UNIVERSITY OF **Cincinnati**

Classical Mechanics

(2$^{\text{nd}}$ Edition)

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Contents

Chapter 1

Vectors

1.1 Displacements

Even though motion in mechanics is best described in terms of vectors, the formal study of vectors began only after the development of electromagnetic theory, when it was realized that they were essential to the problem of describing the electric and magnetic fields. However, vector analysis assumes an even more interesting role in mechanics, where it is used to implement a powerful principle of physics called the principle of **covariance**. This principle was first explicitly stated by Einstein as a fundamental postulate of the special theory of relativity. It requires the laws of physics to be independent of the features of any particular coordinate system, thereby lending a certain depth to the fundamental laws of physics and giving us a way to compare observations of physical phenomena by different observers using different coordinate frames. The great value of vector analysis lies in the fact that it clarifies the meaning of coordinate independence.

We assume that motion in space will, in general, occur smoothly along some curve passing through two given points (an initial position and a final position), but the net effect of the motion is described by a directed line segment beginning at the initial position of the moving body and terminating at its final position, as shown in figure (1.1). If a body moves from the point labeled "i" in figure (1.1) to the point "f" then, no matter what the actual path traced by the body in going from i to f, we define its **displacement** as the directed straight line segment from i to f as shown. This directed line segment has both magnitude (its length) and direction (the arrow pointing from the initial position to the final position) and will be our prototypical **vector**. Thus, roughly speaking, a vector is any physical quantity that has both magnitude and direction in space and it may graphically be represented by a directed line segment. It is important to bear in mind that what defines a displacement is its magnitude and direction, not the actual initial and final points. Two displacements with the same magnitude and direction are identical,
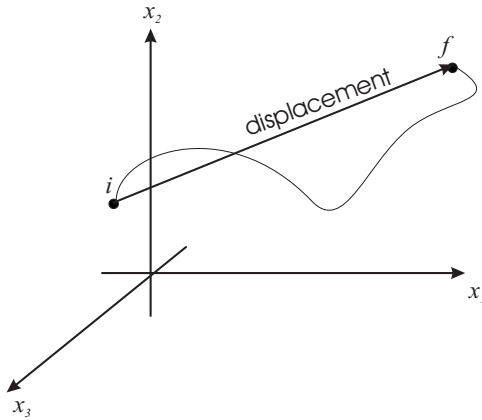
Figure 1.1: Displacement vector

regardless of the initial and final points. So also, what defines a vector is its magnitude and direction and not its location in space.

We must also consider how displacements in particular and vectors in general may be represented algebraically. In a two dimensional plane, we introduce two mutually perpendicular axes intersecting at some point O, the origin, order them in some way calling one the x axis and the other the y axis, and label points by an ordered pair, the coordinates (x, y), where x represents the projection of the point on the x axis and y its projection on the y axis. A more fruitful way to think about this Cartesian coordinate system is to imagine that we have two mutually perpendicular and space filling one parameter families of parallel straight lines in the plane (see figure (1.2). Because the families are space filling, every point will lie on the intersection of one "vertical" and one "horizontal" line. Label a point by the parameter values of the straight lines it lies on. Why is this a better way to think about coordinates? Because it is now easily generalized. Straight lines are not the only curves possible. We could also consider circles of radius r about an origin together with radial lines from the origin, each making an angle θ with some chosen direction [see figure (1.3)]. Every point in the plane lies on the intersection of some circle with some radial line and could therefore be labeled by the pair (r, θ). These, of course, are the familiar polar coordinates. The system is ill defined at the origin because θ cannot be defined there.

The situation is similar in three dimensions, except that the curves are now replaced by surfaces. A coordinate system in three dimensions is a set of three independent, space filling, one parameter families of surfaces relative to which points are labeled. In the Cartesian system this set consists of three mutually perpendicular one parameter families of parallel planes. All points in $\mathbb{R}^3$ will lie on the intersection of a unique set of three
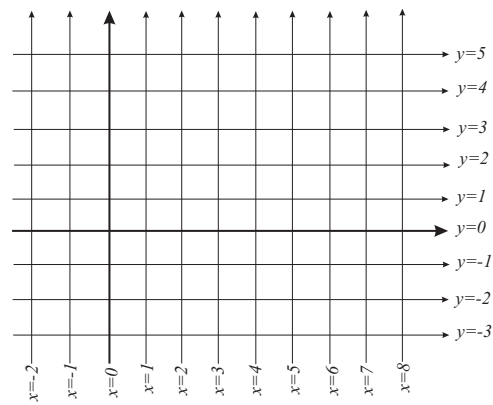
Figure 1.2: Cartesian coordinates in the plane.
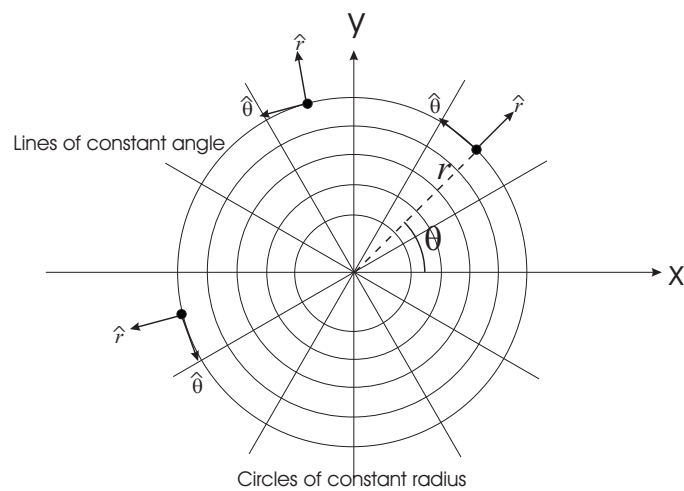
Figure 1.3: Polar coordinates in the plane.

planes and can be represented by an ordered triplet $\vec{r} = (x_1, x_2, x_3)$ consisting of the parameter values of the planes that intersect at the point in question. This is equivalent to the traditional way of viewing the Cartesian system as consisting of three mutually perpendicular straight lines, called coordinate axes, which intersect at a fixed point called the origin. Each axis is normal to a particular family of planes and each coordinate value gives the value of the papameter labeling the members of this family. They are ordered in some way and all points are represented by a correspondingly ordered set of three numbers, an ordered triplet $\vec{r} = (x_1, x_2, x_3)$, each number measuring the distance along the direction specified by one of the axes from the origin to the point in question. Alternatively one could consider a family of right circular cylinders with a common axis of symmetry and two families of mutually perpendicular planes, one of which contains the axis of symmetry of the cylinders and the other is perpendicular to this axis. These lead to the so called cylindrical system. In a spherical spherical system the surfaces are concentric spheres, right circular cones with common axis of symmetry and apex at the center of the spheres and planes containing the axis of symmetry of the cones.

The choice of coordinate system will depend on the "symmetries" of the given problem. For example, it is simpler to work with polar coordinates when examining the motion of the planets about the sun. However, let us confine ourselves to Cartesian systems until they become inconvenient. Although we could consider finite displacements in $\mathbb{R}^3$, it is sufficient and beneficial in the long run to restrict our attention to infinitesimal displacements. Introduce a Cartesian coordinate system and consider two points, i and f that are infinitesimally separated, with coordinates $\vec{r}_i = (x_1, x_2, x_3)$ and $\vec{r}_f = (x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$ respectively. The quantities dx_i represent displacements from i to f along the three (independent) coordinate directions. Let us represent the displacement, $d\vec{r}$, of a body moving from i to f as the ordered collection of these displacements (an ordered triplet),[1]

$$d\vec{r} = (dx_1, dx_2, dx_3). \tag{1.1.1}$$

The numbers dx_i that form the triplet are called the **components** of $d\vec{r}$. The magnitude of $d\vec{r}$, denoted by $|d\vec{r}|$, is given by Pythagoras' theorem as

$$|d\vec{r}| = \sqrt{dx_1^2 + dx_2^2 + dx_3^2} \ . \tag{1.1.2}$$

Its direction can be specified by the angles that $d\vec{r}$ makes with the coordinate axes. Calling these angles α_1, α_2 and α_3 respectively and applying Pythagoras' theorem again we find

$$dx_i = |d\vec{r}| \cos \alpha_i, \quad i \in \{1, 2, 3\}, \tag{1.1.3}$$

The cosines are called the **direction cosines** of the displacement. They are not all independent. Indeed, by substituting (1.1.3) into (1.1.2), one sees that they must satisfy

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[1]The arrow over the $\vec{r}$ indicates that $d\vec{r}$ is not a number but an ordered triplet.
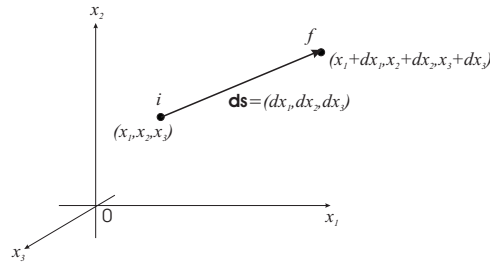
Figure 1.4: Representation of the displacement vector

the constraint

$$\sum_i \cos^2 \alpha_i = 1, \tag{1.1.4}$$

showing that one of the three angles is determined by the other two.

We will sometimes denote the ith component, dx_i, of $d\vec{r}$ by $[d\vec{r}]_i$. The following definitions are natural:

- Two displacements are equal if they have the same magnitude and direction:

$$d\vec{r}_1 = d\vec{r}_2 \Leftrightarrow [d\vec{r}_1]_i = [d\vec{r}_2]_i \tag{1.1.5}$$

- If a is a real number,

$$[a d\vec{r}]_i = a[d\vec{r}]_i \tag{1.1.6}$$

In particular, with $a = -1$, $[-d\vec{r}]_i = -[d\vec{r}]_i$.

- If $d\vec{r}_1$ and $d\vec{r}_2$ are two displacements then their sum is also a displacement given by

$$[d\vec{r}_1 + d\vec{r}_2]_i = [d\vec{r}]_{1,i} + [d\vec{r}]_{2,i} \tag{1.1.7}$$

(This definition can be understood as the algebraic equivalent of the familiar geometric parallelogram law of vector addition.)

Our implicit choice of coordinate system can be made explicit by assigning directions to the coordinate axes as follows: since every straight line is determined by two distinct points, on each axis choose two points one unit away from each other, in the direction of increasing coordinate value. There are only three corresponding displacements, which can be written as[2]

$$\widehat{x}_1 = \widehat{x} = (1,0,0), \quad \widehat{x}_2 = \widehat{y} = (0,1,0) \text{ and } \widehat{x}_3 = \widehat{z} = (0,0,1) \tag{1.1.8}$$

---

[2]Carets, as opposed to arrows, are used to represent any displacement of unit magnitude.

and it is straightforward that, using the scalar multiplication rule (1.1.6) and the sum rule (1.1.7), any displacement $d\vec{r}$ could also be represented as

$$d\vec{r} = dx_1\widehat{x}_1 + dx_2\widehat{x}_2 + dx_3\widehat{x}_3 = \sum_i dx_i\widehat{x}_i. \tag{1.1.9}$$

The $\widehat{x}_i$ represent unit displacements along the of our chosen Cartesian system and the set $\{\widehat{x}_i\}$ is called a **basis**. In $\mathbb{R}^3$, we could use the Cartesian coordinates of any point to represent its displacement from the origin. Displacements in $\mathbb{R}^3$ from the origin

$$\vec{r} = (x_1, x_2, x_3) = \sum_i x_i\widehat{x}_i. \tag{1.1.10}$$

are called **position vectors**.

It is extremely important to recognize that the *representation* of a displacement depends sensitively on the choice of coordinate system whereas the *displacement itself* does *not*. Therefore, we must distinguish between displacements (and, vectors, in general) and their representations. To see why this is important, we first examine how different Cartesian systems transform into one another.

1.2   Linear Coordinate Transformations

Two types of transformations exist between Cartesian frames, *viz.,* translations of the origin of coordinates and rotations of the axes. Translations are just constant shifts of the coordinate origin. If the origin, O, is shifted to the point O' whose coordinates are (x_O, y_O, z_O), measured from O, the coordinates get likewise shifted, each by the corresponding constant,

$$x' = x - x_O, \;\; y' = y - y_O, \;\; z' = z - z_O \tag{1.2.1}$$

But since x_O, y_O and z_O are all constants, such a transformation does not change the representation of a displacement vector,

$$d\vec{r} = (dx, dy, dz) = (dx', dy', dz'). \tag{1.2.2}$$

Representations of displacement vectors are, however, affected by a rotation of the coordinate axes. Let us first consider rotations in two spatial dimensions [see figure (1.5)], where the primed axes are obtained from the original system by a rotation through some angle, θ. The coordinates (x_1, x_2) of a point P in the original system would be (x'_1, x'_2) in the rotated system. In particular, in terms of the length l of the hypotenuse OP of triangle AOP [figure (1.5)], we have

$$x_1 = l\cos(\alpha + \theta) = (l\cos\alpha)\cos\theta - (l\sin\alpha)\sin\theta = x'_1\cos\theta - x'_2\sin\theta$$
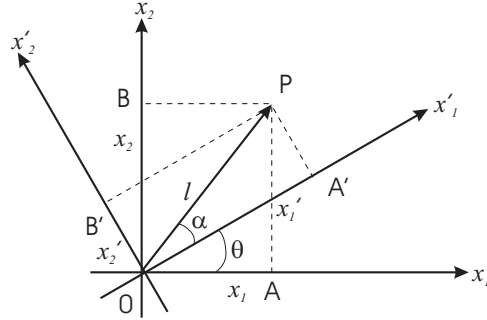
Figure 1.5: Rotations in two space dimensions

$$x_2 = l\sin(\alpha + \theta) = (l\sin\alpha)\cos\theta + (l\cos\alpha)\sin\theta = x_2'\cos\theta + x_1'\sin\theta. \qquad (1.2.3)$$

Inverting these relations gives

$$x_1' = x_1\cos\theta + x_2\sin\theta$$

$$x_2' = -x_1\sin\theta + x_2\cos\theta \qquad (1.2.4)$$

or, in terms of the components of an infinitesimal displacement from P,

$$dx_1' = dx_1\cos\theta + dx_2\sin\theta$$

$$dx_2' = -dx_1\sin\theta + dx_2\cos\theta \qquad (1.2.5)$$

We could also exploit the representation given in (1.1.9) to obtain the same result. Let $\widehat{x}_1$ and $\widehat{x}_2$ designate the directions of the original x_1 and x_2 axes respectively and $\widehat{x}_1'$ and $\widehat{x}_2'$ the directions of the rotated axes. Then, because the displacement itself is independent of the coordinate system, we may write

$$d\vec{r} = dx_1\widehat{x}_1 + dx_2\widehat{x}_2 = dx_1'\widehat{x}_1' + dx_2'\widehat{x}_2' \qquad (1.2.6)$$

Clearly, from figure (1.5)

$$\widehat{x}_1' \;\; = \;\; \cos\theta\widehat{x}_1 + \sin\theta\widehat{x}_2$$

$$\widehat{x}_2' \;\; = \;\; -\sin\theta\widehat{x}_1 + \cos\theta\widehat{x}_2. \qquad (1.2.7)$$

Inserting these into the (1.2.6) we find

$$d\vec{r} = dx_1\widehat{x}_1 + dx_2\widehat{x}_2 \;\; = \;\; dx_1'(\cos\theta\widehat{x}_1 + \sin\theta\widehat{x}_2) + dx_2'(-\sin\theta\widehat{x}_1 + \cos\theta\widehat{x}_2)$$

$$= (dx'_1 \cos\theta - dx'_2 \sin\theta)\widehat{x}_1 + (dx'_1 \sin\theta + dx'_2 \cos\theta)\widehat{x}_2 \quad (1.2.8)$$

A simple comparison now gives

$$dx_1 = dx'_1 \cos\theta - dx'_2 \sin\theta$$

$$dx_2 = dx'_1 \sin\theta + dx'_2 \cos\theta \quad (1.2.9)$$

or, upon inverting the relations,

$$dx'_1 = dx_1 \cos\theta + dx_2 \sin\theta$$

$$dx'_2 = -dx_1 \sin\theta + dx_2 \cos\theta. \quad (1.2.10)$$

It is easy to see that these transformations can also be written in matrix form as

$$\begin{pmatrix} dx'_1 \\ dx'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} \quad (1.2.11)$$

and

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} dx'_1 \\ dx'_2 \end{pmatrix} \quad (1.2.12)$$

Other, more complicated but rigid transformations of the coordinate system can always be represented as combinations of rotations and translations.

1.3   Vectors and Scalars

Definition: A **vector** is a quantity that can be represented in a Cartesian system by an ordered triplet (A_1, A_2, A_3) of components, which transform as the components of an infinitesimal displacement under a rotation of the reference coordinate system. Any vector can always be expressed as a linear combination of basis vectors, $\vec{A} = A_i \widehat{x}_i$.

In two dimensions, a vector may be represented by two Cartesian components $\vec{A} = (A_1, A_2)$, which transform under a rotation of the Cartesian reference system as $(A_1, A_2) \rightarrow (A'_1, A'_2)$ such that

$$\begin{pmatrix} A'_1 \\ A'_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad (1.3.1)$$

Definition: A **scalar** is any physical quantity that *does not* transform (stays **invariant**) under a rotation of the reference coordinate system.

A typical scalar quantity in Newtonian mechanics would be the mass of a particle. The magnitude of a vector is also a scalar quantity, as we shall soon see. It is of great interest to determine scalar quantities in physics because these quantities are not sensitive to particular choices of coordinate systems and are therefore the same for all observers. Other examples of scalars within the context of Newtonian mechanics are temperature and density.

In the Newtonian conception of space and time, time is also a scalar. Because time is a scalar all quantities constructed from the position vector of a particle moving in space by taking derivatives with respect to time are also vectors, therefore

- the velocity: $\vec{v} = \frac{d\vec{r}}{dt}$

- the acceleration: $\vec{a} = \frac{d\vec{v}}{dt}$

- the momentum: $\vec{p} = m\vec{v}$ and

- the force $\vec{F} = \frac{d\vec{p}}{dt}$

are all examples of vectors that arise naturally in mechanics. In electromagnetism, the electric and magnetic fields are vectors. As an example of a quantity that has the appearance of a vector but is not a vector, consider $\mathbf{A} = (x, -y)$. Under a rotation of the coordinate system by an angle θ,

$$
\begin{aligned}
A_1' &= A_1 \cos\theta - A_2 \sin\theta \\
A_2' &= A_1 \sin\theta + A_2 \cos\theta
\end{aligned}
\tag{1.3.2}
$$

which are not consistent with (1.3.1). The lesson is that the transformation properties must always be checked.

1.4 Rotations in two dimensions

Equation (1.3.1) can also be written as follows

$$
A_i' = \sum_j \widehat{R}_{ij} A_j
\tag{1.4.1}
$$

where

$$
\widehat{R}_{ij}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}
\tag{1.4.2}
$$

is just the two dimensional "rotation" matrix. We easily verify that it satisfies the following very interesting properties:

1. If we perform two successive rotations on a vector $\vec{A}$, so that after the first rotation

$$A_i \to A'_i = \sum_j \widehat{R}_{ij}(\theta_1) A_j \tag{1.4.3}$$

and after the second rotation

$$A'_i \to A''_i = \sum_k \widehat{R}_{ik}(\theta_2) A'_k = \sum_k \widehat{R}_{ik}(\theta_2) \widehat{R}_{kj}(\theta_1) A_j \tag{1.4.4}$$

then by explicit calculation it follows that

$$\sum_k \widehat{R}_{ik}(\theta_2) \widehat{R}_{kj}(\theta_1) = \widehat{R}_{ij}(\theta_1 + \theta_2) \tag{1.4.5}$$

so

$$A''_i = \sum_j \widehat{R}_{ij}(\theta_1 + \theta_2) A_j \tag{1.4.6}$$

i.e., the result of two rotations is another rotation. The set of rotation matrices is therefore "closed" under matrix multiplication.

2. The unit matrix, **1**, is the rotation matrix $\widehat{R}(0)$.

3. The transpose of the rotation matrix whose angle is θ is the rotation matrix whose angle is $-\theta$. This follows easily from,

$$\widehat{R}(-\theta) = \begin{pmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \widehat{R}^T(\theta) \tag{1.4.7}$$

Now, using the closure property,

$$R^T(\theta) \cdot R(\theta) = R(-\theta) \cdot R(\theta) = R(0) = \mathbf{1} \tag{1.4.8}$$

Therefore, for every rotation matrix $\widehat{R}(\theta)$ there exists an inverse, $\widehat{R}(-\theta) = \widehat{R}^T$.

4. Matrix multiplication is associative.

The rotation matrices therefore form a group under matrix multiplication.[3] The group elements are all determined by one continuous parameter, the rotation angle θ. This is the commutative group, called $SO(2)$, of 2×2 orthogonal matrices with unit determinant, under matrix multiplication. We will now see that the situation gets vastly more complicated in the physically relevant case of three dimensions.

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[3]Recall the following definitions:

<u>Definition:</u> The pair $(G, *)$ consisting of any set $G = \{g_1, g_2, ...\}$ with a binary operation $*$ defined on it that obeys the four properties

- closure under $*$, *i.e.,* $\forall\ g_1, g_2 \in G\ g_1 * g_2 \in G$
- existence of an identity, *i.e.,* $\exists\ e \in G$ s.t. $\forall\ g \in G,\ g * e = e * g = g$

1.5   Rotations in three dimensions

In two dimensions there is just one way to rotate the axes which, if we introduce a "x_3" axis, amounts to a rotation of the $x_1 - x_2$ axes about it. In three dimensions there are three such rotations possible: the rotation of the $x_1 - x_2$ axes about the x_3 axis, the rotation of the $x_2 - x_3$ axes about the x_1 axis and the rotation of the $x_1 - x_3$ axes about the x_2 axis. In each of these rotations the axis of rotation remains fixed, and each rotation is obviously independent of the others. Thus, we now need 3×3 matrices and may write

$$\widehat{R}^3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.5.1}$$

to represent the rotation of the $x_1 - x_2$ axes as before about the x_3 axis. Under such a rotation only the first and second component of a vector are transformed according to the rule

$$A_i' = \sum_j \widehat{R}_{ij}^3(\theta) A_j \tag{1.5.2}$$

Rotations about the other two axes may be written likewise as follows:

$$\widehat{R}^1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{1.5.3}$$

and[4]

$$\widehat{R}^2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{1.5.4}$$

The general rotation matrix in three dimensions may be constructed in many ways, one of which (originally due to Euler) is canonical:

- first rotate the (x_1, x_2) about the x_3 axis through an angle θ. This gives the new axes (ξ, η, τ) $(\tau \equiv z)$,

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- existence of an inverse *i.e.*, $\forall g \in G \, \exists \, g^{-1} \in G$ s.t. $g * g^{-1} = g^{-1} * g = e$, and
- associativity of $*$, *i.e.*, $\forall \, g_1, g_2, g_3 \in G$, $g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3$

is called a **group**.

Definition: If $\forall \, g_1, g_2 \in G$, $[g_1, g_2] = g_1 * g_2 - g_2 * g_1 = 0$ then the group $(G, *)$ is called a "commutative" or " Abelian" group. $[g_1, g_2]$ is called the commutator of the elements g_1 and g_2.

[4]**Note** the change in sign. It is because we are using a right-handed coordinate system. Convince yourself that it should be so.

- then rotate (ξ, η, z) about the ξ axis through an angle ϕ. This gives the new axes (ξ', η', τ') $(\xi' \equiv \xi)$,

- finally rotate (ξ, η', τ') about the τ' axis through an angle ψ to get (x', y', z').

We get

$$\widehat{R}(\theta, \phi, \psi) = \widehat{R}^3(\psi) \cdot \widehat{R}^2(\phi) \cdot \widehat{R}^3(\theta) \tag{1.5.5}$$

The angles $\{\theta, \phi, \psi\}$ are called the **Euler angles** after the the originator of this particular sequence of rotations.[5] The sequence is not unique however and there are many possible ways to make a general rotation. To count the number of ways, we need to keep in mind that three independent rotations are necessary:

- the first rotation can be performed in one of three ways, corresponding to the three independent rotations about the axes,

- the second rotation can be performed in one of two independent ways: we are not permitted to rotate about the axis around which the previous rotation was performed, and

- the third rotation can be performed in one of two independent ways: again we are not permitted to rotate about the axis around which the previous rotation was performed.

So in all there are $3 \times 2 \times 2 = 12$ possible combinations of rotations that will give the desired general rotation matrix in three dimensions. Note that any scheme you choose will involve three and only three independent angles, whereas only one angle was needed to define the general rotation matrix in two dimensions. The general rotation matrix in n dimensions will require $n(n-1)/2$ angles.

Three dimensional rotation matrices satisfy some interesting properties that we will now outline:

- The product of any two rotation matrices is also a rotation matrix.

- The identity matrix is just the rotation matrix $\widehat{R}(0,0,0)$.

- All three dimensional rotation matrices, like their two dimensional counterparts, obey the condition

$$\widehat{R}^T \cdot \widehat{R} = \mathbf{1} \tag{1.5.6}$$

---

[5]Problem: Show that the general rotation matrix constructed with the rotation sequence of (1.5.5) is

$$\widehat{R}(\theta, \phi, \psi) = \begin{pmatrix} \cos\theta\cos\phi\cos\psi - \sin\theta\sin\psi & \cos\phi\cos\psi\sin\theta + \cos\theta\sin\psi & -\cos\psi\sin\phi \\ -\cos\psi\sin\theta - \cos\theta\cos\phi\sin\psi & \cos\theta\cos\psi - \cos\phi\sin\theta\sin\psi & \sin\phi\sin\psi \\ \cos\theta\sin\phi & \sin\theta\sin\phi & \cos\phi \end{pmatrix}$$

where $\widehat{R}^T$ is the transpose of $\widehat{R}$, *i.e.,*

$$\widehat{R}^T_{ij} = \widehat{R}_{ji} \tag{1.5.7}$$

The transpose of any rotation matrix is also a rotation matrix. It is obtained by applying the separate rotation matrices in reverse order. In the Euler parametrization,[6]

$$\widehat{R}^T(\theta, \phi, \psi) = \widehat{R}(-\psi, -\phi, -\theta) \tag{1.5.8}$$

Therefore, the transpose of a rotation matrix is its inverse.

- Finally, the associative property of matrix multiplication ensures that the product of rotations is associative.

The four properties listed above ensure that three dimensional rotations form a group under matrix multiplication. This the continuous, three parameter group called $SO(3)$ and is the group of all 3×3 orthogonal matrices of unit determinant, under matrix multiplication. The group is not commutative.

Rotations keep the magnitude of a vector invariant. Suppose $\vec{A}$ has components (A_1, A_2, A_3). Under a general rotation the components transform as

$$A'_i = \sum_j \widehat{R}_{ij} A_j \tag{1.5.9}$$

Therefore,

$$\sum_i A'_i A'_i = \sum_{ijk} A_j \widehat{R}^T_{ji} \widehat{R}_{ik} A_k = \sum_{jk} A_j \left(\sum_i \widehat{R}^T_{ji} \widehat{R}_{ik} \right) A_k = \sum_{jk} A_j \delta_{jk} A_k = \sum_j A_j A_j \tag{1.5.10}$$

where δ_{jk} is the Kronecker δ,[7] and in the last step we use the fact that the transpose of $\widehat{R}$ is its inverse. $\sum_i A_i A_i$ is simply the length square of the vector $|\vec{A}|$, or its magnitude squared, *i.e.,*

$$|\vec{A}| = \sqrt{\sum_i A_i A_i} \tag{1.5.11}$$

---

[6]Problem: Verify this explicitly!

[7]Problem: The Kronecker δ is defined by

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

so it is the unit matrix. In fact, δ_{ij} is a "tensor", *i.e.,* it transforms as two copies of a vector under rotations. Show this by showing that

$$\delta'_{ij} = \sum_{lk} \widehat{R}_{il} \widehat{R}_{jk} \delta_{lk} = \delta_{ij}.$$

is invariant under rotations.

1.6   Algebraic Operations on Vectors

We define

- Vector equality:

$$\vec{A} = \vec{B} \Leftrightarrow A_i = B_i, \quad \text{for all } i \tag{1.6.1}$$

- Scalar multiplication:

$$\vec{B} = a\vec{A} \Leftrightarrow B_i = aA_i, \quad \text{for } a \in \mathbb{R} \tag{1.6.2}$$

and

- Vector addition/subtraction:

$$\vec{C} = \vec{A} \pm \vec{B} \Leftrightarrow C_i = A_i \pm B_i \tag{1.6.3}$$

It is easy to show that the results of scalar multiplication, addition and subtraction are vectors (*i.e.,* having the correct transformation properties). Furthermore, there are two ways to define a product between two vectors.

1.6.1   The scalar product

The first is called the scalar (or inner, or dot) product and yields a scalar quantity. If $\vec{A}$ and $\vec{B}$ are two vectors,

$$\vec{A} \cdot \vec{B} = \sum_i A_i B_i \tag{1.6.4}$$

To show that $\vec{A} \cdot \vec{B}$ is a scalar under rotations, consider

$$\sum_i A_i' B_i' = \sum_{ijk} A_j \widehat{R}_{ji}^T \widehat{R}_{ik} B_k = \sum_{jk} A_j \delta_{jk} B_k = \sum_j A_j B_j. \tag{1.6.5}$$

Notice that $|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$.

The basis vectors $\{\widehat{x}_i\}$ satisfy $\widehat{x}_i \cdot \widehat{x}_j = \delta_{ij}$ and the component of a vector $\vec{A}$ along any of the axes can be obtained from the scalar product of $\vec{A}$ with the unit vector in the direction of the axis,

$$A_i = \vec{A} \cdot \widehat{x}_i, \tag{1.6.6}$$

Since $A_i = |\vec{A}| \cos \alpha_i$, it can be used to define the direction cosines,

$$\cos \alpha_i = \frac{\vec{A} \cdot \hat{x}_i}{|\vec{A}|} = \frac{\vec{A} \cdot \hat{x}_i}{\sqrt{\vec{A} \cdot \vec{A}}} \tag{1.6.7}$$

Indeed, if $\hat{u}$ is any unit vector, the component of $\vec{A}$ in the direction of $\hat{u}$ is $A_u = \vec{A} \cdot \hat{u}$. Because the scalar product is invariant under rotations, we prove this by letting α_i be the direction angles of $\vec{A}$ and β_i be the direction angles of $\hat{u}$ in the particular frame in which both $\vec{A}$ and $\hat{u}$ lie in the $x_1 - x_2$ plane (such a plane can always be found). Then $\alpha_3 = \beta_3 = \frac{\pi}{2}$ and

$$\vec{A} \cdot \hat{u} = |\vec{A}| \sum_i \cos \alpha_i \cos \beta_i = |\vec{A}|(\cos \alpha_1 \cos \beta_1 + \cos \alpha_2 \cos \beta_2) \tag{1.6.8}$$

In two dimensions, $\alpha_2 = \frac{\pi}{2} - \alpha_1$ and $\beta_2 = \frac{\pi}{2} - \beta_1$ so

$$\vec{A} \cdot \hat{u} = |\vec{A}|(\cos \alpha_1 \cos \beta_1 + \sin \alpha_1 \sin \beta_1) = |\vec{A}| \cos(\alpha_1 - \beta_1) = |\vec{A}| \cos \theta_u \tag{1.6.9}$$

where θ_u is the angle between $\vec{A}$ and $\hat{u}$, because α_1 and β_1 are the angles made with the x axis. It follows, by Pythagoras' theorem, that A_u is the component of $\vec{A}$ in the direction of $\hat{u}$. In a general coordinate frame, for any two vectors $\vec{A}$ and $\vec{B}$,

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \sum_i \cos \alpha_i \cos \beta_i = |\vec{A}||\vec{B}| \cos \theta_{AB} \tag{1.6.10}$$

where θ_{AB} is the angle between $\vec{A}$ and $\vec{B}$.

1.6.2 The vector product

The second product between two vectors yields another vector and is called the vector (or cross) product. If $\vec{C} = \vec{A} \times \vec{B}$, then

$$C_i = \sum_{jk} \epsilon_{ijk} A_j B_k \tag{1.6.11}$$

where we have introduced the three index quantity called the Levi-Civita tensor (density), defined by[8]

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } \{i,j,k\} \text{ is an even permutation of } \{1,2,3\} \\ -1 & \text{if } \{i,j,k\} \text{ is an odd permutation of } \{1,2,3\} \\ 0 & \text{if } \{i,j,k\} \text{ is not a permutation of } \{1,2,3\} \end{cases} \tag{1.6.12}$$

---

[8]Prove that ϵ_{ijk} transforms as a rank three tensor, *i.e.,* according to *three* copies of a vector. Show that

$$\epsilon'_{ijk} = \sum_{lmn} \widehat{R}_{il} \widehat{R}_{jm} \widehat{R}_{kn} \epsilon_{lmn} = \epsilon_{ijk}$$

provided that the rotation matrices are of unit determinant.
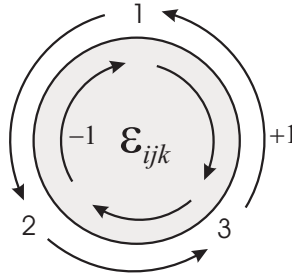
Figure 1.6: Useful way to remember ϵ_{ijk}

An useful mnemonic is shown in figure (1.6). One should check the following identities by direct computation

$$\sum_i \epsilon_{ijk}\epsilon_{irs} = \delta_{jr}\delta_{ks} - \delta_{js}\delta_{kr}$$

$$\sum_{ij} \epsilon_{ijk}\epsilon_{ijs} = 2\delta_{ks}$$

$$\sum_{ijk} \epsilon_{ijk}\epsilon_{ijk} = 3! \tag{1.6.13}$$

Note that the Levi-Civita symbol is antisymmetric under an interchange of its indices, eg., $\epsilon_{ijk} = -\epsilon_{ikj}$ etc. Using the above definition of the cross product, we could write out the components of $\vec{A} \times \vec{B}$ explicitly,

$$\vec{A} \times \vec{B} = (A_2B_3 - A_3B_2, A_3B_1 - A_1B_3, A_1B_2 - A_2B_1), \tag{1.6.14}$$

which is also obtained from the determinant form[9]

$$\vec{A} \times \vec{B} = \det \begin{vmatrix} \widehat{x}_1 & \widehat{x}_2 & \widehat{x}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \tag{1.6.16}$$

It is worth showing that the cross product is a vector. Since the Levi-Civita symbol transforms as a rank three tensor,

$$C_i' = \sum_{j,k} \epsilon_{ijk}' A_j' B_k' = \sum_{l,m,n,,j,k,p,q} R_{il}R_{jm}R_{kn}R_{jp}R_{kq}\epsilon_{lmn}A_pB_q$$

---

[9] The Levi-Civita symbol can be used to define the determinant of any 3×3 matrix as follows: if $\widehat{M}$ is a 3×3 matrix then

$$\det|\widehat{M}| = \sum_{ijk} \epsilon_{ijk}\widehat{M}_{1i}\widehat{M}_{2j}\widehat{M}_{3k} \tag{1.6.15}$$
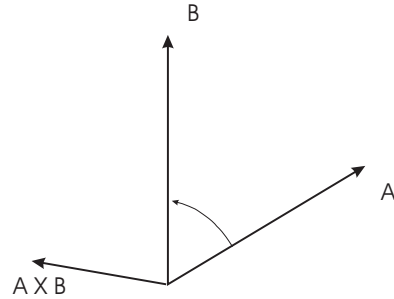
Figure 1.7: The right hand rule

$$= \sum_{l,m,n} R_{il}\epsilon_{lmn}A_m B_n = \sum_{l} R_{il}C_l \qquad (1.6.17)$$

where we have used $\sum_k R_{kn}R_{kq} = \delta_{nq}$ and $\sum_j R_{jm}R_{jp} = \delta_{mp}$.

Notice that $\vec{A} \times \vec{A} = 0$ and that the basis vectors obey $\hat{x}_i \times \hat{x}_j = \epsilon_{ijk}\hat{x}_k$. In a coordinate frame that has been rotated so that both $\vec{A}$ and $\vec{B}$ lie in the $x_1 - x_2$ plane, using $\cos\alpha_2 = \sin\alpha_1$ and $\cos\beta_2 = \sin\beta_1$ together with $\cos\alpha_3 = \cos\beta_3 = 0$, we find that the only non-vanishing component of $\vec{C}$ is C_3 given by

$$C_3 = |\vec{A}||\vec{B}|(\cos\alpha_1 \sin\beta_1 - \sin\alpha_1 \cos\beta_1) = |\vec{A}||\vec{B}|\sin(\beta_1 - \alpha_1) \qquad (1.6.18)$$

If $\beta_1 > \alpha_1$, then C_3 is positive and $\vec{C}$ points along the positive x_3 axis. On the contrary if $\beta_1 < \alpha_1$, then C_3 points along the negative x_3 axis. Because the magnitude of a vector independent of the frame, we conclude: in a general coordinate frame, the vector $\vec{A} \times \vec{B}$ has magnitude

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \left| \sum_{j,k} \epsilon_{ijk}\cos\alpha_j \cos\beta_k \right| = |\vec{A}||\vec{B}|\sin|\theta_{AB}| \qquad (1.6.19)$$

and direction given by the **right-hand rule**, which states that if the fingers of the right hand rotate $\vec{A}$ into $\vec{B}$ then the outstretched thumb points in the direction of $\vec{C}$ (see figure (1.7).

1.7 Vector Spaces

It is easy to verify that the set of all vectors in three dimensional space ($\mathbb{R}^3$), form an Abelian group under vector addition. The unit element is the zero vector and the inverse

of $\vec{A}$ is $-\vec{A}$. Moreover, vector addition inherits its associativity from addition of ordinary numbers and is commutative. The space is also closed under scalar multiplication, since multiplying any vector by a real number gives another vector. Scalar multiplication is also

- associative,

$$a(b\vec{A}) = (ab)\vec{A}, \tag{1.7.1}$$

- distributive over vector addition,

$$a(\vec{A} + \vec{B}) = a\vec{A} + a\vec{B} \tag{1.7.2}$$

- as well as over scalar addition,

$$(a + b)\vec{A} = a\vec{A} + b\vec{B}, \tag{1.7.3}$$

- and admits an identity (1),

$$1(\vec{A}) = \vec{A} \tag{1.7.4}$$

In general, a vector space is any set that is a group under some binary operation (addition) over which multiplication by elements of a field, satisfying the four properties listed above, is defined.[10] Although we have considered only scalar multiplication by real numbers, scalar multiplication by elements of any field (eg. the complex numbers or the rational numbers) is possible in general. The scalar and vector products we have defined are additional structures, not inherent to the definition of a vector space. The vectors we have introduced are geometric vectors in $\mathbb{R}^3$.

1.8   Some Algebraic Identities

We turn to proving some simple but important identities involving the scalar and vector products. The examples given will not be exhaustive, but will serve to illustrate the general

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[10]Additional Definitions:

- A set of vectors, $\{\vec{A}_i\}$, is linearly independent if for scalars a_i,

$$\sum_i a_i \vec{A}_i = 0 \Leftrightarrow a_i = 0 \ \forall \ i.$$

- A set of linearly independent vectors is **complete** if any vector in the vector space may be expressed as a linear combination of its members.

- A complete set of linearly independent vectors is said to form a **basis** for the vector space.

- The set of vectors $\widehat{x}_1 = (1, 0, 0)$, $\widehat{x}_2 = (0, 1, 0)$ and $\widehat{x}_3 = (0, 0, 1)$, form a basis for $\mathbb{R}^3$.

methods used. To simplify notation we will henceforth employ the following convention: if an index is repeated in any expression, it is automatically assumed that the index is to be summed over. Thus we will no longer write the sums explicitly (this is known as the Einstein summation convention).

1. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$.
   We prove this for the components.

   $$[\vec{A} \times \vec{B}]_i = \epsilon_{ijk} A_j B_k = \epsilon_{ikj} A_k B_j = \epsilon_{ikj} B_j A_k = -\epsilon_{ijk} B_j A_k = -[\vec{B} \times \vec{A}]_i$$

   where, in the second step, we have simply renamed the indices by calling $j \leftrightarrow k$ which changes nothing as the indices j and k are summed over. In the next to last step we have used the fact that ϵ_{ijk} is antisymmetric in its indices, so that every interchange of indices in ϵ_{ijk} introduces a negative sign.

2. $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$
   Again take a look at the components,

   $$[\vec{A} \times (\vec{B} \times \vec{C})]_i = \epsilon_{ijk} A_j [\vec{B} \times \vec{C}]_k = \epsilon_{ijk}\epsilon_{klm} A_j B_l C_m$$

   $$= \epsilon_{ijk}\epsilon_{lmk} A_j B_l C_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) A_j B_l C_m$$

   $$= (\vec{A} \cdot \vec{C})B_i - (\vec{A} \cdot \vec{B})C_i$$

3. $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$
   Write everything down in components. The left hand side is

   $$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = \epsilon_{ijk} A_j B_k \epsilon_{ilm} C_l D_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) A_j B_k C_l D_m$$

   $$= (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

   In particular, $(\vec{A} \times \vec{B})^2 = \vec{A}^2\vec{B}^2 \sin^2\theta$, where θ is the angle between $\vec{A}$ and $\vec{B}$.

4. The triple product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is defined by

   $$[\vec{A}, \vec{B}, \vec{C}] = \vec{A} \cdot (\vec{B} \times \vec{C}) = \epsilon_{ijk} A_i B_j C_k$$

   This is a scalar.[11] It satisfies the following properties:

   $$[\vec{A}, \vec{B}, \vec{C}] = [\vec{C}, \vec{A}, \vec{B}] = [\vec{B}, \vec{C}, \vec{A}] = -[\vec{B}, \vec{A}, \vec{C}] = -[\vec{C}, \vec{B}, \vec{A}] = -[\vec{A}, \vec{C}, \vec{B}] \quad (1.8.1)$$

   i.e., the triple product is even under cyclic permutations and otherwise odd. Also $[\vec{A}, \vec{A}, \vec{B}] = 0$. All these properties follow directly from the properties of the Levi-Civita tensor density, ϵ_{ijk}.[12]

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[11]Problem: Verify this!
[12]Problem: Convince yourself that this is so.

5. $(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = [\vec{A}, \vec{B}, \vec{D}]\vec{C} - [\vec{A}, \vec{B}, \vec{C}]\vec{D}$
   The left hand side is just

$$(\vec{A} \times \vec{B}) \times (\vec{C} \times \vec{D}) = \epsilon_{ijk}\epsilon_{jrs}\epsilon_{kmn}A_r B_s C_m D_n = \epsilon_{jrs}(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})A_r B_s C_m D_n$$

$$= (\epsilon_{nrs}A_r B_s D_n)C_i - (\epsilon_{mrs}A_r B_s C_m)D_i$$

$$= [\vec{A}, \vec{B}, \vec{D}]\vec{C} - [\vec{A}, \vec{B}, \vec{C}]\vec{D}$$

1.9   Differentiation of Vectors

1.9.1   Time derivatives

A vector function of time is a vector whose components are functions of time. The derivative of a vector function of time is then defined in a natural way in terms of the derivatives of its components in the Cartesian basis. Let $\vec{A}(t)$ be a vector function of some parameter t, *i.e.*,

$$\vec{A}(t) = (A_1(t), A_2(t), A_3(t)) \tag{1.9.1}$$

The derivative of $\vec{A}(t)$ is another vector function, $\vec{C}(t)$, whose Cartesian components are given by

$$C_i = \frac{dA_i}{dt} \tag{1.9.2}$$

Note that the above definition is "good" only in for the Cartesian components of the vector. This is because the Cartesian basis $\{\hat{x}_i\}$ is rigid, *i.e.*, it does not change in space. In more general coordinate systems, where the basis is not rigid, the derivative of a vector must be handled delicately. We will return to this later. Here, we will convince ourselves that $\vec{C}$ is really a vector. Under a rotation

$$A_i \to A_i' \Rightarrow C_i' = \frac{dA_i'}{dt} = \frac{d}{dt}(\hat{R}_{ij}A_j) = \hat{R}_{ij}\frac{dA_i}{dt} = \hat{R}_{ij}C_j \tag{1.9.3}$$

which shows that $\vec{C}(t)$ has the correct transformation properties, inherited from $\vec{A}(t)$. This justifies the statement that the velocity, momentum, acceleration and force must all be vectors, because they are all obtained by differentiating $\vec{r}(t)$.[13]

---

[13]Problem: Show that

$$\frac{dr}{dt} = \hat{r} \cdot \vec{v}$$
$$\frac{d\hat{r}}{dt} = \frac{\vec{v}}{r} - \frac{\hat{r}}{r}(\hat{r} \cdot \vec{v})$$

where $r = |\vec{r}|$, $\hat{r}$ is the unit position vector and $\vec{v}$ is the velocity vector.

1.9.2 The Gradient Operator

The gradient operator is a vector **differential operator**, whose definition is motivated by a simple geometric fact. Consider some scalar function $\phi(\vec{r})$[14] and the surface in $\mathbb{R}^3$, defined by

$$\phi(\vec{r}) = \phi(x_1, x_2, x_3) = \text{const.}, \tag{1.9.4}$$

so that

$$\phi'(x_1', x_2', x_3') = \phi(x_1, x_2, x_3) \tag{1.9.5}$$

The total differential of $\phi(\vec{r})$ is given by

$$d\phi = \frac{\partial \phi}{\partial x_1} dx_1 + \frac{\partial \phi}{\partial x_2} dx_2 + \frac{\partial \phi}{\partial x_3} dx_3, \tag{1.9.6}$$

which can be re-expressed as

$$d\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3} \right) \cdot (dx_1, dx_2, dx_3) = 0. \tag{1.9.7}$$

The vector (dx_1, dx_2, dx_3) represents an infinitesimal displacement on the surface determined by the equation $\phi(x_1, x_2, x_3) = \text{const.}$ The other vector in the scalar product is called the gradient of the function $\phi(\vec{r})$,

$$\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_2}, \frac{\partial \phi}{\partial x_3} \right) \tag{1.9.8}$$

It has the form of a vector, but we need to check of course that its transformation properties under rotations are those of a vector. We will therefore look at the components of $\vec{\nabla}\phi$:

$$\frac{\partial \phi}{\partial x_i} = \partial_i \phi \rightarrow \partial_i' \phi' = \frac{\partial \phi'}{\partial x_i'} = \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial x_i'} \tag{1.9.9}$$

Now

$$x_i' = \widehat{R}_{ik} x_k \Rightarrow x_j = \widehat{R}_{ji}^T x_i \tag{1.9.10}$$

so

$$\frac{\partial x_j}{\partial x_i'} = \widehat{R}_{ji}^T = \widehat{R}_{ij} \tag{1.9.11}$$

and therefore

$$\partial_i' \phi' = \frac{\partial \phi'}{\partial x_i'} = \widehat{R}_{ij} \partial_j \phi \tag{1.9.12}$$

---

[14] Any scalar function $\phi(\vec{r}, t)$ is called a **scalar field**.

which is indeed the vector transformation law. Hence $\vec{\nabla}\phi$ is a vector if $\phi(\vec{r})$ is a scalar. Now it turns out that $\vec{\nabla}\phi$ has a nice geometric meaning. Because

$$\vec{\nabla}\phi \cdot d\vec{r} = 0 \tag{1.9.13}$$

for all infinitesimal displacements along the surface, it follows that $\vec{\nabla}\phi$, if it is not vanishing, must be *normal* to the surface given by $\phi(\vec{r}) = \text{const}$. Thus, given any surface $\phi(\vec{r}) = \text{const.}$,

$$\widehat{n} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} \tag{1.9.14}$$

is the unit normal to the surface.

Example: Take $\phi(x, y, z) = x^2 + y^2 + z^2$, then $\phi(\vec{r}) = \text{const.}$ represents a sphere centered at the origin of coordinates. The unit normal to the sphere at any point is

$$\vec{\nabla}\phi = \frac{\vec{r}}{r} \tag{1.9.15}$$

where r is the radius of the sphere and $\vec{r}$ is the position vector of the point. The normal to the sphere is therefore in the direction of the radius.

Example: Take $\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, so that $\phi(x, y, z) = 1$ represents an ellipsoid with semi-axes of lengths a, b and c respectively. We find

$$\widehat{n} = (\frac{x}{a}, \frac{y}{b}, \frac{z}{c}) \tag{1.9.16}$$

which is the normal to the ellipsoid at the point (x, y, z).

We see that $\vec{\nabla}$ is just the derivative operator in the Cartesian system, so we can think of it in component form as the collection of derivatives,

$$[\vec{\nabla}]_i = \partial_i. \tag{1.9.17}$$

Now if we introduce the concept of a **vector field** as a vector function of space and time,

$$\vec{A}(\vec{r}, t) = (A_1(\vec{r}, t), A_2(\vec{r}, t), A_3(\vec{r}, t)) \tag{1.9.18}$$

then we can define two distinct operations on $\vec{A}(\vec{r}, t)$ using the scalar and vector products given earlier,

- the **divergence** of a vector field $\vec{A}(\vec{r}, t)$ as

$$\text{div}\,\vec{A} = \vec{\nabla} \cdot \vec{A} = \partial_i A_i \tag{1.9.19}$$

  and

- the **curl** (or **rotation**) of a vector field $\vec{A}(\vec{r}, t)$ as

$$[\vec{\nabla} \times \vec{A}]_i = \epsilon_{ijk}\partial_j A_k \qquad (1.9.20)$$

These turn out to be of fundamental importance in any dynamical theory of fields, eg., electromagnetism. We will understand their physical significance in the following chapters. For now, we only prove a few identities involving the $\vec{\nabla}$ operator. Once again, the examples given are far from exhaustive, their purpose being only to illustrate the method.

1.10 Some Differential Identities

1. $\vec{\nabla} \cdot \vec{r} = 3$

   This follows directly from the definition of the divergence,

$$\vec{\nabla} \cdot \vec{r} = \partial_i x_i = \delta_{ii} = 3$$

2. $\vec{\nabla} \cdot (\phi\vec{A}) = (\vec{\nabla}\phi) \cdot \vec{A} + \phi(\nabla \cdot \vec{A})$

   Expand the l.h.s to get

$$\partial_i(\phi A_i) = (\partial_i \phi)A_i + \phi(\partial_i A_i) = (\vec{\nabla}\phi) \cdot \vec{A} + \phi(\nabla \cdot \vec{A}) \qquad (1.10.1)$$

   As a special case, take $\vec{A} = \vec{r}$, then $\vec{\nabla} \cdot (\vec{r}\phi) = \vec{r} \cdot (\vec{\nabla}\phi) + 3\phi$

3. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) \equiv 0$

   The proof is straightforward and relies on the antisymmetry of ϵ_{ijk}:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = \epsilon_{ijk}\partial_i\partial_j A_k = 0$$

   which follows because $\partial_i\partial_j$ is symmetric w.r.t. $\{ij\}$ while ϵ_{ijk} is antisymmetric w.r.t. the same pair of indices.

4. $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

   Expanding the l.h.s.,

$$
\begin{aligned}
\partial_i(\epsilon_{ijk}A_j B_k) &= \epsilon_{ijk}[(\partial_i A_j)B_k + A_j(\partial_i B_k)] \\[2mm]
&= (\epsilon_{kij}\partial_i A_j)B_k - A_j(\epsilon_{jik}\partial_i B_k) \\[2mm]
&= (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B})
\end{aligned}
$$

5. $\vec{\nabla} \times \vec{r} = 0$

   This also follows from the antisymmetry of the Levi-Civita tensor,

$$\vec{\nabla} \times \vec{r} = \epsilon_{ijk}\partial_j x_k = \epsilon_{ijk}\delta_{jk} = 0$$

6. $\vec{\nabla} \times \vec{\nabla}\phi \equiv 0$

   This is another consequence of the same reasoning as above,

   $$\vec{\nabla} \times \vec{\nabla}\phi = \epsilon_{ijk}\partial_j\partial_k\phi = 0$$

7. $\vec{\nabla} \times (\phi\vec{A}) = (\vec{\nabla}\phi) \times \vec{A} + \phi(\nabla \times \vec{A})$

   Consider the ith component of the l.h.s.,

   $$
   \begin{aligned}
   [\vec{\nabla} \times (\phi\vec{A})]_i &= \epsilon_{ijk}\partial_j(\phi A_k) = \epsilon_{ijk}(\partial_j\phi)A_k + \epsilon_{ijk}\phi(\partial_j A_k) \\
   &= [\vec{\nabla}\phi \times \vec{A}]_i + \phi[\vec{\nabla} \times \vec{A}]_i
   \end{aligned}
   $$

   As a special case, take $\vec{A} = \vec{r}$, then $\vec{\nabla} \times (\vec{r}\phi) = (\vec{\nabla}\phi) \times \vec{r}$.

8. $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2\vec{A}$

   Beginning with,

   $$
   \begin{aligned}
   [\vec{\nabla} \times (\vec{\nabla} \times \vec{A})]_i &= \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_l A_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_l A_m \\
   &= \partial_i(\partial_m A_m) - \partial_j\partial_j A_i = [\vec{\nabla}(\vec{\nabla} \cdot \vec{A})]_i - [\vec{\nabla}^2\vec{A}]_i
   \end{aligned}
   $$

9. $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{\nabla} \cdot \vec{B})\vec{A} - (\vec{\nabla} \cdot \vec{A})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B}$

   Again, beginning with,

   $$
   \begin{aligned}
   [\vec{\nabla} \times (\vec{A} \times \vec{B})]_i &= \epsilon_{ijk}\epsilon_{klm}\partial_j(A_l B_m) = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j(A_l B_m) \\
   &= \partial_j(A_i B_j) - \partial_j(A_j B_i) \\
   &= (\partial_j B_j)A_i + (B_j\partial_j)A_i - (\partial_j A_j)B_i - (A_j\partial_j)B_i \\
   &= (\vec{\nabla} \cdot \vec{B})[\vec{A}]_i - (\vec{\nabla} \cdot \vec{A})[\vec{B}]_i + (\vec{B} \cdot \vec{\nabla})[\vec{A}]_i - (\vec{A} \cdot \vec{\nabla})[\vec{B}]_i
   \end{aligned}
   $$

10. $\vec{\nabla}(\vec{A} \cdot \vec{B}) = (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A} + \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A})$

    Consider the ith component of the last two terms on the right,

    $$
    \begin{aligned}
    [\vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A})]_i &= \epsilon_{ijk}\epsilon_{klm}A_j\partial_l B_m + \epsilon_{ijk}\epsilon_{klm}B_j\partial_l A_m \\
    &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})A_j\partial_l B_m \\
    &= A_j\partial_i B_j - A_j\partial_j B_i + B_j\partial_i A_j - B_j\partial_j A_i
    \end{aligned}
    $$
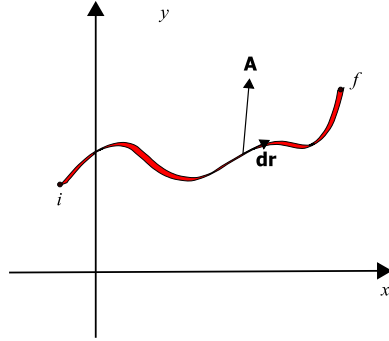
Figure 1.8: The Line Integral

$$= \partial_i(A_j B_j) - A_j \partial_j B_i - B_j \partial_j A_i$$

$$= [\vec{\nabla}(\vec{A} \cdot \vec{B})]_i - (\vec{A} \cdot \vec{\nabla})B_i - (\vec{B} \cdot \vec{\nabla})A_i$$

The stated result follows.

11. A vector $\vec{A}$ is said to be **irrotational** if $\vec{\nabla} \times \vec{A} = 0$ and it is **solenoidal** if $\vec{\nabla} \cdot \vec{A} = 0$. It turns out that $\vec{A} \times \vec{B}$ is solenoidal if both $\vec{A}$ and $\vec{B}$ are irrotational. Begin with

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \epsilon_{ijk} \partial_i (A_j B_k) = \epsilon_{ijk}(\partial_i A_j)B_k + \epsilon_{ijk} A_j(\partial_i B_k)$$

$$= (\vec{\nabla} \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = 0$$

(since both $\vec{A}$ and $\vec{B}$ are irrotational).

There are many more identities which we will encounter along the way and all of them can be proved using the methods above

1.11 Vector Integration

There are three types of integrations involving vector and scalar functions that lead to scalar quantities, *viz.,*

1.11.1 Line Integrals

Line integrals involve integrations along a curve, C, given by $\vec{r} = \vec{r}(t)$, and will quite generally depend upon the curve over which the integration is carried out. The following

are basic possibilities:

$$\int_{i,C}^{f} \phi(\vec{r})ds, \quad \int_{i,C}^{f} \vec{A}(\vec{r})ds, \quad \int_{i,C}^{f} \phi(\vec{r})d\vec{r}, \quad \int_{i,C}^{f} \vec{A}(\vec{r}) \times d\vec{r}, \quad \int_{i,C}^{f} \vec{A}(\vec{r}) \cdot d\vec{r}, \qquad (1.11.1)$$

where $ds = |\vec{v}|dt$ represents the length of an infinitesimal line element on C and $d\vec{r} = \vec{v}dt$ is an infinitesimal displacement along (tangent to) C. Each integral may be defined in the usual way, as the limit of an infinite (Riemann) sum. The first and last integrals yield scalars, the others are vectors. The curve may be open or closed and i and f are the beginning and endpoints of the integration.

If $\phi(\vec{r})$ represents the density of a "wire" laid along C, then the first integral returns the mass of the wire. On the other hand, a well-known example of the last line integral is the work performed by a force $\vec{F}$ in moving a particle along some trajectory, C (see figure (1.8). A particularly interesting case occurs when the vector $\vec{A}$ is the gradient of a scalar function, *i.e.*, $\vec{A} = \vec{\nabla}\phi$. In this case,

$$\int_{i,C}^{f} \vec{A} \cdot d\vec{r} = \int_{i,C}^{f} \vec{\nabla}\phi \cdot d\vec{r} = \int_{i,C}^{f} d\phi = \phi(\vec{r}_f) - \phi(\vec{r}_i) \qquad (1.11.2)$$

showing that the integral depends only on the endpoints and not on the curve C itself. A vector whose line integral is independent of the path along which the integration is carried out is called **conservative**. Every conservative vector then obeys

$$\oint_C \vec{A} \cdot d\vec{r} = 0, \qquad (1.11.3)$$

for *every* closed path. Conversely, any vector that obeys (1.11.3) is expressible as the gradient of a scalar function, for

$$\oint_C \vec{A} \cdot d\vec{r} = 0 \Rightarrow \vec{A} \cdot d\vec{r} = d\phi = \vec{\nabla}\phi \cdot d\vec{r} \qquad (1.11.4)$$

and, since $d\vec{r}$ is arbitrary, it follows that $\vec{A} = \vec{\nabla}\phi$.

One does not need to evaluate its line integral to determine whether or not a vector is conservative. From the fact that the curl of a gradient vanishes it follows that if $\vec{A}$ is conservative then $\vec{\nabla} \times \vec{A} = 0$. The converse is also true, since if $\vec{A}$ is irrotational then $\epsilon_{ijk}\partial_j A_k = 0$ for all i. These are simply integrability conditions for a function ϕ defined by $A_k = \partial_k\phi$, therefore every irrotational vector is conservative and vice versa. The function $-\phi$ is generally called a **potential** of $\vec{A}$.
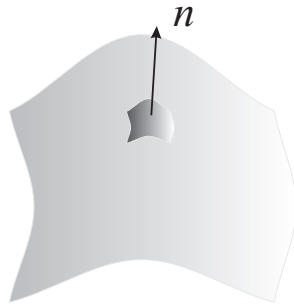
Figure 1.9: The Surface Integral

1.11.2 Surface integrals

Surface integrals also appear in the same three forms, the integration occuring over infinitesimal area elements, $d\vec{S}$, which are assigned the direction of the surface normal, (see figure (1.9) Writing $d\vec{S}$ as $dS\hat{n}$, where $\hat{n}$ is the unit normal to the surface at dS,

$$\int_S dS(\hat{n}\phi), \quad \int_S dS(\hat{n} \times \vec{A}), \quad \int_S dS(\hat{n} \cdot \vec{A}) \tag{1.11.5}$$

where S is some arbitrary (open or closed) surface.

1.11.3 Volume Integrals

We may define volume integrals similarly but because the volume element is a scalar there are only two distinct possibilities,

$$\int_V d^3\vec{r}\ \phi(\vec{r}), \quad \int_V d^3\vec{r}\ \vec{A}(\vec{r}). \tag{1.11.6}$$

1.12 Integral Theorems

The three types of integrals that were defined in the previous section are connected by the following two theorems:[15]

1. **Stokes Theorem**:
$$\oint_C \vec{A} \cdot d\vec{r} = \int_S dS\ \hat{n} \cdot (\vec{\nabla} \times \vec{A}) \tag{1.12.1}$$

---

[15]The proofs of these theorems can be found in any text on mathematical physics. We will leave it to the student to examine the proofs independently.

where C is a **closed** curve, S is a surface bounded by C, and $\widehat{n}$ is normal to the surface element dS, chosen to obey the right hand rule, *i.e.,* if the fingers of the right hand point in the direction of $d\vec{r}$ along the curve then the outstretched thumb determines the choice of the orientation of $\widehat{n}$.

2. **Gauss' theorem**:

$$\oint_S dS\,\widehat{n}\cdot\vec{A} = \int_V d^3\vec{r}\,\vec{\nabla}\cdot\vec{A} \qquad (1.12.2)$$

where S is a **closed** surface, $\widehat{n}$ is the outward directed normal to S and V is the volume bounded S.

While we accept these theorems without proof here, we shall now use then to prove some corollaries that will turn out to be useful in the future.

1.12.1   Corollaries of Stokes' Theorem

We will prove the following three relations:

1. $\oint_C \phi d\vec{r} = \int_S dS(\widehat{n}\times\vec{\nabla}\phi)$

2. $\oint_C d\vec{r}\times\vec{A} = \int_S dS(\widehat{n}\times\vec{\nabla})\times\vec{A}$

where C is a closed curve and S is the surface bounded by C in each case.

The proofs are quite simple. Define the vector $\vec{A} = \vec{a}\phi$, where $\vec{a}$ is an arbitrary, constant vector then by Stokes' theorem,

$$
\begin{aligned}
\oint_C \vec{A}\cdot d\vec{r} &= \vec{a}\cdot\oint \phi d\vec{r} = \int_S dS\widehat{n}\cdot(\vec{\nabla}\times\vec{a}\phi) = \int_S dS\widehat{n}\cdot(\vec{\nabla}\phi\times\vec{a}) \\
&= \vec{a}\cdot\int_S dS(\widehat{n}\times\vec{\nabla}\phi) \qquad\qquad (1.12.3)
\end{aligned}
$$

Thus

$$\vec{a}\cdot\left(\oint \phi d\vec{r} - \int_S dS(\widehat{n}\times\vec{\nabla}\phi)\right) = 0 \qquad (1.12.4)$$

holds for arbitrary vectors implying the first identity.

The second identity may be derived similarly, by using $\vec{B} = \vec{a}\times\vec{A}$ in Stokes' theorem. Then

$$
\begin{aligned}
\oint_C \vec{B}\cdot d\vec{r} &= \oint_C \vec{a}\times\vec{A}\cdot d\vec{r} = -\oint_C (d\vec{r}\times A)\cdot\vec{a} \\
&= \int_S dS\,\widehat{n}\cdot(\vec{\nabla}\times\vec{B}) = \int_S dS\,[(\widehat{n}\times\vec{\nabla})\cdot\vec{B}]
\end{aligned}
$$

$$= \int_S dS \, [(\hat{n} \times \vec{\nabla}) \cdot (\vec{a} \times \vec{A})] \tag{1.12.5}$$

But it is easy to show that[16] $(\hat{n} \times \vec{\nabla}) \cdot (\vec{a} \times \vec{A}) = -[(n \times \vec{\nabla}) \times \vec{A}] \cdot \vec{a}$, therefore

$$\vec{a} \cdot \left[\oint_C (d\vec{r} \times A) - \int_S dS[\hat{n} \times (\vec{\nabla} \times \vec{A})] \right] = 0 \tag{1.12.6}$$

Again $\vec{a}$ was arbitrary, therefore the second identity follows.

1.12.2 Corollaries of Gauss' theorem

We will prove three relations that will be quite helpful to us in the future, *viz.*,

1. $\oint_S dS(\vec{A} \times \hat{n}) = \int_V d^3\vec{r} \, \vec{\nabla} \times \vec{A}$

2. If ϕ and ψ are two arbitrary functions, then

   (a) Green's **first identity**:

   $$\int_V d^3\vec{r} \, [\vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi\vec{\nabla}^2\psi] = \int_S dS \, \hat{n} \cdot \phi\vec{\nabla}\psi \tag{1.12.7}$$

   and

   (b) Green's **second identity**:

   $$\int_V d^3\vec{r} \, [\phi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\phi] = \int_S dS \, \hat{n} \cdot [\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi] \tag{1.12.8}$$

To prove the first corollary we will employ the trick we used to prove the corollaries of Stoke's theorem. For a constant vector $\vec{a}$, let $\vec{B} = \vec{a} \times \vec{A}$ and apply Gauss' law

$$\oint_S dS(\hat{n} \cdot \vec{B}) = \int_V d^3\vec{r} \, \vec{\nabla} \cdot \vec{B} \Rightarrow \oint_S dS[\hat{n} \cdot (\vec{a} \times \vec{A})] = \int d^3\vec{r} \, \vec{\nabla} \cdot (\vec{a} \times \vec{A}) \tag{1.12.9}$$

Developing the last relation, using some of the vector identities we proved earlier, we find

$$\vec{a} \cdot \left[\oint_S dS(\hat{n} \times \vec{A}) - \int_V d^3\vec{r} \, \vec{\nabla} \times \vec{A} \right] = 0 \tag{1.12.10}$$

But since $\vec{a}$ is arbitrary, the identity follows.

To prove Green's two theorems is equally straightforward. Take $\vec{A} = \phi\vec{\nabla}\psi$ and apply Gauss' theorem. Since

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot (\phi\vec{\nabla}\psi) = (\vec{\nabla}\phi) \cdot (\vec{\nabla}\psi) + \phi\vec{\nabla}^2\psi, \tag{1.12.11}$$

---

[16]Problem: Use the properties of the Levi-Civita tensor to show this

it follows that the first of Green's theorems is just Gauss' theorem,

$$\int_V d^3\vec{r}\,[\vec{\nabla}\phi \cdot \vec{\nabla}\psi + \phi\vec{\nabla}^2\psi] = \int_S dS\,\hat{n} \cdot \phi\vec{\nabla}\psi \qquad (1.12.12)$$

To prove the second identity, consider $\vec{B} = \psi\vec{\nabla}\phi$ and again apply Gauss' theorem to get

$$\int_V d^3\vec{r}\,[\vec{\nabla}\psi \cdot \vec{\nabla}\phi + \psi\vec{\nabla}^2\phi] = \int_S dS\,\hat{n} \cdot \psi\vec{\nabla}\phi \qquad (1.12.13)$$

Subtracting the second from the first gives

$$\int_V d^3\vec{r}\,[\phi\vec{\nabla}^2\psi - \psi\vec{\nabla}^2\phi] = \int_S dS\,\hat{n} \cdot [\phi\vec{\nabla}\psi - \psi\vec{\nabla}\phi] \qquad (1.12.14)$$

which is Green's second identity.

This chapter does not do justice to the vast area of vector analysis. On the contrary, most proofs have not been given and many useful identities have been neglected. What has been presented is only an introduction to the material we will immediately need. Consequently, as we progress, expect to periodically encounter detours in which further vector analysis will be presented, often as exercises in the footnotes.

Chapter 2

Newton's Laws and Simple Applications

2.1 Introduction

Theoretical mechanics is concerned with the temporal "evolution" of a physical system, be it a single particle or a complex system of particles that interact among themselves. By "evolution" one simply means a continuous change in "configuration", *i.e.,* a change in some set of parameters that define the system. The fundamental problem is therefore two-fold: on the one hand it is necessary to determine the appropriate parameters that completely define a system within the limits of experimental possibility (disregarding limitations imposed by current technology) and on the other it is necessary to determine the equations (the *dynamical* equations) that govern the evolution of these parameters.

Most realistic physical systems encountered in daily life are complex. Nevertheless we take a point of view, expressed by the Greeks as early as the third century before Christ and later reiterated by Newton, that every complex system can be decomposed into and subsequently reconstructed from elementary constituents which we will call "particles". In its most idealized form a particle is merely a mathematical point endowed with some physical charateristics, which could be for example a "mass", a "charge" etc., but with no discernible geometric structure. These are the so-called "elementary particles". The concept is also useful as an approximation, if the volume of the object we are studying is very small compared to typical volumes over which its evolution occurs. Thus to a reasonable approximation, for example, the earth could be considered a particle if we are interested in its motion about the sun, but it is certainly not a particle if we are interested in its revolution about its North-South axis or in events that occur upon it. Galaxies are particles compared with typical distances over which they may freely evolve but certainly not particles if we are interested in their structural properties. On the other
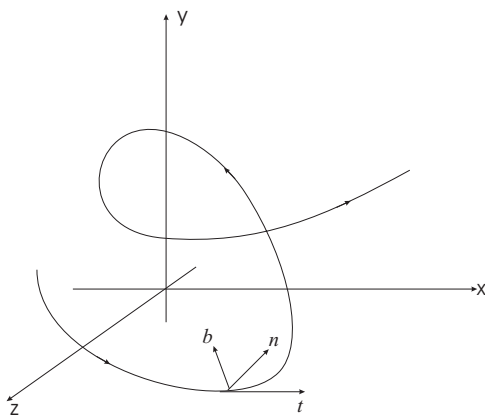
Figure 2.1: A particle trajectory

end of the spectrum, a molecule in a rarified gas may well be considered a particle for most practical purposes when studying the macroscopic properties of the gas as a whole, but we know well that molecules have complex structures which may be probed if one looks close enough. The key to using the concept of a "particle" as an approximation is to compare the volume of the object in consideration to the typical volume of space over which its evolution occurs.

The evolution of particles is the subject of Newton's laws of mechanics. These are known to yield a very accurate description of their evolution if the speeds attained are small compared to the speed of light. When speeds close to the speed of light are reached one must account for "relativistic effects" which can be quite dramatic, but suppressed by powers of v/c where v represents the speed of the body and c the speed of light. The particle **configuration** at any time is defined by its mass, m, its position in space, $\vec{r}$, and its velocity $\vec{v}$. In a nutshell the essential problem or mechanics is: given a certain initial configuration, $\{m_0, \vec{r}_0, \vec{v}_0\}$ at a given (initial) time, t_0, determine the evolution of this configuration over time, *i.e.,* deterimine its position and velocity, $\vec{r} = \vec{r}(t)$, $\vec{v} = \vec{v}(t)$ for all future times.

2.2   The Serret-Frenet description of curves

Our definition of the configuration of a particle it terms of its position and velocity is based on a picture of what the evolution of a particle entails, shown in figure (2.1). A moving particle essentially sweeps out a smooth trajectory (a $C^{(1)}$ curve) in space. Because it has no geometric features of its own, a description of its trajectory is necessarily a complete description of a particle's evolution. Geometrically, a curve is completely characterized at

every point by three unit vectors, *viz.,* the tangent, the normal and the bi-normal, and two scalars, the curvature and torsion, as we now describe. Let $C : \vec{r}(t)$ represent a curve in space and let $s(t)$ be an invertible function representing the distance *along the curve* of the point given by $\vec{r}(t)$ from some fixed point on the curve. We define the unit **tangent vector** to the curve $\vec{r} = \vec{r} \circ t(s)$ as

$$\widehat{t} = \frac{d\vec{r}}{ds} \tag{2.2.1}$$

That $\widehat{t}$ is a unit vector follows from the definition of the path length s,

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = |d\vec{r}| \Rightarrow 1 = \left| \frac{d\vec{r}}{ds} \right| \tag{2.2.2}$$

We also define the unit **normal vector** to the curve by

$$\frac{d\widehat{t}}{ds} = \kappa(s)\widehat{n} \tag{2.2.3}$$

where $\kappa(s)$ is a function of position, called the **curvature** of C. The unit normal $\widehat{n}$ is perpendicular to $\widehat{t}$ because $\widehat{t}$ is a unit vector

$$\widehat{t}^2 = 1 \Rightarrow \widehat{t} \cdot \frac{d\widehat{t}}{ds} = 0 = \kappa(s)\widehat{t} \cdot \widehat{n} \tag{2.2.4}$$

provided that $\kappa(s) \neq 0$. The plane spanned by $\widehat{t}$ and $\widehat{n}$ at any point, P, on the curve is called the **osculating** plane of the curve at P and a circle lying in the osculating plane at P, that touches P, has the same tangent at P as the curve itself and radius equal to the reciprocal of the curvature is called the **osculating circle** at P.

If we define the unit **binormal vector** as

$$\widehat{b} = \widehat{t} \times \widehat{n} \tag{2.2.5}$$

then $\widehat{b}$ is clearly perpendicular to both $\widehat{t}$ and $\widehat{n}$. In this way, $\{\widehat{t}, \widehat{n}, \widehat{b}\}$ form a right handed triad called the **Frenet vectors**. Together they form a basis for three dimensional space, defining a non-static reference frame called the **Frenet frame**. Now since $\widehat{b}$ is a unit vector it follows that $d\widehat{b}/ds$ is perpendicular to $\widehat{b}$ because

$$\widehat{b}^2 = 0 \Rightarrow \widehat{b} \cdot \frac{d\widehat{b}}{ds} = 0. \tag{2.2.6}$$

Moreover, because $\widehat{b} \cdot \widehat{t} = 0$,

$$\frac{d\widehat{b}}{ds} \cdot \widehat{t} + \widehat{b} \cdot \frac{d\widehat{t}}{ds} = \frac{d\widehat{b}}{ds} \cdot \widehat{t} + \kappa(s)\widehat{b} \cdot \widehat{n} = \frac{d\widehat{b}}{ds} \cdot \widehat{t} = 0 \tag{2.2.7}$$
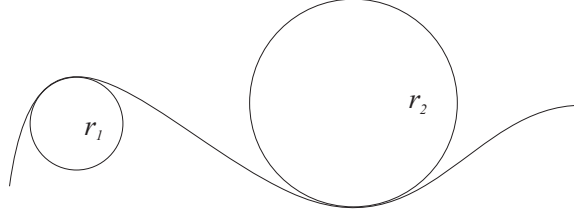
Figure 2.2: Osculating circles for a plane curve

showing that $d\widehat{b}/ds$ is perpendicular to $\widehat{t}$ as well. But in three dimensions $\{\widehat{t}, \widehat{n}, \widehat{b}\}$ is a (right-handed) triad, so $d\widehat{b}/ds$ must be parallel or antiparallel to $\widehat{n}$. We define

$$\frac{d\widehat{b}}{ds} = -\tau(s)\widehat{n} \tag{2.2.8}$$

where $\tau(s)$ is called the **torsion** of C. Finally, expressing (2.5.4) as $\widehat{n} = \widehat{b} \times \widehat{t}$ and taking a derivative we get

$$\frac{d\widehat{n}}{ds} = -\tau(s)\widehat{n} \times \widehat{t} + \kappa(s)\widehat{b} \times \widehat{n} = \tau(s)\widehat{b} - \kappa(s)\widehat{t} \tag{2.2.9}$$

The three equations (2.2.3,2.2.8,2.2.9) relating the vectors of the Frenet frame and their derivatives are called the **Serret-Frenet** formulæ.

We are generically interested in time derivatives, not derivatives w.r.t. the path length as given by the Frenet equations. However, applying the chain rule to express the velocity and acceleration in terms of the Serret-Frenet vectors we get

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\widehat{t}, \Rightarrow |\vec{v}| = \frac{ds}{dt} \ \text{ and } \ \widehat{v} = \widehat{t} \tag{2.2.10}$$

showing that the velocity is tangent to the curve. Taking one more derivative,

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2 s}{dt^2}\widehat{t} + \kappa(s)\left(\frac{ds}{dt}\right)^2 \widehat{n}. \tag{2.2.11}$$

The component of $\vec{a}$ in the direction of $\widehat{t}$ is called the **tangential acceleration** of the particle,

$$a_t = \frac{d^2 s}{dt^2} \tag{2.2.12}$$

and its component in the direction of the normal is called the **centripetal acceleration**,

$$a_n = \kappa\left(\frac{ds}{dt}\right)^2 \tag{2.2.13}$$

Think, for example, of a body performing uniform motion in a circle of radius r.[1] The curvature of a circle of radius r is $1/r$ so the centripetal acceleration is v^2/r, an expression that is already quite familiar. However, we now see that the formula for the centripetal acceleration is quite general, if one thinks of r as the instantaneous radius of curvature as shown pictorially in figure (2.2).

2.3 Galilean Transformations

We have already seen how vectors in general and position vectors in particular transform under rotations of the coordinate system. We may also ask how they transform under **boosts**, *i.e.,* uniform relative motions between coordinate systems. Therefore consider two coordinate systems, S and S' which are such that S' has a constant velocity $\vec{v}_0$ relative to S. Let P be a point in space (the position of some particle at time t, say) that is represented in the coordinate system S by the position vector $\vec{r}$ and in the system S' by the position vector $\vec{r}'$. Suppose for convenience (and without loss of generality) that the origin of the two frames coincide at $t = 0$. If we let $\vec{r}_0$ represent the position of the origin of S' relative to the S at time $t \neq 0$, then we could relate the vectors $\vec{r}$ and $\vec{r}'$ according to [see figure (2.3)]

$$\vec{r} = \vec{r}' + \vec{r}_0 = \vec{r}' + \vec{v}_0 t \qquad (2.3.1)$$

If we also assume that time is absolute, *i.e.,* that any time interval between two events as measured in one frame is identical to the corresponding interval measured in the other, then the two descriptions of the motion of the particle at P are related by

$$t' = t$$

$$\vec{r}' = \vec{r} - \vec{v}_0 t \qquad (2.3.2)$$

---

[1]<u>Problem</u>: Consider a helix defined by the curve

$$\vec{r}(t) = (a \cos \omega t, a \sin \omega t, vt)$$

where a, ω and v are constants, and determine the tangent, normal and binormal vectors, $\hat{t}$, $\hat{n}$ and $\hat{b}$. Verify the Serret-Frenet relations and show that the curvature and torsion are constants given by

$$\kappa = \frac{a\omega^2}{a^2\omega^2 + v^2}$$

$$\tau = \frac{v\omega}{a^2\omega^2 + v^2}$$

Notice that if $v = 0$, so that the curve is a circle (in the $x - y$ plane), then the curvature is just the reciprocal of its radius and the torsion vanishes.
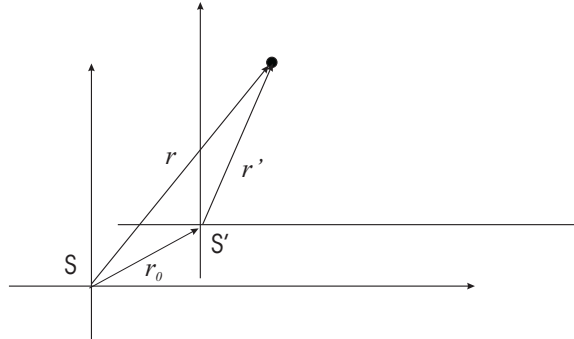
Figure 2.3: Two frames in uniform relative motion

Notice that we are tacitly assuming that S' and S are able to communicate with each other instantaneously. Taking derivatives, it follows that

$$\vec{v}' = \frac{d\vec{r}'}{dt'} = \frac{d\vec{r}'}{dt} = \vec{v} - \vec{v}_0$$

$$\vec{a}' = \frac{d^2\vec{r}'}{dt'^2} = \frac{d^2\vec{r}'}{dt^2} = \vec{a}, \tag{2.3.3}$$

where we make use of the fact that $\vec{v}_0$ is constant. Thus, although the two observers represented by S and S' observe different velocities for the particle, they measure the same acceleration provided that their motion relative to each other is uniform, $\vec{v}_0$ is constant. These are the **Galilean transformations**. Any theory of vector transformations under boosts is called a relativity theory. **Galilean relativity** is described by the transformations in (2.3.2). Soon we will see that these transformations are really a limiting case of **Einstein's relativity** theory in which it is not possible for S and S' to communicate at infinite speeds.

2.4 Newton's Laws

Newton's laws begin with both a definition of a class of observers called **inertial** observers as well as a statement about the "natural" motion of a particle, *i.e.,* the motion of a particle in the absence of any external influence upon it from its environment, as viewed by an inertial observer. We may state the "first" law (also known as the law of **inertia**) as follows:

- 1$^{\text{st}}$ Law: It is always possible to identify a frame, called an **inertial frame**, in which a particle will continue in its state of motion (at rest or in uniform motion in a straight line, $\vec{v} = $ const.) unless compelled to do otherwise by an external force.

In the absence of any external force, the inertial observer can mathematically describe the particle's motion by the statement $d\vec{v}/dt = 0$. It is important to note that there is not enough content in the law for the observer to simultaneously determine both if (s)he is inertial *and* if the particle being observed is "free". Only when one fact is known can the other be determined. If the observer knows herself to be inertial then a particle is free if and only if it moves with a constant velocity relative to her and, conversely, if the particle is known to be free then she is inertial if and only if the particle's velocity relative to her is constant.[2]

If m is the particle mass, which we assume constant, we may introduce the momentum, $\vec{p} = m\vec{v}$, and restate the first law in terms of it as follows: from the point of view of an inertial observer

$$\frac{d\vec{v}}{dt} = 0 = m\frac{d\vec{v}}{dt} \quad \Rightarrow \quad \frac{d\vec{p}}{dt} = 0. \tag{2.4.1}$$

Newton goes on to say that any deviation from this "free" behavior is attributable to the presence of an external force,

- 2$^{\text{nd}}$ Law: Viewed from an inertial frame, the rate of change of momentum is equal to the external, applied "force"

$$\frac{d\vec{p}}{dt} = \vec{F}^{\text{ext}}(\vec{r}, \vec{v}, t) \tag{2.4.2}$$

Notice that because $\vec{p}$ is a vector, so is $\vec{F}^{\text{ext}}$. This law is a *definition* of the term "external force" and does not specify it. $\vec{F}^{\text{ext}}$ must be determined by experiment. Once $\vec{F}^{\text{ext}}$ is known by careful experimentation, then (2.4.2), viewed as a differential equation, gives the particle trajectory subject, as always, to some initial conditions. Every effort is therefore directed to determining the "force law", *i.e.*, the form of $\vec{F}^{\text{ext}}$. Note that "force" has mechanical dimension $[F] = ml/t^2$. In the MKS system, force is measured in "Newtons" and in the CGS system it is measured in "dynes". As examples we know of

1. *Hooke's Law*: The force exerted by a spring, stretched by a distance $|\vec{r}|$,

$$\vec{F} = -k\vec{r},$$

   where $\vec{r}$ represents a displacement along the length of the spring from some position at which the spring is in equilibrium, and k is a constant whose mechanical dimension is $[k] = m/t^2$ or N/m (MKS) and dynes/cm (CGS).

---

[2]The inertial observer in Newtonian mechanics is therefore a primitive concept. Newton asserts that such frames exist, but does not state independently how they are determined (Newton considered observers "moving at a constant velocity relative to the fixed stars" as inertial). The existence of inertial frames is also asserted in Einstein's special theory of relativity without further elaboration, but in the general theory inertial frames may exist only locally, *i.e.*, over infinitesimal regions of space-time.

2. *Newton's law of Gravity*: The gravitational force exerted by a point-like mass m_1 on another point-like mass m_2

$$\vec{F}_{1\to 2} = -G\frac{m_1 m_2}{r^2}\hat{r},$$

where r is the separation between two particles, $\hat{r}$ is the unit vector that points from the particle labeled 1 to the particle labeled 2 and G is "Newtons constant of universal gravitation", with mechanical dimension $[G] = l^3/mt^2$ or N·m²/kg² (MKS) and dyne·cm/g². in the CGS system.

3. *Coulomb's law*: The electrostatic force exerted by a point-like charge q_1 on another point-like charge q_2

$$\vec{F}_{1\to 2} = \frac{1}{4\pi\epsilon_o}\frac{q_1 q_2}{r^2}\hat{r},$$

where r is the separation between the charges, $\hat{r}$ points from the charge labeled 1 to the charge labeled 2 and $C = 1/4\pi\epsilon_0$ is "Coulomb's constant" written in terms of the electric permitivity of the vacuum ϵ_0. The electric permitivity has mechanical dimension $[\epsilon_0]$ =Coulomb²/N · m² (MKS).

4. *Frictional Drag*: The "drag" or resistance offered by a medium to the motion of a body (whose dimensions are much larger than the typical dimensions of the molecules of the medium) inside it,

$$\vec{F}_{\text{drag}} = -k|\vec{v}|^\alpha \hat{v},$$

where k and α are constants, the first representing the strength of the drag and the second its dependence on the velocity, $\vec{v}$ of the body. The exponent α can have no mechanical dimension, but the mechanical dimension of k depends on α: $[k]$ =N·s$^\alpha$/m$^\alpha$. This expression for the drag is valid so long as the speed of the body is not comparable to the speed of sound, c, in the medium. If the $|\vec{v}| \ll c$ then the index α is on the order of unity. α gets larger as the speed of sound is approached, which makes air drag significant at high speeds. The constant k depends on the viscosity of the medium and the geometry of the body.

While the first law can be seen as an positing the existence of the inertial frame and defining it by specifying what is the "natural" motion of a particle observed from such a frame in the absence of any external influence, and the second defines quantitatively the concept of an external influence (or force), the third law of Newton is actually a law in its classical sense:

- 3$^{\text{rd}}$ Law: To every action of (force applied by) one particle on another, there is an equal and contrary reaction of the other on the first, *i.e.,*

$$\vec{F}_{1\to 2} = -\vec{F}_{2\to 1} \tag{2.4.3}$$

This law does not have general applicability and neither does it have an obvious generalization. It is true only when

- the forces between the particles act along the line joining the particles,
- the forces do not depend on the velocities of the particles themselves, and
- all velocities involved are small compared with the speed of light.

It does not, for example, hold for the Lorentz force of electrodynamics.

2.5 Newton's Laws and the Serret Frenet Formulæ

Newton's description of particle dynamics can be neatly connected with the Serret-Frenet description of curves. In general,

$$\vec{v} = \frac{ds}{dt}\widehat{t}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2s}{dt^2}\widehat{t} + \kappa(t)\left(\frac{ds}{dt}\right)^2\widehat{n} \tag{2.5.1}$$

and so

$$\widehat{t} = \widehat{v}, \quad \frac{ds}{dt} = |\vec{v}| \quad \text{and} \quad \frac{d^2s}{dt^2} = \widehat{v}\cdot\vec{a} \tag{2.5.2}$$

It also follows that

$$\vec{v}\times\vec{a} = \kappa|\vec{v}|^3\,\widehat{b} \quad\Rightarrow\quad \kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} \tag{2.5.3}$$

and the binormal vector is

$$\widehat{b} = \frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|}, \tag{2.5.4}$$

therefore using the fact that $\{\widehat{t},\widehat{n},\widehat{b}\}$ form a right handed triad we find the normal vector,

$$\widehat{n} = \frac{(\vec{v}\times\vec{a})\times\widehat{v}}{|\vec{v}\times\vec{a}|}. \tag{2.5.5}$$

Let us compute the torsion from

$$\frac{d\widehat{b}}{ds} = -\tau\widehat{n} \quad\Rightarrow\quad \tau = -\frac{\widehat{n}}{|\vec{v}|}\cdot\frac{d\widehat{b}}{dt}, \tag{2.5.6}$$

where $\widehat{b}$ is given in (2.5.4). Then,[3]

$$\tau = -\frac{\widehat{n}}{|\vec{v}|}\cdot\frac{d}{dt}\left[\frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|}\right]$$

---

[3]<u>Problem</u>: Show that

$$\frac{d}{dt}\frac{1}{|\vec{v}\times\vec{a}|} = -\frac{1}{|\vec{v}\times\vec{a}|^3}[\vec{v}\times(\vec{v}\times\vec{a})\cdot\dot{\vec{a}}]$$

$$= \quad -\frac{\widehat{n}}{|\vec{v}|} \cdot \left[\frac{\vec{v} \times \dot{\vec{a}}}{|\vec{v} \times \vec{a}|} - \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|^3} [\vec{v} \times (\vec{v} \times \vec{a}) \cdot \dot{\vec{a}}] \right] \qquad (2.5.7)$$

and, since $\widehat{n}$ is perpendicular to $\vec{v} \times \vec{a}$, it follows that

$$
\begin{aligned}
\tau &= -\frac{\widehat{n}}{|\vec{v}|} \cdot \frac{\vec{v} \times \dot{\vec{a}}}{|\vec{v} \times \vec{a}|} = -\frac{[(\vec{v} \times \vec{a}) \times \widehat{v}] \cdot (\vec{v} \times \dot{\vec{a}})}{|\vec{v}||\vec{v} \times \vec{a}|^2} \\
&= -\frac{\vec{a} \cdot (\vec{v} \times \dot{\vec{a}})}{|\vec{v} \times \vec{a}|^2} = \frac{\vec{v} \cdot (\vec{a} \times \dot{\vec{a}})}{\kappa^2 |\vec{v}|^6} = \frac{(\vec{v} \times \vec{a}) \cdot \dot{\vec{a}}}{\kappa^2 |\vec{v}|^6},
\end{aligned} \qquad (2.5.8)
$$

where we used $\widehat{n} = (\vec{v} \times \vec{a}) \times \widehat{v}/|\vec{v} \times \vec{a}|$ and the symmetries of the triple product. Notice that the torsion is proportional to the rate of change of the acceleration (called the **jerk**) and does not vanish only if the jerk has a non-vanishing component perpendicular to the plane containing the velocity and acceleration.

One may re-express everything in terms of the Newtonian force, considering that $\vec{a} = \vec{F}/m$, and we now summarize the relationship between the Serret-Frenet description of curves and the Newtonian description of motion:

$$\widehat{t} = \widehat{v}, \quad \frac{ds}{dt} = |\vec{v}|, \quad \frac{d^2 s}{dt^2} = \frac{\widehat{v} \cdot \vec{F}}{m}$$

$$\widehat{n} = \frac{(\vec{v} \times \vec{F}) \times \widehat{v}}{|\vec{v} \times \vec{F}|}$$

$$\widehat{b} = \frac{(\vec{v} \times \vec{F})}{|\vec{v} \times \vec{F}|}$$

$$\kappa = \frac{|\vec{v} \times \vec{F}|}{m|\vec{v}|^3}$$

$$\tau = \frac{(\vec{v} \times \vec{F}) \cdot \dot{\vec{F}}}{m^2 \kappa^2 |\vec{v}|^6} \qquad (2.5.9)$$

The force is therefore responsible for generating both the curvature and the torsion.[4] To generate a curvature $\vec{F}$ must have a component orthogonal to $\vec{v}$ and to generate a torsion the rate of change of $\vec{F}$ should have a component perpendicular to the plane containing

---

[4]Show that

- for a "central force" (eg., Hooke's law, Newton's law of gravity, Coulomb's law), the torsion of the particle trajectory vanishes, and

- for the frictional drag force, both the curvature and the torsion vanish.

What do these facts mean in each case?

$\vec{v}$ and $\vec{F}$. For example, during uniform motion in a circle of radius r the tangential acceleration is zero, implying that $\hat{v} \cdot \vec{F} = 0$ (the force must be perpendicular to the velocity). The curvature of the circle is just $1/r$ and the fourth equation tells us that

$$|\vec{F}_\perp| = \frac{m\vec{v}^2}{r} \tag{2.5.10}$$

which is the well known expression for the (magnitude of the) centripetal force on the particle. In fact, the above formula holds for the component of $\vec{F}$ perpendicular to the velocity at all times, even if the motion is not circular and uniform, provided that r is taken to be the instantaneous radius of curvature of the trajectory and $\vec{v}$ the particle's instantaneous velocity.

2.6 One dimensional motion

We will now examine some simple applications of Newton's laws. But first, let us develop some solution methods. In a single spatial dimension, Newton's second law of motion yields a single equation,

$$\frac{dp}{dt} = F(t, x, v) \tag{2.6.1}$$

where the force may depend quite generally on time, the position and the velocity of the body but not on higher derivatives, such as the acceleration.

