motion

In a particular inertial reference frame S the coordinates of a space ship are given as

$$t = \frac{c}{a_0}\sinh(\frac{a_0}{c}\tau), \quad x = \frac{c^2}{a_0}\cosh(\frac{a_0}{c}\tau), \quad y = z = 0 \tag{9}$$

with a_0 as a constant and τ as a time parameter. Since the motion is linear in the x direction we leave out in the following the coordinates y and z.

a) Give the general definition of the *proper time* (egentid) for a moving body in relativity, and show that the time parameter τ in Eq. (9) is the proper time of the space ship. Similarly give the definition of the *proper acceleration* and show that the constant a_0 is the proper acceleration of the space ship.

b) Determine the velocity $v = v(t)$ and acceleration $a = a(t)$ of the space ship, as registered in the inertial frame S. Express them as a functions of the coordinate time t.

c) A space station has coordinate $x = \frac{c^2}{a_0}$ and is at rest in reference frame S. It sends radio messages to the space ship at regular intervals $t_n, n = 0, 1, 2, ...$ Due to increasing Doppler shifts these are received in the space ship with increasing time difference. Show that only radio signals that are sent before a certain time, $t_n < t_{max}$ will be received in the space ship as long as it moves with constant proper acceleration. What is this limit time t_{max}?

d) Draw the Minkowski diagram of reference frame S with x and ct as coordinate axes and plot the space-time curve (*world line*) of the space ship. Plot also the world line of the space station and indicate the space time path of the radio signal sent from the space station at time t_{max}.

e) The constant proper acceleration can in reality be sustained only as long as the space ship does not run out of fuel. Assume the space ship has a very efficient engine that is based on emission of photons (massless particles). Use conservation of energy and momentum to show that in the rest frame of the space ship the change in its mass due to emission of photons is given by the equation

$$\frac{dm}{d\tau} = -\frac{a_0}{c} m(\tau) \tag{10}$$

(It is useful to consider energy and momentum conservation for photon emission in an infinitesimal time interval $d\tau$.)

f) When the ship finally runs out of fuel, 99% of the original rest mass m_0 has been converted to energy in the form of radiated photons. Find the proper time coordinate τ and velocity v of the space ship when this happens. Find also the relativistic energy $E = \gamma mc^2$ at that time, as registered in reference frame S, and show that it is close to half the original rest frame energy $E_0 = m_0 c^2$.

FYS 3120 Classical Mechanics and Electrodynamics
Midterm Exam, Spring semester 2009

Return of solutions
The problem set is available on the course page from Friday, March 20.
Deadline for return of solutions is Friday, March 27.
Return of solutions to *Ekspedisjonskontoret* in the Physics building.

Language
The solutions may be written in Norwegian or English.

Questions
concerning the text can be posed to Per Øyvind Sollid (office 469) or Jon Magne Leinaas (office 471).
We are available for questions on Friday March 20 and Monday, March 23.

The problem set
consists of 3 problems printed on 4 pages.

PROBLEM 1
Particle in a periodic potential

A particle of mass m moves in a one-dimensional periodic potential

$$V(x) = V_0(\sin x + a \sin^2 x) \tag{1}$$

with x as the coordinate in the direction of motion, a as an external parameter that can be varied, and V_0 as a constant that measures the strength of the potential. We assume both V_0 and a to be positive.

a) Determine the equilibrium points of the potential for different values of a, and indicate which of the equilibrium points that are stable and which ones are unstable. Discuss separately the cases $a < 1/2$ and $a > 1/2$.

b) Illustrate the situation by plotting the potential for the three values $a = 0$, 0.5 and 1. Discuss in what sense the situation changes when a increases through the value $1/2$ and relate this to the results of point a).

c) Give the expression for the Lagrangian of the particle and use Lagrange's equation to find the equation of motion of the particle.

d) Assume the particle performs small oscillations about one of the stable equilibrium points, with coordinate denoted by x_0. We write the postion coordinate as $x = x_0 + \xi$ with $|\xi| << 1$. Show that this condition allows us to simplify the equation of motion so it takes the form of an harmonic oscillator equation for ξ. Determine the oscillation frequency as a function of a for $a < 1/2$ and $a > 1/2$.

e) Find the Hamiltonian $H(x, p)$ of the system, with p as the conjugate momentum of the coordinate x. Explain what is meant by considering $H(x, p)$ as a *phase space potential* and describe how the motion in the two-dimensional phase space is determined by this potential.

f) Make a contour plot of the phase space potential which show equipotential lines for the three different situations $a = 0$, 0.5 and 1. Indicate in the diagrams the direction of motion of the particle.

Discuss, based on the plots, what are the different types of motion of the particle and indicate in the diagrams the location of the limiting curves, called *separatrices*, that separate the different types of motion.

PROBLEM 2
Fermat's principle

Fermat's principle says that a light ray will follow the path between two points that has the minimal *optical length*, which is the path that takes the *shortest time* for light to propagate. In an optical medium the speed of light (c_m) is modified by the index of refraction n, so that

$$c_m = \frac{c}{n} \tag{2}$$

with c as speed of light in vacuum. If the index is not a constant, but varies through the medium this implies that a light ray does not follow a straight line, but is bent.

a) Let us assume that the index of refraction depends on the vertical coordinate y, so that $n = n(y)$. A light ray is sent in the x, y-plane, with x as a horizontal coordinate, between an initial point with coordinates (x_1, y_1) and final point with coordinates (x_2, y_2). Show that Fermat's principle means that the path of the light ray, $y(x)$, between these points gives a minimal value of action integral

$$S[y(x)] = \int_{x_1}^{x_2} n(y)\sqrt{1 + y'^2}dx \ , \qquad y' = \frac{dy}{dx} \tag{3}$$

b) This minimization problem is equivalent to a Lagrange's equation. Formulate this equation and show that it leads to a differential equation for the path $y(x)$ of the light ray that can be written as

$$y'' = [\frac{d}{dy}\ln n(y)](1 + y'^2) \ , \qquad y'' = \frac{d^2y}{dx^2} \tag{4}$$

c) Show that the second order differential equation (4) can be integrated to give the following first order equation

$$\left(\frac{n(y)}{n_0}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \tag{5}$$

with n_0 as a constant.