- If the force depends only on time, the equation of motion can be easily integrated twice. Without loss of generality, let the initial time be set to zero, *i.e.*, $t_0 = 0$. This can always be done because it simply amounts to a resetting of our clock. For constant m,

$$m\frac{dv}{dt} = F(t) \Rightarrow m \int_{v_0}^{v} dv = \int_0^t dt' F(t') \Rightarrow v - v_0 = \int_0^t dt' F(t') \tag{2.6.2}$$

This is the first integral. Noting that $v = dx/dt$, the equation

$$\frac{dx}{dt} = v_0 + \frac{1}{m} \int_0^t dt' F(t') \tag{2.6.3}$$

may be integrated again to give

$$x(t) = x_0 + v_0 t + \frac{1}{m} \int_0^t dt' \int_0^{t'} dt'' F(t'') \tag{2.6.4}$$

where x_0 is the initial position of the particle, which also may be set to zero without loss of generality. This would exploit our freedom to choose an origin. If $F(t)$ is a

constant, independent of time, let $F = ma$. For instance, $a = -g$ could represent the acceleration of a particle in the gravitational field of the earth, close to its surface ($g = 9.81$ m/s^2). Then

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2 \tag{2.6.5}$$

which is the well known kinematical formula.

• If the applied force depends only on the position of the particle, as for example the force exerted by a spring on a body depends only on the displacement of the body from the equilibrium position of the spring (Hooke's law), then a useful trick is to consider the velocity as a function of position, *i.e.*,

$$v(t) = v \circ x(t) = v[x(t)] \tag{2.6.6}$$

and use the chain rule to reexpress the rate of change of momentum as

$$m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = mv\frac{dv}{dx} = F(x) \tag{2.6.7}$$

Integrating this equation once gives

$$m \int_{v_0}^{v} v\,dv = \int_{0}^{x} F(x')dx' \tag{2.6.8}$$

(where we have used our freedom to choose an origin by setting $x_0 = 0$). Performing the integrals gives

$$\frac{1}{2}m(v^2 - v_0^2) = \int_{0}^{x} F(x')dx' \Rightarrow v = \pm\sqrt{v_0^2 + \frac{2}{m}\int_{0}^{x} F(x')dx'} \tag{2.6.9}$$

and rewriting the velocity as dx/dt,

$$\frac{dx}{dt} = \pm\sqrt{v_0^2 + \frac{2}{m}\int_{0}^{x} F(x')dx'} \Rightarrow \int_{0}^{x} \frac{dx'}{\sqrt{v_0^2 + \frac{2}{m}\int_{0}^{x'} F(x'')dx''}} = \pm\int_{0}^{t} dt = \pm t. \tag{2.6.10}$$

If the integral on the left can be performed we obtain $t = t(v_0, x)$, which must be inverted to obtain $x = x(v_0, t)$. Neither of the two steps need be trivial to perform, nevertheless the above constitutes a complete albeit implicit solution in to the problem.

- If the force depends only on the velocity, the second law of motion may also be integrated to obtain an implicit solution, provided that the integrations and inversions can be performed. Consider

$$m\frac{dv}{dt} = F(v) \Rightarrow m\int_{v_0}^{v}\frac{dv'}{F(v')} = t \qquad (2.6.11)$$

If the integral can be done, we get $t = t(v, v_0)$. If the function can be inverted, we would get $v = v(v_0, t)$ which can be integrated once more to obtain the desired solution, so

$$\frac{dx}{dt} = v(v_0, t) \Rightarrow x = x_0 + \int_0^t v(v_0, t')dt' \qquad (2.6.12)$$

would constitute a complete solution to the problem. It is sometimes convenient to combine the trick of considering the velocity as a function of position and writing

$$mv\frac{dv}{dx} = F(v) \Rightarrow \int_{v_0}^{v}\frac{v'dv'}{F(v')} = \frac{x}{m} \qquad (2.6.13)$$

setting $x_0 = 0$. Assuming the integral can be performed, we obtain $v = v(v_0, x)$, which can be integrated once more

$$\frac{dx}{dt} = v(v_0, x) \Rightarrow \int_0^x \frac{dx'}{v(v_0, x')} = t \qquad (2.6.14)$$

again giving a complete implicit solution to the problem, but in a different form.

- In the general case,

$$m\frac{dv}{dt} = F(x, v, t) \qquad (2.6.15)$$

the second law must be treated as the second order differential equation

$$m\frac{d^2x}{dt^2} - F(x, \frac{dx}{dt}, t) = 0 \qquad (2.6.16)$$

Needless to say, the problem can be non-linear and difficult to solve, although in rare cases a solution may be obtained. We will examine one such example in the following pages.

We will now work out some specific examples to illustrate these procedures in problems where all the required steps can be performed and the answer given in closed form. In the worst case scenarios, it is usually possible to integrate the equations numerically. Numerical solutions, while not ideal, are unfortunately what one must appeal to in most realistic physical situations. Exact (closed form) solutions on the other hand serve our intuition best and we will concentrate on these for the most part.
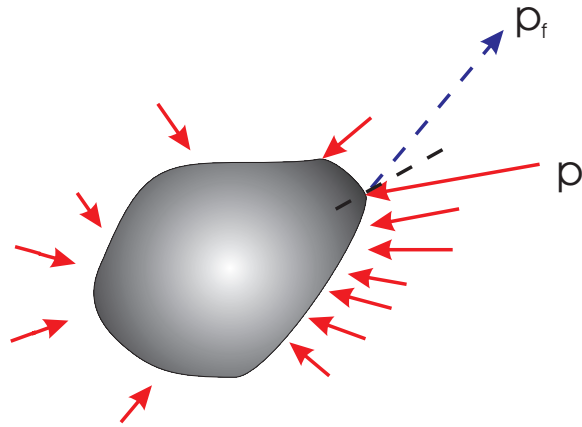
Figure 2.4: The microscopic origin of "drag" in fluids

2.7   Motion in a resisting medium

It is known from experiment that the resistive force applied by a homogeneous fluid on a body moving in it with a velocity $\vec{v}$ relative to the fluid itself is given by

$$F_i = -\sum_j k_{ij} |\vec{v}|^\alpha \widehat{v}^j \qquad (2.7.1)$$

where k_{ij} is a constant matrix with positive real coefficients and $\widehat{v}$ is the unit vector in the direction of the body's motion relative to the fluid. The expression is valid provided that the dimensions of the body are large compared with the molecular dimensions and the speed of the body is smaller than the speed of sound in the medium. It is acceptable provided that $k_{ij}\widehat{v}^j$ transforms as a vector, since $|\vec{v}|^\alpha = (\vec{v}^2)^{\alpha/2}$ transforms as a scalar. The constant matrix k_{ij}, called the drag coefficient, characterizes both the viscosity of the fluid and the geometry of the body. The constant α is generally found to be unity for speeds that are small compared with the speed of sound in the medium, but grows as the speed of sound is approached.

This phenomenological force law can be derived from a microsocopic theory. The idea is the following: if a body moves through a fluid with velocity $\vec{v}$ in a frame in which the fluid is at rest (the molecules have zero average velocity in this frame), we consider the frame of an observer who is instantaneously at rest relative to the body. In this second frame the molecules of the fluid flow past the body with an average velocity $-\vec{v}$. The flux of molecules (number of particles striking the body per unit area per unit time) in "front" would be higher than that of the molecules striking the body from the "rear" (see (2.4). During a collision between a molecule and the body, the molecule transfers some of its

momentum to the body

$$\Delta \vec{p} = \vec{p}_f - \vec{p}_i \tag{2.7.2}$$

Let $\langle \Delta \vec{p}_{\text{mol}} \rangle$ be the average momentum transferred by the molecules in this way to the body, then by conservation of the total momentum

$$\langle \Delta \vec{p}_{\text{tot}} \rangle = \langle \Delta \vec{p}_{\text{body}} + \Delta \vec{p}_{\text{mol}} \rangle = 0 \Rightarrow \langle \Delta \vec{p}_{\text{body}} \rangle = -\langle \Delta \vec{p}_{\text{mol}} \rangle \tag{2.7.3}$$

Because the molecular flux is greater in "front" of the body, more momentum is transferred to it in the "backward" direction than in the "forward" direction and this causes the drag. For example, a more detailed analysis, employing hydrodynamics, of a sphere of radius r_0 moving slowly compared to the speed of sound in the fluid suggests that

$$\vec{F} = -6\pi\eta r_0 \vec{v}. \tag{2.7.4}$$

where η is the viscosity of the fluid. We recognize (2.7.1) with $k_{ij} = (6\pi\eta r_0)\delta_{ij}$ and $\alpha = 1$.

Let us consider motion in one dimension under (2.7.1) with $\alpha = 1$. As the force depends on the only on the velocity, we employ the general approach laid down earlier for this case,

$$m\frac{dv}{dt} = -kv \Rightarrow \int_{v_0}^{v} \frac{dv}{v} = -\frac{k}{m}t \Rightarrow v = v_0 e^{-kt/m} \tag{2.7.5}$$

Integrating once more gives

$$\frac{dx}{dt} = v_0 e^{-kt/m} \Rightarrow x = x_0 + v_0 \int_0^t dt' e^{-kt'/m} \tag{2.7.6}$$

The solution of the second law of motion is therefore

$$x(t) = x_0 - \frac{mv_0}{k}e^{-kt'/m}\Big|_0^t = x_0 + \frac{mv_0}{k}(1 - e^{-kt/m}) \tag{2.7.7}$$

To verify if the answer makes sense, we check some limits. At $t = 0$ we have $v(0) = v_0$ and $x(0) = x_0$, which are just our initial conditions. As $t \to \infty$,

$$v(t) \to 0 \quad \text{and} \quad x(t) \to x_0 + \frac{mv_0}{k} \tag{2.7.8}$$

Although the body comes to rest an infinite time later the distance traveled by it is finite,

$$x(\infty) - x(0) = \frac{mv_0}{k} \tag{2.7.9}$$

The distance traveled depends on the initial velocity (increases with increasing initial velocity), on the viscosity (decreases as the viscosity increases) and on the geometry of the body.
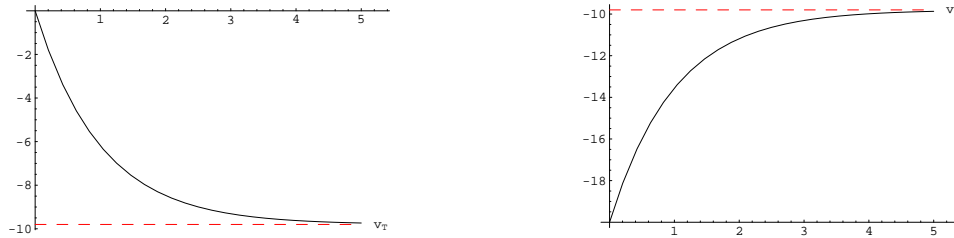
Figure 2.5: The velocity as a function of time: Figure 2.6: The velocity as a function of time: $v_0 = 0$ m/s, $b = 1$ s^{-1} and $g = 9.8$ m/s^2. $v_0 = -20$ m/s, $b = 1$ s^{-1} and $g = 9.8$ m/s^2.

Let us complicate this example a bit by letting the body fall under the influence of gravity (close to the surface of the earth, $F_{\text{grav}} = -mg$) while experiencing the frictional drag of the medium. Newton's second law gives

$$m\frac{dv}{dt} = -kv - mg \qquad (2.7.10)$$

Calling $b = k/m$, this equation can be written as

$$\frac{dv}{dt} = -bv - g \qquad (2.7.11)$$

and integrated $(t_0 = 0)$

$$\int_{v_0}^{v} \frac{dv}{bv + g} = -t \Rightarrow \frac{1}{b}\ln\left(v + \frac{g}{b}\right)\bigg|_{v_0}^{v} = -t \qquad (2.7.12)$$

Fortunately this expression is easy to invert and gives

$$v = -\frac{g}{b} + \left(v_0 + \frac{g}{b}\right)e^{-bt} \qquad (2.7.13)$$

We check that at $t = 0$, $v = v_0$. Furthermore, as $t \to \infty$, $v(t) \to v_T = -g/b$, *i.e.*, the body's velocity approaches a *constant*, the so called terminal velocity (the negative sign indicates that the terminal velocity is in the direction of the gravitational field). The terminal velocity depends directly on the mass of the body and inversely on the drag coefficient. If the initial velocity is less than the terminal velocity, as for example is the case for a parachutist jumping off an aircraft, it will increase in magnitude until the terminal velocity is reached. If on the contrary the initial velocity is larger than the terminal velocity, it will decrease in magnitude to approach the terminal velocity [see figures (2.7)] Next consider integrating the velocity equation one more time,

$$\frac{dx}{dt} = -\frac{g}{b} + \left(v_0 + \frac{g}{b}\right)e^{-bt}$$

$$\Rightarrow \quad x \ = \ x_0 - \frac{g}{b}t - \frac{1}{b}\left(v_0 + \frac{g}{b}\right)e^{-bt}\Big|_0^t$$

$$\Rightarrow \quad x \ = \ x_0 - \frac{g}{b}t + \frac{1}{b}\left(v_0 + \frac{g}{b}\right)(1 - e^{-bt}) \tag{2.7.14}$$

At $t = 0$, $x = x_0$ as required. As $t \to \infty$,

$$x(t) \to x_0 + \frac{1}{b}\left(v_0 + \frac{g}{b}\right) + v_T t \tag{2.7.15}$$

This result makes sense because, in the limit of large t, the body moves with constant velocity equal to its terminal velocity. The effective "initial" position is not the same as x_0, but displaced by $(v_0 b + g)/b^2$. This essentially accounts for the extra distance traveled while the body's speed was larger than its terminal speed.

2.7.1 Drag and the projectile

We can combine the motions examined above by considering a projectile in the presence of drag. The equation of motion is now effectively two dimensional,

$$m\frac{d\vec{v}}{dt} = -k\vec{v} + m\vec{g} \tag{2.7.16}$$

or the (decoupled) set of equations,

$$\frac{dv_x}{dt} \ = \ -bv_x$$

$$\frac{dv_y}{dt} \ = \ -bv_y - g \tag{2.7.17}$$

Suppose that the body has initial velocity

$$\vec{v}_0 = v_0(\cos\theta, \sin\theta) = (v_{0x}, v_{0y}) \tag{2.7.18}$$

where θ is the angle made with the $x-$axis, then applying our solutions,

$$x(t) \ = \ \frac{v_{0x}}{b}\left(1 - e^{-bt}\right)$$

$$y(t) \ = \ v_T t + \left(\frac{bv_{0y} + g}{b^2}\right)(1 - e^{-bt}) \tag{2.7.19}$$

This trajectory is shown in figure (2.7.1). Notice that it is no longer symmetric about the maximum. This is because the equations do not describe a parabola, as they do when
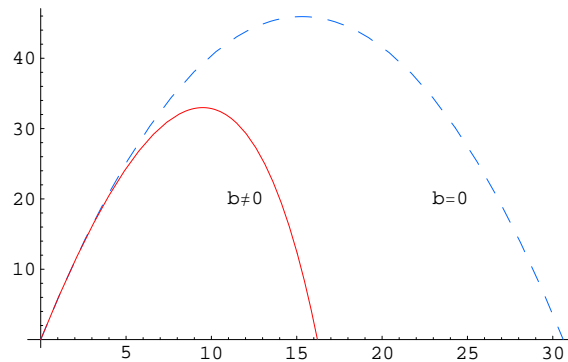
Figure 2.7: Trajectory of a projectile with drag: $\vec{v}_0 = (5, 30)$ m/s, $b = 0.2$ s^{-1} and $g = 9.8$ m/s^2.

$k = 0$. The drag free motion of a projectile must be recovered by taking the limit as $b \to 0$. This is easiest to do via l'Hospital's rule and gives[5],

$$\lim_{b \to 0} x(t) = v_{0x} t$$

$$\lim_{b \to 0} y(t) = v_{y0} t - \frac{1}{2} g t^2 \qquad (2.7.20)$$

2.7.2   Perturbative expansions: an example

Let us calculate the range of the projectile, when b is small, but non-vanishing. We must solve the equation

$$\frac{gt}{b} = \left(\frac{b v_{0y} + g}{b^2} \right) (1 - e^{-bt}) \qquad (2.7.21)$$

This is a transcendental equation, which is impossible to solve analyticaly. However, if bt is small we may reexpress the exponential as a power series,

$$bgt = (v_{0y} b + g) \left(bt - \frac{b^2 t^2}{2!} + \frac{b^3 t^3}{3!} - \dots \right)$$

$$\Rightarrow \quad 0 = v_{0y} b^2 t - (v_{0y} b + g) \left(\frac{b^2 t^2}{2!} - \frac{b^3 t^3}{3!} + \frac{b^4 t^4}{4!} - \dots \right) \qquad (2.7.22)$$

Dividing by $v_{0y} b^2$ and rearranging terms, we get

$$t = (b + \frac{g}{v_{0y}}) \left(\frac{t^2}{2!} - \frac{bt^3}{3!} + \frac{b^2 t^4}{4!} - \dots \right) \qquad (2.7.23)$$

---

[5]Problem: Apply l'Hospital's rule and show that the drag free projectile motion is recovered.

and we want to solve this for t. We now apply a method called "perturbation theory" to solve this equation. The idea is to expand the solution (2.7.19) about its solution in the absence of drag, which of course is only meaningful if the effects of drag are "small". To make precise what we mean by "small", we proceed as follows: if t_0 represents the solution of the equation when $b = 0$, then the parameter

$$\lambda = bt_0 \tag{2.7.24}$$

is dimensionless (recall that $[b] = t^{-1}$). The time scale t_0 will depend on the initial conditions. For small enough b (the drag) the dimensionless quantity λ will also be small and, since λ is dimensionless, its magnitude will be independent of the units used. Any parameter that is small, time independent and free of mechanical dimensions can serve as a "perturbation" parameter. What this means is that we assume that the solution of (2.7.23) has the form

$$t = t_0 + \lambda t_1 + \lambda^2 t_2 + ... = \sum_{j=0}^{\infty} \lambda^j t_j \tag{2.7.25}$$

where t_0 solves (2.7.23) with $b = 0$, *i.e.*,

$$t_0 = \frac{2v_{0y}}{g} \tag{2.7.26}$$

Equation (2.7.23) now becomes

$$
\begin{aligned}
t_0 + \lambda t_1 + \lambda^2 t_2 + ... &= \left(\frac{\lambda}{t_0} + \frac{g}{v_{0y}}\right) \left(\frac{1}{2}(t_0 + \lambda t_1 + ...)^2 - \frac{\lambda}{6t_0}(t_0 + \lambda t_1 + ...)^3 + ...\right) \\
&= \left(\frac{\lambda}{t_0} + \frac{g}{v_{0y}}\right) \left(\frac{t_0^2}{2} + \lambda t_0 t_1 - \frac{\lambda}{6}t_0^2 + ...\mathcal{O}(\lambda^2)\right)
\end{aligned}
\tag{2.7.27}
$$

and we compare terms with the same power of λ to get t_1, t_2 etc. To zeroeth order in λ we get

$$t_0 = \frac{gt_0^2}{2v_{0y}} \Rightarrow t_0 = 0 \ \text{ or } \ t_0 = \frac{2v_{0y}}{g} \tag{2.7.28}$$

which is precisely the same solution we had before, when $b = 0$. To first order

$$t_1 = \frac{t_0}{2} + \frac{gt_0 t_1}{v_{0y}} - \frac{gt_0^2}{6v_{0y}} \Rightarrow t_1 = -\frac{v_{0y}}{3g} \tag{2.7.29}$$

which is the first correction to $b = 0$ result: to first order in lambda we find

$$t \approx t_0 + \lambda t_1 = \frac{2v_{0y}}{g} - \frac{2bv_{0y}^2}{3g^2} = \frac{2v_{0y}}{g}\left[1 - \frac{bv_{0y}}{3g}\right] \tag{2.7.30}$$

One can go on in this way, calculating t_2, t_3, etc., by comparing higher and higher powers of lambda in (2.7.27) and using the previously obtained values of t_0, t_1, t_2, etc. Note that this method is based on the assumption that the series in (2.7.25) converges. This is no longer true when $\lambda \geq 1$, so we must require that

$$\lambda \ll 1 \Rightarrow bt_0 \ll 1 \Rightarrow k \ll \frac{mg}{2v_{0y}} \tag{2.7.31}$$

Thus the validity of the expansion depends on the mass of the body and the $y-$ component of the initial velocity. The approximation gets better for larger masses and smaller initial velocities.

It is now straightforward to also approximate the range of the particle. From the first equation in (2.7.19),

$$
\begin{aligned}
R' &= \frac{v_{0x}}{b}(1 - 1 + bt - \frac{b^2 t^2}{2} + ...)_{t=t_0+\lambda t_1+...} \\
&= v_{0x}(t - \frac{bt^2}{2} + ...)_{t=t_0+\lambda t_1+...} \\
&= v_{0x}\left(\frac{2v_{0y}}{g}\left[1 - \frac{bv_{0y}}{3g}\right] - \frac{b}{2}\frac{4v_{0y}^2}{g^2} + \mathcal{O}(b^2) \right) \\
R' &\approx \frac{2v_{0x}v_{0y}}{g}\left(1 - \frac{4}{3}\frac{bv_{0y}}{g} + ... \right)
\end{aligned}
\tag{2.7.32}
$$

The first term is the range of the drag free projectile. As we would expect, the range of the projectile is diminished by the drag. We find that it is diminished (to first order in b) by[6]

$$\Delta R = R - R' \approx \frac{4}{3}\frac{bv_{0x}v_{0y}^2}{g^2} \tag{2.7.33}$$

2.8   Harmonic motion

A deformed spring generally applies a "restorative" force on any body that is attached to it. The restorative force is directed in such a way as to return the spring to its original

---

[6]Problem: The maximum height attained by the projectile is given by $v_y = 0$, *i.e.*,

$$\frac{g}{b} = \left(v_{0y} + \frac{g}{b}\right)\left(1 - bt + \frac{b^2 t^2}{2} - ...\right)$$

Expand t as before ($t = t_0 + \lambda t_1 + ...$) with $\lambda = bt_0$, where t_0 is the time taken to reach maximum height when $b = 0$. Calculate the time taken to reach maximum height and then the maximum height attained by the projectile.

size and shape. If, for example, one end of the spring is attached to an (unmovable) wall and the other to a body that is free to move, the restorative force will have the effect of returning the body to its original position. Restorative forces can occur in a wide variety of physical situations. For example consider an object floating in a liquid. If the body is pushed deeper into the liquid, we know from experience that it will bounce back, moving up and down until, over a characteristic period of time, it returns to its original position. Under certain conditions that occur frequently enough to be most interesting and that we will discuss later, the magnitude of the restorative force will depend linearly on the displacement of the body from its equilibrium position. In general,

$$F_i = -\sum_j k_{ij} x^j \tag{2.8.1}$$

where k_{ij} is a constant, symmetric matrix of positive, real coefficients. This is a general version of "Hooke's Law".

2.8.1 Harmonic motion in one dimension

Let us consider motion under Hooke's law in one dimension. As the force depends only on the position of the body, we apply the general method developed earlier for this case

$$m\frac{d^2x}{dt^2} = -kx \Rightarrow \frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x \tag{2.8.2}$$

Rewriting the equation as

$$v\frac{dv}{dx} = -\omega^2 x \Rightarrow \frac{1}{2}(v^2 - v_0^2) = -\frac{1}{2}\omega^2 x^2, \tag{2.8.3}$$

where we have taken the equilibrium position of the body to be the origin and v_0 is the velocity at $x = 0$. This gives

$$v = \frac{dx}{dt} = \sqrt{v_0^2 - \omega^2 x^2} \Rightarrow \int_0^x \frac{dx}{\sqrt{v_0^2 - \omega^2 x^2}} = t - t_0, \tag{2.8.4}$$

(since we have no right to fix the time at which the body is at its equilbrium position, t_0 remains arbitrary). The integral is easily solved by substitution and we find

$$x(t) = \frac{v_0}{\omega}\cos(\omega[t - t_0]) = \mathcal{A}\cos(\omega t - \phi_0) \tag{2.8.5}$$

and

$$v(t) = \frac{dx}{dt} = -\omega\mathcal{A}\sin(\omega t - \phi_0) \tag{2.8.6}$$
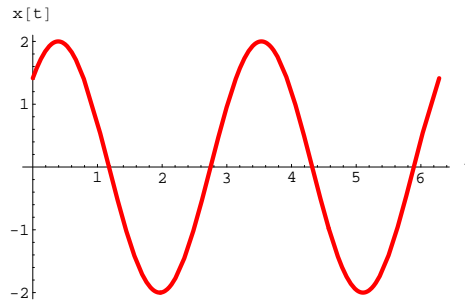
Figure 2.8: Graphic representation of simple harmonic motion

This solution represents "simple harmonic motion", *i.e.*, regular oscillations about a fixed point [shown in figure(2.8)]. It involves two arbitrary constants, *viz.*, A and ϕ_0. The constant A represents the maximum displacement of the body from its equilibrium position and is called the **amplitude** of the oscillations. The second constant is called the **initial phase** of the oscillation. Together with the amplitude, it determines the initial position of the body by the relation

$$x(0) = x_0 = \mathcal{A}\cos(\phi_0) \tag{2.8.7}$$

Both the amplitude and the initial phase may be reexpressed in terms of the initial position and velocity of the motion, thus

$$v(0) = v_0 = \omega\mathcal{A}\sin(\phi_0) \tag{2.8.8}$$

together with (2.8.7) gives

$$\tan(\phi_0) = \frac{v_0}{\omega x_0}, \quad \mathcal{A} = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} \tag{2.8.9}$$

The **period** of the motion is the minimum time taken for the body to return to its position. If we call this time τ, then we require that

$$x(t+\tau) = x(t) \Rightarrow \mathcal{A}\sin[\omega(t+\tau) - \phi_0] = \mathcal{A}\sin(\omega t - \phi_0) \tag{2.8.10}$$

This gives $\omega\tau = 2n\pi, \quad n \in \mathbb{Z} - \{0\}$ and therefore

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{2.8.11}$$

The **frequency** of the oscillations is the number of oscillations that occur in one second, *i.e.*,

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{2.8.12}$$
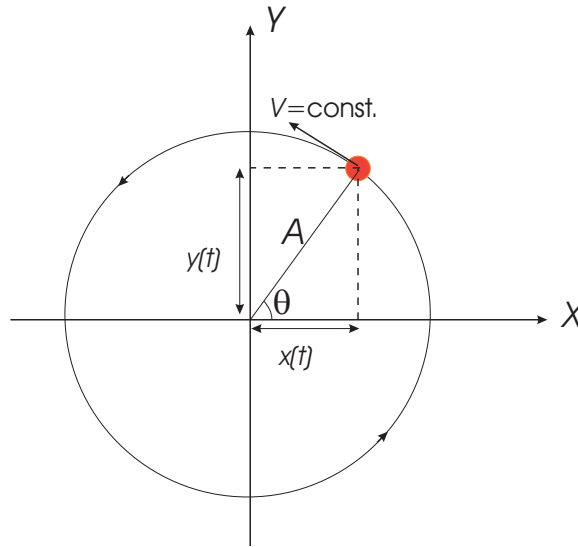
Figure 2.9: Relationship between Uniform Circular motion and Simple Harmonic Motion.

It is sometimes useful to think of simple harmonic motion in terms of the uniform circular motion of a body as shown in figure (2.9). As the angular velocity, $\omega = v/\mathcal{A}$, of the body performing uniform circular motion is constant, the angle, θ, made by the body with the $x-$ axis at any time is given by $\theta(t) = \omega t - \phi_0$, where ϕ_0 is a constant representing the initial angle made by the body with the $x-$ axis. The $x-$coordinate of the body,

$$x(t) = \mathcal{A}\cos\theta(t) = \mathcal{A}\cos(\omega t - \phi_0) \qquad (2.8.13)$$

performs simple harmonic motion about the origin, as does its $y-$coordinate. For this reason, ω is called the **angular frequency** of the motion. It represents the number of radians swept out per second by the equivalent body performing uniform circular motion.

2.8.2 One dimensional oscillations with damping

If in addition to the restoring force of a spring on a body, we must account for the dragging of a medium, Newton's second law should read

$$\frac{d^2x}{dt^2} = -\omega^2 x - bv \qquad (2.8.14)$$

Now the driving force is a function of both x and v. Rewriting the equation as

$$\frac{d^2x}{dt^2} + b\frac{dx}{dt} + \omega^2 x = 0 \qquad (2.8.15)$$

we see that this is a linear, second order differential equation with *constant coefficients*. To solve this equation, it is best to think of the l.h.s. as the action of the differential operator $D^2 + bD + \omega^2$ on the function $x(t)$, where $D = d/dt$. Factoring the operator,

$$(D^2 + bD + \omega^2) = (D - \lambda_+)(D - \lambda_-) \tag{2.8.16}$$

we find

$$\lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4\omega^2}}{2}. \tag{2.8.17}$$

Now because

$$(D - \lambda_+)(D - \lambda_-)x(t) = 0 \tag{2.8.18}$$

and because the operators $(D - \lambda_\pm)$ commute, it follows that either

$$(D - \lambda_+)x(t) = 0 \tag{2.8.19}$$

or

$$(D - \lambda_-)x(t) = 0. \tag{2.8.20}$$

Each first order equation is simple to solve, we find

$$\frac{dx_\pm}{dt} - \lambda_\pm x_\pm = 0 \Rightarrow x_\pm(t) = A_\pm e^{\lambda_\pm t} \tag{2.8.21}$$

where $A_\pm$ are arbitrary constants. The general solution of our problem is a linear combination of these two solutions because we are really interested in the second order equation in (2.8.18). Therefore

$$x(t) = e^{-\frac{bt}{2}} \left[A_+ e^{\sqrt{\frac{b^2}{4} - \omega^2}t} + A_- e^{-\sqrt{\frac{b^2}{4} - \omega^2}t} \right] \tag{2.8.22}$$

provided that the eigenvalues are not degenerate, *i.e.,* $\lambda_+ \neq \lambda_-$. Two cases arise:

- If $\frac{b^2}{4} > \omega^2$, the system is said to be **overdamped**. Calling $\overline{\omega} = \sqrt{\frac{b^2}{4} - \omega^2}$, all solutions take the form

$$x(t) = e^{-bt/2} \left[A_+ e^{\overline{\omega}t} + A_- e^{-\overline{\omega}t} \right] \tag{2.8.23}$$

  The motion is not oscillatory, but decays with time to the equilibrium position of the body, *i.e.,* $x(t \to \infty) \to 0$. The two constants, $A_\pm$, are determined from the initial conditions, $x(0) = x_0$ and $v(0) = v_0$. We find[7]

$$x_0 = A_+ + A_-$$

---

[7]Problem: Obtain explicit expressions for $A_\pm$ in terms of x_0 and v_0 from the equations in (2.8.24)

$$v_0 = -\frac{b}{2}(A_+ + A_-) + \overline{\omega}(A_+ - A_-) \tag{2.8.24}$$

For real solutions, the coefficients $A_\pm$ must be real. The trajectory is shown as the red curve in figure (2.10).

- If $\frac{b^2}{4} < \omega^2$, the system is said to be **underdamped**. The motion is oscillatory, but its amplitude decays to zero with time. We could set $\overline{\omega} = \sqrt{\omega^2 - \frac{b^2}{4}}$ and write

$$x(t) = e^{-\frac{bt}{2}}\left[A_+ e^{i\overline{\omega}t} + A_- e^{-i\overline{\omega}t}\right] \tag{2.8.25}$$

Since the position of the body must be real,

$$x^*(t) = x(t) \Rightarrow A_+^* e^{-i\overline{\omega}t} + A_-^* e^{i\overline{\omega}t} = A_+ e^{i\overline{\omega}t} + A_- e^{-i\overline{\omega}t} \Rightarrow A_- = A_+^* \tag{2.8.26}$$

and therefore

$$x(t) = 2e^{-bt/2}\Re(A_+ e^{i\overline{\omega}t}) \tag{2.8.27}$$

Now A_+, being generally complex, can be written as

$$A_+ = \frac{\mathcal{A}}{2}e^{-i\phi_0} \tag{2.8.28}$$

where $\mathcal{A}$ is real and $-\phi_0$ is the phase of A_+, giving

$$x(t) = \mathcal{A}e^{-bt/2}\Re(e^{i(\overline{\omega}t - \phi_0)}) \tag{2.8.29}$$

Applying de Moivre's formula,[8] we then find the final form of the solution

$$x(t) = \mathcal{A}e^{-bt/2}\cos(\overline{\omega}t - \phi_0) \tag{2.8.30}$$

This expression is similar to that obtained earlier for the simple harmonic oscillator, except that the amplitude is now decreasing exponentially with time,

$$\mathcal{A}(t) = \mathcal{A}e^{-bt/2}, \tag{2.8.31}$$

and the frequency of oscillations,

$$\tau = \frac{2\pi}{\overline{\omega}} = \frac{2\pi}{\sqrt{\omega^2 - \frac{b^2}{4}}} \quad \Rightarrow \quad f = \frac{\sqrt{\omega^2 - \frac{b^2}{4}}}{2\pi}, \tag{2.8.32}$$

---

[8]De Moivre's formula reads
$$e^{i\theta} = \cos\theta + i\sin\theta$$
Show this by expanding the exponential function in a Taylor series about $\theta = 0$ and comparing the real and imaginary parts of the expansion with the Taylor series of the cosine and sine functions respectively.
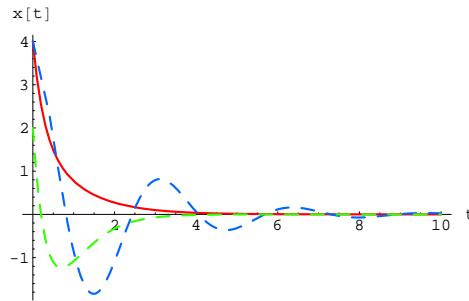
Figure 2.10: Damped Oscillators: overdamping (red), underdamping (blue) and critical damping (green).

is diminished by the damping. As before, the constants $\mathcal{A}$ and ϕ_0 can be obtained from the initial conditions.[9] The underdamped oscillator is shown as the blue curve in figure (2.10).

- A special situation occurs when the roots of the "auxiliary equation" $D^2 + bD + \omega^2 = 0$ are degenerate, *i.e.*, $\lambda_+ = \lambda_- = \lambda$. In this situation (called **critical damping**) the general solution is given by

$$x(t) = (A + Bt)e^{\lambda t} \tag{2.8.33}$$

It occurs when $\omega = b/2$ which means that $\lambda = -b/2$ and therefore

$$x(t) = (A + Bt)e^{-bt/2} \tag{2.8.34}$$

As in the overdamped case, the system does not oscillate but returns to its equilibrium position in time, *i.e.*, $x(t \to \infty) \to 0$. The constants may be determined from the initial conditions,

$$x_0 = x(0) = A, \quad v_0 = v(0) = -\frac{bA}{2} + B. \tag{2.8.35}$$

The critically damped oscillator is the green curve in figure (2.10).

2.8.3   Two dimensional oscillations

Let us consider the force law (2.8.1) in two dimensions. The two equations of motion

$$m\frac{d^2x}{dt^2} = -k_{11}x - k_{12}y$$

---

[9]<u>Problem</u>: Find the constants $\mathcal{A}$ and ϕ_0 in terms of $x_0 = x(0)$ and $v_0 = v(0)$.

$$m\frac{d^2y}{dt^2} = -k_{21}x - k_{22}y \qquad (2.8.36)$$

are coupled, *i.e.,* the motion in any direction is affected by the motion in the other. This is because we have not been clever enough to choose a convenient coordinate system *i.e.,* one in which the motions decouple. To see how this can be done, let us rewrite the equations in matrix form,

$$m\frac{d^2\vec{r}}{dt^2} = -\widehat{k}\vec{r} \qquad (2.8.37)$$

where $\widehat{k}$ is the matrix of elements k_{ij}. Because $\widehat{k}$ is a constant, symmetric matrix it can be diagonalized by a constant orthogonal transformation, $\widehat{S}$. Let the diagonal matrix be $\widehat{k}_D = \widehat{S}\widehat{k}\widehat{S}^{-1}$. Multiplying the equation of motion on the left by $\widehat{S}$,

$$m\frac{d^2}{dt^2}(\widehat{S}\vec{r}) = -(\widehat{S}\widehat{k}\widehat{S}^{-1})(\widehat{S}\vec{r}) \qquad (2.8.38)$$

where on the left we have used the fact that $\widehat{S}$ is constant and on the right we have inserted unity in the form of $\widehat{S}^{-1}\widehat{S}$. Calling $\vec{q} = \widehat{S}\vec{r}$, we get

$$m\frac{d^2\vec{q}}{dt^2} = -\widehat{k}_D\vec{q}. \qquad (2.8.39)$$

But $\widehat{k}_D$ is diagonal, so the equation above is just a set of two decoupled harmonic oscillators whose characteristic angular frequencies are determined by the eigenvalues of the matrix $\widehat{k}$. However, the vector $\vec{q}$ represents the position of the body in a rotated coordinate system. Let us suppose that the two eigenvalues of $\widehat{k}$ are $k_\pm$. Then the equations of motion, in terms of $\vec{q} = (q_+, q_-)$ become

$$\begin{aligned}
\frac{d^2q_+}{dt^2} &= -\frac{k_+}{m}q_+ = -\omega_+^2 q_+ \\
\frac{d^2q_-}{dt^2} &= -\frac{k_-}{m}q_- = -\omega_-^2 q_-
\end{aligned} \qquad (2.8.40)$$

which have the solutions

$$q_\pm = \mathcal{A}_\pm \cos(\omega_\pm t - \phi_\pm) \qquad (2.8.41)$$

The constants $\mathcal{A}_\pm$ and $\phi_\pm$ are related to the initial conditions (four of them because we are in two dimensions; there are two initial conditions per dimension, *viz.,* the initial position and the initial velocity).

But what are $q_\pm$ and $\omega_\pm$? To answer this question we must first determine $\widehat{S}$. Being two dimensional and orthogonal it must have the form

$$\widehat{S} = \begin{pmatrix} \cos\eta & \sin\eta \\ -\sin\eta & \cos\eta \end{pmatrix} \qquad (2.8.42)$$

and we would like to find η. By simply multiplying out the matrices, we determine that for $\widehat{k}_D$ to be diagonal

$$k_{12} \cos 2\eta - \frac{1}{2}(k_{11} - k_{22}) \sin 2\eta = 0 \Rightarrow \eta = \frac{1}{2} \tan^{-1}\left[\frac{2k_{12}}{(k_{11} - k_{22})}\right] \qquad (2.8.43)$$

and that the eigenvalues of $\widehat{k}$ are

$$k_{\pm} = \frac{1}{2}\left[(k_{11} + k_{22}) \pm \sqrt{(k_{11} - k_{22})^2 + 4k_{12}^2}\right] = m\omega_{\pm}^2 \qquad (2.8.44)$$

These are the "**normal modes**" of the system and $f_{\pm} = \omega_{\pm}/2\pi$ are its "normal frequencies". Moreover, since $\vec{q} = \widehat{S}\vec{r}$ we obtain the solution in terms of the old coordinates by inversion,

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} q_+ \\ q_- \end{bmatrix} = \begin{bmatrix} \cos\eta & -\sin\eta \\ \sin\eta & \cos\eta \end{bmatrix} \begin{bmatrix} \mathcal{A}_+ \cos(\omega_+ t - \phi_+) \\ \mathcal{A}_- \cos(\omega_- t - \phi_-) \end{bmatrix} \qquad (2.8.45)$$

We have assumed that the eigenvalues of $\widehat{k}$ are positive, which is true only if $k_{11}k_{22} > k_{12}^2$. It should be clear that the above procedure can be carried out in any number of dimensions. The diagonalizing matrix will always be a member of special orthogonal group $SO(n)$ and therefore involve $n(n-1)/2$ free parameters (the Euler angles).

The method can also be applied to more complex situations, such as the one that occurs in solids. A solid can be thought of as a very large number, N, of molecules each of which oscillates about some equilibrium position. The net force acting upon any molecule is due to its interaction with all the molecules in its neighborhood and can be quite complicated. However, it turns out that Hooke's law is an excellent approximation under normal conditions, so (2.8.1) applies. A generalization of the diagonalization illustrated in the simple two dimensional example above can be applied to reduce the solid to a system of $3N$ *decoupled* harmonic oscillators which, as we have just seen, is easily solved.

2.8.4   Trajectories in the plane

In the present, parametrized form it is difficult to visualize the trajectory of the body in the $x - y$ plane. For this it is sometimes better to determine the trajectory as an implicit function $f(x, y) =$const. We begin with the simpler situation that occurs when $k_{12} = 0$. This gives $\eta = 0$, $\omega_+^2 = k_{11}/m$ and $\omega_-^2 = k_{22}/m$. Therefore

$$x(t) = \mathcal{A}_+ \cos(\omega_+ t - \phi_+), \quad y(t) = \mathcal{A}_- \cos(\omega_- t - \phi_-), \qquad (2.8.46)$$

If we further specialize to the case when $\omega_+ = \omega_- = \omega$, then the motions in the $x-$ and $y-$ directions differ only by amplitude and a phase. If we let $\delta = \phi_+ - \phi_-$ then

$$y = \mathcal{A}_- \cos(\omega t - \phi_+ + \delta)$$

$$= \mathcal{A}_-[\cos(\omega t - \phi_+)\cos\delta - \sin(\omega t - \phi_+)\sin\delta]$$

$$= \frac{\mathcal{A}_-}{\mathcal{A}_+}[x\cos\delta - \sqrt{\mathcal{A}_+^2 - x^2}\sin\delta] \tag{2.8.47}$$

Some special cases are particularly interesting:

- When $\delta = 2n\pi$, $n \in \mathbb{Z}$,

$$y = \frac{\mathcal{A}_-}{\mathcal{A}_+}x \tag{2.8.48}$$

  which is the equation of a straight line in the $x - y$ plane with positive slope determined by the respective amplitudes.

- When $\delta = (2n + 1)\pi$, $n \in \mathbb{Z}$,

$$y = -\frac{\mathcal{A}_-}{\mathcal{A}_+}x, \tag{2.8.49}$$

  which is the equation of a straight line in the $x - y$ plane with negative slope whose magnitude is again determined by the respective amplitudes.

- When $\delta = (2n + 1)\frac{\pi}{2}$, $n \in \mathbb{Z}$,

$$\frac{x^2}{\mathcal{A}_+^2} + \frac{y^2}{\mathcal{A}_-^2} = 1 \tag{2.8.50}$$

  which is the equation of an ellipse whose major/minor axes are along the coordinate axes.

The general case is just as easy to analyze, for now we have

$$(\mathcal{A}_+ y - \mathcal{A}_- x\cos\delta)^2 = \mathcal{A}_-^2(\mathcal{A}_+^2 - x^2)\sin^2\delta. \tag{2.8.51}$$

Expanding the expressions on both sides, dividing by $\mathcal{A}_+^2\mathcal{A}_-^2$ and rearranging terms,

$$\frac{x^2}{\mathcal{A}_+^2} + \frac{y^2}{\mathcal{A}_-^2} - 2\frac{x}{\mathcal{A}_+}\frac{y}{\mathcal{A}_-}\sin\delta\cos\delta = \sin^2\delta \tag{2.8.52}$$

This equation also represents an ellipse, but one whose major/minor axes are rotated away from the coordinate axes. The angle by which the rotation occurs depends again on the amplitudes and the phase difference. If we want to find this angle, all we need to do is to perform a rotation of the original coordinates by an angle θ to a new set (x', y'), subject to the condition that the equation in (2.8.52) when written in terms of (x', y') becomes the standard (canonical) equation of an ellipse. This determines θ, as we now see. Performing the rotation, and expressing (x, y) in terms of (x', y') by inversion we substitute

$$x = x'\cos\theta - y'\sin\theta$$

$$y = x' \sin\theta + y' \cos\theta \tag{2.8.53}$$

into (2.8.52) to obtain

$$x'^2(\mathcal{A}_+^2 \sin^2\theta + \mathcal{A}_-^2 \cos^2\theta + 2\mathcal{A}_+\mathcal{A}_- \sin\theta \cos\theta \cos\delta)+$$

$$y'^2(\mathcal{A}_+^2 \cos^2\theta + \mathcal{A}_-^2 \sin^2\theta - 2\mathcal{A}_+\mathcal{A}_- \sin\theta \cos\theta \cos\delta)+$$

$$x'y'(-(\mathcal{A}_+^2 - \mathcal{A}_-^2)\sin 2\theta - 2\mathcal{A}_+\mathcal{A}_- \cos 2\theta \cos\delta) = \mathcal{A}_+^2 \mathcal{A}_-^2 \sin^2\delta \tag{2.8.54}$$

To get the canonical form, we require

$$\tan 2\theta = \frac{2\mathcal{A}_+\mathcal{A}_- \cos\delta}{\mathcal{A}_-^2 - \mathcal{A}_+^2}, \quad \mathcal{A}_+ \neq \mathcal{A}_- \tag{2.8.55}$$

If $\mathcal{A}_+ = \mathcal{A}_- = \mathcal{A}$ then $\theta = \frac{\pi}{4}$. The equation of the ellipse is then

$$(1 + \cos\delta)x'^2 + (1 - \cos\delta)y'^2 = \mathcal{A}^2 \sin^2\delta \tag{2.8.56}$$

Only the lengths of the major and minor axes are determined by δ in this case. This is no longer true when $\mathcal{A}_+ \neq \mathcal{A}_-$ as seen from (2.8.55), which must be applied otherwise.

2.8.5   Lissajou's figures

The situation gets substantially more complicated when $\omega_+ \neq \omega_-$, but it also gets more interesting. Returning to (2.8.46), we could express t in terms of x,

$$t = \frac{1}{\omega_+}\left[\cos^{-1}\left(\frac{x}{\mathcal{A}_+}\right) + \phi_+\right] \tag{2.8.57}$$

and substitute into the equation for y,

$$y = \mathcal{A}_- \cos\left(\frac{\omega_-}{\omega_+}\left[\cos^{-1}\frac{x}{\mathcal{A}_+} + \phi_+\right] - \phi_2\right) \tag{2.8.58}$$

This makes it obvious that the function $y = y(x)$ depends on the ratio $\frac{\omega_-}{\omega_+}$ and the phase difference δ. One finds that when the ratio of frequencies is a rational number the trajectory closes in the $x - y$ plane. In this case the frequencies are said to be commensurable. When the ratio of frequencies is not a rational number they are said to be incommensurable and the trajectory never closes. Instead, given a sufficiently long time, the body will come arbitrarily close to *every* point within the rectangle of dimensions $2\mathcal{A}_+ \times 2\mathcal{A}_-$. When the frequencies are commensurable, the (closed) trajectories of the body are called Lissajou's figures. Some examples of Lissajou's figures are shown in figure (2.11).

$$\frac{\omega_-}{\omega_+} = 2, \ \delta = \frac{\pi}{4} \qquad\qquad \frac{\omega_-}{\omega_+} = 2, \ \delta = \frac{\pi}{3}$$

$$\frac{\omega_-}{\omega_+} = \frac{4}{3}, \ \delta = \frac{\pi}{3} \qquad\qquad \frac{\omega_-}{\omega_+} = \frac{7}{2}, \ \delta = \frac{\pi}{6}$$
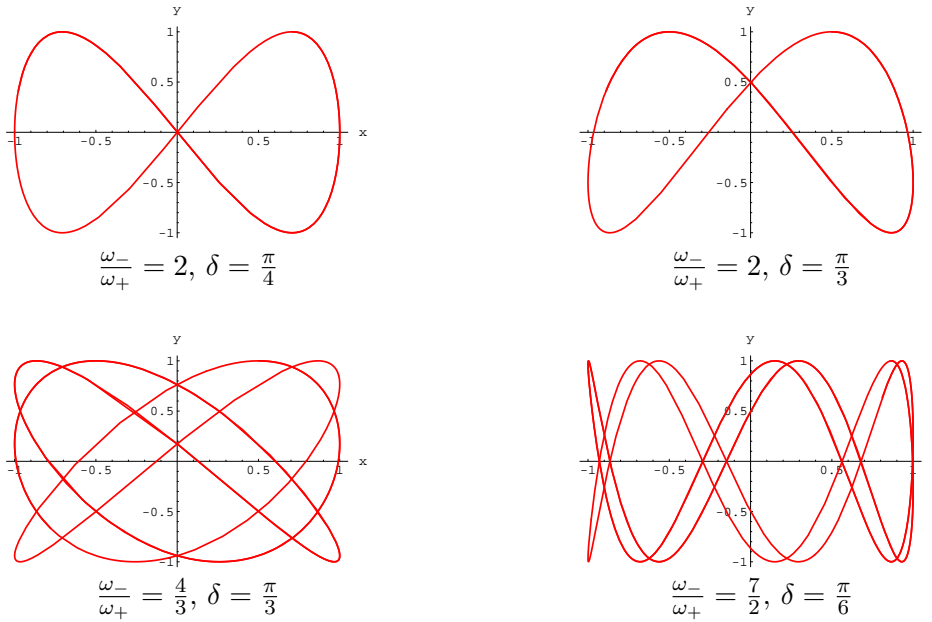
Figure 2.11: Lissajou's figures

2.9 One dimensional free fall

Let us turn our attention to a different problem. Consider two bodies falling freely into one another under the influence of their mutual gravitational interaction. (We could imagine, for instance, that the moon were caused to suddenly stop rotating around the earth, so that at a certain instant (call it $t = 0$) it began to fall freely. How long will it take for the moon to collide with the earth?) Let m_1 and m_2 be the masses of the bodies and let their positions at time t be denoted by $\vec{r}_1(t)$ and $\vec{r}_2(t)$ respectively. Newton's second law applied to each of the bodies in turn tells us that

$$m_1 \frac{d^2 \vec{r}_1}{dt^2} = -G \frac{m_1 m_2 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|^3} \tag{2.9.1}$$

while for the second body

$$m_2 \frac{d^2 \vec{r}_2}{dt^2} = -G \frac{m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \tag{2.9.2}$$

assuming that the masses stay constant during the free fall. Adding the two equations gives

$$\frac{d^2}{dt^2} (m_1 \vec{r}_1 + m_2 \vec{r}_2) = 0 \tag{2.9.3}$$

which tells us that the point whose position is

$$r_{\text{cm}} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \tag{2.9.4}$$

moves with a constant velocity, as would a free body. This point is called the **center of mass** of the system. Note that the center of mass is just a point in space; it is not itself a material body.

Multiplying the first equation by m_2 and the second by m_1, and then subtracting the first from the second gives

$$m_1 m_2 \frac{d^2}{dt^2}(\vec{r}_2 - \vec{r}_1) = -G \frac{m_1 m_2 (m_1 + m_2)(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_2 - \vec{r}_1|^3} \tag{2.9.5}$$

or, if $\vec{r}$ is used to represent the relative position $\vec{r}_2 - \vec{r}_1$ of body 2 w.r.t. body 1, and $M = m_1 + m_2$ is the total mass of the system,

$$\frac{d^2 \vec{r}}{dt^2} = -\frac{GM}{r^2}\hat{r} \tag{2.9.6}$$

The coordinate $\vec{r}$ is called the **relative coordinate**. As the motion is purely radial, *i.e.,* along the line joining the two bodies, it is effectively one dimensional with the force law

$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2}. \tag{2.9.7}$$

It can solved using the techniques laid out earlier in the chapter. We have

$$v \frac{dv}{dr} = -\frac{GM}{r^2} \Rightarrow \frac{1}{2} v^2 = GM \left[\frac{1}{r} - \frac{1}{r_0} \right] \tag{2.9.8}$$

where r_0 represents the position at which the bodies commenced their free fall ($r(0) = r_0$) and we have assumed that $v(0) = 0$. This means that

$$v = \frac{dr}{dt} = -\sqrt{2GM \left[\frac{1}{r} - \frac{1}{r_0} \right]} \Rightarrow -\sqrt{\frac{r_0}{2GM}} \int_{r_0}^{r} dr' \sqrt{\frac{r'}{r_0 - r'}} = t \tag{2.9.9}$$

(we have chosen the negative square root because r is decreasing in time). By making the substitution $r' = r_0 \sin^2 \eta$, we evaluate the integral on the left and arrive at

$$\sqrt{\frac{r_0}{2GM}} \left[\sqrt{r(r - r_0)} - r_0 \sin^{-1} \sqrt{\frac{r'}{r_0}} \right]_{r_0}^{r} = t(r) \tag{2.9.10}$$

or

$$\sqrt{\frac{r_0}{2GM}} \left[\frac{\pi r_0}{2} + \sqrt{r(r - r_0)} - r_0 \sin^{-1} \sqrt{\frac{r}{r_0}} \right] = t(r) \tag{2.9.11}$$

At $r = 0$ we have a collision, because r is the magnitude of the relative coordinate. The time taken for collision is therefore

$$t(0) = \frac{\pi}{2} \left(\frac{r_0^3}{2GM} \right)^{\frac{1}{2}} \tag{2.9.12}$$

We can now answer the question we began with. Suppose the moon were indeed suddenly caused to stop in its orbit. Assuming that its orbit is quasi-circular, its period is given by

$$\tau = \frac{2\pi r_0}{v} = 2\pi \left(\frac{r_0^3}{GM} \right)^{\frac{1}{2}} = 2\sqrt{2}\pi \left(\frac{r_0^3}{2GM} \right)^{\frac{1}{2}}, \tag{2.9.13}$$

where we used the fact that the centripetal acceleration of the moon is caused by the gravitational force on it, *i.e.*,

$$\frac{v^2}{r_0} = \frac{GM}{r_0^2} \Rightarrow v = \sqrt{\frac{GM}{r_0}}. \tag{2.9.14}$$

We can give the time for collision in terms of its orbital period prior to being stopped,

$$t(0) = \frac{\tau}{4\sqrt{2}} \approx 4.95 \text{ days} \tag{2.9.15}$$

(taking $\tau = 28$ days for the moon's orbit about the earth). Of course this is only an academic discussion because if the moon is indeed suddenly stopped in its orbit it can be only due to a catastrophic event such as a collision with another heavenly body, which will not leave the moon intact. Moreover, a freely falling moon would cause such disruptions to the earth's environment as to make life impossible long before the actual collision takes place.

2.10 Systems with variable mass: the rocket

Finally, we address a different sort of system, one in which the mass does not stay constant in time. This kind of system emphasizes the fact that Newton's second law is actually

$$\frac{d\vec{p}}{dt} = \vec{F}^{\text{ext}} \tag{2.10.1}$$

and not the more often quoted $\vec{F} = m\vec{a}$, whose validity relies on the mass of the body being held constant throughout the body's motion.

Consider the rocket, shown schematically in figure (2.12). Rockets contain a combustible material that burns at some rate depending on its design. The gasses produced
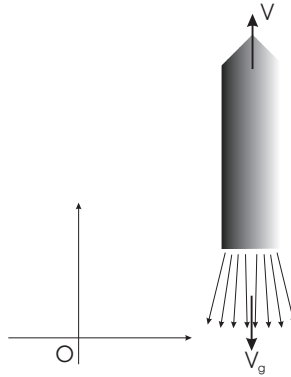
Figure 2.12: The Rocket

by the burning combustible are exhausted from one end, which we call "the bottom", of the rocket, *usually at a fixed rate relative to the rocket*. Let O be an observer situated on earth, relative to whom the velocity of the rocket is $\vec{v}(t)$. Let the average velocity of the gasses relative to this observer be $\vec{v}_g$, so that the velocity of the gasses relative to the rocket itself is $\vec{u} = \vec{v}_g - \vec{v}$, and this is usually constant. At some time t, the momentum of the rocket together with all the combustible material that remains unburned is

$$\vec{p}(t) = m(t)v(t) \tag{2.10.2}$$

An infinitesimal time δt later, a portion $-\delta m$ of the combustible has been burned and now exits the rocket with speed $\vec{v}_g$ while the mass $m + \delta m$ travels upward with a speed $\vec{v} + \delta\vec{v}$. The momentum of the original mass is

$$
\begin{aligned}
\vec{p}(t + \delta t) &= (m + \delta m)(\vec{v} + \delta\vec{v}) - \delta m\vec{v}_g \\
&= m\vec{v} + m\delta\vec{v} + \delta m(\vec{v} - \vec{v}_g) \\
&= p(t) + m\delta\vec{v} - \delta m\vec{u} \tag{2.10.3}
\end{aligned}
$$

where we have neglected the term $\delta m\delta\vec{v}$, which is too small, given that δm and $\delta\vec{v}$ are themselves infinitesimals. The total change in momentum of the system in the time interval δt is therefore

$$
\begin{aligned}
\delta\vec{p}(t) &= \vec{p}(t + \delta t) - \vec{p}(t) = m\delta\vec{v} - \delta m\vec{u} \\
\Rightarrow \frac{d\vec{p}}{dt} &= m\frac{d\vec{v}}{dt} - \frac{dm}{dt}\vec{u} \tag{2.10.4}
\end{aligned}
$$

By Newton's second law this can only be equal to the net external force, therefore

$$m\frac{d\vec{v}}{dt} = \frac{dm}{dt}\vec{u} + \vec{F}^{\text{ext}} \qquad (2.10.5)$$

This is known as the **rocket equation**. Even when $\vec{F}^{\text{ext}} = 0$ there is an effective force on the rocket, which comes about because of the need to conserve momentum and is proportional to the rate at which fuel is being burned and the velocity, relative to the rocket itself, at which the resulting gasses are expelled. Notice that because the fuel is being consumed the rocket is continuously losing mass, so $\delta m < 0$. Therefore the first term on the right hand side is directed *opposite* to the velocity of the gasses relative to the rocket. Thus hot gasses ejected from the base of the rocket cause the rocket itself to rise. If $\vec{F}^{\text{ext}}$ is just the (approximately) constant gravitational field of the earth close to its surface, $\vec{F}^{\text{ext}} = m\vec{g}$, we can integrate once to get

$$\vec{v} = \vec{v}_0 + \vec{u}\ln\left(\frac{m}{m_0}\right) + \vec{g}(t - t_0) \qquad (2.10.6)$$

where m_0 is the mass of the rocket at time t_0. A net zero acceleration (hovering) is possible only if

$$\frac{dm}{dt}\vec{u} = -m\vec{g} \Rightarrow m(t) = m_0 e^{-\frac{g}{u}(t-t_0)} \qquad (2.10.7)$$

i.e., the rate of fuel burning must be adjusted to precisely balance the changing mass of the rocket.

Chapter 3

Conservation Theorems

In the previous chapter we stated Newton's three laws of motion and used Newton's second law to obtain explicit solutions to a few simple problems involving one or two particles subject to certain initial conditions, *i.e.,* the position(s) and velocity(ies) of the particle(s). Unfortunately it is not possible to obtain such simple, closed form solutions in most cases of real physical interest. Can we then say anything about a physical system without referring to a tidy, closed form solution? It turns out that the answer is yes, it is possible to say quite a lot! For this we now examine some very general consequences of Newton's laws, the so-called **conservation theorems**. We will do this by building and examining a generic mechanical system made up of particles with generic interactions. In doing so, we will develop a new set of tools that will later allow us to analyze more complicated problems than would otherwise be possible.

3.1   Single Particle Conservation Theorems

3.1.1   Conservation of momentum

We begin by considering a system consisting of one particle. According to Newton's second law,

$$\frac{d\vec{p}}{dt} = \vec{F}^{\text{ext}} \tag{3.1.1}$$

where $\vec{F}^{\text{ext}}$ is the external force applied to it. Evidently, if $\vec{F}^{\text{ext}} = 0$, there is no external force and

$$\frac{d\vec{p}}{dt} = 0 \Rightarrow \vec{p} = \text{const.} \tag{3.1.2}$$

Thus we have our first conservation theorem:

- When the next external force on a particle is zero, the momentum of the particle is conserved.

3.1.2 Conservation of angular momentum

We define the angular momentum of a single particle about a point (call that point the origin) as

$$\vec{L} = \vec{r} \times \vec{p} \tag{3.1.3}$$

where $\vec{r}$ is the position of the particle with respect to the reference point. Taking a derivative with respect to time,

$$\frac{d\vec{L}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F} \tag{3.1.4}$$

because $\vec{v} \times \vec{p} \equiv 0$. This is the equivalent of Newton's second law for rotational motion. The quantity $\vec{r} \times \vec{F}$ is called the **torque**

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{3.1.5}$$

and

$$\frac{d\vec{L}}{dt} = \vec{\tau} \tag{3.1.6}$$

Our second conservation law follows:

- When the net torque due to all the external forces on a particle vanishes the angular momentum of the particle is conserved.

3.1.3 Work and the conservation of energy

Again, consider what happens when $\vec{F}^{\text{ext}} \neq 0$. Then taking the inner product of the second law with $\vec{v}$,

$$\frac{d\vec{p}}{dt} = \vec{F}^{\text{ext}} \Rightarrow m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \vec{F}^{\text{ext}} \tag{3.1.7}$$

and integrating the resulting equation gives,

$$m \int_1^2 \vec{v} \cdot d\vec{v} = \int_1^2 \vec{F}^{\text{ext}} \cdot \vec{v} \, dt \Rightarrow \frac{1}{2}(m\vec{v}_2^2 - m\vec{v}_1^2) = \int_1^2 \vec{F}^{\text{ext}} \cdot d\vec{r} \tag{3.1.8}$$

The quantity $K = \frac{1}{2}m\vec{v}^2$ is called the **kinetic energy** of the particle, and

$$W_{12} = \int_1^2 \vec{F}^{\text{ext}} \cdot d\vec{r} \tag{3.1.9}$$

is the work done by the external force $\vec{F}^{\text{ext}}$ on the particle in moving it from the point labeled "1" to the point labeled "2". The **power** is the instantaneous rate at which work is done,

$$P = \frac{dW}{dt} = \vec{F}^{\text{ext}} \cdot \frac{d\vec{r}}{dt} = \vec{F}^{\text{ext}} \cdot \vec{v} \tag{3.1.10}$$
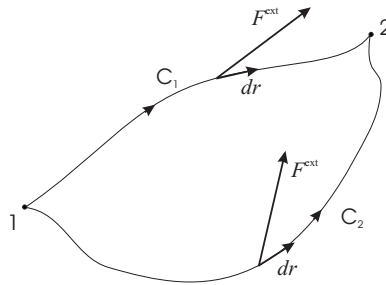
Figure 3.1: Line integrals defining work.

We see from (3.1.8) that

$$K_2 - K_1 = W_{12}, \tag{3.1.11}$$

which states that the change in the kinetic energy of a particle is equal to the work done by the external force acting upon it. This is called the "**work energy theorem**".[1]

Work as defined in (3.1.9) is a line integral

$$W_{12} = \int_1^2 \vec{F}^{\text{ext}} \cdot d\vec{r} = \sum_i \int_1^2 F_i^{\text{ext}} dx_i, \tag{3.1.12}$$

and will, in general, depend on the path along which it is taken. The work energy theorem therefore says that the change in kinetic energy of the particle depends on the line integral *taken along the on the trajectory of the particle as given by Newton's second law.* Thus it would seem that to use the theorem we would first have to solve Newton's equations. Of course, if that were always possible there would be no need of the theorem! Fortunately, in many situations of fundamental importance $\vec{F}^{\text{ext}}$ is such that the line integral does *not* depend on the path taken. This would mean that

$$\int_{1,C_1}^2 \vec{F} \cdot d\vec{r} = \int_{1,C_2}^2 \vec{F} \cdot d\vec{r} \tag{3.1.13}$$

where C_1 and C_2 are arbitrary paths joining the points "1" and "2" as shown in (3.1). If so, then we would find that the line integral over the *loop* from "1" to "2" along path C_1

---

[1]<u>Problem</u>: Consider two inertial frames, S, and S' and suppose that the velocity of S' relative to S is $\vec{u}$. It should be clear that the Kinetic energy is not invariant under a Galilean transformation from S to S'. Neither is the work done invariant under this transformation. Show that the work energy theorem nevertheless has the same form in S' as in S. This is a general principle, called the principle of **covariance**: the laws of physics must have the same form in all inertial frames. Later we will see that the principle of covariance can be extended to non-inertial frames as well.

and from "2" to "1" along path C_2 yields exactly zero because, rewriting (3.1.13) as

$$\int_{1,C_1}^{2} \vec{F} \cdot d\vec{r} = -\int_{2,C_2}^{1} \vec{F} \cdot d\vec{r} \tag{3.1.14}$$

gives

$$\int_{1,C_1}^{2} \vec{F} \cdot d\vec{r} + \int_{2,C_2}^{1} \vec{F} \cdot d\vec{r} = 0 = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} \tag{3.1.15}$$

But C_1 and C_2 were quite arbitrary as were the initial and final points they connected, therefore

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{3.1.16}$$

along *any* closed loop, C.

<u>Definition</u>: If the work done by a force $\vec{F}$,

$$W_{12} = \int_{1,C}^{2} \vec{F} \cdot d\vec{r}, \tag{3.1.17}$$

is independent of the path taken (the curve C) then it is **conservative**. A force $\vec{F}$ is conservative if and only if

$$\oint_C \vec{F} \cdot d\vec{r} = 0 \tag{3.1.18}$$

for any closed curve C.

Conservative forces are of particular interest because, as far as we know, all the fundamental forces of nature are conservative. Moreover, they are easy to work with. If a force is conservative then applying Stokes theorem to (3.1.18) gives

$$\oint_C \vec{F} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0 \tag{3.1.19}$$

But as C is arbitrary, so is S and it follows that $\vec{\nabla} \times \vec{F} = 0$. Thus $\vec{F}$ is conservative if and only if it is rotation free. Of course, every rotation free vector can be expressed as the gradient of a scalar function, therefore we define

$$\vec{F} = -\vec{\nabla}U(\vec{r}) \tag{3.1.20}$$

The work done by the conservative force $\vec{F}$ in moving a particle from one point to another becomes

$$\int_{1,C}^{2} \vec{F} \cdot d\vec{r} = -\int_{1}^{2} \vec{\nabla}U \cdot d\vec{r} = -\int_{1}^{2} dU = U(\vec{r}_1) - U(\vec{r}_2) \tag{3.1.21}$$

$U(\vec{r})$ is called the potential energy function corresponding to $\vec{F}$. If $\vec{F}(\vec{r})$ is known then it can be recovered, up to a constant, from

$$U(\vec{r}) = -\int_{*}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}' \tag{3.1.22}$$

where $*$ represents some fixed point, a "reference" point, of our choosing. In other words, $U(\vec{r})$ is defined only up to a constant. The constant is determined by our choice of $*$, which is itself arbitrary but once chosen must be held fixed (although $U(\vec{r})$ is defined via a line integral, it is well defined *i.e.,* independent of the path from $*$ to $\vec{r}$) because $\vec{F}$ is conservative). Now, by the work energy theorem

$$W_{12} = U(\vec{r_1}) - U(\vec{r_2}) = K_2 - K_1 \Rightarrow K_1 + U(\vec{r_1}) = K_2 + U(\vec{r_2}) \tag{3.1.23}$$

which means that

$$E = K + U = \frac{1}{2}m\vec{v}^2 + U(\vec{r}) \tag{3.1.24}$$

is constant during the motion of the particle. The constant, E, is called the total energy of the particle. Evidently, *the total energy of a particle is defined only up to an additive constant.*

The concept of potential energy and by implication of the total energy of a particle is strictly defined only when the total external force acting upon the particle is conservative. If the reference point $*$ is changed to $*'$ we find that

$$\begin{aligned}
U'(\vec{r}) &= -\int_{*'}^{\vec{r}} \vec{F} \cdot d\vec{r} = -\int_{*'}^{*} \vec{F} \cdot d\vec{r} - \int_{*}^{\vec{r}} \vec{F} \cdot d\vec{r} \\
&= U(\vec{r}) + C \tag{3.1.25}
\end{aligned}$$

where C is the constant given by first integral on the right (because $\vec{F}$ is conservative the integrals do not depend on the path between the endpoints, so they must depend only on the endpoints). This leads to our second conservation theorem:

- When the external force(s) on a particle is (are) conservative, then $\vec{F} = -\vec{\nabla}U(\vec{r})$, for some function $U(\vec{r})$ called the potential energy of the particle,

$$U(\vec{r}) = -\int_{*}^{\vec{r}} \vec{F} \cdot d\vec{r}, \tag{3.1.26}$$

and the total energy, which is the sum of the kinetic energy and the potential energy,

$$E = \frac{1}{2}m\vec{v}^2 + U(\vec{r}) \tag{3.1.27}$$

is conserved throughout its motion.

For a general potential function of the form $U = U(\vec{r}, t)$, taking a time derivative of E in (3.1.27) we see that

$$
\begin{aligned}
\frac{dE}{dt} &= m\vec{v} \cdot \frac{d\vec{v}}{dt} + \frac{dU}{dt} \\
&= m\vec{v} \cdot \frac{d\vec{v}}{dt} + \vec{v} \cdot \vec{\nabla}U(\vec{r}) + \frac{\partial U}{\partial t} \\
&= \vec{v} \cdot \left(m\frac{d\vec{v}}{dt} + \vec{\nabla}U(\vec{r}) \right) + \frac{\partial U}{\partial t} = \left(m\frac{d\vec{v}}{dt} - \vec{F} \right) + \frac{\partial U}{\partial t} \equiv \frac{\partial U}{\partial t}
\end{aligned} \qquad (3.1.28)
$$

where we have used Newton's second law. For conservative forces, $U(\vec{r}, t)$ cannot depend explicitly on time.

3.2 Frictional forces and mechanical energy

Frictional forces generally arise by the interactions of our physical system with the rest of the universe. They are generally complex interactions and can be modelled only phenomenologically. No frictional force, $\vec{f}$, can be conservative because friction always acts in a direction opposite to the velocity and therefore the work done by friction,

$$
W_{12}^f = \int_1^2 d\vec{r} \cdot \vec{f} = -\int_1^2 ds (\hat{v} \cdot \hat{v}) |\vec{f}| = -\int_1^2 ds |\vec{f}| \qquad (3.2.1)
$$

where s is the path length and we have used $d\vec{r}/ds = \hat{v}$, is always negative and can never be independent of the path because the integrand is always positive. For instance, if $|\vec{f}| = $ const., then the work done is just $W_{12} = |\vec{f}| L_{12}$ where L_{12} is the path length between points 1 and 2. On the other hand, if $|\vec{f}| = k|\vec{v}|^\alpha$ then

$$
W_{12}^f = -k \int_1^2 ds |\vec{v}|^\alpha \qquad (3.2.2)
$$

and the rate at which this work is being done is

$$
\frac{dW_{12}^f}{dt} = -k|\vec{v}|^{\alpha+1} \qquad (3.2.3)
$$

Suppose that all the other forces on our particle are conservative, so that we may define a potential energy for the sum of the conservative forces,

$$
U^{\mathrm{cons}}(\vec{r}) = -\int_*^{\vec{r}} \vec{F}^{\mathrm{cons}} \cdot d\vec{r}. \qquad (3.2.4)
$$

Integrate the equations of motion once,

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{f} \cdot \vec{v} + \vec{F}^{\text{cons}} \cdot \vec{v}$$

$$\Rightarrow \quad \frac{1}{2}m(\vec{v}_2^2 - \vec{v}_1^2) = \int_1^2 d\vec{r} \cdot \vec{f} + U^{\text{cons}}(\vec{r}_1) - U^{\text{cons}}(\vec{r}_2)$$

$$\Rightarrow \quad \frac{1}{2}m\vec{v}_2^2 + U^{\text{cons}}(\vec{r}_2) = \frac{1}{2}m\vec{v}_1^2 + U^{\text{cons}}(\vec{r}_1) - k\int_1^2 ds|\vec{v}|^{\alpha} \qquad (3.2.5)$$

The quantity $\mathcal{E} = K + U^{\text{cons}}$, which we will refer to as the *mechanical* energy of the particle is no longer conserved. In fact, it is easy to see that

$$\frac{d\mathcal{E}}{dt} = -k|\vec{v}|^{\alpha+1} \qquad (3.2.6)$$

Now it is a fundamental postulate of physics that the total energy of the universe remains constant. This loss of mechanical energy of our particle would then imply that the particle's energy is being transferred to the environment (the rest of the universe) at a rate that is proportional to some power of the particle's speed.

3.3   Examples of conservative forces

Only two of the forces considered in the pervious chapter are conservative, *viz.,* the restorative force of an ideal spring and the gravitational force acting between two massive bodies. Let us take a brief look at them.

The simple harmonic oscillator: The restorative force exerted by a spring, $\vec{F} = -k\vec{r}$, is conservative. To see this simply take the curl of $\vec{F}$ to get

$$\vec{\nabla} \times \vec{F} = -k\vec{\nabla} \times \vec{r} \equiv 0. \qquad (3.3.1)$$

So we can obtain the potential energy of the simple harmonic oscillator by the integral

$$U(\vec{r}) = -\int_*^{\vec{r}} \vec{F} \cdot d\vec{r} = \frac{1}{2}k\vec{r}^2 \qquad (3.3.2)$$

where we have taken the reference point to be the origin. The total energy of the harmonic oscillator,

$$E = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}k\vec{r}^2, \qquad (3.3.3)$$

is conserved and in one dimension the relation may be thought of as the first of the two integrals that are required to solve Newton's equations. Solving for v gives

$$v = \frac{dx}{dt} = \sqrt{\frac{2E}{m} - \omega^2 x^2} \tag{3.3.4}$$

and we note that if v_0 represents the velocity of the body at $x = 0$ then $v_0^2 = 2E/m$. Thus we recover (2.8.4).

Gravitational free fall: The gravitational force between two particles is conservative. At this point suppose, for simplicity, that particle 1 is somehow fixed at the origin and particle 2 is situated at $\vec{r}$. Then the gravitational force on particle 2 due to 1 is

$$\vec{F} = -G\frac{m_1 m_2}{r^3}\hat{r} \tag{3.3.5}$$

which is easily proved to be conservative ($\vec{\nabla} \times \vec{F} = 0$). The potential energy of particle 2 due to its gravitational interaction with particle 1 is therefore

$$U(\vec{r}) = -\int_*^{\vec{r}} \vec{F} \cdot d\vec{r} = Gm_1 m_2 \int_*^{\vec{r}} \frac{d\vec{r}}{r^2} \cdot \hat{r} = Gm_1 m_2 \int_*^{r} \frac{dr}{r^2} \tag{3.3.6}$$

The most convenient reference point would be out at infinity, *i.e.,* particle 2 is infinitely separated from particle 1, in which case

$$U(\vec{r}) = -\frac{Gm_1 m_2}{r} \tag{3.3.7}$$

The total energy of the particle 2 would be the sum

$$E = \frac{1}{2}m\vec{v}_2^2 - \frac{Gm_1 m_2}{r} \tag{3.3.8}$$

In general we cannot assume that 1 is fixed to the origin (this may be a good approximation when the mass of 1 is very much greater than the mass of 2 – as in, for example, the earth sun system – but even so it is only an approximation.) A more detailed examination of this system of two particles will be given at the end of this chapter.

3.4 The damped and driven oscillator

We have examined the damped and undamped oscillator in the previous chapter. Let us now complicate the problem by adding a time dependent force driving the oscillator. The equation of motion is

$$m\frac{d^2 x}{dt^2} = -kx - b\frac{dx}{dt} + F(t) \tag{3.4.1}$$

which can be written as

$$(D^2 + 2\gamma D + \omega_0^2)x(t) = \frac{F(t)}{m} \tag{3.4.2}$$

where $\gamma = b/2m$ and $\omega_0^2 = k/m$. Again we have a linear differential equation with constant coefficients, but it is not homogeneous because of the driving force on the right.

3.4.1   Fourier Expansion

The function $F(t)$ could be anything, but situations of greatest physical interest occur when $F(t)$ is periodic. It also turns out that, in this case, we can give a complete and general solution to the problem by exploiting a beautiful theorem in mathematics due to Jean-Baptiste Fourier, which says that:

<u>The Fourier Theorem:</u> Any piecewise continuous, periodic function with period τ, which is square integrable, *i.e.*,

$$\int_{t_1}^{t_2} dt |F(t)|^2 < \infty \tag{3.4.3}$$

where $t_2 - t_1 = T$, can be decomposed into the infinite series

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos n\omega t + b_n \sin n\omega t\right), \quad \omega = \frac{2\pi}{\tau} \tag{3.4.4}$$

where a_n and b_n are the "Fourier coefficients" that can be obtained from $F(t)$ according to

$$a_n = \frac{2}{\tau} \int_0^\tau dt\ F(t) \cos n\omega t$$

$$b_n = \frac{2}{\tau} \int_0^\tau dt\ F(t) \sin n\omega t \tag{3.4.5}$$

for all n (including $n = 0$). The nth order term in the expansion is called the nth harmonic of $F(t)$.

We will not prove Fourier's theorem here,[2] but simply examine the idea. It has to do with the fact that the set of functions $\{\sin nx, \cos nx\}$ (for integer values of n) is **"orthonormal"** with respect to an inner product, defined by an integral over one period. In fact, it is easy to show that

$$\frac{1}{\pi} \int_0^{2\pi} dx\ \sin nx \sin mx = \delta_{mn} = \frac{1}{\pi} \int_0^{2\pi} dx\ \cos nx \cos mx$$

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[2] A proof can be found in the book *Fourier Series and Boundary Value Problems* by J.W. Brown and R.V. Churchill, McGraw-Hill Science 7$^{\text{th}}$ edition, (2006).

$$\frac{1}{\pi} \int_0^{2\pi} dx \ \sin nx \cos mx = 0 \tag{3.4.6}$$

for all values of n and m. This property of the sines and cosines is similar to the orthonormality of basis vectors $\hat{x}_1, \hat{x}_2, ...$ in ordinary space, except that the set is now countably infinite and the definition of what we mean by the inner product is different. If we let the variable $x = \omega t = 2\pi t/\tau$, then

$$\frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} dt \ \sin n\omega t \sin m\omega t = \delta_{mn} = \frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} dt \ \cos n\omega t \cos m\omega t$$

$$\frac{\omega}{\pi} \int_0^{\frac{2\pi}{\omega}} dt \ \sin n\omega t \cos m\omega t = 0 \tag{3.4.7}$$

Often the inner product between the functions is denoted by $\langle \ , \ \rangle$, for example if $f(t)$ and $g(t)$ are two piecewise continuous, periodic functions of t with period τ the inner product of f with g is

$$\langle f(t), g(t) \rangle = \frac{2}{\tau} \int_0^\tau dt \ f(t)g(t), \tag{3.4.8}$$

where we have set $2\pi/\omega = \tau$. Carrying the analogy with an ordinary (finite dimensional) vector space even further and using the inner product, we could define the **norm** of the function $f(t)$ by

$$\|f\| = \langle f, f \rangle^{\frac{1}{2}} \tag{3.4.9}$$

Two functions are **orthogonal** if $\langle f, g \rangle = 0$ and a function is normalized if its norm is unity. The distance between two functions is defined to be $d(f, g) = \|f - g\|$ and represents the root mean square distance between their graphs. Suppose that a function $F(t)$ is expressed as

$$F(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t] \tag{3.4.10}$$

Taking the inner product of $F(t)$ with $\sin m\omega t$ and then $\cos m\omega t$ with $m \neq 0$ would give

$$\langle F(t), \sin m\omega t \rangle = \sum_{n=1}^{\infty} b_n \langle \sin n\omega t, \sin m\omega t \rangle = \sum_{n=1}^{\infty} b_n \delta_{nm} = b_m$$

$$\langle F(t), \cos m\omega t \rangle = \sum_{n=1}^{\infty} a_n \langle \cos n\omega t, \cos m\omega t \rangle = \sum_{n=1}^{\infty} a_n \delta_{nm} = a_m \tag{3.4.11}$$

while if $m = 0$,

$$\langle F(t), 1 \rangle = \frac{2}{\tau} \int_0^\tau dt \ F(t) = \frac{1}{2}a_0 \cdot \frac{2}{\tau} \int_0^\tau dt = a_0 \tag{3.4.12}$$

(we see why the first coefficient, a_0, was defined with the extra factor of $1/2$). Thus all the coefficients can be formally defined simply because the set $\{\sin n\omega t, \cos n\omega t\}$ is orthonormal with respect to this inner product. However this does not guarantee that the series faithfully reproduces the function $F(t)$. Formally, this would happen if the sequence S_N defined by

$$S_N(t) = \frac{1}{2}a_0 + \sum_{n=1}^{N} [a_n \cos n\omega t + b_n \sin n\omega t] \tag{3.4.13}$$

were to *converge in the mean* to the function $F(t)$, *i.e.*, if the root mean square distance between the two were to approach zero as N approached infinity, or

$$\lim_{N\to\infty} \|F(t) - S_N(t)\|^2 = \lim_{N\to\infty} \int_0^\tau [F(t) - S_N(t)]^2 = 0 \tag{3.4.14}$$

If this condition holds for a class of periodic functions $F(t)$ then the set $\{\sin n\omega t, \cos n\omega t\}$ would be not simply orthonormal but also **complete** within that class. It turns out that this condition is met if $F(t)$ is piecewise continuous and square integrable, so every square integrable, piecewise continuous and periodic function has a faithful Fourier decomposition.

By a simple transformation,

$$\alpha_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = \tan^{-1}\left(\frac{b_n}{a_n}\right), \tag{3.4.15}$$

which is valid so long as $n \neq 0$ and provided that $a_n \neq 0$, the Fourier series can be put in the form

$$F(t) = \sum_{n=0}^{\infty} \alpha_n \cos(n\omega t - \phi_n) \tag{3.4.16}$$

with $\phi_0 = 0$ and $\alpha_0 = a_0/2$. But (3.4.2) is a linear equation, which means that we can consider its solution, $x(t)$, to be a superposition, $x(t) = \sum_n x_n(t)$, where each $x_n(t)$ is a solution of

$$(D^2 + \gamma D + \omega_0^2)x_0(t) = f_0$$

$$(D^2 + \gamma D + \omega_0^2)x_n(t) = f_n \cos(n\omega t - \phi_n), \tag{3.4.17}$$

where $f_{0,n} = \alpha_{0,n}/m$. There is a standard way to solve such differential equations. Recall that a general solution of an nth order differential equation must have n (integration) constants. Newton's equations are second order, so they require two constants per space dimension, which are related of course to the components of the initial position and velocity of the particle. We already know how to solve the homogeneous equation

$$(D^2 + 2\gamma D + \omega_0^2)x_n(t) = 0, \tag{3.4.18}$$

so here is the trick: first calculate the general solution to (3.4.18). This is the **complimentary function**, $x_c(t)$, and will contain the desired integration constants. It does not depend on n. Then look for a *particular* solution to the inhomogeneous equation, $x_{n,p}(t)$ (this will depend on n). Any solution will do and the sum of $x_c(t)$ and $x_{n,p}(t)$

$$x_n(t) = x_c(t) + x_{n,p}(t) \tag{3.4.19}$$

is also a solution of the inhomogeneous equation. Moreover, because it contains two arbitrary constants, it is also the most general solution.

Since we already have the general solutions of the homogeneous equation (see (2.8.22), we will now look for a particular solution. Try

$$x_{n,p}(t) = B_n \cos(n\omega t - \delta_n), \tag{3.4.20}$$

where B_n and δ_n are to be determined. Inserting this into (3.4.17) gives

$$(\omega_0^2 - n^2\omega^2)B_n \cos(n\omega t - \delta_n) - 2\gamma n\omega B_n \sin(n\omega t - \delta_n) = f_n \cos(n\omega t - \phi_n). \tag{3.4.21}$$

If we expand the trigonometric functions on the left and right of the above equation, we have

$$B_n[(\omega_0^2 - n^2\omega^2)\cos\delta_n - 2n\omega\gamma\sin\delta_n]\cos n\omega t$$

$$+B_n[(\omega_0^2 - n^2\omega^2)\sin\delta_n - 2n\omega\gamma\cos\delta_n]\sin n\omega t$$

$$= f_n \cos\phi_n \cos n\omega t + f_n \sin\phi_n \sin n\omega t \tag{3.4.22}$$

Comparing terms we get

$$B_n[(\omega_0^2 - n^2\omega^2)\cos\delta_n - 2n\omega\gamma\sin\delta_n] = f_n \cos\phi_n$$

$$B_n[(\omega_0^2 - n^2\omega^2)\sin\delta_n - 2n\omega\gamma\cos\delta_n] = f_n \sin\phi_n \tag{3.4.23}$$

which together give ϕ_n in terms of δ_n,

$$\tan\phi_n = \frac{(\omega_0^2 - n^2\omega^2)\tan\delta_n - 2n\omega\gamma}{(\omega_0^2 - n^2\omega^2) - 2n\omega\gamma\tan\delta_n}, \tag{3.4.24}$$

equivalently δ_n in terms of ϕ_n

$$\tan\delta_n = \frac{2n\omega\gamma + (\omega_0^2 - n^2\omega^2)\tan\phi_n}{(\omega_0^2 - n^2\omega^2) - 2n\omega\gamma\tan\phi_n} \tag{3.4.25}$$

which is the relation we were after. We now seek the next relation we need, *i.e.,* an expression for B_n. Using (3.4.25) we find

$$\sin \delta_n = \frac{(\omega_0^2 - n^2\omega^2)\tan \phi_n + 2n\omega\gamma}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4n^2\omega^2\gamma^2}} \ \cos \phi_n \qquad (3.4.26)$$

and

$$\cos \delta_n = \frac{(\omega_0^2 - n^2\omega^2)\tan \phi_n - 2n\omega\gamma}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4n^2\omega^2\gamma^2}} \ \sin \phi_n \qquad (3.4.27)$$

which expressions may be inserted in either of the pair of equations in (3.4.23) to get

$$B_n = \frac{f_n}{\sqrt{(\omega_0^2 - n^2\omega^2)^2 + 4n^2\omega^2\gamma^2}} \qquad (3.4.28)$$

The general solution of the damped and forced oscillator is therefore

$$x(t) = e^{-\gamma t}\left[A_+ e^{\sqrt{\gamma^2 - \omega_0^2}\,t} + A_- e^{-\sqrt{\gamma^2 - \omega_0^2}\,t}\right] + \sum_{n=0}^{\infty} B_n \cos(n\omega t + \delta_n) \qquad (3.4.29)$$

with B_n and δ_n given above for each n.

Notice that, after a very long time, $t \to \infty$, the first term will die away because of the exponential damping term multiplying it. The effect of the first term is therefore negligible after a sufficiently long time (roughly $\tau \sim \gamma^{-1}$), and $x_c(t)$ is said to represent "transient effects". For large times, the most important part of the solution is therefore the particular solution,

$$x(t) \to x_p(t) = \sum_{n=0}^{\infty} B_n \cos(n\omega t - \delta_n). \qquad (3.4.30)$$

Let us study some late time properties of the solution for the case when there are damped oscillations with $\omega_0 > \sqrt{2}\gamma$. The amplitude of the driven oscillations is extreme for those harmonics which satisfy

$$\frac{dB_n}{d(n\omega)} = 0 \Rightarrow 2\gamma^2 - (\omega_0^2 - n^2\omega^2) = 0, \quad n \neq 0 \qquad (3.4.31)$$

i.e., when

$$n_R\,\omega = \omega_R = \sqrt{\omega_0^2 - 2\gamma^2} \qquad (3.4.32)$$

It is easy to show that the second derivative of B_n is negative in this case and therefore ω_R yields a *maximum* amplitude. The total energy of the oscillator is proportional to the

square of the amplitude and therefore ω_R also represents the harmonic for which the total energy of the oscillator is maximum. On the other hand, the kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m\left[\sum_{n=0}^{\infty} n\omega B_n \sin(n\omega t - \delta_n)\right]^2 \tag{3.4.33}$$

It follows that the time averaged kinetic energy is[3]

$$\overline{T} = \frac{1}{4}m\sum_{n=0}^{\infty} n^2\omega^2 B_n^2, \tag{3.4.34}$$

(where we have used

$$\overline{\sin(n\omega t - \delta_n)\sin(m\omega t - \delta_m)} = \frac{1}{2}\delta_{mn}, \tag{3.4.35}$$

which follows from orthonormality) is not maximum for the harmonic ω_R but for

$$\omega_K = n_K\omega = \omega_0. \tag{3.4.36}$$

The total energy of the oscillator and the kinetic energy resonate at different harmonics, an effect of the damping. Physically, this happens because energy is continuously being transferred from the driving mechanism to the damping medium, where it is lost as heat. To see this consider the power delivered by the driving force,

$$\begin{aligned}\overline{F\dot{x}} &= -\sum_{n,m} m\omega\alpha_n B_m \overline{\cos(n\omega t - \phi_n)\sin(m\omega t - \delta_n)} \\ &= \frac{1}{2}\sum_n n\omega\alpha_n B_n \sin(\phi_n - \delta_n). \end{aligned} \tag{3.4.37}$$

It is not difficult to see that the time average of the power transferred to the system by the driving force is maximum for the harmonic ω_K above, *i.e.,* when the kinetic energy is maximum. This is compatible with the fact that the rate at which mechanical energy is lost to the dissipating medium depends on some power of the speed, as shown in 3.2.6.

3.4.2 Green's Function

There is another generic technique for solving linear inhomogeneous differential equations such as the one we have been dealing with, and that is to seek a particular solution of the

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[3]For a periodic function, f, of period τ, we define the time average as

$$\overline{f} = \frac{1}{\tau}\int_0^\tau dt f(t)$$

form

$$x_p(t) = \frac{1}{m} \int_{-\infty}^{\infty} dt' G(t,t') F(t'), \tag{3.4.38}$$

where $G(t,t')$ is a function of two variables, to be determined. Since the differential operator $(D^2 + 2\gamma D + \omega_0^2)$ is translation invariant, the function $G(t,t')$ is expected to be translation invariant as well and this implies that $G(t,t') = G(t-t')$. To solve the differential equation governing $x_p(t)$, $G(t-t')$ must satisfy the condition

$$(D^2 + 2\gamma D + \omega_0^2)G(t-t') = \delta(t-t') \tag{3.4.39}$$

where $\delta(t-t')$ is not a function but a "distribution" called a "Dirac delta function" or simply a $\delta-$function and defined by the condition that

$$\int_{-\infty}^{\infty} \delta(t-t') f(t') = f(t) \tag{3.4.40}$$

for *any* piecewise continuous function, $f(t)$, on the real line. Any function that satisfies an equation of the form (3.4.39), with a $\delta-$ function driving term is a **Green's function**. There are many explicit representations of $\delta-$functions (see Appendix A), but the most convenient one is

$$\delta(t) = \lim_{n\to\infty} \frac{\sin nt}{\pi t}. \tag{3.4.41}$$

It can be written in integral form as

$$\delta(t) = \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} e^{ikt} dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} dk \tag{3.4.42}$$

Now if we let

$$G(t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k) e^{ik(t-t')} dk \tag{3.4.43}$$

then acting on $G(t-t')$ with the our differential operator gives

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} G(k)(-k^2 + 2i\gamma k + \omega_0^2) e^{ik(t-t')} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(t-t')} dk \tag{3.4.44}$$

or

$$G(k) = \frac{1}{-k^2 + 2i\gamma k + \omega_0^2} \tag{3.4.45}$$

and therefore

$$G(t-t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ik(t-t')}}{-k^2 + 2i\gamma k + \omega_0^2} \tag{3.4.46}$$

It is certainly not straightforward to solve for $G(t-t')$. Care must be taken to define the integral to ensure that causality is not violated and causality implies that $G(t-t') = 0$
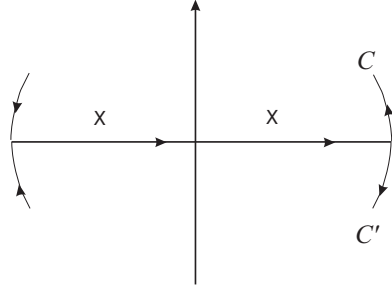
Figure 3.2: Contour in the complex $k-$ plane defining the integral for $G(t - t')$.

when $t < t'$ because the oscillator should not respond to the force before the latter acts on it.

Notice that the integrand has two poles, *viz.*,

$$k^2 - 2i\gamma k - \omega_0^2 = 0 \Rightarrow k = i\gamma \pm \sqrt{-\gamma^2 + \omega_0^2}, \qquad (3.4.47)$$

both of which lie in the upper half plane complex plane. We will define the integral by a contour in the complex $k-$plane in such a way as to ensure that $G(t - t') = 0$ when $t < t'$.

- First consider the **underdamped** case, for which the square root is positive, and let $\overline{\omega} = \sqrt{\omega_0^2 - \gamma^2}$. We define the integral by its value along one of the contours shown in 3.2. Both contours run along the real axis from $-\infty$ to $+\infty$. One of them is closed in the upper half plane and the other in the lower half plane by a semi-circle at infinity. They must be chosen so that the integrals over the semi-circles vanish and do not contribute to the value of the integral:

  - if $t > t'$, we close the contour in the upper half plane and
  - if $t < t'$, we close it in the lower half plane.

Because there are no poles in the lower half plane $G(t - t') = 0$ when $t < t'$ by Cauchy's residue theorem. Moreover, using the residue theorem when $t > t'$, we get

$$G(t - t') = \frac{1}{\overline{\omega}} e^{-\gamma(t-t')} \sin \overline{\omega}(t - t') \qquad (3.4.48)$$

when $t > t'$. We can combine the two cases by introducing the Heaviside function,

$$\Theta(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ 1 & \text{if } t > 0 \end{cases} \qquad (3.4.49)$$

writing

$$G(t - t') = \frac{1}{\overline{\omega}} e^{-\gamma(t-t')} \sin \overline{\omega}(t - t') \Theta(t - t'). \qquad (3.4.50)$$

- For the **critically** damped case there is a double pole, situated at $k = i\gamma/2$. We find straightforwardly that

$$G(t - t') = e^{-\gamma(t-t')}(t - t')\Theta(t - t'). \tag{3.4.51}$$

- Finally, for the **overdamped** case there are two imaginary roots at

$$k = i\left(\gamma \pm \sqrt{\gamma^2 - \omega_0^2}\right) = k_\pm \tag{3.4.52}$$

and closing the contour as before gives

$$G(t - t') = -i\left[\frac{e^{ik_+(t-t')}}{k_+ - k_-} - \frac{e^{ik_-(t-t')}}{k_+ - k_-}\right]\Theta(t - t') \tag{3.4.53}$$

or

$$G(t - t') = \frac{e^{-\gamma(t-t')}}{\sqrt{\gamma^2 - \omega_0^2}}\sinh\sqrt{\gamma^2 - \omega_0^2}\,\Theta(t - t'). \tag{3.4.54}$$

Now that $G(t - t')$ is known in each case, $x_p(t)$ is determined directly by integrating according to (3.4.38). The Green's function is an extremely powerful tool to solve linear, inhomogeneous differential equations with constant coefficients and is straightforwardly extended to partial differential equations in higher dimension.

3.5   Systems of many particles

Extending the arguments given in the previous sections, let us now consider many particles, labeled by integers $n \in \mathbb{N}$ (not to be confused with the integers that label the harmonics in the Fourier series), located at $\vec{r}_n$ and having masses m_n as shown in figure (3.3). Define the total momentum of the system as

$$\vec{p} = \sum_n \vec{p}_n = \sum_n m_n \vec{v}_n \Rightarrow \frac{d\vec{p}}{dt} = \sum_n m_n \frac{d\vec{v}_n}{dt} = \sum_n \vec{F}_n \tag{3.5.1}$$

where $\vec{F}_n$ represents the total external force on the nth particle. This force may be thought of as arising from two sources: (i) the forces, $\vec{F}_n^{\text{ext}}$, on the particle due to the rest of the universe, *i.e.*, the part of the universe that excludes the system of particles itself and (ii) the force on the particle due to the other particles *within* the system. The latter force can be written as the sum of all the forces exerted on the particle by all the *other* particles of the system,

$$\vec{F}_n^{\text{int}} = \sum_{m \neq n} \vec{F}_{m \to n} \tag{3.5.2}$$
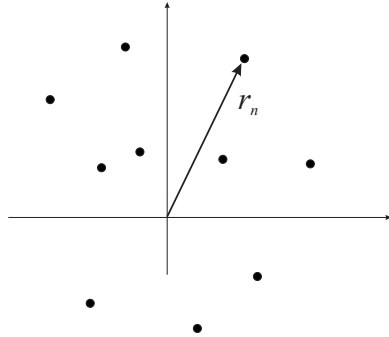
Figure 3.3: Many particle systems.

The first we shall call the "external" force, because it's source is external to the system. The second we call call an "internal" force because it arises from the system itself. Hence the superscripts. The total force on the nth particle is then

$$\vec{F}_n = \vec{F}_n^{\text{ext}} + \sum_{m \neq n} \vec{F}_{m \to n} \tag{3.5.3}$$

where we have used $\vec{F}_{m \to n}$ to signify the force exerted by the mth particle on the nth. Since a particle is presumed not to exert a force on itself, we exclude $m = n$ from the sum. It follows that

$$\frac{d\vec{p}}{dt} = \sum_n \vec{F}_n^{\text{ext}} + \sum_{n, m \neq n} \vec{F}_{m \to n} \tag{3.5.4}$$

and clearly the first term on the right gives the total external force on the system, *i.e.,*

$$\vec{F}^{\text{ext}} = \sum_n \vec{F}_n^{\text{ext}} \tag{3.5.5}$$

while the second term summarizes the effect of the internal forces on the total momentum of the system of particles. Let us evaluate the sum

$$\sum_{n, m \neq n} \vec{F}_{m \to n}$$

making two very reasonable assumptions about the forces between the particles of the system, *viz.,*

- they act along the lines joining the particles, and

- they do not depend on the velocities of the particles.

Under these conditions, we may appeal to Newton's third law which states that

$$\vec{F}_{m \to n} = -\vec{F}_{n \to m}, \tag{3.5.6}$$

to argue that the desired sum is identically zero! The proof may be made by mathematical induction. Considering just two particles we have

$$\sum_{n,m \neq n} \vec{F}_{m \to n} = \vec{F}_{1 \to 2} + \vec{F}_{2 \to 1} \equiv 0 \tag{3.5.7}$$

With three particles there are six terms,

$$\sum_{n,m \neq n} \vec{F}_{m \to n} = \vec{F}_{1 \to 2} + \vec{F}_{1 \to 3} + \vec{F}_{2 \to 1} + \vec{F}_{2 \to 3} + \vec{F}_{3 \to 1} + \vec{F}_{3 \to 2}$$

$$= (\vec{F}_{1 \to 2} + \vec{F}_{2 \to 1}) + (\vec{F}_{1 \to 3} + \vec{F}_{3 \to 1}) + (\vec{F}_{2 \to 3} + \vec{F}_{3 \to 2}) \tag{3.5.8}$$

which when suitably paired are again shown to cancel each other. We will now argue that if the sum vanishes for N particles then it must vanish for $N + 1$ particles. Notice that, for N particles, there are $N(N-1)$ terms in the sum. Suppose then that

$$\sum_{\{n,m \neq n\}=1}^{N} \vec{F}_{m \to n} = 0 \tag{3.5.9}$$

If one more particle were added to we must account for an additional $(N+1)N - N(N-1) = 2N$ terms, which may be written as

$$\sum_{\{n,m \neq n\}=1}^{N+1} \vec{F}_{m \to n} = \sum_{\{n,m \neq n\}=1}^{N} \vec{F}_{m \to n} + (\vec{F}_{1 \to N+1} + \vec{F}_{N+1 \to 1})$$

$$+ (\vec{F}_{2 \to N+1} + \vec{F}_{N+1 \to 2}) + ... (\vec{F}_{N \to N+1} + \vec{F}_{N+1 \to N}) \tag{3.5.10}$$

But all the additional terms vanish by Newton's third law and we once again have a vanishing sum. In fact, it is not difficult to see that

$$\sum_{n,m \neq n} \vec{F}_{m \to n} = \sum_{m < n} (\vec{F}_{m \to n} + \vec{F}_{n \to m}) \equiv 0 \tag{3.5.11}$$

and we conclude that

$$\frac{d\vec{p}}{dt} = \vec{F}^{\text{ext}} \tag{3.5.12}$$

i.e., the rate of change of total momentum depends only on the external applied force (and is independent of the internal forces of the system).

3.5.1 Conservation of momentum.

Let us define the **center of mass** of the system by

$$\vec{p} = M\vec{v}_{\text{cm}} = M\frac{d\vec{r}_{\text{cm}}}{dt} \tag{3.5.13}$$

where $M = \sum_n m_n$ is the total mass of the system. Applying the definition of the total momentum, we see that

$$\vec{p} = \sum_n m_n \vec{v}_n = M\vec{v}_{\text{cm}} \quad \Rightarrow \quad \vec{v}_{\text{cm}} = \frac{\sum_n m_n \vec{v}_n}{\sum_n m_n}, \tag{3.5.14}$$

or

$$\vec{r}_{\text{cm}} = \frac{\sum_n m_n \vec{r}_n}{\sum_n m_n} \tag{3.5.15}$$

gives the location of the center of mass of the system. It is the weighted mean position of the particles of the system, where the masses of the individual particles weight the mean, and moves with the velocity,

$$\vec{v}_{\text{cm}} = \frac{d\vec{r}_{\text{cm}}}{dt} = \frac{\sum_n m_n \vec{v}_n}{\sum_n m_n}. \tag{3.5.16}$$

The center of mass is not necessarily the location of a massive particle but only behaves as a particle of mass equal to the total mass of the system and located at $\vec{r}_{\text{cm}}$. We see this from (3.5.12), which says that

$$\frac{d\vec{p}}{dt} = M\frac{d^2\vec{r}_{\text{cm}}}{dt^2} = \vec{F}^{\text{ext}} \tag{3.5.17}$$

and leads to our first conservation law for a system of particles:

- If the net external force on a system of particles is zero then the total momentum of the system is conserved.

The center of mass of the system then behaves as a free particle moving, according to Newton's first law, with constant velocity. We used this fact to solve the problem of two bodies falling freely under the influence of their mutual gravitational acceleration in the previous chapter and it is regularly used in the analysis of collisions, when the colliding system is isolated.
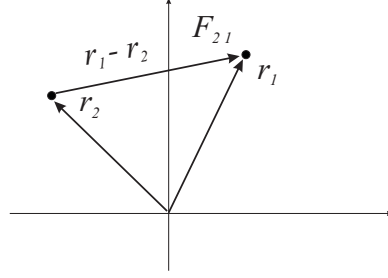
Figure 3.4: A system of two particles

3.5.2   Conservation of angular momentum.

Every particle in the system contributes an angular momentum

$$\vec{L}_n = \vec{r}_n \times \vec{p}_n \tag{3.5.18}$$

to the total angular momentum of the system,

$$\vec{L} = \sum_n \vec{r}_n \times \vec{p}_n \tag{3.5.19}$$

Taking one derivative,

$$
\begin{aligned}
\frac{d\vec{L}}{dt} &= \sum_n \left(\frac{d\vec{r}_n}{dt} \times \vec{p}_n + \vec{r}_n \times \frac{d\vec{p}_n}{dt} \right) = \sum_n \vec{r}_n \times \vec{F}_n \\
&= \sum_n \vec{r}_n \times \vec{F}_n^{\text{ext}} + \sum_{n,m\neq n} \vec{r}_n \times \vec{F}_{m\to n} \\
&= \sum_n \vec{\tau}_n^{\text{ext}} + \sum_{n,m\neq n} \vec{r}_n \times \vec{F}_{m\to n}
\end{aligned}
\tag{3.5.20}
$$

The sum in the first term on the right represents the total external torque on the system. Let us examine the second sum on the right. As before, consider just two particles, then

$$\vec{r}_1 \times \vec{F}_{2\to1} + \vec{r}_2 \times \vec{F}_{1\to2} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{2\to1} \tag{3.5.21}$$

where we have used Newton's third law. But $(\vec{r}_1 - \vec{r}_2)$ is the position vector of 1 relative to 2 [see figure (3.4)] and, if the force $\vec{F}_{2\to1}$ acts along the line joining the particles, *i.e.,* along $(\vec{r}_1 - \vec{r}_2)$, then the sum vanishes. Likewise, for three particles

$$\sum_{n,m\neq n} \vec{r}_n \times \vec{F}_{m\to n} = \vec{r}_1 \times (\vec{F}_{2\to1} + \vec{F}_{3\to1}) + \vec{r}_2 \times (\vec{F}_{1\to2} + \vec{F}_{3\to2})$$

$$+\vec{r}_3 \times (\vec{F}_{1\to3} + \vec{F}_{2\to3}) \tag{3.5.22}$$

and rearranging the terms we find

$$\sum_{n,m\neq n} \vec{r}_n \times \vec{F}_{m\to n} = (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{2\to1} + (\vec{r}_1 - \vec{r}_3) \times \vec{F}_{3\to1}$$

$$+(\vec{r}_2 - \vec{r}_3) \times \vec{F}_{3\to2} \equiv 0 \tag{3.5.23}$$

for the same reason we mentioned before, if all the forces between the particles act along the lines joining them. In fact[4]

$$\sum_{n,m\neq n} \vec{r}_n \times \vec{F}_{m\to n} = \sum_{n<m} (\vec{r}_n - \vec{r}_m) \times \vec{F}_{m\to n} \equiv 0 \tag{3.5.24}$$

It follows that

$$\frac{d\vec{L}}{dt} = \sum_n \vec{\tau}_n^{\text{ext}} = \vec{\tau}^{\text{ext}} \tag{3.5.25}$$

and

- If the net external torque on the system vanishes, the total angular momentum of the system is conserved.

There is an interesting expression for the total angular momentum of a system of particles in terms of the angular momentum of its center of mass. Let $\vec{r}_n$ be the position of the nth particle relative to the laboratory and let $\vec{r}'_n$ be its position relative to the center of mass, i.e., $\vec{r}'_n = \vec{r}_n - \vec{r}_{\text{cm}}$. The total angular momentum of the system can be written as

$$
\begin{aligned}
\vec{L} &= \sum_n \vec{r}_n \times \vec{p}_n = \sum_n m_n (\vec{r}'_n + \vec{r}_{\text{cm}}) \times (\vec{v}'_n + \vec{v}_{\text{cm}}) \\
&= \sum_n m_n (\vec{r}'_n \times \vec{v}'_n + \vec{r}'_n \times \vec{v}_{\text{cm}} + \vec{r}_{\text{cm}} \times \vec{v}'_n + \vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}}) \\
&= \sum_n m_n (\vec{r}'_n \times \vec{v}'_n) + \left(\sum_n m_n \vec{r}'_n\right) \times \vec{v}_{\text{cm}} + \vec{r}_{\text{cm}} \times \left(\sum_n m_n \vec{v}'_n\right) + \vec{r}_{\text{cm}} \times (M\vec{v}_{\text{cm}}) \\
&= \sum_n (\vec{r}'_n \times \vec{p}'_n) + \vec{r}_{\text{cm}} \times \vec{p} \tag{3.5.26}
\end{aligned}
$$

where $\vec{p}'_n = m_n \vec{v}'_n$ is the momentum of the nth particle relative to the center of mass and we have used

$$\sum_n m_n \vec{r}'_n = 0 = \sum_n m_n \vec{v}'_n \tag{3.5.27}$$

---

[4]Problem: Prove this by mathematical induction

which follow directly from the definition of the center of mass. So *the total angular momentum is the sum of the angular momentum of the center of mass about the origin, plus the angular momentum of the system about the center of mass.*

3.5.3   The Work-Energy theorem

While the rate of change of the total momentum of a system of particles depends only on the external force, the rate of change of the momentum of each particle within the system is subject both to the external force acting on it as well as on the internal forces exerted by the other particle of the system upon it, thus

$$m_n \frac{d\vec{v}_n}{dt} = \vec{F}_n = \vec{F}_n^{\text{ext}} + \sum_{m \neq n} \vec{F}_{m \to n} \qquad (3.5.28)$$

Taking the scalar product of each side with $\vec{v}_n$, we get a single scalar equation,

$$m_n \vec{v}_n \cdot \frac{d\vec{v}_n}{dt} = \vec{v}_n \cdot \vec{F}_n \qquad (3.5.29)$$

which may be integrated once, as we did before for a single particle, to obtain the work energy theorem,

$$\frac{1}{2} m_n (\vec{v}_{n,f}^2 - \vec{v}_{n,i}^2) = \int_i^f \vec{F}_n \cdot d\vec{r}_n \qquad (3.5.30)$$

or

$$K_{n,f} - K_{n,i} = W_{n,if} \qquad (3.5.31)$$

Defining the **total** kinetic energy of the system and the **total** work done respectively by

$$K = \sum_n K_n = \frac{1}{2} \sum_n m_n \vec{v}_n^2 \qquad (3.5.32)$$

and

$$W_{if} = \sum_n W_{n,if} = \sum_n \int_i^f \vec{F}_n \cdot d\vec{r}_n \qquad (3.5.33)$$

we find that

$$K_f - K_i = W_{if} \qquad (3.5.34)$$

This, of course, is the work energy theorem for a system of particles. Notice that the total kinetic energy of the system is just the sum of the individual kinetic energies of the particles and that the total work done on the system is the work done *by both the internal and the external forces acting on the system.*

   Just as we had separated the total angular momentum into two parts, the first corresponding to the angular momentum of the center of mass about the origin and the second

corresponding to the angular momentum of the system about the center of mass, so it is also possible to separate the kinetic energy into two pieces, one corresponding to the kinetic energy of the center of mass and the other to the system relative to the center of mass. To see this, write the position vector of each particle as

$$\vec{r}_n = \vec{r}_{\text{cm}} + \vec{r}_n', \quad \vec{v}_n = \vec{v}_{\text{cm}} + \vec{v}_n' \tag{3.5.35}$$

and inserting this into the expression for the total kinetic energy,

$$
\begin{aligned}
K &= \sum_n m_n \vec{v}_n^2 = \frac{1}{2} \sum_n m_n (\vec{v}_{\text{cm}} + \vec{v}_n')^2 \\
&= \frac{1}{2} M \vec{v}_{\text{cm}}^2 + \sum_n m_n \vec{v}_{\text{cm}} \cdot \vec{v}_n' + \frac{1}{2} \sum_n m_n \vec{v}_n'^2 \tag{3.5.36}
\end{aligned}
$$

But once again we appeal to the fact that the center of mass is the weighted mean position of the particles constituting the system so that $\sum_n m_n \vec{r}' = 0 = \sum_n m_n \vec{v}'$. We get

$$K = \frac{1}{2} M \vec{v}_{\text{cm}}^2 + \frac{1}{2} \sum_n m_n \vec{v}_n'^2. \tag{3.5.37}$$

The first term on the right is obviously the kinetic energy of the center of mass. The second is just the kinetic energy of the system as measured by an observer moving with the center of mass.

The total work may also be broken up into two pieces,

$$W_{if} = \sum_n \int_i^f \vec{F}_n^{\text{ext}} \cdot d\vec{r}_n + \sum_{n, m \neq n} \int_i^f \vec{F}_{m \to n} \cdot d\vec{r}_n \tag{3.5.38}$$

If all the external forces acting on the particles of the system are conservative then each integral in the sum of the first term on the right is expressed as

$$\int_i^f \vec{F}_n^{\text{ext}} \cdot d\vec{r}_n = -\int_i^f \vec{\nabla} U_n^{\text{ext}}(\vec{r}_n) \cdot d\vec{r}_n = -\left[U_{n,f}^{\text{ext}} - U_{n,i}^{\text{ext}} \right] \tag{3.5.39}$$

The potential energy of the system, due to the external forces, is

$$U^{\text{ext}} = \sum_n U_n^{\text{ext}} \tag{3.5.40}$$

For the internal forces, consider

$$\sum_{m \neq n} \int_i^f \vec{F}_{m \to n} \cdot d\vec{r}_n. \tag{3.5.41}$$

If the internal forces are also conservative then for every pair (m, n), the force exerted by m on n is the gradient of some function, which we call U_{nm}^{int}, taken at n,

$$\vec{F}_{m \to n} = -\vec{\nabla}_n U_{nm}^{\text{int}}. \tag{3.5.42}$$

Therefore, the second term in (3.5.38) is

$$\sum_{n, m \neq n} \int_i^f \vec{F}_{m \to n} \cdot d\vec{r}_n = -\sum_{n, m \neq n} \int_i^f \vec{\nabla}_n U_{nm}^{\text{int}} \cdot d\vec{r}_n \tag{3.5.43}$$

It generally happens that U_{nm}^{int} depends only on the separation of the particles n and m, i.e.,

$$U_{nm}^{\text{int}}(\vec{r}_n, \vec{r}_m) = U_{nm}^{\text{int}}(|\vec{r}_n - \vec{r}_m|). \tag{3.5.44}$$

This is compatible with the third law, since the $\vec{F}_{m \to n} = \vec{\nabla}_n U_{nm} = -\vec{\nabla}_m U_{mn} = -\vec{F}_{n \to m}$, and it makes the internal forces invariant with respect to Galilean boosts. It also means that

$$
\begin{aligned}
dU_{mn}^{\text{int}} &= \vec{\nabla}_n U_{mn}^{\text{int}} \cdot d\vec{r}_n + \vec{\nabla}_m U_{mn}^{\text{int}} \cdot d\vec{r}_m \\[2mm]
&= \vec{\nabla}_n U_{mn}^{\text{int}} \cdot d(\vec{r}_n - \vec{r}_m) \\[2mm]
&= \vec{\nabla}_n U_{mn}^{\text{int}} \cdot d\vec{r}_{nm} \tag{3.5.45}
\end{aligned}
$$

where $\vec{r}_{nm}$ is the position of n relative to m. Returning to the second term in (3.5.38), first consider just two particles

$$
\begin{aligned}
-\sum_{n, m \neq n} \int_i^f \vec{\nabla}_n U_{nm}^{\text{int}} \cdot d\vec{r}_n &= -\int_i^f \left[\vec{\nabla}_1 U_{12}^{\text{int}} \cdot d\vec{r}_1 + \vec{\nabla}_2 U_{21}^{\text{int}} \cdot d\vec{r}_2 \right] \\[2mm]
&= -\int_i^f \vec{\nabla}_1 U_{12}^{\text{int}} \cdot d\vec{r}_{12} \\[2mm]
&= -\int_i^f dU_{12}^{\text{int}} = U_{12}^{\text{int}}(\vec{r}_i) - U_{12}^{\text{int}}(\vec{r}_f) \tag{3.5.46}
\end{aligned}
$$

For three particles, we have

$$
\begin{aligned}
-\sum_{n, m \neq n} \int_i^f \vec{\nabla}_n U_{nm}^{\text{int}} \cdot d\vec{r}_n &= -\int_i^f \left[\vec{\nabla}_1 (U_{12}^{\text{int}} + U_{13}^{\text{int}}) \cdot d\vec{r}_1 + \vec{\nabla}_2 (U_{21}^{\text{int}} + U_{23}^{\text{int}}) \cdot d\vec{r}_2 \right. \\[2mm]
&\qquad \left. + \vec{\nabla}_3 (U_{31}^{\text{int}} + U_{32}^{\text{int}}) \cdot d\vec{r}_3 \right]
\end{aligned}
$$

$$= -\int_i^f \left[\vec{\nabla}_1 U_{12}^{\text{int}} \cdot d\vec{r}_{12} + \vec{\nabla}_1 U_{13}^{\text{int}} \cdot d\vec{r}_{13} + \vec{\nabla}_2 U_{23}^{\text{int}} \cdot d\vec{r}_{23} \right]$$

$$= -\int_i^f \left[dU_{12}^{\text{int}} + dU_{13}^{\text{int}} + dU_{23}^{\text{int}} \right] \tag{3.5.47}$$

using (3.5.45), and we arrive at (for three particles)

$$-\sum_{n,m\neq n} \int_i^f \vec{\nabla}_n U_{nm}^{\text{int}} \cdot d\vec{r}_n = - \left[U_{12}^{\text{int}} + U_{13}^{\text{int}} + U_{23}^{\text{int}} \right]_i^f \tag{3.5.48}$$

We can already recognize a pattern: for N particles,

$$-\sum_{n,m\neq n} \int_i^f \vec{\nabla}_n U_{nm}^{\text{int}} \cdot d\vec{r}_n = - \left[\sum_{m<n} U_{mn}^{\text{int}} \right]_i^f = -\frac{1}{2} \left[\sum_{m\neq n} U_{mn}^{\text{int}} \right]_i^f \tag{3.5.49}$$

can be proved by induction provided, of course, that all the internal forces are conservative with potentials that depend only on the separation of the particles.

We see that the system of particles also has a potential energy due to the internal forces, which is given by

$$U^{\text{int}} = \sum_{m<n} U_{mn}^{\text{int}} = \frac{1}{2} \sum_{m\neq n} U_{mn}^{\text{int}} \tag{3.5.50}$$

The *total* potential energy of the system, due to both the internal and the external forces is therefore

$$U = U^{\text{ext}} + U^{\text{int}} = \sum_n U_n^{\text{ext}} + \sum_{m<n} U_{mn}^{\text{int}} \tag{3.5.51}$$

The work energy theorem then tells us that

$$K_f - K_i = W_{if} = U_i - U_f \Rightarrow K_i + U_i = K_f + U_f \tag{3.5.52}$$

which, as before, means that there is a quantity, which we call the total energy of the system of particles, that is conserved throughout the motion of the system:

$$E = K + U = \frac{1}{2} \sum_n m_n \vec{v}_n^2 + \sum_n U_n^{\text{ext}}(\vec{r}_n) + \sum_{m<n} U_{mn}^{\text{int}}(|\vec{r}_n - \vec{r}_m|) \tag{3.5.53}$$

We have arrived at the law of conservation of energy: *if the internal and external forces on a system of particles are conservative and if the interactions between the particle depend only on the distance between them then the total energy of the system, which is given by (3.5.53), is conserved.*

We close with the example of two bodies under the influence of their mutual gravitational interaction. This could be, for example, a binary star system. The gravitational force is certainly conservative,

$$\vec{F}_{1\to 2} = -G\frac{m_1 m_2}{r_{12}^3}\vec{r}_{12} \tag{3.5.54}$$

and $\vec{\nabla}_2 \times \vec{F}_{1\to 2} \equiv 0$. Therefore the potential energy is obtained from the integral

$$U_{21} = -\int_*^{\vec{r}_2} \vec{F}_{1\to 2}\cdot d\vec{r}_2 = Gm_1 m_2 \int_*^{\vec{r}_2} \frac{\hat{r}_{12}\cdot d\vec{r}_2}{r_{12}^2} = -\frac{Gm_1 m_2}{r_{12}} \tag{3.5.55}$$

where, in the last step, we have used the fact that the inner product selects only the component of $d\vec{r}_2$ that is parallel to the displacement vector from 1 to 2 and we have chosen the reference point at infinite separation. This potential energy satisfies the conditions of the energy conservation theorem for systems of particles in as much as it depends only on the separation of the particles. Thus, if there are no external forces on the particles, the energy theorem gives the total energy of the system as

$$E = \frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 - \frac{Gm_1 m_2}{|\vec{r}_1 - \vec{r}_2|} \tag{3.5.56}$$

Note that the potential energy is not counted twice.

From another point of view, recall how we had broken this problem up into two decoupled pieces: the motion of its center of mass and the motion of the relative coordinate. The center of mass behaves as a free particle of mass $M = m_1 + m_2$ whereas the relative coordinate behaves as a particle of "reduced" mass, $\mu = m_1 m_2/(m_1 + m_2)$, moving with potential energy

$$U(\vec{r}) = -\frac{G\mu M}{r}, \tag{3.5.57}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$. Treating the two particle system in terms of these new degrees of freedom, we could write the energy as

$$E = \frac{1}{2}M\vec{v}_{\text{cm}}^2 + \frac{1}{2}\mu\vec{v}^2 - \frac{G\mu M}{r} \tag{3.5.58}$$

with

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{v}_2 - \vec{v}_1, \quad \vec{v}_{\text{cm}} = \frac{d\vec{r}_{\text{cm}}}{dt} = \frac{1}{M}(m_1\vec{v}_1 + m_2\vec{v}_2). \tag{3.5.59}$$

It is easy to see by direct substitution that (3.5.58) is identical to (3.5.56).
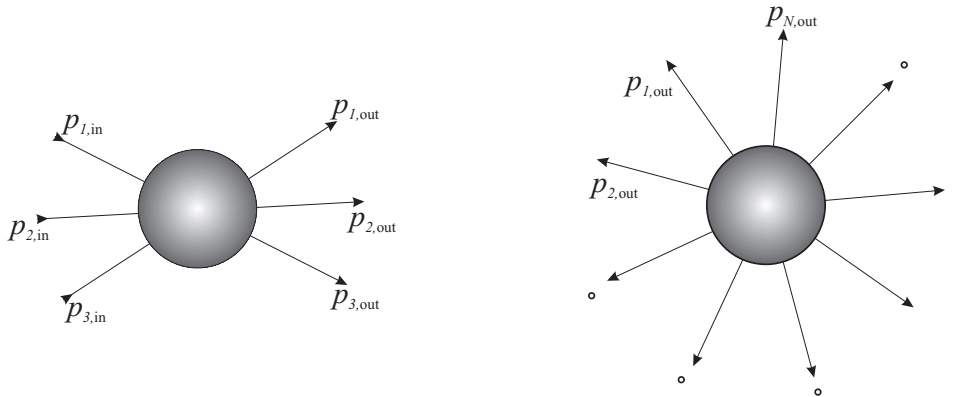
Figure 3.5: Scattering (left) and explosions (right).

3.6 Collisions

The **state** of an N particle system in D dimensions at any time is determined by the positions and momenta of the particles at that time and therefore requires $2DN$ variables. Suppose we know the initial state of some N particle system and we wish to determine its state at some future time, then we must solve DN coupled, second order differential equations governing their motion (Newton's second law) subject to the initial conditions. Experience teaches us that this can be a very difficult task both because coupled differential equations are generally difficult to solve and because such an approach requires a detailed knowledge of the interactions between the particles. In our search for a more tractable approach to this many body problem, we ask if the conservation laws may be put to good use.

Let's confine our attention to isolated systems of particles which come under each other's influence over a relatively short distance (compared with their path lengths) and undergo scattering, or explosions as shown in 3.5. The shaded region in the figure represents the region in which the particles are strongly interacting with each other. We assume that the initial state is known at some early time when the particles are separated by distances large enough so that their mutual interactions can be ignored and fix our attention on the $3N$ momenta of the particles at some future time, when they are again separated by large enough distances as to be effectively free. Our objective will be to recover as much information as possible about the final momenta of the particles, in terms of the known initial state and using only the conservation laws.

If the system is isolated from the rest of the universe then its total momentum and total angular momentum are conserved and, in D dimensions, this gives us D equations for the momentum and $D(D-1)/2$ equations, assuming they are not trivial, for the angular

momentum, a total of $D + D(D-1)/2$ equations relating the particles' initial positions and momenta to their final positions and momenta. If, moreover, we have some information about the mechanical energy of the system (say, that the total mechanical energy, or some fraction of it, is preserved in the final state) then we have one additional condition, leading to $D + D(D-1)/2 + 1$ equations in all. We expect therefore that a direct application of only the conservation laws leaves $\Delta = 2DN - D - D(D-1)/2 - 1$ variables of the state of the system undetermined and conclude that if N is very large, conservation laws alone can tell us little about the evolution of the system – seeing that D is small. However, if N is small the situation is different. For example, consider a system of two particles interacting in effectively one dimension. According to our formula, a straightforward application of the conservation laws would lead to $\Delta = 2$ undetermined variables. These variables would be the final positions of the particles, so we are able to determine the final momenta of the particles without any *à priori* knowledge of the detailed force between them. For two particles in two effective dimensions, $\Delta = 4$ remain undetermined and these can once again be taken to represent the final positions of the two particles.

If we confine our attention to collisions in which the total angular momentum both before and after the collision is identically zero, *i.e.,* for which the angular momentum equations are in fact trivial, and ask only for the final momenta of the particles, we will have at most $D+1$ conditions with which to determine DN final momenta. It follows that the problem of two particles colliding in effectively one dimension continues to be solvable without any *à priori* knowledge of the final state, whereas the collision of two particles in two dimensions requires the knowledge of one final state variable and in D dimensions requires a knowledge of $D-1$ variables of the final state. Let us work out some illustrative examples.

3.6.1   One Dimensional Collisions

In a **plastic** or **completely inelastic** collision, energy is not conserved but the particles stick together in the final state so that the number of momenta to be found is reduced to just the number of dimensions, D. Consider a plastic collision of two particles in one dimension. Let the particles have masses $m_{1,2}$ and let their initial velocities be $v_{1,i}$ and $v_{2,i}$ respectively in some frame attached to the observer, which we'll call the "Laboratory frame". The velocities (being vectors) are *signed* quantities in one dimension. As always we will take velocities "to the right" to be positive and velocities "to the left" to be negative. As the particles stick together in the final state, there will be only one final momentum to be determined: that of the combination. The mass of the particle in the end state is $m_1 + m_2$ and we will call its velocity v_f. Conservation of momentum implies that

$$m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f \qquad (3.6.1)$$

and so

$$v_f = \frac{m_1 v_{1i} + m_2 v_{2i}}{(m_1 + m_2)} \tag{3.6.2}$$

solves the problem: the final velocity is simply the velocity the center of mass. Clearly mechanical energy was not conserved in this collision and we can determine the fraction of the initial mechanical energy that was lost,

$$Q = \frac{K_f}{K_i} = \frac{(m_1 + m_2) v_f^2}{m_1 v_{1i}^2 + m_2 v_{2i}^2}. \tag{3.6.3}$$

We find that

$$1 - Q = \frac{m_1 m_2 (v_{1i} - v_{2i})^2}{(m_1 + m_2)(m_1 v_{1i}^2 + m_2 v_{2i}^2)} > 0 \tag{3.6.4}$$

which shows that $Q < 1$. The lost mechanical energy is actually energy transferred to the environment as "heat" due to frictional forces during the collision, or as radiation if the colliding particles were charged.

A slightly more difficult problem is that of an **elastic** collision in one dimension. Elastic collisions conserve energy, so under the initial conditions of the system above we have two equations (conservation of momentum and conservation of energy) that must be solved for the two final velocities,

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \tag{3.6.5}$$

Combining these two equations we find

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \tag{3.6.6}$$

which serves to eliminate v_{2f} (say). The result is[5]

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i} + \frac{2m_2}{m_1 + m_2} v_{2i}$$

$$v_{2f} = \frac{m_2 - m_1}{m_1 + m_2} v_{2i} + \frac{2m_1}{m_1 + m_2} v_{1i} \tag{3.6.7}$$

Two special cases are of interest: (a) If $m_1 = m_2 = m$ then $v_{1f} = v_{2i}$ and $v_{2f} = v_{1i}$ i.e., the particles simply exchange their velocities, and (b) if $m_2 \ll m_1$ then $v_{1f} \simeq v_{1i}$ and $v_{2f} \simeq 2v_{1i} - v_{2i}$, i.e., the more massive particle simply moves as if it were unaffected by the collision. In particular, if the more massive particle is at rest in the observer's frame

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[5]<u>Problem</u>: Fill in the omitted steps.

then the smaller particle simply "bounces back", $v_{2f} = -v_{2i}$. Alternatively, if the lighter particle is initially at rest it will move at twice the speed of the larger particle after the collision.

If the collision is not elastic, but the fraction of the initial mechanical energy that is preserved in the final state is known to be $Q \le 1$ then the second equation of (3.6.5) is modified to

$$m_1 v_{1f}^2 + m_2 v_{2f}^2 = Q(m_1 v_{1i}^2 + m_2 v_{2i}^2) \tag{3.6.8}$$

and can once again be used, in combination with the conservation of momentum, to determine the final velocities of the particles as above. The general expression is not particularly illuminating and we leave this as an exercise to the interested reader.[6] Collisions between macroscopic bodies will always be **inelastic** and mechanical energy will be converted into heat by frictional forces during the impact.

An important fact to note is that the result in (3.6.7) has the *same form* in *any* reference frame. This can be shown directly by applying a Galilean transformation to the expressions for $v_{1,2f}$ in the Laboratory frame. For a frame S', traveling with a velocity v_0 relative to the Laboratory frame, one has only to substitute

$$\begin{aligned} v'_{1i} &= v_{1i} - v_0, \quad v'_{2i} = v_{2i} - v_0, \\ v'_{1f} &= v_{1f} - v_0, \quad v'_{2f} = v_{2f} - v_0, \end{aligned} \tag{3.6.9}$$

into (3.6.7) to obtain the expressions

$$\begin{aligned} v'_{1f} &= \frac{m_1 - m_2}{m_1 + m_2} v'_{1i} + \frac{2m_2}{m_1 + m_2} v'_{2i} \\ v'_{2f} &= \frac{m_2 - m_1}{m_1 + m_2} v'_{2i} + \frac{2m_1}{m_1 + m_2} v'_{1i}. \end{aligned} \tag{3.6.10}$$

Expressions that are form invariant under some set of transformations are said to be **covariant** under those transformations.

Now because the total momentum and angular momentum of the particles is conserved when there is no external force or torque on the system, the center of mass of the system must move with a constant velocity,

$$v_{\mathrm{cm}} = \frac{m_1 v_{1i} + m_2 v_{2i}}{m_1 + m_2} = \frac{m_1 v_{1f} + m_2 v_{2f}}{m_1 + m_2}, \tag{3.6.11}$$

relative to the Laboratory frame. An observer who is attached to the center of mass and moves along with it is therefore inertial, so what would a collision look like from her point of view? Since the total momentum in this frame must be zero, the conservation of momentum implies that

$$m_1 v'_{1i} + m_2 v'_{2i} = 0 = m_1 v'_{1f} + m_2 v'_{2f} \quad \Rightarrow \quad v'_{2i} = -\frac{m_1}{m_2} v'_{1i}, \quad v'_{2f} = -\frac{m_1}{m_2} v'_{1f}. \tag{3.6.12}$$

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[6]<u>Problem</u>: Determine the final velocities in terms of the initial velocities and the fraction Q.
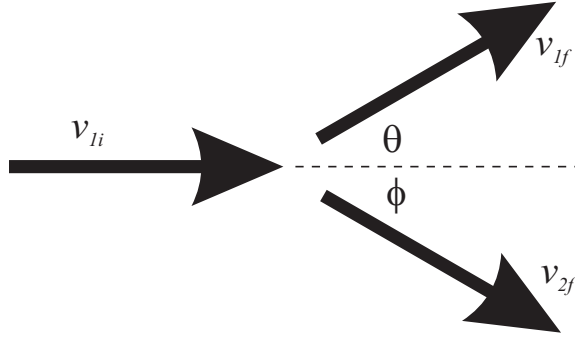
Figure 3.6: Two dimensional collisions from the point of view of particle "2".

If the collision is plastic then, because $v'_{1f} = v'_{2f} = v'_f$ (say), it follows that $v'_f = 0$. If the collision is elastic, we must combine the above equations with (3.6.7) in this frame and we will find $v'_{1f} = -v'_{1i}$ and $v'_{2f} = -v'_{2i}$.

3.6.2 Two Dimensional Collisions

Let us now turn to two dimensional collisions between two particles and take the point of view of the observer who is at rest relative to one of the particles, say particle "2", in the initial state. In component form the conservation of momentum reads (see figure 3.6)

$$m_1 v_{1i} = m_1 v_{1f} \cos\theta + m_2 v_{2f} \cos\phi$$

$$m_1 v_{1f} \sin\theta = m_2 v_{2f} \sin\phi \tag{3.6.13}$$

and if energy is also conserved, it provides a third equation,

$$m_1 v_{1i}^2 = m_1 v_{1f}^2 + m_2 v_{2f}^2. \tag{3.6.14}$$

These three equations are not sufficient to determine the four final state variables, *viz.*, v_{1f}, v_{2f}, $\theta = \cos^{-1}(\widehat{v}_{1i} \cdot \widehat{v}_{1f})$ and $\phi = \cos^{-1}(\widehat{v}_{1i} \cdot \widehat{v}_{2f})$. For the present, imagine that we know one of the angles, say ϕ in figure 3.6. Our strategy will be to determine the final speeds in terms of the angles and then determine the unknown angle, θ. Multiplying the second equation in (3.6.13) by $\cos\phi$ and then using the first we find

$$m_1 v_{1f} \sin\theta \cos\phi = m_2 v_{2f} \sin\phi \cos\phi = m_1 v_{1i} \sin\phi - m_1 v_{1f} \cos\theta \sin\phi \tag{3.6.15}$$

and therefore, collecting terms,

$$m_1 v_{1f} \sin(\theta + \phi) = m_1 v_{1i} \sin\phi \Rightarrow v_{1f} = \frac{v_{1i} \sin\phi}{\sin(\theta + \phi)} \tag{3.6.16}$$
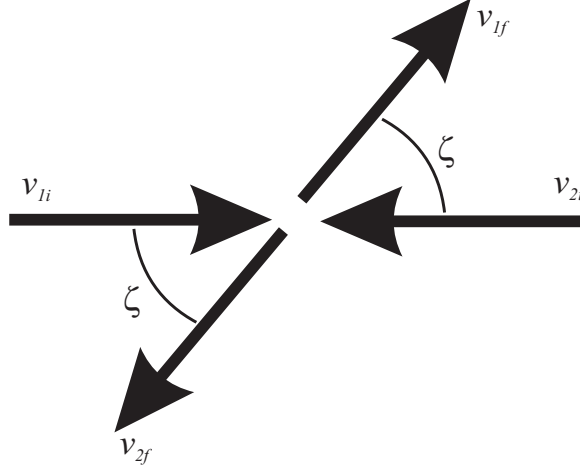
Figure 3.7: Two dimensional collisions from the point of view of the center of mass.

provided of course that $\sin(\theta + \phi) \neq 0$. Now using the second momentum conservation equation we find

$$v_{2f} = \frac{m_1}{m_2} \frac{v_{1f} \sin \theta}{\sin \phi} \Rightarrow v_{2f} = \frac{m_1}{m_2} \frac{v_{1i} \sin \theta}{\sin(\theta + \phi)} \tag{3.6.17}$$

It remains to determine θ, which we can do by using the energy equation. Inserting the values of v_{1f} and v_{2f} as determined above we find

$$\sin^2(\theta + \phi) = \sin^2 \phi + \frac{m_1}{m_2} \sin^2 \theta \tag{3.6.18}$$

and, expanding both sides and simplifying the resulting expressions, we arrive at

$$\tan \theta = \frac{\sin 2\phi}{\frac{m_1}{m_2} - \cos 2\phi}. \tag{3.6.19}$$

Notice that if $m_1 = m_2$ then $\tan \theta = \cot \phi$ which implies that $\theta + \phi = \pi/2$, *i.e.,* a two dimensional collision between equal masses will scatter the objects at right angles to one another, in the frame in which one of them is initially at rest.

What is the view from the center of mass frame? In this case the initial and final total momentum as measured in this frame must vanish exactly, which yields

$$\vec{v}_{2i}' = -\frac{m_1}{m_2} \vec{v}_{1i}', \quad \vec{v}_{2f}' = -\frac{m_1}{m_2} \vec{v}_{1f}' \tag{3.6.20}$$

which implies that the initial and final velocities are anti-parallel. If the collision is also elastic then energy conservation will lead to

$$\left(1 + \frac{m_1}{m_2}\right) v_{1i}^{\prime}{}^2 = \left(1 + \frac{m_1}{m_2}\right) v_{1f}^{\prime}{}^2 \tag{3.6.21}$$

which means that $v_{1f}^{\prime} = v_{1i}^{\prime}$ and so $v_{2f}^{\prime} = v_{2i}^{\prime}$, i.e., the particles simply bounce off with the same speeds as their respective initial speeds. But what about the direction? Since the final velocities must be anti-parallel there is only one angle in the final state, as shown in figure 3.7 but there isn't enough information to determine it.

The final velocities in any frame can always be transformed into any other frame by simply applying Galilei transformations. In particular, the simple results obtained in the center of mass frame can be transformed to another frame by using $\vec{v}_{1i} = \vec{v}_{1i}^{\prime} + \vec{v}_{cm}$, etc. As an example, let us transform from the center of mass frame to a Laboratory frame in which particle "2" is at rest and compare what we obtain with our previous results. Let ζ be the angle made in the center of mass frame, as indicated in figure 3.7. In the Laboratory frame, $\vec{v}_{2i} = 0$,

$$\vec{v}_{cm} = \frac{m_1 \vec{v}_{1i}}{m_1 + m_2} \tag{3.6.22}$$

and

$$\vec{v}_{1i}^{\prime} = \vec{v}_{1i} - \vec{v}_{cm} = \frac{m_2}{m_1 + m_2}\, \vec{v}_{1i}$$

$$\vec{v}_{2i}^{\prime} = -\vec{v}_{cm} = -\frac{m_1}{m_1 + m_2}\, \vec{v}_{1i}$$

$$\vec{v}_{1f}^{\prime} = v_{1i}^{\prime}(\cos\zeta, \sin\zeta), \quad \vec{v}_{2f}^{\prime} = v_{2i}^{\prime}(-\cos\zeta, -\sin\zeta), \tag{3.6.23}$$

where, in the last equation, we used $v_{1f}^{\prime} = v_{1i}^{\prime}$ and $v_{2f}^{\prime} = v_{2i}^{\prime}$. Therefore

$$\vec{v}_{1f} = \vec{v}_{1f}^{\prime} + \vec{v}_{cm} = \frac{m_2 v_{1i}}{m_1 + m_2}(\cos\zeta + m_1/m_2, \sin\zeta)$$

$$\vec{v}_{2f} = \vec{v}_{2f}^{\prime} + \vec{v}_{cm} = \frac{m_1 v_{1i}}{m_1 + m_2}(-\cos\zeta + 1, \sin\zeta) \tag{3.6.24}$$

and we can now relate the angles θ and ϕ in figure 3.6 to ζ by

$$\tan\theta = \frac{v_{1fy}}{v_{1fx}} = \frac{\sin\zeta}{\frac{m_1}{m_2} + \cos\zeta}, \quad \tan\phi = \frac{v_{2fy}}{v_{2fx}} = \cot(\zeta/2) \tag{3.6.25}$$

The second equation above says that $\phi = (\pi - \zeta)/2$ and with this we verify (3.6.19) by direct substitution into the first.[7]

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[7] Problem: Also verify (3.6.17).

3.7   The Virial Theorem

The virial theorem relates the average kinetic energy of the particles of a many body system to the so-called **virial**, which is closely related to their average potential energy in cases of physical interest. Consider a system of particles which have momenta $\vec{p}_n$ and positions $\vec{r}_n$ and define the quantity

$$S = \sum_n \vec{r}_n \cdot \vec{p}_n \tag{3.7.1}$$

then the rate of change of S is given by

$$\frac{dS}{dt} = \sum_n (\dot{\vec{r}}_n \cdot \vec{p}_n + \vec{r}_n \cdot \dot{\vec{p}}_n) \tag{3.7.2}$$

Consider the average rate of change,

$$\left\langle \frac{dS}{dt} \right\rangle_\tau = \left\langle \sum_n \dot{\vec{r}}_n \cdot \vec{p}_n \right\rangle_\tau + \left\langle \sum_n \vec{r}_n \cdot \dot{\vec{p}}_n \right\rangle_\tau \tag{3.7.3}$$

where the average is defined over some time interval, say $[0, \tau]$, in the usual way,

$$\left\langle \frac{dS}{dt} \right\rangle_\tau = \frac{1}{\tau} \int_0^\tau dt \frac{dS}{dt} = \frac{1}{\tau} \int_0^\tau dS(t). \tag{3.7.4}$$

By definition then,

$$\left\langle \frac{dS}{d\tau} \right\rangle_\tau = \frac{S(\tau) - S(0)}{\tau}. \tag{3.7.5}$$

If the mechanical state of the system is periodic, *i.e.,* the particles of the system return to their positions and momenta at $t = 0$ after an interval an interval of time τ, then it follows that

$$\left\langle \frac{dS}{dt} \right\rangle_\tau = 0 \tag{3.7.6}$$

On he other hand, if the motion is not periodic, assume that neither the positions nor the momenta ever approach infinity during the evolution of the system. This makes sense if the system is bound, and as a consequence $S(\tau) - S(0)$ is always finite. If we consider very large times, taking $\tau \to \infty$, then clearly

$$\lim_{\tau \to \infty} \left\langle \frac{dS}{dt} \right\rangle_\tau \to 0 \tag{3.7.7}$$

But this means that over the time interval τ (one period for periodic systems and infinite for systems that are not periodic but bounded)

$$\sum_n \langle \dot{\vec{r}}_n \cdot \vec{p}_n \rangle_\tau = -\sum_n \langle \vec{r}_n \cdot \dot{\vec{p}}_n \rangle_\tau \tag{3.7.8}$$

Now $\vec{F}_n = \dot{\vec{p}}_n$ and $\vec{p}_n \cdot \dot{\vec{r}}_n = 2K_n$. Therefore the left hand side of the equation is twice the time averaged kinetic energy and we have

$$\langle K \rangle_\tau = -\frac{1}{2} \langle \sum_n \vec{r}_n \cdot \vec{F}_n \rangle_\tau \qquad (3.7.9)$$

The average on the right hand side is called the **Virial**, and the relationship between the time averaged kinetic energy and the Virial is called the **Virial Theorem**. Now $\vec{F}_n$ represents the *total* force on the particle labeled n. If all the forces, internal and external are conservative, so is $\vec{F}_n$ and we can write

$$\vec{F}_n = -\vec{\nabla}_n U_n. \qquad (3.7.10)$$

The time averaged kinetic energy becomes

$$\langle K \rangle_\tau = \frac{1}{2} \langle \sum_n \vec{r}_n \cdot \vec{\nabla}_n U_n \rangle_\tau \qquad (3.7.11)$$

To understand its significance, consider what this means for a bound, self-gravitating system of particles such as, say, a galaxy or even a group of galaxies (stars are small compared to the distance between them and may be considered particles over galactic scales). Thus we have to consider only the internal potential energies of interactions, so

$$\begin{aligned}
\frac{1}{2} \sum_n \vec{r}_n \cdot \vec{\nabla}_n U_n &= \frac{1}{2} \sum_{n,m \neq n} \vec{r}_n \cdot \vec{\nabla}_n U_{nm}(|\vec{r}_n - \vec{r}_m|) \\
&= \frac{1}{4} \sum_{n,m \neq n} (\vec{r}_n - \vec{r}_m) \cdot \vec{\nabla}_n U_{nm}(|\vec{r}_n - \vec{r}_m|) \qquad (3.7.12)
\end{aligned}$$

where we have used $\vec{\nabla}_n U_{nm} = -\vec{\nabla}_m U_{mn}$. Now suppose that

$$U_{nm}(|\vec{r}_n - \vec{r}_m|) = k_{nm} |\vec{r}_n - \vec{r}_m|^{\alpha+1} \qquad (3.7.13)$$

where k_{nm} is a constant depending on the properties of particles m and n and $\alpha \neq -1$. For instance, if the interactions are purely gravitational, $k_{mn} = -GM_n M_m$ where M_n and M_m are the masses of particles n and m respectively and $\alpha = -2$. We find

$$(\vec{r}_n - \vec{r}_m) \cdot \vec{\nabla}_n U_{nm} = (\alpha + 1) U_{nm} \qquad (3.7.14)$$

and so, using

$$U^{\text{int}} = \frac{1}{2} \sum_{n,m \neq n} U_{nm}, \qquad (3.7.15)$$

we find that

$$\langle K \rangle_\tau = \frac{1}{2} \langle \sum_n \vec{r}_n \cdot \vec{\nabla}_n U_n \rangle_\tau = \frac{\alpha+1}{2} \langle U^{\text{int}} \rangle_\tau. \tag{3.7.16}$$

In particular, when $\alpha = -2$ the time averaged kinetic energy is precisely minus one half of the time averaged potential energy of the system![8] On the other hand, the time averaged kinetic energy is precisely equal to the time averaged potential energy of a harmonic oscillator.

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[8]Problem: Determine the Virial for the the Yukawa interaction,

$$U_{nm}(|\vec{r}_n - \vec{r}_m|) = \frac{k_{nm}}{|\vec{r}_n - \vec{r}_m|} e^{-\alpha|\vec{r}_n - \vec{r}_m|}$$

where α is independent of m, n.

Chapter 4

Newtonian Gravity

We have seen that once the dependence of a force on the position, velocity and time has been experimentally determined then the problem of calculating the trajectory of a particle under the influence of the force is a mathematical problem of fundamental interest. If an exact solution is obtained there are usually two free parameters per space dimension, which can be thought of as representing the initial position and velocity of the particle. Sometimes, a more fruitful way to think about these constants is in terms of conserved quantities. The allowed motions can then be classified by these conserved quantities. We shall soon see how this comes about, but first let us examine the gravitational force in greater detail.

4.1 The force law

Gravitation was the first of the fundamental forces to be discovered and for which a "force law" was given. The force law was first given by Newton after many years of studying Kepler's (1571-1630) three laws of planetary motion and Tyco Brahe's (1546-1601) excellent observational data on the same.[1] We have already mentioned Newton's law of gravitation before. It says simply that the force of attraction exerted by a body of mass m_1 on another of mass m_2 located respectively at $\vec{r}_1$ and $\vec{r}_2$ is given by

$$\vec{F}_{1\to 2} = -G\frac{m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) \qquad (4.1.1)$$

This force is always attractive because mass is positive.

A most remarkable fact about the gravitational force is that it depends on the masses of the particles, in the same way as the electromagnetic force depends on the charges. m_1

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[1]Kepler was initially Tycho Brahe's assistant and carried on his work after Tycho Brahe's death in 1601.

and m_2 can therefore be thought of as gravitational "charges" and there is no *à priori* reason to expect that they are precisely the inertial masses of the particles, *i.e.,* the masses that appear on the left hand side of Newton's second law of motion. Experimentally, however, one finds that they the gravitational mass and the inertial mass are identical up to about one part in 10^{11} and it is necessary therefore to assume this as a principle. The simple statement that gravitational and inertial masses are the same is called the "weak equivalence principle". A stronger form of the equivalence principle was given by Einstein in making the transition from Newtonian gravity, summarized by the weak equivalence principle and (4.1.1), and General Relativity, which is a theory about non-inertial frames.

As a consequence of the weak equivalence principle, the equations of motion for a particle, say particle 2, do not involve its inertia,

$$\frac{d^2\vec{r}_2}{dt^2} = -\frac{Gm_1}{r_{12}^2}\hat{r}_{12} = \vec{g}(\vec{r}_2) \tag{4.1.2}$$

where $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$ and $\vec{g}(\vec{r})$ denotes the gravitational acceleration. One could even conclude that gravity is not a force in the traditional Newtonian sense, since motion under its influence is kinematical: when a gravitational field is present, the kinematical equation

$$\frac{d^2\vec{r}}{dt^2} = 0 \tag{4.1.3}$$

should be replaced by the more general

$$\frac{d^2\vec{r}}{dt^2} = \vec{g}(\vec{r}) \tag{4.1.4}$$

where $\vec{g}(\vec{r})$ would be a property of the space(time) in which the particle is moving. This kinematical equation would not lead to "motion in a straight line with constant velocity", as required by Newton's first law of motion. However, the "straight line" can be thought of as the shortest distance between two points in a flat space(time), so the presence of matter (more generally, energy) in space must bend the space(time) in its neighborhood, causing the shortest distance between two points to no longer be a straight line.[2] Particle trajectories between two points would then be geodesics, not necessarily straight lines. General Relativity is a theory of how space(time) is curved by the presence of matter (energy). We will ignore this modified view of gravity and take the Newtonian approach, treating gravity as a force. It is a very good approximation as long as the gravitational force is weak and the speeds involved are small compared with the speed of light. The gravitational acceleration, $\vec{g}(\vec{r})$ is called the "gravitational field vector". It can be defined, for an arbitrary distribution of matter by two equations, which we now examine.

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[2]Trajectories that minimize the distance between two points are called geodesics.

4.2 Two properties of the gravitational field

Suppose we have a distribution of point particles, each labeled by an integer n, located at $\vec{r}_n$ and having a mass m_n. Consider a test particle of mass M and located at $\vec{r}$ in the neighborhood of this distribution. Assuming that the net force on the test body is the sum of forces exerted by each of the particles in the distribution on it, we have

$$\vec{F} = -G\sum_{n=1}^{N}\frac{Mm_n(\vec{r}-\vec{r}_n)}{|\vec{r}-\vec{r}_n|^3} \tag{4.2.1}$$

This would give the gravitational acceleration of our test body (also known as the gravitational field vector) at the point $\vec{r}$ as

$$\vec{g}(\vec{r}) = -G\sum_{n=1}^{N}\frac{m_n(\vec{r}-\vec{r}_n)}{|\vec{r}-\vec{r}_n|^3} \tag{4.2.2}$$

If the distribution can be considered continuous (this means that the average separation between the atoms/molecules that make up the system is on the order of the typical atomic/molecular size), then we can divide the system into infinitesimal pieces, each of mass $dm(\vec{r}')$ and add up the contribution of each infinitesimal mass to the gravitational field vector at $\vec{r}$ as we did before, except that now the sum is continuous and not discrete, *i.e.,* it becomes an integral over distribution,

$$\vec{g}(\vec{r}) = -G\int_D dm(\vec{r}')\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \tag{4.2.3}$$

We can express the mass of the infinitesimal pieces in terms of the density of the distribution: $dm(\vec{r}') = \rho(\vec{r}')d^3\vec{r}'$ and reexpress the field vector as a volume integral over the distribution,

$$\vec{g}(\vec{r}) = -G\int_{V_D} d^3\vec{r}'\frac{\rho(\vec{r}')(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3} \tag{4.2.4}$$

We have made a rather important assumption in all of the above: we assumed that the gravitational force on a body due to a distribution is the simple sum of forces exerted on it by elements of the distribution. In other words, the interaction between an pair of bodies does not affect the interaction of any of the bodies in the pair and a third body in their neighborhood. This assumption fails in general relativity, which is a non-linear theory of the gravitational field. If the field is sufficiently weak, however, superposition of forces is a very good approximation.

It is easy to see from (4.1.2) that the gravitational field vector due to a single massive body satisfies

$$\vec{\nabla}\times\vec{g} = 0 \tag{4.2.5}$$

This can be proved by simply taking the curl of the left hand side of (4.1.2). Of course it will hold for the gravitational field vector of a distribution too as long as the resulting $\vec{g}(\vec{r})$ is obtained by simply adding the contributions from elementary masses in the distribution (superposition). Thus, for the field vector due to a point mass, m, located at $\vec{r}'$,

$$\vec{\nabla} \times \vec{g} = -Gm\vec{\nabla} \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = -Gm \left[\frac{\vec{\nabla} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} - (\vec{r} - \vec{r}') \times \vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|^3} \right] \quad (4.2.6)$$

Now

$$[\vec{\nabla} \times (\vec{r} - \vec{r}')]_i = \epsilon_{ijk} \partial_j (x_k - x_k') = \epsilon_{ijk} \delta_{jk} \equiv 0 \quad (4.2.7)$$

by the antisymmetry of the Levi-Civita symbol, and

$$\left[\vec{\nabla} \frac{1}{|\vec{r} - \vec{r}'|^3} \right]_i = \partial_i \left[\sum_j (x_j - x_j')^2 \right]^{-3/2} = -3 \left[\sum_j (x_j - x_j')^2 \right]^{-5/2} (x_j - x_j')\delta_{ij}$$

$$= -\left[\frac{3(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^5} \right]_i \quad (4.2.8)$$

so

$$\vec{\nabla} \times \vec{g} = -3Gm(\vec{r} - \vec{r}') \times \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^5} = 0. \quad (4.2.9)$$

Because $\vec{g}$ is irrotational it can be expressed as the gradient of a scalar function and we can write

$$\vec{g} = -\vec{\nabla}\phi \quad (4.2.10)$$

(the minus sign is inserted for later convenience). The scalar function ϕ is called the **gravitational potential**. The gravitational field has not three but just one (continuous) degree of freedom since all of the three components of $\vec{g}$ are obtained from the gradient of a single scalar field. Knowing $\vec{g}$ we can determine ϕ: for a point particle, of mass m,

$$\phi(\vec{r}) = -\frac{Gm}{|\vec{r} - \vec{r}'|} \quad (4.2.11)$$

and, because we can superpose the gravitational field (at least, in the Newtonian approximation), we can also superpose the gravitational potential. It follows that for a discrete distribution, made up of masses m_n located at $\vec{r}_n$,

$$\phi(\vec{r}) = -G \sum_n \frac{m_n}{|\vec{r} - \vec{r}_n|} \quad (4.2.12)$$

and for a continuous distribution characterized by the density $\rho(\vec{r})$,

$$\phi(\vec{r}) = -G \int_{V_D} d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} \tag{4.2.13}$$

The gravitational force on a mass M located in the neighborhood of the distribution can be expressed in terms of the gravitational potential,

$$\vec{F} = M\vec{g} = -M\vec{\nabla}\phi \tag{4.2.14}$$

The gravitational force is obviously conservative. This means that we can define the potential energy of a body of mass M in the neighborhood of a distribution of masses as

$$U_M(\vec{r}) = -\int_*^{\vec{r}} \vec{F} \cdot d\vec{r} = M \int_*^{\vec{r}} \vec{\nabla}\phi \cdot d\vec{r} = M[\phi(\vec{r}) - \phi(*)] \tag{4.2.15}$$

where $*$ is used to represent the standard fixed point to which the potential energy is referred.

A second property of the gravitational field,

$$\vec{\nabla} \cdot \vec{g} = -4\pi G\rho \tag{4.2.16}$$

follows from Gauss' theorem. Again, if we can prove this for a point mass, then it will hold true for arbitrary mass distributions, simply because of superposition. Consider a closed surface, S and let us begin by evaluating

$$\oint_S dS\, \hat{n} \cdot \vec{g} = -Gm \oint_S dS\, \frac{\hat{n} \cdot (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} = -Gm \oint_S dS\, \frac{\cos\theta}{|\vec{r} - \vec{r}\,'|^2} \tag{4.2.17}$$

where θ is the angle between the normal to the surface and the outward radial vector from the charge. Now $dS\,\hat{n} \cdot (\vec{r} - \vec{r}\,')/|\vec{r} - \vec{r}\,'|$ is just the projection of the surface area dS on the sphere of radius $|\vec{r} - \vec{r}\,'|$, therefore the integrand is simply the solid angle subtended by the projection of dS on the sphere of that radius at the charge q. If we sum up these infinitesimal solid angles, then two cases may arise: (a) the original surface S does not enclose the mass m (see figure(4.1)), or (b) the mass is enclosed by the surface S (see figure (4.2)). In case (a), the net result of summing up the solid angles subtended at m will be identically zero due to the two equal and opposite contributions from region I, where $\cos\theta$ is negative, and from region II, where $\cos\theta$ is positive. The contribution from region I is negative, whereas the contribution from region II is positive and equal in magnitude to the contribution from region I. In case (b), the infinitesimal solid angles add up to give precisely the total solid angle of 4π. Thus we find that

$$\oint_S dS\, \hat{n} \cdot \vec{g} = -4\pi Gm_{\text{in}} \tag{4.2.18}$$
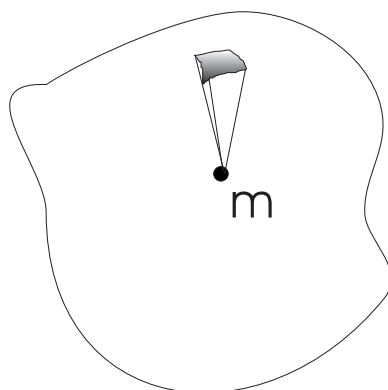
Figure 4.1: Gauss' Law: Mass not enclosed by S
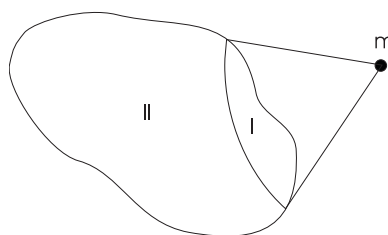
Figure 4.2: Gauss' Law: Mass enclosed by S

where the suffix in m_{in} is to indicate that only the charge *contained within* the surface S contributes. Equation (4.2.18) is the integral form of Gauss' law. Exploiting the fact that the gravitational field due to a distribution of masses is a simple superposition of the gravitational fields due to the individual masses, we may directly write down the integral form of Gauss' law for a discrete distribution

$$\oint_S dS \, \widehat{n} \cdot \vec{g} = -4\pi G \sum_{n=1}^{N} m_{n,\text{in}} = -4\pi G M_{\text{in}} \tag{4.2.19}$$

where M_{in} is the total mass contained within the surface, and for a continuous volume distribution

$$\oint_S dS \, \widehat{n} \cdot \vec{g} = -4\pi G \int_V d^3\vec{r}' \, \rho(\vec{r}') = -4\pi G M_{\text{in}}. \tag{4.2.20}$$

The differential form of Gauss' law can now be obtained by exploiting Gauss' theorem,

$$\oint_S dS \, \widehat{n} \cdot \vec{g} = \int_V d^3\vec{r} \, \vec{\nabla} \cdot \vec{g} = -4\pi G \int_V d^3\vec{r} \rho(\vec{r}). \tag{4.2.21}$$

from which eq. (4.2.16) follows directly.[3]

4.3   Simple Applications of Gauss' Law

In its integral form, Gauss' law is useful to determine the gravitational field vector of highly symmetric distributions. Otherwise, the using the integral form of Gauss' law to determine the gravitational field is not recommended. The idea is that if the mass distribution is sufficiently symmetric and a closed surface that mimics the symmetry of the distribution is chosen, the integral becomes trivial to evaluate.

4.3.1   Point mass.

The symmetry of a point charge is spherical so, for a Gaussian surface, choose a sphere with the mass as its center (see figure (4.3)). By the symmetry, we expect that the gravitational field will point along the unit radial vector and toward the center, so $\vec{g} = -g\widehat{r}$ and,

---

[3]**Note:** Outside of masses, Gauss' reads simply reads $\vec{\nabla} \cdot \vec{g} = 0$. Thus, this is also the equation outside a single point mass. Placing a point mass at the origin of coordinates and assuming spherical symmetry, prove that

$$\vec{g} = -\frac{Gm\widehat{r}}{r^2}$$

is a solution of $\vec{\nabla} \cdot \vec{g} = 0$. Where does the constant m come from?
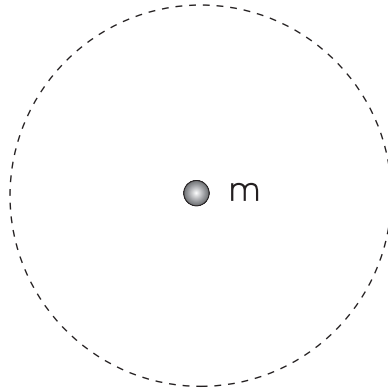
Figure 4.3: Gaussian surface for a point mass.

moreover, that its magnitude will be constant *on the surface of the sphere.* The unit radial vector, $\hat{r}$, is also the normal to the sphere, so

$$\oint_S dS\; \hat{n}\cdot\vec{g} = -g \oint_S dS = -4\pi r^2 g = -4\pi G m \Rightarrow \vec{g} = -\frac{Gm\hat{r}}{r^2} \qquad (4.3.1)$$

This argument may, of course, be extended to any spherically symmetric charge distribution. At points outside it, a spherical charge distribution will therefore behave as if it were a point charge situated at the center of the distribution.

4.3.2 Spherical charge distribution.

Consider a spherical mass distribution of radius R located so that its center is the origin of coordinates and for which we know the density, $\rho(\vec{r})$, which may or may not be constant. We are interested in the gravitational field both inside the sphere and outside it. To find the gravitational field inside the distribution, consider a spherical gaussian surface of radius $r < R$ (shown as S_1 in figure (4.4) with the same center as the distribution. Again, by the symmetry, we expect that the gravitational field at points on the gaussian surface will point radially inward, $\vec{g} = -g\hat{r}$ and that the magnitude of the gravitational field, g, will depend only on the distance from the center, $g = g(r)$, so that it is constant on the gaussian sphere. Therefore,

$$\oint_{S_1} dS\; \hat{n}\cdot\vec{g} = -g \oint_{S_1} dS = -4\pi r^2 g = -4\pi G \int_{S_1} d^3\vec{r}\rho(\vec{r}) \Rightarrow \vec{g} = -G\frac{\hat{r}}{r^2} \int_{S_1} d^3\vec{r}\rho(\vec{r}) \;\;(4.3.2)$$
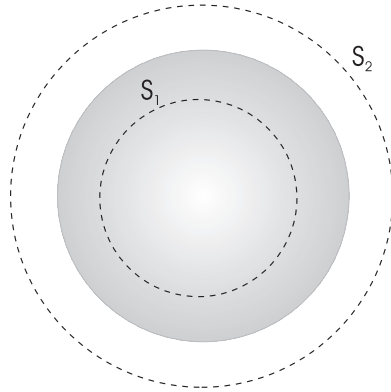
Figure 4.4: Gaussian surfaces for a massive sphere.

The integral on the right is just the mass contained within the gaussian surface. For instance, if $\rho(\vec{r}) = \rho_0$ is constant then performing the integral on the right gives

$$\vec{g} = -\frac{4\pi G \rho_0 \vec{r}}{3}, \tag{4.3.3}$$

which is the gravitational field inside the sphere at a distance $r < R$ from the center. It decreases in magnitude toward the center becoming zero at the center itself. To find the field outside the sphere, consider the gaussian sphere S_2 in figure (4.4), or radius $r > R$. In this case

$$\oint_{S_2} dS\, \hat{n} \cdot \vec{g} = -g \oint_{S_2} dS = -4\pi r^2 g = -4\pi G \int_{S_2} d^3 \vec{r} \rho(\vec{r}) \Rightarrow \vec{g} = -\frac{GM_{\text{tot}} \hat{r}}{r^2} \tag{4.3.4}$$

where M_{tot} is the total mass of the sphere (which is also the mass within the gaussian surface). For points outside the sphere, the gravitational field is precisely the same as it would be if the source were a point (instead of the sphere) of the same mass as the sphere and situated at its center. Therefore, as far as the exterior gravitational field is concerned, one could replace a sphere by an equal point-like mass located at its center. The gravitational field vector is continuous at the surface of the spherical distribution.

4.3.3 Spherical shell.

An extension of the previous example is the spherical shell shown in figure (4.5) with inner radius a and outer radius b. There are three regions of interest: (a) the hollow portion of the shell, (b) the shell itself and (c) the exterior of the shell. In each of these regions we draw gaussian spheres, shown as S_1, S_2 and S_3 respectively in figure (4.5). As usual,
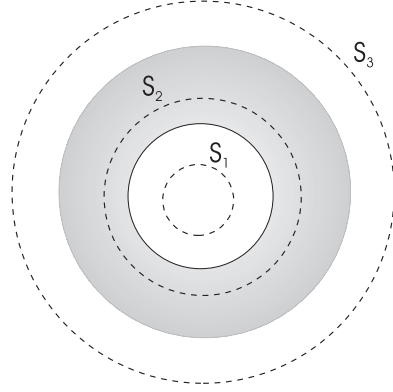
Figure 4.5: Gaussian surfaces for a massive spherical shell.

spherical symmetry implies that the gravitational field is directed radially inward and that its magnitude depends only on the distance from the center. Therefore, inside the hollow portion $(r < a)$ and on the gaussian sphere S_1,

$$\oint_{S_1} dS \, \widehat{n} \cdot \vec{g} = -g \oint_{S_1} dS = -4\pi r^2 g = -4\pi G \int_{S_1} d^3\vec{r}\rho(\vec{r}) = 0 \Rightarrow \vec{g} = 0 \qquad (4.3.5)$$

i.e., the gravitational field vanishes at all points here. Inside the shell itself $(a < r < b)$,

$$\oint_{S_2} dS \, \widehat{n} \cdot \vec{g} = -g \oint_{S_2} dS = -4\pi r^2 g = -4\pi G \int_{S_2} d^3\vec{r}\rho(\vec{r}) \Rightarrow \vec{g} = -G\frac{\widehat{r}}{r^2} \int_{S_2} d^3\vec{r}\rho(\vec{r}) \quad (4.3.6)$$

If the density is constant

$$\vec{g} = -\frac{4\pi G\rho_0\widehat{r}}{r^2}(r^3 - a^3) \qquad (4.3.7)$$

Outside the shell $(r > b)$ an argument similar to the one we made for the sphere tells us that

$$\vec{g} = -\frac{GM_{\text{tot}}\widehat{r}}{r^2} \qquad (4.3.8)$$

where M_{tot} is the total mass of the shell.

4.3.4   Infinite line of constant linear mass density (cosmic string).

The symmetry of an infinite line of mass (infinite string) is that of a right circular cylinder. For a Gaussian surface, choose an infinite cylinder with the line of charge along its axis (see figure (4.6).
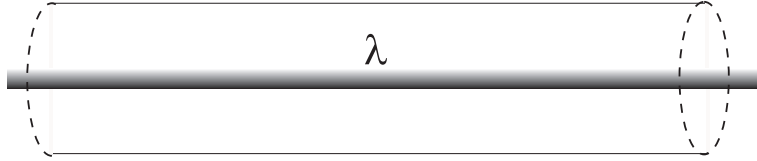
Figure 4.6: Gaussian surface for an infinite, massive string.

Let ρ represent the radius of the cylinder and $\widehat{\rho}$ the unit radial vector. By the symmetry, we expect that the gravitational field will point along the unit radial vector and toward the axis of the cylinder, so $\vec{g} = -g\widehat{\rho}$ and, moreover, that its magnitude will be constant *on the surface of the cylinder, i.e.,* $g = g(\rho)$. We take the length of the string to be L (with the understanding that the limit as $L \to \infty$ is to be taken in the end), so that if λ represents the linear mass density on the string then the total mass of the string is $M = \lambda L$. The unit radial vector $\widehat{\rho}$ is also normal to the cylinder, so

$$\oint_S dS\ \widehat{n} \cdot \vec{g} = -g \oint_S dS = -2\pi\rho L g = -4\pi G M = -4\pi G \lambda L \Rightarrow \vec{g} = -2G\lambda\frac{\widehat{\rho}}{\rho} \qquad (4.3.9)$$

What do we mean by an "infinite" string? Clearly, no such object exists in fact. If the string is finite, of length L, then its symmetry is not really cylindrical and the problem becomes more complicated by the presence of edges. An infinite string is the approximation in which these edge effects can be neglected, *i.e.,* when the point at which the field is measured is close to the line so that $\rho/L << 1$.

4.3.5 Infinite sheet of constant areal mass density: (domain wall)

Choose a "pill-box", *i.e.,* a cylinder closed at its two ends as shown in figure (4.7) for a Gaussian surface. By the planar symmetry of the mass distribution, we expect the gravitational field to be normal to the sheet at all points and pointing inward, therefore only the integrations over the upper and lower ends of the pill-box will yield non-vanishing contributions. Furthermore, as the magnitude of the gravitational field may depend at most on the perpendicular distance from the sheet and as we can arrange the pill-box so that its two faces are equidistant from the sheet, the contributions from these two faces will be identical. If ΔS is the area of each face, integrating over the pill-box shaped Gaussian surface then gives

$$\oint_S dS\ \widehat{n} \cdot \vec{g} = -2g\Delta S = -4\pi G M_{\text{in}} = -4\pi G \sigma \Delta S \Rightarrow \vec{g} = -2\pi G \sigma \widehat{n} \qquad (4.3.10)$$

where $\widehat{n}$ is normal to the sheet.
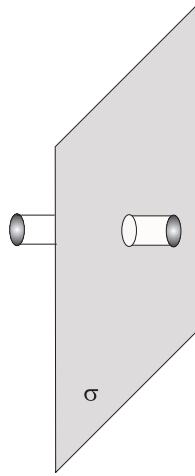
Figure 4.7: A "pill-box" Gaussian surface appropriate for an infinite, massive sheet.

4.4 The Poisson and Laplace Equations

If Gauss' Law is combined with the fact that $\vec{g}$ is irrotational, we find that

$$\vec{\nabla} \cdot \vec{g} = -\vec{\nabla} \cdot \vec{\nabla}\phi = -\vec{\nabla}^2\phi = -4\pi G\rho, \qquad (4.4.1)$$

or

$$\vec{\nabla}^2\phi = 4\pi G\rho. \qquad (4.4.2)$$

This is Poisson's equation. It is the starting point for finding configurations of the gravitational field *i.e.,* given any matter distribution in space, the problem of finding $\vec{g}$ is reduced to the boundary value problem of finding the appropriate solution to (4.4.2). Often we are interested in finding the gravitational field outside of the sources; in this case the matter density, $\rho(\vec{r})$ vanishes and the scalar potential satisfies Laplace's equation

$$\vec{\nabla}^2\phi = 0. \qquad (4.4.3)$$

Both Poisson's equation and Laplace's equation are second order, elliptic partial differential equations that are usually satisfied in some subset of, or all of, space. Suppose we call the region over which it is satisfied R and suppose that R is open and connected having some piecewise smooth boundary, ∂R.

The solution to (4.4.2) becomes unique only when boundary conditions, *i.e.,* conditions satisfied by ϕ on ∂R, are imposed. Boundary conditions are an additional set of equations imposed on ϕ and its derivatives at the boundary and an important question becomes

what boundary conditions are required to have a unique solution to a given problem. In general, in order to obtain a unique solution to any partial differential equation of order k in n dimensions, one requires to specify the function and $k - 1$ normal derivatives of the function on ∂R. These are the **Cauchy** conditions (sometimes referred to as Cauchy "data"). Thus, typically two kinds of boundary conditions ensure unique solutions of Poisson's equation:

- the **Dirichlet** condition specifies the value of $\phi(\vec{r})$ on ∂R and

- the **Neumann** condition specifies the normal derivative, $\widehat{n} \cdot \vec{\nabla}\phi(\vec{r})$, on the boundary, where $\widehat{n}$ is the unit normal to ∂R.

In practice, however, either one or the other *may* suffice to completely determine the unique solution.

If there are no non-trivial boundaries, the gravitational potential is generally required to fall off to zero at infinity and the solution can be no different from the solution we have already obtained,

$$\phi(\vec{r}) = -G \int_{V_D} d^3\vec{r}' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}, \tag{4.4.4}$$

where the integration is performed over the charge distribution D. But this means that the right hand side of the above equation should obey (4.4.2). Taking the Laplacian of the solution gives

$$\vec{\nabla}^2 \phi = -G \int_{V_D} d^3\vec{r}' \rho(\vec{r}') \vec{\nabla}^2 \frac{1}{|\vec{r} - \vec{r}'|} \tag{4.4.5}$$

and we see that

$$\vec{\nabla}^2 \frac{1}{|\vec{r} - \vec{r}'|} = \partial_i \frac{x_i - x_i'}{|\vec{r} - \vec{r}'|^3} = 0, \tag{4.4.6}$$

provided that $\vec{r} \neq \vec{r}'$. However, it is badly defined at $\vec{r} = \vec{r}'$ and this is precisely the limit that supports the integral above to give a non-zero right hand side. We thus encounter the three dimensional version of the "$\delta-$function", introduced in the previous chapter. If we call

$$\vec{\nabla}^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi\delta^3(\vec{r} - \vec{r}'), \tag{4.4.7}$$

then, because

$$\vec{\nabla}^2 \phi(\vec{r}) = 4\pi G \rho(\vec{r}), \tag{4.4.8}$$

the distribution $\delta^3(\vec{r} - \vec{r}')$ must have the following property

$$\int d^3\vec{r}' f(\vec{r}')\delta^3(\vec{r} - \vec{r}') = f(\vec{r}) \tag{4.4.9}$$

for any function, $f(\vec{r})$. In particular, taking $f(\vec{r}') = 1$, we should have

$$\int d^3\vec{r}' \delta(\vec{r} - \vec{r}') = 1 \tag{4.4.10}$$

In the parlance of the previous chapter,

$$G(\vec{r} - \vec{r}') = -\frac{1}{4\pi|\vec{r} - \vec{r}'|} \tag{4.4.11}$$

is the Green's function of the Laplacian.[4]

We can convince ourselves of (4.4.7) by applying Gauss' theorem,

$$-\frac{1}{4\pi}\int d^3\vec{r}\,\vec{\nabla}^2\left(\frac{1}{r}\right) = -\frac{1}{4\pi}\oint_S d\vec{S}\cdot\vec{\nabla}\left(\frac{1}{r}\right) \tag{4.4.12}$$

where S is a closed surface bounding the region of integration. Clearly, only the radial component of the surface normal will contribute to the integral. In other words, given any arbitrary bounding surface, only the projection of this surface on a sphere of radius r is relevant. This means that, without loss of generality, we can take S to be a sphere and write the integral as

$$-\frac{1}{4\pi}\oint_S d\vec{S}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \frac{1}{4\pi}\oint_S d\theta d\phi r^2\left(\frac{1}{r^2}\right) = 1 \tag{4.4.13}$$

as required.

Naturally, the Green's function in (4.4.11) is only defined up to the addition of an arbtrary solution of Laplace's equation, *i.e.,* more generally $G(\vec{r} - \vec{r}')$ can be written as

$$G(\vec{r} - \vec{r}') = -\frac{1}{4\pi|\vec{r} - \vec{r}'|} + \mathcal{Q}(\vec{r} - \vec{r}') \tag{4.4.14}$$

where $\vec{\nabla}^2\mathcal{Q}(\vec{r} - \vec{r}') = 0$. Likewise, in principle any arbitrary solution, $\Phi(\vec{r})$, of Laplace's equation may be added to $\phi(\vec{r})$ in (4.4.4) to yield another solution of Poisson's equation.

---

[4]<u>Problem</u>: This function can be obtained from first principles as we obtained the Green's function for the driven harmonic oscillator. Using the following representation of the $\delta-$function,

$$\delta^3(\vec{r} - \vec{r}') = \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot(\vec{r} - \vec{r}')}$$

the integral expression

$$G(\vec{r} - \vec{r}') = \int \frac{d^3\vec{k}}{(2\pi)^3} G(\vec{k}) e^{i\vec{k}\cdot(\vec{r} - \vec{r}')}$$

for $G(\vec{r} - \vec{r}')$, find an expression for $G(\vec{k})$. Exploit spherical symmetry to reduce the resulting integral for $G(\vec{r} - \vec{r}')$ to a one dimensional integral defined in the complex $|\vec{k}| = k$ plane and define the integral by its principal value along the contour closed in the upper half plane.

This is the analogue of adding a complimentary function to the particular solution in order to get the most general solution of an inhomogeneous, one dimensional differential equation, as we did for the driven harmonic oscillator. In that problem, the arbitrary constants in the complimentary function were determined by the initial conditions. Likewise, the boundary conditions will determine $\Phi(\vec{r})$ up to an additive constant and, conversely, an appropriate $\Phi(\vec{r})$ can always be found so that the sum $\phi(\vec{r}) + \Phi(\vec{r})$ satisfies any given set of boundary conditions.

The $\delta-$function provides a convenient way to represent the mass density of a discrete distribution. Notice how the distribution is essentially zero except at a countable number of points, the points of support, where it is infinite. In fact, by (4.4.9), it is "sufficiently infinite" for its integral to be finite and non-zero. Therefore, we may define the mass density of a discrete distribution as

$$\rho(\vec{r}) = \sum_{n=1}^{N} m_n \delta^3(\vec{r} - \vec{r}_n) \tag{4.4.15}$$

Inserting this into the solution for $\phi(\vec{r})$ in (4.2.13) gives

$$\phi(\vec{r}) = -G \int_{V_D} d^3\vec{r}\,' \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = -G \sum_{n=1}^{N} \int_{V_D} d^3\vec{r}\,' \frac{m_n \delta^3(\vec{r}\,' - \vec{r}_n)}{|\vec{r} - \vec{r}\,'|} = -G \sum_{n=1}^{N} \frac{m_n}{|\vec{r} - \vec{r}_n|} \tag{4.4.16}$$

exactly as given in (4.2.12). The use of $\delta-$functions to represent the charge density of a discrete distribution serves to unify our description of charge distributions.

Methods to solve Poisson's equation and Laplace's equation in the presence of more sophisticated boundary conditions are well developed, but they will not be discussed further here. We will turn to the problem of describing the motion of a body in a given gravitational field instead. Therefore in the next chapter we give a detailed description of the motion of test bodies under the action of a class of forces called "central". The gravitational force is one member of this class.

Chapter 5

Motion under a Central Force

5.1 Symmetries

When symmetries are present it is always convenient to select a coordinate system that is adapted to them. We will be interested principally in problems with rectangular, spherical or axial symmetry. In these cases it is convenient to turn respectively to Cartesian coordinates, Spherical coordinates or Cylindrical coordinates respectively. Later we will learn to work with general coordinate systems, but then we will introduce more powerful techniques.

5.1.1 Spherical Coordinates

Thus, for example, if we know that the source charge distribution and boundary conditions are spherically symmetric it is convenient to work in spherical coordinates, defined in terms of the cartesian coordinate system by the transformations

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \tag{5.1.1}$$

and the inverse transformations

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{5.1.2}$$

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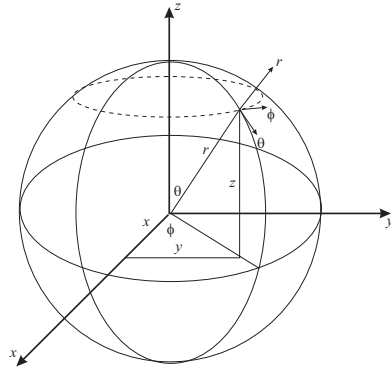
Figure 5.1: Spherical Coordinates

These transformations are of course valid only away from the origin. Recall that unit vectors representing the directions of increasing r, θ and φ are given by

$$
\begin{aligned}
\widehat{r} &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\
\widehat{\theta} &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta) \\
\widehat{\varphi} &= (-\sin\varphi, \cos\varphi, 0)
\end{aligned}
\tag{5.1.3}
$$

and that

$$
\begin{aligned}
\frac{\partial\widehat{r}}{\partial\theta} &= \widehat{\theta}, \quad \frac{\partial\widehat{r}}{\partial\varphi} = \widehat{\varphi}\sin\theta \\
\frac{\partial\widehat{\theta}}{\partial\theta} &= -\widehat{r}, \quad \frac{\partial\widehat{\theta}}{\partial\varphi} = \widehat{\varphi}\cos\theta \\
\frac{\partial\widehat{\varphi}}{\partial\theta} &= 0, \quad \frac{\partial\widehat{\varphi}}{\partial\varphi} = -\widehat{r}\sin\theta - \widehat{\theta}\cos\theta
\end{aligned}
\tag{5.1.4}
$$

The unit vectors in the original Cartesian system may be given in terms of these unit vectors as follows

$$
\begin{aligned}
\widehat{x} &= (\widehat{x}\cdot\widehat{r})\widehat{r} + (\widehat{x}\cdot\widehat{\theta})\widehat{\theta} + (\widehat{x}\cdot\widehat{\varphi})\widehat{\varphi} = \widehat{r}\sin\theta\cos\varphi + \widehat{\theta}\cos\theta\cos\varphi - \widehat{\varphi}\sin\varphi \\
\widehat{y} &= (\widehat{y}\cdot\widehat{r})\widehat{r} + (\widehat{y}\cdot\widehat{\theta})\widehat{\theta} + (\widehat{y}\cdot\widehat{\varphi})\widehat{\varphi} = \widehat{r}\sin\theta\sin\varphi + \widehat{\theta}\cos\theta\sin\varphi + \widehat{\varphi}\cos\varphi \\
\widehat{z} &= (\widehat{z}\cdot\widehat{r})\widehat{r} + (\widehat{z}\cdot\widehat{\theta})\widehat{\theta} + (\widehat{z}\cdot\widehat{\varphi})\widehat{\varphi} = \widehat{r}\cos\theta - \widehat{\theta}\sin\theta
\end{aligned}
\tag{5.1.5}
$$

Of interest to us is the Laplacian in these coordinates, so let us begin by evaluating the gradient operator. Transforming its definition in Cartesian coordinates,

$$
\vec{\nabla} = \widehat{x}\frac{\partial}{\partial x} + \widehat{y}\frac{\partial}{\partial y} + \widehat{z}\frac{\partial}{\partial z}
\tag{5.1.6}
$$

to spherical coordinates by using the transformations just given, we find[1]

$$\vec{\nabla} = \widehat{r}\frac{\partial}{\partial r} + \frac{\widehat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\widehat{\varphi}}{r\sin\theta}\frac{\partial}{\partial \varphi} \tag{5.1.7}$$

and taking the inner product[2] we obtain the Laplacian in spherical coordinates

$$\vec{\nabla}^2 = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 \tag{5.1.8}$$

The Euclidean distance between two points may be given in terms of r, θ and φ by transforming its expression in Cartesian coordinates,

$$d\vec{r}^2 = dx^2 + dy^2 + dz^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{5.1.9}$$

The volume element must account for the Jacobian of the transformation from the Cartesian system to the spherical system,

$$\mathbb{J} = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} \tag{5.1.10}$$

i.e.,

$$\int d^3\vec{r} = \int dxdydz = \int drd\theta d\varphi \det(\mathbb{J}) = \int (r^2\sin\theta)drd\theta d\phi \tag{5.1.11}$$

gives the volume of spheres.

5.1.2 Cylindrical coordinates

Cylindrical coordinates are defined by the following transformations from a Cartesian system:

$$\rho = \sqrt{x^2 + y^2}$$
$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z \tag{5.1.12}$$

where, we have assumed that the axial symmetry is about the "z" axis. The inverse transformations are simple to obtain

$$x = \rho\cos\varphi$$

---

[1]Problem: Show this.

[2]Problem: Show this. Note that the gradient is a vector *operator*, so be careful to take appropriate derivatives when necessary while taking the inner product.
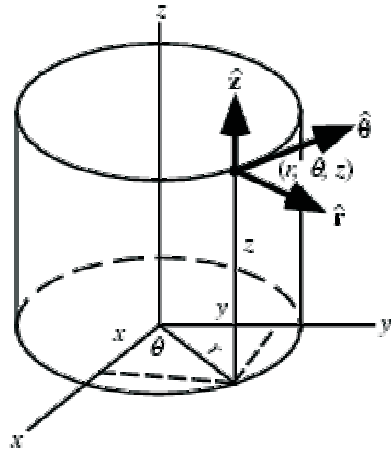
Figure 5.2: Cylindrical Coordinates

$$y = \rho \sin \varphi$$
$$z = z \tag{5.1.13}$$

and this time the system is badly defined along the entire $z-$ axis. Nevertheless, for points away from the $z-$axis, we may define the unit vectors

$$\widehat{\rho} = (\cos \varphi, \sin \varphi, 0)$$
$$\widehat{\varphi} = (- \sin \varphi, \cos \varphi, 0)$$
$$\widehat{z} = (0, 0, 1) \tag{5.1.14}$$

(ρ and φ are now just polar coordinates in the $x - y$ plane) which satisfy

$$\frac{\partial \widehat{\rho}}{\partial \varphi} = \widehat{\varphi}, \quad \frac{\partial \widehat{\varphi}}{\partial \varphi} = -\widehat{\rho} \tag{5.1.15}$$

(all other derivatives vanish). The Cartesian unit vectors can be expressed in terms of the cylindrical ones as we did in the case of spherical symmetry,

$$\widehat{x} = \widehat{\rho} \cos \varphi - \widehat{\varphi} \sin \varphi$$
$$\widehat{y} = \widehat{\rho} \sin \varphi + \widehat{\varphi} \cos \varphi$$
$$\widehat{z} = \widehat{z} \tag{5.1.16}$$

and the gradient operator can be transformed to the cylindrical system as before to get

$$\vec{\nabla} = \widehat{\rho} \partial_\rho + \frac{\widehat{\varphi}}{\rho} \partial_\varphi + \widehat{z} \partial_z \tag{5.1.17}$$

giving

$$\vec{\nabla}^2 = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{1}{\rho^2}\partial_\varphi^2 + \partial_z^2 \qquad (5.1.18)$$

The Euclidean distance between two points in these coordinates is

$$d\vec{r}^2 = d\rho^2 + \rho^2 d\varphi^2 + dz^2 \qquad (5.1.19)$$

Again, the volume element must account for the Jacobian of the transformation from the Cartesian system to the spherical system,

$$\int d^3\vec{r} = \int dx dy dz = \int d\rho d\varphi dz \det(\mathbb{J}) = \int \rho d\rho d\varphi dz \qquad (5.1.20)$$

gives the volume of cylinders.

5.2 Central Forces

A central force is any force of the form

$$\vec{F}(\vec{r}) = F(r)\hat{r} \qquad (5.2.1)$$

where $\hat{r}$ is the unit outgoing radial vector from some fixed point, called the origin of the force. The magnitude of the force depends only on the distance from the fixed point so that it is the same at all points on the surface of a sphere with its center at the origin of force. Central forces have the following two important properties:

1. Every central force is conservative. This can be seen by verifying that $\vec{\nabla} \times \vec{F} = 0$,

$$\vec{\nabla} \times \vec{F} = (\vec{\nabla}F) \times \hat{r} + F(\vec{\nabla} \times \hat{r}) \qquad (5.2.2)$$

   both terms of which vanish, the first because $\vec{\nabla}F(r) = F'(r)\hat{r}$ is in the radial direction and the second because $\hat{r}$ is irrotational. It follows there exists a (potential energy) function, $\phi = \phi(r)$, such that $F = -\vec{\nabla}\phi(r)$.

2. Motion under a central force is torsion free. This is seen by evaluating rate of change of $\dot{\vec{F}}$. Using

$$\frac{dr}{dt} = \hat{r} \cdot \vec{v}, \quad \frac{d\hat{r}}{dt} = \frac{\vec{v}}{r} - \frac{\hat{r}}{r}(\hat{r} \cdot \vec{v}) \qquad (5.2.3)$$

   we determine

$$\dot{\vec{F}}(r) = F'(r)(\hat{r} \cdot \vec{v})\hat{r} + \frac{F(r)}{r}[\vec{v} - (\hat{r} \cdot \vec{v})\hat{r}] \qquad (5.2.4)$$

   and therefore

$$\vec{F} \times \dot{\vec{F}} = \frac{F^2(r)}{r}(\hat{r} \times \vec{v}) \qquad (5.2.5)$$

showing that $\vec{v} \cdot (\vec{F} \times \dot{\vec{F}}) = 0$. This leads to the important conclusion that the motion will always occur in the plane determined by the initial velocity and the force itself.