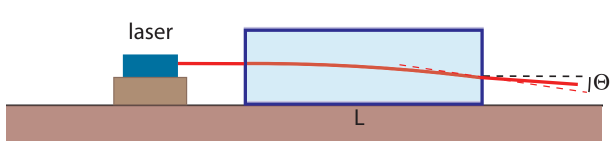


Figure 1: A laser beam is sent through a container with a sugar solutions. Due to variations in the strength of the solution the index of refraction decreases with height. This gives rise to a bending of the beam with θ as deflection angle.

We consider in the following the physical situation where a light ray is sent through a container with a strong sugar solution. The container has the length L in the x direction. Due to the effect of gravity the strength of the solution varies with height, and this gives rise to a variable index of refraction of the form

$$n(y) = n_0 \, e^{-\alpha y} \tag{6}$$

with n_0 and α as a constants.

d) Assume a light beam is sent in the horizontal direction (x direction) into the container. The point of entering we give coordinates $x_1 = y_1 = 0$. Explain why Eq.(5) shows that the beam is deflected in the direction of *increasing* strength of the solution, that means here downwards.

e) Show that Eq.(5) i satisfied if we assume the following relation between y and x along the light path

$$e^{-\alpha y} = \frac{1}{\cos \alpha x} \tag{7}$$

f) At the end of the container (inside the container at $x = L$) the light beam is deflected by an angle θ relative to the incoming beam. Find an expression for the deflection angle in terms of L and α.

g) After leaving the container the angle of the beam is θ'. Explain why θ' is different from θ and give an expression for θ'.

PROBLEM 3
Particle in circular orbit

An electron is circulating with constant speed in the accelerator ring LEP at CERN. The circumfer-

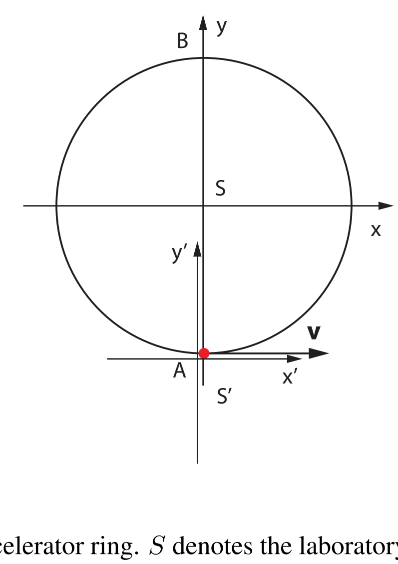


Figure 2: An electron circulates in an accelerator ring. S denotes the laboratory frame where the ring is at rest and with the center of the ring as origin. S' is the instantaneous rest frame of the electron, which is a t the origin of this reference frame when the electron is at the point A of the ring.

ence of the ring is 27 km, and we assume it here to be completely circular with radius R. The speed of the electron corresponds to a gamma factor $\gamma = 10^5$.

The laboratory frame S is the rest frame of the accelerator ring, and we assume that in the corresponding Cartesian coordinate frame the ring lies in the x, y plane with the center of the ring at the origin. Since the electron is restricted to move in this plane we in the following simply neglect the z coordinate and treat space-time as three-dimensional, with coordinates (ct, x, y).

a) Determine the velocity v of the electron in the lab frame S relative to the speed of light. What is the period T of circulation and the circular frequency ω? Find the acceleration a of the electron in the reference frame S.

b) Explain what is meant by the *proper time* τ of the particle and relate it to the coordinate time t of the reference frame S ? What is the period of circulation T_τ measured in proper time?

What is meant by the *proper acceleration* a_0 of the electron? Find the value of the electron's proper acceleration.

In the following we assume at an instant where the particle is located at the point $(x, y) = (0, -R)$ (point A in the figure) the time coordinates are $t = \tau = 0$.

c) Give the expressions for the coordinates of the electron's world line in S, $x^\mu(\tau)$, $\mu = 0, 1, 2$. (Express them as functions of proper time τ, radius R and circular frequency ω.) Give the corresponding expressions for the components of the four-velocity $U^\mu(\tau)$ and the four-acceleration $A^\mu(\tau)$.

A second inertial frame S' is introduced, which is the *instantaneous inertial rest frame* of the electron at time $t = 0$. The coordinate axes of S and S' are parallel and the electron is at the origin of Cartesian coordinate system of S' at the instant $t' = 0$.

d) Explain what is meant by the instantaneous inertial rest frame, and give the transformation between the Cartesian coordinates of the two inertial frames S and S'.

e) At time $t' = 0$ the accelerator ring defines a deformed circle in S'. Show that it is an ellipse and determine the lengths of the long and short axes.

f) What are the coordinates $x'^\mu(\tau)$, $\mu = 0, 1, 2$ of the electron's world line in reference system S'? Make a graphical representation of the trajectory in the x', y' plane. (A different scale for the two directions of the plane may be used.)

g) In reference frame S' the magnitude of the acceleration a' of the electron will change with its position in the accelerator ring. Where on the ring will it have its maximum value? Explain your answer - detailed calculation is not needed.

Part III

Final Exams 2004-2009

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 3120/FYS 4120 Klassisk mekanikk og elektromagnetisme,
FYS 202 - Klassisk teoretisk fysikk I
Eksamensdag: Onsdag 9. juni 2004
Tid for eksamen: kl. 14:30 (3timer)
Oppgavesettet er på 3 sider
Tillatte hjelpemidler: Lommekalkulator
Øgrim: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formelsamling FYS 3120/4120

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene

OPPGAVE 1
Kule i sylinder.