Another way to arrive at the last conclusion is to consider Newton's equations in spherical coordinates. Taking a time derivative of the position vector

$$\vec{r} = r\widehat{r} \tag{5.2.6}$$

and using (5.1.4), gives

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\widehat{r} + r\frac{d\widehat{r}}{dt} = \dot{r}\widehat{r} + r\dot{\theta}\widehat{\theta} + r\sin\theta\dot{\varphi}\widehat{\varphi}. \tag{5.2.7}$$

Taking a derivative once more,

$$
\begin{aligned}
\frac{d^2\vec{r}}{dt^2} &= (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\widehat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\widehat{\theta} \\
&\quad + (2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta + r\ddot{\varphi}\sin\theta)\widehat{\varphi}.
\end{aligned}
\tag{5.2.8}
$$

Now according to Newton's second law

$$m\frac{d^2\vec{r}}{dt^2} = F(r)\widehat{r} \tag{5.2.9}$$

assuming that the force is central. If we orient our coordinate system in such a way that the initial velocity is in the $x - y$ plane ($\theta = \frac{\pi}{2}$) *i.e.*, $\dot{\theta}(t = 0) = 0$, we find (from the $\widehat{\theta}$ component)

$$\ddot{\theta}(t = 0) = 0 \tag{5.2.10}$$

i.e., the motion will stay forever in the $x - y$ plane. This greatly simplifies the problem because it means that motion under a central force is effectively two dimensional. The relevant two components of the equation of motion give

$$m(\ddot{r} - r\dot{\varphi}^2) = F(r)$$

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \tag{5.2.11}$$

Of course, if the initial velocity of the particle is such that $\dot{\varphi}(t = 0) = 0$ then the second equation gives $\ddot{\varphi}(t = 0) = 0$, which means that the motion is forever radial or one dimensional, but this is not the most general situation because it involves a restriction on the initial velocity. On the other hand, because it is always possible to orient the $x - y$ plane so that it coincides with the plane defined by the radial vector and the (arbitrary) initial velocity the motion will always be at most two dimensional.

The second of (5.2.11) can be integrated quite easily and we find

$$\frac{2\dot{r}}{r} = -\frac{\ddot{\varphi}}{\dot{\varphi}} \Rightarrow \ln r^2 = \ln \frac{L}{m\dot{\varphi}}, \tag{5.2.12}$$

where we have written the integration constant in the form L/m. A more convenient way to express the solution is

$$mr^2\dot{\varphi} = L, \tag{5.2.13}$$

which should make it apparent that L is, in fact, the angular momentum of the body. Solving for $\dot{\varphi}$ and inserting the solution into the first equation in (5.2.11) gives

$$m\ddot{r} - \frac{L^2}{mr^3} = F \tag{5.2.14}$$

which is an equation for $r = r(t)$. The second term on the right, $-L^2/mr^3$, is the mass times the centripetal acceleration required to keep the body moving around the center of the force. The equation can be integrated once,

$$m\ddot{r} = m\dot{r}\frac{d\dot{r}}{dr} = F + \frac{L^2}{mr^3} \quad \Rightarrow \quad \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \int F\,dr = E \tag{5.2.15}$$

where E is also an integration constant, which will be recognized as the total energy of the body . The first term on the left is the radial kinetic energy. The second is the rotational kinetic energy and the last term is the potential energy associated with the central force,

$$\phi(r) = -\int^r F(r')dr'. \tag{5.2.16}$$

Thus the quantity

$$V(r) = \phi(r) + \frac{L^2}{2mr^2} \tag{5.2.17}$$

has the form of an effective potential energy in (5.2.15). To understand its origin, turn again to the equation of motion (5.2.14) and write it as

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3} \tag{5.2.18}$$

The extra "force" on the right hand side is directed outward and corresponds to the fictitious centrifugal force that an observer on the rotating body would feel. It corresponds to a potential

$$\phi'(r) = -\int^r dr' \frac{L^2}{mr'^3} = \frac{L^2}{2mr^2} \tag{5.2.19}$$

showing that $V(r)$ is the effective potential energy measured by the observer who is attached to the orbiting body. The energy equation,

$$\dot{r} = \sqrt{\frac{2}{m}[E - V(r)]}, \tag{5.2.20}$$

can be formally integrated, to give

$$\int_{r_0}^{r} \frac{dr'}{\sqrt{E - V(r')}} = \sqrt{\frac{2}{m}}(t - t_0) \tag{5.2.21}$$

Of course, the problem is to perform the integral on the left. If this is done, we obtain the radius of the body from the force center as a function of time, $r = r(t)$, and then $\varphi = \varphi(t)$ directly from the first equation in (5.2.11). Thus,

$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{L}{mr^2} \Rightarrow \varphi - \varphi_0 = \frac{L}{m}\int_{t_0}^{t}\frac{dt'}{r^2(t')} \tag{5.2.22}$$

In principle, this solves the central force problem.

Parametric solutions, with time as the parameter, are difficult to visualize. If possible, in two dimensions it's desirable to obtain the solution in function form. In this particular case, it means that we should seek a solution of the form $r = r(\theta)$. rewrite (5.2.20) as

$$\dot{r} = \frac{dr}{d\varphi}\dot{\varphi} = \frac{L}{mr^2}\frac{dr}{d\varphi} = \sqrt{\frac{2}{m}[E - V(r)]} \tag{5.2.23}$$

or

$$\int_{r_0}^{r}\frac{dr/r^2}{\sqrt{E - V(r)}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0) \tag{5.2.24}$$

where (r_0, φ_0) are the polar coordinates of the object at the initial time. If we could solve the integral on the left we would get a relationship between r and φ, which would have to be inverted to obtain $r = r(\varphi)$.

5.3 Inverse square force

For an inverse square central force,

$$F(r) = \frac{k}{r^2}, \tag{5.3.1}$$

where k is positive for a repulsive force and negative for an attractive force (such as gravity) the potential energy is

$$\phi(r) = -\int^{r} dr' F(r') = \frac{k}{r} \tag{5.3.2}$$

and the integral equation becomes

$$\int_{r_0}^{r} \frac{dr/r^2}{\sqrt{E - \frac{k}{r} - \frac{L^2}{2mr^2}}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0) \tag{5.3.3}$$

Now if we substitute $u = 1/r$, we have

$$-\int_{u_0}^{u} \frac{du}{\sqrt{E - ku - \frac{L^2 u^2}{2m}}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0) \tag{5.3.4}$$

The integral on the left is easily solved by completion of squares and further substitutions. Completing squares first, express the equation as

$$-\int_{u_0}^{u} \frac{du}{\sqrt{\left(E + \frac{mk^2}{2L^2}\right) - \left(\frac{Lu}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\frac{k}{L}\right)^2}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0) \tag{5.3.5}$$

Evidently, for real solutions, we require

$$\mathcal{E} = E + \frac{mk^2}{2L^2} > 0 \tag{5.3.6}$$

so let

$$w = \frac{Lu}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\frac{k}{L}, \quad dw = \frac{Ldu}{\sqrt{2m}} \tag{5.3.7}$$

Then

$$-\int_{w_0}^{w} \frac{dw'}{\sqrt{\mathcal{E} - w'^2}} = (\varphi - \varphi_0) \tag{5.3.8}$$

and the integral on the left (substitute $w = \mathcal{E}\cos\eta$) is solved to give

$$\cos^{-1}\frac{w}{\mathcal{E}} - \cos^{-1}\frac{w_0}{\mathcal{E}} = (\varphi - \varphi_0) \tag{5.3.9}$$

or

$$w = \frac{Lu}{\sqrt{2m}} + \sqrt{\frac{m}{2}}\frac{k}{L} = \sqrt{\mathcal{E}}\cos(\varphi - \overline{\varphi}_0) \tag{5.3.10}$$

where φ_0 has been redefined to accommodate the constant term arising in the integral on the left. Solving for u we find

$$u = \frac{1}{r} = \frac{m|k|}{L^2}\left[-\text{sgn}(k) + \sqrt{1 + \frac{2EL^2}{mk^2}}\cos(\varphi - \overline{\varphi}_0)\right] \tag{5.3.11}$$

where $\text{sgn}(k)$ is just the sign of the constant k^3. The solution has the form

$$\frac{\alpha}{r} = -\text{sgn}(k) + \varepsilon \cos\varphi \qquad (5.3.12)$$

where

$$\alpha = \frac{L^2}{m|k|}, \quad \varepsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}$$

Consider first the case of an attractive inverse square central force, for which $k < 0$. The solution,

$$\frac{\alpha}{r} = 1 + \varepsilon \cos(\varphi - \overline{\varphi}_0), \qquad (5.3.13)$$

is described by two parameters which depend on the strength of the central force, k, and the two integration constants, *viz.*, the total energy and the angular momentum.

A very special case occurs when $E = -mk^2/2L^2$, for then $\varepsilon = 0$ and the motion becomes circular, with radius $\alpha = L^2/m|k|$. This result can also be obtained by setting $\ddot{r}$ in (5.2.14) to zero,

$$\ddot{r} = 0 = -\frac{|k|}{r^2} + \frac{L^2}{mr^3} \quad \Rightarrow \quad r = \frac{L^2}{m|k|} = \alpha. \qquad (5.3.14)$$

The energy of such an orbit is just the effective potential energy,

$$E = V(r)|_{r=\alpha} = \left[-\frac{|k|}{r} + \frac{L^2}{2mr^2} \right]_{r=\alpha} = -\frac{mk^2}{2L^2} \qquad (5.3.15)$$

In general the motion is a conic section. In the following section (detour) we obtain the equations of conic sections in the form (5.3.13). This will allow us to interpret the constants α and ε and so to describe the motion in detail as it depends on the two integration constants E and L.

5.3.1   Conic sections

A **conic section** is a curve that is formed by the intersection of a cone with a plane, as shown in figure (5.3).

- The **circle** is just the locus of points whose distance from a fixed point called the **center**) is constant. If we locate the fixed point at the origin of coordinates and call the constant distance from the fixed point a, then the equation of a circle is simply $r = a$.

---

[3]$\text{sgn}(k)$ is defined by

$$\text{sgn}(k) = \begin{cases} 1 & k > 0 \\ -1 & k < 0 \end{cases}.$$
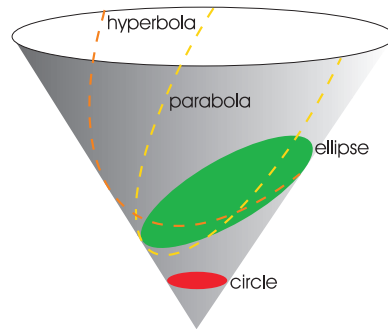
Figure 5.3: Conic sections.

- The **ellipse** is the locus of all points which are such that the sum of their distances from two fixed points (called the **foci**) is constant (see figure (5.4). Imagine that we place the origin of coordinates at one of the foci, say F, and measure the polar angle φ as shown in the figure. From the figure we see that

$$r + r' = 2a \qquad (5.3.16)$$

where we set the value of the constant to be $2a$. It is also clear that

$$
\begin{aligned}
r'^2 &= [2a\varepsilon + r\cos\varphi]^2 + r^2\sin\varphi^2 \\
&= r^2 + 4a^2\varepsilon^2 + 4a\varepsilon r\cos\varphi \qquad (5.3.17)
\end{aligned}
$$

Therefore, the equation of the ellipse should be

$$r + \sqrt{r^2 + 4a^2\varepsilon^2 + 4a\varepsilon r\cos\varphi} = 2a \qquad (5.3.18)$$

which can be simplified to give

$$r^2 + 4a^2 - 4ar = r^2 + 4a^2\varepsilon^2 + 4a\varepsilon r\cos\varphi \qquad (5.3.19)$$

or

$$4a^2(1 - \varepsilon^2) = 4ar(1 + \varepsilon\cos\varphi), \qquad (5.3.20)$$

which can be put in the canonical form

$$\frac{\alpha}{r} = \frac{a(1 - \varepsilon^2)}{r} = 1 + \varepsilon\cos\varphi \qquad (5.3.21)$$

We see that $\varphi = 0$ represents the point of closest approach, the **periapsis**, to the focus serving as the force center, with $r = a(1 - \varepsilon)$. On the other hand, $\varphi = \pi$ represents the point farthest from the force center, the **apoapsis**, with $r = a(1 + \varepsilon)$.
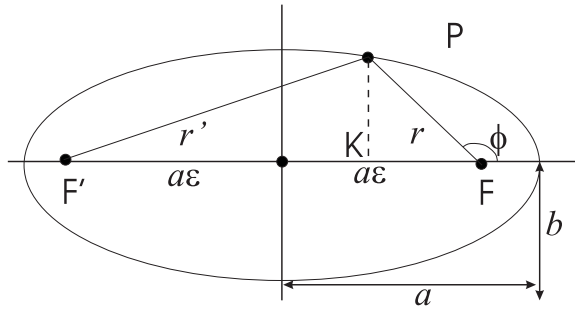
Figure 5.4: The ellipse.

- The **parabola** is the locus of points whose distance from a fixed point (called the **focus**) is equal to the distance from a fixed line (the **directrix**). Referring to figure (5.5), and once again placing the origin of coordinates at the (single) focus of the parabola, we find

$$r = a - r \cos \varphi \qquad (5.3.22)$$

or

$$\frac{a}{r} = 1 + \cos \varphi \qquad (5.3.23)$$

The periapsis has $\varphi = 0$, with $r = a/2$.

- The **hyperbola** is the locus of points which are such that the *difference* of their distances from two fixed points (called the **foci**) is constant (see figure (5.6). As before, we pick the origin at one of the foci, then referring to the figure we see that the constraint can be written as

$$r' - r = 2a \qquad (5.3.24)$$

Again because

$$r'^2 = r^2 \sin^2 \varphi + (2a\varepsilon - r \cos \varphi)^2 = r^2 + 4a^2\varepsilon^2 - 4ar\varepsilon \cos \varphi \qquad (5.3.25)$$

we have

$$\sqrt{r^2 + 4a^2\varepsilon^2 - 4ar\varepsilon \cos \varphi} - r = 2a$$

$$\Rightarrow \qquad 4a^2\varepsilon^2 - 4ar\varepsilon \cos \varphi = 4a^2 + 4ar\varepsilon$$

$$\Rightarrow \qquad \frac{\alpha}{r} = \frac{a(\varepsilon^2 - 1)}{r} = 1 + \varepsilon \cos \varphi \qquad (5.3.26)$$

The periapsis has $\varphi = 0$ with $r = a(\varepsilon - 1)$.
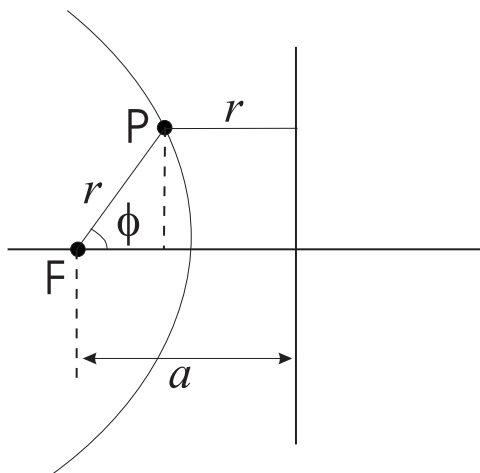
Figure 5.5: The parabola.
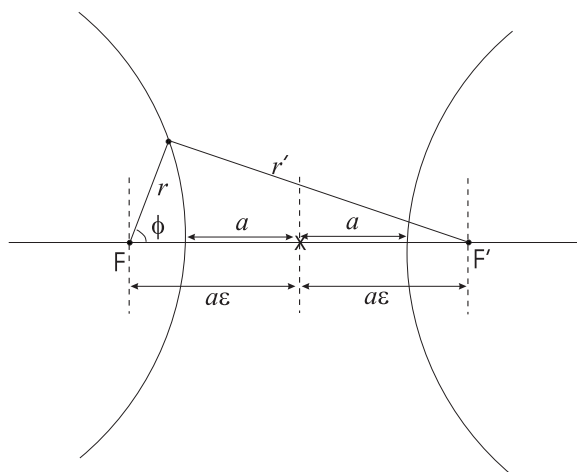
Figure 5.6: The Hyperbola.

In all cases, the conic section depends on two parameters, *viz.*, (a) the constant α and the **eccentricity** $\varepsilon \geq 0$. The circle has zero eccentricity, for the ellipse the eccentricity is less than unity, for the parabola it is precisely unity and for the hyperbola it is greater than unity.

5.3.2 Analysis of solutions

Returning to our general solutions in (5.3.13) with $k < 0$, we see immediately that

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{mk^2}}, \quad \text{and}$$

$$a|1 - \varepsilon^2| = \frac{L^2}{m|k|} \quad \Rightarrow \quad a = \left| \frac{k}{2E} \right|, \quad E \neq 0 \tag{5.3.27}$$

so that the geometric parameters (α and ε) are given completely by the physical parameters, $|k|$, E and L of the system. The constant $\overline{\varphi}_0$ is seen to determine the orientation of the $x-$axis relative to the conic section and may always be chosen to be identically zero. Several cases arise. We have already seen that the orbit is circular if $E = -mk^2/2L^2$. For other values of E we also have:

- Case $E < 0$: If the total energy is negative, then $\varepsilon < 1$ and the orbit is elliptical. The length of the semi-major axis is

$$2a = \left| \frac{k}{E} \right| \tag{5.3.28}$$

  and the length of the semi-minor axis is obtained by looking at the point P located on the intersection of the curve and the vertical axis. For this point, $r = r'$ gives $r = r' = a$, $\cos \phi_P = -\varepsilon$ and

$$2b = 2a\sqrt{1 - \varepsilon^2} = \sqrt{\frac{2L^2}{m|E|}} \tag{5.3.29}$$

  The orbit is closed and the motion periodic.

- Case $E = 0$: If $E = 0$ then $\varepsilon = 1$ and the motion is parabolic, with

$$a = \frac{L^2}{m|k|} \tag{5.3.30}$$

  The orbit is open, not closed. It is a limiting case, the boundary between closed, elliptic orbits and open, hyperbolic orbits. The orbiting body returns after an infinite time.
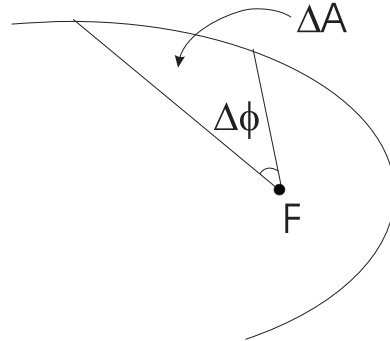
Figure 5.7: Kepler's second law.

- Case $E > 0$: When the total energy is positive the eccentricity is larger than unity and the orbit is hyperbolic. The orbiting body never returns to the neighborhood of the force center, F.

Finite period orbits occur only when $E < 0$, *i.e.,* when the motion is elliptical. We can think of the circle as an ellipse of zero eccentricity, therefore for circular orbits of radius $a = |k/2E|$, there must be a precise relationship between the energy and the angular momentum, $L^2 = mk^2/2|E|$.

5.3.3   Kepler's laws

Kepler's first law simply states that the orbits of the planets about the sun are elliptical with the sun at one of the foci. We have already shown that this must be the case for periodic orbits.

The second law states that the areas swept out by the radius vectors from the sun to the planet in equal time intervals are equal. Putting this statement in the language of calculus, we can say that

$$\frac{dA}{dt} = \text{constant} \tag{5.3.31}$$

where A is the area swept out by the radius vectors from the sun to the planet. This is not difficult to prove and is actually a statement about the conservation of angular momentum! Referring to the figure (5.7),

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\varphi}{dt} = \frac{L}{2m} \tag{5.3.32}$$

which is a constant of the motion.

Kepler's third law gives the relationship between the period of the orbit and the length of the semi-major axis. It states that the square of the period is proportional to the cube of the semi-major axis, *i.e.*, $\tau^2 \propto a^3$, where τ is the period, or the time taken to complete one orbit. Proving this takes a bit of work, but this is the law that motivated Newton to describe gravity by an inverse square central force. Begin by noting that integrating the second law over an entire period gives the area of the ellipse

$$A = \frac{L\tau}{2m} \tag{5.3.33}$$

However, the area of the ellipse is obtained by integrating the expression

$$dA = \frac{1}{2}r^2 d\varphi \tag{5.3.34}$$

over the entire ellipse. Using the solutions obtained,

$$A = \frac{\alpha^2}{2} \int_0^{2\pi} \frac{d\varphi}{(1 + \varepsilon \cos \varphi)^2} = \pi a^2 \sqrt{1 - \varepsilon^2} \tag{5.3.35}$$

Therefore,

$$\frac{L\tau}{2m} = \pi a^2 \sqrt{1 - \varepsilon^2} \quad \Rightarrow \quad \tau = \pi \sqrt{\frac{mk^2}{2|E|^3}}, \tag{5.3.36}$$

the right hand side of which can be expressed in terms of the length of the semi-major axis, using (5.3.27),

$$\tau^2 = \frac{4\pi^2 m}{k} a^3 \tag{5.3.37}$$

For the gravitational force, the constant k is proportional to the mass of the body, which is just a statement of the weak equivalence principle. The proportionality constant between the square of the radius and the cube of the semi-major axis is therefore independent of the mass of the orbiting body. For instance, for a star-planet system, assuming that the mass of the star, $M_\odot$, is very much larger than the mass of the planet so that it is for all practical purposes fixed,

$$k = GM_\odot m \quad \Rightarrow \quad \tau^2 = \frac{4\pi^2}{GM_\odot} a^3 \tag{5.3.38}$$

When the mass of the planet is not negligible compared to the mass of the star, we may still use the description of the motion as given above, provided we interpret the results consistently. In that case, the coordinate r should be understood as the magnitude of the relative coordinate, $\vec{r} = \vec{r}_p - \vec{r}_\odot$, the mass of the orbiting body, m, should be replaced by the reduced mass, μ, of the system and the mass of the star by the total mass. The

force center is then the center of mass of the system. Because no external forces act on an isolated system, the center of mass moves as though it were a free particle. In other words, the star-planet system may move as a whole freely through space while each member of the system orbits about the center of mass.

5.4   Other examples of central forces

The inverse square force is by no means the only physically important example of a central force. The force exerted by a spring that is fixed at one end, for example, is a central force for which $F(r) = -k(r - r_0)$, where r_0 is the equilibrium length of the spring and k is the spring constant. The negative sign indicates that the force is attractive so long as $r > r_0$ (the spring is stretched) and repulsive when $r < r_0$. The corresponding potential energy is evidently

$$\phi(r) = -\int_{r_0}^{r} F(r)dr = \frac{1}{2}k(r - r_0)^2. \tag{5.4.1}$$

Another important example of a central force is the Yukawa force,

$$F(r) = \frac{k}{r^2}e^{-r/r_0}\left[1 + \frac{r}{r_0}\right], \tag{5.4.2}$$

which is often used to describe the effective weak interactions between mesons. It is approximately inverse square so long as $r/r_0 \ll 1$, but decreases rapidly at large distances, *i.e.,* when $r/r_0 \gg 1$. It is deemed a "short-range" force, as opposed to the inverse-square force, which is considered "long range". The potential energy corresponding to this force is the Yukawa potential,

$$\phi(r) = -\int_{\infty}^{r} F(r)dr = \frac{k}{r}e^{-r/r_0}. \tag{5.4.3}$$

There are also generalizations of the inverse square force, with $F(r) = k/r^n$, where n is a positive integer. The potential energy is

$$\phi(r) = -\int_{\infty}^{r} F(r)dr = \frac{k}{(n-1)r^{n-1}} \tag{5.4.4}$$

For $n > 2$, such terms arise, for example, as general relativistic corrections to Newton's inverse square law of gravity, but they are not so interesting on their own.

For each of the forces above, our starting point will be (5.2.24). As an example, consider Hooke's Law. For simplicity, take $r_0 = 0$, then the effective potential in this case is

$$V(r) = \frac{L^2}{2mr^2} + \frac{1}{2}kr^2 \tag{5.4.5}$$

and (5.2.24) determines the two dimensional motion according to

$$\int_{r_0}^{r} \frac{dr/r^2}{\sqrt{E - \frac{L^2}{2mr^2} - \frac{1}{2}kr^2}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0). \tag{5.4.6}$$

The problem is to evaluate the integral on the left. It is convenient to make the substitution $u = 1/r^2$, which gives

$$-\frac{1}{2}\int_{u_0}^{r} \frac{du}{\sqrt{Eu - \frac{L^2 u^2}{2m} - \frac{k}{2}}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0) \tag{5.4.7}$$

or, completing squares,

$$-\frac{1}{2}\int_{u_0}^{u} \frac{du}{\sqrt{(\frac{mE^2}{2L^2} - \frac{k}{2}) - (\sqrt{\frac{m}{2}}\frac{E}{L} - \frac{Lu}{\sqrt{2m}})^2}} = \sqrt{\frac{2m}{L^2}}(\varphi - \varphi_0) \tag{5.4.8}$$

Evidently, for real solutions,

$$\mathcal{E} = \frac{mE^2}{2L^2} - \frac{k}{2} > 0 \tag{5.4.9}$$

and substituting $w = \sqrt{\frac{m}{2}}\frac{E}{L} - \frac{Lu}{\sqrt{2m}}$ gives

$$\frac{1}{2}\int_{w_0}^{w} \frac{dw}{\sqrt{\mathcal{E} - w^2}} = (\varphi - \varphi_0) \Rightarrow w = -\sqrt{\mathcal{E}}\cos 2(\varphi - \overline{\varphi}_0) \tag{5.4.10}$$

where we have redefined φ_0 to accommodate the additional constant obtained by integrating the left hand side. In terms of r, we find a solution of the form

$$\frac{\alpha^2}{r^2} = 1 + \varepsilon \cos 2(\varphi - \overline{\varphi}_0) \tag{5.4.11}$$

where

$$\alpha = \frac{L}{\sqrt{mE}}, \quad \varepsilon = \sqrt{1 - \frac{kL^2}{mE^2}} \tag{5.4.12}$$

and circular orbits are possible if $E = (kL^2/m)^{1/2}$ (*i.e.,* when $\varepsilon = 0$), in which case the radius of a circular orbit is $r = (L^2/mk)^{1/4}$. This is the minimum permissible energy for two dimensional motion.[4]

The solution in (5.4.11) describes an ellipse provided that $\varepsilon < 1$. This can be made explicit by making the transformations $x = r\cos\varphi$, $y = r\sin\varphi$ and taking $\overline{\varphi}_0 = 0$ for

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[4]Problem: Determine the energy and radius of a circular orbit in terms of the angular momentum using the condition $\ddot{r} = 0$ in (5.2.14).

convenience (this amounts only to choosing an orientation of the axes for the Cartesian system so that the periapsis occurs at $\varphi = 0$). We find

$$\frac{\alpha^2}{r^2} = 1 + \varepsilon \cos 2\varphi \quad \Rightarrow \quad \frac{x^2}{\alpha^2/(1+\varepsilon)} + \frac{y^2}{\alpha^2/(1-\varepsilon)} = 1, \qquad (5.4.13)$$

showing that the center of the force is at the center of the ellipse.[5]

Kepler's second law is true for elliptical orbits caused by *any* central force because it is just a restatement of the law of conservation of momentum. The period, however, does depend on the particular form of the force. In this case, it turns out to be independent of the length of the major axis. We find,

$$A = \frac{L\tau}{2m} = \frac{\alpha^2}{2} \int_0^{2\pi} \frac{d\varphi}{(1 + \varepsilon \cos 2\varphi)} = \frac{\pi \alpha^2}{\sqrt{1 - \varepsilon^2}} = \frac{\pi L}{\sqrt{km}} \qquad (5.4.14)$$

gives

$$\tau = 2\pi \sqrt{\frac{m}{k}}, \qquad (5.4.15)$$

which is precisely the period of one dimensional oscillations.

5.5   Stability of Circular Orbits

We have seen that the equations describing the motion of an object under the action of a central force are

$$m(\ddot{r} - r\dot{\varphi}^2) = F(r)$$

$$2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0 \qquad (5.5.1)$$

We also discovered that the second is integrable, giving a conshtant of the motion, the angular momentum, and that it can be combined with the first to give one equation for the radial motion

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3} \qquad (5.5.2)$$

---

[5]Problem: By transforming to Cartesian coordinates, show that the orbit given by

$$\frac{a(1 - \varepsilon^2)}{r} = 1 + \varepsilon \cos \varphi$$

$(\varepsilon < 1)$ is an ellipse, but that the force center is located at one focus of the ellipse in this case. Show also that the distance from the center of the ellipse to the focus is $a\varepsilon$. **Hint:** Choose a convenient orientation of the Cartesian axes by selecting $\overline{\varphi}_0$, then show that the equation, written in Cartesian coordinates, is the equation of an ellipse, horizontally shifted a distance $-a\varepsilon$.

Now it is quite clear that, no matter what the form of $F(r)$, there will always be circular orbits so long as $\vec{F}$ is attractive: simply solve

$$F(r) = -\frac{L^2}{mr^3} \tag{5.5.3}$$

to get the radius of the circle. This is of course just the condition that the effective potential has an extremum at r_0 because

$$V'(r_0) = \phi'(r_0) - \frac{L^2}{mr_0^3} = 0 \Rightarrow F(r_0) = -\frac{L^2}{mr^3}. \tag{5.5.4}$$

Let us examine the stability of these circular orbits, first in general and then for some particular cases. An orbit will be considered stable if it does not run away, or change drastically when it is perturbed by some external influence. We will make this intuitive idea more precise below.

Suppose that the solution to (5.5.3) is r_0, then imagine that we perturb the orbit slightly so that $r = r_0 + x$, where $x/r_0 \ll 1$. Evidently, $\ddot{r} = \ddot{x}$ and the equation of motion becomes

$$m\ddot{x} = F(r_0 + x) + \frac{L^2}{m(r_0 + x)^3} \tag{5.5.5}$$

Since $x \ll r_0$, we expand in a Taylor series about $x = 0$, retaining only the first order term in x/r_0. Then

$$m\ddot{x} = F(r_0) + \frac{L^2}{mr_0^3} + F'(r_0)x - \frac{3L^2 x}{mr_0^4} + \ldots \tag{5.5.6}$$

The first two terms on the right vanish because r_0 is the radius of the circular orbit [see (5.5.3)]. Then the equation for the perturbation, x, becomes

$$m\ddot{x} = -\left(\frac{3L^2}{mr_0^4} - F'(r_0)\right) x = -\kappa x \tag{5.5.7}$$

and may be compared to the equation of a one dimensional harmonic oscillator, with force constant κ. We understand these solutions quite well. If the force constant is positive there are oscillations about $x = 0$, but if the force constant is negative then x may grow exponentially, without bound. The perturbations continue small when the κ is positive and that is therefore our condition for stability. It cannot have escaped you that we can think of this condition as the requirement that the effective potential has a well defined minimum at $r = r_0$ (not simply an extremum) because

$$V''(r_0) = -F'(r_0) + \frac{3L^2}{mr_0^4} \tag{5.5.8}$$

which we have just seen must be *larger* than zero for stability.

As an example, if

$$F(r) = -\frac{k}{r^n} \tag{5.5.9}$$

the radius of the circular orbits would be given by

$$\frac{k}{r_0^n} = \frac{L^2}{mr_0^3} \Rightarrow r_0^{n-3} = \frac{mk}{L^2} \tag{5.5.10}$$

and in particular when $n = 2$ we have

$$r_0 = \frac{L^2}{mk} \tag{5.5.11}$$

Are these circular orbits stable? Let us examine the condition for stability:

$$\frac{3L^2}{mr_0^4} - \frac{nk}{r_0^{n+1}} > 0 \Rightarrow r_0^{n-3} > \frac{nmk}{3L^2} \tag{5.5.12}$$

or simply (inserting the value of r_0 found above) $n < 3$. In this case, any perturbation causes the body to oscillate about the stable circular orbit with an angular frequency of

$$\omega = \frac{L}{mr_0^2}\sqrt{3-n} \tag{5.5.13}$$

The solution for x, up to this order of approximation is then

$$x = A_1 \cos \omega t = A_1 \cos \left(\sqrt{3-n}\,\varphi\right) \tag{5.5.14}$$

where we used (5.2.13) and $r = r_0$. Clearly, for *closed* orbits we must ask for

$$\sqrt{3-n} = \frac{p}{q} = \beta \tag{5.5.15}$$

where p/q is a rational number. It this way the perturbation x would return to itself after q rotations through an angle of 2π. Therefore, for rational β and

$$F(r) = -\frac{k}{r^{3-\beta^2}} \tag{5.5.16}$$

the circular orbits are not only stable against small perturbations, but they are also closed.

5.5.1 Bertand's Theorem

If we seek more stringent conditions on the force law, so orbits remain closed even when the deviations from circularity are large (*i.e.,* if we wish to beyond the linear approximation) then the analysis is more complicated. It is convenient to rewrite the radial equation of motion in (5.2.11) in terms of $u(\varphi) = 1/r(\varphi)$ as

$$\frac{d^2u}{d\varphi^2} + u = -\frac{m}{L^2}\frac{d\phi(u)}{du} \stackrel{\text{def}}{=} Q(u) \tag{5.5.17}$$

Perfectly circular orbits are recovered when $u'(\varphi) = 0$, or $u_0 = Q(u_0)$. First consider small deviations from circularity, $u = u_0 + u(\varphi)$; in the linear approximation,

$$\frac{d^2y}{d\varphi^2} + u_0 + y = Q(u_0) + yQ'(u_0) \tag{5.5.18}$$

or

$$\frac{d^2y}{d\varphi^2} + [1 - Q'(u_0)]y = 0 \tag{5.5.19}$$

whose solution is clearly $y = A\cos(\beta\varphi)$, where A is an integration constant and $\beta^2 = 1 - Q'(u_0)$ must be greater than zero for a stable orbit and rational for closed orbits. This is just the result we had before.[6]

To go beyond the linear approximation, we retain higher order terms in the series expansion of $Q(u)$,

$$\frac{d^2y}{d\varphi^2} + \beta^2 y = \frac{1}{2!}Q''(u_0)y^2 + \frac{1}{3!}Q'''(u_0)y^3 + \dots \tag{5.5.20}$$

Now the behavior of $y(\varphi)$ to linear order is the first term of a Fourier expansion in $\beta\varphi$. We therefore seek a closed orbit solution by including more terms of the Fourier series, taking

$$y(\varphi) = A_0 + A_1\cos\beta\varphi + A_2\cos 2\beta\varphi + A_3\cos 3\beta\varphi + \dots \tag{5.5.21}$$

The coefficients A_0 and A_2 are of higher order (smaller) than A_1, as can be seen in the approach to circularity. Moreover, A_3 will be smaller than A_0 and A_2, A_4 will be smaller than A_3 and so on (we will see this shortly). Therefore, in the expansion (5.5.20), if we include terms up to $\cos(3\beta\varphi)$ in y (third order), then we include only terms up to $\cos(2\beta\varphi)$ in y^2 and $\cos(\beta\varphi)$ in y^3. Make use of the fact that the set

$$\frac{1}{\sqrt{2}}, \quad \left\{\cos(n\beta\varphi) \mid n \in \mathbb{Z}^+\right\}$$

---

[6]Taking $F(r) = k/r^n$, determine (a) $Q(u)$, (b) the radius of circular orbits and (c) β. Confirm the earlier results.

is an orthonormal and complete basis for even, periodic functions, with inner product

$$\langle \cos(n\beta\varphi), \cos(m\beta\varphi) \rangle \overset{\text{def}}{=} \frac{\beta}{\pi} \int_0^{2\pi/\beta} d\varphi \left[\cos(n\beta\varphi), \cos(m\beta\varphi)\right] = \delta_{mn}, \qquad (5.5.22)$$

to reduce higher powers of $\cos(n\beta\phi)$ and products as, for example,

$$\cos^2(\beta\varphi) = \frac{1}{2}\langle \cos^2 \beta\varphi, 1 \rangle + \sum_n \langle \cos^2 \beta\varphi \cos(n\beta\varphi) \rangle \cos(n\beta\varphi) = \frac{1}{2}\left(1 + \cos(2\beta\varphi)\right)$$

$$\cos^3(\beta\varphi) = \frac{1}{4}\left(3\cos(\beta\varphi) + \cos(3\beta\varphi)\right)$$

$$\cos(\beta\varphi)\cos(2\beta\varphi) = \frac{1}{2}\left(\cos(\beta\varphi) + \cos(3\beta\varphi)\right) \qquad (5.5.23)$$

and so on. Carefully expanding (5.5.20), we find

$$A_0 = \frac{A_1^2}{4\beta^2}Q''(u_0)$$

$$A_2 = -\frac{A_1}{12\beta^2}Q''(u_0)$$

$$A_3 = -\frac{1}{16\beta^3}\left[A_1 A_2 Q''(u_0) + \frac{A_1^3}{12}Q'''(u_0)\right] \qquad (5.5.24)$$

as well as one constraint,

$$-A_0 A_1 Q''(u_0) - \frac{1}{2}A_1 A_2 Q''(u_0) - \frac{1}{8}A_1^3 Q'''(u_0) = 0. \qquad (5.5.25)$$

We have already seen that $F(r) = -k/r^{3-\beta^2}$ is necessary for stable orbits. This means that

$$Q(u) = \frac{mk}{L^2}u^{1-\beta^2} \qquad (5.5.26)$$

and therefore the inverse radius of circular orbits is given by $u_0^{\beta^2} = mk/L^2$. Then

$$Q''(u_0) = -\frac{\beta^2(1-\beta^2)}{u_0}$$

$$Q'''(u_0) = \frac{\beta^2(1-\beta^4)}{u_0^2}, \qquad (5.5.27)$$

and inserting (5.5.24) and (5.5.27) into (5.5.25) shows that β must satisfy the (much stronger) condition

$$\beta^2(1-\beta^2)(4-\beta^2) = 0 \qquad (5.5.28)$$
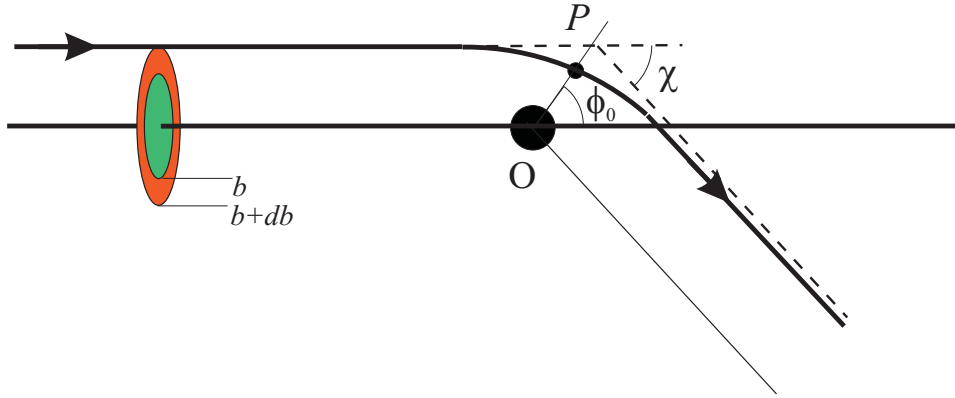
Figure 5.8: Scattering in the CM frame for an attractive central force.

For $\beta \neq 0$, *i.e.,* for non-vanishing deviations from a circular orbit, the only possibilities are $\beta^2 = 3 - n = 1, 4$ meaning that $n = 2, -1$ respectively, which are the inverse square force law and Hooke's law.[7]

5.6   Scattering by a Central Force

So far we have considered only bound orbits. Let us now consider unbound, hyperbolic orbits about a force center as shown in figure 5.8, where O is the scattering center (the center of mass) and P represents the (fictitious) particle of (reduced) mass $\mu = m_1 m_2/(m_1 + m_2)$ and coordinate $\vec{r} = \vec{r}_2 - \vec{r}_1$ of body "2" relative to body "1" at the point of closest approach (the periapsis).

The equation of motion is given in (5.5.17), with m replaced by μ and we consider an attractive, inverse square force for which $Q(u) = \frac{\mu|k|}{L^2}$. The solution possesses a positive energy, E, so $\varepsilon > 1$ in

$$u = \frac{1}{\alpha} \left[1 + \varepsilon \cos(\varphi - \overline{\varphi}_0) \right]. \tag{5.6.1}$$

Extremizing u we find that

$$u' = -\frac{\varepsilon}{\alpha} \sin(\varphi - \overline{\varphi}_0) = 0 \;\Rightarrow\; \varphi = \overline{\varphi}_0, \;\; u''(\overline{\varphi}_0) < 0, \tag{5.6.2}$$

so $\overline{\varphi}_0$ maximizes u (u has no minimum) and minimizes r, showing that $\overline{\varphi}_0$ is the angular coordinate of the periapsis. To find $\overline{\varphi}_0$ we refer to figure 5.8, noting that, at early times,

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[7]Notice that $A_0/A_1 \sim A_2/A_1 \sim A_1/u_0 \ll 1$ by assumption and $A_3/A_1 \sim (A_1/u_0)^2$ which is smaller still. This justifies the approximation made.

$u(\pi) = 0$ implies that $\overline{\varphi}_0 = \cos^{-1}(1/\varepsilon)$. Thus the radial distance of the periapsis from the scattering center will be $r_P = \alpha/(1+\varepsilon) = a(\varepsilon - 1)$. The total deflection angle is given by the condition that $u = 0$ and $\varphi \neq \pi$, *i.e.*, referring to figure 5.8,

$$\varphi_\pm - \overline{\varphi}_0 = \pm \cos^{-1}\left(-\frac{1}{\varepsilon}\right) \quad \Rightarrow \quad \varphi_\pm = \cos^{-1}\left(\frac{1}{\varepsilon}\right) \pm \cos^{-1}\left(-\frac{1}{\varepsilon}\right). \tag{5.6.3}$$

The solution with the positive sign yields π, which represents the situation at early times, so we pick the second solution

$$\chi = \varphi_- = \cos^{-1}\left(\frac{1}{\varepsilon}\right) - \cos^{-1}\left(-\frac{1}{\varepsilon}\right) = -2\sin^{-1}\left(\frac{1}{\varepsilon}\right), \tag{5.6.4}$$

as representative of the situation at late times. χ is the scattering angle. It depends exclusively on the eccentricity.[8] Larger eccentricities are scattered through smaller angles, but because the eccentricity depends on the energy and the angular momentum,

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{\mu k^2}}, \tag{5.6.5}$$

the higher the total energy and total angular momentum, the smaller will be the deflection angle.

5.6.1   Differential Cross-Section

A problem of physical interest concerns the deflection of a beam of particles by a scattering center, as in the case, for example, of Rutherford scattering. Typically, one is interested in the differential scattering cross-section, which is the ratio of the number of particles scattered in a solid angle between Ω and $\Omega + d\Omega$ per unit time by the total number of incident particles per unit area per unit time (flux),

$$d\sigma = \frac{\text{\# of particles scattered in } d\Omega}{\text{incident flux}} \tag{5.6.6}$$

The mechanical dimension of the differential cross section is area. It may be thought of as the area, transverse to the relative velocity, within which the interacting objects must meet in order to scatter through a given solid angle. The differential solid angle is $d\Omega = 2\pi \sin\chi\, d\chi$. If v_∞ is the initial relative speed far from the scattering center, so that the initial relative velocity is $\vec{v} = v_\infty \hat{x}$ in figure 5.8, we express the angular momentum and the total energy as

$$L = mv_\infty b, \quad E = \frac{1}{2}\mu v_\infty^2. \tag{5.6.7}$$

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[8]<u>Problem:</u> Perform a similar analysis for a repulsive central force, $Q = -\frac{\mu|k|}{L^2}$. Show that $\chi = +2\sin^{-1}(1/\varepsilon)$.

For particles of the same incident energy, the scattering angle χ depends only on the angular momentum and therefore only on b, then the number particles scattered between χ and $d\chi$ will pass through a ring of radius between b and $b + db$ (again, refer to the figure), so $d\sigma = 2\pi b\, db$, *i.e.*,

$$\frac{d\sigma}{d\Omega} = \frac{b}{|\sin\chi|}\left|\frac{db}{d\chi}\right| \tag{5.6.8}$$

and the total cross-section is defined as

$$\sigma_T = \int_\Omega d\sigma = 2\pi \int_0^\pi d\chi\, b \left|\frac{db}{d\chi}\right|. \tag{5.6.9}$$

From σ_T, we may also define a total cross sectional radius, a, by equating σ_T to the area of a disk of radius a, *i.e.*, $\sigma_T \stackrel{\text{def}}{=} \pi a^2$.

For the attractive, inverse square force we have examined above, because $\varepsilon = -1/\sin(\chi/2)$ it follows that

$$\frac{d\varepsilon^2}{d\chi} = \frac{2\mu^2 v_\infty^4 b\, db/d\chi}{k^2} = -\frac{\cos\chi/2}{\sin^3\chi/2} \tag{5.6.10}$$

and therefore

$$\frac{d\sigma}{d\Omega} = \frac{b}{|\sin\chi|}\left|\frac{db}{d\chi}\right| = \left(\frac{k\csc^2\chi/2}{4E}\right)^2 \tag{5.6.11}$$

The total differential cross section is infinite! This is because the force we have considered is long range. In the Rutherford experiment, it is screened by the electrons in the atom.

5.6.2 Dynamical "Friction" (Chandrashekar)*

As our treatment above has dealt exclusively with the relative coordinate, we have had little explicit information regarding the motion of the masses themselves. Let us close this chapter by addressing the motion of the individual masses. Conservation of momentum requires that

$$m_1\frac{d\vec{v}_1}{dt} + m_2\frac{d\vec{v}_2}{dt} = 0 \tag{5.6.12}$$

so if we set $\vec{v} = \vec{v}_2 - \vec{v}_1$, then it follows that the rate of change of each velocity depends only on the rate of change of $\vec{v}$,

$$\frac{d\vec{v}_1}{dt} = -\frac{m_2}{M}\frac{d\vec{v}}{dt}, \quad \frac{d\vec{v}_2}{dt} = \frac{m_1}{M}\frac{d\vec{v}}{dt} \tag{5.6.13}$$

As before, we denote by b the impact parameter and $\vec{v}_- = v_\infty \hat{x}$ the velocity as $t \to -\infty$, so $L = \mu b v_\infty$ and $E = \frac{1}{2}\mu v_\infty^2$ and both, being conserved, have the same values as $t \to +\infty$. Thus in the infinite future $\vec{v}$ will be

$$\vec{v}_+ = v_\infty\langle\cos\chi, \sin\chi\rangle = v_\infty\langle 1 - \frac{2}{\varepsilon^2}, -\frac{2}{\varepsilon}\sqrt{\varepsilon^2 - 1}\rangle \tag{5.6.14}$$
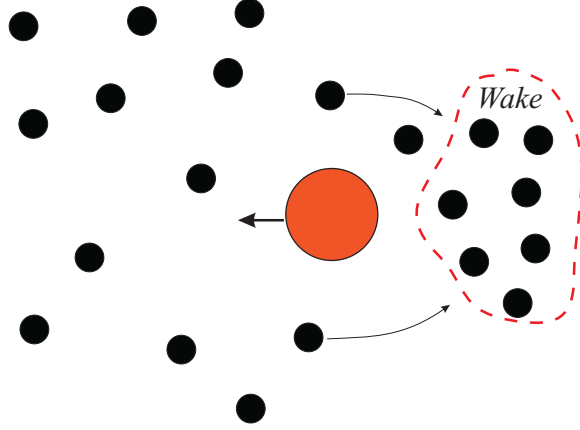
Figure 5.9: Formation of a wake when an object moves through a sea of force centers.

which we write as

$$v_{+\parallel} = v_\infty \left(1 - \frac{2}{\varepsilon^2} \right)$$

$$v_{+\perp} = v_\infty \left(-\frac{2}{\varepsilon} \sqrt{\varepsilon^2 - 1} \right) \tag{5.6.15}$$

$v_\parallel$ represents the velocity component parallel to the initial velocity (the x axis) and $v_\perp$ represents the velocity component perpendicular to the initial velocity (the y axis). The component of $\vec{v}_+$ parallel to the initial velocity "slows" relative to the initial velocity and the slowing effect is pronounced for small eccentricities. The total change in the parallel component during the collision is $\Delta v_\parallel = -2v_\infty/\varepsilon^2$, which translates into the following changes in the parallel velocity component for the two masses m_1 and m_2:

$$\Delta v_{1\parallel} = \frac{2m_2 v_\infty}{M\varepsilon^2}, \quad \Delta v_{2\parallel} = -\frac{2m_1 v_\infty}{M\varepsilon^2} \tag{5.6.16}$$

(we will not be concerned with the transverse direction for the present).

Now imagine that m_2 is passing through a homogeneous sea of force centers (as an astrophysical application, for example, think of m_2 as a black hole racing through a galaxy and its dark matter halo or a globular cluster). Because of the assumed homogeneity there will be no deflection in the transverse direction on the average (this is why we ignored it above) but, as the object moves through the "medium", scattering will occur as shown in figure 5.9, resulting in an overdensity of scattering centers (a wake) *behind* the moving object. This region of higher density exerts a drag on the moving object.

We now single out $\vec{v}_2$, call it $\vec{v}_M$ and its associated mass M. Let the force centers be labeled by by their velocities $\vec{v}$, along with a distribution function $2\pi b f(\vec{v}) db d^3 \vec{v}$ giving the number, dN, of force centers with velocities between $\vec{v}$ and $\vec{v} + d\vec{v}$, masses m and impact parameters between b and $b + db$. The rate at which the mass M encounters these force centers is

$$2\pi b \left(\frac{v_\infty}{\lambda} \right) f(\vec{v}) db d^3 \vec{v} \tag{5.6.17}$$

where λ is the mean distance between them. For each encounter, there is a change in $v_{M\parallel}$ given by (5.6.16). The average rate of change of $\vec{v}_M$ due to encounters with dN force centers is therefore

$$d \left(\frac{dv_{M\perp}}{dt} \right) = 0$$

$$d \left(\frac{dv_{M\parallel}}{dt} \right) = -\frac{4\pi \mu v_\infty^2}{\lambda(m+M)} \left[1 + \frac{2EL^2}{\mu k^2} \right]^{-1} b f(\vec{v}) db d^3 \vec{v}$$

$$= -\frac{4\pi \mu v_\infty^2}{\lambda(m+M)} \left[1 + \frac{\mu^2 v_\infty^4 b^2}{k^2} \right]^{-1} b f(\vec{v}) db d^3 \vec{v} \tag{5.6.18}$$

Perform the b integration from zero up to some maximum value to get

$$d \left(\frac{dv_{M\parallel}}{dt} \right) = -\frac{2\pi k^2}{\lambda \mu (M+m) v_\infty^2} \ln \left(1 + \frac{b_{\max}^2 \mu^2 v_\infty^4}{k^2} \right) f(\vec{v}) d^3 \vec{v} \tag{5.6.19}$$

A reasonable choice for $b_{\max}$ is the size of the system of force centers in which the mass M is moving. In practice, the argument of the natural logarithm is very large and is usually replaced by a constant, Λ^2. As v_∞ is actually $|\vec{v}_M - \vec{v}|$ and $v_\perp$ undergoes no change, we can integrate over the distribution writing the above equation as

$$\frac{d\vec{v}_M}{dt} \approx -\frac{4\pi k^2}{\lambda \mu (m+M)} \ln \Lambda \int_0^{v_M} \frac{\vec{v}_M - \vec{v}}{|\vec{v}_M - \vec{v}|^3} f(\vec{v}) d^3 \vec{v} \tag{5.6.20}$$

For an isotropic distribution,

$$\frac{d\vec{v}_M}{dt} \approx -\frac{16\pi^2 k^2}{\lambda \mu (M+m)} \ln \Lambda \frac{\vec{v}_M}{|\vec{v}_M|^3} \int_0^{v_M} f(v) v^2 dv \tag{5.6.21}$$

and if the interactions are *gravitational* then $k = G(m+M)\mu$ gives

$$\frac{d\vec{v}_M}{dt} \approx -\frac{16\pi^2 G^2 (m+M)\mu}{\lambda} \ln \Lambda \frac{\vec{v}_M}{|\vec{v}_M|^3} \int_0^{v_M} f(v) v^2 dv \tag{5.6.22}$$

The integral is taken over all force centers in the distribution with speeds up to v_M. The effect causes a gradual slowing down of the mass M as if it were moving in a viscous field. However, it has nothing to do with friction and is purely due to the central force, as switching it off by setting $k = 0$ would elliminate it.[9]

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[9]Problem: If the central force is repulsive, what sort of effect may one expect? Why?

Chapter 6

Motion in Non-Inertial Reference Frames

So far we have examined motion only from the point of view of the "inertial" observer. Recall that while we were discussing Newton's laws we defined such an observer (frame) as one in which the trajectory of a "free" body is a straight line. There is some ambiguity here, as the observer must either know independently of her(is) frame that a body is free, and then determine which frames are inertial, or s(he) must (independently) determine whether her(is) frame is inertial and subsequently whether the body is "free". The concept of a free body is therefore intimately tied with the concept of an inertial frame. The problem is more severe when we realize that the ambiguity creeps into the definition of a force in the second law as that which gives rise to deviations from the first law.

This issue was addressed generally only in the early part of the twentieth century by Einstein. We do not here delve into its resolution, but we assume that it is always possible to find a family of inertial frames. However, it may often be impossible for a given observer to actually occupy such a frame. For example, in order to describe the motion of a particle on the surface of the earth, the most convenient frame is one that is attached to the earth itself. But the earth rotates both about itself and the sun, and the entire solar system rotates about the galaxy while also oscillating about the galactic plane. The earth, convenient though it may be as a reference frame, is certainly not an inertial.

6.1   Newton's second law in an accelerating frame

Since we will assume that inertial frames exist, let us compare our description of motion two frames S and S', in which S is inertial. Suppose that S' is non-inertial, having an

acceleration $\vec{a}(t)$ relative to S. It is clear that, instantaneously,

$$\vec{r}\,' = \vec{r} - \vec{R} \tag{6.1.1}$$

where $\vec{r}(t)$ and $\vec{r}\,'(t)$ represent the positions of a particle, P, relative to S and S', and $\vec{R}$ is the position of S' relative to S at time t. Taking derivatives,

$$\begin{aligned} \vec{v}\,' &= \vec{v} - \vec{v}_R \\[4pt] \vec{a}\,' &= \vec{a} - \vec{a}_R \end{aligned} \tag{6.1.2}$$

Now, Newton's laws are applicable in the unprimed frame, so if $\vec{F}^t ext ext$ is the force acting on the particle at P, its motion, as described in S would be given by Newton's second law

$$m\vec{a} = \vec{F}^{\text{ext}} \Rightarrow m\vec{a}\,' = \vec{F}^{\text{ext}} - m\vec{a}_R. = \vec{F}^{\text{ext}} + \vec{f} \tag{6.1.3}$$

(Notice that, if the acceleration of S' were zero as measured by S, then Newton's second law would hold in S' as well. Therefore, all frames moving at a constant velocity relative to an inertial frame are also inertial.) In general, we must account the additional term $f = -m\vec{a}_R$ that appears on the right hand side. It has the mechanical dimension of force and we will refer to it as a "fictitious" force, $\vec{f}$. One recognizes a fictitious force by the fact that it is multiplied by the mass of the particle itself, so that the acceleration of the body measured in S', due to the fictitious force, is independent of its mass,

$$\frac{d^2 \vec{r}\,'}{dt^2} = \frac{\vec{F}^{\text{ext}}}{m} - \vec{a}_R. \tag{6.1.4}$$

In other words: motion under the influence of purely fictitious forces is kinematical (the mass, or inertial, of the body plays no role in the equations of motion)! An example of such a force is obviously gravity itself, since the weak equivalence principle ensures that the mass appearing on the left hand side of Newton's equation is the same as the mass appearing the the gravitational force law.

6.2   Rotating Frames

In considering the earth itself as a reference frame, the dominant acceleration of an observer fixed relative to the surface of the earth is her(is) centripetal acceleration due to the rotation of the earth about its north-south axis. As a special, but nevertheless very interesting case, let us describe the motion of a body in a frame that is rotating with respect to an inertial frame, but not translating relative to it. As before, let S be an inertial frame and S' be a non-inertial frame, rotating about some axis OP as shown in figure (6.1). Any vector $\vec{A}$ may be described in S or S'
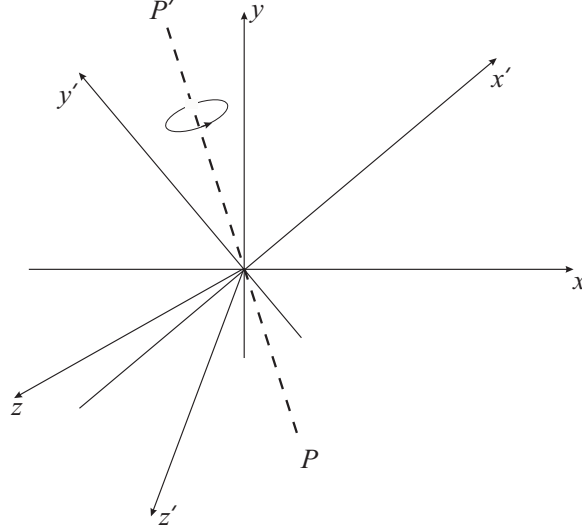
Figure 6.1: S' rotates about the axis PP'

$$\vec{A} = A_i \widehat{e}_i = A'_i \widehat{e}'_i \tag{6.2.1}$$

where $\widehat{e}_i$ form a basis in S and $\widehat{e}'_i$ a basis in S'. We will choose an orthonormal basis in each frame so that $\widehat{e}_i \cdot \widehat{e}_j = \delta_{ij} = \widehat{e}'_i \cdot \widehat{e}'_j$. Now in the inertial frame S,

$$\frac{d\vec{A}}{dt} = \frac{dA_i}{dt}\widehat{e}_i \tag{6.2.2}$$

whereas if the inertial observer were to refer to the basis $\widehat{e}'_i$, since the frame S' is rotating with respect to S, s(he) would describe it as

$$\frac{d\vec{A}}{dt} = \frac{dA'_i}{dt}\widehat{e}'_i + A'_i\frac{d\widehat{e}'_i}{dt} \tag{6.2.3}$$

Now the observer who is rotating sees the basis $\widehat{e}'_i$ as fixed so that her(is) description of the rate of change of A is therefore,

$$\frac{d'\vec{A}}{dt} = \frac{dA'_i}{dt}\widehat{e}'_i \tag{6.2.4}$$

where we use the prime in the derivative to indicate that it is the rate of change as measured by the non-inertial observer. It follows that

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + A'_i\frac{d\widehat{e}'_i}{dt} \tag{6.2.5}$$

Let us concentrate on the difference between the derivatives: $A_i' d\widehat{e}_i'/dt$. Because the rate of change of each basis vector in S' is itself a vector, it can be expanded in the S' basis, *i.e.*,

$$\frac{d\widehat{e}_i'}{dt} = a_{ij}\widehat{e}_j' \tag{6.2.6}$$

But since the basis vectors in S' are orthonormal

$$\widehat{e}_i' \cdot \widehat{e}_j' = \delta_{ij} \Rightarrow \frac{d\widehat{e}_i'}{dt} \cdot \widehat{e}_j' = -\widehat{e}_i' \cdot \frac{d\widehat{e}_j'}{dt} \Rightarrow a_{ij} = -a_{ji}, \quad \forall\, i,j \,\in\, \{1,2,3\}. \tag{6.2.7}$$

This means that the matrix a_{ij} is antisymmetric, and therefore has only 3 independent components (as opposed to nine for a general real 3×3 matrix). We could write it as

$$\widehat{a} = \begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{bmatrix} \tag{6.2.8}$$

Define the vector $\vec{\omega}$ with components

$$[\vec{\omega}]_i = \frac{1}{2}\epsilon_{ijk}a_{jk} \tag{6.2.9}$$

in the S' basis. Inverting this definition for a_{ij}, we find

$$\epsilon_{lmi}\omega_i = \frac{1}{2}\epsilon_{lmi}\epsilon_{ijk}a_{jk} = \frac{1}{2}(\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{mj})a_{jk} = a_{lm}, \tag{6.2.10}$$

where we use the fact that $\widehat{a}$ is antisymmetric, or

$$a_{lm} = \epsilon_{lmk}\omega_k. \tag{6.2.11}$$

Then

$$\frac{d\widehat{e}_i'}{dt} = a_{ij}\widehat{e}_j' = \epsilon_{ijk}\omega_k\widehat{e}_j' = \vec{\omega} \times \widehat{e}_i' \tag{6.2.12}$$

The last step is not obvious, but it is straightforward to prove as follows: using the fact that the nth component of $\widehat{e}_i'$ is δ_{in} in the rotating basis, the right hand side is

$$\vec{\omega} \times \widehat{e}_i' = (\epsilon_{jkn}\omega_k\delta_{in})\widehat{e}_j' = \epsilon_{ijk}\omega_k\widehat{e}_j'. \tag{6.2.13}$$

It follows that (6.2.5) can be written in the form

$$\frac{d\vec{A}}{dt} = \frac{d'\vec{A}}{dt} + \vec{\omega} \times \vec{A} \tag{6.2.14}$$

or, viewed as an operator, the derivative transforms from one frame to the other as

$$\frac{d}{dt}\ldots = \left[\frac{d'}{dt} + \omega \ \times \ \right]\ldots. \tag{6.2.15}$$

This general result is valid for *any* vector $\vec{A}$[1] and in particular for the position vector of a particle, *i.e.*,

$$\frac{d\vec{r}}{dt} = \frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r} \tag{6.2.16}$$

If we take another derivative, to obtain the acceleration in the S frame,

$$\begin{aligned} \frac{d^2\vec{r}}{dt^2} &= \frac{d}{dt}\left[\frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r}\right] \\ &= \frac{d'}{dt}\left[\frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r}\right] + \omega \times \left[\frac{d'\vec{r}}{dt} + \vec{\omega} \times \vec{r}\right] \end{aligned} \tag{6.2.17}$$

we find

$$\frac{d^2\vec{r}}{dt^2} = \frac{d'^2\vec{r}}{dt^2} + 2\vec{\omega} \times \frac{d'\vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d'\vec{\omega}}{dt} \times \vec{r} \tag{6.2.18}$$

This is called **Coriolis' theorem**. All the time derivatives on the right are relative to the rotating observer. This means that if the particle with position vector $\vec{r}$ is subject to a force $\vec{F}$ as measured in S, then Newton's second law reads

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F} \tag{6.2.19}$$

or, written in the rotating frame,

$$\frac{d'^2\vec{r}}{dt^2} + 2\vec{\omega} \times \frac{d'\vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d'\vec{\omega}}{dt} \times \vec{r} = \frac{\vec{F}}{m}. \tag{6.2.20}$$

It follows, according to (6.1.3), that we must associate

$$\vec{f} = -m\left[2\vec{\omega} \times \frac{d'\vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d'\vec{\omega}}{dt} \times \vec{r}\right] \tag{6.2.21}$$

with the fictitious force that appears to act on the particle when it is described in the non-inertial frame S'. The first term is called the **Corriolis** force. The second is the **centrifugal** force and the last term, which turns out to have no particular name, is

------

[1]Notice that

$$\frac{d\vec{\omega}}{dt} = \frac{d'\vec{\omega}}{dt}$$

since $\vec{\omega} \times \vec{\omega} = 0$.

relevant only if the angular velocity of S' is changing in time. This is because, as we argue below, $\vec{\omega}$ is the instantaneous angular velocity of S' relative to S.

Although $\vec{\omega}$ was introduced in somewhat of an *ad hoc* manner in the previous section, it can easily be assigned a physical significance. To do so, we must first show that it possesses the transformation properties under rigid rotations that would make it a vector. But this is obvious because from (6.2.6) it follows that a_{ij} transforms as two copies of a vector (*i.e.,* is a rank two tensor),

$$a'_{ij} = \widehat{R}_{il}\widehat{R}_{jm}a_{lm} \tag{6.2.22}$$

and, because the Levi-Civita symbol is a rank three tensor, (6.2.9) ensures that $\vec{\omega}$ transforms as a vector. It is even easier to see that $\vec{\omega}$ is the angular velocity of the frame S'. Let $\vec{A}$ be any one of the basis vectors in the S' frame. Since the basis vectors $\widehat{e}'_i$ are not moving relative to the S' observer, $d'\widehat{e}'_i/dt = 0$. We then have

$$\frac{d\widehat{e}'_i}{dt} = \vec{\omega} \times \widehat{e}'_i, \tag{6.2.23}$$

which is precisely the equation for a vector that is rotating with an angular velocity $\vec{\omega}$.

6.3 Motion near the surface of the earth.

For an observer fixed relative to the surface of the rotating earth, it is natural to apply not Newton's second law, but Corriolis' theorem in describing the motion of a body. Thus, the effective equations of motion take into account the fictitious forces that must be added to compensate for her(is) rotation,

$$\frac{d'^2\vec{r}}{dt^2} + 2\vec{\omega} \times \frac{d'\vec{r}}{dt} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d'\vec{\omega}}{dt} \times \vec{r} = \frac{\vec{F}}{m} + \vec{g} \tag{6.3.1}$$

where we have restricted our attention to bodies moving close to the surface of the earth (at heights that are small compared to the earth's radius) and $\vec{F}$ now incorporates all forces apart from the gravitational pull of the earth on the body.

6.3.1 Deflection of a freely falling particle

Examine the diagram in figure (6.2). For an observer at some latitude λ we erect a right handed, rotatings coordinate system as follows: the $\widehat{z}$ axis points "up" along the outward radius from the center of the earth, the $\widehat{x}$ axis points along the observer's longitude toward the south and the $\widehat{y}$ axis is directed east. We have seen that the vector $\vec{\omega}$ is actually the angular velocity about the north-south axis. Now the magnitude of $\vec{\omega}$ is exceedingly small:

$$|\vec{\omega}| = \frac{2\pi}{365.25 \times 24 \times 3600} \approx 7.27 \times 10^{-7} \text{ rads/s} \tag{6.3.2}$$
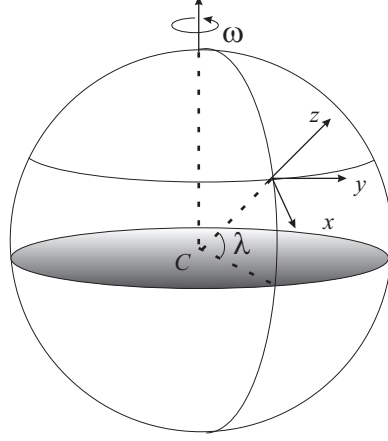
Figure 6.2: The earth as a rotating frame

so we may neglect effects on the order of $|\omega|^2$ and, in particular, the centrifugal term. In terms of λ

$$\vec{\omega} = \omega(-\cos\lambda, 0, \sin\lambda) \tag{6.3.3}$$

and we let the instantaneous velocity of a body as measured in this rotating frame be

$$\vec{v}' = \frac{d'\vec{r}}{dt} = (\dot{x}, \dot{y}, \dot{z}). \tag{6.3.4}$$

(the overdot now refers to the derivative in the rotating frame). It follows that the Coriolis term is proportional to $\vec{\omega} \times \vec{v}' = \omega(-\dot{y}\sin\lambda, \dot{x}\sin\lambda + \dot{z}\cos\lambda, -\dot{y}\cos\lambda)$. Consider no forces but gravity acting on the body and take gravity's acceleration to be a constant vector, $\vec{g} = (0, 0, -g)$. Let us henceforth drop the primes. Because all unprimed derivatives have been eliminated, no ambiguity arises. Putting all this together, Coriolis' theorem reads,

$$m\ddot{x} = 2m\omega\dot{y}\sin\lambda$$

$$m\ddot{y} = -2m\omega[\dot{x}\sin\lambda + \dot{z}\cos\lambda]$$

$$m\ddot{z} = -mg + 2m\omega\dot{y}\cos\lambda \tag{6.3.5}$$

If the initial velocity has no component in the east-west or north-south direction then evidently both $\dot{x}$ and $\dot{y}$ will be of order ω. This means that we can approximate the equations above by

$$m\ddot{x} \approx 0$$

$$m\ddot{y} \approx -2m\omega\dot{z}\cos\lambda$$

$$m\ddot{z} \approx -mg \tag{6.3.6}$$

to be consistent with our neglect of terms that are second order in ω, such as the centrifugal term. This system of equations is easy to solve: the last equation says that

$$z(t) = z_0 - \frac{1}{2}gt^2 \tag{6.3.7}$$

assuming that $\dot{z}_0 = 0$. Inserted into the second equation this gives

$$y = y_0 + \frac{1}{3}\omega gt^3 \cos\lambda \tag{6.3.8}$$

and, evidently, $x = x_0$. Notice that there is deflection of the falling body along the *easterly* direction,[2] of magnitude

$$\Delta y = \frac{1}{3}\omega gt^3 \cos\lambda \tag{6.3.9}$$

The deflection is maximum at the equator and vanishes at the pole, in agreement with what one would naturally expect. If the body falls through a height h then

$$t = \sqrt{\frac{2h}{g}} \Rightarrow \Delta y = \frac{1}{3}\omega\cos\lambda\sqrt{\frac{8h^3}{g}} \tag{6.3.10}$$

The result does not depend on the mass of the body because the forces causing the deflection are fictitious.

6.3.2 Motion of a projectile

Take a projectile that is fired from the origin in the easterly direction and making an angle of α with the $y-$axis, so that the initial conditions read:

$$\vec{r}_0 = 0, \quad \vec{v}_0 = v_0(0, \cos\alpha, \sin\alpha) \tag{6.3.11}$$

Because $v_{0x} = 0$, $\dot{x}$ will always be of order ω and because $\ddot{y}$ is driven by an omega dependent force, changes to $\dot{y}$ will be of order ω, that is, $\dot{y} = v_0\cos\alpha + \mathcal{O}(\omega)$. Thus, up to the first order in ω our equations in (6.3.5) read

$$m\ddot{x} \approx 2m\omega v_0 \cos\alpha\sin\lambda$$

---

[2]Problem: Erect a suitable right handed coordinate system in the southern hemisphere and show that a deflection of the same magnitude occurs in the *westerly* direction.

$$m\ddot{y} \approx -2m\omega\dot{z}\cos\lambda$$

$$m\ddot{z} = -mg + 2m\omega v_0\cos\alpha\cos\lambda \tag{6.3.12}$$

The first and last equations are immediately solved:

$$x(t) \approx \omega v_0 t^2 \cos\alpha\sin\lambda$$

$$z(t) \approx v_0 t\sin\alpha + \left(-\frac{1}{2}g + \omega v_0\cos\alpha\cos\lambda\right)t^2 \tag{6.3.13}$$

Up to the first order, only the first term in $\dot{z}(t)$ is relevant in the equation for $\ddot{y}$. The second term would contribute to $\mathcal{O}(\omega^2)$. Therefore

$$\ddot{y} \approx -2\omega\cos\lambda(v_0\sin\alpha - gt) \tag{6.3.14}$$

which gives

$$\dot{y}(t) = v_0\cos\alpha - 2\omega\left(v_0 t\sin\alpha - \frac{1}{2}gt^2\right)\cos\lambda$$

$$y(t) = v_0\cos\alpha t - \omega\left(v_0 t^2\sin\alpha - \frac{1}{3}gt^3\right)\cos\lambda \tag{6.3.15}$$

Now the deflection is southerly and given as a function of time by (6.3.13). To find the total deflection we require the time taken by the projectile to strike the earth, which is the non-vanishing solution of $z(t) = 0$,

$$t = \frac{2v_0\sin\alpha}{g - 2\omega v_0\cos\alpha\cos\lambda} \approx \frac{2v_0\sin\alpha}{g}\left[1 + \frac{2\omega v_0}{g}\cos\alpha\cos\lambda + \ldots\right] \tag{6.3.16}$$

The total deflection, to order ω is then

$$\Delta x = \frac{4\omega v_0^3}{g^2}\sin\lambda\cos\alpha\sin^2\alpha \tag{6.3.17}$$

It is vanishing for a projectile fired along the equator.

If the projectile is fired in the southerly direction and making an angle of α with the $x-$ axis, our initial conditions would read

$$\vec{r}_0 = 0, \quad \vec{v}_0 = v_0(\cos\alpha, 0, \sin\alpha) \tag{6.3.18}$$

Applying the same arguments as in the previous example, Corriolis' theorem approximated up to the first order in ω gives

$$\ddot{x} = 0$$

$$\ddot{y} = -2\omega[\dot{x}\sin\lambda + \dot{z}\cos\lambda]$$

$$\ddot{z} = -g \tag{6.3.19}$$

With our initial conditions,

$$x(t) = v_0 t \cos\alpha, \quad z(t) = v_0 t \sin\alpha - \frac{1}{2}gt^2 \tag{6.3.20}$$

and

$$\ddot{y} = -2\omega[v_0\cos\alpha\sin\lambda + (v_0\sin\alpha - gt)\cos\lambda] \tag{6.3.21}$$

Integrating this equation, subject to the initial conditions we find

$$\dot{y}(t) = -2\omega\left[v_0 t \sin(\alpha + \lambda) - \frac{1}{2}gt^2\cos\lambda\right] \tag{6.3.22}$$

and

$$y(t) = -\omega\left[v_0 t^2(\sin(\alpha + \lambda) - \frac{1}{3}gt^3\cos\lambda\right] \tag{6.3.23}$$

When the time for the projectile to strike the earth,

$$t = \frac{2v_0\sin\alpha}{g}, \tag{6.3.24}$$

is substituted into the expression for $y(t)$ gives the total deflection as

$$\Delta y = -\frac{4\omega v_0^3}{3g^2}\sin^2\alpha\left(\sin\alpha\cos\lambda + 3\cos\alpha\sin\lambda\right). \tag{6.3.25}$$

It is in the *westerly* direction.[3] Notice that the tendency is to rotate the plane of the projectile in a clockwise direction in the northern hemisphere and in a counterclockwise direction in the southern hemisphere.

This has an interesting consequence for hurricanes and tornadoes, as illustrated in figure (6.3). When they form, regions of high pressure surround a region of low pressure. As a result of the pressure differential, there is a radial flow of air mass into the region of low pressure. The clockwise deflection of the trajectories by the Corriolis force leads to a counter-clockwise rotation of the air mass about the center of the hurricane (or tornado) as shown in the figure.

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[3]Problem: Examine the projectile in the southern hemisphere, with similar initial conditions.
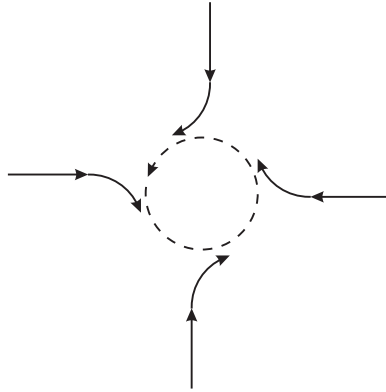
Figure 6.3: The rotation of hurricanes and cyclones in the northern hemisphere.

6.3.3   The Foucault Pendulum

Because of the Coriolis force, the plane of oscillation of a simple pendulum will precess or rotate so that the pendulum actually sweeps out a cone. This is the Foucault pendulum, named after the french physicist Leon Foucault who used it to demonstrate the rotation of the earth and the Corolis effect. The Foucault pendulum was first demonstrated to the public in 1851, but it is known that the Italian physicist Vicenzo Viviani had experimented with a very similar device as early as 1661. The Foucault pendulum, however, was the first dynamical proof of the earth's rotation.

It is not difficult for us now to see how this comes to be. As before, we apply Corriolis' theorem up to the first order in ω [see figure (6.4)]. Take the origin to be the equilbrium of the oscillation. The external force is just

$$\vec{F} = m\vec{g} + \vec{T} \tag{6.3.26}$$

where $\vec{g} = (0, 0, -g)$ and $\vec{T} = (T_x, T_y, T_z)$ is the tension in the string.   If the angle, η made with the vertical is small, then $T_z = T \cos\eta = mg \approx T$ and therefore,

$$
\begin{aligned}
T_x &\approx -T \sin\eta \cos\varphi = -T\left(\frac{l'}{l}\right)\left(\frac{x}{l'}\right) = -T\frac{x}{l} \\
T_y &\approx -T \sin\eta \sin\varphi = -T\left(\frac{l'}{l}\right)\left(\frac{y}{l'}\right) = -T\frac{y}{l}
\end{aligned}
\tag{6.3.27}
$$

[see figure (6.5). We take $\dot{z} \approx 0$, assuming small oscillations, so that Corriolis' equations read

$$m\ddot{x} = 2m\omega\dot{y}\sin\lambda - T\frac{x}{l}$$
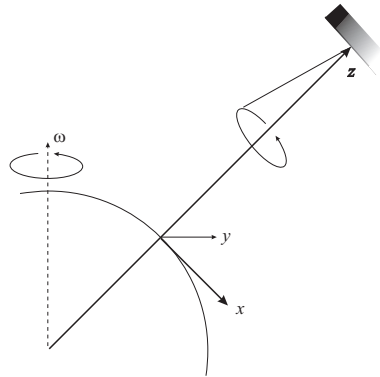
Figure 6.4: The Foucault Pendulum
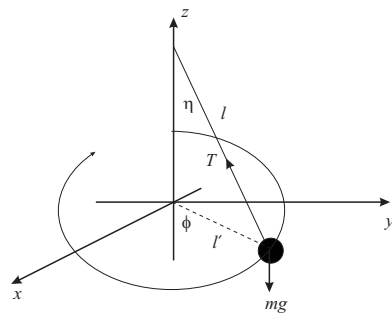
Figure 6.5: The Foucault Pendulum: components of the tension

$$m\ddot{y} = -2m\omega\dot{x}\sin\lambda - T\frac{y}{l}$$

$$m\ddot{z} = 2m\omega\dot{y}\cos\lambda - mg + T \tag{6.3.28}$$

It is convenient to define the variable $u = x + iy$ and write the first two equations as the following complex equation in u

$$m\ddot{u} = -2im\omega\dot{u}\sin\lambda - T\frac{u}{l} \tag{6.3.29}$$

which can be written in the form of the linear equation

$$m\ddot{u} + i\beta\dot{u} + \alpha^2 u = 0 \tag{6.3.30}$$

where $\beta = 2m\omega\sin\lambda$ and $\alpha^2 = T/l$. The roots of the auxiliary equation are

$$\lambda_\pm = -\frac{i}{2m}\left[\beta \pm \sqrt{\beta^2 + 4m\alpha^2}\right] \tag{6.3.31}$$

Define the quantities $\omega_0 = \alpha/\sqrt{m} = \sqrt{T/ml} \approx \sqrt{g/l}$, which is independent of the earth's rotation, and $\gamma = \beta/2m = \omega\sin\lambda$, which vanishes if $\omega = 0$. The general solution is of the form

$$u(t) = e^{-i\gamma t}\left[Ae^{i\sqrt{\omega_0^2 + \gamma^2}} + Be^{-i\sqrt{\omega_0^2 + \gamma^2}}\right] \tag{6.3.32}$$

To understand it, imagine for a moment that the earth does not rotate, *i.e.*, that $\omega = 0$, then our solution reads

$$u_0(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \tag{6.3.33}$$

which represents a simple oscillation of frequency ω_0. But, when γ is small (we are interested only in terms up to the first order in ω), we can neglect the γ^2 occurring inside the radicals in (6.3.32) so that

$$u(t) \approx e^{-i\gamma t}u_0(t). \tag{6.3.34}$$

Recalling that $u(t) = x(t) + iy(t)$, this can be put into the suggestive form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \approx \begin{bmatrix} \cos\gamma t & \sin\gamma t \\ -\sin\gamma t & \cos\gamma t \end{bmatrix} \begin{bmatrix} x_0(t) \\ y_0(t) \end{bmatrix}, \tag{6.3.35}$$

showing that the plane of the simple oscillations that would occur in the absence of the earth's rotation is subject to a time dependent rotation at constant angular velocity $\gamma = \omega\sin\lambda$.

Chapter 7

Rigid Bodies

A rigid body is defined as a system of particles for which the relative distance between any two particles is constant. There are, of course, no truly rigid bodies in nature because being rigid implies that the body cannot admit any deformations whatsoever, and we know of no such system of particles: every body may be deformed, at least minimally, by the application of external forces. The existence of purely rigid bodies is also prohibited by special relativity because a truly rigid body would transmit information at an infinite speed, which contradicts the postulate that no signal can travel faster than the speed of light.

How many degrees of freedom would a rigid body possess? Consider any one particle in the rigid body. To locate this particle relative to any coordinate system requires three coordinates (or one position vector). Now consider a second particle within the body: because its distance from the first is fixed it must lie on the surface of a sphere of a fixed radius from the first, so we now require only two coordinates to locate this particle. A third particle is at fixed distances from the first and the second particles, so it lies on the curve of intersection of two spheres centered about them. This calls for just one additional coordinate to locate its position. Once the positions of the first three particles are fixed the positions of the remaining particles are completely determined in terms of them by the constraints, therefore a rigid body in three dimensions possesses six degrees of freedom.[1]

Solids admit strong interactions between their elementary constituents that do a pretty good job of resisting deformation and "rigidity" entails taking those inter-molecular interactions resisting deformations to be *effectively infinite* or, more realistically, much larger than the external applied forces. Our objective in this chapter is to describe the motion in three dimensions of a system of particles that is effectively rigid.

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[1]<u>Problem</u>: Repeat this argument to show that in D dimensions a rigid body will have $\frac{D(D+1)}{2}$ degrees of freedom.
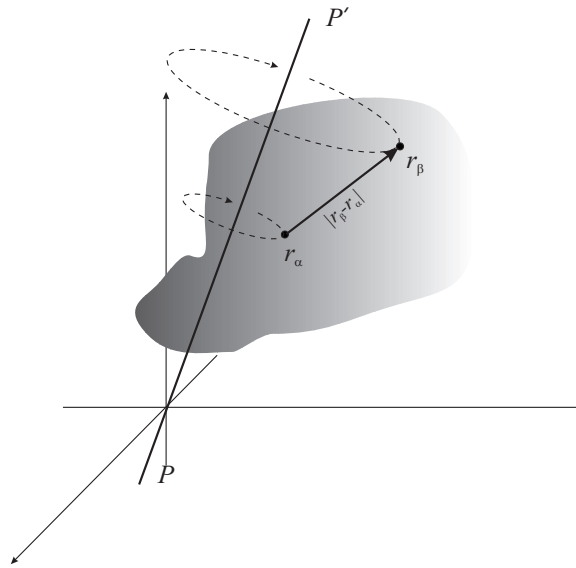
Figure 7.1: A rigid body rotating about an arbitrary axis

7.1   Equations of motion

In keeping with our definition of what constitutes a rigid body, let's consider a system of particles whose separations are fixed, *i.e.*, $|\vec{r}_\alpha - \vec{r}_\beta|$ is constant for any two particles, labeled α and β. (We will use greek letters to designate the particles of the body so as to avoid confusion with the components of vectors, for which we use the roman alphabet as usual.) If the body rotates, its rigidity forces all the particles to move with the same angular velocity. This simple observation makes it convenient to separate the motion of the body into a rotational piece and translational one. Let S be an inertial frame which we assume is attached to the laboratory and will henceforth refer to as the laboratory frame. Let S' be a frame whose orgin is fixed to the body, but which is not rotating relative to S. Thus, in S', the velocity of the particle labeled α is

$$\vec{v}_\alpha' = \vec{\omega} \times \vec{r}_\alpha' \tag{7.1.1}$$

Notice that $\vec{\omega}$ is the same for all particles because the system is rigid. After all, if $\vec{\omega}_\alpha \neq \vec{\omega}_\beta$ for two particles α and β then the distance between α and β would change which would violate our condition that tha body is rigid. Now, if the frame S' has a velocity $\vec{v}$ relative to S then the velocity of the αth particle in the laboratory system will be

$$\vec{v}_\alpha = \vec{v} + \vec{\omega} \times \vec{r}_\alpha' \tag{7.1.2}$$

The total momentum of the system is therefore

$$\vec{P} = \sum_{\alpha} m_{\alpha} \vec{v}_{\alpha} = M\vec{v} + \vec{\omega} \times \sum_{\alpha} m_{\alpha} \vec{r}'_{\alpha} \tag{7.1.3}$$

where $M = \sum_{\alpha} m_{\alpha}$ is the total mass of the body and

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + \sum_{\alpha} m_{\alpha} \frac{d}{dt}(\vec{\omega} \times \vec{r}'_{\alpha}) = \vec{F}^{\text{ext}}. \tag{7.1.4}$$

If the origin of S' is located at the center of mass then the second term is clearly vanishing and the equation says that the center of mass moves as a particle of mass equal to the mass of the body and subject to the total external force to which the body is subjected. This equation says nothing about the rotation of the body about the center of mass.

From another point of view, imagine yet another frame S'' whose origin coincides with that of S' but which is rotating with the body. Because their origins coincide, the position of the αth particle in the body with respect to S'' is just $\vec{r}'_{\alpha}$ but, with respect to S'' every particle is at rest, therefore

$$\frac{d'\vec{r}'_{\alpha}}{dt} = 0 \Rightarrow \frac{d\vec{r}'_{\alpha}}{dt} - \vec{\omega} \times \vec{r}'_{\alpha} = 0 \tag{7.1.5}$$

according to (6.2.16). But if $\vec{r}$ is the position of S' relative to the laboratory frame S, then $\vec{r}_{\alpha} = \vec{r} + \vec{r}'_{\alpha}$ and

$$\vec{v}_{\alpha} = \vec{v} + \frac{d\vec{r}'_{\alpha}}{dt} = \vec{v} + \vec{\omega} \times \vec{r}'_{\alpha} \tag{7.1.6}$$

which is the same, of course, as (7.1.2). This leads to the equations of motion in (7.1.5).

7.2 The Inertia Tensor

The equations of motion in (7.1.5) split naturally into translational and rotational pieces. So do the kinetic energy and angular momentum as we now see. The kinetic energy of the body is the sum of kinetic energies of its constituents, so

$$K = \frac{1}{2} \sum_{\alpha} m_{\alpha} \vec{v}_{\alpha}^2 = \frac{1}{2} \sum_{\alpha} m_{\alpha} (\vec{v} + \vec{\omega} \times \vec{r}'_{\alpha})^2 \tag{7.2.1}$$

Expanding the square,

$$K = \frac{1}{2} \left[M\vec{v}^2 + 2\sum_{\alpha} m_{\alpha} \vec{v} \cdot (\vec{\omega} \times \vec{r}'_{\alpha}) + \sum_{\alpha} m_{\alpha} (\vec{\omega} \times \vec{r}'_{\alpha})^2 \right]. \tag{7.2.2}$$

If we take S' (commonly referred to as the "body system") to lie at the center of mass of the rigid body then the second term vanishes because $\sum_\alpha m_\alpha \vec{r}_\alpha' \equiv 0$. On the other hand,

$$(\vec{\omega} \times \vec{r}_\alpha') \cdot (\vec{\omega} \times \vec{r}_\alpha') = \epsilon_{ijk}\epsilon_{ipq}\omega_j r_{\alpha,k}'\omega_p r_{\alpha,q}'$$

$$= \vec{\omega}^2 \vec{r}_\alpha'^2 - (\vec{\omega} \cdot \vec{r}_\alpha')^2 \tag{7.2.3}$$

giving

$$K = \frac{1}{2}M\vec{v}_{\text{cm}}^2 + I_{ij}^{\text{cm}}\omega_i\omega_j \tag{7.2.4}$$

where

$$I_{ij} = \sum_\alpha m_\alpha(\vec{r}_\alpha'^2 \delta_{ij} - \vec{r}_{\alpha,i}'\vec{r}_{\alpha,j}') \tag{7.2.5}$$

is called the **Inertia Tensor** of the rigid body. The Kinetic energy in (7.2.4) breaks up into a translational piece, which is just the translational kinetic energy of the center of mass, behaving as a particle having the entire mass of the body, and a rotational piece, which is the kinetic energy of the rigid body about the center of mass and is determined by the angular velocity and the inertia tensor. Notice that while the definition of the inertia tensor is quite generally given by (7.2.5), the simplified expression for the kinetic energy depends on the fact that the body system has its origin at the center of mass. Therefore, in that expression, we have labeled the inertia tensor by the superscript "cm" (for 'center of mass"). That the inertia tensor is a rank two, symmetric tensor is clear by inspection.

The angular momentum of the rigid body can also be written in terms of the inertia tensor. Thus,

$$\vec{L} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha m_\alpha(\vec{r} + \vec{r}_\alpha') \times (\vec{v} + \vec{\omega} \times \vec{r}_\alpha') \tag{7.2.6}$$

Again, if the origin of S' is located at the center of mass, we could neglect all terms in which a sum of the form $\sum_\alpha m_\alpha \vec{r}_\alpha'$ appears, because this sum vanishes identically by definition. It follows that the expression for the angular momentum on the right hand side expands into

$$\vec{L} = M\vec{r}_{\text{cm}} \times \vec{v}_{\text{cm}} + \sum_\alpha m_\alpha \vec{r}_\alpha' \times (\vec{\omega} \times \vec{r}_\alpha') \tag{7.2.7}$$

or, in components,

$$L_i = [\vec{r}_{\text{cm}} \times \vec{P}]_i + I_{ij}^{\text{cm}}\omega_j \tag{7.2.8}$$

which is naturally interpreted as the angular momentum of the center of mass about S plus the angular momentum of the body around the center of mass. Once again, the latter is determined by the inertia tensor.

7.3   Computing the Inertia Tensor: examples

In the continuum limit, when the average separation between the constituents is comparable to their size one can turn the sum into an integral by imagining the constituents as infinitesimal elements of mass $dm(\vec{r})$. In this case,

$$I_{ij} = \int_{\text{Vol}} dm(\vec{r})(\vec{r}^2 \delta_{ij} - \vec{r}_i \vec{r}_j) \tag{7.3.1}$$

where the integral is over the volume of the solid. This can be made more explicit, if the material is characterized by a density $\rho(\vec{r})$. Then

$$I_{ij} = \int_{\text{Vol}} d^3\vec{r}\, \rho(\vec{r})(\vec{r}^2 \delta_{ij} - \vec{r}_i \vec{r}_j) \tag{7.3.2}$$

The diagonal elements of I_{ij} are called the **moments of inertia** about the respective axes.[2] The off-diagonal elements are the **products of inertia**. We evaluate this integral in two cases.

7.3.1   Homogeneous sphere

Let us begin with the simple example of of a homogeneous ($\rho = $ const.) sphere. We wish to calculate its inertia tensor about the center [see figure (7.2)]. First consider the piece that is proportional to δ_{ij}. The integral is best evaluated in spherical coordinates and gives

$$4\pi\rho \int_0^R r^4 dr \delta_{ij} = \frac{4}{5}\pi\rho R^5 \delta_{ij} \tag{7.3.3}$$

For the other piece, we'll need to calculate each component separately. There are only six independent ones to compute because of the symmetry, *viz.*, I'_{11}, I'_{12}, I'_{13}, I'_{22}, I'_{23} and I'_{33}.

---

[2]When the rotational motion can be reduced to two dimensional motion as, for example, during rigid rotations, all particles move in planes perpendicular to a *fixed* axis of rotation whose direction is, say, $\hat{u}$, the moment of inertia *about this axis* is defined simply as the scalar $I^{(\hat{u})} = \sum_\alpha m_\alpha \vec{r}_\alpha'^2$, where $\vec{r}_\alpha'$ represents the *perpendicular displacement from the axis*. This definition arises by projecting the inertia tensor in (7.2.5) along the (rigid) axis: $I^{(\hat{u})} = I_{ij} u_i u_j$. Taking the angular velocity about the axis to point along it, *i.e.*, taking $\vec{\omega} = \omega\hat{u}$, the kinetic energy becomes

$$K = \frac{1}{2} I_{ij}\omega_i\omega_j = \frac{1}{2}(I_{ij}u_iu_j)\omega^2 = \frac{1}{2}I^{(\hat{u})}\omega^2$$

and the angular momentum about the axis is defined as

$$L_i^{(\hat{u})} = (\vec{L}\cdot\hat{u})u_i = (I_{jk}u_j\omega_k)u_i = (I_{jk}u_ju_k)\omega u_i = I^{(\hat{u})}\omega_i.$$

This simplification, however, is useful only for motions which are effectively two dimensional.
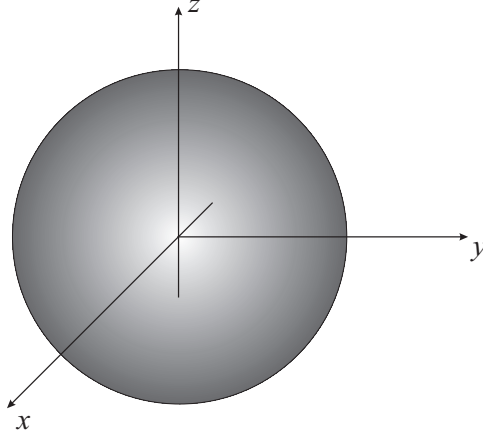
Figure 7.2: Inertia tensor of a sphere about its center

Thus, using $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$ and $z = r \cos\theta$, we get

$$I'_{11} = \rho \int d^3\vec{r}\, x^2 = \rho \int_0^{2\pi} d\varphi \cos^2\varphi \int_0^\pi d\theta \sin^3\theta \int_0^R dr\, r^4 = \frac{4}{15}\pi\rho R^5$$

$$I'_{22} = \rho \int d^3\vec{r}\, y^2 = \rho \int_0^{2\pi} d\varphi \sin^2\varphi \int_0^\pi d\theta \sin^3\theta \int_0^R dr\, r^4 = \frac{4}{15}\pi\rho R^5$$

$$I_{33} = \rho \int d^3\vec{r}\, z^2 = \rho \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin\theta \cos^2\theta \int_0^R dr\, r^4 = \frac{4}{15}\pi\rho R^5 \qquad (7.3.4)$$

and find that all the products of inertia are vanishing. The result is that the inertia tensor is diagonal,

$$I_{ij} = \frac{8}{15}\pi\rho R^5 \delta_{ij} = \frac{2}{5}MR^2\delta_{ij}, \qquad (7.3.5)$$

where $M = (4\pi R^3/3)\rho$ is the mass of the sphere. The inertia tensor turned out diagonal because of the rotational symmetry of the sphere. Let us evaluate the moment of inertia of an object that has manifestly no rotational symmetry.

7.3.2   Homogeneous cube

We will calculate the inertia tensor for the uniform cube, about one of its corners. To be specific we select the corner shown in figure (7.3). Once again, calculating the diagonal term is straightforward if we use Cartesian coordinates:

$$\rho \int_0^L dx\, dy\, dz\, (x^2 + y^2 + z^2)\delta_{ij} = \frac{\rho}{3}\left[x^3 yz + xy^3 z + xyz^3\right]_0^L \delta_{ij} = \rho L^5 \delta_{ij} \qquad (7.3.6)$$
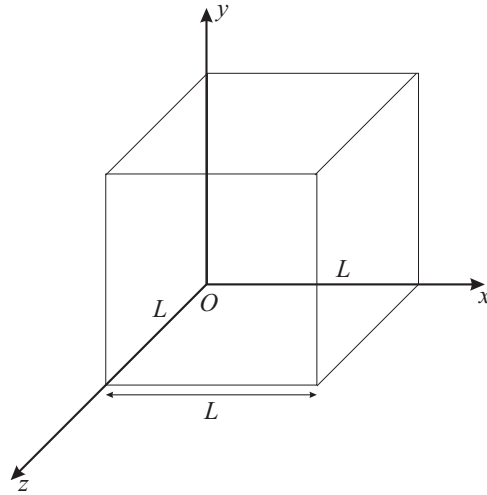
Figure 7.3: Inertia tensor of a cube about one corner

On the other hand,

$$I'_{ij} = \rho \int_0^L \prod_k dx_k (x_i x_j) = \begin{cases} \frac{\rho}{4} L^5, & \forall \ i \neq j \\ \frac{\rho}{3} L^5, & i = j \end{cases} \tag{7.3.7}$$

Thus, using $M = \rho L^3$, we find

$$I = M \begin{bmatrix} \frac{2}{3}L^2 & -\frac{1}{4}L^2 & -\frac{1}{4}L^2 \\ -\frac{1}{4}L^2 & \frac{2}{3}L^2 & -\frac{1}{4}L^2 \\ -\frac{1}{4}L^2 & -\frac{1}{4}L^2 & \frac{2}{3}L^2 \end{bmatrix} \tag{7.3.8}$$

The fact that all the moments of inertia are the same reflects the fact that rotations about the x, y and z axes are all equivalent. This is due to the rectangular symmetry. The existence of non-vanishing products of inertia, on the other hand, indicates that the none of the three Cartesian axes is an "axis of symmetry" of the cube (*i.e.*, the mass is not distributed evenly about the axes).

The question that arises is whether it is possible to find the axes of symmetry, defined as the axes for which all the products of inertia vanish. One can imagine that rotations about symmetry axes are much simpler to analyze because if I is diagonal it can be written

in the form $I = I_i \delta_{ij}$ (no sum) and then the kinetic energy reduces to

$$K = \frac{1}{2}I_{ij}\omega_i\omega_j = \frac{1}{2}\sum_i I_i \omega_i^2 \qquad (7.3.9)$$

and

$$L_i = I_{ij}\omega_j = I_i\omega_i \qquad (7.3.10)$$

We see that the purely mathematical problem of diagonalizing a symmetric matrix is actually in this case the physical problem of determining the axes of symmetry of the body.

The "axes of symmetry" are just the eigenvectors of the inertia tensor. The eigenvalues are called the "principal moments of inertia" and the eigenvectors are called the "principal axes". In many systems of high symmetry, the principal axes are determined by inspection. For the cube they are not so evident, so let us determine the principal axes, passing through the origin. We must diagonalize the matrix

$$\widehat{I} = \begin{bmatrix} \frac{2}{3}\alpha & -\frac{1}{4}\alpha & -\frac{1}{4}\alpha \\ -\frac{1}{4}\alpha & \frac{2}{3}\alpha & -\frac{1}{4}\alpha \\ -\frac{1}{4}\alpha & -\frac{1}{4}\alpha & \frac{2}{3}\alpha \end{bmatrix}, \qquad \alpha = ML^2 \qquad (7.3.11)$$

The secular equation, $|\widehat{I} - \lambda \mathbf{1}| = 0$ is

$$\left(\frac{2}{3}\alpha - \lambda\right)^2 - \frac{\alpha^3}{32} - \frac{3}{16}\alpha^2 \left(\frac{2}{3}\alpha - \lambda\right) = 0 \qquad (7.3.12)$$

and one can easily check that the eigenvalues are

$$\lambda = \left\{\frac{1}{6}\alpha, \ \frac{11}{12}\alpha\right\} \qquad (7.3.13)$$

where the second is degenerate. The diagonalized inertia tensor is then

$$\widehat{I}_D = \begin{bmatrix} \frac{1}{6}\alpha & 0 & 0 \\ 0 & \frac{11}{12}\alpha & 0 \\ 0 & 0 & \frac{11}{12}\alpha \end{bmatrix} \qquad (7.3.14)$$

To find the principal axes (the eigenvectors), we use the eigenvalue equation ($\widehat{v}_n = (u_n, v_n, w_n)$)

$$\widehat{I}\widehat{v}_n = \lambda_n \widehat{v}_n \qquad (7.3.15)$$

which gives

$$
\frac{2\alpha}{3}u_n - \frac{\alpha}{4}v_n - \frac{\alpha}{4}w_n = \lambda_n u_n
$$

$$
-\frac{\alpha}{4}u_n + \frac{2\alpha}{3}v_n - \frac{\alpha}{4}w_n = \lambda_n v_n
$$

$$
-\frac{\alpha}{4}u_n - \frac{\alpha}{4}v_n + \frac{2\alpha}{3}w_n = \lambda_n w_n \tag{7.3.16}
$$

For $\lambda_n = \alpha/6$ we find

$$
\widehat{v}_{\alpha/6} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7.3.17}
$$

For the degenerate eigenvalue, $11\alpha/12$, the principal axes lie in the plane

$$
u + v + w = 0, \tag{7.3.18}
$$

passing through the origin with normal given precisely by $\widehat{v}_{\alpha/6}$. Thus $11\alpha/12$ is the moment of inertia of a cube rotating about any axis passing through the origin and in the plane perpendicular to $\widehat{v}_{\alpha/6}$. It should be obvious that the principal axes for the cube are obtainable by a rotation of the original axes. We now prove that this is true in general: the inertia tensor may always be diagonalized by a rotation of the original system.

Consider a rotation of the original coordinate system so that

$$
\vec{r}_\alpha \to \vec{r}_\alpha' = \widehat{R}\vec{r}_\alpha \tag{7.3.19}
$$

Because $\widehat{I}$ is a second rank tensor, under the rotation above it transforms as

$$
I_{ij} \to I_{ij}' = \sum_{lm} R_{il}R_{jm}I_{lm} \tag{7.3.20}
$$

or

$$
\widehat{I} = \widehat{R}\, I\, \widehat{R}^T. \tag{7.3.21}
$$

Now what condition must be satisfied so that I_{ij}' is diagonal? We should have

$$
I_i'\delta_{ij} = \sum_{lm} R_{il}R_{jm}I_{lm} \Rightarrow \sum_{j} I_i'\delta_{ij}R_{nj}^T = \sum_{jlm} R_{il}R_{nj}^T R_{jm}I_{lm} \tag{7.3.22}
$$

So we see that

$$
I_i'R_{ni}^T = \sum_{lm} R_{il}\delta_{nm}I_{lm} = \sum_{l} R_{il}I_{ln} \tag{7.3.23}
$$

and therefore

$$\sum_l R_{il}(I_{ln} - I'_i \delta_{ln}) = 0 \tag{7.3.24}$$

which is possible if and only if the secular equation $|\widehat{I} - I_i \mathbf{1}| = 0$ holds because $|\widehat{R}| = 1$. This is the eigenvalue condition which yields the principal moments of inertia. Therefore the principal axes are determined by a rotation of the original system.

Let us return to the example of the cube and find the rotation matrix that takes the $x-$ axis to the diagonal principal axis of the cube. We will see that it diagonalizes the inertia tensor obtained before. We want to take the unit vector $\widehat{x} = (1, 0, 0)$ to the unit vector

$$\widehat{x}' = \frac{1}{\sqrt{3}}(1, 1, 1)$$

and for the $\widehat{y}'$ and $\widehat{z}'$ axes we choose *any* axes in the plane perpendicular to $\widehat{x}'$. For instance, choose

$$\widehat{y}' = \frac{1}{\sqrt{2}}(-1, 1, 0) \tag{7.3.25}$$

and

$$\widehat{z}' = \widehat{x}' \times \widehat{y}' = \frac{1}{\sqrt{6}}(-1, -1, 2) \tag{7.3.26}$$

Recalling that the basis vectors transform as

$$R_{ik}\widehat{e}_k = \widehat{e}'_i \tag{7.3.27}$$

we can take an inner product of both sides with $\widehat{e}_j$ to get

$$R_{ik}\widehat{e}_k \cdot \widehat{e}_j = R_{ik}\delta_{kj} = R_{ij} = \widehat{e}'_i \cdot \widehat{e}_j \tag{7.3.28}$$

which gives

$$\widehat{R} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix} \tag{7.3.29}$$

and it is easy to check that

$$\widehat{I}_D = \widehat{R}\,\widehat{I}\,\widehat{R}^T = \begin{bmatrix} \frac{\alpha}{6} & 0 & 0 \\ 0 & \frac{11}{12}\alpha & 0 \\ 0 & 0 & \frac{11}{12}\alpha \end{bmatrix} \tag{7.3.30}$$

as expected.

7.4   The parallel axis theorem

Although it is convenient to consider a body system with its origin at the center of mass because of the simplification in the expressions for the angular momentum and kinetic energy that ensue, it is necessary also to consider body systems that do not have their origins at the center of mass. How does the inertia tensor behave under translations? Consider a translation of the origin by $\vec{\xi}$, then $\vec{r}_\alpha \to \vec{r}'_\alpha = \vec{r}_\alpha + \vec{\xi}$ and

$$
\begin{aligned}
I_{ij} \to I'_{ij} &= \sum_\alpha m_\alpha \left[(\vec{r}_\alpha + \vec{\xi})^2 \delta_{ij} - (\vec{r}_{\alpha,i} + \xi_i)(\vec{r}_{\alpha,j} + \xi_j) \right] \\
&= \sum_\alpha m_\alpha \left[(\vec{r}_\alpha^2 + \vec{\xi}^2 + 2\vec{r}_\alpha \cdot \vec{\xi}) \delta_{ij} - (\vec{r}_{\alpha,i}\vec{r}_{\alpha,j} + \vec{r}_{\alpha,i}\xi_j + \vec{r}_{\alpha,j}\xi_i + \xi_i \xi_j) \right]
\end{aligned}
$$

$$(7.4.1)$$

Collecting terms,

$$
I'_{ij} = I_{ij} + M(\vec{\xi}^2 \delta_{ij} - \xi_i \xi_j) \tag{7.4.2}
$$

where we have made use of the fact that $\vec{r}_\alpha$ is measured from the center of mass, so $\sum_\alpha m_\alpha \vec{r}_\alpha = 0$ and therefore I_{ij} represents the inertia tensor about the center of mass. This is "Steiner's parallel axis theorem", which can also be stated as follows:

Steiner's parallel axis theorem: The inertia tensor about any body system is the sum of the inertia tensor about the center of mass and the inertia tensor of the center of mass about the new body system.[3]

7.5   Dynamics

Let us return to the dynamics of the rigid body. We have seen that the kinetic energy and angular momentum have relatively simple expressions in terms of the inertia tensor and in particular if the body system is taken with its origin at the center of mass. Yet, how is the position of the body described? It is utterly redundant to specify the positions of *all* the particles making up the body because their relative positions are constrained. In fact, the body being rigid, the position vectors of any two distinct points on the body determine completely the position vectors of all the other points on it, therefore only six coordinates are required to describe a rigid body! Three of those coordinates may be thought of as determining the position of the center of mass and the remaining three as determining the orientation of the body in space. The coordinates of the center of mass

---

[3]For rigid rotations, the parallel axis theorem takes the more familiar form:

$$
I'^{(\hat{u})} = I'_{ij} u_i u_j = I_{ij} u_i u_j + M h_\perp^2 = I_{cm}^{(\hat{u})} + M h_\perp^2
$$

where $h_\perp$ represents the perpendicular distance of the axis from the center of mass.

are straightforward to give. To specify the orientation of the body we consider a system chosen arbitrarily but fixed rigidly to the body so that it rotates with the body relative to the center of mass system. The orientation of the body is then the orientation of the attached system relative to the center of mass system (not rotating). The orientation is specified by the time dependent parameters of a rotation matrix,

$$\widehat{R} = \widehat{R}(\theta_1(t), \theta_2(t), \theta_3(t)) \tag{7.5.1}$$

Thus our configuration space is coordinatized by $(\vec{r}_{\text{cm}}, \theta_1, \theta_2, \theta_3)$. If $\vec{r}_\alpha$ represents the position of a particle in the center of mass system,

$$\frac{d\vec{r}_\alpha}{dt} = \vec{\omega} \times \vec{r}_\alpha. \tag{7.5.2}$$

Since the change is a pure rotation,

$$\delta\vec{r}_{\alpha,i} = -\delta R_{ik}(t)\vec{r}_{\alpha,k} \tag{7.5.3}$$

it follows that

$$\frac{d\vec{r}_\alpha}{dt} = -\frac{dR_{ik}}{dt}\vec{r}_{\alpha,k} \tag{7.5.4}$$

But, again, infinitesimally,

$$\delta R_{ik} = \delta\theta_j (\widehat{J}_j)_{ik} \tag{7.5.5}$$

where $\widehat{J}_j$ is a generator of the rotation group, $[\widehat{J}_j]_{ik} = \epsilon_{jik}$, where ϵ_{ijk} is the Levi-Civita tensor. Therefore

$$\delta R_{ik} = -\delta\theta_j \epsilon_{ijk} \Rightarrow \frac{dR_{ik}}{dt} = -\epsilon_{ijk}\frac{d\theta_j}{dt} \tag{7.5.6}$$

This gives

$$\frac{d\vec{r}_\alpha}{dt} = \epsilon_{ijk}\frac{d\theta_j}{dt}\vec{r}_{\alpha,k} \tag{7.5.7}$$

Comparing with (7.5.2) we see that

$$\omega_i = d\theta_i/dt, \tag{7.5.8}$$

which defines each component of the angular velocity as the rate of change of the rotation angle about the corresponding center of mass axis.

Simple counting should indicate that the three equations of motion in (7.1.4) are not sufficient to describe the motion of the rigid body. Three more equations of motion are required because there are six independent coordinates in all and these are the "torque" equations, so let us consider the angular momentum of the body. Using (7.2.6)

$$\frac{d\vec{L}}{dt} = \sum_\alpha \vec{r}_\alpha \times \vec{F}_\alpha = \vec{\tau}^{\text{ext}} \tag{7.5.9}$$

from which we get

$$\frac{d\vec{L}}{dt} = \vec{r}_{\text{cm}} \times \vec{F}^{\text{ext}} + \widehat{I}_{\text{cm}} \, \frac{d\vec{\omega}}{dt} = \vec{\tau}^{\text{ext}} \tag{7.5.10}$$

or

$$\widehat{I}_{\text{cm}} \, \frac{d\vec{\omega}}{dt} = \vec{\tau}'^{\text{ext}} \tag{7.5.11}$$

where $\vec{\tau}'^{\text{ext}}$ represents the torque about the center of mass. These equations, together with (7.1.4) constitute a complete description of the motion of a rigid body.[4]

Finally, let us rewrite the torque equation in terms of the time derivative appropriate to a body frame that is rotating with the body. Using (6.2.16) we find

$$I_{ij}\frac{d'\omega_j}{dt} + \epsilon_{ijk}\omega_j I_{kr}\omega_r = \tau_i'^{\text{ext}}, \tag{7.5.12}$$

but we can put this in a more elegant form by multiplying both sides by ϵ_{ilm} and summing over i,

$$\epsilon_{ilm}\left[I_{ij}\frac{d'\omega_j}{dt} + \epsilon_{ijk}\omega_j I_{kr}\omega_r\right] = \epsilon_{ilm}\tau_i'^{\text{ext}} \tag{7.5.13}$$

Now if the axes coincide with the principal axes, then $I_{ij} = I_i\delta_{ij}$ and

$$\epsilon_{ilm}I_i\frac{d'\omega_i}{dt} + (\delta_{lj}\delta_{mk} - \delta_{lk}\delta_{jm})\omega_j I_{kr}\omega_r = \epsilon_{ilm}\tau_i'^{\text{ext}} \tag{7.5.14}$$

or

$$\epsilon_{ilm}I_i\frac{d'\omega_i}{dt} + \omega_l I_m\omega_m - \omega_m I_l\omega_l = \epsilon_{ilm}\tau_i'^{\text{ext}} \tag{7.5.15}$$

and with appropriate renaming of the indices,

$$(I_i - I_j)\omega_i\omega_j - \epsilon_{ijk}\left(I_k\frac{d'\omega_k}{dt} - \tau_k\right) = 0. \tag{7.5.16}$$

These three equations (one for every pair (i,j)) are called **Euler's equations** for rotational motion. They are just a restatement of the torque equation as we have seen.

Instead of writing the equations of motion in terms of rotations about the the three Cartesian axes, as we have just done, one could also consider rotations parameterized by the Euler angles (ϕ, θ, ψ) that were introduced in the first chapter. This is just a change in variables, of course, and the problem is to find the values of ω_i in terms of (ϕ, θ, ψ)

---

[4]For rigid rotations,

$$\frac{dL^{(\widehat{u})}}{dt} = \widehat{u} \cdot \vec{\tau}'^{\text{ext}}$$

completes the system of equations required to describe the motion.

and $(\dot{\phi}, \dot{\theta}, \dot{\psi})$. This is easy, for all we need to do is compare the expressions for a rotation matrix, $\widehat{R}$, in two parameterizations,

$$\widehat{R} = e^{\psi\widehat{J}_3}e^{\theta\widehat{J}_1}e^{\phi\widehat{J}_3} = e^{\theta_i\widehat{J}_i} \tag{7.5.17}$$

Taking a derivative of $\widehat{R}$ we find in the Euler parameterization

$$\frac{d\widehat{R}}{dt} = (\dot{\psi}\widehat{J}_3 + \dot{\theta}e^{\psi\widehat{J}_3}\widehat{J}_1 e^{-\psi\widehat{J}_3} + \dot{\phi}e^{\psi\widehat{J}_3}e^{\theta\widehat{J}_1}\widehat{J}_3 e^{-\psi\widehat{J}_1}e^{-\theta\widehat{J}_3})\widehat{R} \tag{7.5.18}$$

and in the Cartesian parameterization

$$\frac{d\widehat{R}}{dt} = \dot{\theta}_i\widehat{J}_i\widehat{R} \tag{7.5.19}$$

To simplify the expression we obtained for the derivative of $\widehat{R}$ in the Euler parameterization, we need to use the Baker-Campbell-Hausdorf formula:

$$e^{\widehat{A}}\ \widehat{B}\ e^{-\widehat{A}} = \widehat{B} + [\widehat{A}, \widehat{B}] + \frac{1}{2!}[\widehat{A}, [\widehat{A}, \widehat{B}]] + \dots \tag{7.5.20}$$

Then

$$
\begin{aligned}
e^{\psi\widehat{J}_3}\widehat{J}_1 e^{-\psi\widehat{J}_3} &= \widehat{J}_3 + \psi[\widehat{J}_3, \widehat{J}_1] + \frac{\psi^2}{2!}[\widehat{J}_3, [\widehat{J}_3, \widehat{J}_1]] + \dots \\
&= \widehat{J}_1 - \psi\widehat{J}_2 - \frac{\psi^2}{2!}\widehat{J}_1 + \dots
\end{aligned}
\tag{7.5.21}
$$

where we used the algebra of the three dimensional rotation group,

$$[\widehat{J}_i, \widehat{J}_j] = -\epsilon_{ijk}\widehat{J}_k \tag{7.5.22}$$

Continuing for a few terms we find

$$e^{\psi\widehat{J}_3}\widehat{J}_1 e^{-\psi\widehat{J}_3} = \cos\psi\widehat{J}_1 - \sin\psi\widehat{J}_2 \tag{7.5.23}$$

and likewise

$$e^{\theta\widehat{J}_1}\widehat{J}_3 e^{-\theta\widehat{J}_1} = \cos\theta\widehat{J}_3 + \sin\theta\widehat{J}_2$$

$$e^{\psi\widehat{J}_3}\widehat{J}_2 e^{-\psi\widehat{J}_3} = \cos\psi\widehat{J}_2 + \sin\psi\widehat{J}_1 \tag{7.5.24}$$

Putting it all together, we find

$$\dot{\theta}_1\widehat{J}_1 + \dot{\theta}_2\widehat{J}_2 + \dot{\theta}_3\widehat{J}_3 = \dot{\psi}\widehat{J}_3 + \dot{\theta}(\cos\psi\widehat{J}_1 - \sin\psi\widehat{J}_2) + \dot{\phi}(\cos\theta\widehat{J}_3 + \sin\theta[\cos\psi\widehat{J}_2 + \sin\psi\widehat{J}_1]) \tag{7.5.25}$$

so that comparing terms

$$\dot{\theta}_1 = \omega_1 = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$

$$\dot{\theta}_2 = \omega_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

$$\dot{\theta}_3 = \omega_3 = \dot{\phi}\cos\theta + \dot{\psi} \tag{7.5.26}$$

relates the angular velocity that we have used throughout to the time derivatives of the Euler angles.

Chapter 8

Mechanical Waves

In physics, a mechanical **wave** is a disturbance or deformation of a continuous medium that propagates through it, transferring energy from one point in the medium to another with little or (ideally) no mass transport of the medium itself. The energy of the wave consists in the kinetic and potential energies of the particles of the medium, which are displaced from their equilibrium positions as the wave propagates through it. A mechanical wave may be **transverse** or **longitudinal**, depending on the direction of the displacement from equilibrium. For transverse waves the displacement of the particles of the medium occurs perpendicular to the direction of propagation of the wave itself and for longitudinal waves the displacement occurs parallel to the direction of the propagation. Classic examples of the two kinds of waves are waves propagating in a string (transverse) and sound waves in air (longitudinal). In this chapter we develop the basic theory of waves in continuous media. Waves, however, will henceforth be a recurring theme, turning up again and again both as we learn of new symmetries (relativity) and as we we continue to deepen our understanding of the mathematical methods of mechanics and the physics of continuous media.

8.1   The Wave Equation

It is best to begin with the simplest example of a wave: a deformation in one dimension, produced in a string. Imagine an ideal, by which we will mean perfectly elastic, string of length L that is either fixed at one end (as indicated by the wall in figure 8.1) or free and held at the other. By a jerk of the hand, one can induce a deformation in the string, which we know, by common experience, will travel through the string beginning at the hand and moving toward the fixed end at a constant speed. One also notices that the actual *shape* of the deformation in the string does not change as it propagates through the string, assuming that there is no loss of energy, so let us consider two observers, one stationary
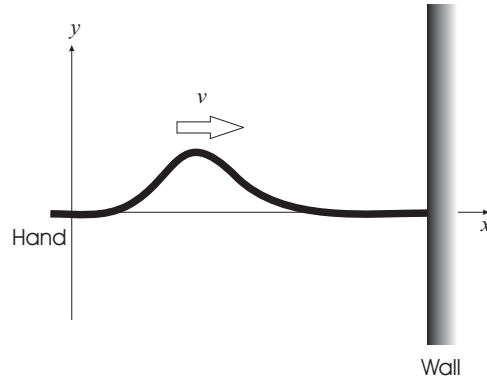
Figure 8.1: A deformation traveling in a string.

with respect to the string (medium) and the other moving with the deformation.

If the string is laid out along the $x-$axis, the deformation or wave function will be represented by each observer as some function of (t, x) and (t, x'), *i.e.,*

$$y = y(t, x), \quad y' = y'(t, x'), \tag{8.1.1}$$

where x' represents the coordinate of the observer moving with the deformation and x the coordinate of stationary observer, and we assume that the two origins coincide at $t = 0$. However, because the moving observer is stationary relative to the deformation, we expect that y' can have no explicit dependence on t, *i.e.,* $y' = y'(x')$.

Suppose that the deformation travels through the string with speed v, then Galilean relativity tells us that

$$x' = x \mp vt, \quad y' = y, \tag{8.1.2}$$

where the negative sign applies for a wave traveling to the right and the positive applies for a wave traveling to the left. It follows that

$$y = y(x - vt), \tag{8.1.3}$$

so the wave function can depend on space and time only in a very specific combination. This, in turn, implies that the wave function must satisfy a particular equation, called the wave equation. Since y could be *any* function of its argument, the argument must be dimension free. Therefore it becomes necessary to introduce a constant, k, with dimension of l^{-1}, and consider instead

$$y_k(t, x) = y_k(kx - \omega t), \tag{8.1.4}$$

where $\omega = kv$. The constant k we just introduced, with mechanical dimension l^{-1}, is called the **wave number** and the constant ω, whose mechanical dimension is t^{-1}, is the **angular**

frequency. The physical significance of k and ω will become clearer as we progress. Of course if the wave were moving from right to left (instead of from left to right) then the same arguments would lead to a wave function of the form

$$y_k(t, x) = y_k(kx + \omega t). \tag{8.1.5}$$

It is easy to see, by repeatedly taking derivatives and using the chain rule, that the wave function must satisfy the so-called **wave equation**,

$$\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0, \tag{8.1.6}$$

in either case. For a physical wave or pulse, this equation must follow directly from Newton's laws. Note, however, the equation itself is *not* invariant under Galilean transformations[1] and holds *only in the rest frame of the medium*. We will show how this comes about in strings and in fluid media below.

We end this section by pointing out that the equation is linear and this means that any linear combination (a **superposition**) of solutions is also a solution of the wave equation. In particular, a solution may contain waves traveling to the left and to the right,

$$y_k(t, x) = a_k f(kx - \omega t) + b_k g(kx + \omega t), \tag{8.1.7}$$

where a_k and b_k are constants and more generally, solutions of differing wave numbers may also be superposed to get

$$y(t, x) = \sum_k \left[a_k f(kx - \omega t) + b_k g(kx + \omega t) \right]. \tag{8.1.8}$$

Superposition is a powerful tool in the development of solutions of the wave equation.

8.2   The Wave Equation from Dynamics

Let us now demonstrate how the arguments of the previous section, which employed only kinematical concepts via the Galilean relativity principle, have led in fact to the correct form of the wave equation. We will do this by obtaining the wave equation from dynamics, employing Newton's laws. It will turn out that dynamical considerations will not only recover the wave equation but also relate the wave velocity to the macroscopic properties of the medium.

8.2.1   Waves in Strings

Consider first an element of string of *unstretched* length Δx and mass per unit length μ as shown in 8.2. It's mass is then $\Delta m = \mu \Delta x$. The two tensions acting on either edge of

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[1]<u>Problem</u>: Show that the wave equation is not invariant under Galilean transformations.
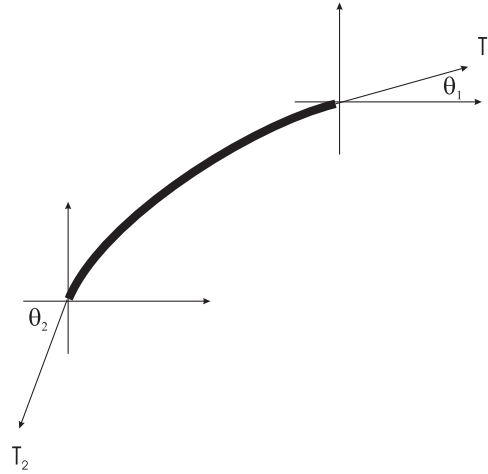
Figure 8.2: An element of string

it have $x-$ and $y-$components that we now write as

$$\sum T_x = T_1 \cos \theta_1 - T_2 \cos \theta_2$$

$$\sum T_y = T_1 \sin \theta_1 - T_2 \sin \theta_2 \tag{8.2.1}$$

The string element is able only to vibrate in the vertical direction (this is a transverse wave) therefore there can be no net force in the $x-$direction,

$$\sum T_x = T_1 \cos \theta_1 - T_2 \cos \theta_2 = 0 \Rightarrow T_2 = T_1 \frac{\cos \theta_1}{\cos \theta_2} \tag{8.2.2}$$

Inserting this into the equation for the $y-$component of the net tension we find

$$\sum T_y = T_1 \cos \theta_1 (\tan \theta_1 - \tan \theta_2) = T_1 \cos \theta_1 \left[\left. \frac{\partial y}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial y}{\partial x} \right|_x \right] \tag{8.2.3}$$

using the familiar definition of the tangent function. This is the net force in the $y-$direction, so by Newton's second law,

$$\mu \Delta x \frac{\partial^2 y}{\partial t^2} = T_1 \cos \theta_1 \left[\left. \frac{\partial y}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial y}{\partial x} \right|_x \right] \tag{8.2.4}$$

Now consider the approximation in which the amplitude of oscillation (of our tiny bit of string) is not large, so that the angles $\theta_{1,2}$ are small. Then $\cos \theta_{1,2} \approx 1$ and, from the $\sum T_x$
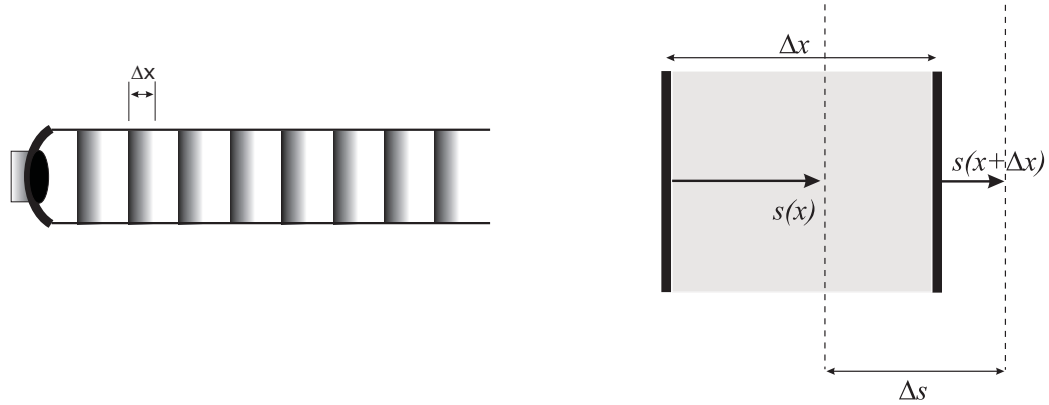
Figure 8.3: Sound waves in continuous media

equation, we see that $T_1 \approx T_2 \equiv T$ (say). Newton's second law now reads,

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\left[\left. \frac{\partial y}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial y}{\partial x} \right|_x \right]}{\Delta x} \tag{8.2.5}$$

Take the limit as $\Delta x \to 0$ to get

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu} \frac{\partial^2 y}{\partial x^2} \tag{8.2.6}$$

which, when compared to the wave equation, will be recognized as the equation of a wave propagating with a speed

$$v = \sqrt{\frac{T}{\mu}} \tag{8.2.7}$$

in the string.

8.2.2   Sound Waves in Media

Consider a cylindrical chamber of cross section A with a driver on the left, which produces compressions and rarefactions of a medium in the chamber (see figure 8.3), and otherwise open. Due to the disturbance produced by the driver, layers of the medium oscillate about their mean (undisturbed) positions. Let the displacement of a layer from its undisturbed position, say x, be denoted by $s(x,t)$. In the absence of the driver, the displacement of the element is just $s(x,t) \equiv 0$. Consider some volume element of the medium and let Δx be its thickness, which we will eventually take to zero. Owing to the difference in pressure

on the sides of the element, the net force in the $x-$direction experienced by this element is

$$F_x = [p(x) - p(x + \Delta x)]A \tag{8.2.8}$$

Since the mass of the element is $\Delta m = \rho A \Delta x$, where ρ is the mass density and $A\Delta x = \Delta V$ is its volume, we have

$$\rho \Delta x \frac{\partial^2 s}{\partial t^2} = [p(x) - p(x + \Delta x)] \implies \rho \frac{\partial^2 s}{\partial t^2} = -\frac{\partial p}{\partial x} \tag{8.2.9}$$

It's worth introducing the so-called Bulk Modulus of the medium (a bulk property, which will play the role of the tension in the string). It is defined by

$$B = -V \left(\frac{\partial p}{\partial V} \right)_{\text{process}} \tag{8.2.10}$$

where (p, V) are the pressure and volume of some region. The Bulk modulus of any medium depends on the process (that is the reason for the subscript in the above equation) and represents the response of the medium to stress by measuring the rate at which pressure changes relative to volume. Eq. (8.2.10) means that the change in pressure within the element that has been compressed by ΔV is given in terms of the bulk modulus by

$$\Delta p = -B \frac{\Delta V}{V} \tag{8.2.11}$$

If $s(x)$ represents the displacement from equilibrium of a fluid layer, originally situated at x, then the change in volume of the element upon stress will be

$$\Delta V = A[s(x + \Delta x) - s(x)] \tag{8.2.12}$$

The pressure change within the element is

$$\Delta p = -B \left[\frac{s(x + \Delta x) - s(x)}{\Delta x} \right] \tag{8.2.13}$$

Taking the limit as $\Delta x \to 0$ gives the change in pressure at any fluid layer in terms of the spatial rate at which the fluid layer is displaced in the direction of the traveling wave:

$$\Delta p(x) = -B \frac{\partial s}{\partial x} \tag{8.2.14}$$

The actual pressure at any layer is therefore

$$p(x) = p_0 + \Delta p = p_0 - B \frac{\partial s}{\partial x} \tag{8.2.15}$$

where p_0 represents the unstressed pressure in the medium. Thus,

$$\frac{\partial p}{\partial x} = -B\frac{\partial^2 s}{\partial x^2}. \tag{8.2.16}$$

But, dividing (8.2.9) by Δx and taking the limit as $\Delta x \to 0$, we get

$$\rho\frac{\partial^2 s}{\partial t^2} = -\frac{\partial p}{\partial x} \tag{8.2.17}$$

and therefore,

$$\frac{\partial^2 s}{\partial t^2} = \frac{B}{\rho}\frac{\partial^2 s}{\partial x^2} \tag{8.2.18}$$

which is the wave equation. Comparison shows that the speed of sound is given by

$$v = \sqrt{\frac{B}{\rho}} \tag{8.2.19}$$

Let's see how it works for the propagation of sound in an ideal gas.

Sound traveling through dry air, for example, is essentially an adiabatic process even though it may seem that the surrounding air would act as a reservoir. The sound wave causes oscillations in pressure but the oscillations are fast enough that energy cannot be transferred in the form of heat from compressed regions to rarified regions in such a way as to keep the temperature constant. For an adiabatic process, the effective equation of state is

$$pV^\gamma = \text{const.} \Rightarrow \left(\frac{\partial p}{\partial V}\right)_{\text{ad}} = -\gamma\frac{p}{V} \Rightarrow B = \gamma p \tag{8.2.20}$$

Using $p = nkT = \rho kT/m$, where m is the average molecular mass,

$$v = \sqrt{\frac{\gamma p}{\rho}} = \sqrt{\frac{\gamma kT}{m}} \tag{8.2.21}$$

For the (diatomic) "molecule of air" $\gamma = 7/5$ and $m = 28.8$ u. We find $v \approx 347.1$ m/s. For an isothermal propagation on the other hand we should have

$$B = -V\left(\frac{\partial p}{\partial V}\right)_T = p \to v = \sqrt{\frac{kT}{m}} \tag{8.2.22}$$

which is smaller than the result for adiabatic propagation by about 15.5%.

Although the wave equation is given in terms of the displacement, s, of layers of the medium, we can give it in terms of the pressure in the medium. For example, a sinusoidal wave

$$s(x,t) = s_{\max}\sin(kx - \omega t), \tag{8.2.23}$$

using (8.2.15), gives

$$\Delta p = -Bks_{\max}\cos(kx - \omega t) \tag{8.2.24}$$

Substituting $B = \rho v^2$ and $kv = \omega$ gives

$$\Delta p = -\rho v \omega s_{\max}\cos(kx - \omega t) \tag{8.2.25}$$

Notice that it is $90°$ out of phase with the displacement.[2] Still, the maximum value of the pressure is related to the maximum of the displacement,

$$\Delta p_{\max} = \rho v \omega s_{\max} \tag{8.2.26}$$

as one expects.

8.3   Energy Transfer

8.3.1   Waves in Strings

The energy contained by a unit length of the string is the sum of its kinetic and its potential energies. For the kinetic energy we could write (per unit length)

$$\frac{dK}{dx} = \frac{1}{2}\mu v_y^2 = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \tag{8.3.1}$$

For the potential energy, we identify the force on a unit length of the string from the right hand side of the wave equation, *i.e.*,

$$F = T\frac{\partial^2 y}{\partial x^2}. \tag{8.3.2}$$

This force is directed opposite to the displacement (in the y direction) and so the potential energy (per unit length) is

$$\frac{dU}{dx} = -\int^y(-F)d\overline{y} = \int^y T\frac{\partial^2\overline{y}}{\partial x^2}d\overline{y} = T\int^y \overline{y}'\frac{\partial\overline{y}'}{\partial\overline{y}}d\overline{y} = \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2 \tag{8.3.3}$$

where $y' = \partial y/\partial x$. The total energy per unit length of the string is therefore

$$\frac{dE}{dx} = \frac{1}{2}\mu\left[\left(\frac{\partial y}{\partial t}\right)^2 + v^2\left(\frac{\partial y}{\partial x}\right)^2\right] \tag{8.3.4}$$

---

[2]Problem: Give a qualitative argument for why this makes sense, *i.e.*, what do you expect the particles of the medium are doing very close to a minimum of the displacement and why would this lead to a maximum pressure?

where we used $T = \mu v^2$. The precise value of this energy per unit length would depend on the wave-form $y(t, x)$, but it follows that the energy contained between x_1 and x_2 can be given by

$$E = \frac{1}{2} \int_{x_1}^{x_2} dx \, \mu \left[\left(\frac{\partial y}{\partial t} \right)^2 + v^2 \left(\frac{\partial y}{\partial x} \right)^2 \right] \tag{8.3.5}$$

where y represents the deformation of the string by the wave.

As a simple but useful example consider a right moving, sinusoidal wave given by

$$y(t, x) = y_{\max} \sin(kx - \omega t) \tag{8.3.6}$$

and note that

$$\frac{\partial y}{\partial t} = -\omega y_{\max} \cos(kx - \omega t), \quad \frac{\partial y}{\partial x} = k y_{\max} \cos(kx - \omega t) \tag{8.3.7}$$

then

$$\frac{dE}{dx} = \frac{1}{2} \mu y_{\max}^2 (\omega^2 + v^2 k^2) \cos^2(kx - \omega t) \tag{8.3.8}$$

but, because $v = \omega/k$, we find that the energy contained in a segment from x_1 to x_2 of the string

$$E = \mu \omega^2 y_{\max}^2 \int_{x_1}^{x_2} \cos^2(kx - \omega t) dx \tag{8.3.9}$$

In particular, over one entire wavelength,

$$E_\lambda = \mu \omega^2 y_{\max}^2 \int_a^{a+\lambda} \cos^2 \left(\frac{2\pi x}{\lambda} - \omega t \right) dx = \frac{1}{2} \mu \omega^2 y_{\max}^2 \lambda \tag{8.3.10}$$

where a is *any* starting point on the string. As the wave propagates, the energy transferred per second is then

$$P = \frac{E_\lambda}{\tau} = \frac{1}{2} \mu \omega^2 y_{\max}^2 \frac{\lambda}{\tau} = \frac{1}{2} \mu \omega^2 y_{\max}^2 v \tag{8.3.11}$$

because τ is the period. Notice that the kinetic and potential energies contribute equally to the total energy over a wavelength,

$$K_\lambda = U_\lambda = \frac{1}{4} \mu \omega^2 y_{\max}^2 \lambda \tag{8.3.12}$$

as they do for the simple harmonic oscillator over one period.

8.3.2 Sound Waves

The energy contained by a unit volume of the medium is the sum of its kinetic and its potential energies. For the kinetic energy we could write (per unit volume)

$$\frac{dK}{dV} = \frac{1}{2}\rho v_s^2 = \frac{1}{2}\rho \left(\frac{\partial s}{\partial t}\right)^2 \tag{8.3.13}$$

For the potential energy, we identify the force on a unit volume of the medium from the right hand side of the wave equation, *i.e.*,

$$F = B\frac{\partial^2 s}{\partial x^2}. \tag{8.3.14}$$

This force is directed opposite to the displacement and so the potential energy (per unit volume) is

$$\frac{dU}{dV} = -\int^s (-F)d\overline{s} = \int^s B\frac{\partial^2 \overline{s}}{\partial x^2}ds = B\int^s \overline{s}'\frac{\partial \overline{s}'}{\partial s}ds = \frac{1}{2}B\left(\frac{\partial s}{\partial x}\right)^2 \tag{8.3.15}$$

where $s' = \partial s/\partial x$ as before. The total energy per unit volume is therefore

$$\frac{dE}{dV} = \frac{1}{2}\rho\left[\left(\frac{\partial s}{\partial t}\right)^2 + v^2\left(\frac{\partial s}{\partial x}\right)^2\right], \tag{8.3.16}$$

where we used $B = \rho v^2$, and gives the energy contained in any volume V of the medium containing the travelling wave

$$E = \frac{1}{2}\int_V dV\rho\left[\left(\frac{\partial s}{\partial t}\right)^2 + v^2\left(\frac{\partial s}{\partial x}\right)^2\right]. \tag{8.3.17}$$

The precise value of this energy per unit length would depend on the wave-form $s(x,t)$.

Again consider right-moving sinusoidal waves,

$$s(x,t) = s_{\max}\sin(kx - \omega t) \tag{8.3.18}$$

traveling in the positive $x-$direction, and note that

$$\frac{\partial s}{\partial t} = -\omega s_{\max}\cos(kx - \omega t), \quad \frac{\partial s}{\partial x} = k s_{\max}\cos(kx - \omega t) \tag{8.3.19}$$

then

$$\frac{dE}{dV} = \frac{1}{2}\rho s_{\max}^2(\omega^2 + v^2 k^2)\cos^2(kx - \omega t) \tag{8.3.20}$$

but, because $v = \omega/k$ and because we can write $dV = A\,dx$, we find that the energy contained in a cylindrical region of cross-sectional area A between x_1 to x_2 of the medium is

$$E = (\rho A)\omega^2 s_{\text{max}}^2 \int_{x_1}^{x_2} \cos^2(kx - \omega t)\,dx \qquad (8.3.21)$$

(assuming a constant density). In particular, over one entire wavelength we find

$$E_\lambda = (\rho A)\omega^2 s_{\text{max}}^2 \int_{a}^{a+\lambda} \cos^2\left(\frac{2\pi x}{\lambda} - \omega t\right) dx = \frac{1}{2}(\rho A)\omega^2 s_{\text{max}}^2 \lambda \qquad (8.3.22)$$

where a is *any* starting point. As the wave propagates, the energy transferred per second through the medium by the sound way is then

$$P = \frac{E_\lambda}{\tau} = \frac{1}{2}(\rho A)\omega^2 s_{\text{max}}^2 \frac{\lambda}{\tau} = \frac{1}{2}(\rho A)\omega^2 s_{\text{max}}^2 v \qquad (8.3.23)$$

because τ is the period. As before, the kinetic and potential energies contribute equally to the total energy over an entire wavelength,

$$K_\lambda = U_\lambda = \frac{1}{4}(\rho A)\omega^2 s_{\text{max}}^2 \lambda. \qquad (8.3.24)$$

8.4   Solutions of the Wave Equation

A direct and standard approach to solving linear partial differential equations of the above form is to apply the method of "separation of variables". In applying this method one assumes that the solution is "separable" *i.e.,* that it can be expressed as

$$y(t, x) = a(t)b(x) \qquad (8.4.1)$$

and therefore it is important to bear in mind that solutions that do not separate as above, if they exist, will not be obtained by this method. Inserting the *ansatz* into the wave equation gives

$$\frac{d^2 a(t)}{dt^2} b(x) = v^2 a(t)\frac{d^2 b(x)}{dx^2}, \quad \text{or} \quad \frac{1}{a(t)}\frac{d^2 a(t)}{dt^2} = \frac{v^2}{b(x)}\frac{d^2 b(x)}{dx^2} \qquad (8.4.2)$$

where, we have divided by $a(t)b(x)$ to obtain the last form. Now the left hand side of the equation depends only on t whereas the right hand side depends only on x. They may be equal to one another only if they are each constant,

$$\frac{1}{a(t)}\frac{d^2 a(t)}{dt^2} = \frac{v^2}{b(x)}\frac{d^2 b(x)}{dx^2} = -\omega^2. \qquad (8.4.3)$$

We have called the constant $-\omega^2$ so that (i) we may have oscillatory solutions, hence the negative sign, and (ii) the definition of ω in the previous sections is consistent with our solutions. Thus we have

$$\frac{d^2a(t)}{dt^2} + \omega^2 a(t) = 0 = \frac{d^2b(x)}{dx^2} + k^2 b(x) \tag{8.4.4}$$

where we have made use of the fact that $\omega = kv$. The solutions are therefore

$$a(t) = Ae^{i\omega t} + Be^{-i\omega t}, \quad b(x) = Ce^{ikx} + Be^{-ikx}, \tag{8.4.5}$$

where A, B, C, D are complex coefficients and hence $y(t, x)$ has four terms

$$y(t, x) = A_{++}e^{i(kx+\omega t)} + A_{+-}e^{i(kx-\omega t)} + A_{-+}e^{-i(kx-\omega t)} + A_{--}e^{-i(kx+\omega t)} \tag{8.4.6}$$

involving eight real degrees of freedom (two per complex coefficient). However, we have applied no boundary or initial conditions in obtaining these solutions and such conditions will greatly reduce the number of degrees of freedom. For example, we already know that the solution must be real (since $y(t, x)$ represents some measurable property such as a strng's displacement or the pressure variations in a fluid) and so the coefficients must satisfy the conditions

$$A_{++} = A^*_{--} = A_<, \quad A_{+-} = A^*_{-+} = A_> \tag{8.4.7}$$

where we have used $<$ to indicate left moving waves and $>$ to indicate right movers. The number of degrees of real freedom have thus been reduced to four and $y(t, x)$ can be written as

$$y(t, x) = 2\Re \left[A_< e^{i(kx+\omega t)} + A_> e^{i(kx-\omega t)} \right] \tag{8.4.8}$$

where $\Re$ stands for "the real part of". In all our subsequent analysis we will stick with the complex expression as far as possible and take the real part only when it is absolutely necessary. This will simplify the algebra considerably, while not affecting the final result in any way.

Notice that the phase of the left moving piece is $\phi_< = kx + \omega t$ and that of the right moving piece is $\phi_> = kx - \omega t$. The phases are constant when $x = \mp \frac{\omega}{k}t = \mp vt$ respectively. Thus $\mp v$ is called the **phase velocity** of the plane waves. It is the velocity with which plane wave fronts travel in space. Holding t fixed (imagine taking a photograph of the wave), we notice that the wave function repeats itself spatially every λ units, where

$$k\lambda = 2n\pi \tag{8.4.9}$$

for integer n. The wave number is therefore related to the **wavelength**, λ, according to $k = 2\pi/\lambda$. On the other hand, viewing the same spatial point (*i.e.,* holding x fixed) the wave function is seen to repeat itself in a time τ, which satisfies

$$\omega\tau = 2n\pi \tag{8.4.10}$$

for integer n and so the angular frequency is related to the **period**, τ, according to $\omega = 2\pi/\tau$. The **frequency** of the wave is related to the angular frequency by $f = 1/\tau = \omega/2\pi$.

8.5   Boundary Conditions and Particular Solutions

We must now consider what boundary conditions are to be satisfied by the wave. Boundary conditions are crucial as they are what determine the actual wave forms encountered in any given physical situation. As examples, we will consider two common types of particular solutions below.

8.5.1   Standing Waves

A "standing wave" is a solution of the wave equation that is required to possess certain characteristics at the boundary. In the case of (one dimensional) waves in string, the boundary would correspond two distinct points on the string; in the case of (one dimensional) sound waves, the boundary would correspond to two two-dimensional surfaces perpendicular the direction of propagation. At each boundary we may require either (i) that the wave function vanishes there or (ii) that the wave function achieves a maximum there. Thus three possibilities occur: (a) the wave vanishes at both boundaries (b) it vanishes at one boundary but is maximum at the other and (c) it is maximum at both boundaries. Standing waves are formed by musical instruments. Every string instrument, for example produces a standing wave in accordance with boundary condition (a) because the strings are held fixed at two ends. On the other hand, wind instruments may be designed to obey the boundary conditions of types (b) or (c). For instance, a clarinet has one closed end, where the sound waves are required to vanish, and therefore obeys condition (b) whereas flutes and oboes have both ends open and the sound waves achieve their maximum value at both ends, so they obey condition (c).

We will begin with cases (a) and (b). For convenience, choose the point at which the wave vanishes as the origin of coordinates and the other at a distance L from it, then the condition

$$y(t,0) = A_< e^{i\omega t} + A_<^* e^{-i\omega t} + A_> e^{-i\omega t} + A_>^* e^{i\omega t} = 0 \tag{8.5.1}$$

implies that

$$A_< = -A_>^* \tag{8.5.2}$$

and inserting this into the wave equation gives

$$y(t,x) = 2\Re \left[A_< e^{i(kx+\omega t)} - A_<^* e^{i(kx-\omega t)} \right] \tag{8.5.3}$$

Our solution is now given in terms of one complex coefficient (two real degrees of freedom). Suppose we let the complex coefficient $A_< = -\frac{1}{4}\mathcal{A}e^{i\delta}$ where $\mathcal{A}$ and δ are real. Then

$$
\begin{aligned}
y(t, x) &= -\frac{1}{2}\mathcal{A}\,\Re\left[e^{i(kx+\omega t+\delta)} - e^{i(kx-\omega t-\delta)}\right] \\
&= -\frac{1}{2}\mathcal{A}\,\Re\left[e^{ikx}\left(e^{i(\omega t+\delta)} - e^{-i(\omega t+\delta)}\right)\right] \\
&= -\frac{1}{2}\mathcal{A}\,\Re\left[2ie^{ikx}\sin(\omega t + \delta)\right] \\
&= \mathcal{A}\sin(kx)\sin(\omega t + \delta).
\end{aligned}
\tag{8.5.4}
$$

$\mathcal{A}$ is the **amplitude** of the wave (it is the maximum value of $y(t,x)$) and δ is the **initial phase**. Because the variables (t, x) in $y(t, x)$ do not occur in the combination $x \pm vt$, this wave does not "travel", *i.e.*, it does not propagate in the medium. That is why it is called a standing wave.

If the wave function is required to vanish at $x = L$ as well, we see from (8.5.4) that it is only possible if

$$
\sin(kL) = 0 \Rightarrow k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}
\tag{8.5.5}
$$

for some positive integer n. The angular frequency is therefore

$$
\omega_n = k_n v = \frac{n\pi v}{L}
\tag{8.5.6}
$$

On the other hand, if $y(t, x)$ achieves its maximum at $x = L$ (instead of vanishing there) then the appropriate condition to apply is $y'(t, L) = 0$, where the prime refers to a derivative with respect to x, and

$$
\cos(kL) = 0 \Rightarrow k_n = \frac{2\pi}{\lambda_n} = (2n - 1)\frac{\pi}{2L},
\tag{8.5.7}
$$

again for a positive integer, n. This gives

$$
\omega_n = (2n - 1)\frac{\pi v}{2L}
\tag{8.5.8}
$$

Naturally, equation (8.5.4) does not represent the only possible wave form. Since the wave equation is linear we may superpose solutions with arbitrary (ω dependent) coefficients and phases, so the most general solution is obtained from

$$
y(t, x) = \sum_n \mathcal{A}_n \sin(k_n x)\cos(\omega_n t + \delta_n)
\tag{8.5.9}
$$
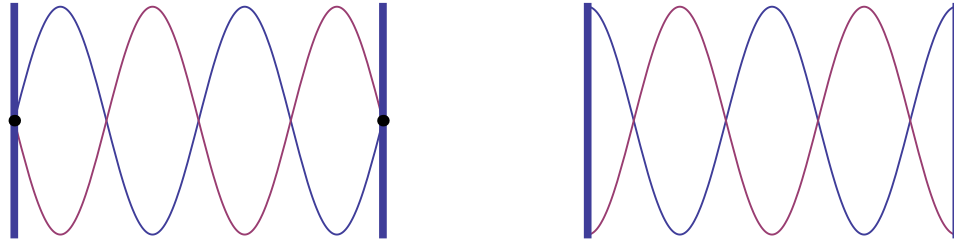
Figure 8.4: Dirichlet (left) and Neumann (right) conditions on both ends of a string

The lowest allowed frequency, given by $n = 1$, is called the **fundamental frequency** and all allowed higher frequencies are called **harmonics** of the fundamental frequency. The fundamental frequency is set both by the length L as well as the properties of the medium, but the longest allowed wavelength is set only by the length L. Thus, the longest possible wavelength in a string of length L is $\lambda_1 = 2L$ but for a clarinet of the same length it is $\lambda_1 = 4L$.

We must turn now to case (c). The condition $y'(0, t) = 0$ leads to

$$A_< = A_>^* \tag{8.5.10}$$

so taking $A_< = \frac{1}{4}\mathcal{A}e^{i\delta}$ (as before) leads to

$$y(t, x) = \mathcal{A}\cos kx \cos(\omega t + \delta) \tag{8.5.11}$$

and applying the open boundary conditions at $x = L$ as well will yield (8.5.5). Everything that has been said for case (a) will also hold. In particular the most general waveform can be obtained by superposition according to

$$y(t, x) = \sum_n \mathcal{A}_n \cos(k_n x)\cos(\omega_n t + \delta_n) \tag{8.5.12}$$

where the sum is over all harmonics.

The two kinds of conditions we have imposed are, respectively, special cases of the **Dirichlet** boundary conditions, for which the wave function is specified at the boundary, and the **Neumann** boundary condition, for which the value of the normal derivative of the wave function is given at the boundary. The unspecified amplitude, $\mathcal{A}$, and initial phase, δ, would be determined by initial conditions which specify the wave function and its first time derivative at some (initial) time.

8.5.2   Traveling Wave at an Interface

Consider two strings of different densities attached to one another at (say) $x = 0$ and suppose that a continuous wave of some frequency, ω, is incident on the joint and moving from

left to right. Alternatively, consider sound waves passing from a region of some density to another of a different density across a plane boundary. What stationary solutions of the wave equation will be differentiable across the boundary? We will see that a portion of the incident wave is reflected at the boundary and a portion of it is transmitted across the boundary. The solution we seek will have the form

$$y(t,x) = \begin{cases} 2\Re\left[A_<e^{i(kx+\omega t)} + A_>e^{i(kx-\omega t)}\right] & x < 0 \\[2mm] 2\Re\left[B_>e^{i(k'x-\omega't)}\right] & x > 0 \end{cases} \qquad (8.5.13)$$

i.e., no left moving wave in the region $x > 0$ is allowed. The phase and amplitude of the incoming wave are set by the initial conditions, so we do not attempt to find $A_>$. We are after the amplitude of the reflected and transmitted waves, respectively $A_<$ and $B_>$. If we compare $y(t,0^-)$ with $y(t,0^+)$ we see immediately that the wave function will be continuous at $x = 0$ if and only if $\omega = \omega'$ and

$$B_> = A_> + A_<^*. \qquad (8.5.14)$$

Notice in particular that the frequency remains the same across the boundary but the wavelength may change. This can only happen because the speed of the wave is different in the two media and since $\omega = kv = \omega' = k'v'$ it must then follow that $k/k' = v'/v$. Furthermore, for the wave function to be differentiable at $x = 0$ we require

$$k(A_<^* - A_>) = -k'B_> \qquad (8.5.15)$$

and solving the two conditions for the reflection and transmission amplitudes we find

$$A_< = \frac{k - k'}{k + k'}A_>^*,$$
$$B_> = \frac{2kA_>}{k + k'}. \qquad (8.5.16)$$

The first of (8.5.16) tells us that the phase of $A_<$ is equal to the phase of $A_>$ if $k > k'$ (or $v' > v$) and differs from it by π if $k < k'$ (or $v > v'$). Assuming that the tension, in the case of the string, or the bulk modulus, in the case of fluid media, is the same on both sides of the boundary, the speeds of the wave in either region will depend inversely on the square root of the density in that region. A phase change will occur when $\mu' > \mu$ ($\rho' > \rho$) but no phase change will occur when $\mu' < \mu$ ($\rho' < \rho$). Thus the reflected wave undergoes a phase change of π when the wave is incident on a boundary with a region of higher density and no phase change when the boundary leads to a region of lower density. The second condition says that the phase of the transmitted wave is always the same as the phase of the incident wave.

In our simple example of a sinusoidal wave traveling from left to right, we saw that the energy transferred across the string per second is proportional to $\mu \omega^2 v y_{\max}^2$. Thus the ratio of the power transferred by the reflected wave and the incident wave in the left portion of the string will be

$$\frac{P_{\text{ref}}}{P_{\text{inc}}} = \left| \frac{A_<}{A_>} \right|^2 = \left(\frac{k - k'}{k + k'} \right)^2 = R, \tag{8.5.17}$$

where we replaced $y_{\max}$ by the amplitudes for the incident and reflected waves respectively. R is called the reflection coefficient. Likewise, the ratio of the power transmitted by the outgoing wave in the right portion of the string to the incident wave (in the left) is

$$\frac{P_{\text{tr}}}{P_{\text{inc}}} = \frac{\mu' v'}{\mu v} \left| \frac{B_>}{A_>} \right|^2 \tag{8.5.18}$$

since $\omega = \omega'$. Again, because the string tension in both regions must be the same,

$$\mu v^2 = \mu' v'^2 \Rightarrow \frac{\mu' v'}{\mu v} = \frac{v}{v'} = \frac{k'}{k}, \tag{8.5.19}$$

where we used $v = \omega / k$ and the fact that $\omega = \omega'$ in the last step above. Therefore,

$$\frac{P_{\text{tr}}}{P_{\text{inc}}} = \sqrt{\frac{\mu'}{\mu}} \frac{4k^2}{(k + k')^2} = \frac{k'}{k} \frac{4k^2}{(k + k')^2} = \frac{4kk'}{(k + k')^2} = T, \tag{8.5.20}$$

and T is called the transmission coefficient. Note that

$$R + T = 1, \tag{8.5.21}$$

which simply restates the law of conservation of energy.

8.6   The Doppler Effect

The Doppler effect consists of the change in the measured frequency of a source, whenever the source or the observer or both the source and the observer are moving with respect to the medium in which the wave travels. Recall that the wave equation is not invariant under Galilean transformations, having the form (8.1.6) only in the rest frame of the medium. The medium's rest frame thus becomes a canonical reference frame, with respect to which all quantities must be measured.
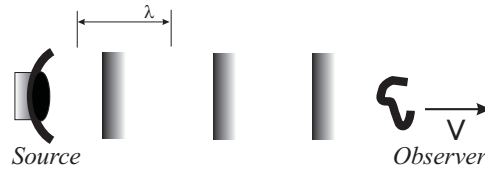
Figure 8.5: Doppler effect with moving observer.
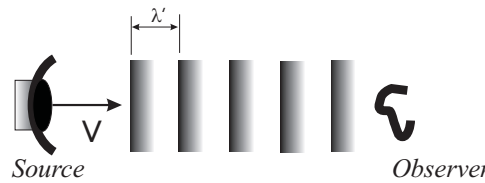
Figure 8.6: Doppler effect with moving observer.

8.6.1 Stationary Source, Moving Observer

Let us first consider the situation depicted in 8.5 in which the source is stationary with respect to the medium and the observer is moving either away from or toward it, with a speed v_O relative to it. Consider two successive wave fronts and suppose that the observer has received the first at some time, t_0. If the observer were stationary relative to the medium the next front would arrive τ seconds later, where τ is the period of the wave. However, since the observer is moving, the next front must cover a greater or lesser distance to the observer and therefore takes a longer or shorter time (than τ) to arrive. Thus, while the wavelength received by the observer is the same as the wavelength in the medium, the received frequency will differ from its frequency in the medium. Let λ and f represent respectively the wavelength and frequency of the wave in the medium. Because $\lambda f = v$ is required for the wave equation, it is only possible to change f while holding λ fixed if the speed of sound changes in such a way that $\lambda f' = v'$. But, according to the Galilean principle of relativity, the speed of sound relative to the moving observer is $v' = v \mp v_O$, so

$$f' = \frac{v'}{\lambda} = \frac{v \mp v_O}{\lambda} = \left(1 \mp \frac{v_O}{v}\right) f, \tag{8.6.1}$$

where we used $\lambda f = v$, so the frequency received is lower if the observer is moving away from the source and higher if he is moving toward it.

8.6.2   Moving Source, Stationary Observer

If the source is moving with a speed v_S relative to the medium, as shown in 8.6, then neither the frequency of the emitted wave nor its wavelength will end up being those that would be emitted had the source been stationary with respect to the medium. Yet, in both cases, the wave would travel at the same speed, v, in the medium. If unprimed quantities refer to the properties of the wave emitted by a stationary source and primed quantities to the wave emitted by the moving source we should have $\lambda f = v = \lambda' f'$. It is easy to compute the change in wavelength if one again thinks in terms of wave fronts. Because the source has moved in the time between the emission of successive wave fronts the effective wavelength in the medium will be

$$\lambda' = \lambda \mp v_S \tau \tag{8.6.2}$$

where v_S is the source velocity and the negative sign applies to the situation in which the source is moving in the direction of the wave, the positive sign if it is moving in the opposite direction. Since the speed of the wave in the medium is v, the observed frequency becomes

$$f' = \frac{v}{\lambda'} = \frac{v}{\lambda \mp v_S \tau} = \frac{f}{\left(1 \mp \frac{v_S}{v}\right)} \tag{8.6.3}$$

where we used $\lambda f = v$. The received frequency is therefore higher if the source is moving toward the observer and lower if it is moving away.

8.6.3   Generalizations

Both of these results generalize easily to the case in which the observer is moving at some angle with the direction of the wave. Let $\hat{u}$ represent the direction in which the wave is traveling and $\vec{v}_O$ represent the observer's velocity, then

$$f' = \begin{cases} \left(1 - \frac{\vec{v}_O \cdot \hat{u}}{v}\right) f, & \text{moving observer} \\[2em] \frac{f}{\left(1 - \frac{\vec{v}_S \cdot \hat{u}}{v}\right)}, & \text{moving source.} \end{cases} \tag{8.6.4}$$

Moreover, the two effects can be combined so that if both the source and the observer are moving relative to the medium with velocities, respectively, $\vec{v}_S$ and $\vec{v}_O$,

$$f' = \frac{1 - \frac{\vec{v}_O \cdot \hat{u}}{v}}{\left(1 - \frac{\vec{v}_S \cdot \hat{u}}{v}\right)} f, \tag{8.6.5}$$

which is directly recovered by combining the two arguments above.

8.7 Superposition

In the preceeding examples we superposed waves propagating in a medium with a definite frequency but traveling in opposite directions. In principle, though, there may be waves of many different frequencies propagating simultaneously in the medium in either direction. The superposition principle then tells us that the resulting wave will be the simple sum of the individual waves and this leads to

$$y(t, x) = \Re \left[\sum_k A(k) e^{i(kx - \omega t)} \right], \tag{8.7.1}$$

where the sum is over a discrete set of wave-numbers, $A_>(k)$ represents the amplitude of the waves with wave-number k and ω represents the angular frequency. In this sum, k is allowed to be negative in order to account for both right and left moving waves and $\omega = |k|v$. Furthermore, if $A(k)$ is decomposed into a real amplitude and a phase, $A(k) = \mathcal{A}_k e^{i\delta_k}$, then we can rewrite the sum as

$$y(t, x) = \Re \left[\sum_k \mathcal{A}_k e^{i(kx - \omega t + \delta_k)} \right]. \tag{8.7.2}$$

It is worth considering some examples.

8.7.1 Interference

We have already seen that the solution representing two waves of the same frequency, amplitude and initial phase, traveling in opposite directions is given by the standing wave

$$y(t, x) = \mathcal{A} \sin(kx) \cos(\omega t + \delta). \tag{8.7.3}$$

If the two waves were to be traveling in the same direction, somehow arriving at a point P with differing phases, say δ_1 and δ_2, then we expect the resulting wave at P to be

$$
\begin{aligned}
y(t, x) &= \mathcal{A} \, \Re \left[e^{i(kx - \omega t + \delta_1)} + e^{i(kx - \omega t + \delta_2)} \right] \\
&= \mathcal{A} \, \Re \left\{ e^{i(kx - \omega t)} \left[e^{i\delta_1} + e^{i\delta_2} \right] \right\} \tag{8.7.4}
\end{aligned}
$$

Suppose $\bar{\delta}$ represents the mean phase, $\bar{\delta} = (\delta_1 + \delta_2)/2$, then introducing the phase difference $\Delta\delta = \delta_1 - \delta_2$, we have $\delta_1 = \bar{\delta} + (\Delta\delta)/2$ and $\delta_2 = \bar{\delta} - (\Delta\delta)/2$. Expressed in terms of the mean phase and the phase difference,

$$y(t, x) = 2\mathcal{A} \cos\left(\frac{\Delta\delta}{2} \right) \Re \, e^{i(kx - \omega t + \bar{\delta})} \tag{8.7.5}$$
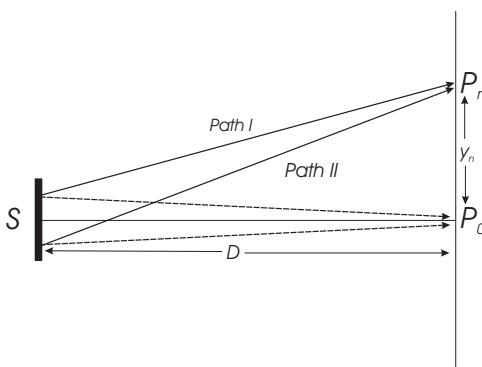
Figure 8.7: Superposition of two waves.

represents a plane wave with the same wavelength and frequency, and an initial phase equal to the mean phase of the two waves, but with an amplitude that depends on the phase difference. In the extreme, if the phase difference is an odd integer multiple of π then the waves completely annihilate one another and if it is an integer multiple of 2π then they enhance one another giving a resulting wave of twice the original amplitude.

The phase difference may be introduced by using two identical sources that are out of phase with one another. Alternatively, it may be introduced by having the wave traverse paths of differing lengths before arriving at P. Consider two plane waves of the same frequency, wavelength and initial phase, traveling along two different paths toward the same point, P, as shown in figure 8.7. In the figure, S represents a common source for the waves, ensuring that the frequency, wavelength and initial phase are the same for both and, for simplicity, we also assume that the amplitudes are the same. The waves may be produced from the single source by introducing two slits in an opaque screen a distance d apart. Let x_1 and x_2 represent the path lengths of I and II respectively. By superposition, the resulting wave at P at any time t will be

$$
\begin{aligned}
y(t,x) &= \mathcal{A}\Re\left[e^{i(kx_1-\omega t+\delta)} + e^{i(kx_2-\omega t+\delta)}\right] \\
&= \mathcal{A}\Re\left[e^{ikx_1} + e^{ikx_2}\right] e^{-i(\omega t+\delta)}
\end{aligned}
\tag{8.7.6}
$$

Suppose $\overline{x}$ represents the mean path length, $\overline{x} = (x_1 + x_2)/2$ and $\Delta x = x_1 - x_2$ the path difference, then we have $x_1 = \overline{x} + \Delta x/2$ and $x_2 = \overline{x} - \Delta x/2$. Inserting this into the expression for the resultant, we find

$$
y(t,x) = 2\mathcal{A}\cos\left(\frac{k\Delta x}{2}\right)\Re\left[\mathcal{A}e^{i(k\overline{x}-\omega t+\delta)}\right]
\tag{8.7.7}
$$
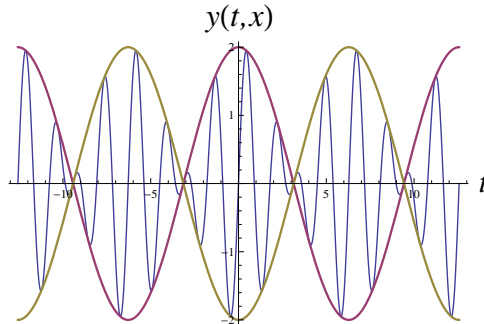
Figure 8.8: Beats.

The wave at P therefore behaves as a traveling wave with the same frequency, wavelength and initial phase as the source, but its amplitude at the point P is sensitive to the difference Δx in the path lengths traveled by the waves. Indeed it is always zero when the path difference is an odd multiple of one half the wavelength,

$$\frac{k\Delta x}{2} = (2n+1)\frac{\pi}{2} \Rightarrow \Delta x = (2n+1)\frac{\lambda}{2}, \qquad (8.7.8)$$

and always a maximum, which is twice the amplitude of the original waves, when the path difference is a multiple of the wavelength,

$$\frac{k\Delta x}{2} = n\pi \Rightarrow \Delta x = n\lambda. \qquad (8.7.9)$$

and so two waves traveling in a medium can, under certain conditions, completely destroy each other or perfectly enhance each other. Superposition between waves of the same frequency and wavelength is commonly referred to as the phenomenon of **interference**. Where the amplitude of the resulting wave is enhanced over the amplitudes of the original waves, we say that **constructive** interference has occurred. Consversely, **destructive** interference occurs where the amplitude of the resulting wave is diminished.

8.7.2   Beats

What if the wavelengths and frequencies of the component waves are not the same? Consider two waves of differing wavelength and frequency but (for simplicity) having the same amplitude and initial phase and traveling in the same direction. The resulting wave will be

$$y(t,x) \;\; = \;\; \mathcal{A}\Re\, e^{i\delta}\left[e^{i(kx-\omega t)} + e^{i(k'x-\omega' t)}\right]$$

$$= \quad 2\mathcal{A}\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right] \Re\, e^{i(\overline{k}x - \overline{\omega}t + \delta)} \tag{8.7.10}$$

where, as before, we have introduced $\overline{k} = \frac{1}{2}(k + k')$, $\Delta k = k - k'$ and likewise for ω. The resultant is therefore made up of two traveling waves: One wave travels with the average wavelength and frequency of the component waves (this is the frequency perceived by the listener) and the other is a cosine wave which oscillates with a (lower) frequency, $\frac{1}{2}\Delta\omega$, and modulates the amplitude of the first. The modulation wave is called the **beat**. Because both waves are traveling in the same medium we know that $\omega/k = v = \omega'/k'$, where v is the speed of the wave in that medium. Therefore the **beat** or **group** velocity, which may be defined as the rate at which a constant amplitude travels through space, is $v_g = \Delta\omega/\Delta k$ and will be the same as the wave velocity. The frequency of intensity fluctuation will be $\Delta\omega/4\pi$. This is called the **beat frequency**, which, in this particular case, ends up being half the difference in the frequencies of the component waves.

In all the cases analyzed above, we only considered waves traveling with the same velocity. Our reasoning was that the properties of the medium determine the velocity of a traveling wave and we tacitly assumed that the properties of the medium are insensitive to the wavelength and frequency of the traveling wave, making the velocity of propagation the same for *all* frequencies. There is, however, no fundamental principle that ensures this to be the case and in fact a more detailed analysis shows that it is only an approximation. Materials in which the velocity of wave propagation is the same for all wavelengths are called **non-dispersive**. Most materials are **dispersive**, *i.e.,* they will cause waves of differing wavelengths to travel at different speeds. If we account for this dependence of v on k then the frequency $\omega(k) = v(k)k$ will no longer be linearly related to k and then

$$v_g(k) = \frac{\Delta\omega}{\Delta k} \neq \frac{\overline{\omega}}{\overline{k}} = \overline{v}(k). \tag{8.7.11}$$

In other words, the beat velocity will differ from the phase velocity. The two are related; assuming that Δk and $\Delta\omega$ are small, $\overline{v}(k) \approx v(k) = \omega(k)/k$ and

$$v_g(k) = \frac{d\omega(k)}{dk} = v(k) + k\frac{dv(k)}{dk} \tag{8.7.12}$$

It can be shown that energy and momentum is transferred through the medium with the group velocity and therefore it is the group velocity that we measure.

Examples of dispersion in materials abound. For example, the polarizability of a medium changes rapidly when the photon energies approach the energies of quantum transitions (because of the possibility of resonant absorption), which affects the dielectric constant and hence the refractive index in a frequency dependent way. Dispersion in the ionosphere as the frequency of the radio waves approaches the plasma frequency, is the

reason why AM radio is accessible at great distances. The changing refractive index of the ionosphere leads to "total internal reflection" so that the waves are reflected back to earth.

8.7.3 Wave Packets

If the wave-numbers are allowed to vary continuously, instead of discretely, then the superposition sum in (8.7.2) should be replaced by an integral over k,

$$y(t, x) = \Re \int_{-\infty}^{\infty} dk \mathcal{A}(k) e^{i(kx - \omega(k)t + \delta_k)}. \tag{8.7.13}$$

where the amplitudes must now be understood to carry the appropriate dimensional information, since $\mathcal{A}(k)dk$ represents the amplitude of waves with wavenumber lying between k and $k + dk$. $\mathcal{A}(k)$ is the Fourier transform of the wave function and is known as the **spectral distribution**. A single wave, propagating with wavenumber k_0 and amplitude $\mathcal{A}_0$, for example, can be recovered if the spectral distribution is $\mathcal{A} = \mathcal{A}_0 \delta(k - k_0)$. We have accounted for dispersion in (8.7.13) by allowing $\omega(k)$ to be a general function of k. If the spectral distribution is not a delta function then $y(t, x)$ is called a **wave packet**.

While plane waves are of infinite extent, wave packets can be "localized" if the spectral distribution is peaked around some value, say k_0, and vanishes rapidly about the peak. Suppose that the integral receives a significant contribution only in a finite interval $[k_0 - \Delta k, k_0 + \Delta k]$ about k_0. Then

$$y(t, x) \approx \Re \, e^{i(k_0 x - \omega(k_0)t)} \int_{-\Delta k}^{\Delta k} d\widetilde{k} \{\mathcal{A}(k_0) + \mathcal{A}''(k_0)\widetilde{k}^2\} e^{i\widetilde{k}(x - \omega'(k_0)t) + i\delta_k}, \tag{8.7.14}$$

where $\widetilde{k} = k - k_0$. The term we have been able to take out of the integral is a plane wave, propagating at the phase velocity $v = \omega(k_0)/k_0$. The integral term should be viewed as the beat since the condition $\widetilde{k} < \Delta k \ll k_0$ ensures that the exponential outside the integral oscillates rapidly compared with the exponential inside integral. Coherent propagation of the amplitude will occur when $x - \omega'(k_0)t = $ const., *i.e.*, when

$$\frac{dx}{dt} = \omega'(k_0) = v_g(k_0) = \left. \frac{d\omega(k)}{dk} \right|_{k=k_0} \tag{8.7.15}$$

This is the group velocity, as we saw earlier, and it is related to the phase velocity by (8.7.12).

Let us close with the simple example of a "gaussian" wave packet, for which the spectral distribution is of the form

$$\mathcal{A}(k) = a e^{-b(k - k_0)^2}, \tag{8.7.16}$$

$$y(0,x)$$
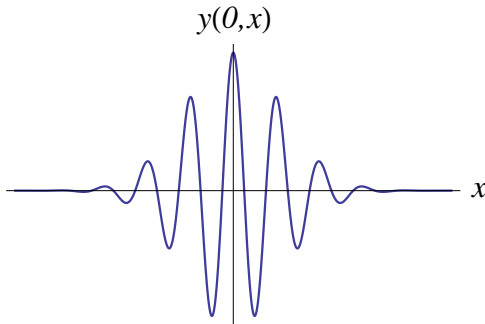
Figure 8.9: Spatial form of the Gaussian Wavepacket at $t = 0$

where a and b are constants, and let $\delta_k = 0$ and $\omega(k) = \sigma k^2$ (the group velocity is therefore

$$v_g(k) = \frac{d\omega}{dk} = 2\sigma k. \tag{8.7.17}$$

Integrating (8.7.13), the solution is given as

$$
\begin{aligned}
y(t,x) \;=\;& \sqrt{\frac{\pi a^2}{r}}\,\exp\left[-\frac{b}{4r^2}(x - v_g(k_0)t)^2\right] \;\times \\
& \times \cos\left[-\frac{\sigma t x^2}{4r^2} + \frac{b^2}{r^2}(k_0 x - \omega_0 t) + \tan^{-1}\frac{\sigma t}{b}\right]
\end{aligned}
\tag{8.7.18}
$$

where $r = \sqrt{b^2 + \sigma^2 t^2}$. The wave packet is localized, with a variance of $\sigma = \sqrt{2r^2/b}$. Its spatial distribution at $t = 0$ is shown in figure 8.9. As t increases, the peak travels (to the right) at the group velocity, $v_g(k_0)$, but its amplitude decreases according to $t^{-1/2}$ (when $t \gg b/\sigma$) and its variance increases as t, causing the packet to spread about the mean.

Chapter 9

The Calculus of Variations

We are now going to embark on a different approach to mechanics than the one we have taken so far. The new approach involves looking for some quantity that appears to be optimized by the physical system. Once the quantity is discovered, the mathematical techniques that we introduce in this chapter allow us to compute the equations of motion of the system. It should always be remembered that the equations of motion are no different than those that would follow from a careful application of Newton's laws. The fundamental principles of mechanics, as embodied in Newton's three laws do not change, only our way of thinking about them and the techniques used to execute what in the end is Newton's program are enlarged and made more powerful. Yet, the new approach is not just about formalism. It also provides a significantly deeper insight into Newton's laws themselves. It makes manifest the underlying symmetries of the mechanical system and relates the conservation laws to the symmetries. A famous theorem by the English mathematician Emmy Noether showed that the relationship between symmetries of the quantity that is optimized and conservation laws obeyed by the system is a profound one. We will explore Noether's theorem in considerable depth as we go along.

One of the most significant uses of the new formalism will be the ease with which it is possible to incorporate constraints: for instance, one may be interested in the motion of a body subject to the action of some known forces but constrained to move only along some surface. While not impossible to handle, such problems are far from easy if all that were available to us were the methods that have already been introduced. The techniques we are about to learn will significantly simplify them. Because of the relationship they bear with underlying symmetries and because they are able to handle constrained systems so naturally, the approach finds its most powerful use in the treatment of field theories.

The new techniques are based upon the "calculus of variations", which we examine in this chapter. This chapter should be thought of as an essential mathematical detour. From the mathematical point of view, the problem is one of extremization. Only, this time
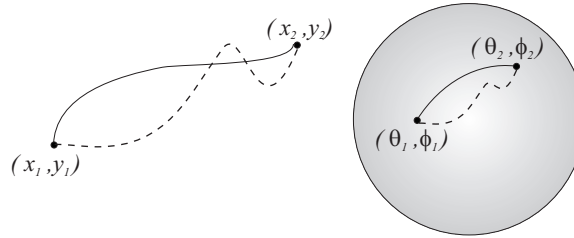
Figure 9.1: The path length in a plane and on the surface of a two sphere

we are interested not in extremizing functions on space but real valued maps from a subset of the space of functions on a manifold to the reals. Such maps are called **functionals**.

9.1   Functionals

Roughly speaking, a functional is a real or complex valued function that takes functions (or, more generally, elements of a vector space) as its argument. Thus, the domain of a functional is a set of functions and its range is a subset of the reals). As an example of a functional, let $f(x)$ be any square integrable function on $\mathbb{R}$, then F defined by

$$F[f] = \int_{-\infty}^{\infty} dx \|f(x)\|^2 \tag{9.1.1}$$

is a functional on the set of square integrable functions on $\mathbb{R}$ because it associates every element in its domain to one and only one real number, the norm of f.

Of more immediate interest is the distance between two points in the plane along some curve given by $y = y(x)$. Using Pythagoras' theorem, it can be written as the integral

$$S[y] = \int_1^2 \sqrt{dx^2 + dy^2} = \int_{x_1}^{x_2} dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{9.1.2}$$

To each curve connecting the points 1 and 2, which are held fixed, there is a unique real number representing its path length (distance). Different curves, specified by different functions $y(x)$ may present the same path length, but functionals (like functions) are not required to be one to one. In this case we have a functional representing the path length on the curve $y = y(x)$ in $\mathbb{R}^2$. Extremizing this functional, if we could do such a thing, would give us the curve with the shortest path length between 1 and 2. We could generalize to the surface of a two sphere. If the latitudes are specified by θ and the longitudes by φ, so
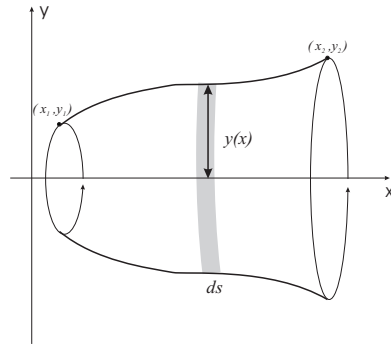
Figure 9.2: The surface of revolution generated by a curve $y = y(x)$

that a curve can be given by a function of the form $\varphi = \varphi(\theta)$, then

$$S[\varphi] = \int_1^2 r\sqrt{d\theta^2 + \sin^2\theta d\varphi^2} = r\int_1^2 d\theta\sqrt{1 + \sin^2\theta\left(\frac{d\varphi}{d\theta}\right)^2} \qquad (9.1.3)$$

where r is the radius of the sphere.

Another example of a functional is the area of the surface of revolution generated by a curve $y = y(x)$, rotated about the $x-$axis. The problem would be to find the curve that generates the surface of minimum area. If the curve begins at (x_1, y_1) and ends at (x_2, y_2), the area of the surface can be obtained by integrating infinitesimal circular strips as shown in figure (9.2). The area is then

$$A[y] = \int_1^2 2\pi y ds = 2\pi \int_{x_1}^{x_2} dx\ y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \qquad (9.1.4)$$

While these examples may appear to be of geometrical interest only, they are in fact of physical interest as well. The first is connected to Newton's first law, according to which a free particle will choose the shortest distance between two points (the straight line). Newton's first law may therefore be viewed as an optimization problem in which the quantity that is optimized is the path length. The advantage of thinking about it in this way is that it can easily be generalized to curved spaces in which straight lines may be impossible. The "straight line" is replaced with the "geodesic", the path of minimum length.

The second above is a problem that is intimately connected with the formation of soap bubbles between two rings. Minimizing the surface area would give us the shape of the bubble. In this genre of problems, consider a flexible cable, suspended between two fixed points as in a suspension bridge. If the length of the cable is fixed, we could write

an expression for its gravitational potential energy as a sum over the potential energies infinitesimal lengths, ds, each located at a height y from some reference level,

$$U = \int_1^2 \rho g y \, ds = \rho g \int_{x_1}^{x_2} dx \; y(x) \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{9.1.5}$$

Minimizing the potential energy would tell us the shape of the cable in equilibrium.

Again, consider a fixed volume of water in a cylindrical container that is rotating with a fixed angular velocity ω. Due to the rotation, the surface of the water will not be planar but will be such as to minimize the potential energy of the water in the combined gravitational-centrifugal force field. The potential energy of an infinitesimal volume of water of mass $\delta m = \rho dV$ can be written as

$$\delta U = \delta m g y - \frac{1}{2} \delta m \omega^2 r^2 \tag{9.1.6}$$

since $\vec{F}_c = \delta m \omega^2 \vec{r}$ represents the centrifugal force on the element if it is located a distance r from the axis of rotation. Using cylindrical coordinates in which the y axis is the axis of rotations, the total potential energy is

$$U = \rho \int_0^R r dr \int_0^{2\pi} d\varphi \int_0^{y(r)} dy (gy - \frac{1}{2}\omega^2 r^2) \tag{9.1.7}$$

where $y = y(r)$ is understood to represent points on the surface of the water. Because of axial symmetry, we may perform the φ integration right away. Performing also the y integration gives

$$U[y] = 2\pi\rho \int dr \; r \left[gy^2(r) - \frac{1}{2}\omega^2 r^2 y(r) \right] \tag{9.1.8}$$

Minimizing this energy shows that the surface of the water is parabolic.

As a final example, a problem of considerable historical interest is determining the trajectory of a particle falling in a constant gravitational field in such a way that that the *time* taken for it to travel between two points is minimized. We can setup the problem as follows: the time taken to cover an infinitesimal length of the trajectory is

$$dt = \frac{ds}{v} \tag{9.1.9}$$

where v is its instantaneous velocity. Conservation of energy tells us that

$$\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv^2 + mgy \tag{9.1.10}$$
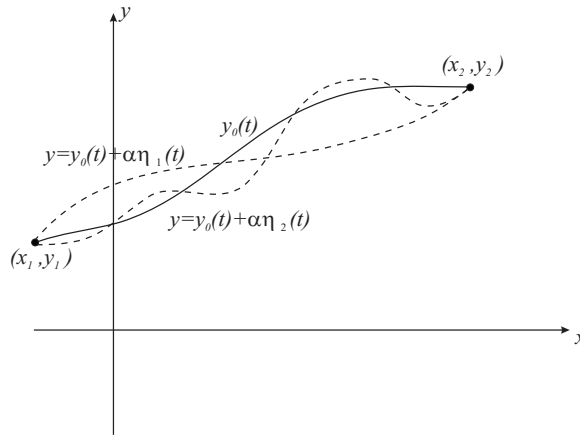
Figure 9.3: Deformations of a curve $y = y(t)$, holding the endpoints fixed.

where y_1 represents its initial height and v_1 its initial speed. For convenience, take $v_1 = 0$, so the particle begins at rest at (x_1, y_1), then

$$v = \sqrt{2g(y_1 - y)} \tag{9.1.11}$$

and

$$T[y] = \int_1^2 dt = \int_{x_1}^{x_2} dx \sqrt{\frac{1 + y'^2}{2g(y_1 - y)}} \tag{9.1.12}$$

is the time taken between the initial and final points. This is known as the "Brachistochrone" problem. It is the curve of fastest descent and has a stellar history in which some of the greatest minds in physics and mathematics were involved. It was first posed by Galileo (who incorrectly stated that the solution was a circle, and later addressed by the likes of Newton, the brothers Bernoulli, Leibnitz and l'Hospital). We will later address a similar problem (not the same!), which is relevant to geometric optics when it is treated according to Fermat's principle. Fermat's principle states that the trajectories followed by "particles" of light (the principle was enunciated before the wave theory of light was generally recognized as correct) are such as to extremize the time taken by the particles to travel between two points. . Much later, when we address the theory of special relativity, Fermat's principle will be return, but now it will also describe the relativistic dynamics of ordinary matter.

9.2   Euler's equation for extrema

With the motivation provided by the examples in the previous section, we want to obtain
the necessary and sufficient conditions for the existence of an extremum of a functional
which has the form

$$S[y] = \int_1^2 dt f(y(t), \dot{y}(t); t) \tag{9.2.1}$$

We are using t for our independent variable and $y(t)$ as the test function. Suppose that
a trajectory that extremizes $S[y]$ exists and call it $y_0(t)$. Then consider a one parameter
deformation of the trajectory $y_0(t)$ of the form

$$y(\alpha, t) = y_0(t) + \alpha \eta(t), \tag{9.2.2}$$

and $\eta(t)$ is any differentiable function obeying

$$\eta(t_1) = \eta(t_2) = 0 \tag{9.2.3}$$

In other words, the the endpoints are held fixed by the deformations, but otherwise they
are arbitrary [see figure (9.3)]. The functional

$$S[y; \alpha] = \int_1^2 dt f(y(\alpha, t), \dot{y}(\alpha, t); t) \tag{9.2.4}$$

is now also a *function* of α. The necessary condition for an extremum is

$$\left. \frac{\partial S[y; \alpha]}{\partial \alpha} \right|_{\alpha=0} = 0 \tag{9.2.5}$$

but we must understand how to take this derivative. Expanding $f(y_0, \dot{y}_0; t)$ in a Taylor
series, we have

$$f(y, \dot{y}; t) = f(y_0, \dot{y}_0; t) + \alpha \left[\eta \left. \frac{\partial f}{\partial y} \right|_{\alpha=0} + \dot{\eta} \left. \frac{\partial f}{\partial \dot{y}} \right|_{\alpha=0} \right] + \ldots \tag{9.2.6}$$

so that

$$\frac{\partial S[y; \alpha]}{\partial \alpha} = \int_1^2 dt \left[\left. \frac{\partial f}{\partial y} \right|_{\alpha=0} + \dot{\eta} \left. \frac{\partial f}{\partial \dot{y}} \right|_{\alpha=0} \right]. \tag{9.2.7}$$

With the understanding that all partial derivatives of f are taken at $\alpha = 0$, we write this
as

$$\frac{\partial S[y; \alpha]}{\partial \alpha} = \int_1^2 dt \left[\frac{\partial f}{\partial y} \eta + \frac{\partial f}{\partial \dot{y}} \dot{\eta} \right]. \tag{9.2.8}$$

The second term can be integrated by parts

$$\frac{\partial S[y;\alpha]}{\partial \alpha}\bigg|_{\alpha=0} = \int_1^2 dt \left[\eta \frac{\partial f}{\partial y} - \eta \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right] + \left[\eta(t)\frac{\partial f}{\partial \dot{y}}\right]_1^2, \qquad (9.2.9)$$

but because the deformations we are considering vanish at the boundaries ($\eta(t_1) = 0 = \eta(t_2)$), the necessary condition for an extremum becomes

$$\frac{\partial S[y;\alpha]}{\partial \alpha}\bigg|_{\alpha=0} = \int_1^2 dt \left[\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\eta(t) = 0. \qquad (9.2.10)$$

The deformations, $\eta(t)$, are arbitrary, therefore the integral can vanish only if

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \qquad (9.2.11)$$

This necessary condition for the existence of an extremum is known as **Euler's equation**. It is equivalent to the usual condition for the existence of an extremum of a simple function of a single variable in elementary calculus. If the functional S depends on more than one independent function, there is one Euler equation for each such independent function.

The left hand side of (9.2.11) is known as the **first functional derivative** of the $S[y]$ or the **Euler derivative** of f, and denoted by

$$\frac{\delta S[y]}{\delta y(t)} \stackrel{\text{def}}{=} \frac{\partial f}{\partial y(t)} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{y}(t)}\right). \qquad (9.2.12)$$

so that a necessary condition for the existence of an extremum is a vanishing first functional derivative,

$$\frac{\delta S[y]}{\delta y(t)} = 0 \qquad (9.2.13)$$

An alternative (quick and dirty) approach to recovering Euler's equation that we shall use henceforth is to consider a "variation" of S under variations of the function $y(t)$ as follows,

$$\delta S[y] = \int_1^2 dt \left(\frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial \dot{y}}\delta \dot{y}\right) \qquad (9.2.14)$$

where $\delta y = \alpha \eta$. The variations commute with the time derivative of y, *i.e.*,

$$\delta \frac{dy}{dt} = \frac{d}{dt}\delta y, \quad i.e., \quad \left[\delta, \frac{d}{dt}\right]y(t) = 0 \qquad (9.2.15)$$

Integrating by parts (as we did before) and discarding the boundary term because the variations vanish on the boundary, gives

$$\delta S[y] = \int_1^2 dt \left(\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial \dot{y}}\right)\delta y = 0, \qquad (9.2.16)$$

which imply Euler's equation by the same argument we used earlier. Let's apply this equation to determine the extrema of the functionals we developed earlier.