En liten kule ruller på innsiden av en hul sylinder som vist på figuren. Bevegelsen foregår hele

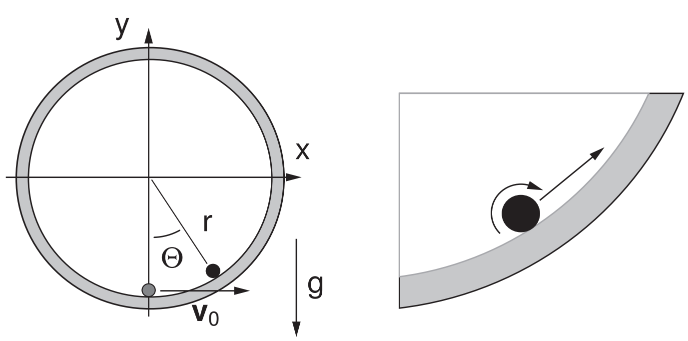


Fig 1a   Fig 1b

tiden i et vertikalplan (x, y-planet) under påvirkning av tyngden. Den indre radius til sylinderen betegnes r og massen til kula er m. Utgangshastigheten for kula betegnes v_0 ved det tidspunkt den befinner seg i bunnen av sylinderen. Vi antar at hastigheten er stor nok til at kula beskriver fullstendige omløp og at den hele tiden har kontakt med sylinderen.

I første del av oppgaven ser vi bort fra at kula har en liten radius og ser også bort fra at den har et treghetsmoment om massesenteret (Fig. 1a).

a) Velg generalisert koordinat og sett opp Lagrangefunksjonen til kula.

b) Sett opp Lagranges ligning og vis at den har form som en pendelligning.

c) Hva er den minste utgangshastigheten v_0 som kula kan ha hvis den hele tiden skal beholde kontakten med underlaget. Uttrykk svaret ved r og m.

d) Ta hensyn til at kula har en liten radius a ($a << r$) og et treghetsmoment $I = (2/5)ma^2$ om en akse gjennom massesenteret (Fig 1b). Hva blir nå det korrekte uttrykk for Lagrangefunksjonen og hva er nå den minste utgangshastigheten kula kan ha for ikke å miste kontakten med sylinderen?

OPPGAVE 2
Elektron i magnetfelt

Et elektron beveger seg i en sirkelbane under påvirkning av et konstant $\mathbf{B}$-felt. Vi tenker oss

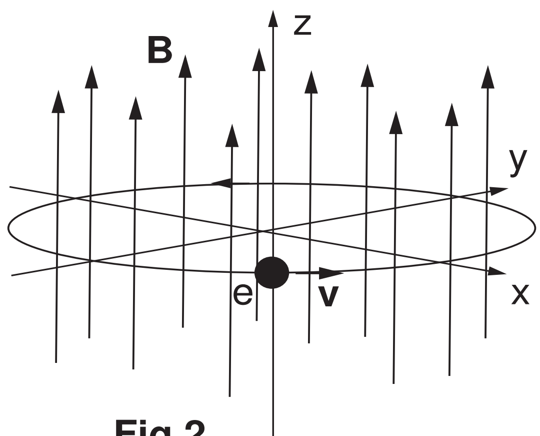


Fig 2

$\mathbf{B}$-feltet rettet langs z-aksen og elektronbanen i $x, y,$-planet med origo i sentrum av sirkelbanen. Radius i sirkelbanen er $r = 3.0m$ og hastigheten er $v = 0.9998c$ med $c = 3.0 \times 10^8 m/s$ som lyshastigheten. Elektronmassen er oppgitt til $m_0 = 9.1 \times 10^{-31} kg$ og ladningen til $e = -1.6 \times 10^{-19} C$.

a) Bestem elektronets γ-faktor og relativistiske masse m.

b) Hva er akselerasjonen til elektronet i laboratoriesystemet og hva er egenakselerasjonen (akselerasjonen målt i et momentant inertielt hvilesystem)?

c) Finn styrken på magnetfeltet B. (SI-enheten som benyttes for magnetisk feltstyrke er Tesla, $1T = 1kgs^{-2}A^{-1} = 1kgs^{-1}C^{-1}$.)

d) Finn styrken på magnetfeltet B' og det elektriske feltet E' i elektronets momentane hvilesystem. Hvilken retning har disse feltene?

OPPGAVE 3
Akselerert myon

Et myon er en ustabil ladet partikkel, og kan sees på som en tyngre utgave av elektronet. Det har en midlere levetid som vi betegner med τ_0. Vi studerer i oppgaven et myon som skapes i laboratoriet ved tid $t = 0$ og som beveger seg rettlinjet under påvirkning av et konstant elektrisk felt $\mathbf{E}$. Hastigheten er $\mathbf{v} = 0$ ved $t = 0$. Hvilemassen til myonet betegner vi m_0 og ladningen e.

a) Den relativistiske utgave av Newtons 2. lov er

$$\frac{d}{dt}\mathbf{p} = \mathbf{F} \tag{1}$$

med $\mathbf{p}$ som den relativistiske impuls og $\mathbf{F}$ som kraften som virker på partikkelen. Benytt denne til å finne størrelsen til myonets relativistiske impuls p og dets relativistiske masse m som funksjoner av laboratorietiden t. (Vi ser bort fra strålingsfeltets påvirkning på partikkelen.)

b) Vis at levetiden til myonet målt i laboratoriet er

$$t_0 = \frac{1}{k}\sinh(k\tau_0) \ , \quad k = \frac{eE}{m_0 c} \tag{2}$$

c) Den relativistiske formen på Larmors formel er

$$P = \frac{\mu_0 e^2}{6\pi c}{\mathbf{a}_0}^2 \tag{3}$$

hvor $\mathbf{a}_0$ er myonets egenakselerasjon og P er den utstrålte effekt fra partikkelen. P er en Lorentz-invariant størrelse, dvs. uttrykket (3) er gyldig i ethvert inertialsystem. Benytt formelen til å finne energien som myonet utstråler i løpet av sin levetid.