9.3   Examples

9.3.1   Geodesics

Consider first the functional

$$S[y] = \int_{x_1}^{x_2} dx \ \sqrt{1 + y'^2} \tag{9.3.1}$$

which gives the path length between two points on a trajectory specified by $y = y(x)$. Applying Euler's equation gives a differential equation for the function $y(x)$,

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = -\frac{d}{dx}\frac{y'}{\sqrt{1 + y'^2}} = 0 \tag{9.3.2}$$

which means that

$$\frac{y'}{\sqrt{1 + y'^2}} = c, \tag{9.3.3}$$

where c is some constant. Solving shows that the solution is of the form $y = mx + d$, where $m = c/(1 - c^2)$ and d is another arbitrary constant. This is the equation of a straight line, as we expected. The two constants (the slope and the $y-$intercept) are determined by the end points of course. An alternative approach would be to expand the derivative getting

$$\left[\frac{1}{\sqrt{1 + y'^2}} - \frac{y'^2}{(1 + y'^2)^{3/2}}\right] y'' = \frac{y''}{(1 + y'^2)^{3/2}} = 0 \tag{9.3.4}$$

which is only possible if $y''(x) = 0$. This gives the solution $y = mx + d$. Paths given by functions of the form $y = y(x)$ are very special if $y(x)$ is a function because it cannot turn into itself as the function cannot be one to many.

   The generalization of this problem to the problem of the geodesic on a sphere can also be solved exactly. Our action functional was

$$S = r \int_{1}^{2} d\theta \sqrt{1 + \sin^2 \theta \varphi'^2} \tag{9.3.5}$$

where the prime now refers to a derivative with respect to φ. Euler's equation is

$$-\frac{d}{d\theta}\left(\frac{\sin^2 \theta \varphi'}{\sqrt{1 + \sin^2 \theta \varphi'^2}}\right) = 0 \tag{9.3.6}$$

which is not dissimilar from the problem of the geodesic in $\mathbb{R}^2$. The solution is however different. We have

$$\frac{\sin^2 \theta \varphi'}{\sqrt{1 + \sin^2 \theta \varphi'^2}} = \alpha \tag{9.3.7}$$

where α is some constant, and simplifying

$$\varphi' = \frac{\alpha}{\sqrt{\sin^4\theta - \alpha^2\sin^2\theta}} \tag{9.3.8}$$

or

$$\varphi = \int \frac{\alpha\csc^2\theta d\theta}{\sqrt{1 - \alpha^2\csc^2\theta}} \tag{9.3.9}$$

The integral on the right can be solved by the substitution $x = \alpha\cot\theta$, which gives

$$\varphi + \beta = \int \frac{dx}{\sqrt{1 - \alpha^2 - x^2}} = \sin^{-1}\frac{x}{\sqrt{1 - \alpha^2}} = \sin^{-1}\left(\frac{\cot\theta}{a}\right) \tag{9.3.10}$$

where $a = \sqrt{1 - \alpha^2}/\alpha$. The solution is

$$\cot\theta = a\sin(\varphi + \beta) \tag{9.3.11}$$

To understand what it means, expand the right hand side to rewrite the equation as

$$\cos\theta = a(\sin\varphi\cos\beta + \cos\varphi\sin\beta)\sin\theta \tag{9.3.12}$$

and view the sphere as embedded in $\mathbb{R}^3$, where

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta. \tag{9.3.13}$$

The solution, rewritten in terms of the Cartesian coordinates in the embedding space is

$$z = Ax + By \tag{9.3.14}$$

where A and B are constants of the integration, related in an obvious way to a and β. We recognize this as the equation of a plane passing through the origin, *i.e.*, the center of the sphere! So the geodesic on a sphere is a great circle.[1]

_____

[1] Problem: Show that the equation for a geodesic on a cylinder is

$$\varphi = mz + \varphi_0$$

which is the equation of a helix. By "unwrapping" the cylinder, it turns into a plane. What does the geodesic look like in the plane?

9.3.2   Minimum surface of revolution

Our next problem was to determine the curve whose surface of revolution is minimum. For this we had to extremize the functional

$$A[y] = 2\pi \int_{x_1}^{x_2} dx\; y\sqrt{1 + y'^2}. \tag{9.3.15}$$

The Euler equation for the area functional is

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = \sqrt{1 + y'^2} - \frac{d}{dx}\frac{yy'}{\sqrt{1 + y'^2}} = 0 \tag{9.3.16}$$

Expand the derivative

$$\sqrt{1 + y'^2} - \frac{yy'' + y'^2}{\sqrt{1 + y'^2}} + \frac{yy'^2 y''}{(1 + y'^2)^{3/2}} = 0 \tag{9.3.17}$$

and simplify to see that this gives the simple equation

$$(1 + y'^2)^2 - (1 + y'^2)y'^2 - yy'' = 1 + y'^2 - yy'' = 0. \tag{9.3.18}$$

One may employ the following trick to solve this equation, even though it looks difficult. Divide by $(1 + y'^2)^{3/2}$ and multiply by y' to write it as

$$\frac{y'}{\sqrt{1 + y'^2}} - \frac{yy'y''}{(1 + y'^2)^{3/2}} = \frac{d}{dx}\frac{y}{\sqrt{1 + y'^2}} = 0, \tag{9.3.19}$$

which means that

$$\frac{y}{\sqrt{1 + y'^2}} = c \Rightarrow y' = \frac{dy}{dx} = \pm\sqrt{\frac{y^2}{c^2} - 1} \tag{9.3.20}$$

where c is an arbitrary constant. The solution is, of course

$$\int \frac{dy}{\sqrt{\frac{y^2}{c^2} - 1}} = \pm x + d \tag{9.3.21}$$

where d is yet another constant of integration. The solution is obtained by the substitution $y = c\cosh\eta$

$$y(x) = c\cosh\left(\frac{x + d}{c}\right) \tag{9.3.22}$$

The shape of a cable on a suspension bridge is given by the same solution because the functional minimizing the potential energy is the same.

9.3.3 The rotating bucket

We have argued that the shape of the surface of water that is rotating in a bucket is determined by the functional

$$U[y] = 2\pi\rho \int_0^R dr \; r \left[gy^2 - \frac{1}{2}\omega^2 r^2 y \right] \qquad (9.3.23)$$

This functional has no derivatives of $y(r)$, so Euler's equation reads

$$y(r) = \frac{1}{4g}\omega^2 r^2 \qquad (9.3.24)$$

which is the equation of a parabola whose steepness is determined by $\omega^2/4g$.

9.3.4 The Brachistochrone

We turn finally to the Brachistochrone, which involves minimizing the functional

$$T[y] = \int_{x_1}^{x_2} dx \sqrt{\frac{1 + y'^2}{2g(y_1 - y)}}. \qquad (9.3.25)$$

and leads to

$$\frac{1}{2}\frac{\sqrt{1 + y'^2}}{(y_1 - y)^{3/2}} - \frac{d}{dx}\left(\frac{y'}{\sqrt{(1 + y'^2)(y_1 - y)}} \right) = 0 \qquad (9.3.26)$$

Expanding the drivative term,

$$\frac{1}{2}\frac{\sqrt{1 + y'^2}}{(y_1 - y)^{3/2}} - \frac{y''}{\sqrt{(1 + y'^2)(y_1 - y)}} + \frac{y'^2 y''}{\sqrt{y_1 - y}(1 + y'^2)^{3/2}}$$
$$- \frac{y'^2}{2\sqrt{1 + y'^2}(y_1 - y)^{3/2}} = 0 \qquad (9.3.27)$$

and simplifying

$$\frac{1}{2}(1 + y'^2)^2 - y''(y_1 - y)(1 + y'^2) + y'^2 y''(y_1 - y) - \frac{1}{2}y'^2(1 + y'^2) = 0$$

$$\Rightarrow \quad 1 + y'^2 - 2(y_1 - y)y'' = 0 \qquad (9.3.28)$$

To solve this equation, it's convenient to eliminate y_1 by defining $z = y_1 - y$ so that the equation we must solve is

$$1 + z'^2 + 2zz'' = 0 \qquad (9.3.29)$$

and divide by $z^{3/2}(1 + z'^2)^{3/2}$. This gives

$$\frac{1}{z^{3/2}\sqrt{1 + z'^2}} + \frac{2z''}{\sqrt{z}(1 + z'^2)^{3/2}} = -\frac{d}{dx}\frac{1}{\sqrt{z(1 + z'^2)}} = 0 \qquad (9.3.30)$$

which, when solved for z' gives

$$z' = \pm\sqrt{\frac{c}{z} - 1} \Rightarrow \int \frac{\sqrt{z}\,dz}{\sqrt{c - z}} = x + d \qquad (9.3.31)$$

The solution to this integral is best obtained in parameterized form by making the substitution $z = c\sin^2\theta$ to rewrite the integral as

$$2c\int \sin^2\theta = x + d \Rightarrow x = \frac{c}{2}(2\theta - \sin 2\theta) - d \qquad (9.3.32)$$

Our solution, in parameterized form is therefore

$$x = \frac{c}{2}(2\theta - \sin 2\theta), \quad y = y_1 + \frac{c}{2}(1 - \cos 2\theta) \qquad (9.3.33)$$

which is a cycloid reflected on the $x-$axis and vertically shifted by y_1 .

9.4 Functional Derivatives

The Euler derivative is also called the first functional derivative of $S[y]$. It is worth rewriting our definition of this derivative in a way that is more adapted to extending it to higher order functional derivatives. To this end, define

$$\delta S[y] = \int dt \frac{\delta S[y]}{\delta y(t)}\eta(t) = \left.\frac{\partial S[y;\alpha]}{\partial \alpha}\right|_{\alpha=0} \qquad (9.4.1)$$

and compare the left hand side of the definition to (9.2.10) to see that it is identical to (9.2.12). In this suggestive form, however, it is easy to see how higher functional derivatives are to be defined. A natural extension is

$$\delta^n S[y] = \int dt_1 dt_2 ... dt_n \frac{\delta^n S[y]}{\delta y(t_1)\delta y(t_2)...\delta y(t_n)}\eta(t_1)\eta(t_2)...\eta(t_n) \overset{\text{def}}{=} \left.\frac{\partial^n S[y;\alpha]}{\partial \alpha^n}\right|_{\alpha=0}.$$
$$(9.4.2)$$

It turns out that higher order functional derivatives are *ultralocal differential operators*, *i.e.,* operators that are proportional to Dirac $\delta-$function(s). The second functional derivative defines an "eigenvalue" (Sturm-Liouville) problem for $\eta(t)$ and whether the extrema we have found are minima or maxima depends on whether or not $\delta^2 S[y] > 0$, *i.e.,* on the

eigenvalue spectrum of this differential operator. If the spectrum is positive definite the extremum is a minimum.

Let us use the definition in (9.4.2) to determine the second order functional derivative of (9.2.1). Expanding $f(y + \alpha\eta, \dot{y} + \alpha\dot{y}; t)$ up to second order in α,

$$f(y, \dot{y}; t) = f(y_0, \dot{y}_0; t) + \alpha \left[\eta \frac{\partial f}{\partial y} + \dot{\eta} \frac{\partial f}{\partial \dot{y}} \right]_{\alpha=0} + \frac{\alpha^2}{2} \left[\eta^2 \frac{\partial^2 f}{\partial y^2} + \eta\dot{\eta} \frac{\partial^2 f}{\partial y \partial \dot{y}} + \dot{\eta}^2 \frac{\partial^2 f}{\partial \dot{y}^2} \right]_{\alpha=0} + \dots \tag{9.4.3}$$

so

$$\left. \frac{\partial^2 S[y + \alpha\eta]}{\partial \alpha^2} \right|_{\alpha=0} = \int_i^f dt \left[\eta^2 \frac{\partial^2 f}{\partial y^2} + \eta\dot{\eta} \frac{\partial^2 f}{\partial y \partial \dot{y}} + \dot{\eta}^2 \frac{\partial^2 f}{\partial \dot{y}^2} \right] \tag{9.4.4}$$

Now notice that in the middle term $\eta\dot{\eta} = \dot{\eta^2}/2$ allows us to integrate the second term by parts and drop a total derivative (boundary term) because η is required to vanish there. Therefore

$$\left. \frac{\partial^2 S[y + \alpha\eta]}{\partial \alpha^2} \right|_{\alpha=0} = \int_i^f dt \left[\eta^2 \left\{ \frac{\partial^2 f}{\partial y^2} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial^2 f}{\partial y \partial \dot{y}} \right) \right\} + \dot{\eta}^2 \frac{\partial^2 f}{\partial \dot{y}^2} \right] \tag{9.4.5}$$

Also integrating the last term once by parts allows us to write the entire expression as

$$\left. \frac{\partial^2 S[y + \alpha\eta]}{\partial \alpha^2} \right|_{\alpha=0} = \int_i^f dt \; \eta \left[\left\{ \frac{\partial^2 f}{\partial y^2} - \frac{1}{2} \frac{d}{dt} \left(\frac{\partial^2 f}{\partial y \partial \dot{y}} \right) - \frac{d}{dt} \left(\frac{\partial^2 f}{\partial \dot{y}^2} \frac{d}{dt} \right) \right\} \right] \eta \tag{9.4.6}$$

and now we're able to compare it with (9.4.2). Let's apply this to a simple, but useful type of functional for which $f(y, \dot{y}; t) = \dot{y}^2/2 - v(y)$. In this case, a straightforward calculation reveals

$$\left. \frac{\partial^2 S[y + \alpha\eta]}{\partial \alpha^2} \right|_{\alpha=0} = -\int_i^f dt \; \eta(t) \left[\frac{\partial^2 v}{\partial y^2} + \frac{d^2}{dt^2} \right] \eta(t) \tag{9.4.7}$$

or ($y_1 = y(t_1)$ and $y_2 = y(t_2)$)

$$\frac{\delta^2 S}{\delta y_1 \delta y_2} = - \left[\frac{d^2}{dt_1^2} + \frac{\partial^2 v(y_1)}{\partial y_1^2} \right] \delta(t_1 - t_2) \tag{9.4.8}$$

Consider the eigenvalue equations for this operator,

$$- \left[\frac{\partial^2 v}{\partial y^2} + \frac{d^2}{dt^2} \right]_{y=y_0} \phi_n(t) = \lambda_n \phi_n(t), \tag{9.4.9}$$

where $y_0(t)$ represents an extreme value of the functional, *i.e.* a solution to Euler's equation and $\phi_n(t)$ represent a complete set of orthonormal eigenfunctions. Let $\phi_n(t)$ be normalized in the interval $[t_i, t_f]$ and vanishing at the boundaries, so that we can set

$$\eta(t) = \sum_n c_n \phi_n(t) \tag{9.4.10}$$

then if we let $S[y] = S[y_0] + \delta S[y_0] + \delta^2 S[y_0] + \ldots$,

$$\delta^2 S[y_0] = \sum_n \lambda_n c_n^2 \tag{9.4.11}$$

To make our example simpler still, take $v''(y)$ to be ignorable, then our eigenvalue equation becomes

$$\frac{d^2 \phi_n}{dt^2} + \lambda_n \phi_n = 0 \tag{9.4.12}$$

Solutions that vanish at the t_i and t_f can be found. To simplify the algebra, take $t_i = 0$ and $t_f = T$, so that the only solutions that vanish at the end points are

$$\phi_n = A \sin \sqrt{\lambda_n} t \tag{9.4.13}$$

where

$$\lambda_n = \frac{n^2 \pi^2}{T^2}, \quad n \in \mathbb{N} \cup \{0\} \tag{9.4.14}$$

The eigenvalues are all non-negative, so $\delta^2 S[y_0] > 0$ so, in this case, the extremum of the action, given by Eulers equations, will represent a minimum. More generally, eg., if $v''(y)$ were not ignorable, the result would depend on $v(y)$.

9.5   An alternate form of Euler's equation

There is an alternate form of Euler's equation, which is particularly useful when $f(y, \dot{y})$ does not explicitly depend on t. It can be obtained directly from (9.2.11) by writing

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} \dot{y} + \frac{\partial f}{\partial \dot{y}} \ddot{y} \tag{9.5.1}$$

and substituting

$$\frac{d}{dt} \left(\frac{\partial f}{\partial \dot{y}} \dot{y} \right) = \frac{\partial f}{\partial \dot{y}} \ddot{y} + \dot{y} \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{y}} \right) \tag{9.5.2}$$

we find

$$\frac{d}{dt} \left[f - \frac{\partial f}{\partial \dot{y}} \dot{y} \right] = \frac{\partial f}{\partial t} + \left[\frac{\partial f}{\partial y} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{y}} \right) \right] \dot{y}. \tag{9.5.3}$$

But by Euler's equation, the term in square brackets on right hand side vanishes, therefore

$$\frac{\partial f}{\partial t} - \frac{d}{dt} \left[f - \frac{\partial f}{\partial \dot{y}} \dot{y} \right] = 0 \tag{9.5.4}$$

and if f does not depend explicitly on time,

$$f - \frac{\partial f}{\partial \dot{y}} \dot{y} = \text{constant} \tag{9.5.5}$$

summarizes the content of (9.2.11). One can immediately see how useful this is as it involves just a first order equation.

9.6   Functionals involving several functions

Our considerations so far are easily generalized to functionals of several dependent variables,

$$S[x_1, x_2, ...x_n] = \int_1^2 dt \ f(x_1(t), ..., x_n(t), \dot{x}_1(t), ..., \dot{x}_n(t), t) \tag{9.6.1}$$

Such functionals are more adapted to the requirements of mechanics, where even two dimensional trajectories may fold in on themselves and cannot be represented as functions, but more generally are not constrained to surfaces. As we know, in mechanics it is more useful to consider parameterized curves, $\vec{r} = \vec{r}(t)$. We can use the same variational techniques as we did before. Once more, requiring

$$\delta S = \int \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \delta x_i + \frac{\partial f}{\partial \dot{x}_i} \delta \dot{x}_i \right) = 0 \tag{9.6.2}$$

gives

$$\delta S = 0 \Rightarrow \int dt \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_i} \right) \delta x_i = 0 \tag{9.6.3}$$

which is possible if and only if each term vanishes because the variations $\delta x_i(t)$ are all arbitrary and independent. Therefore we have n Euler equations, one for each dependent variable,

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_i} = 0, \quad i \in \{1, ..., n\} \tag{9.6.4}$$

which may, or may not be coupled, dependending on the functions f.[2]

As an example, consider the parameterized curve in $\mathbb{R}^3$, given by the three functions, $x_1(t)$, $x_2(t)$ and $x_3(t)$. In this case, the path length can be written as

$$S = \int_{t_1}^{t_2} dt \ \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}. \tag{9.6.5}$$

There are therefore three Euler equations, one for each function. We have,

$$\frac{\partial f}{\partial x_i} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_i} = -\frac{d}{dt} \frac{\dot{x}_i}{\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}} = 0 \tag{9.6.6}$$

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[2]Problem: Show that if f has no explicit dependence on t, then

$$f - \sum_i \frac{\partial f}{\partial \dot{y}_i} \dot{y}_i = 0$$

Note that this is just one equation, whereas there are as many Euler's equations as dependent functions, assuming no external constraints. This is the "alternative" form of the previous section.

The vector $\vec{v} = (\dot{x}_1, \dot{x}_2, \dot{x}_3)$ is just the "velocity" and therefore these equations tell us that $\hat{v}$ is a constant vector. Since the direction of the velocity vector is constant, the trajectory is a straight line. We can put the equations in a more suggestive form if we recognize that

$$\frac{ds}{dt} = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2} \tag{9.6.7}$$

and therefore that (9.6.6) can also be written as

$$\frac{d\hat{t}}{ds} = 0 \tag{9.6.8}$$

where $\hat{t}$ is the unit tangent vector to the curve. In the Serret-Frenet description of curves, this is saying that the curvature is zero. Note that minimizing the path length gives no information about the *magnitude* of the velocity and this means that there are really only *two* and not three independent equations of motion. One can trace this to "reparametrization invariance", which means that any transformation $t \to t' = f(t)$, with $x_i \to x_i' = x_i$, leaves the functional S invariant so that the parameter t itself has no physical significance. We will be looking for functionals whose extremization leads to Newton's laws of motion. However, because Newton's first law already requires a free body to possess a constant velocity (therefore both *magnitude* and direction), this functional is not a good starting point.

9.7   Constraints

Often it happens that it is necessary to consider additonal constraints on a system. For example, consider the equation of the geodesic on sphere. The way we solved the problem was to think of the sphere *intrinsically i.e.,* without reference to the three dimensional space, $\mathbb{R}^3$, in which it is embedded. This is a very useful way to think about curved spaces, but it is not necessary because every such space can also be thought of as embedded in some higher dimensional space, $\mathbb{R}^n$.[3] For example, we could think of the sphere as embedded in $\mathbb{R}^3$. We already know that the shortest distance between two points in $\mathbb{R}^3$ is a straight line, but it is far from so on the sphere! To get the shortest distance *along* the sphere we must impose an additional condition, a *constraint*, which explicitly requires that all points on the trajectory lie on it, *i.e.,* we must also require

$$x_1^2(t) + x_2^2(t) + x_3^2(t) = r^2 \tag{9.7.1}$$

where r is the radius of the sphere.

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[3]Note that the embedding space does not necessarily possess just one dimension more than the embedded space.

In general, when constraints are present, we need to impose additional conditions of the form

$$g(x_1, x_2, ..., \dot{x}_1, ..., \dot{x}_n) = 0 \tag{9.7.2}$$

The presence of constraints implies that the variations we make in deriving the Euler equations are not all independent and therefore $\delta S = 0$ does not imply that the Euler equations hold separately for each dependent variable. Let us first review how constraints are handled in simple problems not involving functionals.

Consider a cylinder of radius r and length l. How can the surface area be minimized while keeping the volume of the cylinder fixed?. (For example, you may be asked to design a cylindrical container of fixed volume in such a way as to minimize the material used in its construction.) For any r and l, the surface area of the cylinder (including its ends) is $A(r, l) = 2\pi r l + 2\pi r^2$ and its volume is $V(r, l) = \pi r^2 l$. In this simple example, we could solve the constraint directly,

$$l = \frac{V}{\pi r^2} \tag{9.7.3}$$

and rewrite A as a function of r (subject to the constraint)

$$A(r) = \frac{2V}{r} + 2\pi r^2 \tag{9.7.4}$$

Then it is easy to see that $r = \sqrt[3]{V/2\pi}$ gives the radius that minimizes the surface area. Notice that solving the constraint first reduces the number of independent variables (in this case, r and l). It also complicates the function to be extremized. Sometimes it may be difficult to solve the constraint equation(s); what then do we do? We follow a method introduced by and named after the mathematician Joseph Louis **Lagrange**.

To motivate Lagrange's approach, consider having to extremize a function S of two variables, x_i, $i \in \{1, 2\}$, subject to a constraint of the form $g(x_i) = 0$. The extremization condition for S is

$$\delta S = \sum_i \frac{\partial S}{\partial x_i} \delta x_i = 0 \tag{9.7.5}$$

but because the variables are not all independent on account of the constraint, we cannot set each first partial derivative to zero independently. Instead, noting that $g(x_i) = 0$, vary the constraint also to get

$$\delta g(x_i) = 0 = \sum_i \frac{\partial g}{\partial x_i} \delta x_i \tag{9.7.6}$$

Of course, this means that

$$\delta x_2 = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2} \delta x_1 \tag{9.7.7}$$

Inserting this into the variation of S, we find

$$\delta S = \left[\frac{\partial S}{\partial x_1} - \frac{\partial S}{\partial x_2} \frac{\partial g/\partial x_1}{\partial g/\partial x_2} \right] \delta x_1 \qquad (9.7.8)$$

or

$$\frac{\partial S}{\partial x_1} \left(\frac{\partial g}{\partial x_1} \right)^{-1} = \frac{\partial S}{\partial x_2} \left(\frac{\partial g}{\partial x_2} \right)^{-1} \qquad (9.7.9)$$

However, because the left hand side involves only derivatives with respect to x_1 and the right only derivatives with respect to x_2, the two must be equal to some constant, $-\lambda$. In particular,

$$\frac{\partial S}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} = 0, \quad i \in \{1, 2\} \qquad (9.7.10)$$

and, if we call $\mathcal{S} = S + \lambda g$, then not only do we have

$$\frac{\partial \mathcal{S}}{\partial x_i} = 0, \quad i \in \{1, 2\} \qquad (9.7.11)$$

but also

$$\frac{\partial \mathcal{S}}{\partial \lambda} = 0, \qquad (9.7.12)$$

which reproduces the constraint $g(x_i) = 0$. The three equations (9.7.11) and (9.7.12) are sufficient to solve for the three variables x_i and λ and constitute a general solution of the extremzation problem when constraints are present. The extra variable that was introduced (λ) has come to be called a Lagrange multiplier. The method is easily extended (by mathematical induction) to the case of n variables and any number $m < n$ of constraints. We need one Lagrange multiplier for each constraint.

Lagrange thus turned a problem originally involving n variables satisfying $m < n$ constraints (our simple example involves two variables and one constraint) into a problem involving $n + m$ variables and no constraints by introducing the **Lagrange multipliers**. In our toy example, instead of considering $A(r, l)$ we consider the function

$$\mathcal{A} = A(r, l) + \lambda (V(r, l) - V_0) \qquad (9.7.13)$$

where λ is a new variable (called a Lagrange multiplier) that multiplies the constraint and V_0 is the desired volume of the container. Notice that if all variables are independent, then $\mathcal{A}$ is extreme only if its first derivative with respect to each variable vanishes

$$\frac{\partial \mathcal{A}}{\partial r} = 2\pi (l + 2r + \lambda r l) = 0$$

$$\frac{\partial \mathcal{A}}{\partial l} = 2\pi \left(r + \frac{\lambda}{2} r^2 \right) = 0$$

$$\frac{\partial \mathcal{A}}{\partial \lambda} = \pi r^2 l - V_0 = 0. \tag{9.7.14}$$

We have three equations for the three variables r, l, λ. The last equation makes the constraint explicit. The second equation, when solved for λ gives $\lambda = -2/r$ and when this value of λ is inserted into the first equation we obtain $l = 2r$. With this value of l, the last equation gives precisely $r = \sqrt[3]{V_0/2\pi}$. The cost of using Lagrange multipliers is that we have had to solve three equations simultaneously, whereas we would have had to solve just one equation if the constraint were solved at the start and l replaced by an appropriate function of r. In general, when there are $m < n$ constraints in the system, the use of Lagrange multipliers introduces an additional $2m$ equations. The lesson is that if the constraints can be solved at the start, it is best to do so and work with independent variables. Lagrange multipliers should be used only if it is impossible to solve the constraints.

Solving constraints may be done in one of two ways: either one uses the constraint equations to directly eliminate one or more variables and reexpress the function to be extremized in terms of the remaining (now independent) variables or, what is most frequently the case, finding an entirely new set of (independent) variables in terms of which which to express the function to be extremized. For example, (9.7.1) can be solved by introducing the angles θ and φ, in terms of which

$$x_1 = r \sin\theta \cos\varphi, \quad x_2 = r \sin\theta \sin\varphi, \quad x_3 = r \cos\theta. \tag{9.7.15}$$

These are the transformations to spherical coordinates, but here r is constant. By construction $x_1^2 + x_2^2 + x_3^2 = r^2$ is automatic and the distance function can now be expressed in terms of the angles as

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \tag{9.7.16}$$

The problem of finding geodesics on the sphere in terms of θ and φ has already been solved.

Consider the problem of finding the dimensions of a box for which the energy of a quantum particle in its ground state and confined to its interior is minimum, holding the volume of the box fixed. The ground state energy of a quantum particle in a box of dimensions a b and c is

$$E(a, b, c) = \frac{\pi^2 \hbar^2}{2m} \left[\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right] \tag{9.7.17}$$

and we must impose the constraint $g(a, b, c) = abc - V_0 = 0$. We have a function of three variables and one constraint. Using the method of Lagrange multipliers, we would introduce one Lagrange multiplier (for the constraint) and extremize the function

$$\mathcal{E}(a, b, c, \lambda) = E(a, b, c) + \lambda g(a, b, c), \tag{9.7.18}$$

treating it as a function of four independent variables. Thus,

$$\frac{\partial \mathcal{E}}{\partial a} = -\frac{\pi^2 \hbar^2}{ma^3} + \lambda bc = 0$$

$$\frac{\partial \mathcal{E}}{\partial b} = -\frac{\pi^2 \hbar^2}{mb^3} + \lambda ac = 0$$

$$\frac{\partial \mathcal{E}}{\partial c} = -\frac{\pi^2 \hbar^2}{mc^3} + \lambda ab = 0$$

$$\frac{\partial \mathcal{E}}{\partial \lambda} = abc - V_0 = 0 \tag{9.7.19}$$

Multiplying the first equation by a, the second by b and the last by c we find

$$\lambda abc = \frac{\pi^2 \hbar^2}{ma^2} = \frac{\pi^2 \hbar^2}{mb^2} = \frac{\pi^2 \hbar^2}{mc^2} \tag{9.7.20}$$

i.e., $a = b = c$ and the box is a cube. Further, taking second derivatives makes it clear that the energy is minimized, so the box that minimizes the ground state energy is a cube of side $a = \sqrt[3]{V_0}$.[4]

A still more interesting example is the following. For a system of identical particles obeying Pauli's exclusion principle, the probability of a given arrangement possessing n_i particles in state i of energy ε_i and degeneracy g_i is

$$W = \prod_i \frac{g_i!}{n_i!(g_i - n_i)!}. \tag{9.7.21}$$

If the total number of particles, $N = \sum_i n_i$ and the total energy of the system, $E = \sum_i n_i \varepsilon_i$, are together held fixed, we can show using Lagrange's method that the maximum probability occurs for the arrangement

$$n_i = \frac{g_i}{e^{\lambda_1 + \lambda_2 \varepsilon_i} + 1} \tag{9.7.22}$$

where $\lambda_{1,2}$ are the Lagrange multipliers corresponding to the first and second constraints respectively. It's simpler to examine the log of the probability function

$$\ln W \approx \sum_i [g_i \ln g_i - n_i \ln n_i - (g_i - n_i) \ln(g_i - n_i)] \tag{9.7.23}$$

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[4]Problem: The ground state energy of a quantum particle confined to a cylindrical container of radius r and length l is

$$E(r,l) = \frac{\hbar^2}{2m}\left(\frac{c^2}{r^2} + \frac{\pi^2}{l^2}\right)$$

where c is a constant. If the volume of the cylinder is held fixed, show that the ground state energy is minimized when

$$\frac{r}{l} = \frac{c}{\sqrt{2}\pi}$$

where we have used Stirling's formula

$$\ln N! \approx N \ln N - N \tag{9.7.24}$$

Now introduce the constraints via Lagrange multipliers, consider

$$\mathcal{W} = \sum_i [g_i \ln g_i - n_i \ln n_i - (g_i - n_i) \ln(g_i - n_i) - \lambda_1 n_i - \lambda_2 n_i \varepsilon_i] \tag{9.7.25}$$

and vary with respect to n_i,

$$\delta \ln \mathcal{W} = \sum_i [- \ln n_i + \ln(g_i - n_i) - \lambda_1 - \lambda_2 \varepsilon_i] \delta n_i = 0 \tag{9.7.26}$$

gives

$$\frac{n_i}{g_i - n_i} = e^{\lambda_1 + \lambda_2 \varepsilon_i} \Rightarrow n_i = \frac{g_i e^{-\lambda_1 - \lambda_2 \varepsilon_i}}{e^{-\lambda_1 - \lambda_2 \varepsilon_i} + 1} \tag{9.7.27}$$

or

$$n_i = \frac{g_i}{e^{\lambda_1 + \lambda_2 \varepsilon_i} + 1} \tag{9.7.28}$$

The Lagrange multipliers are related to the fermi energy, E_F, and the temperature, T, of the Fermi gas. Comparing with the Fermi distribution we see that $\lambda_2 = 1/kT$ and $\lambda_1 = -E_F/kT$.

We can apply the same arguments to the problem of extremizing functionals with constraints. Consider a functional of two functions,

$$S[x_1, x_2] = \int_1^2 dt f(x_1, x_2, \dot{x}_1, \dot{x}_2, t) \tag{9.7.29}$$

where $x_1(t)$ and $x_2(t)$ are subject to some constraint $g(x_1, x_2) = 0$. Varying S

$$\delta S = \int_1^2 dt \left[\left(\frac{\partial f}{\partial x_1} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_1} \right) \delta x_1 + \left(\frac{\partial f}{\partial x_2} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_2} \right) \delta x_2 \right] = 0 \tag{9.7.30}$$

but each term in brackets cannot be separately set to zero because of the constraint. Instead we consider

$$\delta g(x_1, x_2) = \frac{\partial g}{\partial x_1} \delta x_1 + \frac{\partial g}{\partial x_2} \delta x_2 = 0 \tag{9.7.31}$$

which means that

$$\delta x_2 = -\frac{\partial g/\partial x_1}{\partial g/\partial x_2} \delta x_1 \tag{9.7.32}$$

which can be inserted into the variation of S to give

$$\delta S = \int_1^2 dt \left[\left(\frac{\partial f}{\partial x_1} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_1} \right) - \left(\frac{\partial f}{\partial x_2} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}_2} \right) \frac{\partial g/\partial x_1}{\partial g/\partial x_2} \right] \delta x_1 = 0 \tag{9.7.33}$$

Thus,

$$\left(\frac{\partial f}{\partial x_1} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}_1}\right)\left(\frac{\partial g}{\partial x_1}\right)^{-1} = \left(\frac{\partial f}{\partial x_2} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}_2}\right)\left(\frac{\partial g}{\partial x_2}\right)^{-1} \tag{9.7.34}$$

But the left hand side of the above equation involves only functional derivatives with respect to x_1, whereas the right hand side only functional derivatives with respect to x_2, so they must each be equal to some function $-\lambda(t)$,

$$\left(\frac{\partial f}{\partial x_1} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}_1}\right) + \lambda(t)\left(\frac{\partial g}{\partial x_1}\right) = 0$$

$$\left(\frac{\partial f}{\partial x_2} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}_2}\right) + \lambda(t)\left(\frac{\partial g}{\partial x_2}\right) = 0 \tag{9.7.35}$$

along with the constraint $g(x_1, x_2) = 0$. The function $\lambda(t)$ is determined from these three equations. It is a Lagrange multiplier function. In fact if we have n dependent variables and $m < n$ constraints of the form $g_l(x_1, ..., x_n) = 0$, then there are $n + m$ equations of the form

$$\left(\frac{\partial f}{\partial x_i} - \frac{d}{dt}\frac{\partial f}{\partial \dot{x}_i}\right) + \lambda(t)\left(\frac{\partial g}{\partial x_i}\right) = 0, \quad i \in \{1, ..., n\}$$

$$g_l(x_1, ..., x_n) = 0, \quad l \in \{1, ..., m\} \tag{9.7.36}$$

all of which can be derived from varying the modified functional

$$S \to \mathcal{S} = \int_1^2 dt\left[f(x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n, t) + \sum_{l=1}^{m}\lambda_l(t)g_l(x_1, ..., x_n)\right] \tag{9.7.37}$$

with respect to the x_i and λ_l, treating all of them as independent variations.[5]

Constraints are classified according to their form. If x_i are the dependent variables of the system and the constraints can be expressed in the form

$$g(x_i, t) = 0 \tag{9.7.38}$$

then they are called **holonomous**. If they cannot be expressed in the form (9.7.38) then they are **non-holonomous**. For example the requirement that an object may move only on the surface of a sphere is the holonomous constraint $g(x, y, z) = x^2 + y^2 + z^2 - r^2 = 0$. On the other hand, requiring the object to always reside *inside* the sphere is the non-holonomous constraint $g(x, y, z) = x^2 + y^2 + z^2 - r^2 \leq 0$. Velocity dependent constraints such as

$$g(x_i, \dot{x}_i, t) = 0 \tag{9.7.39}$$

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[5]Problem: Use mathematical induction to show this!

are holonomous if they can be integrated to the form (9.7.38).

Constraints may or may not depend explicitly on time. Holonomic constraints that *do not* depend explicitly on time are called **scleronomous**. Holonomic constraints that *do* depend explicitly on time are **rheonomous**. Thus if we require a bead to slide along a wire spanning some *fixed* curve in space the holonomous constraint would be scleronomous. But if the curve itself were to change in time in some externally prescribed fashion then the constraint would be rheonomous. The key words here are "externally prescribed". If the curve were to change *as a consequence of the beads motion* so that the wire were itself a part of the system then the constraint would be scleronomous because this time dependence would not be explicit. We will have more to say about constraints as we proceed to address problems of physical interest.

Chapter 10

The Lagrangian

So far we have applied the variational principle to problems primarily of geometric interest. However, it has played a most crucial role in the development of physics. For instance, Fermat in 1740 derived the laws of reflection and refraction (of light) from such a principle. Newton's laws of motion can be formulated in terms of a variational principle and the variational principle makes their generalization to relativistic (special and general) systems straightforward. In this chapter we will learn to formulate Newton's laws in terms of a variational principle, but first let us examine Fermat's derivation of the laws of reflection and refraction, not simply because of its historic importance but also because it will return when we consider mechanics in the context of special relativity.

We will discuss the concept of symmetries and prove Noether's theorems, in fulfillment of a promise made in the previous chapter to show how how the symmetries of a mechanical system relate to the conservation laws. Finally we will examine an important reformulation in terms of the "Hamiltonian" and the Poisson brackets which played a key role in the development of quantum mechanics. When the Lagrangian describing the mechanical system has no explicit time dependence (equivalently, when the action describing the system is time translation invariant), the Hamiltonian is nothing but the energy of the mechanical system.

10.1   Fermat's least time principle

In 1740, Fermat postulated that light always travels along the path that minimizes the time taken by it to get from one point to another. This has come to be known as "Fermat's least time principle". We will see now how it can be used to derive Snell's laws of geometric optics. Consider a ray of light that travels from A to B upon reflection at some point P on a reflecting surface, as shown in figure (10.1)]. Because we are considering reflection, the ray of light travels in a single medium throughout its path and the speed of light is
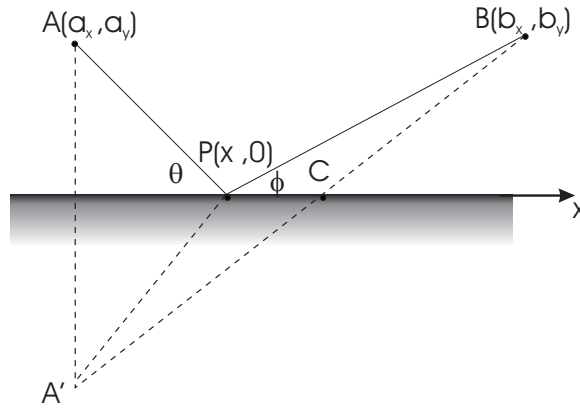
Figure 10.1: Law of reflection

constant. It follows that the least time principle is equivalent in this case to the shortest distance principle. The distance a ray of light would have to travel in going from A to B via P (refer to the figure) is

$$d(x) = \sqrt{(x - a_x)^2 + a_y^2} + \sqrt{(b_x - x)^2 + b_y^2} \tag{10.1.1}$$

Extremizing the function $d(x)$,

$$d'(x) = \frac{(x - a_x)}{\sqrt{(x - a_x)^2 + a_y^2}} - \frac{(b_x - x)}{\sqrt{(b_x - x)^2 + b_y^2}} = 0 \tag{10.1.2}$$

which implies that

$$\cos\theta = \cos\phi \tag{10.1.3}$$

or $\theta = \phi$, which is the law of reflection.

The situation is slightly more complicated when we consider refraction because the ray of light travels through two media (at least) with different velocities [see figure (10.2)]. Therefore the times taken for the paths are respectively

$$t_{\mathrm{AP}} = \frac{\sqrt{(x - a_x)^2 + a_y^2}}{v_1}$$

$$t_{\mathrm{PB}} = \frac{\sqrt{(b_x - x)^2 + b_y^2}}{v_2} \tag{10.1.4}$$
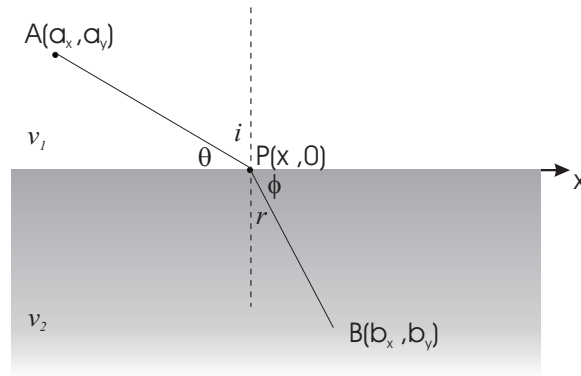
Figure 10.2: Law of refraction

and, according to the least time principle, the quantity to be minimized is

$$t_{\mathrm{AB}}(x) = t_{\mathrm{AP}} + t_{\mathrm{PB}} = \frac{\sqrt{(x - a_x)^2 + a_y^2}}{v_1} + \frac{\sqrt{(b_x - x)^2 + b_y^2}}{v_2}. \tag{10.1.5}$$

We find

$$t'_{\mathrm{AB}} = \frac{\cos\theta}{v_1} - \frac{\cos\phi}{v_2} = 0 \tag{10.1.6}$$

or

$$\frac{\sin i}{\sin r} = \frac{v_1}{v_2} \tag{10.1.7}$$

which we recognize as Snell's law, if we identify the ratio $n_{12} = v_1/v_2$ as the refractive index of medium 2 relative to medium 1. This is quite a prediction, because we know experimentally that light bends *toward* the normal when passing from a rarer to a denser medium and, in this case, $n_{12} > 1$. This implies that the seed of light is smaller in a denser medium. In particular, the speed of light is maximum in the vacuum. This behavior is opposite that of the speed of sound and contrary to the predictions of Newton's corpuscular theory of light, which were set forth by Newton himself *after* Fermat enunciated his least time principle. We know of course that it is Fermat's principle that gives the correct result.

In general, Fermat's principle can be formulated in terms of a functional to be extremized. Let $n_{(}\vec{r}) = c/|\vec{v}(\vec{r})|$, where $\vec{r} = (x_1, x_2, x_3)$ is the position vector of any point P, represent the refractive index of a medium (or set of media) relative to the vacuum (c is the speed of light in the vacuum), then the time taken for light to travel from point A to

point B is the functional

$$T[x_1, x_2, x_3] = \int_A^B \frac{ds}{|\vec{v}(\vec{r})|} = \frac{1}{c} \int_A^B n(\vec{r})ds \tag{10.1.8}$$

According to Fermat's principle, the path followed by a ray of light is the one for which $T[x_1, x_2, x_3]$ is minimum. The functional $T[x_1, x_2, x_3]$ is called the **optical path length** from A to B.

10.2 The variational principle of mechanics

In the previous chapter we saw that minimization of the path length yields straight line trajectories but not Newton's first law. This was because reparameterization invariance implied that Euler's equations cannot determine the speed of the particle, only the curvature of its trajectory. To recover the entire content of the first law from Euler's equations, we could introduce a constraint requiring that the speed is held fixed, but the constraint would be non-holonomous and non-holonomous constraints are usually difficult to deal with. Moreover, this cannot be the correct approach because the constraint could not be imposed in the presence of external forces and we are, in the end, after a variational principle for a particle or system of particles in the presence of external and internal forces. We would therefore encounter a situation in which the functional describing the motion of a free particle cannot be obtained as the limit of a more general functional as the external forces are made to vanish. Therefore, we ask instead if there is a functional whose extremization would yield

$$m\ddot{x}_i = 0, \quad i \in \{1, 2, 3\} \tag{10.2.1}$$

(the equations of motion for a free particle) without imposing additional constraints. It is not difficult to see that the equations $\ddot{x}_i = 0$ extremize the functional

$$S' = \int_1^2 dt\, \frac{1}{2} \sum_i \dot{x}_i^2, \tag{10.2.2}$$

therefore, since we want $m\ddot{x}_i = 0$, or more precisely $dp_i/dt = 0$, we simply multiply the functional S' by the mass of the particle, taking instead the action

$$S = \int_1^2 dt\, \frac{1}{2}m \sum_i \dot{x}_i^2 = \int_1^2 dt\, T(\dot{x}_i). \tag{10.2.3}$$

where we recognize $T(\dot{x}_i)$ to be the kinetic energy of the particle.

Notice that the motion of a free particle is not interpreted as optimizing any geometric quantity (such as the path length) but rather the average kinetic energy of the particle.

However, it turns out that it is actually an approximation of a quantity that does have a geometric meaning, the "proper time" of the particle. This is the time measured on a clock that moves with the particle. For observers relative to whom the speed of the particle is slow compared with the speed of light the proper time approximates to the absolute, Galilean, time up to corrections of order v^2/c^2. The functional (10.2.3) represents the first correction term.

Including the action of forces on the particle (Newton's second law) just as easy. Modify S by introducing an additional function

$$S = \int_1^2 dt \ [T(\dot{x}_i) + Z(x_i, \dot{x}_i, t)] \tag{10.2.4}$$

and determine Z. The Euler Lagrange equations are then

$$- m\ddot{x}_i + \frac{\partial Z}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial Z}{\partial \dot{x}_i}\right) = 0. \tag{10.2.5}$$

We will consider only forces that are derivable from a potential, $F_i = -\partial_i U$ (conservative forces). If the force does not depend on the velocity then the potential is a function only of position, $U = U(\vec{r})$. According to Newton's second law, $m\ddot{x}_i = -\partial_i U$, therefore we must take $Z(x_i \dot{x}_i, t) = -U(x_i)$. Therefore, when all the forces acting on the particle are conservative, extremizing the functional

$$S = \int_1^2 dt \ [T(\dot{x}_i) - U(x_i)] \tag{10.2.6}$$

then gives precisely Newton's second law of motion. As is often the case in physics, we raise the statement to the status of a fundamental law, applicable to *all* physical systems and generalizing when necessary even Newton's three laws of motion, leaving it to experiment to determine its the limits if any of its validity. Thus we arrive at Hamilton's principle (which has yet to be challenged):

Hamilton's principle: Of all possible paths along which a dynamical system may evolve, from one configuration to another within a fixed time interval, the actual path taken is the one that extremizes the functional (10.2.6).

The quantity $\mathcal{L} = T - U$ is called the **Lagrangian** of the system and S is called the **action** functional,

$$S = \int_1^2 dt \ \mathcal{L}(x_i, \dot{x}_i, t) \tag{10.2.7}$$

When a system consists of more than one particle, the kinetic energy is the total kinetic energy, *i.e.,* the simple sum of kinetic energies of the individual particles and the potential energy is the *total* potential energy (external plus internal) of the system.

When there are constraints on a system, the coordinates and velocities are not all independent. In the case of holonomous constraints this is resolved by choosing a set of **generalized coordinates** that solve the constraints. For an unconstrained n particle system there are $3n$ coordinates, x_i. Imposing r constraints reduces the number of independent coordinates to $3n - r$. If the constraints can be solved so that the Cartesian coordinates, or a subset of them, are expressed in terms of $3n - r$ generalized coordinates, $q_a \in \{q_1, q_2, \ldots, q_{3n-r}\}$ then we will have $2n$ relations of the form

$$x_i = x_i(q_1, q_2, ..., q_{3n-r}), \quad \dot{x}_i = \sum_{a=1}^{3n-r} \frac{\partial x_i}{\partial q_a} \dot{q}_a \tag{10.2.8}$$

The Lagrangian may be expressed in terms of the geneneralized coordinates without constraints. If it is impossible to find generalized coordinates (because it turns out to be exceedingly difficult to solve the constraints) then the Lagrangian may be written in terms of the original coordinates, x_i, with the constraints introduced via Lagrange multipliers. One should use Lagrange multipliers only as a last resort. Solving and eliminating the constraints first is *always preferred*, when possible.

When the constraints are of the form

$$g_\alpha(x_i, \dot{x}_i, t) = 0 \tag{10.2.9}$$

one should first check if they are integrable. For example, the constraint

$$\sum_i A_i(\vec{x}, t) \frac{dx_i}{dt} + B(\vec{x}, t) \tag{10.2.10}$$

is integrable if and only if

$$\sum_i A_i(\vec{x}, t) dx_i + B(\vec{x}, t) dt = df(\vec{x}, t) \tag{10.2.11}$$

where f is some function of x_i and t. This means that

$$A_i = \frac{\partial f}{\partial x_i} \quad B = \frac{\partial f}{\partial t} \tag{10.2.12}$$

for some function $f(\vec{x}, t)$ and it follows that

$$\frac{\partial A_i}{\partial t} = \frac{\partial B}{\partial x_i} \tag{10.2.13}$$

If these conditions fail the constraint is not holonomous.

10.3   Examples

The one dimensional harmonic oscillator is described by the potential function $U(x) = \frac{1}{2}kx^2$. Its Lagrangian is therefore

$$\mathcal{L} = T(\dot{x}) - U(x) = \frac{1}{2}\left(m\dot{x}^2 - kx^2\right) \tag{10.3.1}$$

with the action

$$S = \frac{1}{2}\int_1^2 dt(m\dot{x}^2 - kx^2). \tag{10.3.2}$$

The Euler equation,

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = -kx - m\ddot{x} = 0 \tag{10.3.3}$$

yields the well known equation of motion $m\ddot{x} = -kx$.

The gravitational potential energy of a system of two particles of mass m_1 and m_2 is $U(r) = -Gm_1m_2m/|\vec{r}_1 - \vec{r}_2|$. The Lagrangian therefore is

$$\mathcal{L} = \frac{1}{2}m_1\dot{\vec{r}}_1^{\ 2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\ 2} + \frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|}, \tag{10.3.4}$$

and the action,

$$S = \int_1^2 dt\left[\frac{1}{2}m_1\dot{\vec{r}}_1^{\ 2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\ 2} + \frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|}\right] \tag{10.3.5}$$

when extremized gives the Euler equations

$$\frac{\partial \mathcal{L}}{\partial x_{n,i}} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{n,i}}\right) = 0 \tag{10.3.6}$$

or

$$m_1\ddot{\vec{r}}_1 = -\frac{Gm_1m_2(\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$m_2\ddot{\vec{r}}_2 = -\frac{Gm_1m_2(\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^3} \tag{10.3.7}$$

which together represent the familiar Newton's law of universal gravitation.

Consider the simple pendulum shown in figure (10.3). Begin by using the Cartesian coordinates of the bob, in terms of which

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \tag{10.3.8}$$
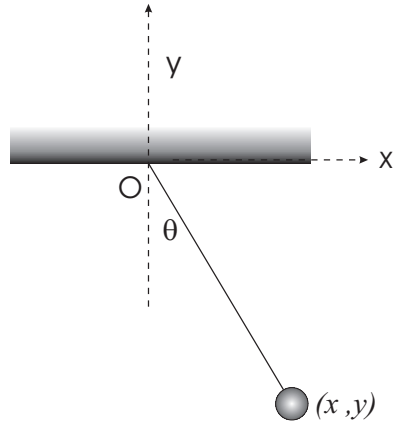
Figure 10.3: The simple pendulum

and $U(y) = mgy$. There is one holonomous constraint on the system, which is that the length of the cord is held fixed during the motion

$$g(x, y) = \sqrt{x^2 + y^2} - l. \tag{10.3.9}$$

So we have a choice. We could incorporate the constraint in an action,

$$S_\lambda = \int_1^2 dt \left[\frac{1}{2} m(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(t)(\sqrt{x^2 + y^2} - l) \right], \tag{10.3.10}$$

or we could choose to solve the constraints by reparameterizing the problem via the invertible transformation

$$x = l \sin \theta, \quad y = l \cos \theta \tag{10.3.11}$$

in which case we calculate

$$T = \frac{1}{2} m l^2 \dot{\theta}^2 \tag{10.3.12}$$

and therefore we could write a action functional of just one dependent function

$$S = \int_1^2 dt \left[\frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos \theta \right] \tag{10.3.13}$$

The equations of motion for this system are

$$- mgl \sin \theta - ml^2 \ddot{\theta} = 0 \quad \Rightarrow \ddot{\theta} = -\frac{g}{l} \sin \theta \tag{10.3.14}$$
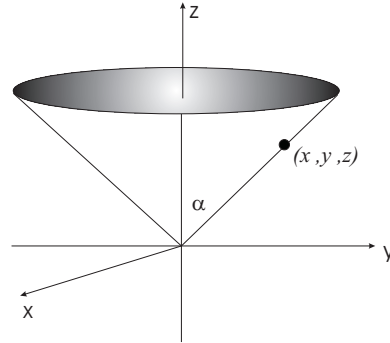
Figure 10.4: Particle on a cone

(one equation) whereas the equations of motion in the constrained system are

$$m\ddot{x} - \frac{\lambda x}{\sqrt{x^2 + y^2}} = 0$$

$$m\ddot{y} + mg - \frac{\lambda y}{\sqrt{x^2 + y^2}} = 0$$

$$\sqrt{x^2 + y^2} = l \tag{10.3.15}$$

(three equations for three unknown functions).

For another example, consider a particle moving on a cone in a constant gravitational field. Suppose that the angle of the cone is α as shown in figure (10.4) Again, we have a choice: we could use the Cartesian coordinates, in terms of which

$$S_\lambda = \int_1^2 dt \left[\frac{1}{2} m \sum_i (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz + \lambda(t)(z - \sqrt{x^2 + y^2}\cot\alpha) \right] \tag{10.3.16}$$

where we have used the fact that $U = mgz$ and, for the constraint, the equation of the cone

$$z = \sqrt{x^2 + y^2 + z^2}\cos\alpha \Rightarrow z = \sqrt{x^2 + y^2}\cot\alpha \tag{10.3.17}$$

This gives rise to the following four equations

$$m\ddot{x} = -\frac{\lambda x \cot\alpha}{\sqrt{x^2 + y^2}}$$

$$m\ddot{y} = -\frac{\lambda y \cot\alpha}{\sqrt{x^2 + y^2}}$$

$$m\ddot{z} = -mg + \lambda$$
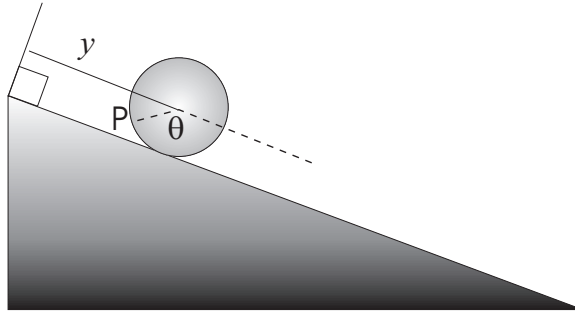
Figure 10.5: A disk rolling down an inclined plane without slipping

$$z = \sqrt{x^2 + y^2} \cot \alpha. \tag{10.3.18}$$

(four equations for four unknown functions). Alternatively we could solve the constraint by reparameterizing as follows:

$$x = \rho \cos \varphi$$

$$y = \rho \sin \varphi$$

$$z = \rho \cot \alpha \tag{10.3.19}$$

in which case we end up with the action

$$S = \int_1^2 dt \left[\frac{1}{2} m(\dot{\rho}^2 \csc^2 \alpha + \rho^2 \dot{\varphi}^2) - mg\rho \cot \alpha \right] \tag{10.3.20}$$

from which we get two Euler equations

$$\ddot{\rho} - \rho \sin^2 \alpha \dot{\varphi}^2 = -g \sin \alpha \cos \alpha$$

$$\frac{d}{dt}(m\rho^2 \dot{\varphi}) = 0 \tag{10.3.21}$$

which is a much simpler problem in just two unknown functions. The last equation will be recognized as conserving angular momentum.

Consider a disk rolling down a plane without slipping, as shown in figure (10.5) Let's use the coordinates (y, θ) as shown in the figure. We choose y to be the distance of the center of the disk, measured along a line parallel to the plane of the incline, from the left edge and θ is the angle made by the radius to a fixed point P on the disk with this line.

The kinetic energy of the disk is the kinetic energy of its center of mass together with its rotational kinetic energy

$$T = \frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 \tag{10.3.22}$$

and its potential energy is just gravitational, $U = mgh = mg(L - y)\sin\alpha$, where L is the length of the incline and α is its angle.

However, this is not enough because we must also incorporate the constraint that the disk does not slip. This means that $dy = Rd\theta$ where R is the radius of the disk. We could include a constraint,

$$S_\lambda = \int_1^2 dt \left[\frac{1}{2}M\dot{y}^2 + \frac{1}{2}I\dot{\theta}^2 - mg(L-y)\sin\alpha + \lambda(y - y_0 - R\theta) \right] \tag{10.3.23}$$

and solve the three equations,

$$M\ddot{y} = mg\sin\alpha + \lambda$$

$$I\ddot{\theta} = -R\lambda$$

$$y = y_0 + R\theta, \tag{10.3.24}$$

for the three unknown functions, $y(t)$, $\theta(t)$ and $\lambda(t)$. Alternatively, we could incorporate the constraint directly:

$$T = \frac{1}{2}(MR^2 + I)\dot{\theta}^2 = \frac{1}{2}I'\dot{\theta}^2$$

$$U = mg(L - y_0 - R\theta) \tag{10.3.25}$$

so that

$$S = \int_1^2 dt \left[\frac{1}{2}I'\dot{\theta}^2 - mg(L - y_0 - R\theta)\sin\alpha \right] \tag{10.3.26}$$

giving the single Euler equation

$$I'\ddot{\theta} = mgR\sin\alpha, \tag{10.3.27}$$

for $\theta(t)$.[1]

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[1]Suggestion: It's a good exercise to go through mechanics problems from elementary textbooks, say on Calculus based first year Physics, and set up Lagrangians describing the systems with careful attention to the constraints. Verify, in each case, that the equations of motion obtained from the traditional approach ($m\vec{a} = \vec{F}$, $I\vec{\alpha} = \vec{\tau}$) are equivalent to the Euler equations extremizing the Lagrangians.

10.4   Symmetries and Noether's theorems

E. Noether's two theorems, the first relating the conservation laws to the existence of global symmetries (finite, continuous group symmetries of the action) and the second pointing out the existence of identities that result from local symmetries (infinite, continuous group symmetries of the action), profoundly influenced the course of physics in the 20th century, leading eventually to the discovery of gauge theories in particle physics. In this section we will prove both Noether's first and most well-known theorem in general, as well as her second theorem but only for a special class of local transformations. Both proofs will be given in the context of particle mechanics, but their extension to "field" theories is straightforward.

Consider a general action,

$$S = \int_1^2 dt \; \mathcal{L}(q_i, \dot{q}_i; t), \tag{10.4.1}$$

where $\mathcal{L}(q, \dot{q}; t)$ includes all constraints (solved via generalized coordinates or not) and consider a transformation of time, say $t \to t' = t + \delta t = t + \epsilon(t)$ and the coordinates, $q(t) \to q'(t') = q(t) + \delta q(t)$. The action is invariant under this transformation if $\delta S = 0$. So far we have only been concerned with variations in which $\delta t = 0$. When $\delta t \neq 0$, the total change in the coordinates is the sum of two parts, the first arising due to a change in the argument, t, of the coordinates and the second due to a change in the *functional form* of $q(t)$, which is what we used up to now and which, to avoid confusion with the total change in $q(t)$, we will henceforth represent by $\delta_0 q(t)$:

$$\delta_0 q(t) = q'(t) - q(t) = \alpha \eta(t) \tag{10.4.2}$$

where $\eta(t)$ are arbitrary functions of t, earlier taken to vanish at the boundaries. Note that a variation of the functional form of $q_i(t)$ and differentiation with respect to time are independent operations, which can be performed in any order, *i.e.,*

$$[\delta_0, \frac{d}{dt}] = 0. \tag{10.4.3}$$

Thus, for $\delta t \neq 0$,

$$q(t) \to q'(t') = q(t') + \delta_0 q(t') = q(t + \epsilon) + \delta_0 q(t) \tag{10.4.4}$$

where, in the second term on the right, we have ignored a second order correction, given that the transformations are infinitesimal. For each coordinate in $\mathcal{L}$, we can write

$$q'(t') = q(t) + \epsilon(t)\dot{q}(t) + \delta_0 q(t), \tag{10.4.5}$$

which gives

$$\delta q(t) = \epsilon(t)\dot{q}(t) + \delta_0 q(t) \tag{10.4.6}$$

where $\delta q(t)$ is the total variation of $q(t)$.[2] In the same way, noting that

$$\dot{q}'(t') = \frac{d}{dt'}q'(t') = \frac{dt}{dt'}\frac{d}{dt}[q(t) + \epsilon(t)\dot{q}(t) + \delta_0 q(t)]$$

$$= (1 - \dot{\epsilon}(t))[\dot{q}(t) + \epsilon(t)\ddot{q}(t) + \dot{\epsilon}(t)\dot{q}(t) + \delta_0\dot{q}(t)] \tag{10.4.7}$$

we find, to first order in the variations,

$$\dot{q}'(t') = \dot{q}(t) + \epsilon(t)\ddot{q}(t) + \delta_0\dot{q}(t), \tag{10.4.8}$$

or

$$\delta\dot{q}(t) = \epsilon(t)\ddot{q}(t) + \delta_0\dot{q}(t). \tag{10.4.9}$$

The change induced in S by the variations just described is

$$\delta S = \int_1^2 [\delta(dt)\mathcal{L}(q_i, \dot{q}_i; t) + dt\delta\mathcal{L}(q_i, \dot{q}_i; t)] \tag{10.4.10}$$

Now

$$t \to t' = t + \epsilon(t) \Rightarrow dt \to dt' = [1 + \dot{\epsilon}(t)]dt \tag{10.4.11}$$

or

$$\delta(dt) = \dot{\epsilon}(t)dt \tag{10.4.12}$$

and therefore

$$\delta S = \int_1^2 dt \left[\dot{\epsilon}(t)\mathcal{L} + \epsilon(t)\frac{\partial\mathcal{L}}{\partial t} + \sum_i \frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \sum_i \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\delta\dot{q}_i \right]. \tag{10.4.13}$$

Substituting the total variations of $q(t)$ and $\dot{q}(t)$, given in (10.4.6) and (10.4.9), we find

$$\delta S = \int_1^2 dt \left[\dot{\epsilon}\mathcal{L} + \epsilon\frac{\partial\mathcal{L}}{\partial t} + \sum_i \frac{\partial\mathcal{L}}{\partial q_i}(\epsilon\dot{q}_i + \delta_0 q_i) + \sum_i \frac{\partial\mathcal{L}}{\partial \dot{q}_i}(\epsilon\ddot{q}_i + \delta_0\dot{q}_i) \right] \tag{10.4.14}$$

which expression, collecting terms proportional to ϵ, simplifies to

$$\delta S = \int_1^2 dt \left[\frac{d}{dt}(\epsilon\mathcal{L}) + \sum_i \frac{\partial\mathcal{L}}{\partial q_i}\delta_0 q_i + \sum_i \frac{\partial\mathcal{L}}{\partial \dot{q}_i}\delta_0\dot{q}_i \right]. \tag{10.4.15}$$

---

[2]**Problem**: Show that the total variation of $q_i(t)$ does not commute with the differentiation with respect to time,

$$\left[\delta, \frac{d}{dt}\right]q_i(t) \neq 0.$$

The last term in the integrand of (10.4.15) can be re-expressed using,

$$\sum_i \left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta_0 q_i \right) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right)\delta_0 q_i \right] \tag{10.4.16}$$

and we find

$$\delta S = \int_1^2 dt \left[\sum_i \mathcal{E}_i \delta_0 q_i + \frac{d}{dt}\left(\epsilon \mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta_0 q_i \right) \right] \tag{10.4.17}$$

where $\mathcal{E}_i$ is the Euler derivative of $\mathcal{L}$. If the variations are required to vanish on the boundaries, then the surface term vanishes and the action is invariant ($\delta S = 0$) provided that

$$\sum_i \int_1^2 dt \mathcal{E}_i \delta_0 q_i = 0 \tag{10.4.18}$$

This will be recognized as precisely (9.2.16). The action is also invariant under the transformations that do not vanish on the boundaries, provided that each term vanishes, *i.e.*, both $\mathcal{E}_i = 0$ (Euler's equations) *and*

$$\frac{d}{dt}\left(\epsilon \mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta_0 q_i \right) = 0 \tag{10.4.19}$$

hold true simultaneously. This is saying that the quantity in brackets is a constant of the motion, so it is a conservation law. Let us first re-express the conserved quatity above in terms of the total variations, using (10.4.6), as

$$\left(\mathcal{L} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i \right)\epsilon + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i = \text{const.} \tag{10.4.20}$$

and suppose that the variations result from some global transformations which form a finite, continuous group and depend on r constant parameters, $\delta\omega_a$, $a \in \{1, 2, ..., r\}$, according to

$$\delta t = \epsilon(t) = \mathcal{G}_a(t)\delta\omega_a \Rightarrow \mathcal{G}_a = \frac{\delta\epsilon(t)}{\delta\omega_a}$$

$$\delta q_i(t) = \mathcal{G}_{ia}(q,t)\delta\omega_a, \Rightarrow \mathcal{G}_{ia} = \frac{\delta q_i}{\delta\omega_a}. \tag{10.4.21}$$

The $\mathcal{G}$'s are called the generators of the transformation. Inserting these into (10.4.21), we find that the r quantities

$$j_a = \left(\mathcal{L} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i \right)\mathcal{G}_a + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\mathcal{G}_{ia} \tag{10.4.22}$$

are conserved on the physical trajectories, *i.e.,* so long as Euler's equations are obeyed. This is Noether's first and most famous theorem. Below are some simple but very special examples of how it works in the context of particle mechanics.

- Time translations: Take a translation of time by a constant, $t \to t' = t + \delta\omega$ and $q_i(t) \to q_i'(t') = q_i(t)$ (trajectories do not depend on the choice of the time origin) so $\delta q_i = 0$. This is a one parameter group of transformations in which $\mathcal{G} = 1$, $\mathcal{G}_i = 0$. If S is invariant under time translations, $\delta S = 0$ and

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \tag{10.4.23}$$

  is a constant of the motion. $\mathcal{H}$ is called the "Hamiltonian" of the system and it can, under certain conditions, be interpreted as the energy of the system. Energy conservation is thus a consequence of time translation invariance.

- Spatial translations: Constant translations of r coordinates, $q_i \to q_i + a_i$ holding time fixed, $t \to t'(t) = t$ form an r dimensional Abelian group with $\delta\omega_i = a_i = \delta q_i$ and $\epsilon(t) = 0$, or $\mathcal{G}_i = 0$ and $\mathcal{G}_{ij} = \delta_{ij}$. If the action is invariant under spatial translations of some or all of the N coordinates then for those coordinates,

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{10.4.24}$$

  is conserved. p_i is called the **momentum** of the system "conjugate" to q_i.[3] Thus momentum conservation in any direction is a consequence of space translation invariance of the action. A coordinate that does not appear in the Lagrangian *i.e.,* upon which the Lagrangian does not explicitly depend is called **cyclic** or **ignorable**. Naturally, the momentum conjugate to a cyclic coordinate is conserved.

- Spatial Rotations: Finally, let the coordinates q_i represent the usual Cartesian coordinates, and use x_i instead to denote them. General rotations of the coordinate

---

[3]In the simple example in which

$$\mathcal{L} = \sum_i \frac{1}{2} m \dot{x}_i^2 - U(x_i)$$

we find that

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m \dot{x}_i$$

which is precisely the momentum of the system. We generalize this to any system by simply *declaring* $\partial \mathcal{L} / \partial \dot{q}_i$ to be the momentum of the system conjugate to the (possibly generalized) coordinate q_i. The Hamiltonian is generally expressed in terms of the generalized momentum as

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}.$$

system may be written as products of rotations about the individual axes, of which there are three. If we consider an infinitesimal rotation, for example, about the $z-$axis we have

$$\mathbb{R}_{ij}^z = \delta_{ij} + \delta\theta^3 \mathbb{U}_{ij}^3 \tag{10.4.25}$$

where

$$\mathbb{U}_{ij}^3 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\epsilon_3]_{ij} \tag{10.4.26}$$

is the "generator" of rotations about the $z-$axis. Here $[\epsilon_k]_{ij} = \epsilon_{kij}$ is, of course, the Levi-Civita tensor, and, for example, $[\epsilon_3]_{ij}$ is to be thought of as a matrix $\hat{\epsilon}_3$, whose components are given by the Levi Civita symbol ϵ_{3ij}. For an infinitesimal rotation about the $x-$ axis we would have

$$\mathbb{R}_{ij}^x = \delta_{ij} + \delta\theta^1 \mathbb{U}_{ij}^1 \tag{10.4.27}$$

where

$$\mathbb{U}_{ij}^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = [\epsilon_1]_{ij} \tag{10.4.28}$$

and, finally about the y axis,

$$\mathbb{R}_{ij}^y = \delta_{ij} + \delta\theta^2 \mathbb{U}_{ij}^2 \tag{10.4.29}$$

where

$$\mathbb{U}_{ij}^2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = [\epsilon_2]_{ij} \tag{10.4.30}$$

An arbitrary infinitesimal rotation of the coordinates x^i of our particle would therefore take the form (sum over repeated indices)

$$\delta x_i = \delta\theta^k [\epsilon_k]_{ij} x_j \tag{10.4.31}$$

and the time parameter remains unchanged, $t' = t$ or $\delta t = 0$. The parameters of the transformation are the angles θ^k and

$$\mathcal{G}_{ik} = \frac{\delta x_i}{\delta\theta^k} = \epsilon_{kij} x_j \tag{10.4.32}$$

It follows from Noether's theorem that, if the action for the particle is invariant under spatial rotations, the quantity

$$L_k = -\epsilon_{kij} p_i x_j = (\vec{r} \times \vec{p})_k \tag{10.4.33}$$

(which will be recognized as the angular momentum of the particle) is conserved. Conservation of angular momentum is a consequence of invariance of the action under spatial rotations.

Since the currents derived from (10.4.22) are conserved only on physical trajectories, they are sometimes said to be **weakly** conserved and Noether's first theorem is said to lead to **weak** conservation laws.

Noether's second theorem deals with actions that are invariant under transformations belonging to an infinite, continuous group, *i.e.,* transformations that are not just global but **local**, in which the variations ϵ and δq_i depend of a finite number, r, of *functions* and their derivatives. It states that in general, and not simply on (physical) trajectories for which Euler's equations hold, the local symmetries imply a set of r identities between the Euler derivatives of the Lagrangian.

According to (10.4.17), the action is invariant under the transformations if

$$\int_1^2 dt \left[\sum_i \mathcal{E}_i \delta_0 q_i + \frac{d}{dt} \left(\epsilon \mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \delta_0 q_i \right) \right] = 0, \tag{10.4.34}$$

where the $\mathcal{E}_i$ are the Euler derivatives of $\mathcal{L}$. For simplicity we will restrict ourselves here to a special set of transformations, in which the variations depend on r functions $\delta\omega^a(t)$ and their first derivatives only,

$$\begin{aligned} \epsilon(t) &= \mathcal{G}_a \delta\omega^a(t) \\ \delta q_i(t) &= \mathcal{G}_{ia} \delta\omega^a(t) + \mathcal{T}_{ia} \delta\dot{\omega}^a(t). \end{aligned} \tag{10.4.35}$$

Because the $\delta\omega^a$ are arbitrary functions of t, we can choose them so that they and their derivatives vanish at the boundaries. In that case, the surface terms (total derivatives) automatically vanish and invariance implies that

$$\sum_i \int_1^2 dt\, \mathcal{E}_i(\delta q_i - \epsilon \dot{q}_i) = \sum_i \int_1^2 dt\, \mathcal{E}_i \left[(\mathcal{G}_{ia} - \dot{q}_i \mathcal{G}_a)\delta\omega^a + \mathcal{T}_{ia}\delta\dot{\omega}^a \right] = 0. \tag{10.4.36}$$

Integrating the last term by parts, we find

$$\sum_i \int_1^2 dt\, \left[\mathcal{E}_i(\mathcal{G}_{ia} - \dot{q}_i \mathcal{G}_a) - \frac{d}{dt}(\mathcal{E}_i \mathcal{T}_{ia}) \right] \delta\omega^a, \tag{10.4.37}$$

which, for arbitrary $\delta\omega^a$, is only possible if

$$\sum_i \left[\mathcal{E}_i(\mathcal{G}_{ia} - \dot{q}_i \mathcal{G}_a) - \frac{d}{dt}(\mathcal{E}_i \mathcal{T}_{ia}) \right] = 0 \tag{10.4.38}$$

These r identities of Noether's second theorem are trivially satisfied on physical trajectories ($\mathcal{E}_i = 0$) but they hold irrespective of the Euler equations (they are **strong**) and, if they are non-trivial, show that the Euler equations are not all independent. Although we have carried out the calculation here for the special transformations in (10.4.35), the theorem itself is quite general and the same procedure may be employed when higher derivatives are present. Naturally, more complicated local symmetries lead to more complicated identities, but it all cases the existence of r identities implies that only $(n - r)$ of the equations of motion are independent. Thus, like external constraints, local symmetries of the action reduce the number of physical degrees of freedom in the system. As we saw earlier, for example, the local reparameterization invariance of the path length functional in (9.6.5) implies that there are only two instead of three independent equations of motion.

Chapter 11

The Hamiltonian

The Lagrangian formulation adds considerably to our understanding of mechanics, both by contributing to its philosophical underpinnings and by introducing a substantial amount of new mathematical structure, which, as we have seen, can be exploited to gain a more profound understanding of the general properties of a physical system. The Hamiltonian formulation of mechanics is an alternative to the Lagrangian formulation. While not adding in any way to the physics, it provides new mathematical structure and more powerful ways of working with the physical principles that have already been established.

11.1   Legendre Transformations

From Noether's first theorem, an action that is invariant under time translations describes a system that conserves the Hamiltonian

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L}, \quad \frac{d\mathcal{H}}{dt} = 0. \tag{11.1.1}$$

But, because

$$\begin{aligned}
\frac{d\mathcal{L}}{dt} &= \frac{\partial \mathcal{L}}{\partial q_i} \dot{q}_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \\
&= \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i \right) + \frac{\partial \mathcal{L}}{\partial t}
\end{aligned} \tag{11.1.2}$$

(where we have used Euler's equation), (11.1.1) implies that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \dot{q}_i - \mathcal{L} \right) = -\frac{\partial \mathcal{L}}{\partial t} = 0. \tag{11.1.3}$$

and it follows also that $\mathcal{L}$ can have no explicit dependence on time. For example, take a single particle of mass m moving in a potential $U(\vec{r})$ and let it be described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{x}_i^2 - U(x_i) \tag{11.1.4}$$

where a summation over repeated indices is understood. We find

$$\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = m\dot{x}_i \tag{11.1.5}$$

which is just the ith component of the momentum of the particle! The Hamiltonian is

$$\mathcal{H} = m\dot{x}_i^2 - \frac{1}{2}m\dot{x}_i^2 + U(x_i) = \frac{1}{2}m\dot{x}_i^2 + U(x_i) \tag{11.1.6}$$

which will be recognized as the total energy of the particle. Because $\mathcal{L}$ does not explicitly depend on time, the total energy of such a system is conserved.

Now we had defined the "generalized momentum" conjugate to the generalized coordinate q_i by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{11.1.7}$$

If this system of equations is invertible then we may solve it to get the velocities in terms of the momenta and coordinates, $\dot{q}_i = \dot{q}_i(q, p, t)$. Once this is done, the Hamiltonian can be written in the form

$$\mathcal{H} = p_i\dot{q}_i(q, p, t) - \mathcal{L}. \tag{11.1.8}$$

and it is easy to show that $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$ is independent of the velocities. The right hand side of (11.1.8) is a Legendre transformation of $\mathcal{L}$, taking us from a function of the coordinates and velocities, $\mathcal{L}(q_i, \dot{q}_i, t)$, to a function of the coordinates and momenta, $\mathcal{H}(q_i, p_i, t)$.

Legendre transformations are performed often in physics. For example in Thermodynamics, going from the internal energy to the free energy of a system involves a Legendre transformation. The first law of Thermodynamics,

$$dU = TdS - pdV,$$

shows that U is naturally viewed as a function of (S, V) and then $T = (\partial U/\partial S)_V$. By defining the (Helmholz) free energy as $F = U - TS$, we make a Legendre transformation taking $U(S, V) \to F(T, V)$, as one can check from

$$dF = dU - TdS - SdT = -pdV - SdT.$$

This enables us to deal more effectively with situations in which the temperature and volume are easy to control or measure. For example, while the work done by a thermodynamic system during an isentropic process is equal to the decrease in its internal energy, the work done during an isothermal process is equal to the decrease in its *free energy*. Another useful quantity with dimensions of energy is the enthalpy, H, which involves a Legendre transformation from $U(S, V) \to H(S, p)$. Again, $p = -\partial U / \partial V$ implies that the desired Legendre transformation must be $H = U + pV$,

$$dH = dU + pdV + V\,dp = TdS + V\,dp$$

In the same spirit, a double Legendre transformation would take us to the Thermodynamic Potential or Gibbs free energy, $G(T, p) = U - TS + pV$.

$$dG = -SdT + V\,dp$$

The Gibbs free energy stays constant during any process that is both isobaric and isothermal, such as a phase transition.

11.2   The Canonical equations of motion

The Legendre transformation that takes us from the Lagrangian description to the Hamiltonian replaces $\dot{q}_i$ with p_i so that $\mathcal{H}$ can be thought of as a function only of the coordinates and momenta, not the velocities. The coordinates and momenta must be treated independently and, assuming there are n independent coordinates, they define a space that is $2n$ dimensional. This space of coordinates and momenta is called **phase space**.

   The first step in performing the transformation involves using the definition of p_i in (11.1.7) to obtain the velocities, $\dot{q}_i$, in terms of the coordinates and momenta, *i.e.,* we solve (11.1.7) for $\dot{q}_i(q_i, p_i, t)$. In the next step we must replace the replace $\dot{q}_i$ everywhere on the right hand side of (11.1.8) to recover $\mathcal{H}(q_i, p_i, t)$. By the inverse function theorem, however, the first step is only possible if

$$\det|W_{ij}| = \det\left|\frac{\partial p_i}{\partial \dot{q}_j}\right| = \det\left|\frac{\partial^2 \mathcal{L}}{\partial \dot{q}_i \partial \dot{q}_j}\right| \neq 0. \tag{11.2.1}$$

The matrix W_{ij} is called the **Hessian**. When the determinant of the Hessian is non-vanishing the Lagrangian system is called **regular**, otherwise it is **singular**. When the Lagrangian system is singular a Hamiltonian may be defined only on a subset of the original phase space. This Hamiltonian may subsequently be extended over the entire phase space by explicitly introducing constraints in much the same way as we did earlier by means of Lagrange multipliers. The procedure was originally outlined by Dirac and later elaborted upon by Bergmann. We will outline the Dirac-Bergmann algorithm at the

end of this chapter but, for the most part, we consider regular Lagrangian systems in these notes.

From the defining equations and the definition of the generalized momenta follow the equations of motion in terms of the Hamiltonian. Notice that from the definition of $\mathcal{H}$ in (11.1.8),

$$
\begin{aligned}
d\mathcal{H} &= \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial t} dt - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \\
&= \dot{q}_i dp_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial t} dt,
\end{aligned}
\tag{11.2.2}
$$

where we used the definition of the generalized momenta. This shows that $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$ and therefore

$$
d\mathcal{H} = \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial t} dt.
\tag{11.2.3}
$$

Comparing the two expressions we see that

$$
\begin{aligned}
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
\frac{\partial \mathcal{L}}{\partial q_i} &= -\frac{\partial H}{\partial q_i} \\
\frac{\partial \mathcal{L}}{\partial t} &= -\frac{\partial H}{\partial t}
\end{aligned}
\tag{11.2.4}
$$

But, by Euler's equations,

$$
\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) = \dot{p}_i
\tag{11.2.5}
$$

so we find that the set of equations in (11.2.4) can also be written as

$$
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial \mathcal{L}}{\partial t} = -\frac{\partial H}{\partial t}.
\tag{11.2.6}
$$

These are known as the **Hamilton equations** (of motion). Using them we find that

$$
\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{H}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{H}}{\partial p_i}\dot{p}_i = \frac{\partial \mathcal{H}}{\partial t}
\tag{11.2.7}
$$

which says that any time dependence of the Hamiltonian must be *explicit.*

The Hamilton equations are also called the **canonical equations of motion** and the description of the motion by them is called Hamiltonian dynamics. If there are n generalized coordinates, there must be $2n$ pairs (q_i, p_i) and $2n$ canonical equations. The canonical equations are coupled, first order differential equations in time whose integration

involves $2n$ constants or initial values. This is the same number of constants that are required for Lagrangian dynamics because, although there are just n Euler equations they are second order in time. The space formed by all the generalized coordinates is called the **configuration space** of the system. The space formed by all the generalized coordinates and momenta is called the **phase space** or **momentum state space** of the system. The space formed by all the generalized coordinates and their corresponding velocities is called the **velocity state space** of the system.

11.3   Poisson Brackets

Observables in the Hamiltonian description will consist of functions of the coordinates, momenta and time, $A = A(q_i, p_i, t)$. Given two observables A and B we define the Poisson Brackets of A and B as

$$\{A, B\}_{\text{PB}} \quad \overset{\text{def}}{=} \quad \sum_k \left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right) \tag{11.3.1}$$

The Poisson brackets have several features of considerable interest. In particular, if we take $A = q_i$ and $B = p_j$, we find that

$$\begin{aligned}
\{q_i, p_j\}_{\text{PB}} &= \sum_l \delta_{il} \delta_{lj} = \delta_{ij} \\
\{q_i, q_j\}_{\text{PB}} &= 0 = \{p_i, p_j\}_{\text{PB}}
\end{aligned} \tag{11.3.2}$$

These are called the "fundamental" Poisson brackets. Again, if we consider the time rate of change of any observable A,

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i} \dot{q}_i + \frac{\partial A}{\partial p_i} \dot{p}_i + \frac{\partial A}{\partial t} \tag{11.3.3}$$

and insert the canonical equations of motion, we find that

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial \mathcal{H}}{\partial q_i} + \frac{\partial A}{\partial t} \tag{11.3.4}$$

or

$$\frac{dA}{dt} = \{A, \mathcal{H}\}_{\text{PB}} + \frac{\partial A}{\partial t}. \tag{11.3.5}$$

An observable that satisfies $dA/dt = 0$ is called a **constant of the motion**. We see that for an observable not explicitly dependent on time to be a constant of the motion it must have vanishing Poisson brackets with $\mathcal{H}$. On the other hand, if A is a constant of the motion but does explicitly depend on time then $\partial A/\partial t = -\{A, \mathcal{H}\}_{\text{PB}}$.

We might take A to be just q_i or p_i, to find an alternative description of the canonical equations of motion

$$\dot{q}_i = \{q_i, \mathcal{H}\}_{\text{PB}}, \quad \dot{p}_i = \{p_i, \mathcal{H}\}_{\text{PB}}, \tag{11.3.6}$$

in terms of Poisson brackets. Although the Poisson Bracket appears to be merely a convenient notation, one should really think of it as a binary operation between observables which turns out to be of central importance in analytical mechanics. This will become clearer as we go along.

From now on, we will drop the subscript "PB" on the curly brackets unless they are required for clarity. Let's list some of the properties of the Poisson Bracket. Each of these can be proved by starting from definition (11.3.1).

1. It is **antisymmetric**,

$$\{A, B\} = -\{B, A\} \tag{11.3.7}$$

2. It is **linear**: if a and b are constants then

$$\{aA + bB, C\} = a\{A, C\} + b\{B, C\} \tag{11.3.8}$$

3. For any constant, c, and any observable A, $\{c, A\} = 0$.

4. It satisfies the **Jacobi Identity**,

$$\{A, \{B, C\}\} + \{C\{A, B\}\} + \{B\{C, A\}\} = 0 \tag{11.3.9}$$

   (cyclic permtations of the observables).

5. It obeys the following **product rule**,

$$\{AB, C\} = \{A, C\}B + A\{B, C\} \tag{11.3.10}$$

If only the first four conditions are obeyed by some binary operation between observables then it is called a "generalized Poisson Bracket".

Two facts concerning observables are worthy of note. The first has to do with the construction of new invariants of the motion from old ones and the second has to do with the approximate solution of the evolution equations for any function on phase space.

- Knowing any two constants of the motion, it is possible in principle to construct new constants of the motion as follows: take $C = \mathcal{H}$ in the Jacobi identity,

$$\{\{A, B\}, \mathcal{H}\} + \{\{\mathcal{H}, A\}, B\} + \{\{B, \mathcal{H}\}, A\} = 0$$
$$\Rightarrow \quad \{\{A, B\}, \mathcal{H}\} + \{\frac{\partial A}{\partial t}, B\} - \{\frac{\partial B}{\partial t}, A\} = 0$$

$$\Rightarrow \quad \{\{A, B\}, \mathcal{H}\} + \{\frac{\partial A}{\partial t}, B\} + \{A, \frac{\partial B}{\partial t}\} = 0$$

$$\Rightarrow \quad \{\{A, B\}, \mathcal{H}\} + \frac{\partial}{\partial t}\{A, B\} = 0 \qquad\qquad (11.3.11)$$

showing that $\{A, B\}$ is also a constant of the motion.[1] In practice this method is not very useful because the Poisson brackets of A and B often give trivial functions or even constants.

- If an A does not depend explicitly on time and is well behaved, one can use the Poisson brackets to find a formal solution to its evolution equation

$$\frac{dA}{dt} = \{A, \mathcal{H}\}. \qquad\qquad (11.3.12)$$

Expand A as a Taylor series

$$A(t) = \sum_{n=0}^{\infty} \frac{A^{(n)}(t_0)}{n!}(t - t_0)^n \qquad\qquad (11.3.13)$$

where $A^{(n)}$ is the nth time derivative of A, and notice that

$$A^{(1)}(t_0) = \{A, \mathcal{H}\}(t_0)$$

$$A^{(2)}(t_0) = \{A^{(1)}, \mathcal{H}\}(t_0) = \{\{A, \mathcal{H}\}, \mathcal{H}\}(t_0)$$

$$A^{(3)}(t_0) = \{A^{(1)}, \mathcal{H}\}(t_0) = \{\{\{A, \mathcal{H}\}, \mathcal{H}\}, \mathcal{H}\}(t_0)$$

$$\dots$$

$$A^{(n)}(t_0 = \{\dots\{\{A, \mathcal{H}\}, \mathcal{H}\}, \dots \ n \text{ times}\}(t_0) \qquad\qquad (11.3.14)$$

giving the following (formal) solution for the observable A

$$A(t) = \sum_{n=0}^{\infty} \frac{1}{n!}\{\dots\{\{A, \mathcal{H}\}, \mathcal{H}\}, \dots \ n \text{ times}\}(t_0)(t - t_0)^n. \qquad\qquad (11.3.15)$$

In all but the simplest cases, this method cannot be applied to obtain exact solutions. However, it is quite useful in perturbation theory.

---

[1] Problem: Prove that

$$\{\frac{\partial A}{\partial t}, B\} + \{A, \frac{\partial B}{\partial t}\} = \frac{\partial}{\partial t}\{A, B\}$$

11.4 Examples

Consider the one dimensional system whose Hamiltonian is

$$\mathcal{H} = \frac{p^2}{2} - \frac{1}{2q^2} \tag{11.4.1}$$

which you can think of as a particle of unit mass moving in the potential of a central, inverse *cube* force. The equations of motion for the particle are

$$\dot{q} = \{q, \mathcal{H}\} = p, \quad \dot{p} = \{p, \mathcal{H}\} = -\frac{1}{q^3} \tag{11.4.2}$$

which also give the (Euler) equation

$$\ddot{q} = -\frac{1}{q^3}. \tag{11.4.3}$$

The quantity $D = pq/2 - \mathcal{H}t$ is a constant of the motion [2]

$$\frac{dD}{dt} = \{D, \mathcal{H}\} + \frac{\partial D}{\partial t} = \frac{p}{2}\left\{q, \frac{p^2}{2}\right\} + \frac{q}{2}\left\{p, -\frac{1}{2q^2}\right\} - \mathcal{H} = 0, \tag{11.4.4}$$

whose origin lies in the invariance of the system under a scale transformation,

$$q \to Q = \lambda q, \quad t \to \tau = \lambda^2 t, \quad p \to P = \frac{p}{\lambda}. \tag{11.4.5}$$

Invariance can be verified either from the canonical equations of motion, which remain unchanged in form,

$$\begin{aligned}
\frac{dq}{dt} &= \lambda\frac{dQ}{d\tau} = p = \lambda P \Rightarrow \frac{dQ}{d\tau} = P \\
\frac{dp}{dt} &= \lambda^3\frac{dP}{d\tau} = -\frac{1}{q^3} = -\frac{\lambda^3}{Q^3} \Rightarrow \frac{dP}{d\tau} = -\frac{1}{Q^3}
\end{aligned} \tag{11.4.6}$$

or from the action for this system

$$S = \int_1^2 dt \mathcal{L}(q, \dot{q}, t) = \int_1^2 dt \left(\frac{\dot{q}^2}{2} + \frac{1}{2q^2}\right). \tag{11.4.7}$$

---

[2]Problem: Generalize this: show that

$$D = \frac{\vec{p} \cdot \vec{r}}{n} - \mathcal{H}t$$

is a constant of the motion for a Hamiltonian of the form $\mathcal{H} = |\vec{p}|^n - ar^{-n}$ where a is any constant.

where now only the transformations of q and t need be specified. Noether's first theorem then implies that

$$j = \left(\mathcal{L} - \frac{\partial\mathcal{L}}{\partial\dot{q}}\dot{q}\right)\frac{\delta t}{\delta\lambda} + \frac{\partial\mathcal{L}}{\partial\dot{q}}\frac{\delta q}{\delta\lambda} = p\frac{\delta q}{\delta\lambda} - \mathcal{H}\frac{\delta t}{\delta\lambda} \tag{11.4.8}$$

is conserved. Taking an infinitesimal scaling, $\lambda = 1 + \delta\lambda$, and evaluating j, we find

$$j = pq - 2\mathcal{H}t = 2D. \tag{11.4.9}$$

The simple harmonic oscillator is such a simple physical system that the iterative method that was described in the previous section can actually be used to obtain an exact solution. We have

$$\mathcal{L} = \frac{1}{2}(m\dot{x}^2 - kx^2)$$

$$p = \frac{\partial\mathcal{L}}{\partial\dot{x}} = m\dot{x} \Rightarrow \dot{x} = \frac{p}{m}$$

$$\mathcal{H} = p\dot{x} - \mathcal{L} = \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{11.4.10}$$

The equations of motion are

$$\dot{p} = \{p, \mathcal{H}\} = \frac{k}{2}\{p, x^2\} = -kx$$

$$\dot{x} = \{x, \mathcal{H}\} = \frac{1}{2m}\{x, p^2\} = \frac{p}{m} \tag{11.4.11}$$

To find $x = x(t)$, we need to compute

$$x^{(2)} = \{\dot{x}, \mathcal{H}\} = \{\{x, \mathcal{H}\}, \mathcal{H}\} = \frac{1}{m}\{p, \mathcal{H}\} = -\frac{k}{m}x$$

$$x^{(3)} = \{x^{(2)}, \mathcal{H}\} = -\frac{kp}{m^2}$$

$$x^{(4)} = \{x^{(3)}, \mathcal{H}\} = +\frac{k^2x}{m^2}$$

$$\dots \tag{11.4.12}$$

and so on. If we call set $t_0 = 0$ and let $x_0 = x(0)$ and $p_0 = p(0)$ then (11.3.15) will give the series solution

$$x(t) = x_0 + \frac{p_0}{m}t - \frac{kx_0}{m}t^2 - \frac{kp_0}{m^2}t^3 + \frac{k^2x_0}{m^2}t^4 + \dots$$
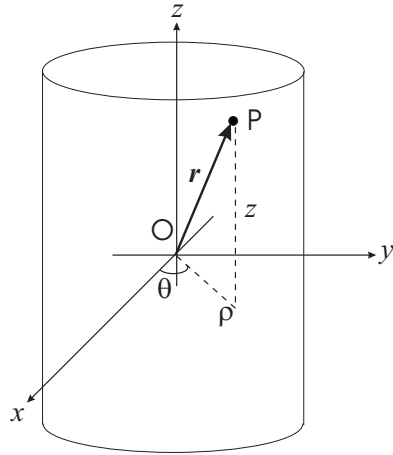
Figure 11.1: Particle on a cylinder acted upon by a central force.

$$
\begin{aligned}
&= x_0 \left[1 - \frac{k}{m}t^2 + \frac{k^2}{m^2}t^4 + \ldots \right] + \frac{p_0}{\sqrt{mk}} \left[\left(\frac{k}{m}\right)^{1/2} t - \left(\frac{k}{m}\right)^{3/2} t^3 + \ldots \right] \\
&= x_0 \cos \omega t + \frac{p_0}{\sqrt{mk}} \sin \omega t
\end{aligned}
\tag{11.4.13}
$$

where $\omega = \sqrt{k/m}$ is the angular frequency of the oscillator. We can put this in standard form by letting

$$
x_0 = \mathcal{A} \cos \phi, \quad p_0 = \sqrt{mk}\mathcal{A} \sin \phi
\tag{11.4.14}
$$

which gives

$$
x(t) = \mathcal{A} \cos(\omega t - \phi),
\tag{11.4.15}
$$

where

$$
\phi = \tan^{-1}\left(\frac{p_0}{x_0\sqrt{mk}}\right), \quad \mathcal{A} = \sqrt{x_0^2 + \frac{p_0^2}{mk}}
\tag{11.4.16}
$$

are respectively the initial phase and amplitude of the oscillations.

Next, consider a particle acted upon by an attractive central force, which proportional to its distance from the origin, but moving on a cylinder as shown in figure (11.1) It is not difficult to see that

$$
T = \frac{m}{2}(\rho^2\dot{\theta}^2 + \dot{z}^2)
$$

$$
U = \frac{1}{2}kr^2 = \frac{1}{2}k(\rho^2 + z^2)
$$

$$\mathcal{L} = \frac{m}{2}(\rho^2\dot{\theta}^2 + \dot{z}^2) - \frac{k}{2}(\rho^2 + z^2) \tag{11.4.17}$$

and there are no constraints, because they have been solved by using cylindrical coordinates and treating ρ as a constant. The Lagrangian equations of motion are

$$m\rho^2\ddot{\theta} = 0$$

$$m\ddot{z} = -kz \tag{11.4.18}$$

whose solutions are clearly

$$z = \mathcal{A}\cos(\omega t + \phi), \quad \theta = \theta_0 + \frac{Lt}{m\rho^2} \tag{11.4.19}$$

where $\mathcal{A}$ and ϕ are respectively the amplitude and initial phase of an oscillation along the z axis with angular frequency $\sqrt{k/m}$, and L is the angular momentum of the motion around the cylinder. To compute the Hamiltonian of this system, we determine the generalized momenta

$$p_z = \frac{\partial\mathcal{L}}{\partial\dot{z}} = m\dot{z}, \quad p_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m\rho^2\dot{\theta}, \tag{11.4.20}$$

invert them to get the velocities as functions of the coordinates and momenta

$$\dot{z} = \frac{p_z}{m}, \quad \dot{\theta} = \frac{p_\theta}{m\rho^2} \tag{11.4.21}$$

and then use (11.1.8) to get the Hamiltonian

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{p_\theta^2}{2m\rho^2} + \frac{k}{2}(\rho^2 + z^2) \tag{11.4.22}$$

(The term $k\rho^2/2$ is just a constant which may be discarded by resetting the standard fixed point which determines the zero of energy.) The Hamilton equations are

$$\dot{z} = \{z, \mathcal{H}\} = \frac{p_z}{m}, \quad \dot{p}_z = \{p_z, \mathcal{H}\} = -kz,$$

$$\dot{\theta} = \{\theta, \mathcal{H}\} = \frac{p_\theta}{m\rho^2}, \quad \dot{p}_\theta = 0 \tag{11.4.23}$$

(the last says that p_θ is the conserved angular momentum). From our previous example, we recognize the structure of the Poisson brackets for (z, p_z) and can write the solution for $z(t)$ directly. It is precisely what we had in (11.4.19). To find the solution for $\theta(t)$, we note that

$$\ddot{\theta} = \{\dot{\theta}, \mathcal{H}\} = \frac{1}{m\rho^2}\{p_\theta, \mathcal{H}\} = 0 \tag{11.4.24}$$

and it follows that *all* higher derivatives of $\theta(t)$ vanish. Therefore, calling $p_\theta = L$, the solution is

$$\theta(t) = \theta_0 + \frac{Lt}{m\rho^2} \tag{11.4.25}$$

as in (11.4.19).[3]

11.5 The Dirac-Bergmann Algorithm for Singular Systems

So far we have restricted our attention to regular Lagrangian systems. However, by Noether's theorem we know that this excludes systems with local symmetries and other constrained Lagrangian systems . Singular Lagrangian systems are in fact more common (and important) than one may think, although they make their most spectacular appearance in field theories as manifestations of local, internal symmetries. In this section we examine how one can set up the Hamiltonian for singular systems with a finite number of degrees of freedom (particle mechanics). The treatment can be extended to systems with an infinite number of degrees of freedom (field theories) and is due to Dirac and Bergman.

What follows is an algorithm, called the Dirac-Bergman algorithm, for constructing the Hamiltonian of constrained systems. The basic idea is to check, in a systematic fashion, whether or not the equations of motion are consistent. In order to arrive at the Hamiltonian we start with the definition of the momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{11.5.1}$$

and if the Hessian is invertible then these relations can be solved for the velocities to give $\dot{q}_i = \dot{q}_i(q, p)$. However, if the Hessian is not invertible then some of the velocities cannot be expressed in terms of the momenta and it will be possible to solve the equations only for a subset of velocities. Suppose that there are $(n-r)$ of these and that r velocities that cannot be obtained in terms of the momenta. We will use letters from the beginning of the roman alphabet to label the $n-r$ coordinates whose velocities are expressed in terms of momenta and letters from the beginning of the greek alphabet to label the r coordinates whose velocities cannot be so expressed. From (11.5.1), we will have $(n-r)$ relations of the form

$$\dot{q}_a = f_a(q_i, p_a, \dot{q}_\alpha) \tag{11.5.2}$$

corresponding to the $(n-r)$ velocities that have been determined in terms of momenta. Inserting these relations into (11.5.1) will give

$$p_i = g_i(q_i, f_a, \dot{q}_\alpha) \tag{11.5.3}$$

---

[3]<u>Problem</u>: Use (11.3.15) to recover the kinematical equations.

When $i = a$ the g_a are just the momenta p_a. The remaining p_α cannot depend on the $\dot{q}_\alpha$ because, if they did then we could solve for more of the velocities, replacing them with momenta. Thus we will have r relations of the form

$$p_\alpha = g_\alpha(q_i, p_a) \tag{11.5.4}$$

These relations will be called the **primary constraints**. One now defines the canonical Hamiltonian for singular systems by

$$\mathcal{H}_c = p_i \dot{q}_i - \mathcal{L} \tag{11.5.5}$$

where it is understood that whenever $i = a$, $\dot{q}_a$ is to be replaced by $f_a(q_i, p_a, t)$ and whenever $i = \alpha$, p_α is replaced by $g_\alpha(q_i, p_a)$. It is easy to see that $\mathcal{H}_c$ is independent of $\dot{q}_i$ for *all* i, simply because of the definition of the momenta in (11.5.1). Therefore,

$$\mathcal{H}_c = p_a \dot{q}_a + g_\alpha \dot{q}_\alpha - \mathcal{L} \tag{11.5.6}$$

and we have the following derivatives

$$\frac{\partial \mathcal{H}_c}{\partial q_i} = \frac{\partial g_\alpha}{\partial q_i} \dot{q}_\alpha - \frac{\partial \mathcal{L}}{\partial q_i}$$
$$\frac{\partial \mathcal{H}_c}{\partial p_a} = \dot{q}_a + \frac{\partial g_\alpha}{\partial p_a} \dot{q}_\alpha \tag{11.5.7}$$

These are $(2n - r)$ equations of motion in all.[4] Of course, this is a consequence of the r primary constraints in (11.5.4), which restrict the motion to a $(2n - r)$ dimensional subspace of the full phase space. For a solution of the Euler-Lagrange equations, $\dot{p}_i = \partial \mathcal{L} / \partial q_i$, *i.e.*,

$$\dot{p}_i = -\frac{\partial \mathcal{H}_c}{\partial q_i} + \frac{\partial g_\alpha}{\partial q_i} \dot{q}_\alpha$$
$$\dot{q}_a = +\frac{\partial \mathcal{H}_c}{\partial p_a} - \frac{\partial g_\alpha}{\partial p_a} \dot{q}_\alpha \tag{11.5.8}$$

Our goal is to extend the Hamiltonian to the entire phase space in such a way that the canonical equations of motion appear the same as they do for canonical systems. To do

---

[4]Because $\mathcal{H}_c$ cannot depend on the velocities and because the momenta p_α can be exchanged in favor of q_i and p_a, follow the same steps as before with $\mathcal{H}_c = \mathcal{H}_c(q_i, p_a,) \Rightarrow$

$$
\begin{aligned}
d\mathcal{H}_c &= \frac{\partial \mathcal{H}_c}{\partial q_i} dq_i + \frac{\partial \mathcal{H}_c}{\partial p_a} dp_a + \frac{\partial \mathcal{H}_c}{\partial t} dt \\
&= \left(\frac{\partial g_\alpha}{\partial p_a} \dot{q}_\alpha + \dot{q}_a \right) dp_a + \left(\frac{\partial g_\alpha}{\partial q_i} \dot{q}_\alpha - \frac{\partial \mathcal{L}}{\partial q_i} \right) dq_i - \frac{\partial \mathcal{L}}{\partial t}
\end{aligned}
$$

Comparing the two expressions gives precisely the $(2n - r)$ equations of motion listed.

so, we first consider $\mathcal{H}_c$ to be a restriction to the desired subspace of another Hamiltonian $\mathcal{H}$ that is defined on the *entire* phase space. Now $\mathcal{H}$, being defined on the entire phase space would be of the form $\mathcal{H} = \mathcal{H}(q_i, p_i, t)$ and it cannot be uniquely defined *i.e.*, many Hamiltonians on the entire phase space may have the same restriction, $\mathcal{H}_c$, on the subspace. How then can we construct $\mathcal{H}$?

An arbitrary function on the phase space is called **weakly vanishing** if it vanishes identically on a restriction, Γ_p, of the phase space. One denotes a weakly vanishing function, F, by saying that $F \approx 0$, *i.e.*,

$$F|_{\Gamma_p} = 0 \Leftrightarrow F \approx 0. \tag{11.5.9}$$

If both F and its gradient vanish on Γ_p, then F is said to be **strongly vanishing**, *i.e.*,

$$\left.\begin{array}{c} F|_{\Gamma_p} = 0 \\[2mm] \left.\frac{\partial F}{\partial q_i}\right|_{\Gamma_p} = 0 = \left.\frac{\partial F}{\partial p_i}\right|_{\Gamma_p} \end{array}\right\} \Leftrightarrow F \simeq 0 \tag{11.5.10}$$

where the symbol $\simeq$ has been used to denote "strongly" vanishing. Suppose $F(q_i, p_i)$ is a weakly vanishing function, then

$$
\begin{aligned}
\delta F|_{\Gamma_p} &= \left. \frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial p_i}\delta p_i \right|_{\Gamma_p} \\
&= \left. \frac{\partial F}{\partial q_i}\delta q_i + \frac{\partial F}{\partial p_a}\delta p_a + \frac{\partial F}{\partial p_\alpha}\delta p_\alpha \right|_{\Gamma_p} \\
&= \left(\frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial p_\alpha}\frac{\partial g_\alpha}{\partial q_i} \right)\delta q_i + \left(\frac{\partial F}{\partial p_a} + \frac{\partial F}{\partial p_\alpha}\frac{\partial g_\alpha}{\partial p_a} \right)\delta p_a \tag{11.5.11}
\end{aligned}
$$

must also be (weakly) vanishing, which means that

$$
\frac{\partial F}{\partial q_i} + \frac{\partial F}{\partial p_\alpha}\frac{\partial g_\alpha}{\partial q_i} \approx 0
$$
$$
\frac{\partial F}{\partial p_a} + \frac{\partial F}{\partial p_\alpha}\frac{\partial g_\alpha}{\partial p_a} \approx 0 \tag{11.5.12}
$$

For g_α now substitute $G_\alpha = p_\alpha - g_\alpha(q_i, p_a)$. Then, ignoring terms that are proportional to G_α itself (which weakly vanishes), we can rewrite the conditions as

$$
\frac{\partial}{\partial q_i}\left(F - G_\alpha \frac{\partial F}{\partial p_\alpha} \right) \approx 0
$$
$$
\frac{\partial}{\partial p_i}\left(F - G_\alpha \frac{\partial F}{\partial p_\alpha} \right) \approx 0 \tag{11.5.13}
$$

where, in the last equation, we replace the index "a" by "i", which we can do because

$$\frac{\partial G_\alpha}{\partial p_\beta} = \delta_{\alpha\beta}.$$

We have proved a useful theorem: if $F \approx 0$ then

$$F - G_\alpha \frac{\partial F}{\partial p_\alpha} \simeq 0. \tag{11.5.14}$$

We want to apply this idea to the desired Hamiltonian, $\mathcal{H}(q_i, p_i)$.

Because $\mathcal{H}(q_i, p_i)$ is required to coincide with $\mathcal{H}_c$ on Γ_p, the function $\mathcal{H} - \mathcal{H}_c \approx 0$ (*i.e.*, is weakly vanishing) and therefore

$$\mathcal{H} - \mathcal{H}_c - G_\alpha \frac{\partial \mathcal{H}}{\partial p_\alpha} \simeq 0. \tag{11.5.15}$$

because $\mathcal{H}_c$ does not depend on p_α. It follows that

$$\frac{\partial \mathcal{H}_c}{\partial q_i} \approx \frac{\partial}{\partial q_i}\left(\mathcal{H} - G_\alpha \frac{\partial \mathcal{H}}{\partial p_\alpha}\right) \overset{\text{def}}{=} \frac{\partial \widetilde{\mathcal{H}}}{\partial q_i}$$

$$\frac{\partial \mathcal{H}_c}{\partial p_i} \approx \frac{\partial}{\partial p_i}\left(\mathcal{H} - G_\alpha \frac{\partial \mathcal{H}}{\partial p_\alpha}\right) \overset{\text{def}}{=} \frac{\partial \widetilde{\mathcal{H}}}{\partial p_i} \tag{11.5.16}$$

Inserting these into (11.5.8) we find

$$\dot{p}_i \approx -\frac{\partial \widetilde{\mathcal{H}}}{\partial q_i} - \frac{\partial G_\alpha}{\partial q_i}\dot{q}_\alpha \approx \{p_i, \widetilde{\mathcal{H}} + \dot{q}_\alpha G_\alpha\}$$

$$\dot{q}_i \approx \frac{\partial \widetilde{\mathcal{H}}}{\partial p_i} + \frac{\partial G_\alpha}{\partial p_i}\dot{q}_\alpha \approx \{q_i, \widetilde{\mathcal{H}} + \dot{q}_\alpha G_\alpha\}. \tag{11.5.17}$$

where the brackets are calculated treating the q_i and p_i as independent, and

$$\widetilde{\mathcal{H}} = \left(\mathcal{H} - G_\alpha \frac{\partial \mathcal{H}}{\partial p_\alpha}\right)$$

is defined on the *entire* phase space. The strong equivalence between $\mathcal{H}$ and $\mathcal{H}_c$ allows us to use $\mathcal{H}_c$ instead of $\widetilde{\mathcal{H}}$ in the equations and we find

$$\dot{p}_i \approx \{p_i, \mathcal{H}_c + \dot{q}_\alpha G_\alpha\}$$
$$\dot{q}_i \approx \{q_i, \mathcal{H}_c + \dot{q}_\alpha G_\alpha\}. \tag{11.5.18}$$

the canonical equations have been returned to their natural form in terms of Poisson brackets. They show clearly how G_α, although weakly vanishing , do influence the dynamics. The price we pay is that the equations of motion hold only weakly and the constraints must eventually be imposed.

We will now address two issues:

- we have assumed that the defining equations are solved so that it is possible to express $G_\alpha = p_\alpha - g_\alpha(q_i, p_a) \approx 0$. In practice this may be a very tedious, if not impossible task and sometimes it may not even be the desirable thing to do. Is it possible to carry out the program above if the constraints are to be used instead in some arbitrary form Φ_α, not necessarily G_α?

- The equations still contain the r undetermined functions $\dot{q}_\alpha$. We will shortly see that if constraints in some arbitrary form other than G_α are used then the $\dot{q}_\alpha$ must be replaced by other functions, η_α (they are the analogues of the Lagrange multipliers): in general, how many of them can be determined?

First, let us rewrite (11.5.18) in terms of arbitrary constraints $\Phi_\alpha(q_i, p_i) \approx 0$, where $G_\alpha \approx 0$ is a solution. The functional form of Φ_α is ambiguous since, for example, if $\Phi_\alpha \approx 0$ then $\Phi_\alpha^2 \simeq 0$, so we require that Φ_α is minimal in the sense that a weakly vanishing function should be strongly equivalent to a linear combination of the constraints defining the constraint surface. Differentiating Φ_α w.r.t. q_i and p_a,

$$\frac{\partial \Phi_\alpha}{\partial q_i} + \frac{\partial \Phi_\alpha}{\partial p_\beta}\frac{\partial g_\beta}{\partial q_i} \approx 0$$

$$\frac{\partial \Phi_\alpha}{\partial p_a} + \frac{\partial \Phi_\alpha}{\partial p_\beta}\frac{\partial g_\beta}{\partial p_a} \approx 0 \tag{11.5.19}$$

The matrix $\mathcal{V}_{\alpha\beta} = \partial \Phi_\alpha / \partial p_\beta$ must be non-degenerate, otherwise we could obtain some constraints involving only q's, which is impossible. This means that we could express

$$\frac{\partial G_\beta}{\partial q_i} = V_{\beta\alpha}^{-1}\frac{\partial \Phi_\alpha}{\partial q_i}$$

$$\frac{\partial G_\beta}{\partial p_a} = V_{\beta\alpha}^{-1}\frac{\partial \Phi_\alpha}{\partial p_a} \tag{11.5.20}$$

Insert these into (11.5.18) to get

$$\dot{p}_i = \{p_i, \mathcal{H}_c\} - \dot{q}_\alpha V_{\alpha\beta}^{-1}\frac{\partial \Phi_\beta}{\partial q_i}$$

$$\dot{q}_i = \{q_i, \mathcal{H}_c\} + \dot{q}_\alpha V_{\alpha\beta}^{-1}\frac{\partial \Phi_\beta}{\partial p_i} \tag{11.5.21}$$

then, with $\mu_\alpha = \dot{q}_\beta V_{\beta\alpha}^{-1}$ and the **primary Hamiltonian**

$$\mathcal{H}_p = \mathcal{H}_c + \mu_\alpha \Phi_\alpha, \tag{11.5.22}$$

we find that the time development of any function, $A(q_i, p_i)$, on phase space is determined by

$$\dot{A} = \{A, \mathcal{H}_p\} + \frac{\partial A}{\partial t}. \tag{11.5.23}$$

The μ_α play the role of multipliers, as the λ_l did in the Lagrangian theory. We will now ask how many of them can be determined.

Consistency requires that the constraints should be preserved in time, *i.e.*,

$$\dot{\Phi}_\alpha = \{\Phi_\alpha, \mathcal{H}_p\} = \{\Phi_\alpha, \mathcal{H}_c\} + \mu_\beta\{\Phi_\alpha, \Phi_\beta\} \equiv 0 \tag{11.5.24}$$

Let us call $M_{\alpha\beta} = \{\Phi_\alpha, \Phi_\beta\}$ and $K_\alpha = \{\Phi_\alpha, \mathcal{H}_c\}$ then the μ_α are solutions of the r equations

$$M_{\alpha\beta}\mu_\beta = -K_\alpha \tag{11.5.25}$$

and much will depend on the matrix $\widehat{M}$ and the vector K_α.

- Suppose $\widehat{M}$ is non-degenerate and $K_\alpha \neq 0$, then

$$\mu_\alpha = M_{\alpha\beta}^{-1}K_\beta \tag{11.5.26}$$

  The μ_α are all fixed and the equations can be solved without ambiguity.

- If $\widehat{M}$ is non-degenerate, but $K_\alpha \equiv 0$ then there are only the trivial solutions, $\mu_\alpha = 0$.

- Suppose $\widehat{M}$ is degenerate and $K_\alpha \neq 0$. If $\widehat{M}$ has rank $m < r$, then m of the μ's are determined and the remaining equations determine a certain number, $l \leq r - m$, of new relations which are all independent of the previous constraints Φ_α. If such new relations appear, they further constrain the system and are called **secondary constraints**. They restrict the motion to a hypersurface that is of even smaller dimension than the original Γ_p.

- Finally, if $\widehat{M}$ is degenerate and $K_\alpha \equiv 0$, non trivial solutions may exist. If m is the rank of $\widehat{M}$ then $r - m$ multipliers are weakly fixed.

What is important to bear in mind is that there are situations in which new constraints, the secondary constraints, may emerge. If l new constraints appear, the new constraint hypersurface is the surface defined by the $r + l$ weak relations

$$\Phi_\alpha \approx 0, \quad \alpha \in \{1, ...r\}$$

$$\chi_\rho \approx 0, \quad \rho \in \{1, ...l\} \tag{11.5.27}$$

For consistency, we must now require that the secondary constraints are preserved in time, which implies the *new* set of relations

$$\{\chi_\rho, \mathcal{H}_c\} + \mu_\beta\{\chi_\rho, \Phi_\beta\} \approx 0 \tag{11.5.28}$$

where the weak equality obviously refers to the new hypersurface determined by the $r + l$ relations. in (11.5.27). Again, these relations are either fullfilled or lead to new and

independent constraints, the tertiary constraints that, together with the primary and secondary constraints, define a new hypersurface of even lower dimension. In the latter case, one carries out the iterative procedure again, as before, on this new hypersurface. The process ends after a fininte number of steps. This is the Dirac-Bergman algorithm.

11.5.1   Dirac Bracket

One of the important goals of constrained dynamics is to count the true physical degrees of freedom. Let us distinguish between two kinds of constraints: a constraint, Ψ, is **first class** if it Poisson commutes weakly with all the other constraints

$$\{\Psi, \Phi_\alpha\} \approx 0 \tag{11.5.29}$$

otherwise it is **second class**. Let Υ_ρ denote all second class constraints on a system, then

$$C_{\rho\lambda} = \{\Upsilon_\rho, \Upsilon_\lambda\} \tag{11.5.30}$$

is non-degenerate and we may define a **Dirac bracket** as follows:

$$\{F, G\}_* = \{F, G\} - \{F, \Upsilon_\rho\} C_{\rho\lambda}^{-1} \{\Upsilon_\lambda, G\} \tag{11.5.31}$$

The Dirac bracket, satisfies all the conditions of Poisson bractet, *viz.,*

$$\{F, G\}_* = -\{G, F\}_*$$

$$\{F, GH\}_* = \{F, G\}_* H + G\{F, H\}_*$$

$$\{F, \{G, H\}_*\}_* + \{H, \{F, G\}_*\}_* + \{G, \{H, F\}_*\}_* = 0 \tag{11.5.32}$$

and also

$$\{F, \Upsilon_\alpha\}_* = 0 \tag{11.5.33}$$

for any F and Υ_α. Furthermore, the brackets are unchanged if the set $\{\Upsilon_\lambda\}$ is replaced by any $\{\Upsilon_\lambda'\}$ for which the equations $\Upsilon_\lambda \approx 0$ and $\Upsilon_\lambda' \approx 0$ define the same constrained hypersurface.

We state without proof the following result: if there are n coordinate functions, r first class constraints and s second class constraints, the true dynamical degrees of freedom of the system are

$$f = n - r - \frac{s}{2} \tag{11.5.34}$$

This reflects the fact that two second class constraints are required to eliminate one degree of freedom. It can be proved that the Dirac bracket is equivalent to the Poisson bracket calculated using a (reduced) set of unconstrained canonical variables. It is the Dirac bracket that gets turned into the commutator upon quantization.

11.5.2   Examples

For our first example, let us examine a simple system described by the Lagrangian

$$\mathcal{L} = a(x\dot{y} - y\dot{x}) - V(x, y), \tag{11.5.35}$$

where a is a positive, real constant. The configuration space consists of the coordinates (x, y), but there is no term that is quadratic in the velocities. This does not prevent us from writing the Euler-Lagrange equations of motion,

$$-2a\dot{y} + \frac{\partial V}{\partial x} = 0 = 2a\dot{x} + \frac{\partial V}{\partial y}. \tag{11.5.36}$$

However, if we compute the momenta,

$$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = -ay, \quad p_y = \frac{\partial \mathcal{L}}{\partial \dot{y}} = ax \tag{11.5.37}$$

we find that they do not contain the velocities and therefore neither $\dot{x}$ nor $\dot{y}$ can be eliminated from the Lagrangian. Therefore the two relations above should be thought of as primary constraints, $\Phi_1 = p_x + ay \approx 0$ and $\Phi_2 = p_y - ax \approx 0$. We have the canonical Hamiltonian,

$$\mathcal{H}_c = p_x \dot{x} + p_y \dot{y} - \mathcal{L} = V(x, y) \tag{11.5.38}$$

and the primary Hamiltonian must be defined with the help of two Lagrange multipliers, μ_1 and mu_2,

$$\mathcal{H}_p = \mathcal{H}_c + \mu_1(p_x + ay) + \mu_2(p_y - ax) = \mu_1(p_x + ay) + \mu_2(p_y - ax) + V(x, y). \tag{11.5.39}$$

The canonical equations of motion will now read read

$$\dot{x} = \mu_1, \quad \dot{y} = \mu_2$$

$$\dot{p}_x = a\mu_2 - \frac{\partial V}{\partial x}, \quad \dot{p}_y = -a\mu_1 - \frac{\partial V}{\partial x}, \tag{11.5.40}$$

but consistency of our primary constraints requires that

$$\dot{\Phi}_1 = \dot{p}_x + a\dot{y} = 2a\mu_2 - \frac{\partial V}{\partial x} \approx 0 \tag{11.5.41}$$

and

$$\dot{\Phi}_2 = \dot{p}_y - a\dot{x} = -2a\mu_1 - \frac{\partial V}{\partial y} \approx 0. \tag{11.5.42}$$

These are not, however, secondary constraints because they can be solved to determine $\mu_{1,2}$. We find

$$\mu_1 \approx -\frac{1}{2a}\frac{\partial V}{\partial y}, \quad \mu_2 \approx \frac{1}{2a}\frac{\partial V}{\partial x} \tag{11.5.43}$$

and so we end up with a system of just two primary constraints. The Hamiltonian can be rewritten by replacing $\mu_{1,2}$ for their values determined above,

$$\mathcal{H}_p = -\frac{1}{2a}\frac{\partial V}{\partial y}(p_x + ay) + \frac{1}{2a}\frac{\partial V}{\partial x}(p_y - ax) + V(x,y) \tag{11.5.44}$$

The canonical equations are now

$$\dot{x} \approx -\frac{1}{2a}\frac{\partial V}{\partial y}, \quad \dot{y} \approx \frac{1}{2a}\frac{\partial V}{\partial x}$$

$$\dot{p}_x \approx -\frac{1}{2}\frac{\partial V}{\partial x}, \quad \dot{p}_y \approx -\frac{1}{2}\frac{\partial V}{\partial y} \tag{11.5.45}$$

and the first pair will be recognized as the Euler-Lagrange equations.

The primary constraints are not first class but second class; in fact it is easy to compute the matrix

$$C_{\lambda\rho} = \{\Phi_\lambda, \Phi_\rho\} = 2a\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{11.5.46}$$

whose inverse appears in the definition of the Dirac bracket between observables. For instance, the non-vanishing fundamental Dirac brackets for this system are

$$\{x,y\}_* = -\frac{1}{2a}$$

$$\{x,p_x\}_* = \{y,p_y\}_* = \frac{1}{2}$$

$$\{p_x, p_y\}_* = -\frac{a}{2} \tag{11.5.47}$$

They are replaced by commutators in Dirac's quantization scheme. This system had two configuration space variables and two second class constraints, giving it $f = 2 - 2/2 = 1$ degrees of freedom.

Next consider the example of the pendulum described in the previous chapter by the Lagrangian

$$\mathcal{L} = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - mgy + \lambda(t)(\sqrt{x^2 + y^2} - l) \tag{11.5.48}$$

The configuration space consists of the three coordinates (x, y, λ). Computing the momenta,

$$p_x = m\dot{x}, \quad p_y = m\dot{y}, \quad p_\lambda = 0, \tag{11.5.49}$$

we see that only $\dot\lambda$ cannot be determined in terms of momenta. There is one primary constraint, *i.e.*, $\Phi = p_\lambda \approx 0$, so we define the primary Hamiltonian using a multiplier μ

$$
\begin{aligned}
\mathcal{H}_p \;&=\; \mathcal{H}_c + \mu\Phi_\lambda = p_x\dot x + p_y\dot y + \mu p_\lambda - \mathcal{L} \\[2mm]
&=\; \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy + \mu p_\lambda - \lambda(\sqrt{x^2+y^2} - l) \qquad (11.5.50)
\end{aligned}
$$

and the canonical equations are

$$
\dot x \approx \frac{p_x}{m}
$$

$$
\dot y \approx \frac{p_y}{m}
$$

$$
\dot\lambda \approx \mu
$$

$$
\dot p_x \approx \frac{\lambda x}{\sqrt{x^2+y^2}}
$$

$$
\dot p_y \approx -mg + \frac{\lambda y}{\sqrt{x^2+y^2}}
$$

$$
\dot p_\lambda \approx \sqrt{x^2+y^2} - l \qquad (11.5.51)
$$

The last of course is the consistency condition on the primary constraint. We encounter the third possibility listed before, the primary constraint $p_\lambda \approx 0$ gives rise to secondary constraints

$$
\chi_1 = \sqrt{x^2+y^2} - l \approx 0
$$

$$
\chi_2 = xp_x + yp_y \approx 0
$$

$$
\chi_3 = \frac{p_x^2 + p_y^2}{m} - mgy + \lambda l \approx 0 \qquad (11.5.52)
$$

and

$$
\{\chi_3, \mathcal{H}_p\} = \chi_4 = -3gp_y + \mu l \approx 0 \Rightarrow \mu \approx 3gp_y/l. \qquad (11.5.53)
$$

when consistency is required order by order according to the Dirac-Bergman algorithm. One can check easily that all the constraints are second class. There are four of them, so the number of dynamical degrees of freedom of this problem is $f = 3 - 4/2 = 1$. We already know that this is the angle made with the vertical (or horizontal) axis.[5]

---

[5]<u>Problem</u>: Evaluate the matrix $C_{\rho\lambda}$ defined earlier and compute the Dirac brackets between pairs of $\{x, y, \lambda, p_x, p_y, p_\lambda\}$.

For a final example, consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{q}_1^2 + \dot{q}_1 q_2 + (1 - \alpha)q_1\dot{q}_2 + \frac{\beta}{2}(q_1 - q_2)^2 \tag{11.5.54}$$

The configuration space now consists of the coordinates (q_1, q_2) and the conjugate momenta

$$\begin{aligned} p_1 &= \dot{q}_1 + q_2 \\ p_2 &= (1 - \alpha)q_1 \end{aligned} \tag{11.5.55}$$

so $\dot{q}_2$ cannot be determined in terms of the momenta. The canonical Hamiltonian is determined to be

$$\mathcal{H}_c = \frac{1}{2}(p_1 - q_2)^2 - \frac{\beta}{2}(q_1 - q_2)^2 \tag{11.5.56}$$

and the second equation in (11.5.55) is a primary constraint, Φ, so we define the primary Hamiltonian

$$\begin{aligned} \mathcal{H}_p &= \mathcal{H}_c + \mu\Phi \\[2mm] &= \frac{1}{2}(p_1 - q_2)^2 - \frac{\beta}{2}(q_1 - q_2)^2 + \mu[p_2 - (1 - \alpha)q_1] \end{aligned} \tag{11.5.57}$$

giving the canonical equations as

$$\begin{aligned} \dot{q}_1 &= p_1 - q_2, \quad \dot{q}_2 = \mu \\ \dot{p}_1 &= \beta(q_1 - q_2) + \mu(1 - \alpha), \quad \dot{p}_2 = (p_1 - q_2) - \beta(q_1 - q_2) \end{aligned} \tag{11.5.58}$$

The primary constraint then leads to the following secondary constraints,

$$\begin{aligned} \chi_1 &= \alpha(p_1 - q_2) - \beta(q_1 - q_2) \approx 0 \\[2mm] \chi_2 &= \beta[(p_1 - q_2) - \alpha(q_1 - q_2)] + (\alpha^2 - \beta)\mu \approx 0 \end{aligned} \tag{11.5.59}$$

If $\alpha^2 = \beta$ then it is straightforward to show that all three constraints are first class and μ cannot be determined.[6] The system has *no* degrees of freedom. If $\alpha^2 \neq \beta$ then $\chi_2 \approx 0$ may be used to determine μ,

$$\mu \approx \frac{\beta}{(\beta - \alpha^2)}[(p_1 - q_2) - \alpha(q_1 - q_2)] \tag{11.5.60}$$

and the remaining two constraints have the following Poisson bracket

$$\{\Phi, \chi_1\}_{\text{P.B}} = \alpha^2 - \beta \tag{11.5.61}$$

Consider two special cases:

---

[6]<u>Problem</u>: Work out the Poisson algebra of the constraints and show that all the constraints are first class if $\alpha^2 = \beta$.

- If $\alpha = 0$ ($\beta \neq 0$) then we find that $\mu \approx p_1 - q_2$ and

$$\mathcal{H}_p = \frac{1}{2}(p_1 - q_2)^2 - \frac{\beta}{2}(q_1 - q_2)^2 + (p_1 - q_2)(p_2 - q_1). \tag{11.5.62}$$

Thus, from the constraints, $p_2 \approx q_1$ and $q_1 \approx q_2$ the equations of motion become

$$\dot{q}_1 \approx \dot{q}_2 \approx \dot{p}_1 \approx \dot{p}_2 \approx \mu \approx (p_1 - q_2) \tag{11.5.63}$$

in this special case.

- If $\beta = 0$ ($\alpha \neq 0$) then $\mu = 0$ and

$$\mathcal{H}_p = \frac{1}{2}(p_1 - q_2)^2. \tag{11.5.64}$$

We find the equations of motion $\dot{q}_2 \approx \dot{p}_1 \approx 0$ and

$$\dot{q}_1 \approx \dot{p}_2 \approx p_1 - q_2 \approx 0 \tag{11.5.65}$$

where the last weak equality follows because of the primary constraint.

There are two second class constraints in general and this implies that the system has one degree of freedom.[7]

---

[7]<u>Problem</u>: Evaluate the matrix $C_{\rho\lambda}$ defined earlier and compute the Dirac brackets between pairs of $\{q_1, q_2, p_1, p_2\}$ for the general case in which $\alpha^2 \neq \beta$.

Chapter 12

Canonical Transformations

Hamilton's equations do not simplify the problems of mechanics over Euler's equations. It is true that Hamilton's equations are first order, but there are twice as many of them and they are also coupled differential equations that tend to maintain the same level of difficulty as their second order counterparts. Where then is the new mathematical structure useful? We have already seen that it lends itself particularly to perturbation approaches. Its main advantage, however, lies in the fact that it seems to provide a natural framework for the application of mechanics in various other areas of physics. For example, it provides the basis for the formulation of statistical mechanics and quantum mechanics. It also often allows for a certain kind of transformation of the fundamental variables of the theory to more transparent variables (these are the canonical transformations we will discuss in this chapter). Often, new variables can shed new light on a problem and in some cases provide the clues that lead to its solution.

12.1   Hamilton's equations from a Variational Principle

Let us return to Hamilton's $2n$ first order equations governing the dynamics of physical systems. In terms of generalized coordinates and momenta, they are

$$
\begin{aligned}
\dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i} \\
\dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i}.
\end{aligned}
\tag{12.1.1}
$$

Recall that the generalized momenta are defined in terms of the Lagrangian

$$
p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}
\tag{12.1.2}
$$

and the Legendre transformation leading from $\mathcal{L}$ to $\mathcal{H}$ requires the system to be regular, *i.e.,* that the determinant of the Hessian is non-vanishing. The Legendre transformation can be undone and the Lagrangian expressed as

$$\mathcal{L} = p_i \dot{q}_i - \mathcal{H}(p_i, q_i) \qquad (12.1.3)$$

where p_i are expressed in terms of the velocities. Euler's equations are then obtained from a variation of the action

$$S = \int_1^2 dt [p_i \dot{q}_i - \mathcal{H}(p_i, q_i)] \qquad (12.1.4)$$

holding the variations of $q_i(t)$ fixed at the endpoints. This is sufficient in the Lagrangian formulation.

At this point we may ask if it is possible to use the action in the form (12.1.3), with $\mathcal{L}$ viewed as a function of coordinates and momenta, to directly obtain Hamilton's equation from a variational principle. Because, in Hamilton's formulation, the momenta are *à priori* independent of the coordinates (unlike the velocities in Lagrange's formultion), the condition that the variations should vanish at the boundary means that we must explicitly hold both p_i and q_i fixed there. It is easy to see that we then get

$$\frac{\partial \mathcal{L}}{\partial p_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{p}_i} = 0$$
$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \qquad (12.1.5)$$

But, since $\mathcal{H}$ contains no time derivatives of the coordinates and momenta, these imply preciesly Hamilon's equations

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}. \qquad (12.1.6)$$

Note, however, that it appears we have had to pay a price by enlarging the number of coordinates whose variations are required to vanish at the boundaries from n to $2n$. This is not really so because by keeping p_i also fixed at the boundaries we avoid a contribution from a term of the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{p}_i}\delta p_i\right) \qquad (12.1.7)$$

but, because the Lagrangian does not contain derivatives of the momenta, such a term never appears and holding p_i fixed at the end points is redundant. Indeed, a careful look at the derivation of Euler's equation reveals that one only needs to get rid of the term

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\delta q_i\right) \qquad (12.1.8)$$

so only q_i must be held fixed at the boundaries as before.

12.2 The Generating Function

While the canonical equations of motion do not provide a simpler setting for mechanical problems than Euler's equations, they do suggest an alternative approach. To motivate this approach, let us consider a system system, in which the Lagrangian does not depend on one of the genealized coordinates (say q_k), *i.e.,* $\mathcal{L} = \mathcal{L}(q_1, \ldots, \widehat{q_k}, \ldots q_n, \dot{q}_i, t)$. Spatial translation invariance of the action would then imply that the momentum, p_k, conjugate to q_k is a constant of the motion, $\dot{p}_k = 0$. Notice that, in this case, the Hamiltonian is also independent of q_k, $\mathcal{H} = \mathcal{H}(q_1, \ldots, \widehat{q_k}, \ldots, q_n, p_i)$. Coordinates that do not apear explicitly in the Lagrangian (or Hamiltonian) are called cyclic. If *all* the coordinates are cyclic then $\mathcal{H} = \mathcal{H}(p_i)$ and

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = 0, \quad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i} = \omega_i \quad \text{(constants)} \tag{12.2.1}$$

(the ω_i being functions of only the p_i) and it follows that

$$q_i(t) = q_{i,0} + \omega_i(t - t_0) \tag{12.2.2}$$

constitutes a complete solution to the problem.

This leads us to wonder if there *always* exists a set of generalized coordinates and momenta that are cyclic. This amounts to turning the observation in the previous paragraph around to ask if every solution should in fact correspond to precisely such a set. So far we have considered only coordinate transformations (also known as **point transformations**),

$$Q_i = Q_i(q_j, t), \tag{12.2.3}$$

but the Hamiltonian formulation allows us to consider more general transformations involving *all of* phase space,

$$q_i \to Q_i = Q_i(q_j, p_j, t)$$

$$p_i \to P_i = P_i(q_j, p_j, t) \tag{12.2.4}$$

because it treats the coordinates and momenta on an equal footing, as independent variables.

Not all such transformations will preserve the form of the canonical equations of motion. We therefore define

Canonical Transformations are transformations in phase-space that preserve Hamilton's equations.

To see what the definition implies, let the transformation in (12.2.4) take the Hamiltonian to

$$\mathcal{H}(q_i, p_i, t) \rightarrow \mathcal{K}(Q_i, P_j, t). \tag{12.2.5}$$

If it preserves the canoncial equations of motion, we must have

$$\dot{Q}_i = \frac{\partial \mathcal{K}}{\partial P_i}$$

$$\dot{P}_i = -\frac{\partial \mathcal{K}}{\partial Q_i}. \tag{12.2.6}$$

But because Hamilton's equations are derivable from an action principle it is sufficient (but not necessary) for both the action

$$S = \int_1^2 dt \, [p_i \dot{q}_i - \mathcal{H}(q_i, p_i, t)] \tag{12.2.7}$$

as well as the action

$$S' = \int_1^2 dt \left[P_i \dot{Q}_i - \mathcal{K}(Q_i, P_i, t) \right]. \tag{12.2.8}$$

to be stationary under variations, *i.e.*, $\delta S = 0 = \delta S'$. A sufficient (but again, not necessary) condition for this to hold is that

$$\lambda[p_i \dot{q}_i - \mathcal{H}(q_i, p_i, t)] = P_i \dot{Q}_i - \mathcal{K}(Q_i, P_i, t) + \frac{dF}{dt} \tag{12.2.9}$$

where λ is some constant and $F = F$ is any arbitrary function, given either in terms of the old variables or the new. It is called the **generating function** of the transformation. If some canonical transformations are known then F may be obtained simply by comparing the two sides of (12.2.9). Viewing F as a function of the q_i and p_i,

$$\lambda[p_i \dot{q}_i - \mathcal{H}] = P_j \left[\frac{\partial Q_j}{\partial q_i} \dot{q}_i + \frac{\partial Q_j}{\partial p_i} \dot{p}_i + \frac{\partial Q_j}{\partial t} \right] - \mathcal{K} + \frac{\partial F}{\partial q_i} \dot{q}_i + \frac{\partial F}{\partial p_i} \dot{p}_i + \frac{\partial F}{\partial t} \tag{12.2.10}$$

and comparing terms we find

$$\begin{aligned} \lambda p_i &= P_j \frac{\partial Q_j}{\partial q_i} + \frac{\partial F}{\partial q_i} \\ 0 &= P_j \frac{\partial Q_j}{\partial p_i} + \frac{\partial F}{\partial p_i} \\ \lambda \mathcal{H} &= K - P_j \frac{\partial Q_j}{\partial t} - \frac{\partial F}{\partial t} \end{aligned} \tag{12.2.11}$$

These are sufficient to determine $F(q_i, p_i, t)$. However, it is also true that any choice of F will determine a canonical transformation. It turns out that F is useful only when in can be expressed in special ways in terms of functions depending on half of the variables from the old set and half from the new set. We now consider four particularly useful possibilities.

(Type I) $F = F_1(q_i, Q_i, t)$:
Suppose that we take the generating function to depend on the old coordinates and the new coordinates, but not the momenta (old or new). Beginning with (12.2.9), we have

$$\lambda[p_i\dot{q}_i - \mathcal{H}] = P_i\dot{Q}_i - K + \frac{\partial F_1}{\partial q_i}\dot{q}_i + \frac{\partial F_1}{\partial Q_i}\dot{Q}_i + \frac{\partial F_1}{\partial t} \qquad (12.2.12)$$

which, upon comparison, yields

$$
\begin{aligned}
\lambda p_i &= \frac{\partial F_1}{\partial q_i} \\
P_i &= -\frac{\partial F_1}{\partial Q_i} \\
\lambda\mathcal{H} &= \mathcal{K} - \frac{\partial F_1}{\partial t}
\end{aligned}
\qquad (12.2.13)
$$

These are $2n$ equations which determine the transformations as follows. The first equation expresses p_i in terms of the variables q_i, Q_i and t. It can be inverted to obtain $Q_i = Q_i(q_j, p_j, t)$. Once the new coordinates are known functions of the old phase-space variables, the second equation gives the new momenta. The last equation returns the new Hamiltonian, $\mathcal{K}$ (jokingly referred to as the "Kamiltonian"), in terms of the old. Notice that they are equal up to a scaling and a partial time derivative of F, $i.e.$, they are identical upto a scale if F has no explicit time dependence.

(Type II) $F = F_2(q_i, P_i, t) - P_iQ_i$:
If we take a generating function of this form then

$$\lambda[p_i\dot{q}_i - \mathcal{H}] = -\mathcal{K} - \dot{P}_iQ_i + \frac{\partial F_2}{\partial q_i}\dot{q}_i + \frac{\partial F_2}{\partial P_i}\dot{P}_i + \frac{\partial F_2}{\partial t} \qquad (12.2.14)$$

gives

$$
\begin{aligned}
\lambda p_i &= \frac{\partial F_2}{\partial q_i} \\
Q_i &= \frac{\partial F_2}{\partial P_i} \\
\lambda\mathcal{H} &= \mathcal{K} - \frac{\partial F_2}{\partial t}.
\end{aligned}
\qquad (12.2.15)
$$

They are used to obtain the transformations as follows. The first equation gives p_i in terms of q_i, P_i and t. It can be inverted to obtain $P_i = P_i(q_j, p_j, t)$. Once the new momenta are known as functions of the old phase-space variables, the second equation gives the new coordinates. As before, the new Hamiltonian is identical to the old upto a scale if F_2 is not explicitly time dependent.

(Type III) $F = F_3(p_i, Q_i, t) + \lambda p_i q_i$:
For a generating function of this form,

$$\lambda[-\dot{p}_i q_i - \mathcal{H}] = -\mathcal{K} + P_i \dot{Q}_i + \frac{\partial F_3}{\partial p_i}\dot{p}_i + \frac{\partial F_3}{\partial Q_i}\dot{Q}_i + \frac{\partial F_3}{\partial t} \tag{12.2.16}$$

and so

$$
\begin{aligned}
\lambda q_i &= -\frac{\partial F_3}{\partial p_i} \\
P_i &= -\frac{\partial F_3}{\partial Q_i} \\
\lambda \mathcal{H} &= \mathcal{K} - \frac{\partial F_3}{\partial t}.
\end{aligned}
\tag{12.2.17}
$$

Thus, the first equation allows us to obtain $Q_i = Q_i(q_j, p_j, t)$ by inversion and the second equation then gives $P_i = P_i(q_j, p_j, t)$. The relationship between $\mathcal{H}$ and $\mathcal{K}$ is the same as before.

(Type IV) $F = F_4(p_i, P_i, t) + \lambda p_i q_i - P_i Q_i$:
For a generating function of this form,

$$\lambda[-\dot{p}_i q_i - \mathcal{H}] = -\mathcal{K} - \dot{P}_i Q_i + \frac{\partial F_4}{\partial p_i}\dot{p}_i + \frac{\partial F_4}{\partial P_i}\dot{P}_i + \frac{\partial F_3}{\partial t} \tag{12.2.18}$$

and so

$$
\begin{aligned}
\lambda q_i &= -\frac{\partial F_4}{\partial p_i} \\
Q_i &= \frac{\partial F_4}{\partial P_i} \\
\lambda \mathcal{H} &= \mathcal{K} - \frac{\partial F_4}{\partial t}.
\end{aligned}
\tag{12.2.19}
$$

The first equation allows us to obtain $P_i = P_i(q_j, p_j, t)$ by inversion, the second then gives $Q_i = Q_i(q_j, p_j, t)$ and the relationship between $\mathcal{H}$ and $\mathcal{K}$ is similar to what we had before.

It's easy to see that invertible coordinate transformations are a particular class of canonical transformations generated by a transformation of type II with

$$F_2(q_i, P_i, t) = P_i f_i(q_j, t) \tag{12.2.20}$$

and $\lambda = 1$. Using the equations appropriate to this case, we see that

$$\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} = P_j \frac{\partial f_j}{\partial q_i} \\ Q_i &= \frac{\partial F_2}{\partial P_i} = f_j(q_i, t) \end{aligned} \qquad (12.2.21)$$

Clearly, the transformations of the coordinates must be invertible, *i.e.,* the Jacobian matrix $J_{ij} = \partial_i f_j$ must be invertible. In that case, inverting the first equation gives the new momenta

$$P_i = J_{ij}^{-1}(q, t) p_j. \qquad (12.2.22)$$

In particular, the transformation generated by

$$F_2(q_i, P_i, t) = q_i P_i \qquad (12.2.23)$$

exchanges is the identity transformation. On the other hand, the following generating function of type I,

$$F_1(q_i, Q_i, t) = q_i Q_i \qquad (12.2.24)$$

with $\lambda = 1$ exchanges coordinates and momenta,

$$\begin{aligned} p_i &= \frac{\partial F_1}{\partial q_i} = Q_i \\ P_i &= -\frac{\partial F_1}{\partial Q_i} = -q_i, \end{aligned} \qquad (12.2.25)$$

thereby emphasizing the fact that the coordinates and momenta are on an equal footing in the Hamiltonian approach.

12.3 Examples

The Harmonic Oscillator: Consider how a canonical transformation may actually help solve a problem by taking the case of a simple harmonic oscillator, whose Hamiltonian is given by

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \qquad (12.3.1)$$

The fact that the Hamiltonian is a quandratic function of both the coordinates and the momenta suggests a transformation of the form

$$\begin{aligned} p &= f(P)\cos Q \\ q &= \frac{f(P)}{m\omega}\sin Q \end{aligned} \qquad (12.3.2)$$

We want to determine $f(P)$ so that it this is a canonical transformation, therefore let us find a generating function for it. Eliminating $f(P)$, we see that

$$p = m\omega q \cot Q. \tag{12.3.3}$$

Therefore the generating function must be of type I and, in fact, to recover (12.3.3) we must have

$$F_1(q, Q, t) = \frac{1}{2}m\omega q^2 \cot Q. \tag{12.3.4}$$

The generating function will now determine $f(P)$. Using the equations appropriate to a type I generating functional we find

$$
\begin{aligned}
p &= \frac{\partial F_1}{\partial q} = m\omega q \cot Q \\
P &= -\frac{\partial F_1}{\partial Q} = \frac{1}{2}m\omega q^2 \csc^2 Q
\end{aligned}
\tag{12.3.5}
$$

From the first equation one finds that

$$Q = \tan^{-1} \frac{m\omega q}{p}, \tag{12.3.6}$$

and similarly from the second,

$$P = \frac{1}{2m\omega} \left(p^2 + m^2\omega^2 q^2 \right). \tag{12.3.7}$$

Putting (12.3.6) and (12.3.7) together

$$
\begin{aligned}
\cos Q &= \frac{p}{\sqrt{p^2 + m^2\omega^2 q^2}} = \frac{p}{\sqrt{2m\omega P}} \\
\sin Q &= \frac{m\omega q}{\sqrt{p^2 + m^2\omega^2 q^2}} = \frac{m\omega q}{\sqrt{2m\omega P}},
\end{aligned}
\tag{12.3.8}
$$

which may be inverted to get

$$
\begin{aligned}
q &= \sqrt{\frac{2P}{m\omega}} \sin Q \\
p &= \sqrt{2m\omega P} \cos Q
\end{aligned}
\tag{12.3.9}
$$

Clearly then, $f(P) = \sqrt{2m\omega P}$. Because $F_1(q, Q, t)$ is time independent, the new Hamiltonian is no different from the original. Re-expressed in terms of the new coordinates we find

$$\mathcal{K} = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) = \omega P \tag{12.3.10}$$

Q is therefore a cyclic coordinate and P is a constant related to the total energy, E, of the oscillator. Because Q is cyclic, $Q(t) = \omega t + Q_0$ and therefore

$$q(t) = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + Q_0) \qquad (12.3.11)$$

is the desired solution. While the solution was obvious after the canonical transformation, we must bear in mind that *finding* the transformation that leads to a maximal number of cyclic coordinates is generally a difficult problem, which can, as in this case, be a more difficult enterprise than obtaining a solution of Euler's (or the canonical) equations for the system. Nevertheless it is a novel approach to problem solving.[1]

Another interesting canonical transformation for the harmonic oscillator is

$$Q = \sqrt{\frac{m\omega}{2}} \left(q + \frac{ip}{m\omega} \right), \quad P = i\sqrt{\frac{m\omega}{2}} \left(q - \frac{ip}{m\omega} \right) \qquad (12.3.12)$$

To verify that this is a canonical transformation it is only necessary to check the fundamental Poisson brackets: the only non-vanishing bracket is

$$\{Q, P\} = \frac{im\omega}{2} \left\{ q + \frac{ip}{m\omega}, q - \frac{ip}{m\omega} \right\} = \{q, p\} = 1 \qquad (12.3.13)$$

Because the transformation is time independent, the Hamiltonian for the harmonic oscillator becomes

$$\mathcal{K} = \mathcal{H} = -i\omega Q P \qquad (12.3.14)$$

and the equations of motion read

$$\dot{Q} = -i\omega Q, \quad \dot{P} = i\omega P. \qquad (12.3.15)$$

It is now not clear what the Lagrangian system leading to this form of $\mathcal{K}$ might correspond to because of the impossibility of exchanging the new momentum for a velocity. Nevertheless, we may ask what generating function is responsible for this transformation. Applying the conditions in (12.2.11) we find

$$\frac{\partial F}{\partial q} = \lambda p - \sqrt{\frac{m\omega}{2}} P = \left(\lambda - \frac{1}{2} \right) p - \frac{im\omega q}{2}$$

---

[1]The quantization procedure is sadly not invariant under canonical transformations and there can be differing quantizations of the same system, depending on the phase-space variables chosen. For example, the quantum harmonic oscillator quantized in the (q, p) variables is known to have energy eigenvalues $E_n = \left(n + \frac{1}{2} \right) \hbar\omega$. Consider the time independent Schroedinger's equation for the same oscillator in the (Q, P) system,

$$E\psi = -i\hbar\omega \frac{\partial \psi}{\partial Q} \Rightarrow \psi = e^{i\frac{EQ}{\hbar\omega}}$$

Because Q is a compact coordinate of period 2π, the wave function is single valued only if $E_n = n\hbar\omega$ for whole numbers n, since P and therefore E is non-negative. Notice the absence of the vacuum energy $\frac{1}{2}\hbar\omega$.

$$\frac{\partial F}{\partial p} = -\frac{i}{\sqrt{2m\omega}}P = \frac{1}{2}\left(q - \frac{ip}{m\omega}\right) \tag{12.3.16}$$

Integrating the first equation gives

$$F = \left(\lambda - \frac{1}{2}\right)pq - \frac{im\omega q^2}{4} + f(p) \tag{12.3.17}$$

where $f(p)$ is as yet undetermined. To find an equation for $f(p)$ take a derivative with respect to p,

$$\frac{\partial F}{\partial p} = \left(\lambda - \frac{1}{2}\right)q + f'(p) \tag{12.3.18}$$

and compare it with the second of the equations obtained using (12.2.11). They are compatible if $\lambda = 1$ and

$$f'(p) = -\frac{ip}{2m\omega} \Rightarrow f(p) = -\frac{ip^2}{4m\omega} + \text{const.} \tag{12.3.19}$$

and so

$$F(q,p) = \frac{1}{2}qp - \frac{im\omega q^2}{4} - \frac{ip^2}{4m\omega} + \text{const.} \tag{12.3.20}$$

It is not in any of the standard forms given earlier, but can be put in one of them. For example, if we want to express the generating functional in terms of $F_2(q, P)$, we could eliminate p in favor of q and P. But first, recall that

$$F_2(q, P) = PQ + F = \frac{1}{2}(qp + QP) \tag{12.3.21}$$

Then, by direct substitution we find that

$$F_2(q, P) = \sqrt{2m\omega}qP + \frac{i}{2}(P^2 - m\omega q^2) \tag{12.3.22}$$

generates the transformation.

It has probably become clear that while whether or not a transformation is canonical has little to do with the physical system, the usefulness of a particular transformation depends on the particularities of the system. Consider the transformation

$$Q = \ln(1 + \sqrt{q}\cos p), \quad P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p \tag{12.3.23}$$

It is not difficult to show that the only non-vanishing Poisson bracket is $\{Q, P\} = 1$, so it is a canonical transformation. To find a generating function for this transformation we must solve

$$\lambda p = P\frac{\partial Q}{\partial q} + \frac{\partial F}{\partial q} = \sin p \cos p + \frac{\partial F}{\partial q}$$

$$0 = P\frac{\partial Q}{\partial p} + \frac{\partial F}{\partial p} = -2q\sin^2 p + \frac{\partial F}{\partial p}$$
$$\mathcal{H} = \mathcal{K} \tag{12.3.24}$$

(the last follows because the transformation is time independent). As before, when the solution of the first,

$$F(q,p) = \lambda pq - q\sin p\cos p + f(p), \tag{12.3.25}$$

is inserted into the second we get

$$-2q\sin^2 p + \lambda q - q\cos^2 p + q\sin^2 p + f'(p) = (\lambda - 1)q + f'(p) = 0 \tag{12.3.26}$$

which is solved by $\lambda = 1$ and $f(p) = \text{const}$. The generating function is

$$F(q,p) = pq - q\sin p\cos p \tag{12.3.27}$$

and we will now express it in terms of $F_3(p,Q)$ via

$$F_3(p,Q) = F(q,p) - pq \tag{12.3.28}$$

where (q,P) have been eliminated in favor of (p,Q). It is not difficult to see that

$$q = (e^Q - 1)^2\sec^2 p, \quad P = 2e^Q(e^Q - 1)\tan p \tag{12.3.29}$$

which gives

$$F_3(p,Q) = -(e^Q - 1)^2\tan p \tag{12.3.30}$$

from which the transformations in (12.3.29) could be obtained by taking derivatives appropriate to the type III generating function.

As a final example, consider the transformation

$$Q = \ln\left(\frac{1}{q}\sin p\right), \quad P = q\cot p. \tag{12.3.31}$$

We may verify that it is a canonical transformation by simply evaluating the fundamental Poisson brackets. We find that the only non-vanishing bracket is

$$\{Q,P\} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{1}{q}(q\csc^2 p) - \cot^2 p \equiv 1 \tag{12.3.32}$$

so the transformation is canonical. To find a generating function for this transformation we must solve (12.2.11),

$$\lambda p = P\frac{\partial Q}{\partial q} + \frac{\partial F}{\partial q} = -\cot p + \frac{\partial F}{\partial q}$$

$$0 = P\frac{\partial Q}{\partial p} + \frac{\partial F}{\partial p} = -q\cot^2 p + \frac{\partial F}{\partial p}$$
$$\mathcal{H} = \mathcal{K} \tag{12.3.33}$$

(the last follows because the transformation is time independent). Integrating the first equation

$$F(q,p) = \lambda pq + q\cot p + f(p) \tag{12.3.34}$$

where $f(p)$ is an arbitrary function of p. Inserting this into the second equation implies that

$$\lambda q - q\csc^2 p + f'(p) - q\cot^2 p = (\lambda - 1)q + f'(p) = 0 \tag{12.3.35}$$

Evidently, $\lambda = 1$ and $f(p) = $ const. solves this equation. Therefore the transformation is generated by

$$F(q,p) = pq + q\cot p \tag{12.3.36}$$

Let us put this in the form $F_4(q, Q)$ by eliminating (q, Q) in favor of (p, P). We find

$$q = P\tan p, \quad Q = \ln\left(\frac{\cos p}{P}\right) \tag{12.3.37}$$

which give

$$F_4(p, P) = F - pq + PQ = P\left[1 + \ln\left(\frac{\cos p}{P}\right)\right], \tag{12.3.38}$$

from which (12.3.37) are obtained by taking the derivatives appropriate to generating functions of type IV.

12.4   The Symplectic Approach

If a canonical transformation is independent of time it is called a **restricted** canonical transformation. Let

$$P_i = P_i(q_j, p_j), \quad Q_i = Q_i(q_j, p_j) \tag{12.4.1}$$

be a restricted transformation and let $\mathcal{K}(Q, P)$ be the Hamiltonian of the system expressed in terms of the new phase space coordinates. Assume that the transformations are invertible then

$$\dot{Q}_i = \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j \tag{12.4.2}$$

and therefore

$$\{Q_i, \mathcal{K}\} = \frac{\partial Q_i}{\partial q_j}\{q_j, \mathcal{H}\} + \frac{\partial Q_i}{\partial p_j}\{p_j, \mathcal{H}\} \tag{12.4.3}$$

or,

$$\frac{\partial \mathcal{K}}{\partial P_i} = \frac{\partial Q_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} \tag{12.4.4}$$

But, because $\mathcal{K} \equiv \mathcal{H}$, we could rewrite the left hand side of the equation to get

$$\frac{\partial \mathcal{H}}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial \mathcal{H}}{\partial p_j}\frac{\partial p_j}{\partial P_i} = \frac{\partial Q_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} \qquad (12.4.5)$$

This shows (by comparing terms) that

$$\frac{\partial q_j}{\partial P_i} = -\frac{\partial Q_i}{\partial p_j}$$
$$\frac{\partial p_j}{\partial P_i} = +\frac{\partial Q_i}{\partial q_j} \qquad (12.4.6)$$

Doing the same things for the second set of canonical equations (for $\dot{P}_i$), we get[2]

$$\frac{\partial p_j}{\partial Q_i} = -\frac{\partial P_i}{\partial q_j}$$
$$\frac{\partial q_j}{\partial Q_i} = +\frac{\partial P_i}{\partial p_j} \qquad (12.4.7)$$

The conditions (12.4.6) and (12.4.7) are called the **direct conditions** for a restricted canonical transformation.

We have shown that the direct conditions are necessary for a transformation to be a canonical transformation, we now show that they are sufficient.[3] Suppose that the direct conditions hold, then

$$\begin{aligned}\{Q_i, Q_j\} &= \frac{\partial Q_i}{\partial q_k}\frac{\partial Q_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial Q_j}{\partial q_k} \\ &= \frac{\partial p_k}{\partial P_i}\frac{\partial Q_j}{\partial p_k} + \frac{\partial q_k}{\partial P_i}\frac{\partial Q_j}{\partial q_k} = \frac{\partial Q_j}{\partial P_i} \equiv 0 \qquad (12.4.8)\end{aligned}$$

where we have used the direct conditions. It is straightforward also that $\{P_i, P_j\} \equiv 0$. It remains to evaluate

$$\begin{aligned}\{Q_i, P_j\} &= \frac{\partial Q_i}{\partial q_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial Q_i}{\partial p_k}\frac{\partial P_j}{\partial q_k}. \\ &= \frac{\partial Q_i}{\partial q_k}\frac{\partial q_k}{\partial Q_j} + \frac{\partial Q_i}{\partial p_k}\frac{\partial p_k}{\partial Q_j} = \frac{\partial Q_i}{\partial Q_j} \equiv \delta_{ij} \qquad (12.4.9)\end{aligned}$$

where use has once again been made of the direct conditions. We have thus proved that a time independent transformation in phase space implies and is implied by the direct conditions.

---

[2]Problem: Do it!

[3]If a mathematical statement says that condition A implies the condition B (if A then B or $A \Rightarrow B$) then A is "*sufficient*" for B and B is "*necessary*" for A. The statement $A \Leftrightarrow B$ says that both A and B are necessary and sufficient for each other: they are identical statements.

The restricted transformations can be put in a very elegant and suggestive form if we write them out in the language of matrices. Let $\vec{\xi}$ be a "position vector" in phase space,

$$\vec{\xi} = (q_1, q_2, ...q_n, p_1, p_2, ..., p_n) \qquad (12.4.10)$$

Let us think of it as a column matrix and designate an arbitrary component of this vector by ξ_α (greek indices), where $\alpha \in \{1, 2, ..., n, n+1, ..., 2n\}$. Evidently, $\xi_i = q_i$ if $i \leq n$ and $\xi_{n+i} = p_i$. Consider the $2n \times 2n$ matrix $\widehat{\omega}$ defined by[4]

$$\widehat{\omega}_{(2n \times 2n)} = \begin{bmatrix} 0 & +\mathbf{1}_{n \times n} \\ -\mathbf{1}_{n \times n} & 0 \end{bmatrix}, \qquad (12.4.12)$$

which satisfies the conditions

$$\widehat{\omega} \cdot \widehat{\omega} = -\mathbf{1}_{2n \times 2n}, \quad \widehat{\omega} \cdot \widehat{\omega}^T = \widehat{\omega}^T \cdot \widehat{\omega} = +\mathbf{1}_{2n \times 2n}, \qquad (12.4.13)$$

where

$$\widehat{\omega}^T = \begin{bmatrix} 0 & -\mathbf{1}_{n \times n} \\ +\mathbf{1}_{n \times n} & 0 \end{bmatrix} \qquad (12.4.14)$$

is the transpose of $\widehat{\omega}$. In terms of indices these properties are expressed as

$$\widehat{\omega}_{\alpha\gamma}\widehat{\omega}_{\gamma\beta} = -\delta_{\alpha\gamma} \qquad (12.4.15)$$

and

$$\widehat{\omega}_{\alpha\gamma}\widehat{\omega}_{\beta\gamma} = \widehat{\omega}_{\gamma\alpha}\widehat{\omega}_{\gamma\beta} = \delta_{\alpha\beta}. \qquad (12.4.16)$$

Acting on $\vec{\xi}$,

$$[\widehat{\omega} \cdot \vec{\xi}]_\alpha = \widehat{\omega}_{\alpha\beta}\xi_\beta, \qquad (12.4.17)$$

$\widehat{\omega}$ takes q_i to p_i and p_i to $-q_i$, and the direct conditions for a restricted canonical transformation take on a simple form when they are written in terms of $\widehat{\omega}$:

$$\frac{\partial \eta_\alpha}{\partial \xi_\beta} = \widehat{\omega}_{\alpha\gamma}\widehat{\omega}_{\beta\delta}\frac{\partial \xi_\delta}{\partial \eta_\gamma}. \qquad (12.4.18)$$

Poisson brackets can be expressed in terms of $\widehat{\omega}$. The fundamental Poisson brackets are

$$\{\xi_\alpha, \xi_\beta\} = \widehat{\omega}_{\alpha\beta} \qquad (12.4.19)$$

---

[4]It is sometimes more convenient to think of the matrix in terms of its components:

$$\widehat{\omega}_{\alpha\beta} = \begin{cases} \delta_{\alpha\beta} & \beta - \alpha = n \\ -\delta_{\alpha\beta} & \alpha - \beta = n \ . \\ 0 & \text{otherwise} \end{cases} \qquad (12.4.11)$$

and, in fact, given any two functions on phase space, $A(\xi)$ and $B(\xi)$, their Poisson brackets

$$\{A, B\} = \frac{\partial A}{\partial \xi_i} \frac{\partial B}{\partial \xi_{n+i}} - \frac{\partial A}{\partial \xi_{n+i}} \frac{\partial B}{\partial \xi_i} \tag{12.4.20}$$

may also be expressed as

$$\{A, B\} = \widehat{\omega}_{\gamma\delta} \frac{\partial A}{\partial \xi_\gamma} \frac{\partial B}{\partial \xi_\delta}. \tag{12.4.21}$$

In particular, suppose we make a time independent transformation $\xi \to \eta$, then treating the new coordinates as functions on phase space and using the above equation it follows that

$$\{\eta_\alpha, \eta_\beta\} = \widehat{\omega}_{\gamma\delta} \frac{\partial \eta_\alpha}{\partial \xi_\gamma} \frac{\partial \eta_\beta}{\partial \xi_\delta} \tag{12.4.22}$$

If the transformation is a restricted canonical transformation, then

$$
\begin{aligned}
\{\eta_\alpha, \eta_\beta\} &= = \widehat{\omega}_{\gamma\delta} \frac{\partial \eta_\alpha}{\partial \xi_\gamma} \frac{\partial \eta_\beta}{\partial \xi_\delta} \\
&= \widehat{\omega}_{\gamma\delta} \widehat{\omega}_{\alpha\sigma} \widehat{\omega}_{\gamma\lambda} \frac{\partial \xi_\lambda}{\partial \eta_\sigma} \frac{\partial \eta_\beta}{\partial \xi_\delta} \\
&= \widehat{\omega}_{\alpha\sigma} \delta_{\delta\lambda} \frac{\partial \xi_\lambda}{\partial \eta_\sigma} \frac{\partial \eta_\beta}{\partial \xi_\delta} \\
&= \widehat{\omega}_{\alpha\sigma} \frac{\partial \xi_\delta}{\partial \eta_\sigma} \frac{\partial \eta_\beta}{\partial \xi_\delta} \\
&= \widehat{\omega}_{\alpha\sigma} \frac{\partial \eta_\beta}{\partial \eta_\sigma} \\
&= \widehat{\omega}_{\alpha\sigma} \delta_{\beta\sigma} \\
&= \widehat{\omega}_{\alpha\beta} \tag{12.4.23}
\end{aligned}
$$

where we have used $\widehat{\omega} \cdot \widehat{\omega}^T = \mathbf{1}$. It follows that a restricted canonical transformation, $\xi \to \eta$ preserves the form of $\widehat{\omega}$, i.e.,

$$\widehat{\omega}_{\alpha\beta} = \widehat{\omega}_{\gamma\delta} \frac{\partial \eta_\alpha}{\partial \xi_\gamma} \frac{\partial \eta_\beta}{\partial \xi_\delta} \tag{12.4.24}$$

Obviously the converse also holds true because the steps in the above proof are all reversible.

Let us put the condition in matrix form. Call $\widehat{J}$ the Jacobian matrix,

$$\widehat{J}_{\alpha\beta} = \frac{\partial \eta_\alpha}{\partial \xi_\beta} \tag{12.4.25}$$

then our necessary (and sufficient, since it is trivial to reverse the above steps) condition for the transformation to be canonical is that

$$\widehat{\omega} = \widehat{J} \, \widehat{\omega} \, \widehat{J}^T \tag{12.4.26}$$

where $\widehat{J}^T$ is the transpose of $\widehat{J}$. A restricted canonical transformation preserves $\widehat{\omega}$. In the theory of continuous groups, the matrix $\widehat{\omega}$ is called the **symplectic form** and transformations that preserve the symplectic form are called **symplectic transformations**. Restricted canonical transformations are therefore symplectic transformations and (12.4.26) is called the **symplectic condition**. As an example, the transformation

$$\eta_\alpha = J_{\alpha\beta}\xi_\beta \tag{12.4.27}$$

where $\xi = (q,p)$, $\eta = (Q,P)$ and $\widehat{J}$ is a two dimensional rotation matrix, satisfies the symplectic condition and the transformation is therefore canonical.[5]

We have thus proved: Restricted Canonical Transformations $\Leftrightarrow$ Direct Conditions $\Leftrightarrow$ Symplectic Condition.

12.5 Infinitesimal Transformations

Let us now focus only infinitesimal canonical transformations,

$$\xi_\alpha \to \eta_\alpha(\xi) = \xi_\alpha + \delta\xi_\alpha \tag{12.5.1}$$

i.e.,

$$q_i \to Q_i = q_i + \delta q_i, \quad p_i \to P_i = p_i + \delta p_i. \tag{12.5.2}$$

A suitable generating function for the transformation, which we take for convenience to be of type II, would be of the form

$$F_2 = q_i P_i + \varepsilon \mathcal{G}(q_i, P_i, t) \tag{12.5.3}$$

(we are not restricting ourselves to time independent transformations but only to infinitesimal transformations, and ε is the small parameter of the transformation). From the equations appropriate to type II generating functions, we have

$$\begin{aligned} Q_i &= \frac{\partial F_2}{\partial P_i} = q_i + \varepsilon\frac{\partial \mathcal{G}}{\partial P_i} \\ p_i &= \frac{\partial F_2}{\partial q_i} = P_i + \varepsilon\frac{\partial \mathcal{G}}{\partial q_i} \end{aligned} \tag{12.5.4}$$

If we restrict ourselves to the first order in ε, then we could replace P_i in $\mathcal{G}$ above by p_i because the difference is already of order ε. Then

$$Q_i = q_i + \varepsilon\frac{\partial \mathcal{G}}{\partial p_i}$$

---

[5]Problem: Prove this.

$$P_i = p_i - \varepsilon \frac{\partial \mathcal{G}}{\partial q_i} \tag{12.5.5}$$

Now we will show that *any* infinitesimal transformation (not simply a restricted transformation) obeys the symplectic condition. To do so, it's convenient to return to the notation of the previous section and rewrite these conditions as

$$\eta_\alpha = \xi_\alpha + \varepsilon \, \widehat{\omega}_{\alpha\gamma} \frac{\partial \mathcal{G}}{\partial \xi_\gamma} \tag{12.5.6}$$

The Jacobian matrix of the transformation is then

$$J_{\alpha\beta} = \frac{\partial \eta_\alpha}{\partial \xi_\beta} = \delta_{\alpha\beta} + \varepsilon \, \widehat{\omega}_{\alpha\gamma} \frac{\partial^2 \mathcal{G}}{\partial \xi_\beta \partial \xi_\gamma} \tag{12.5.7}$$

Consider then (upto linear order in ε for consistency)

$$
\begin{aligned}
\widehat{J}\,\widehat{\omega}\,\widehat{J}^T &= J_{\alpha\gamma}\widehat{\omega}_{\gamma\delta}J^T_{\delta\beta} \\
&= \left(\delta_{\alpha\gamma} + \varepsilon\,\widehat{\omega}_{\alpha\kappa}\frac{\partial^2\mathcal{G}}{\partial\xi_\gamma\partial\xi_\kappa}\right)\widehat{\omega}_{\gamma\delta}\left(\delta_{\beta\delta} + \varepsilon\,\widehat{\omega}_{\beta\sigma}\frac{\partial^2\mathcal{G}}{\partial\xi_\delta\partial\xi_\sigma}\right) \\
&= \widehat{\omega}_{\alpha\beta} + \varepsilon\left(\widehat{\omega}_{\alpha\kappa}\widehat{\omega}_{\gamma\beta}\frac{\partial^2\mathcal{G}}{\partial\xi_\gamma\partial\xi_\kappa} + \widehat{\omega}_{\alpha\delta}\widehat{\omega}_{\beta\sigma}\frac{\partial^2\mathcal{G}}{\partial\xi_\delta\partial\xi_\sigma}\right) + \mathcal{O}(\varepsilon^2) \\
&= \widehat{\omega}_{\alpha\beta} + \varepsilon\left(\widehat{\omega}_{\alpha\kappa}\widehat{\omega}_{\gamma\beta}\frac{\partial^2\mathcal{G}}{\partial\xi_\gamma\partial\xi_\kappa} - \widehat{\omega}_{\alpha\delta}\widehat{\omega}_{\sigma\beta}\frac{\partial^2\mathcal{G}}{\partial\xi_\sigma\partial\xi_\delta}\right) + \mathcal{O}(\varepsilon^2) \\
&\approx \widehat{\omega}_{\alpha\beta} + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{12.5.8}
$$

where we have made use of the antisymmetry of $\widehat{\omega}$ (*i.e.*, $\widehat{\omega}_{\beta\sigma} = -\widehat{\omega}_{\sigma\beta}$) in the last step. We have shown that the symplectic condition is necessary for a *general*, infinitesimal canonical transformation. To show that it is sufficient we begin with the symplectic condition

$$\widehat{\omega}_{\alpha\beta} = J_{\alpha\gamma}\widehat{\omega}_{\gamma\delta}J^T_{\delta\beta} \tag{12.5.9}$$

and prove that the infinitesimal transformation is canonical. So let

$$\eta_\alpha = \xi_\alpha + \varepsilon \, \chi_\alpha \tag{12.5.10}$$

and consider $\{\eta_\alpha, \eta_\beta\}$ upto $\mathcal{O}(\varepsilon)$. We have

$$
\begin{aligned}
\{\eta_\alpha, \eta_\beta\} &= \{\xi_\alpha + \varepsilon\chi_\alpha, \xi_\beta + \varepsilon\chi_\beta\} = \widehat{\omega}_{\alpha\beta} + \varepsilon(\{\xi_\alpha, \chi_\beta\} + \{\chi_\alpha, \xi_\beta\}) \\
&= \widehat{\omega}_{\alpha\beta} + \varepsilon\left(\widehat{\omega}_{\alpha\gamma}\frac{\partial\chi_\beta}{\partial\xi_\gamma} - \widehat{\omega}_{\beta\gamma}\frac{\partial\chi_\alpha}{\partial\xi_\gamma}\right) + \mathcal{O}(\varepsilon^2)
\end{aligned}
\tag{12.5.11}
$$

However, for an infinitesimal transformation of the form (12.5.10),

$$J_{\alpha\beta} = \delta_{\alpha\beta} + \varepsilon \frac{\partial\chi_\alpha}{\partial\xi_\beta} \tag{12.5.12}$$

so the symplectic condition implies that

$$\widehat{\omega}_{\alpha\beta} = \left(\delta_{\alpha\gamma} + \varepsilon \, \frac{\partial \chi_\alpha}{\partial \xi_\gamma} \right) \widehat{\omega}_{\gamma\delta} \left(\delta_{\beta\delta} + \varepsilon \, \frac{\partial \chi_\beta}{\partial \xi_\delta} \right) \tag{12.5.13}$$

which requires that

$$\frac{\partial \chi_\alpha}{\partial \xi_\gamma} \widehat{\omega}_{\gamma\beta} + \widehat{\omega}_{\alpha\delta} \frac{\partial \chi_\beta}{\partial \xi_\delta} = 0 \tag{12.5.14}$$

or, using the antisymmetry of $\widehat{\omega}$, and relabeling the indices

$$\widehat{\omega}_{\alpha\gamma} \frac{\partial \chi_\beta}{\partial \xi_\gamma} - \widehat{\omega}_{\beta\gamma} \frac{\partial \chi_\alpha}{\partial \xi_\gamma} = 0 \tag{12.5.15}$$

It follows that $\{\eta_\alpha, \eta_\beta\} = \widehat{\omega}_{\alpha\beta}$ *i.e.*, the transformation is canonical.

We have thus proved: General Infinitesimal Canonical Transformations $\Leftrightarrow$ Symplectic Condition. It follows that every finite transformation obtained by exponentiating an infinitesimal canonical transformation (*i.e.*, a finite transformation which is connected to the identity) is a canonical transformation.

12.6   Hamiltonian as the generator of time translations

We have just seen that an infinitesimal canonical transformation can always be written in the form

$$q_i \to Q_i = q_i + \delta q_i, \quad \delta q_i = \varepsilon \{q_i, \mathcal{G}\}$$

$$p_i \to P_i = p_i + \delta p_i, \quad \delta p_i = \varepsilon \{p_i, \mathcal{G}\} \tag{12.6.1}$$

where $\mathcal{G}$ is the generator of the transformation and ε is an infinitesimal parameter. Now $\mathcal{G}(q_i, p_i, t)$ is just about *any* function of phase space and time. In particular, suppose $\mathcal{H}(q_i, p_i, t)$ is the Hamiltonian and $\varepsilon = \delta t$, an infinitesimal shift in time. We then have the canonical transformation

$$\delta q_i = \delta t \{q_i, \mathcal{H}\}$$

$$\delta p_i = \delta t \{p_i, \mathcal{H}\} \tag{12.6.2}$$

These are just the Hamiltonian equations of motion, and we have arrived at an alternative point of view: time evolution is just the continuous unfolding of a canonical transformation generated by the Hamiltonian. In other words, the configuration of the system (coordinates and momenta) at any time, t, can be obtained by a series of infinitesimal

canonical transformations, which is equivalent to a single finite transformation connected to the identity. Conversely, there is a finite transformation connected to the identity that takes us from any configuration at t to the initial configuration at t_0. This prompts us to seek a canonical transformation to new coordinates, Q_i, and momenta, P_i, that are constants of the motion. This is the subject of the following chapter.

Chapter 13

Hamilton-Jacobi Theory

We have seen that an important objective in studying canonical transformations is the possible simplification of a mechanical problem. We also ended the last chapter showing that the Hamiltonian is the generator of time translations and suggested that therefore it might be possible to find a canonical transformation that takes us from the configuration at any time "t" to the configuration at the initial time, "t_0".This transformation would yield $Q_i = \alpha_i$, $P_i = \beta_i$, where α_i and β_i are the initial conditions that we must in any case impose upon the system. If it were possible to determine the transformation then by inversion we would obtain the solution of the problem at any future time. This is tantamount to solving the problem so, by conservation of difficulty, we expect that it will be no easier a task than solving the system by any of the other means already at our disposal.

13.1   The Hamilton-Jacobi equation

Suppose that a transformation, generated by F, does precisely what we want then the Hamiltonian, $\mathcal{H}$, is transformed into $\mathcal{K}$ according to

$$\mathcal{K} = \mathcal{H} + \frac{\partial \mathcal{F}}{\partial t}. \tag{13.1.1}$$

But because the new coordinates and momenta are constants of the motion, we must have

$$\dot{Q}_i = \frac{\partial \mathcal{K}}{\partial P_i} = 0$$

$$\dot{P}_i = -\frac{\partial \mathcal{K}}{\partial Q_i} = 0 \tag{13.1.2}$$

which implies that $\mathcal{K} = \mathcal{K}(t)$ is a function at most of time. Now of course this dependence on time can be removed by simply defining

$$F' = F - \int \mathcal{K}(t)dt \tag{13.1.3}$$

which does not change the equations of motion, so in all generality we can take $\mathcal{K} \equiv 0$. Now suppose we express F in the form

$$F = F_2(q_i, P_i, t) - Q_i P_i \tag{13.1.4}$$

so that

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$Q_i = \frac{\partial F_2}{\partial P_i} \tag{13.1.5}$$

and

$$\mathcal{K} = \mathcal{H}\left(q_i, \frac{\partial F_2}{\partial q_i}, t\right) + \frac{\partial F_2}{\partial t} \equiv 0 \tag{13.1.6}$$

This can be viewed as an equation for the generating function F_2. It is a partial differential equation in $(n + 1)$ variables, called the "**Hamilton-Jacobi**" equation. Its solution is called "**Hamilton's principal function**" and is generally denoted by "S" (we shall use S for F_2 from now on). If S can be found we obtain $S = S(q_i, t)$, which has no apparent dependence on the new momenta! This is not true, of course, because the integration involves $(n + 1)$ constants, n of which can be associated with the P_i. Indeed

$$S = S(q_i, \beta_i, \beta_{n+1}, t) \tag{13.1.7}$$

What about the last constant, β_{n+1}. How do we know which of the $(n + 1)$ constants are related to the new momenta? This is easy to resolve because the Hamilton-Jacobi equation involves only derivatives of S,

$$\mathcal{H}\left(q_i, \frac{\partial S}{\partial q_i}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{13.1.8}$$

so that $S' = S + \beta$ is a solution if S is solution and β is a constant. Thus, one of the constants, call it β_{n+1} is additive and we write

$$S = S(q_i, \beta_i, t) + \beta_{n+1} \tag{13.1.9}$$

Of course, β_{n+1} is completely irrelevant because only derivatives of the generating functional are physically meaningful. So we could take

$$S = S(q_i, \beta_i; t) \tag{13.1.10}$$

and call $\beta_i = P_i$. This choice is completely arbitrary. One could just as well choose $P_i = \gamma_i(\beta_1, ...\beta_n)$ for the momenta and everything that follows would proceed in the same way. All that is important is that the P_i are constants. Then

$$Q_i = \frac{\partial S}{\partial \beta_i} \qquad (13.1.11)$$

are the new coordinates. But they too are constants of the motion, so

$$Q_i = \alpha_i = \frac{\partial S}{\partial \beta_i}, \qquad (13.1.12)$$

which equations can be integrated to obtain $q_i = q_i(\alpha_i, \beta_i, t)$.

Consider

$$\frac{dS}{dt} = \frac{\partial S}{\partial q_i}\dot{q}_i + \frac{\partial S}{\partial t} \qquad (13.1.13)$$

and note that

$$p_i = \frac{\partial S}{\partial q_i} \qquad (13.1.14)$$

while

$$\mathcal{H} = \mathcal{K} - \frac{\partial S}{\partial t} = -\frac{\partial S}{\partial t} \qquad (13.1.15)$$

so

$$S = \int dt[p_i\dot{q}_i - \mathcal{H}] + \text{const.} \qquad (13.1.16)$$

We have just shown that Hamilton's principal function differs from the indefinite integral of the Lagrangian by at most a constant. Now the Lagrangian approach is a statement about the definite integral leading to the Euler-Lagrange equations of motion. Likewise, the Hamilton-Jacobi theory is a statement that the indefinite integral of the Lagrangian leads to the Hamilton-Jacobi equation.

13.2   Two examples

Consider the free particle in one dimension. We choose this example as the simplest, most transparent example of the application of the Hamilton-Jacobi equation. Later we'll work more complicated examples. The Hamiltonian is just

$$\mathcal{H} = \frac{p^2}{2m} \qquad (13.2.1)$$

so the Hamilton Jacobi equation is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{\partial S}{\partial t} = 0 \qquad (13.2.2)$$

and we look for a solution of the form

$$S = S_0(q) + \beta t. \tag{13.2.3}$$

Inserting this ansatz into the Hamilton Jacobi equation gives

$$\frac{\partial S_0}{\partial q} = \pm\sqrt{2m\beta} \tag{13.2.4}$$

so the solution is

$$S(q) = \pm\sqrt{2m\beta}\,q - \beta t + \gamma \tag{13.2.5}$$

where γ is the (irrelevant) additive constant we mentioned before. We associate β with the new momenta P, and

$$Q = \alpha = \frac{\partial S}{\partial \beta} = \sqrt{\frac{m}{2\beta}}\,q - t \tag{13.2.6}$$

which can be inverted to give

$$q = \sqrt{\frac{2\beta}{m}}(t + \alpha) \tag{13.2.7}$$

Again,

$$p = \frac{\partial S}{\partial q} = \sqrt{2m\beta} \tag{13.2.8}$$

is a constant (the initial momentum p_0) and $\beta = p_0^2/2m = E$. On the other hand,

$$q = \frac{p_0}{m}(t + \alpha) \Rightarrow q_0 = \sqrt{\frac{2\beta}{m}}\,\alpha \Rightarrow \alpha = \frac{mq_0}{p_0} \tag{13.2.9}$$

Note that β is identified with the total energy of the system.

   The one dimensional simple Harmonic oscillator provides another transparent example of the Hamilton-Jacobi approach. In this case,

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kq^2 \tag{13.2.10}$$

and the Hamilton-Jacobi equation becomes

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}m\omega^2 q^2 + \frac{\partial S}{\partial t} = 0 \tag{13.2.11}$$

Again, seek a solution of the form $S = S_0(q) - \beta t$ so that

$$\frac{dS_0}{dq} = \pm\sqrt{2m\beta}\sqrt{1 - \frac{m\omega^2 q^2}{2\beta}} \tag{13.2.12}$$

and

$$S_0(q) = \pm\sqrt{2m\beta} \int dq \sqrt{1 - \frac{m\omega^2 q^2}{2\beta}} + \gamma \tag{13.2.13}$$

where γ is the additive constant we expect. Setting it to zero and soling the integral gives

$$S(q) = \sqrt{m\beta} \left[\frac{q}{\sqrt{2}} \sqrt{1 - \frac{m\omega^2 q^2}{2\beta}} + \sqrt{\frac{\beta}{m\omega^2}} \sin^{-1} \sqrt{\frac{m\omega^2}{2\beta}} q - \beta t \right] \tag{13.2.14}$$

Now $P = \beta$ and

$$Q = \alpha = \frac{\partial S}{\partial \beta} = -t + \frac{1}{\omega} \sin^{-1} \sqrt{\frac{m\omega^2}{2\beta}} q \tag{13.2.15}$$

Therefore

$$q = \sqrt{\frac{2\beta}{m\omega^2}} \sin(\omega t + \phi) \tag{13.2.16}$$

where $\phi = \alpha\omega$. Likewise

$$p = \frac{\partial S}{\partial q} = \sqrt{2m\beta} \sqrt{1 - \frac{m\omega^2 q^2}{2\beta}} = \sqrt{2m\beta} \cos(\omega t + \phi) = m\dot{q} \tag{13.2.17}$$

Finally, what is the connection between the constants (α, β) and the initial conditions (q_0, p_0)? we have

$$q_0 = \sqrt{\frac{2\beta}{m\omega^2}} \sin\phi$$

$$p_0 = \sqrt{2m\beta} \cos\phi \tag{13.2.18}$$

Therefore

$$\alpha = \frac{1}{\omega} \tan^{-1} \frac{m\omega q_0}{p_0} \tag{13.2.19}$$

and

$$\beta = \frac{p_0^2}{2m} + \frac{1}{2} m\omega^2 q_0^2 = E \tag{13.2.20}$$

As before, we see that β is identified with the total energy of the system.

13.3   Hamilton's Characteristic Function

In the previous examples we separated Hamilton's principal function into two parts, the first depending only on the coordinates and the second on time. This is always possible when the Hamiltonian is does not explicitly depend on time. To see that this is so, we the principal function as

$$S = W(q_i, \beta_i) - \beta_n t \tag{13.3.1}$$

and plug it into the Hamilton-Jacobi equation to get

$$\mathcal{H}\left(q_i, \frac{\partial S}{\partial q_i}\right) = \beta_n \tag{13.3.2}$$

which certainly admits time-independent solutions. $W(q_i, \beta_i, t)$ is called "**Hamilton's Characteristic Function**" and, moreover, β_n is the total energy of the system (recall that $d\mathcal{H}/dt = \partial\mathcal{H}/\partial t \equiv 0$). The solution is of the form

$$W = W(q_1, ...q_n; \beta_1, ...\beta_{n-1}) - \beta_n \tag{13.3.3}$$

and the equations of the canonical transformation are

$$p_i = \frac{\partial W}{\partial q_i}, \quad Q_i = \alpha_i = \frac{\partial W}{\partial \beta_i} - t\delta_{i,n}. \tag{13.3.4}$$

We can ascribe a meaning to the characteristic function as well, just as we had for the principal function:

$$\frac{dW}{dt} = \frac{\partial W}{\partial q_i}\dot{q}_i + \frac{\partial W}{\partial t} = p_i\dot{q}_i \tag{13.3.5}$$

Thus

$$W = \int p_i dq_i \tag{13.3.6}$$

Even though we have yet again reformulated the dynamical problem in terms of a single non-linear partial differential equation in n varaibles as opposed to the n ordinary second order equations of the Lagrangian theory or the $2n$ first order equations of the Hamiltonian theory, the Hamilton-Jacobi theory does not necessarily provide any simplificaton over the other two approaches. However, in some problems, the Hamilton-Jacobi theory can be useful and we will now examine two classes of problems in which it can be directly integrated.

13.4   Separability

A coordinate q_i is **separable** if and only if the priciple function can be split into two parts, one depending on q_i annd the other depending on all the coordinates *except q_i, i.e.,*

$$S = S(q_i, \beta_1, ... \beta_n; t) + S'(q_1, ..., \widehat{q_i}, ..., q_n, \beta_1, ..., \beta_n; t) \tag{13.4.1}$$

(where the caret signifies that q_i does not appear in S') **and** the Hamilton-Jacobi equation also splits into two parts, one for q_i and the other for the remaining coordinates,

$$\mathcal{H}_i \left(q_i, \frac{\partial S}{\partial q_i}, \beta_1, ..., \beta_n, t \right) + \frac{\partial S}{\partial t} = 0$$

$$\mathcal{H}' \left(q_1, ..., \widehat{q_i}, ... q_n, \frac{\partial S'}{\partial q_k}, ..., \frac{\partial S'}{\partial q_n}, \beta_1, ..., \beta_n, t \right) + \frac{\partial S'}{\partial t} = 0 \tag{13.4.2}$$

If *all* the coordinates are separable then

$$S = \sum_i S_i(q_i, \beta_1, ..., \beta_n; t) \tag{13.4.3}$$

where S_i dependes only on q_i **and**

$$\mathcal{H} \left(q_1, ..., q_n, \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_n}, t \right) = \sum_i \mathcal{H}_i \left(q_i, \frac{\partial S}{\partial q_i}, \beta_1, ... \beta_n; t \right) \tag{13.4.4}$$

When this happens, and if the Hamiltonian does not depend explicityly on time, the problem can be solved systematically in the following way. Let

$$S_i = W_i(q_i, \beta_1, ..., \beta_{n-1}) - \beta_n t \tag{13.4.5}$$

then

$$\mathcal{H} \left(q_1, ..., q_n, \frac{\partial S}{\partial q_1}, ..., \frac{\partial S}{\partial q_1} \right) = \sum_i \beta_i \tag{13.4.6}$$

and because $\mathcal{H}$ splits according to our assumptions,

$$\mathcal{H}_i \left(q_i, \frac{\partial W_i}{\partial q_i}, \beta_1, ..., \beta_n \right) = \beta_i \tag{13.4.7}$$

This is a set of uncoupled, ordinary differential equations for the W_i, which can be solved exactly in principle. The separability of the Hamilton-Jacobi equations is both a property of the dynamical system as well as the coordinates in which it is described. No simple criterion can be given to indicate what coordinate system leads to a separable Hamilton-Jacobi equation for a given system. However, if

- the Hamiltonian is conserved and

- takes the form

$$\mathcal{H} = \frac{1}{2}(p_i - a_i)T_{ij}^{-1}(p_j - a_j) + V(q)$$

  and

- the set of coordinates form an orthogonal system, so that $\widehat{T}$ is diagonal

then Hamilton's characteristic function is completely separable if and only if

- Each a_i is a function only of q_i,

- The potential function is a sum of potential functions of the form $V_i(q_i)$

- there exists an $n \times n$ matrix Φ with elements $\Phi_{ij}(q_i)$ such that

$$\Phi_{1j}^{-1} = \frac{1}{T_{jj}}.$$

These are the "**Staeckel**" conditions for separability.