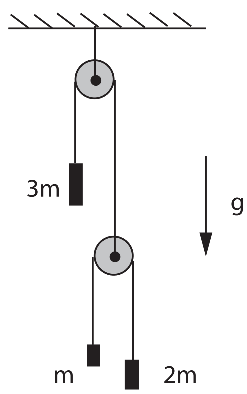
UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 3120/FYS 4120 Klassisk mekanikk og elektromagnetisme,
Eksamensdag: Tirsdag 7. juni 2005
Tid for eksamen: kl. 14:30 (3timer)
Oppgavesettet er på 3 sider
Tillatte hjelpemidler: Godkjent kalkulator
Øgrim og Lian eller Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formelsamling FYS 3120/4120

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene

OPPGAVE 1
Trinser og klosser



Figur 1

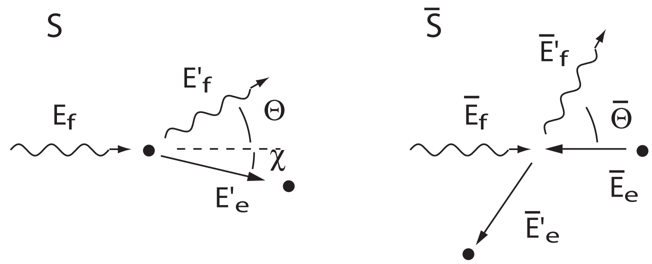
Et mekanisk system er sammensatt av tre klosser og to trinser, som vist i Figur 1. Klossene har masser, $m_1 = 3m$, $m_2 = m$ og $m_3 = 2m$. Trinsene er like, med masse m, radius R og treghetsmoment $I = \frac{1}{2}mR^2$. Systemet beveger seg under påvirkning av tyngdekraften. Snorene ruller på trinsene uten å gli. Ved tidspunktet $t = 0$ er alle deler av systemet i ro, og systemet beveger seg deretter fritt under påvirkning av tyngden.

a) Hvor mange frihetsgrader har systemet? Velg passene (generaliserte) koordinater og sett opp Lagrangefunksjonen for systemet.

b) Sett opp Lagranges ligninger med de valgte koordinater og løs ligningene.

c) Finn akselerasjonen til de tre klossene (uttrykt ved tyngdeakselerasjonen g).

OPPGAVE 2
Compton-spredning.



Figur 2a Figur 2b

Et foton med energi $E_f = 100\,keV$ spres mot et fritt elektron som ligger i ro i laboratoriesystemet. Etter spredningen har fotonet energi E'_f, og retning som danner vinkelen θ i forhold til retningen til det innkommende fotonet. Elektronets hvile-energi er $E_e = m_e c^2 = 0.51 MeV$, og etter spredningen har det en energi som vi betegner med E'_e. Elektronet blir spredd i en retning som danner vinkelen χ i forhold til retningen til det innkommende fotonet.

Vi studerer denne prosessen både i labsystemet S (Figur 2a) og i massesentersystemet $\bar{S}$ (Figur 2b). I massesentersystemet markerer vi alle variable med strek over, f.eks. er energien til det innkommende fotonet $\bar{E}_f$.

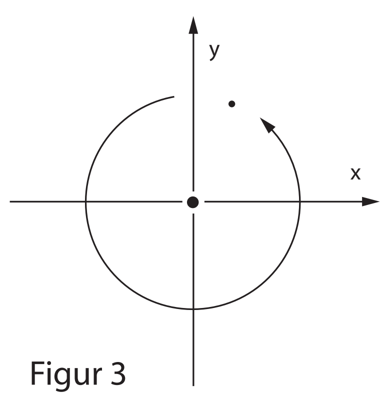
a) Hva menes med massesentersystemet? Benytt transformasjonsformlene for impuls og energi til å bestemme den relative hastighet mellom laboratoriesystemet og massesentersystemet.

b) Forklar hvorfor det innkommende og det utgående foton må ha samme energi i massesentersystemet. Finn denne energien.

c) Når $\theta = 90^\circ$ hva er da energien til det utgående fotonet i labsystemet? Hva er den tilsvarende energi til det utgående elektronet?

OPPGAVE 3
Klassisk atommodell

I en enkel klassisk modell av hydrogenatomet beveger det negativt ladete elektronet seg i en



Figur 3

sirkulær bane om den positivt ladete kjernen. Elektronet har en liten masse $m_e = 9.1 \cdot 10^{-31} kg$ og ladning $q = -1.60 \cdot 10^{-19} C$. Kjernen er mye tyngre og vi kan regne at den hele tiden ligger i ro. Kjerneladningen er like stor som elektronets, men med motsatt fortegn. Radius i elektronbanen setter vi lik Bohr-radius $a_0 = 0.53 \cdot 10^{-10} m$. Vi velger baneplanet som x,y-planet med kjernen i origo, som vist på Figur 3. I første omgang ser vi bort fra strålingseffekter og antar et elektronet beveger seg bare under påvirkning av Coulombfeltet fra kjernen. Både kjernen og elektronet ser vi på som punktpartikler. Vi ser bort fra partiklenes egenspinn, og vi antar at ikke-relativistiske uttrykk kan benyttes.

Permittiviteten i vakuum har verdien $\epsilon_0 = 8.85 \cdot 10^{-12} F/m$ (eller $8.85 \cdot 10^{-12} C^2 N^{-1} m^{-2}$), permeabiliteten i vakuum er $\mu_0 = 4\pi \cdot 10^{-7} N/A^2$ og lyshastigheten er $c = 3.00 \cdot 10^8 m/s$

a) Bestem sirkelfrekvensen ω til elektronet i banen.

b) Benytt Larmors strålingsformel til å finne utstrålt energi pr. tidsenhet.

c) Energibevaring tilsier at den utstrålte energi må komme fra elektronets kinetiske og potensielle energi. Anta at energitapet gir en langsom reduksjon i radius til elektronbanen. Finn den tidsderiverte av radius, for $r = a_0$, og gi utfra det et estimat på levetiden til et slikt klassisk atom.

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 3120/FYS 4120 Classical mechanics and electrodynamics
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Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formula Collection FYS 3120/4120

This paper is available also in Norwegian (Bokmål or Nynorsk) language.

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PROBLEM 1
Composite system
A composite mechanical system, shown in Fig. 1, consists of two parts. Part A is a cylinder with

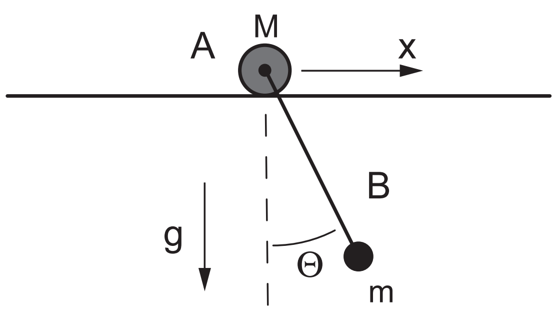


Figure 1:

mass M, radius R and moment of inertia about the symmetry axis $I = \frac{1}{2}MR^2$. The cylinder rolls without sliding on a horizontal plane. Part B is a pendulum with the pendulum rod attached to the symmetry axis of the cylinder. It oscillates without friction about the point of attachment in the plane orthogonal to the symmetry axis of the cylinder. The pendulum rod has length L and we consider it as massless. The mass of the pendulum bob is $m = M/2$.

Use in the following the horizontal displacement x of the cylinder and the pendulum angle θ as generalized coordinates for the composite system.

a) Find the Lagrangian of the system expressed as a function of the generalized coordinates and their time derivatives.

b) Examine if there are cyclic coordinates, and find constants of motion.

c) Formulate Lagrange's equations for the system and assume the following initial conditions: $x = 0$, $\dot{x} = 0$, $\theta = \theta_0$, $\dot{\theta} = 0$ at time $t = 0$. Assume $\theta_0 << 1$ and simplify the equations by using a small angle approximation. Show that the system in this case will perform small oscillations of the form $\theta(t) = \theta_0 \cos \omega t$ and determine the oscillation frequency ω. What is the corresponding expression for $x(t)$?

PROBLEM 2
Accelerated charge

An electron, with charge e, moves in a constant electric field $\mathbf{E}$. The motion is determined by the relativistic Newton's equation

$$\frac{d}{dt}\mathbf{p} = e\mathbf{E} \tag{1}$$

where $\mathbf{p}$ denotes the relativistic momentum $\mathbf{p} = m_e \gamma \mathbf{v}$, with m_e as the electron rest mass, $\mathbf{v}$ as the velocity and $\gamma = 1/\sqrt{1-(v/c)^2}$ as the relativistic gamma factor. We assume the electron to move along the field lines, that is, there is no velocity component orthogonal to $\mathbf{E}$.

a) Show that the electron has a constant proper acceleration $\mathbf{a}_0 = e\mathbf{E}/m_e$, which is the acceleration in an instantaneous rest frame of the electron.

b) Show that if $v = 0$ at time $t = 0$, then γ depends on time t as

$$\gamma = \sqrt{1 + \kappa^2 t^2} \tag{2}$$

and find κ expressed in terms of a_0.

c) Show that if we write $\gamma = \cosh \kappa\tau$ then τ is the proper time of the electron.

As a reminder we give the following functional relations:

$$\cosh^2 x - \sinh^2 x = 1\,, \quad \frac{d}{dx}\cosh x = \sinh x\,, \quad \frac{d}{dx}\sinh x = \cosh x \tag{3}$$

PROBLEM 3
Rotating dipole

A thin rigid rod of length ℓ rotates in a horizontal plane (the x,y-plane) as shown in Fig. 2. At the two end points there are fixed charges of opposite sign, $+q$ and $-q$. The rod is rotating with constant angular frequency ω. This gives rise to a time dependent electric dipole moment

$$\mathbf{p}(t) = q\ell(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j}) \tag{4}$$

a) Use the general expression for the radiation fields of an electric dipole (see the formula collection of the course) to show that the magnetic field in the present case can be written as

$$\mathbf{B}(\mathbf{r},t) = B_0(r)(\cos\theta\sin(\omega(t-\frac{r}{c}))\,\mathbf{i} - \cos\theta\cos(\omega(t-\frac{r}{c}))\mathbf{j} - \sin\theta\sin(\omega(t-\frac{r}{c}) - \phi)\,\mathbf{k} \tag{5}$$

with (r, θ, ϕ) as the polar coordinates of $\mathbf{r}$. Find the expression for $B_0(r)$.

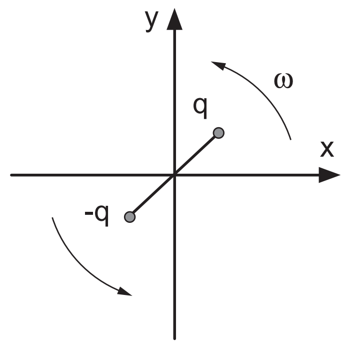


Figure 2:

What is the general relation between the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$ in the radiation zone? (A detailed expression for $\mathbf{E}(\mathbf{r}, t)$ is not needed.)

b) Show that radiation in the x-direction is linearly polarized. What is the polarization of the radiation in the z-direction?

c) Find the time-averaged expression for the energy density of the radiation. In what direction has the radiated energy its maximum?

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Eksamen i: FYS 3120/FYS 4120 Klassisk mekanikk og elektrodynamikk
Eksamensdag: Mandag 4. juni 2007
Tid for eksamen: kl. 14:30 (3timer)
Oppgavesettet er på 3 sider
Tillatte hjelpemidler: Godkjent kalkulator
Øgrim og Lian eller Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formula Collection FYS 3120/4120

Kontroller at oppgavesettet er komplett før du begynner å besvare spørsmålene.

OPPGAVE 1
To sammenbundne legemer
Et mekanisk system er sammensatt av en liten kloss og en kule forbundet med en snor. Klossen

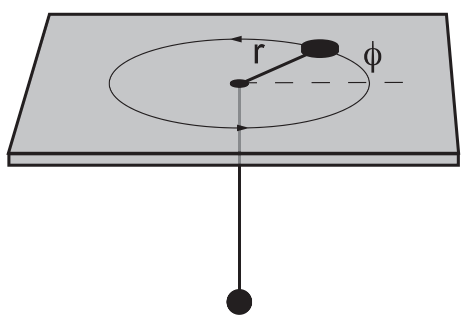


Figure 1:

kan gli friksjonsløst på et horisontalt bord. Snoren er ført langs bordet gjennom et hull til kula som bare beveger seg vertikalt, slik det er vist i Fig. 1. Massen til klossen er $2m$ med m som kulas masse. Vi regner snora som masseløs og uelastisk og at den hele tiden er strukket. Den har lengde d. Klossen regner vi som tilstrekkelig liten til at treghetsmomentet om massemiddelpunktet er neglisjerbart.

a) Benytt polarkoordinatene (r, ϕ) til klossen på bordet som generaliserte koordinater og vis at Lagrangefunksjonen til det sammensatte systemet har formen

$$L = \frac{3}{2}m\dot{r}^2 + mr^2\dot{\phi}^2 + mg(d-r) \qquad (1)$$

b) Sett opp Lagranges ligninger. Hva menes med at vinkelkoordinaten er syklisk? Sett opp uttrykket for den tilhørende bevegelseskonstant l, og benytt det til å redusere bevegelsesligningene til én ligning, i den radielle variabelen r. Hvilken fysisk tolkning har l?

c) Vis v.h.a. den radielle ligningen at det finnes en stabil situasjon hvor klossen går i sirkelbane med konstant radius r_0. Anta at r avviker litt fra r_0, $r = r_0 + \rho$ med $|\rho| << r_0$. Vis at den radielle bevegelsen blir små svingninger om r_0, mens klossen sirkulerer i planet. Hva er frekvensen for de små oscillasjonene om r_0?

OPPGAVE 2
Sirkulerende elektron

Et elektron går i sirkelbane i et konstant magnetfelt (vinkelrett på banen) i en syklotron. Radius i banen er $R = 10m$ og den relativistiske gammafaktoren til elektronet er $\gamma = 100$. Massen til elektronet er $m_e = 9.1 \times 10^{-31} kg$, elektronladningen er $e = -1.6 \times 10^{-19} C$ og lyshastigheten $c = 3.0 \cdot 10^8 m/s$.

a) Hvor stor er elektronets fart uttrykt ved lyshastigheten. Finn også vinkelhastigheten ω og akselerasjonen a til elektronet målt i laboratoriesystemet (dvs. i inertialsystemet hvor syklotronen er i ro).

Benytt i det følgende koordinater hvor sirkelbanen til elektronet ligger i x,y-planet med sentrum av sirkelen i origo. Anta at magnetfeltet er rettet langs den positive z-aksen.

b) Sett opp uttrykkene for elektronets 4-vektorkoordinater x^μ ($\mu = 0, 1, 2, 3$), som funksjon av R, ω, γ og egentiden τ. Bestem også 4-hastigheten og 4-akselerasjonen. Hvor stor er egenakselerasjonen a_0 (akselerasjonen målt i det momentane hvilesystemet) uttrykt ved a?

c) Anta vi studerer bevegelsen i elektronets momentane hvilesystem. Hva er feltstyrken til magnetfeltet $\mathbf{B}'$ og til det elektriske feltet $\mathbf{E}'$ i dette referansesystemet uttrykt ved magnetfeltet $\mathbf{B}$ og elektronhastigheten $\mathbf{v}$ i laboratoriesystemet? (Benytt de generelle uttrykk for Lorentztransformasjon av elektromagnetiske felter.) Sjekk at bevegelsesligningen er oppfylt i det momentane hvilesystemet når den er oppfylt i labsystemet, ved å benytte de uttrykkene som er funnet for egenakselerasjonen og for de transformerte feltene. (Vær oppmerksom på at på vektorform peker akselerasjonene $\mathbf{a}$ og $\mathbf{a}_0$ i de to inertialsystemene i samme retning.)

OPPGAVE 3
Oscillerende strøm

I en sirkelformet strømsløyfe med radius a går det en oscillerende strøm på formen $I = I_0 \cos \omega t$. Strømsløyfen ligger i x,y-planet. Vi benytter betegnelsene $\mathbf{e}_x$, $\mathbf{e}_y$ og $\mathbf{e}_z$ for de kartesiske enhetsvektorene i x, y- og z-retningene, for å kunne reservere $\mathbf{j}$ for strømtettheten. Strømsløyfen regnes å være ladningsnøytral.

a) Forklar hvorfor det elektriske dipolmomentet $\mathbf{p}$ til strømsløyfen er lik null, og vis at det magnetiske momentet har tidsavhengigheten $\mathbf{m}(t) = m_0 \cos \omega t \, \mathbf{e}_z$, med m_0 som en konstant. Bestem m_0 uttrykt ved a og I_0.

Vi minner om de generelle uttrykkene for strålingsfeltet fra en magnetisk dipol,

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi c r} \ddot{\mathbf{m}}_{ret} \times \mathbf{n} \, ; , \quad \mathbf{B}(\mathbf{r}, t) = -\frac{1}{c} \mathbf{E}(\mathbf{r}, t) \times \mathbf{n} \tag{2}$$

hvor $\mathbf{m}_{ret} = \mathbf{m}(t - r/c)$ og $\mathbf{n} = \mathbf{r}/r$. Anta i det følgende at vi studerer feltene langt fra strømsløyfen (i strålingssonen) hvor uttrykkene (2) er gyldige.

b) Sett opp uttrykket for strålingsfeltene for punkter på x-aksen langt fra kilden, og vis at de har form av elektromagnetiske bølger som forplanter seg i x-retningen bort fra strømsløyfen. Hva slags polarisasjon har bølgene?

c) Benytt det generelle uttrykket for Poyntings vektor $\mathbf{S}$ til å finne den utstrålte effekt pr. romvinkelenhet, $\frac{dP}{d\Omega}$, i x-retningen. Hvor stor er den utstrålte effekt i z-retningen?

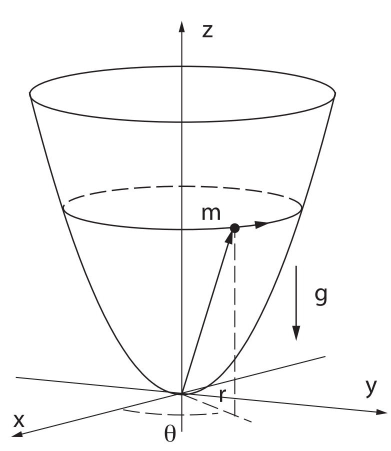
UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 3120 Classical Mechanics and Electrodynamics
Day of exam: Tuesday June 3, 2008
Exam hours: 3 hours, beginning at 14:30
This examination paper consists of 3 pages
Permitted materials: Calculator
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formula Collection FYS 3120
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Particle on a constrained surface
A particle moves on a parabolic surface given by the equation $z = (\lambda/2)(x^2 + y^2)$ where z is the Cartesian coordinate in the vertical direction, x and y are orthogonal coordinates in the horizontal plane and λ is a constant. The particle has mass m and moves without friction on the surface under influence of gravitation. The gravitational acceleration g acts in the negative z-direction. The particle's position is given by the polar coordinates (r, θ) of the *projection* of the position vector into the x, y plane.



a) Show that the Lagrangian for this system is

$$L = \frac{1}{2}m[(1 + \lambda^2 r^2)\dot{r}^2 + r^2\dot{\theta}^2 - g\lambda r^2] \quad (1)$$

and find Lagrange's equations for the particle.

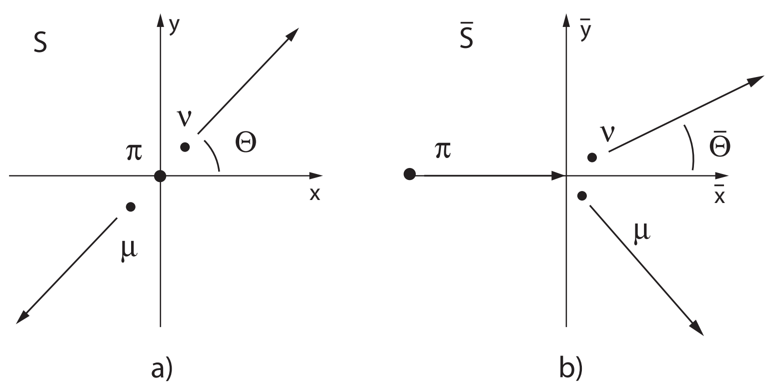
b) Use the fact that there is a cyclic coordinate to show that the equations can be reduced to a single equation in the radial variable r. What is the condition for the particle to move in a circle with radius $r = r_0$?

c) Assume that the path of the particle deviates little from the circular motion so that $r = r_0 + \rho$, where ρ is small. Show that under this condition the radial equation can be reduced to a harmonic oscillator equation for the small variable ρ and determine the corresponding frequency. Give a qualitative description of the motion of the particle.

PROBLEM 2

Particle decay

Pi-mesons (pions) are unstable elementary particles. We consider here a decay process of a charged pion π^+ into a muon μ^+ and a neutrino ν_μ. The masses of the particles are $m_\pi = 273 m_e$ and $m_\mu = 207 m_e$, with $m_e = 0.51 MeV/c^2$ as the electron (rest) mass. (The standard energy unit in particle physics, eV = electron volt is used. The speed of light is as usual represented by c.) The mass of the neutrino is so small that the particle can be regarded as massless.



In the figure the decay process is shown both in the rest frame S of the pion, and in the laboratory frame $\bar{S}$. In this frame the pion moves with the velocity $v = (4/5)c$ along the x axis. To distinguish the variables of the two reference frames S and $\bar{S}$ we mark the variables of the latter with a "bar", so that for example the angle of the neutrino relative to the x axis in S is θ and the corresponding angle in $\bar{S}$ is $\bar{\theta}$.

a) We study first the process in the rest frame S. Set up the equations for conservation of relativistic energy and momentum and use them to determine the energy and (the absolute value of) the momentum of the muon and of the neutrino in this reference system. (Use MeV as unit for energy and MeV/c as unit for momentum.)

b) Use the transformation formula for relativistic 4-momenta to determine the energy of the muon and of the neutrino in the lab frame $\bar{S}$.

c) In the rest frame S all directions for the neutrino momentum are equally probable. Show that this means that in the lab frame $\bar{S}$ the probability is 0.5 for finding the neutrino in a direction with angle $\bar{\theta} < 36.9^o$.

PROBLEM 3
Electric dipole radiation

An electron (with charge e and mass m) is moving with constant speed in a circle under the influence of a constant magnetic field $\mathbf{B}_0$. The magnetic field is directed along the z axis while the motion of the electron takes place in the x, y plane. We assume the motion of the electron to be non-relativistic.

Since the electron is accelerated it will radiate electromagnetic energy and thereby loose kinetic energy when no energy is added to the particle.

a) By use of Larmor's radiation formula, find an expression for the radiated energy per unit time expressed in terms of the radius r of the electron orbit and the cyclotron frequency $\omega = -eB_0/m$.

b) Show that the radius of the electron orbit is slowly reduced with an exponential form of the time dependence, $r = r_0\, e^{-\lambda t}$, and determine λ.

c) The electromagnetic fields produced by the moving charge are essentially electric dipole radiation fields. What is the electric dipole moment of the circulating electron? Give the expressions for the radiation fields $\mathbf{E}(z,t)$ and $\mathbf{B}(z,t)$ on the z axis far from the electron. Show that they correspond to a propagating wave, with direction away from the electron, and determine the form of polarization of the wave.

Expressions found in the formula collection of the course may be useful for this problem.

UNIVERSITETET I OSLO

Det matematisk-naturvitenskapelige fakultet

Exam in: FYS 3120 Classical Mechanics and Electrodynamics
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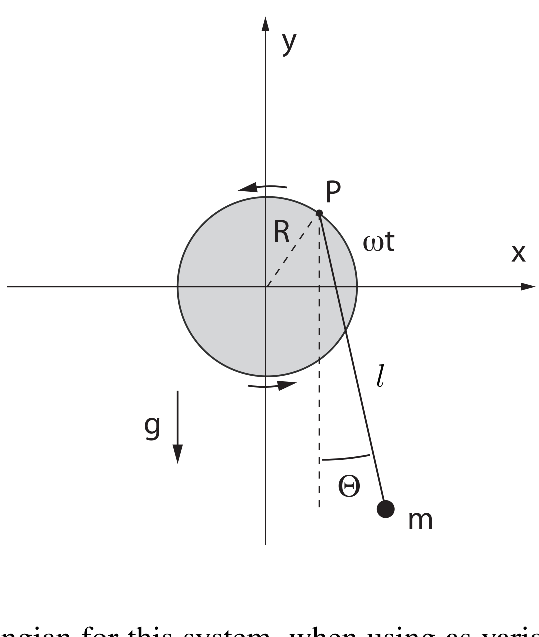
Øgrim og Lian or Angell og Lian: Størrelser og enheter i fysikken
Rottmann: Matematisk formelsamling
Formula Collection FYS 3120
Language: The solutions may be written in Norwegian or English depending on your own preference.

Make sure that your copy of this examination paper is complete before you begin.

PROBLEM 1
Pendulum attached to a rotating disk
A pendulum is attached to a circular disk of radius R, as illustrated in Fig. 1. The end of the pendulum rod is fixed at a point P on the circumference of disk. The disk is vertically oriented and it rotates with a constant angular velocity ω. The pendulum consists of a rigid rod of length l which we consider as massless and a pendulum bob of mass m. The pendulum oscillates freely about the point P under the influence of gravity.



a) Show that the Lagrangian for this system, when using as variable the angle θ of the pendulum rod relative to the vertical direction, has the form

$$L = m[\frac{1}{2}l^2\dot{\theta}^2 + lR\omega\sin(\theta - \omega t)\dot{\theta} + gl\cos\theta + \frac{1}{2}R^2\omega^2 - gR\sin\omega t] \quad (1)$$

b) Formulate Lagrange's equation for the system and write it as a differential equation for θ.

For $\omega = 0$ the equation reduces to a standard pendulum equation. Assume in the following ω to be non-vanishing, but sufficiently small so the ω-dependent contribution to the equation of motion can be viewed as a small periodic perturbation to the pendulum equation. In that case there are solutions corresponding to small oscillations, $|\theta| << 1$, which are modified by the perturbation.

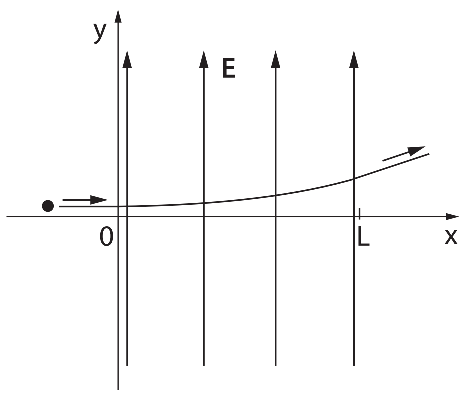
c) Show that under assumption that $|\theta|$ and ω are sufficiently small the equation of motion for the pendulum can be approximated by the equation for a *driven* harmonic oscillator, subject to a periodic force. Show that it has a solution of the form $\theta(t) = \theta_o \cos\omega t$ and determine the amplitude θ_0 in terms of the parameters of the problem.

Based on this solution can you give a more precise meaning to the phrase "sufficiently small ω" as the condition for $\theta_o \cos\omega t$ to be a good approximation to a solution of the full equation of motion?

PROBLEM 2

Charged particle in a constant electric field

A particle with charge q and rest mass m moves with relativistic speed through a region $0 < x < L$ where a constant electric field $\mathbf{E}$ is directed along the y-axis, as indicated in the figure. The particle enters the field at $x = 0$ with momentum $\mathbf{p}_0$ in the direction orthogonal to the field. The relativistic energy at this point is denoted $\mathcal{E}_0$. (Note that we write the energy as $\mathcal{E}$ to avoid confusion with the electric field strength E.)



a) Use the equation of motion for a charged particle in an electric field to determine the time dependent momentum $\mathbf{p}(t)$ and relativistic energy $\mathcal{E}(t)$ (without the potential energy) of the particle inside in the electric field. What is the relativistic gamma factor $\gamma(t)$ expressed as a function of coordinate time t?

b) Find the velocity components $v_x(t)$ and $v_y(t)$ and explain the relativistic effect that the velocity in the x-direction decreases with time even if there is no force acting in this direction.

c) Show that the proper time $\Delta\tau$ spent by the particle on the transit through the region $0 < x < L$ is proportional to the length L, $\Delta\tau = \alpha L$, and determine α.

d) What is the transit time Δt through the region when measured in coordinate time?

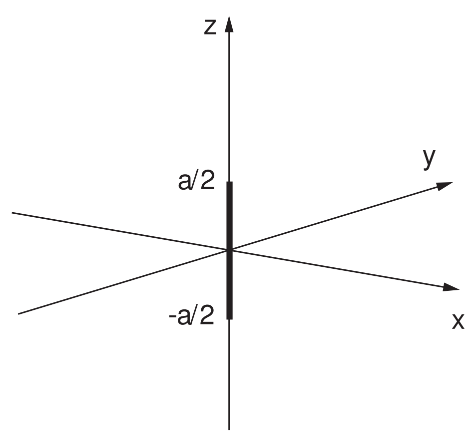
We remind about the integration formula $\int dx \frac{1}{\sqrt{1+x^2}} = \operatorname{arc\,sinh} x + C$.

PROBLEM 3
Radiation from a linear antenna

A so-called *half-wave center-fed* antenna is formed by a thin linear conductor of length a. It is oriented along the z-axis as shown in the figure. An alternating current is running in the antenna, of the form

$$I(z,t) = I_0 \cos\frac{\pi z}{a}\cos\omega t\ , \quad -a/2 < z < a/2 \tag{2}$$

In the following $\lambda(z,t)$ denotes the linear charge density of the antenna (charge per unit length). At time $t = 0$ the antenna is charge neutral, so that $\lambda(z,0) = 0$.



a) Show that the charge density and current satisfy the relation

$$\frac{\partial\lambda}{\partial t} + \frac{\partial I}{\partial z} = 0 \tag{3}$$

and find λ as a function of z and t.

b) Show that the electric dipole moment of the antenna has the form

$$\mathbf{p}(t) = p_0 \sin\omega t\ \mathbf{k} \tag{4}$$

with $\mathbf{k}$ as the unit vector along the z-axis, and determine the constant p_0.

c) Use the expressions for electric dipole radiation to determine the electric and magnetic fields in a point at a large distance r from the antenna on the x-axis. What is the type of polarization of the radiation from the antenna in this direction?