13.5   Periodic motion and Action-Angle Variables

A very important class of physical problems is one in which the system is periodic. Consider a general, periodic, conservative system in one dimension. Since $\mathcal{H}$ does not depend explicitly pn time, we will make use of the characteristic function W, and the Hamilton-Jacobi equation reads

$$\mathcal{H}\left(q, \frac{\partial W}{\partial q}\right) = \beta_1 \tag{13.5.1}$$

Now there are two kinds of periodicities that we will consider:

- Motion in which a subset of the phase space variables, some coordinates together with their conjugate momenta, return to their initial values after fixed time intervals, τ. If all the phase-space variables do this, the phase-space diagram for the system is a compact surface in phase-space and the entire state of the system is periodic, as shown in figure (13.1). An obvious example of this kind of motion is the Harmonic oscillator.

- A more limited situation would arise if only the momenta (or a subset of them, instead of both the coordinates and their conjugate momenta) are periodic as shown in figure (13.2). In the figure the momentum returns to its original value after every coordinate interval Δq. This is typical of a compact coordinate. The rotation of a rigid body is a good example of this type of periodicity and it is often referred to simply as "rotation".
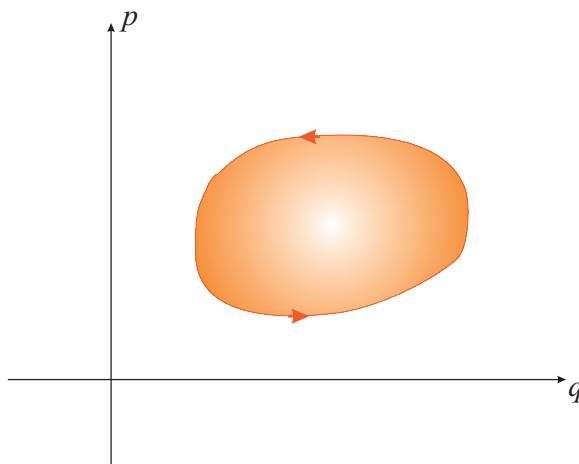
Figure 13.1: The phase space occupied by the system is a compact surface
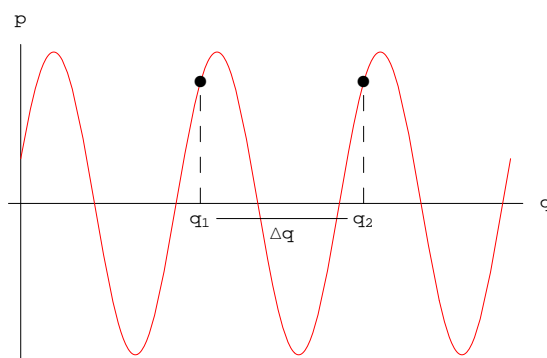
Figure 13.2: The phase space occupied by the system is not compact

Both kinds of periodicities can occur in the same system under different initial conditions. For example, take the pendulum. For smaller energies, the motion is periodic in the first sense. However, if the pendulum's energy is increased beyond $2mgl$, where l is its length and m the mass of the bob, then rotational motion will occur.

For either type of motion, introduce the "action" variable

$$J = \oint p dq \tag{13.5.2}$$

where the interval is to be carried out over one full period. From the Hamilton-Jacobi equation we determine $W = W(q, \beta_1)$ and

$$p = \frac{\partial W}{\partial q} = p(q, \beta_1) \tag{13.5.3}$$

so we find

$$J = \oint p(q, \beta_1) dq = J(\beta_1) \tag{13.5.4}$$

i.e., the variable J depends on β_1 alone and the equation can be inverted to give $\beta_1 = \beta_1(J)$. This can be reinserted into W to obtain W as a function of (q, J), *i.e.*, $W = W(q, J)$. Now introduce the "angle" variable

$$\phi(q, J) = \frac{\partial W}{\partial J} \tag{13.5.5}$$

and consider the principal function $S = W(q, J) - \beta_1(J)t$. We find the coordinate conjugate to J

$$Q_J = \frac{\partial S}{\partial J} = \phi - \frac{\partial \beta_1}{\partial J} t \tag{13.5.6}$$

Now, by construction, $\dot{Q}_J \equiv 0$, so that

$$\dot{\phi} = \frac{\partial \beta_1}{\partial J} = \nu(J) \tag{13.5.7}$$

is constant and depends on J alone. Thus $\phi = \nu t + \phi_0$ and the defining equation (13.5.5) can be inverted to recover $q = q(\phi, J)$, which of course is the desired solution.

In and of themselves, the variables ϕ and J give no particular advantage in solving a given problem. They do not generally make the problem more tractable. Their merit lies in the physical interpretation that can be given to them. Consider the change in ϕ as q goes through a complete period: because

$$\delta\phi = \frac{\partial \phi}{\partial q} \delta q \tag{13.5.8}$$

(J is a constant of the motion) it follows that over an entire period

$$\Delta\phi = \oint \frac{\partial\phi}{\partial q}dq = \oint \frac{\partial^2 W}{\partial J\partial q}dq = \frac{d}{dJ}\oint \frac{\partial W}{\partial q}dq \equiv 1 \tag{13.5.9}$$

This means that ν is the frequency of the motion. Moreover ν is given by

$$\nu = \frac{d\beta_1}{dJ} = \frac{d\mathcal{H}}{dJ} \tag{13.5.10}$$

so we can determine the frequency of the motion *without* actually solving the problem, if it is known *a priori* that the motion is periodic.

For an example, the Hamilton-Jacobi equation for the one dimensional Harmonic oscillator,

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{1}{2}kq^2 = \beta \tag{13.5.11}$$

implies that

$$p = \sqrt{2m\beta}\sqrt{1 - \frac{kq^2}{2\beta}} \tag{13.5.12}$$

which gives the action variable

$$J = \sqrt{2m\beta}\oint dq\sqrt{1 - \frac{kq^2}{2\beta}} = 2\pi\beta\sqrt{\frac{m}{k}} \tag{13.5.13}$$

Thus

$$\beta = \mathcal{H} = \frac{J}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{13.5.14}$$

To completely solve the problem (*i.e.,* to determine $\phi(q, J)$) we would have to solve the Hamilton-Jacobi equation to obtain $W(q, J)$ but, as we see, it is not necessary to do so if our interest is simply to determine the frequency.

13.6 Further Examples

Consider the simple projectile, described by the Hamiltonian

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy \tag{13.6.1}$$

where y is the height measured, say, from the surface of the earth. Because the Hamilton is time independent, choose $S = W(x, y) - \beta_2 t$, then the equation for the characteristic function is

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + mgy = \beta_2 \tag{13.6.2}$$

Let us now see if we can use the separation of variables, writing $W = W_x(x) + W_y(y)$, so that

$$\frac{1}{2m}\left[\left(\frac{dW_x}{dx}\right)^2 + \left(\frac{dW_y}{dy}\right)^2\right] + mgy = \beta_2 \tag{13.6.3}$$

Therefore,

$$\frac{1}{2m}\left(\frac{dW_y}{dy}\right)^2 + mgy = \beta_2 - \frac{1}{2m}\left(\frac{dW_x}{dx}\right)^2 = \beta_1 \tag{13.6.4}$$

because the left hand side depends only on x and the right hand side depends only on y implies that they can only be each equal to a constant. The x equation is directly integrated to give

$$W_x(x) = \sqrt{2m(\beta_2 - \beta_1)}\, x + \text{const.} \tag{13.6.5}$$

The y equation gives

$$W_y(y) = \sqrt{2m\beta_1}\int dy\sqrt{1 - \frac{mgy}{\beta_1}} = -\frac{2}{3}\sqrt{2m\beta_1}\frac{\beta_1}{mg}\left(1 - \frac{mgy}{\beta_1}\right)^{3/2} + \text{const.} \tag{13.6.6}$$

Our separated solution therefore reads

$$S(x, y, t) = \sqrt{2m(\beta_2 - \beta_1)}\, x - \frac{2}{3}\sqrt{2m\beta_1}\frac{\beta_1}{mg}\left(1 - \frac{mgy}{\beta_1}\right)^{3/2} - \beta_2 t + \gamma \tag{13.6.7}$$

where γ will be recognized as our trivial additive constant. We can now determine $x(t)$ and $y(t)$ of course by simply using the canonical equations

$$Q_1 = \alpha_1 = \frac{\partial S}{\partial \beta_1}$$

$$Q_2 = \alpha_2 = \frac{\partial S}{\partial \beta_2} \tag{13.6.8}$$

The second equation gives the solution for $x = x(t)$

$$x(t) = \sqrt{\frac{2}{m}(\beta_2 - \beta_1)}(t + \alpha_2) \tag{13.6.9}$$

and when this solution is inserted into the first equation, we find after a little algebra that

$$y(t) = \frac{\beta_1}{mg}\left[1 - \frac{mg^2}{2\beta_1}(t + \alpha_1 + \alpha_2)^2\right]. \tag{13.6.10}$$

We already know that the constant β_2 represents the total energy of the system, so $\beta_2 = E = \frac{1}{2}mv_0^2$. To extract the meaning of the constants we must apply some initial conditions: take

$$\vec{r}_0 = 0, \quad \vec{v}_0 = v_0(\cos\theta, \sin\theta) \tag{13.6.11}$$

then

$$\frac{2(\beta_2 - \beta_1)}{m}\alpha_2^2 = 0$$

$$\frac{\beta_1}{mg}\left(1 - \frac{mg^2}{2\beta_1}(\alpha_1 + \alpha_2)^2\right) = 0$$

$$\sqrt{\frac{2(\beta_2 - \beta_1)}{m}} = v_0 \cos\theta$$

$$g(\alpha_1 + \alpha_2) = -v_0 \sin\theta \qquad (13.6.12)$$

Together, the first and the third imply that $\alpha_2 = 0$. The last gives

$$\alpha_1 = -\frac{v_0 \sin\theta}{g} \qquad (13.6.13)$$

and the second requires that

$$\beta_1 = \frac{1}{2}mv_0^2 \sin^2\theta \qquad (13.6.14)$$

In fact, with this value of β_1, the third equation would then give precisely $\beta_2 = E$ as expected. Therefore, our solution reads

$$x(t) \;=\; v_0 \cos\theta\; t$$

$$y(t) \;=\; v_0 \sin\theta\; t - \frac{1}{2}gt^2 \qquad (13.6.15)$$

where we get the final answer after inserting the values of the constants above.

Our second example is the central force problem discussed at length in a previous chapter. The Hamiltonian for this problem is

$$\mathcal{H} = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2\theta}\right) + \phi(r) \qquad (13.6.16)$$

where $\phi(r)$ is the potential energy. As before, we choose $S = W(r, \theta, \varphi) - \beta_3 t$ so that the equation for the characteristic function becomes

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial W}{\partial \varphi}\right)^2\right] + \phi(r) = \beta_3 \qquad (13.6.17)$$

and we seek separable solutions, *i.e.*, we take $W(r, \theta, \varphi) = W_r(r) + W_\theta(\theta) + W_\varphi(\varphi)$. Then

$$\left(\frac{dW_r}{dr}\right)^2 + \frac{1}{r^2}\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{dW_\varphi}{d\varphi}\right)^2 + 2m\phi(r) = 2m\beta_3 \qquad (13.6.18)$$

which equation we will separate as follows, writing

$$r^2 \left[\left(\frac{dW_r}{dr} \right)^2 + 2m\phi(r) - 2m\beta_3 \right] = -\left(\frac{dW_\theta}{d\theta} \right)^2 - \frac{1}{\sin^2\theta} \left(\frac{dW_\varphi}{d\varphi} \right)^2 = -\beta_2 \qquad (13.6.19)$$

where we used the fact that the left hand side is a function only of r and the right is a function only of (θ, ϕ). Thus we get the radial equation

$$\left(\frac{dW_r}{dr} \right)^2 + \frac{\beta_2}{r^2} + 2m\phi(r) = 2m\beta_3 \qquad (13.6.20)$$

The remaining equation can be expressed as

$$\left(\frac{dW_\varphi}{d\varphi} \right)^2 = \beta_1$$

$$\left(\frac{dW_\theta}{d\theta} \right)^2 + \frac{\beta_1}{\sin^2\theta} = \beta_2 \qquad (13.6.21)$$

where β_1 is another constant. We will seek a solution that lies wholly in the equatorial plane, i.e., having $W_\theta =$const., $\theta = \pi/2$ and $\beta_1 = \beta_2$. Then

$$W_\varphi = \pm\sqrt{\beta_2} \, \phi \, + \text{const.} \qquad (13.6.22)$$

and our radial equation is solved by

$$W_r = \pm\sqrt{2m} \int dr \sqrt{\beta_3 - \frac{\beta_2}{2mr^2} - \phi(r)} + \text{const.}, \qquad (13.6.23)$$

giving Hamilton's principal function,

$$S(r, \varphi, t) = \sqrt{2m} \int dr \sqrt{\beta_3 - \frac{\beta_2}{2mr^2} - \phi(r)} + \sqrt{\beta_2} \, \phi \, - \beta_3 t + \text{const.}, \qquad (13.6.24)$$

where we have chosen the positive sign. Thus we find

$$\alpha_3 = \frac{\partial S}{\partial \beta_3} = \sqrt{\frac{m}{2}} \int \frac{dr}{\sqrt{\beta_3 - \frac{\beta_2}{2mr^2} - \phi(r)}} - t$$

$$\alpha_2 = -\frac{1}{2\sqrt{2m}} \int \frac{dr/r^2}{\sqrt{\beta_3 - \frac{\beta_2}{2mr^2} - \phi(r)}} + \frac{\varphi}{2\sqrt{\beta_2}} \qquad (13.6.25)$$

If we let $\alpha_2 = \varphi_0/(2\sqrt{\beta_2})$, then the second equation above reads

$$\sqrt{\frac{2m}{\beta_2}}(\varphi - \varphi_0) = \int \frac{dr/r^2}{\sqrt{\beta_3 - \frac{\beta_2}{2mr^2} - \phi(r)}} \qquad (13.6.26)$$

which compares favorably with the second equation in (5.2.24) if β_2 is identified with the square of the angular momentum, $\beta_2 = L^2$. Furthermore, using the fact that β_3 is just the total energy, E, and rearranging terms we find that the first equation of (13.6.25) is

$$\int \frac{dr}{\sqrt{\frac{2}{m}\left[E - V(r)\right]}} = t + \alpha_3 \tag{13.6.27}$$

where $V(r)$ is the effective potential used in (5.2.17). This is then precisely the first equation in (5.2.24) and so we have recovered the equations of motion for the central force problem.

Chapter 14

Special Relativity

The goal of any relativity theory is to relate measurements performed in one inertial frame of reference to those performed in any other. Galilean relativity is based on the concepts of an universal or "absolute" space and an "absolute" time, by which is meant that measurements of spatial distances and time intervals are observer (or frame) independent provided that the spatial distances are measured by a simultaneous measurements of the endpoints. Toward the end of the nineteenth century, however, Maxwell's formulation of electromagnetism, which was completed in 1865, had exposed certain fundamental inconsistencies between the new and extremely successful electromagnetic theory and the Galilean conception of space and time. Today we know that Galilean transformations cease to yield results that agree with experiment when the relative velocity of the two frames being compared is a significant fraction of the speed of light.

In 1887, A. Michelson and E. Morley were able to provide convincing evidence, by means of a very clever and now famous experiment named after them, that the speed of light is the same in all directions and that light does not require a medium in which to travel. At the time their experiment was performed such a medium was assumed to exist because electromagnetic waves were not considered to be different from other well-known mechanical waves (eg. sound) and all mechanical waves were known to require a medium in which to propagate. The putative medium in which light traveled was dubbed the **luminiferous aether** and was thought to pervade all of space. When wave propagation occurs in a medium, the frame that is at rest relative to it assumes a special place in the theory and the "speed" of the wave is its speed as measured in this frame. Thus, the speed of sound in air at STP is approximately $c = 343$ m/s in the frame of the air. An inertial observer moving relative to the medium with velocity $\vec{v}$ in the direction of the wave propagation or opposite it would observe that the speed of the wave is $c \mp v$, in accordance with the principles of Galilean relativity. This change in speed is responsible for the Doppler effect for moving observers. The Earth travels at around 30 km/s around

the sun, so its velocity relative to the aether would vary over the course of a year. The Michelson and Morley experiment was designed to measure this relative velocity, but the results were null and the speed of light was found to be the same for propagation in all directions relative to the observer, indicating that the aether was absent. If no such medium exists then the wave speed could be *the same for all inertial observers*, in other words, a universal constant of nature. This agreed with Maxwell's theory of electromagnetism, by which electromagnetic waves propagate in a vacuum at a speed that depends only on fundamental constants, *viz.,* the permitivity and permeability of space. Motivated by the Michelson-Morley result and by Maxwell's theory, A. Einstein recognized in 1905 that the failure of Galilean relativity at high relative velocities is a consequence of the breakdown of the concepts of "absolute" space and "absolute" time mentioned above. When they are abandoned and replaced by the experimental requirement that the speed of light is the same *in all inertial frames*, we arrive at a dramatically new conception of space and time and therefore of mechanics as well. This modification is known as Einstein's "special" theory of relativity, or simply **Special Relativity** and is the topic of this chapter.

We introduce Einstein's theory in this chapter. We will not dwell much on the questions and experiments that led up to it, neither shall we concern ourselves too much with the apparent paradoxes (there are many, all of them safely resolved). It is assumed that the reader has had some exposure to the topic, so we rather concentrate on the precise mathematical formulation of the theory and on setting up a framework that will be useful for the objectives of these notes.

14.1   The Principle of Covariance

It is a general principle that the laws of physics must be the same in all inertial frames. If this were not true, there would be no way to compare the measurements of one (inertial) observer with those of any other.

Mathematically, the fundamental laws of physics would be same in all inertial frames of reference if the equations describing them have the same form in all inertial frames, that is, if the set of transformations that relate one inertial frame to another would, when applied to the two sides of any fundamental equation of physics, transform each side in precisely the same way as the other. This is the principle of **covariance** and equations that have this property are said to be **covariant**. To further elaborate on this idea, we recall that the transformations that relate two inertial frames will in turn determine the transformation properties of physical quantities such as velocity, acceleration, etc. If they leave a physical quantity the same in every inertial frame then that quantity is an **invariant** or a **scalar**. Other quantities may not remain invariant but they will transform in a prescribed way. Covariance requires that both sides of the fundamental equations

must have the same transformation properties. Thus a scalar quantity can only be related to another scalar quantity, a vector to a vector and so on.

We will see below that Newton's laws are covariant under Galilean transformations but Maxwell's equations are not. This signals an incompatibility between mechanics and electromagnetism, and incompatibilities always indicate that modifications to one or both theories are required at a fundamental level. While it is possible that both theories are wrong, it is more fruitful at first to accept one as correct and modify the other so to make *its* equations covariant under the transformations that are compatible with the first. Given the fundamental agreement between the predictions of electromagnetism and the experiment of Michaelson and Morley, Einstein chose the transformations that preserve the form of Maxwell's equations over the Galilean transformations of Newtonian mechanics. The result is a new formulation of classical mechanics that accounts for the fact that the speed of light is a finite and a universal constant of nature. In the end, of course, only experiment can decide which theory is correct and, indeed, in the years that followed Einstein's 1905 paper, it has resoundingly confirmed his choice.

14.1.1   Galilean tranformations

We are familiar with Galilean relativity, which we may conveniently think of as two sets of transformations *viz.,* the "boosts"

$$\vec{r} \to \vec{r}' = \vec{r} - \vec{v}t, \;\; t \to t' = t \tag{14.1.1}$$

(provided that the frames are coincident at $t = 0$) and spatial rotations

$$\vec{r} \to \vec{r}' = \widehat{R}\,\vec{r}, \;\; t \to t' = t \tag{14.1.2}$$

where $\widehat{R}$ is a rotation matrix (see figure 2). The second of (14.1.1) expresses the absoluteness of time intervals, as $dt' = dt$ is the same for all inertial observers. To see that spatial intervals are also absolute one must remember that the measurement of a distance involves a *simultaneous* measurement of the endpoints and therefore one has

$$|d\vec{r}'|_{dt'=0} = |d\vec{r} - \vec{v}dt|_{dt'=dt=0} = |d\vec{r}|. \tag{14.1.3}$$

Consider a single particle within a collection of N particles with interactions between them. If we label the particles by integers, Newton's equations describing the evolution of a single particle, say particle n, may be written as,

$$m_n \frac{d^2\vec{r}_n}{dt^2} = \vec{F}_n^{ext} + \vec{F}_n^{int} = \vec{F}_n^{ext} + \sum_{m \neq n} \vec{F}_{m \to n}^{int}, \tag{14.1.4}$$

where $\vec{F}_{m \to n}^{int}$ represents the (internal) force that particle m exerts over particle n. Assume that the external forces are invariant under Galilean boosts, $\vec{F}_n'^{ext} = \vec{F}_n^{ext}$, and that the
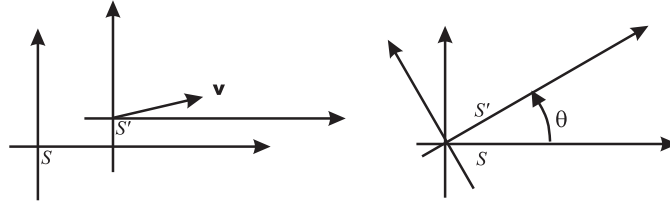
Figure 14.1: Boosts and rotations

internal forces are derivable from a potential that depends only on the spatial distance between the particles, *i.e.*,

$$\vec{F}_n^{int} = -\vec{\nabla}_n \Phi_n^{int} = -\sum_{m \neq n} \vec{\nabla}_n \Phi_{nm}(|\vec{r}_n - \vec{r}_m|). \tag{14.1.5}$$

This is compatible with the third law (of action and reaction) and it also makes the internal forces invariant under Galilean boosts. To see that this is so, specialize to just one space dimension and write the transformations in the following form (we are making this more complicated than it really is so as to introduce methods that will be useful in more complicated situations)

$$\begin{bmatrix} dt' \\ dx' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -v & 1 \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix} \tag{14.1.6}$$

and the inverse transformations as

$$\begin{bmatrix} dt \\ dx \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ v & 1 \end{bmatrix} \begin{bmatrix} dt' \\ dx' \end{bmatrix}. \tag{14.1.7}$$

We can now read off

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'} \frac{\partial}{\partial t} + \frac{\partial x}{\partial t'} \frac{\partial}{\partial x} = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \tag{14.1.8}$$

and

$$\frac{\partial}{\partial x'} = \frac{\partial t}{\partial x'} \frac{\partial}{\partial t} + \frac{\partial x}{\partial x'} \frac{\partial}{\partial x} = \frac{\partial}{\partial x}. \tag{14.1.9}$$

Therefore

$$\frac{\partial}{\partial x'_n} \Phi_{nm}(|x'_n - x'_m|) = \frac{\partial}{\partial x_n} \Phi_{nm}(|x_n - x_m|), \tag{14.1.10}$$

as claimed and the r.h.s. of Newton's equations are invariant. Moreover $dt' = dt$ and the transformation is linear so that the l.h.s. of Newton's equations is also invariant under these transformations. The equations of Newtonian dynamics are therefore invariant under Galilean boosts.

14.1.2 Lorentz Transformations

In electrodynamics, on the other hand, in free space one typically ends up with the wave equation,

$$\Box_x \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \vec{\nabla}^2 \psi = 0, \tag{14.1.11}$$

where c is the speed of light in the vacuum and ψ is the "wave function", which can be the electromagnetic scalar or vector potential. Now it is an experimental fact that the speed of light in a vacuum is the same for all inertial observers. However, then (14.1.11) is not invariant under Galilean transformations. Using the transformations in (14.1.6) and (14.1.7) we have

$$\frac{\partial^2}{\partial t'^2} = \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \left(\frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \right) \tag{14.1.12}$$

and

$$\vec{\nabla}'^2 = \vec{\nabla}^2. \tag{14.1.13}$$

Plugging this into the wave equation, we find

$$\frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 \to \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 + \frac{2\vec{v}}{c^2} \cdot \vec{\nabla} \frac{\partial}{\partial t} + \frac{1}{c^2} (\vec{v} \cdot \vec{\nabla})(\vec{v} \cdot \vec{\nabla}), \tag{14.1.14}$$

but only the first two terms on the r.h.s. correspond to the wave-equation and, moreover, there is no known kinetic transformation of the wave-function that can return the wave equation to its original form,[1] so we must conclude that the electromagnetic wave-equation is not invariant under Galilean transformations. This signals an incompatibility between electromagnetism and Newtonian mechanics therefore, by the principle of covariance, one of the two must be modified. As we now know, Maxwell's theory was preferred over Newtonian mechanics, which leads us to ask: what are the transformations that keep Maxwell's equations covariant? Once we have answered this question we will be in a position to address the problem of constructing a theory of mechanics that is indeed covariant under them.

To answer the first question, assume that the transformations that relate two inertial frames continue to be linear (as the Galilean transformations are) and think of the wave-equation as made up of two distinct parts: the second order differential operator, "$\Box_x$", and the wave function, ψ, each transforming in its own way under the above transformations.

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[1]Problem: Show that, on the contrary, the Schroedinger equation is invariant under Galilean transformations if they are supplemented with the following kinetic transformation of the wave-function:

$$\psi \to \psi' = e^{-\frac{i}{\hbar}(\vec{p}\cdot\vec{r} - Et)} \psi$$

where $\vec{p} = m\vec{v}$ and $E = m\vec{v}^2/2$. What does this mean?

For covariance, we will require "$\Box_x$", to transform as a scalar (invariant). Let us work with Cartesian systems and consider some general transformations of the form

$$t \quad \rightarrow \quad t' = t'(t, \vec{r}),$$

$$\vec{r} \quad \rightarrow \quad \vec{r}' = \vec{r}'(t, \vec{r}). \tag{14.1.15}$$

They must be

1. one-to-one: so that observers may be able to uniquely relate observations, and

2. invertible: so that the transformations can be made from any observer to the other – there is no preferred observer.

Our functions must therefore be bijective. As we have assumed that the transformations are linear, they will have the form

$$t' \quad = \quad -\frac{1}{c^2}(L_{00}t + \sum_i L_{0i}x_i),$$

$$x_i' \quad = \quad L_{i0}t + \sum_j L_{ij}x_j. \tag{14.1.16}$$

The reason for this peculiar definition of the coeffcents will become clear later. For now let us only note that the L's are some constants that we would like to evaluate. In matrix form the transformations could be written as

$$\begin{bmatrix} dt' \\ dx_i' \end{bmatrix} = \begin{bmatrix} -\frac{L_{00}}{c^2} & -\frac{L_{0j}}{c^2} \\ L_{i0} & L_{ij} \end{bmatrix} \begin{bmatrix} dt \\ dx_j \end{bmatrix}. \tag{14.1.17}$$

The matrix on the r.h.s. is really a 4×4 matrix and L_{ij} represents a 3×3 matrix of purely spatial transformations. It must be invertible because the transformation is required to be bijective. For example, $L_{00} = -c^2$ and $L_{0i} = 0 = L_{i0}$. The resulting transformations are purely spatial, transforming $x_i \rightarrow x_i' = \sum_j L_{ij}x_j$ and leaving $t \rightarrow t' = t$ unchanged. Clearly, therefore, the wave-operator,

$$\Box_x \rightarrow \Box_x' = \partial_{t'}^2 - \vec{\nabla}'^2 = \partial_t^2 - \vec{\nabla}'^2, \tag{14.1.18}$$

is a scalar if and only if L_{ij} is a spatial rotation, because only then will $\vec{\nabla}'^2 = \vec{\nabla}^2$.

More interesting are the "boosts", which involve inertial observers with relative velocities. Now $L_{i0} \neq 0 \neq L_{0i}$. Consider relative velocities along the x direction and the transformation

$$\begin{bmatrix} dt' \\ dx_1' \\ dx_2' \\ dx_3' \end{bmatrix} = \begin{bmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dt \\ dx_1 \\ dx_2 \\ dx_3 \end{bmatrix}. \tag{14.1.19}$$

Notice that we have set $x_2' = x_2$ and $x_3' = x_3$. This is because we assumed that space is homogeneous and isotropic so that a boost in the x_1 direction has no effect on the orthogonal coordinates x_2 and x_3. We can consider then only the effective two dimensional matrix

$$\begin{bmatrix} dt' \\ dx' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix} \tag{14.1.20}$$

(where $x_1 := x$). Thus we find the inverse transformation

$$\begin{bmatrix} dt \\ dx \end{bmatrix} = \frac{1}{\|\|} \begin{bmatrix} \delta & -\beta \\ -\gamma & \alpha \end{bmatrix} \begin{bmatrix} dt' \\ dx' \end{bmatrix}, \tag{14.1.21}$$

where $\|\|$ represents the determinant of the transformation, $\|\| = \alpha\delta - \beta\gamma$ and we have

$$\begin{aligned}
\frac{\partial}{\partial t'} &= \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \frac{1}{\|\|}\left(+\delta\frac{\partial}{\partial t} - \gamma\frac{\partial}{\partial x}\right) \\
\frac{\partial}{\partial x'} &= \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = \frac{1}{\|\|}\left(-\beta\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right),
\end{aligned} \tag{14.1.22}$$

turning our wave-operator into

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 &= \frac{1}{\|\|^2}\left(\frac{1}{c^2}\left(+\delta\frac{\partial}{\partial t} - \gamma\frac{\partial}{\partial x}\right)^2 - \left(-\beta\frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x}\right)^2\right) \\
&= \frac{1}{\|\|^2}\left((\delta^2/c^2 - \beta^2)\frac{\partial^2}{\partial t^2} - (\alpha^2 - \gamma^2/c^2)\frac{\partial^2}{\partial x^2}\right. \\
&\qquad\qquad\left. -2(\alpha\beta - \gamma\delta/c^2)\frac{\partial^2}{\partial t\partial x}\right). \tag{14.1.23}
\end{aligned}$$

If it is to remain form invariant, the right hand side above has to look the same in the frame S and we need to set

$$\begin{aligned}
\frac{\delta^2}{c^2} - \beta^2 &= \frac{\|\|^2}{c^2}, \\
\alpha^2 - \frac{\gamma^2}{c^2} &= \|\|^2, \\
\alpha\beta - \frac{\gamma\delta}{c^2} &= 0. \tag{14.1.24}
\end{aligned}$$

We have four unknowns and three constraints, so there is really just one parameter that determines all the unknowns. It is easy to find. Note that setting

$$\delta = \|\| \cosh\eta, \qquad \beta = \frac{\|\|}{c}\sinh\eta \tag{14.1.25}$$

solves the first of these equations, as

$$\alpha = \|\|\cosh\omega, \qquad \gamma = c\|\|\sinh\omega \tag{14.1.26}$$

solves the second. The last equation is then a relationship between η and ω. It implies that

$$\sinh\eta\cosh\omega - \sinh\omega\cosh\omega = \sinh(\eta - \omega) = 0 \to \eta = \omega. \tag{14.1.27}$$

Our boost in the x direction therefore looks like

$$\begin{bmatrix} dt' \\ dx' \end{bmatrix} = \|\| \begin{bmatrix} \cosh\eta & \frac{1}{c}\sinh\eta \\ c\sinh\eta & \cosh\eta \end{bmatrix} \begin{bmatrix} dt \\ dx \end{bmatrix}. \tag{14.1.28}$$

We notice that $\|\|$ is not determined. We will henceforth take it to be unity.

What is the meaning of the parameter η? Consider a test body having a velocity u as observed in the S frame. Its velocity as measured in the S' frame would be (the velocity does not transform as a vector)

$$u' = \frac{dx'}{dt'} = \frac{(\cosh\eta)dx + c(\sinh\eta)dt}{(\cosh\eta)dt + \frac{1}{c}(\sinh\eta)dx}. \tag{14.1.29}$$

Dividing by $(\cosh\eta)dt$ we find

$$u' = \frac{u + c\tanh\eta}{1 + \frac{u}{c}\tanh\eta}. \tag{14.1.30}$$

Now suppose that the body is at rest in the frame S. This would mean that $u = 0$. But, if S' moves with a velocity v relative to S, we can say that S should move with velocity $-v$ relative to S'. Therefore, because the test body is at rest in S, its velocity relative to S' should be $u' = -v$. Our formula gives

$$u' = -v = c\tanh\eta \to \tanh\eta = -\frac{v}{c}. \tag{14.1.31}$$

This in turn implies that

$$\cosh\eta = \frac{1}{\sqrt{1 - v^2/c^2}}, \qquad \sinh\eta = -\frac{v/c}{\sqrt{1 - v^2/c^2}}, \tag{14.1.32}$$

so we can write the transformations in a recognizable form

$$\begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}, \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \end{aligned}$$

$$y' = y,$$

$$z' = z. \tag{14.1.33}$$

Notes:

- These are the Lorentz transformations of the special theory of relativity.[2] They reduce to Galilean transformations when $v/c \ll 1$.

- Because $\tanh \eta \in (-1, 1)$ it follows that the transformations are valid only for $v < c$. The velocity of light is the *limiting* velocity of material bodies and observers. There exists no transformation from the rest frame of light to the rest frame of a material body.

- In general the matrix $\hat{L}$ is made up of boosts *and* rotations. Rotations do not, in general, commute with boosts and two boosts can lead to an overall rotation.

- Lorentz transformations keep the interval

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{14.1.34}$$

invariant[3] *i.e.*, the same for all observers. The interval ds is known as the proper distance and ds/c is known as the proper time (it's not difficult to see that when $d\vec{r} = 0$, $ds/c = dt$ *i.e.*, it is the time measured on a clock that is stationary in the frame). Like the proper distance, the proper time is an invariant. The transformations that keep an interval like (14.1.34) invariant form the Lie group $SO(3, 1)$.

14.2 Elementary consequences of Lorentz transformations

Our transformations mix up space and time, so there is no way for it but to consider both time and space as part of a single entity: "space-time". This is a four dimensional manifold. A point in space-time is called an event and involves not just its spatial location but also the time at which the event occurred.

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[2]For the very curious: Lorentz transformations can be put in four categories:

  – Proper orthochronous: $L_+^\uparrow$ with $\|\|\| = +1, L_{00}/c^2 \leq -1$

  – Proper non-orthochronous: $L_+^\downarrow$ with $\|\|\| = +1, L_{00}/c^2 \geq +1$

  – Improper orthochronous: $L_-^\uparrow$ with $\|\|\| = -1, L_{00}/c^2 \leq -1$

  – Improper non-orthochronous: $L_-^\downarrow$ with $\|\|\| = -1, L_{00}/c^2 \geq +1$

What we have are therefore proper orthochronous transformations.

[3]<u>Problem</u>: Prove this by direct substitution.

14.2.1   Simultaneity

The single most important consequence of the Lorentz transformations is that the concept of "simultaneity" is no longer absolute. Consider two events that are spatially separated, but occur at the same time as measured in the frame of an observer, S. Thus $dx \neq 0$ but $dt = 0$. According to (14.1.33),

$$dt' = \frac{-vdx/c^2}{\sqrt{1 - v^2/c^2}} \neq 0 \tag{14.2.1}$$

Thus events that are regarded as simultaneous in one frame are not so regarded in another frame, which is moving relative to the first.[4]

14.2.2   Length Contraction

Another interesting consequence is that length measurements of objects that are moving relative to an observer are smaller than measurements performed in the frame in which the objects are at rest. (The rest frame of a body is called the "proper" frame of the body). To understand how this comes about, one must recognize that to correctly measure the spatial distance between two points, their positions must be ascertained simultaneously. Let S be the frame in which the body is at rest and let S' be an observer moving at velocity v relative to S. Since a measurement of the body's length involves a simultaneous measurement of its endpoints, we should have $dt' = 0$. By the Lorentz transformations, this means that $dt = vdx/c^2$ and therefore

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}} = dx\sqrt{1 - v^2/c^2}. \tag{14.2.2}$$

But dx represents the length of the body as measured in its proper frame, so its length as measured by S' *i.e.*, dx', is "contracted" by a factor of $\sqrt{1 - v^2/c^2}$.

14.2.3   Time Dilation

Measurements of time intervals are also naturally observer dependent. Let S be the proper frame of a clock, which is moving relative to an observer S' with a velocity v. Being stationary in S, we might say that $dx = 0$ and $dt = ds/c$ represents the proper time intervals of the clock. The Lorentz transformation then tells us that time intervals read off by S' are related to proper time intervals according to

$$dt' = \frac{dt}{\sqrt{1 - v^2/c^2}} \tag{14.2.3}$$

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[4]<u>Problem</u>: Show that "future" and "past" are absolute; *i.e.,* show that if event "1" occurs in the past of event "2" in any inertial frame then "1" will occur in the past of "2" in *every* reference frame.
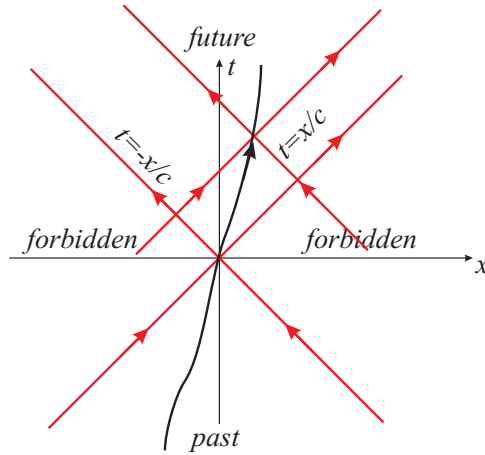
Figure 14.2: The light cone

This is known as "time dilation". Physically, this may be understood by noticing that while the time interval is measured in S **colocally** (at the same place), it is not so in S'. The clock in S appears to be "running slow" to the observer S'. This was in fact predicted in 1897 (long before Einstein's theory) by Louis Larmor who noticed the effect for electrons orbiting the nucleus of atoms.

It is often valuable to understand these phenomena in terms of world (space-time) diagrams. Thus, in figure (14.2) we show a two dimensional universe with the $y-$axis representing time from the point of view of some inertial observer, S. The red lines represent the path of light rays emanating from the origin in the upper half plane and terminating at the origin in the lower half plane. Consider a particle whose path, represented by the black curve, passes through the origin. (This can be arranged by resetting the origin of space and time). At no instant on this path may its slope, dt/dx, be less than or even equal to $1/c$, otherwise our particle would be traveling faster than or at the speed of light at that instant. Therefore the path lies wholly between the boundaries provided by the lines $t = \pm x/c$. This is the light cone. The region within the light cone and to the future is called the future light cone, the region within the light cone and in the past is called the past light cone and the regions on the left and right sides of the light cone are forever forbidden to the particle in the sense that it can *never* physically reach them. At any moment in time, the particle may only receive information from (and thus be influenced by) events within its own past light cone. Thus a fundamental role of Relativity is to restrict the domain of causal influence on any event. However, notice that as our particle travels along its world line, regions that were previously inaccessible begin to fall within its past light cone and become accessible as shown.
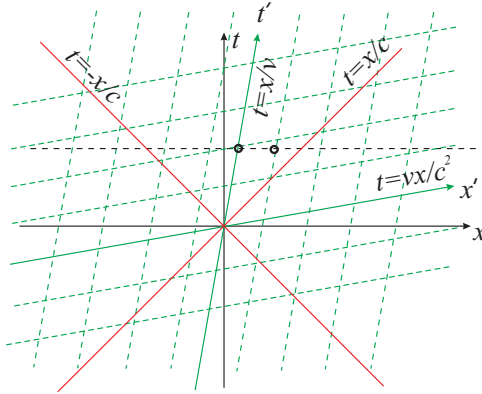
Figure 14.3: Two frames compared to each other

Figure (14.3) shows two inertial frames drawn in the same diagram. Let the (t, x) coordinate system represent an observer S and consider what the reference frame of an observer moving at velocity v relative to S might look like. The t' axis is the axis for which $x' = 0$, *i.e.*, in the (t, x) frame it is given by the straight line $t = x/v$ as shown in green. On the other hand, the x' axis is the one for which $t' = 0$, *i.e.*, it is given by $t = vx/c^2$, also shown in green in the figure. Consider two events that are spatially separated but occur simultaneously in S. These are represented by small circles on a horizontal ($t = $ const.) line. We see immediately that they do not fall on the same $t' = $ const. line. This graphically encapsulates the relativity of simultaneity. One can similarly visualize both length contraction and time dilation by projecting respectively on the x and t directions.[5]

14.2.4   Velocity Addition

Let us also recall the so-called law of "composition of velocities". Consider a particle whose velocity is being measured in two frames S and S'. Suppose that frame S' has a speed v in the positive $x-$direction relative to S then how do the particle velocities, as measured by S and S' relate, to one another? By definition, the velocity measured by S' will be

$$u'_x = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - v\,dx/c^2} = \frac{u_x - v}{1 - u_x v/c^2}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy\sqrt{1 - v^2/c^2}}{dt - v\,dx/c^2} = \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$$

---

[5]<u>Problem</u>: Do this!

$$u'_z = \frac{dz'}{dt'} = \frac{dz\sqrt{1 - v^2/c^2}}{dt - vdx/c^2} = \frac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} \tag{14.2.4}$$

and they all reproduce the Galilean result when $c \to \infty$.

14.2.5 Aberration

Finally, we can compare directions in space, as measured by two different observers. For example, if the particle is moving in the $x - y$ plane ($u_z = 0$), let us see how two observers may describe its direction of motion. According to observer S' the (tangent of the) angle made with the positive $x-$axis will be

$$\tan \theta' = \frac{u'_y}{u'_x} = \frac{u_y\sqrt{1 - v^2/c^2}}{u_x - v} = \frac{u \sin \theta \sqrt{1 - v^2/c^2}}{u \cos \theta - v} \tag{14.2.5}$$

where θ is measured in S and u is the particle speed as measured in S. Notice that it depends on the speed of the particle as well as the relative speed of the frames. If the "particle" were a photon, *i.e.*, in the case of light propagation, $u = c$ and

$$\tan \theta' = \frac{\sin \theta \sqrt{1 - v^2/c^2}}{\cos \theta - v/c} \tag{14.2.6}$$

This is the formula for light **aberration**.[6],[7]

14.3 Tensors on the fly

One lesson that we learn is that we must work with the position vectors of *events* and these are "four-vectors", *i.e.*, vectors having one time and three space components. It is no longer useful or even correct to think of space and time as separate entities because the Lorentz transformations mix the two. Continuing with a Cartesian system, label the coordinates as follows:

$$x^\mu = (x^0, x^i): \quad \mu \in \{0, 1, 2, 3\}, \quad x^0 = t, \quad x^i = x_i. \tag{14.3.1}$$

---

[6]Problem: Show that the formula can be simplified to

$$\tan \frac{\theta'}{2} = \tan \frac{\theta}{2}\sqrt{\frac{1 - v/c}{1 + v/c}}$$

[7]Problem: Show that for small angles, in the limit $v/c \to 0$ and up to first order in v/c, the aberration angle $\Delta\theta = \theta - \theta'$ is given by

$$\Delta\theta \approx \frac{v}{c}\sin\theta$$

Let us be particular about the position of the indices as superscripts, distinguishing be-
tween superscripts and subscripts (soon we will see that this is important) and consider a
displacement, dx^μ, in frame S letting the corresponding displacement in frame S' be dx'^μ.
By our transformations we know that

$$dx^\mu \to dx'^\mu = \sum_\nu L^\mu{}_\nu dx^\mu, \tag{14.3.2}$$

where $L^\mu{}_\nu$ is precisely the matrix we derived earlier for the special case of boosts in the x
direction. In that case

$$
\begin{aligned}
L^0{}_0 &= -L_{00}/c^2 = \cosh\eta, \\
L^0{}_1 &= -L_{01}/c^2 = \sinh\eta/c, \quad L^0{}_i = 0 \;\; \forall \;\; i \in \{2,3\}, \\
L^1{}_0 &= L_{10} = c\sinh\eta, \quad L^i{}_0 = 0 \;\; \forall \;\; i \in \{2,3\}, \\
L^1{}_1 &= L_{11} = \cosh\eta, \quad L^i{}_j = \delta^i_j \;\; \forall \;\; i,j \in \{2,3\},
\end{aligned}
\tag{14.3.3}
$$

where δ^i_j is the usual Kronecker $\hat\delta$ (unit matrix),

$$\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{14.3.4}$$

In spacetime, we may set up a vector space V by defining a set of four unit vectors,
$\{\widehat{u}_{(\mu)}\}$, called a **tetrad frame**, spanning V, so that an arbitrary proper displacement in
space-time can be expressed as $d\vec{s} = \sum_\mu dx^\mu \widehat{u}_{(\mu)}$. Since the displacement itself should not
depend on the observer, it follows from (14.3.2) that under a Lorentz transformation

$$\widehat{u}_{(\mu)} \to \widehat{u}'_{(\mu)} = \sum_\alpha \widehat{u}_{(\alpha)} (L^{-1})^\alpha{}_\mu \tag{14.3.5}$$

A vector is any object of the form $\vec{A} = \sum_\mu A^\mu \widehat{u}_{(\mu)}$, with four "contravariant" components,
A^μ, each of which transforms as dx^μ (so that $\vec{A}$ is also observer independent), *i.e.*,

$$A^\mu \to A'^\mu = \sum_\nu L^\mu{}_\nu A^\nu. \tag{14.3.6}$$

It is both customary and useful to think of a vector in terms of its components, but it
is somewhat inconvenient to explicitly write out the summation (Σ) every time we have
sum over components. We notice, however, that only repeated indices get summed over;
therefore we will use Einstein's convention and drop the symbol Σ, but now with the
understanding that repeated indices, occurring in pairs in which one member appears
"up" (as a superscript) and the other "down" (as a subscript), automatically implies a
sum. Thus, for example, we would write the above transformation of contravariant vectors
as

$$A^\mu \to A'^\mu = L^\mu{}_\nu A^\nu. \tag{14.3.7}$$

Notice that the derivative operator does not transform as dx^μ, but according to the inverse transformation. In other words:

$$\frac{\partial}{\partial x^\mu} := \partial_\mu \rightarrow \frac{\partial}{\partial x'^\mu} := \partial'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha. \tag{14.3.8}$$

But since $\partial x'^\mu/\partial x^\alpha = L^\mu{}_\alpha$, and

$$\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\mu}{\partial x^\beta} = \delta^\alpha{}_\beta = (L^{-1})^\alpha{}_\mu L^\mu{}_\beta, \tag{14.3.9}$$

it follows that

$$\partial'_\mu = \frac{\partial x^\alpha}{\partial x'^\mu}\partial_\alpha = \partial_\alpha(L^{-1})^\alpha{}_\mu, \tag{14.3.10}$$

so the derivatives of a scalar function, $\partial_\mu\phi(x)$, are not the components of a vector. Nevertheless, we see that

$$dx'^\mu\partial'_\mu\phi'(x') = (L^{-1})^\beta{}_\mu L^\mu{}_\alpha dx^\alpha\partial_\beta\phi(x) = \delta^\beta{}_\alpha dx^\alpha\partial_\beta\phi(x) = dx^\mu\partial_\mu\phi(x) \tag{14.3.11}$$

is invariant.

Given any vector space, V, one can one can consider the space of all linear maps from V to the real numbers, $i.e.$, maps of the form $\vec{\omega} : V \rightarrow \mathbb{R}$,

$$\vec{\omega}(\vec{A}) \overset{\text{def}}{=} \vec{\omega}\cdot\vec{A} \in \mathbb{R}$$

satisfying

$$\vec{\omega}(a\vec{A} + b\vec{C}) = a\vec{\omega}(\vec{A}) + b\vec{\omega}(\vec{C}) \tag{14.3.12}$$

where $\vec{A}$ and $\vec{C}$ are in V, and a and a are real numbers. One may now define the sum of two linear maps by

$$(a\vec{\omega} + b\vec{\eta})(\vec{A}) = a\vec{\omega}(\vec{A}) + b\vec{\eta}(\vec{A}) \tag{14.3.13}$$

then it is easy to see that these maps themselves form a vector space, called the **dual** vector space, $^*V$. Given the tetrad $\{\hat{u}_{(\mu)}\}$, spanning V, we could introduce a basis for the dual vector space, $\{\hat{\theta}^{(\mu)}\}$, by requiring that

$$\hat{\theta}^{(\nu)}(\hat{u}_{(\mu)}) = \hat{\theta}^{(\nu)}\cdot\hat{u}_{(\mu)} = \delta^\nu_\mu \tag{14.3.14}$$

For the inner product to remain invariant, it must hold that, under a Lorentz transformation,

$$\hat{\theta}^{(\mu)} \rightarrow \hat{\theta}'^{(\mu)} = L^\mu{}_\alpha\hat{\theta}^{(\alpha)}. \tag{14.3.15}$$

Any member of the dual vector space, $\vec{\omega}$ can now be expressed as $\vec{\omega} = \omega_\mu\hat{\theta}^{(\mu)}$. ω_μ are called the "covariant" components of $\vec{\omega}$. They will transform as

$$\omega_\mu \rightarrow \omega'_\mu = \omega_\alpha(L^{-1})^\alpha{}_\mu, \tag{14.3.16}$$

so that, given any vector, $\vec{A}$, and any dual vector, $\vec{\omega}$, one forms a scalar

$$- \vec{\omega} \cdot \vec{A} = -\omega_\mu A^\mu. \tag{14.3.17}$$

This is the four dimensional dot product, the analogue of the three dimensional dot product we are familiar with. The simplest example of a dual vector is the gradient of a scalar function:

$$\nabla = \widehat{\theta}^{(\mu)} \partial_\mu \phi(x), \tag{14.3.18}$$

as we saw earlier.

We could generalize the concept of vectors to tensors by simply defining a rank $(0,n)$ tensor to be an multilinear map from a tensor product (an ordered collection) of vectors to $\mathbb{R}$, *i.e.,* $\mathbb{T} : V \otimes V \ldots \otimes V$ (n times) $\to \mathbb{R}$.[8] A basis for $\mathbb{T}$ will evidently be $\widehat{\theta}^{(\mu_1)} \otimes \widehat{\theta}^{(\mu_2)} \ldots \otimes \widehat{\theta}^{(\mu_n)}$ and we could express $\mathbb{T}$ as

$$\mathbb{T} = T_{\mu_1 \mu_2 \ldots \mu_n} \widehat{\theta}^{(\mu_1)} \otimes \widehat{\theta}^{(\mu_2)} \ldots \otimes \widehat{\theta}^{(\mu_n)}, \tag{14.3.19}$$

Its covariant components will transform as n copies of a dual vector,

$$T'_{\mu\nu\lambda\ldots} = T_{\alpha\beta\gamma\ldots}(L^{-1})^\alpha{}_\mu (L^{-1})^\beta{}_\nu (L^{-1})^\gamma{}_\lambda \ldots \tag{14.3.20}$$

Similarly, we could define a rank $(m,0)$ tensor to be an multilinear map from a tensor product of dual vectors to $\mathbb{R}$, *i.e.,* $\mathbb{T} : {}^*V \otimes {}^*V \ldots \otimes {}^*V$ (m times) $\to \mathbb{R}$. Following the same reasoning as before, a basis for $\mathbb{T}$ will be $\widehat{u}_{(\mu_1)} \otimes \widehat{u}_{(\mu_2)} \ldots \otimes \widehat{u}_{(\mu_m)}$ and we could express $\mathbb{T}$ as

$$\mathbb{T} = T^{\mu_1 \mu_2 \ldots \mu_m} \widehat{u}_{(\mu_1)} \otimes \widehat{u}_{(\mu_2)} \ldots \otimes \widehat{u}_{(\mu_m)} \tag{14.3.21}$$

so that its contravariant coponents will transform as m copies of a vector,

$$T^{\mu\nu\lambda\ldots} = L^\mu{}_\alpha L^\nu{}_\beta L^\lambda{}_\gamma T^{\alpha\beta\gamma\ldots} \tag{14.3.22}$$

More generally, we define "mixed" tensors as multilinear maps from a tensor product of vectors *and* dual vectors, $\mathbb{T} : {}^*V \otimes {}^*V \ldots \otimes {}^*V$ (m times) $\otimes V \otimes V \ldots \otimes V$ (n times) $\to \mathbb{R}$ and express it as

$$\mathbb{T} = T^{\mu\nu\ldots}{}_{\lambda\kappa\ldots} \widehat{u}_{(\mu)} \otimes \widehat{u}_{(\nu)} \ldots \otimes \widehat{\theta}^{(\lambda)} \otimes \widehat{\theta}^{(\kappa)} \ldots, \tag{14.3.23}$$

with V and ${}^*V$ appearing in any order in the product (the above is simply one example). In this case, the tensor is said to have rank (m,n). Thus vectors and dual vectors are but special cases of tensors: vectors are tensors of rank $(1,0)$ and dual vectors are tensors of rank $(0,1)$. Just as we think of vectors and dual vectors in terms of their components, we will also think of tensors in terms of their components. Thus we will speak of contravariant, covariant and mixed tensors according to their components.

---

[8]A multilinear map acts lineraly on all its arguments.

There is a one to one relationship between the covariant and contravariant tensors: for every covariant tensor we can find a contravariant tensor and vice-versa. To see how this comes about, let us rewrite the proper distance (14.1.34) in a slightly different way, using matrix notation as follows:

$$ds^2 = -d\vec{s} \cdot d\vec{s} = -(\widehat{u}_{(\mu)} \cdot \widehat{u}_{(\nu)})dx^\mu dx^\nu = -\eta_{\mu\nu}dx^\mu dx^\nu, \qquad (14.3.24)$$

or

$$\widehat{u}_{(\mu)} \cdot \widehat{u}_{(\nu)} = \eta_{\mu\nu} \qquad (14.3.25)$$

where, according to (14.1.34), $\eta_{\mu\nu}$ is the matrix: $\mathrm{diag}(-c^2, 1, 1, 1)$ *i.e.,*

$$\hat{\eta} = \eta_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \qquad (14.3.26)$$

It is a covariant tensor of rank two as we see from its transformation properties and is called the Minkowski metric. Given that ds^2 is invariant, we must have

$$- ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \rightarrow \eta'_{\alpha\beta}dx'^\alpha dx'^\beta = \eta'_{\alpha\beta}L^\alpha{}_\mu L^\beta{}_\nu dx^\mu dx^\nu = \eta_{\mu\nu}dx^\mu dx^\nu, \qquad (14.3.27)$$

which implies that

$$\eta_{\mu\nu} = L^\alpha{}_\mu L^\beta{}_\mu \eta'_{\alpha\beta}, \qquad (14.3.28)$$

or, by taking inverses,

$$\eta'_{\alpha\beta} = (L^{-1})^\mu{}_\alpha (L^{-1})^\nu{}_\beta \eta_{\mu\nu}. \qquad (14.3.29)$$

However, $\eta_{\mu\nu}$ is required to be an invariant tensor in Special Relativity, $\eta'_{\mu\nu} \equiv \eta_{\mu\nu}$, and this can be used in conjunction with (14.3.28) to derive expressions for the matrices $\hat{L}$. It is an alternative way of deriving the Lorentz transformations through the so-called generators of the transformation (see Appendix B). Now the metric is invertible ($\|\hat{\eta}\| \neq 0$), with inverse

$$\hat{\eta}^{-1} = \eta^{\mu\nu} = \begin{bmatrix} -\frac{1}{c^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (14.3.30)$$

and, by construction, $\eta^{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu{}_\nu$. (It may be easily shown that the inverse metric $\eta^{\mu\nu}$ transforms as

$$\eta'^{\alpha\beta} = L^\alpha{}_\mu L^\beta{}_\nu \eta^{\mu\nu} = \eta^{\alpha\beta}, \qquad (14.3.31)$$

i.e., according the rule for a contravariant tensor of rank two.)

In (14.3.24), $\eta_{\mu\nu}$ acts upon two contravariant vectors (two factors of dx^μ) to create a scalar, the proper distance between two events. But we had seen that invariants are constructed from the product of contravariant tensors and covariant tensors. Thus we expect that $\eta_{\mu\nu}dx^\nu$ should transform as a *covariant* vector. In general, consider a contravariant vector A^μ and construct the quantity

$$A_\mu = \eta_{\mu\nu}A^\nu. \tag{14.3.32}$$

How does it transform? We see that

$$A_\mu \to A'_\mu = \eta_{\mu\nu}L^\nu{}_\alpha A^\alpha = \eta_{\mu\nu}L^\nu{}_\alpha \eta^{\alpha\gamma}\eta_{\gamma\lambda}A^\lambda = (\eta_{\mu\nu}L^\nu{}_\alpha \eta^{\alpha\gamma})(\eta_{\gamma\lambda}A^\lambda), \tag{14.3.33}$$

where we have used $\eta^{\alpha\gamma}\eta_{\gamma\lambda} = \delta^\alpha{}_\lambda$. But notice that (14.3.28) implies the identity

$$\eta_{\alpha\beta} \;=\; \eta_{\mu\nu}L^\mu{}_\alpha L^\nu{}_\beta \to \eta^{\gamma\alpha}\eta_{\alpha\beta} = \eta_{\mu\nu}\eta^{\gamma\alpha}L^\mu{}_\alpha L^\nu{}_\beta$$

$$\to \delta^\gamma{}_\beta \;=\; (\eta_{\nu\mu}L^\mu{}_\alpha \eta^{\alpha\gamma})L^\nu{}_\beta = (L^{-1})^\gamma{}_\nu L^\nu{}_\beta$$

$$\to (L^{-1})^\gamma{}_\nu \;=\; \eta_{\nu\mu}L^\mu{}_\alpha \eta^{\alpha\gamma}. \tag{14.3.34}$$

Therefore (14.3.33) reads

$$A'_\mu = A_\gamma (L^{-1})^\gamma{}_\mu, \tag{14.3.35}$$

which is the transformation of a covariant vector. The Minkowski metric therefore maps contravariant vectors to covariant vectors. In the same way it maps contravariant tensors to covariant tensors:

$$T_{\alpha_1,\alpha_2,\ldots\alpha_n} = \eta_{\alpha_1\beta_1}\eta_{\alpha_2\beta_2}\ldots\eta_{\alpha_n\beta_n}T^{\beta_1,\beta_2,\ldots\beta_n}. \tag{14.3.36}$$

Likewise, the inverse metric $\eta^{\mu\nu}$ maps covariant vectors to contravariant vectors, *i.e.*, the quantity A^μ defined by

$$A^\mu = \eta^{\mu\nu}A_\nu, \tag{14.3.37}$$

transforms as a contravariant vector.[9] Therefore, it maps covariant tensors to contravariant tensors:

$$T^{\alpha_1,\alpha_2,\ldots\alpha_n} = \eta^{\alpha_1\beta_1}\eta^{\alpha_2\beta_2}\ldots\eta^{\alpha_n\beta_n}T_{\beta_1,\beta_2,\ldots\beta_n}. \tag{14.3.38}$$

This relationship between covariant tensors and contravariant tensors is why we originally defined the boosts as in (14.1.17). Thus, $L^\mu{}_\nu = \eta^{\mu\alpha}L_{\alpha\nu}$ which gives

$$L^0{}_0 \;=\; \eta^{00}L_{00} = -L_{00}/c^2,$$

---

[9]Problem: Show this!

$$
\begin{aligned}
L^0{}_i &= \eta^{00} L_{0i} = -L_{0i}/c^2, \\[2mm]
L^i{}_0 &= \eta^{ij} L_{j0} = L_{i0}, \\[2mm]
L^i{}_j &= \eta^{ik} L_{kj} = L_{ij}.
\end{aligned}
\tag{14.3.39}
$$

Moreover, there is a natural way to define the (invariant) magnitude of a four-vector, A^μ. It is simply

$$
A^2 = -A^\mu A_\mu = -\eta_{\mu\nu} A^\mu A^\nu = -\eta^{\mu\nu} A_\mu A_\nu,
\tag{14.3.40}
$$

which is the equivalent of the familiar way of defining the magnitude of an ordinary three-vector.[10] For example, the familiar operator $\square_x$ can be written as

$$
\square_x = -\eta^{\mu\nu} \partial_\mu \partial_\nu = -\partial^2,
\tag{14.3.41}
$$

in which form it is manifestly a scalar. We see once again that the basic difference between Newtonian space and Lorentzian space-time is that, in the case of the former, space and time do not mix and both are absolute. In this case it is sufficient to consider only spatial distances and Pythagoras' theorem ensures that the metric is just the Kronecker δ (with three positive eigenvalues), so there is no need to distinguish between covariant and contravariant indices. In the case of a Lorentzian space-time an observer's measurements of space and time are not independent, neither is absolute and so one is forced to consider the "distance" between *events* in space-time. The metric, $\eta_{\mu\nu}$, for space-time has signature $(-1, 3)$ *i.e.*, it has one negative eigenvalue and three positive eigenvalues.

For an arbitrary boost specified by a velocity $\vec{v} = (v_1, v_2, v_3) = (v^1, v^2, v^3)$, we find the following Lorentz transformations:

$$
\begin{aligned}
L^0{}_0 &= \frac{1}{\sqrt{1 - \vec{v}^2/c^2}} = \gamma, \\[3mm]
L^i{}_0 &= \gamma v^i, \\[3mm]
L^0{}_i &= \frac{\gamma v_i}{c^2}, \\[3mm]
L^i{}_j &= \delta^i{}_j + (\gamma - 1)\frac{v^i v_j}{\vec{v}^2}.
\end{aligned}
\tag{14.3.42}
$$

These are most easily derived using (14.3.28) and the fact that $\eta'_{\mu\nu} = \eta_{\mu\nu}$. They reduce to the transformations we had earlier for a boost in the $x-$ direction, for then $\vec{v} = (v, 0, 0)$

---

[10]When $A^2 < 0$ the vector points within the light cone and is said to be "time-like". When $A^2 > 0$ it points outside the light cone and is called "space-like" and when $A^2 = 0$ the vector A is "light-like" or "null", pointing along the light cone.

and

$$\hat{L} = \begin{bmatrix} \gamma & -\frac{\gamma v}{c^2} & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{14.3.43}$$

which leads to precisely the transformations in (14.1.33). A more compact way to write (14.3.42) is

$$t' = \gamma[t - \frac{(\vec{v} \cdot \vec{r})}{c^2}]$$

$$\vec{r}' = \vec{r} - \gamma \vec{v} t + (\gamma - 1)\frac{\vec{v}}{v^2}(\vec{v} \cdot \vec{r}) \tag{14.3.44}$$

for a general $\vec{v}$.

Spatial volume elements are not invariant under Lorentz transformations. We can make a rough argument for this as follows: suppose that the volume measured by the proper observer is dV then the observer moving relative to this observer with a velocity $\vec{v}$ will observe the length dimension in the direction of motion contracted according to (14.2.2) and all perpendicular length dimensions will remain unchanged, so we expect $dV' = dV/\gamma$. A more precise treatment follows by mimicking the argument for length contraction. Consider the transformation form $(t,x) \to (t'x')$

$$dt' = L^0{}_0 dt + L^0{}_j dx^j, \quad dx'^i = L^i{}_0 dt + L^i{}_j dx^j \tag{14.3.45}$$

with $dt' = 0$ because length measurements must be made subject to a simultaneous measurement of the endpoints in every frame. Therefore $dt = -L^0{}_j dx^j/\gamma$ and

$$dx'^i = \left(-\frac{1}{\gamma}L^i{}_0 L^0{}_j + L^i{}_j\right) dx^j = \left(\delta^i_j + \frac{(1-\gamma)}{\gamma}\frac{v^i v_j}{v^2}\right) dx^j, \tag{14.3.46}$$

so taking the Jacobian of the transformation gives

$$d^3\vec{r} \to d^3\vec{r}' = d^3\vec{r}\left|\frac{\partial x'^i}{\partial x^j}\right| = d^3\vec{r}/\gamma. \tag{14.3.47}$$

The four dimensional volume element, $d^4 x$, is invariant for proper Lorentz transformations.

A consequence of the Lorentz transformation of volume is that the three dimensional δ function, $\delta^{(3)}(\vec{r} - \vec{r}_0)$, which is defined according to

$$\int d^3\vec{r} \delta^{(3)}(\vec{r} - \vec{r}_0) = 1 \tag{14.3.48}$$

cannot be invariant either. If we require the defining integral to remain invariant then

$$\delta^{(3)}(\vec{r} - \vec{r}_0) \to \delta'^{(3)}(\vec{r}' - \vec{r}_0') = \gamma \delta^{(3)}(\vec{r} - \vec{r}_0). \tag{14.3.49}$$

The four dimensional delta function, $\delta^{(4)}(x - x_0)$, will, however, be invariant.

14.4 Waves and the Relativistic Doppler Effect

Maxwell's equations for the electromagnetic field, A_μ, in Lorentz gauge read

$$\Box_x A_\mu = j_\mu. \tag{14.4.1}$$

In the absence of sources, this is just the wave equation with c being the speed of propagation; in one spatial dimension

$$\left[\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right] A_\mu(t, x) = 0 \tag{14.4.2}$$

and a typical solution will look like a linear combination of plane waves of varying amplitudes and frequencies,

$$A_\mu^{(k)}(t, x) = A_\mu^{(0)}(k, \omega)e^{i(kx-\omega t)}, \tag{14.4.3}$$

subject to $k^2 - \omega^2/c^2 = 0$. Because A_μ transforms as a vector, the exponent must transform as a scalar, *i.e.*, $kx - \omega t$ must be an invariant. This is only possible if $k_\mu = (-\omega, k)$ transforms as a covariant vector,

$$\omega' = \gamma(\omega - vk), \quad k' = \gamma(k - v\omega/c^2) \tag{14.4.4}$$

i.e., $k^\mu = (\omega/c^2, k)$ transforms as $x^\mu = (t, x)$. In particular, the first relation tells us that

$$f' = f\sqrt{\frac{1 - v/c}{1 + v/c}}, \tag{14.4.5}$$

which is the expression for the Doppler shifting of light in the frame of an observer moving with a velocity v, taken as positive when the observer is traveling in the direction of the propagating light wave. Thus an observer moving "away from" the source sees a **red-shifting** of the light, *i.e.*, a shifting toward lower frequencies, and an observer moving toward the source sees a **blue-shifting**, *i.e.*, a shifting toward higher frequencies. If the observer's speed is small compared to the speed of light, the linear approximation of (14.4.5) gives

$$f' \approx \left(1 - \frac{v}{c}\right)f, \tag{14.4.6}$$

which should be compared with the Doppler shifting for ordinary mechanical waves that propagate in a medium. We get the observed wavelengths either directly, by requiring $\lambda f = c = \lambda'f'$, or by using the second relation in (14.4.4),

$$\lambda' = \lambda\sqrt{\frac{1 + v/c}{1 - v/c}}. \tag{14.4.7}$$

The "redshift" factor is defined as

$$z = \frac{\lambda' - \lambda}{\lambda} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1. \tag{14.4.8}$$

and $z \approx \frac{v}{c}$ when $v \ll c$. Because all inertial observers are equivalent in special relativity there is no separate effect for "moving sources" as there is in the case of mechanical waves. Yet, one may wonder why there is an effect at all, considering that light requires no medium in which to travel and its speed in all reference frames is the same. The Doppler effect for light originates in time dilation.

14.5   Dynamics in Special Relativity

The relativistic point particle extremizes its "proper time" (this can be thought of as a generalization of Fermat's principle, which was originally enunciated for the motion of light "corpuscles"),

$$\mathcal{S}_p = -mc^2 \int d\tau = -mc \int_1^2 \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu} = -mc^2 \int_1^2 dt\sqrt{1 - \frac{\vec{v}^2}{c^2}} \tag{14.5.1}$$

where $d\tau = ds/c = \frac{1}{c}\sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}$ is the proper time and the constant "mc^2" is chosen so that $\mathcal{S}_\mathrm{p}$ has the dimension of action (or angular momentum: J·s). One sees quite easily that this action principle reduces to Hamilton's principle (with zero potential energy, $V = 0$) when the velocity of the particle relative to the observer is small compared with the velocity light, for then

$$\sqrt{1 - \frac{\vec{v}^2}{c^2}} \approx 1 - \frac{1}{2}\frac{\vec{v}^2}{c^2} \tag{14.5.2}$$

which, when inserted into (14.5.1) gives

$$\mathcal{S}_p \approx \int_1^2 dt \left[\frac{1}{2}m\vec{v}^2 - mc^2\right] \tag{14.5.3}$$

The second term is, of course, just a constant (later to be identified with the rest mass energy of the particle) and can be dropped without affecting either the equations of motion or the conservation laws. The first term is the non-relativistic kinetic energy of the particle and the action is therefore just that of a free non-relativistic point particle.

The momentum conjugate to x^i is

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{mv_i}{\sqrt{1 - \vec{v}^2/c^2}} = \gamma m v_i, \tag{14.5.4}$$

which reduces to $p_i = mv_i$ when $|\vec{v}| \ll c$, and Euler's equations give

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}(\gamma m\vec{v}) = \frac{d}{dt}\frac{m\vec{v}}{\sqrt{1 - \vec{v}^2/c^2}} = 0 \qquad (14.5.5)$$

which are the equations of motion of the particle. The Lagrangian does not depend explicitly on time, so we expect that the Hamiltonian is the total energy and is conserved,

$$E = \mathcal{H} = p_i\dot{x}^i - \mathcal{L} = \frac{m\vec{v}^2}{\sqrt{1 - \vec{v}^2/c^2}} + mc^2\sqrt{1 - \vec{v}^2/c^2} = \frac{mc^2}{\sqrt{1 - \vec{v}^2/c^2}} \qquad (14.5.6)$$

The quantity

$$m_R = \frac{m}{\sqrt{1 - \vec{v}^2/c^2}} \qquad (14.5.7)$$

is generally called the "relativistic mass" or simply "mass" of the particle, whereas the parameter m we used initially is called the "rest" mass of the particle and can be thought of as its mass when measured in its proper frame ($\vec{v} = 0$). We have just obtained the popular Einstein relation,

$$E = m_R c^2. \qquad (14.5.8)$$

Notice that the energy of the particle is not zero in the rest frame. In this frame the particle possesses an energy, $E = mc^2$, which is exclusively associated with its proper (rest) mass. Furthermore, expanding E in powers of $\vec{v}$ we find

$$E = mc^2 + \frac{1}{2}m\vec{v}^2 + \frac{3}{8}m\frac{\vec{v}^4}{c^2} + \ldots \qquad (14.5.9)$$

The second term is the Newtonian kinetic energy and the higher order terms are all corrections to the Newtonian expression. The kinetic energy, K, in special relativity is defined via the relation

$$E = K + mc^2, \qquad (14.5.10)$$

so it is what remains after its rest mass energy is subtracted from its total energy.

The Hamiltonian is obtained by a Legendre transformation of the Lagrangian and is expressed in terms of the momenta and coordinates but not the velocities. This is easily accomplished by noting that (14.5.4) gives

$$\vec{p}^2 = \frac{m^2\vec{v}^2}{1 - \vec{v}^2/c^2} \rightarrow \frac{\vec{v}}{c} = \frac{\vec{p}}{\sqrt{\vec{p}^2 + m^2c^2}} \qquad (14.5.11)$$

Thus we get

$$1 - \frac{\vec{v}^2}{c^2} = \frac{m^2c^2}{\vec{p}^2 + m^2c^2}, \qquad (14.5.12)$$

which, when inserted into (14.5.6), gives another well known result,

$$\mathcal{H} = E = \sqrt{\vec{p}^2 c^2 + m^2 c^4}. \tag{14.5.13}$$

Again we recover the rest mass energy, $E = mc^2$ when we set $\vec{p} = 0$.

Let us note that the momentum

$$p_i = m\gamma v_i = m\frac{dt}{d\tau}\frac{dx_i}{dt} = m\frac{dx_i}{d\tau} \tag{14.5.14}$$

is quite manifestly the spatial component of the four-vector[11]

$$p_\mu = m\frac{dx_\mu}{d\tau} \equiv mU_\mu. \tag{14.5.15}$$

where $U^\mu = dx^\mu/d\tau$ is the "four velocity" of the particle. The quantity p_μ is called its "four momentum". Its time component is

$$p_0 = -mc^2\frac{dt}{d\tau} = -\frac{mc^2}{\sqrt{1 - \vec{v}^2/c^2}} = -E \tag{14.5.16}$$

so we see that the spatial momentum and the energy are components of one four-vector momentum,

$$p^\mu = m\frac{dx^\mu}{d\tau}, \quad p^0 = \frac{E}{c^2}, \quad p^i = \frac{mv^i}{\sqrt{1 - \vec{v}^2/c^2}} \tag{14.5.17}$$

Formula (14.5.13) for the energy is now seen to result from a purely kinematic relation, because

$$p^2 = \eta_{\mu\nu}p^\mu p^\mu = m^2\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -m^2\left[\frac{ds}{d\tau}\right]^2 = -m^2 c^2 \tag{14.5.18}$$

(the kinematic relation being, of course $U_\mu U^\mu = -c^2$, *remember it*). Therefore, expanding the l.h.s.,

$$p^2 = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2 c^2, \quad \Rightarrow \quad E^2 = \vec{p}^2 c^2 + m^2 c^4 \tag{14.5.19}$$

Interestingly, taking the square root allows for both positive and negative energies but we have chosen the positive sign, thereby excluding negative energy free particles by *fiat*.

Euler's equations as given in (14.5.5) are not in a manifestly covariant form. They can, however, be put in such a form if we multiply by γ, expressing them as

$$\gamma\frac{d\vec{p}}{dt} = \frac{dt}{d\tau}\frac{d\vec{p}}{dt} = \frac{d\vec{p}}{d\tau} = 0 \tag{14.5.20}$$

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[11]Problem: Convince yourself that $p^\mu = mdx^\mu/d\tau$ is indeed a four-vector under Lorentz transformations. Remember that the proper time, τ, is a scalar.

This is the equation of motion for a *free* particle, so the r.h.s. is zero. The l.h.s. transforms as the spatial components of a four-vector and we need not worry about the transformation properties of the r.h.s., since it vanishes. In the presence of an external force the r.h.s. should not vanish and the principle of covariance requires that both sides of the equations of motion should transform in the same way under Lorentz transformations. Let us tentatively write a covariant equation of motion as

$$\frac{dp^{\mu}}{d\tau} = f^{\mu} \tag{14.5.21}$$

where f^{μ} is a four-vector. It must be interpreted as the relativistic equivalent of Newton's force. If m is constant then

$$f^{\mu} = m\frac{dU^{\mu}}{d\tau}$$

and, because $U^2 = -c^2$, the "four force" must satisfy one constraint, *i.e.*,

$$f \cdot U = f^{\mu}U_{\mu} = 0, \tag{14.5.22}$$

which means that not all its components are independent. But what is the connection between f^{μ} and the familiar concept of the Newtonian force, which we will call $\vec{F}_N$? To find it consider the proper frame, $\overline{S}$, of the particle (quantities in this instantaneous rest frame will be represented by an over-bar). In this frame $\tau = \bar{t}$, $\bar{p}^0 = m$ and $\bar{p}^i = 0$. It follows that the time component of the l.h.s of (14.5.21) is zero (assuming m is constant) and therefore so is $\overline{f}^0$. The spatial part of the force equation then reads

$$\frac{d\bar{p}^i}{d\bar{t}} = m\bar{a}^i = \overline{f}^i, \tag{14.5.23}$$

where $\bar{a}^i$ is the particle's acceleration relative to $\overline{S}$, generally referred to as its **proper acceleration**. Naturally, we identify F_N^i with $\overline{f}^i$ or, equivalently, with $m\bar{a}^i$. We will discuss the connection between the proper acceleration, $\bar{a}^i$, and the acceleration as measured in S,

$$a^i = \frac{d^2 x^i}{dt^2}$$

in a later section. For the present take

$$\overline{f}^{\mu} = (0, \vec{F}_N). \tag{14.5.24}$$

To determine f^{μ} in an arbitrary frame we only need to perform a boost because f^{μ} is a genuine four-vector. Therefore, in a frame in which the instantaneous velocity of the particle is $\vec{v}$, we find in particular that

$$f^0 = \gamma\frac{\vec{v} \cdot \vec{F}_N}{c^2} \tag{14.5.25}$$

i.e.,

$$\frac{dE}{dt} = \vec{v} \cdot \vec{F}_N \tag{14.5.26}$$

The equation says that the rate of energy gain (loss) of the particle is simply the power transferred to (or from) the system by the external Newtonian forces. The same boost also gives the spatial components of the relativistic force in an arbitrary frame as

$$\vec{f} = \vec{F}_N + (\gamma - 1)\frac{\vec{v}}{v^2}(\vec{v} \cdot \vec{F}_N) \tag{14.5.27}$$

and we notice that the component of $\vec{f}$ perpendicular to the velocity is equal to the corresponding component of the Newtonian force, $\vec{f}_\perp = \vec{F}_{N\perp}$. However the component of the force in the direction of motion is enhanced over the same component of the Newtonian force by the factor of γ, *i.e.*, $\vec{f}_\parallel = \gamma \vec{F}_{N\parallel}$. Our expression also has the non-relativistic limit $(\gamma \approx 1)$ $\vec{f} \approx \vec{F}_N$, as it should.

We have given two forms of the action for the massive point particle in (14.5.1) although we have concentrated so far on the last of these. The first form is actually quite interesting,

$$\mathcal{S}_p = -mc\int_1^2 \sqrt{-\eta_{\mu\nu}dx^\mu dx^\nu}. \tag{14.5.28}$$

If λ is any parameter describing the particle trajectories then we could write this as

$$\mathcal{S}_p = -mc\int_1^2 d\lambda\sqrt{-\eta_{\mu\nu}U^\mu_{(\lambda)}U^\nu_{(\lambda)}} \tag{14.5.29}$$

where

$$U^\mu_{(\lambda)} = \frac{dx^\mu(\lambda)}{d\lambda} \tag{14.5.30}$$

is tangent to the trajectories $x^\mu(\lambda)$ and λ is an arbitrary parameter. The action is therefore reparameterization invariant and all that we have said earlier corresponds to a particular choice of λ $(= t)$. This is like fixing a "gauge", to borrow a term from electrodynamics.[12],[13]

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[12]<u>Problem</u>: Starting from (14.5.28), treat all the coordinates of an event, x^μ, on the same footing (instead of singling out one of them – time – as a parameter) and define

$$p^{(\lambda)}_\mu = \frac{\partial\mathcal{L}}{\partial U^\mu_{(\lambda)}}.$$

Show that

$$\mathbb{H} = p^{(\lambda)}_\mu U^\mu_{(\lambda)} - \mathcal{L} = 0.$$

This is a consequence of reparameterization invariance.

[13]<u>Problem</u>: The square-root Lagrangian in (14.5.28) is inconvenient for the quantization of the free particle (or working out the statistical mechanics of free, relativistic particles) and a quadratic form,

The relativistic Hamilton-Jacobi equation is obtained by replacing the momenta, p_μ, by $\partial S/\partial x^\mu$ in (14.5.19),

$$\eta^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\right)\left(\frac{\partial S}{\partial x^\nu}\right) = -m^2 c^2 \tag{14.5.31}$$

We could define $S = S' - mc^2 t$ and write the equation in terms of S',

$$\frac{1}{2m}(\vec{\nabla}S')^2 - \left(\frac{\partial S'}{\partial t}\right) - \frac{1}{2mc^2}\left(\frac{\partial S'}{\partial t}\right)^2 = 0 \tag{14.5.32}$$

In this form the limit $c \to \infty$ compares directly with the expected Hamiltonian-Jacobi equation for the free non-relativistic particle.

14.6 Conservation Laws

We will now consider a system of relativistic particles and define the total particle momentum as

$$p^\mu = \sum_n p_n^\mu = \sum_n m_n \frac{dx_n^\mu}{d\tau_n} = \sum_n m_n \gamma_n v_n^\mu \tag{14.6.1}$$

The rate at which each particle's four-momentum changes will depend on the net force acting upon it, according to

$$\frac{dp_n^\mu}{d\tau_n} = f_n^\mu, \tag{14.6.2}$$

but this is an inconvenient form of the equation of motion, particularly when dealing with many particles, because it describes the rate of change with respect to the proper time of the particle, which itself depends on its motion. Making use of the fact that $\gamma_n = dt/d\tau_n$, let's rewrite this equation in the form

$$\frac{dp_n^\mu}{dt} = \gamma_n^{-1} f_n^\mu \tag{14.6.3}$$

and therefore also

$$\frac{dp^\mu}{dt} = \sum_n \frac{dp_n^\mu}{dt} = \sum_n \gamma_n^{-1} f_n^\mu. \tag{14.6.4}$$

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similar to the non-relativistic one, is preferable. This can be achieved by introducing an auxiliary function, χ, together with the action,

$$S = -\int d\lambda \left[\chi \eta_{\mu\nu} U_{(\lambda)}^\mu U_{(\lambda)}^\nu + \chi^{-1} m^2\right]$$

and treating x^μ and χ as independent functions with respect to which the action is to be extremized. Show that one obtains the expected equations of motion.

Concentrate, for the moment, on the spatial components only and, as before, let $\vec{f}_n$ be made up of two parts, *viz.*, (i) an "external" force, $\vec{f}_n^{\text{ext}}$, acting on the particle and (ii) an "internal" force, $\vec{f}_n^{\text{int}}$ acting on it due to its interactions with all the other particles within the system. Then

$$\vec{f}_n^{\text{int}} = \sum_{m \neq n} \vec{f}_{m \to n} \tag{14.6.5}$$

and it follows that

$$\frac{d\vec{p}}{dt} = \frac{d}{dt} \sum_n \vec{p}_n = \sum_{n, m \neq n} \gamma_n^{-1} \vec{f}_{m \to n} + \sum_n \gamma_n^{-1} \vec{f}_n^{\text{ext}} \tag{14.6.6}$$

For low particle velocities $\gamma_n \approx 1$ for all n and $\vec{f}_n \approx \vec{F}_n^N$, where $\vec{F}^N$ is the Newtonian force. Conservation of particle momentum in the absence of external forces then follows in this limit by Newton's third law.

What about the internal forces in a fully relativistic scenario? In principle, no effect experienced at any world point x can have originated at a world point x' outside its past light cone, *i.e.*, at a time earlier than $t - |\vec{r} - \vec{r}'|/c^{14}$ because the speed of light is assumed to be the maximal speed at which information or influence can travel. Instantaneous particle interactions, in particular forces that depend only on the spatial distances between the particles (which are typically employed in Newtonian physics), cannot have the desired Lorentz transformation properties and therefore are impossible in the context of special relativity. This, of course, becomes relevant only for very high particle velocities because the change in position of one particle can influence another particle only after the information has had the time to propagate through the distance that separates the particles, which itself changes appreciably during this time if the relative velocities are high, making the interactions of particles depend in a complicated way on their motions. This is already evident from the expression $\sum_{n,m \neq n} \gamma_n^{-1} \vec{f}_{m \to n}$ describing the effects of the internal forces. One gets around this difficulty by imagining that the particles are immersed in a set of dynamical **fields** whose disturbances propagate in spacetime with a speed not exceeding the speed of light.

A "field" should be thought of as a potential (or set of potentials) associated with each point of spacetime. Disturbances in these potentials transfer energy and momentum from one event to another. Every field is realized by a function (or set of functions) with definite Lorentz transformation properties and one can have many kinds of fields, eg. scalar fields, vector fields, etc., depending on how the field transforms. Particle-particle interactions are then described in terms of **local interactions** of the particles with the fields, which involve an exchange of energy and momentum between the two at the world point of the particle. This exchange causes disturbances in the fields, which then propagate

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[14]This is called **retardation**.

through spacetime and, at a later time, exchange energy and momentum (locally) with other particles in the system. In this way fields act as mediators of inter-particle forces. A familiar example of this would be electrically charged particles in an electromagnetic field. The electromagnetic field is responsible for energy and momentum transfer between the particles via their electromagnetic interaction.

This picture is only consistent if the field that is responsible for mediating the interaction carries energy and momentum in its own right. We then define the total four momentum of the system by

$$P^\mu = p^\mu + \pi_{\mathrm{f}}^\mu \tag{14.6.7}$$

where π_{f}^μ represents the field momentum and assert the following:

- In the absence of external forces the total four momentum of the system, which consists of the momentum of the particles and the field, is conserved.

This is a natural generalization of the non-relativistic statement about momentum conservation and, in fact, follows from Noether's theorem by space-time translation invariance, as we will see later. It is worth understanding why any generalization of the conservation law for momentum *must* involve the entire four-vector momentum if it is to be a covariant statement. It can be argued as follows: let ΔE and $\Delta \vec{P}$ represent the change in energy and momentum respectively of the system in some inertial frame S. In some other frame, S', we represent these quantities by $\Delta E'$ and $\Delta \vec{P}'$ respectively. Being components of a four vector, they are connected by the Lorentz transformation,

$$\Delta P'^i = \frac{\gamma v^i}{c^2} \Delta E + \left(\delta_j^i + (\gamma - 1) \frac{v^i v_j}{v^2} \right) \Delta P^j$$

$$\Delta E' = \gamma \left(\Delta E + \vec{v} \cdot \Delta \vec{P} \right). \tag{14.6.8}$$

It follows that if $\Delta \vec{P}$ vanishes (the spatial momentum is conserved) in S then it will vanish in S' if and only if ΔE vanishes as well and if ΔE vanishes (the total energy is conserved) in S then it will vanish in S' if and only if $\Delta \vec{P}$ also vanishes. Therefore energy and momentum conservation go hand in hand in Einstein's theory of relativity and one cannot be had without the other. This is a most remarkable fact. In Newtonian mechanics the two conservation laws are distinct: momentum conservation requires the absence external forces and a sufficient condition for the conservation of energy is that *all* forces acting on the system are conservative. No such condition appears in the relativistic version of the conservation law, which must therefore *always* hold provided that the momentum of the particles and fields are consistently taken into account.

Note that neither the total field momentum nor the total particle momentum is separately conserved since momentum may be exchanged between the two. This implies that

the interaction forces between the particles do **not** satisfy Newton's third law *i.e.,* action is **not** equal and opposite to reaction.

It is often convenient to define the **center of momentum** frame in complete analogy with the non-relativistic case by setting the spatial components of the total momentum in that frame to zero, *i.e.,*

$$P_{\text{cm}}^{\mu} = \left(\frac{E_{\text{cm}}}{c^2}, \vec{0} \right) = (M_{\text{cm}}, \vec{0}), \tag{14.6.9}$$

which also defines the total rest mass, M_{cm}, of the system. The rest mass energy, $M_{\text{cm}}c^2$, contains all the rest energies of the particles that make up the system. It *also* contains their kinetic energy relative to the center of mass as well as the energies of their interactions with one another and of the fields involved in these interactions. In other words, the rest mass of the system contains the entire internal energy of the system and *it is conserved.* We know of four kinds of elementary fields, each with its characteristic interactions. From weakest to strongest they are the gravitational field, the fields associated with the weak interaction, the electromagnetic field and the fields associated with the strong interaction or chromodynamics. The gravitational field is associated with spacetime itself and its description is unique. All the other fields are special cases of a single family of theories called "gauge theories". These will be discussed in a following chapters. The momentum in a generic frame, S, can be obtained by a Lorentz transformation and will involve only M_{cm} and the velocity of the center of mass relative to the Laboratory, $\vec{v}_{\text{cm}}$,

$$P^0 = M_{\text{cm}}\gamma_{\text{cm}}, \quad \vec{P} = M_{\text{cm}}\gamma\vec{v}_{\text{cm}}. \tag{14.6.10}$$

Thus in every physical system consisting of interacting particles, the center of mass will behave as a single particle with an effective mass (equal to M_{cm}).

14.7   Relativistic Collisions

The preceding discussion leads directly to the topic of collisions between relativistic particles. In this section we will briefly examine such collisions, assuming that whatever fields are present diminish rapidly enough to zero that the field contribution to the total momentum can be ignored when the particles are sufficiently far apart. We will then consider "free" incoming particles and "free" outgoing particles as we did earlier and conserve momentum according to $\sum_n p_{ni}^{\mu} = \sum_n p_{nf}^{\mu}$ as before, but this time taking care with the relativistic factors.

First consider a collision in which two incoming bodies with momenta

$$p_{1i} = \left(m_1 + \frac{K_1}{c^2}, \vec{p}_1 \right), \quad p_{2i} = \left(m_2 + \frac{K_2}{c^2}, \vec{p}_2 \right) \tag{14.7.1}$$

stick together to form a body of mass m_f, with momentum

$$p_f = \left(m_f + \frac{K_f}{c^2}, \vec{p}_f \right), \tag{14.7.2}$$

where we used the definition of the Kinetic energy, $E = mc^2 + K$. Conservation of momentum means that

$$m_1 + m_2 + \frac{K_1}{c^2} + \frac{K_2}{c^2} = m_f + \frac{K_f}{c^2}$$

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_f, \tag{14.7.3}$$

which three equations (the collision is planar) are sufficient to determine m_f and $\vec{p}_f$ in terms of the initial data. Such a collision is best viewed in the center of momentum frame in which $\vec{p}_f = 0$ and $m_f = M_{\text{cm}}$. Then, in this frame $p_{\text{cm}}^\mu = (M_{\text{cm}}, \vec{0})$

$$m_1 + m_2 + \frac{K_1'}{c^2} + \frac{K_2'}{c^2} = M_{\text{cm}}, \quad \vec{p}_1' = -\vec{p}_2' \tag{14.7.4}$$

where we used primes to denote quantities measured in the c.m.s. The first equation gives the effective mass, *i.e.*, the mass-energy in the center of momentum frame. The second simply defines the center of momentum frame. If the velocity of the center of momentum frame as measured in the the Laboratory frame is v_{cm}, then

$$\vec{v}_{\text{cm}} = \frac{\vec{p}_{\text{cm}} c}{\sqrt{\vec{p}_{\text{cm}}^2 + M_{\text{cm}}^2 c^2}} = \frac{(\vec{p}_1 + \vec{p}_2) c}{\sqrt{(\vec{p}_1 + \vec{p}_2)^2 + M_{\text{cm}}^2 c^2}} \tag{14.7.5}$$

and an appropriate boost recovers the solutions in that frame.

Consider a collision in which two incident particles, "1" and "2", give rise to two outgoing particles, "3" and "4" (we label the particles differently because, in relativistic collisions, particle physics processes *may* cause one set of incoming particles to be transformed into a wholly different set of outgoing particles and we wish to allow for this possibility) and suppose particle "2" is at rest in the Laboratory frame. Let the $x-$ axis lie along the motion of "1" and let θ and ϕ be the angles made by "3" and "4" respectively with the $x-$axis, as in 14.4. Conserving the four momentum gives

$$m_1 + m_2 + \frac{K_1}{c^2} = m_3 + m_4 + \frac{K_3}{c^2} + \frac{K_4}{c^2}$$

$$p_1 = p_3 \cos\theta + p_4 \cos\phi$$

$$p_3 \sin\theta - p_4 \sin\phi = 0 \tag{14.7.6}$$
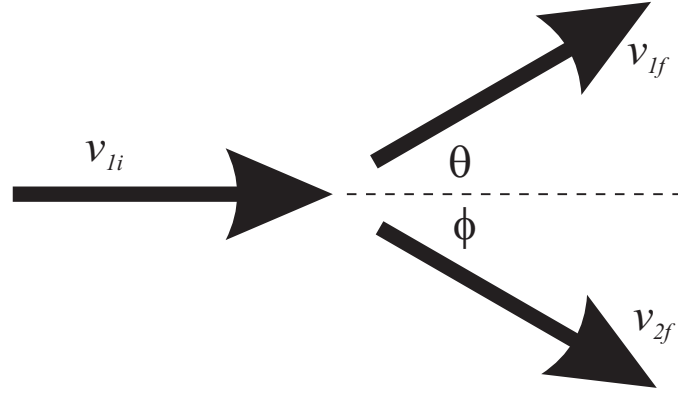
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Figure 14.4: Two dimensional collision

Note that in general

$$p^2c^2 + m^2c^4 = E^2 = (mc^2 + K)^2 \Rightarrow p^2 = 2mK + \frac{K^2}{c^2} \qquad (14.7.7)$$

so these equations are to be solved for p_3, p_4 and one of the angles. As before, the other angle must be specified. Our strategy will be similar to the one we followed for non-relativistic collisions. Multiply the third equation in (14.7.6) by $\cos\phi$ and use the second to find

$$p_3 \sin\theta \cos\phi = p_4 \sin\phi \cos\phi = p_1 \sin\phi - p_3 \cos\theta \sin\phi. \qquad (14.7.8)$$

This gives

$$p_3 \sin(\theta + \phi) = p_1 \sin\phi \Rightarrow p_3 = \frac{p_1 \sin\phi}{\sin(\theta + \phi)} \qquad (14.7.9)$$

and inserting the result into the second equation of (14.7.6)

$$p_4 = \frac{p_1 \sin\theta}{\sin(\theta + \phi)}. \qquad (14.7.10)$$

Finally, to determine θ, we must insert the above two formulae into the energy equation, but the energy equation is much more complicated than its non-relativistic counterpart! A welcome algebraic simplification occurs if the outgoing two particles fly off symmetrically, *i.e.,* with $\theta = \phi$ in the laboratory frame. If, moreover, the incident and outgoing particles have the same mass, m, then
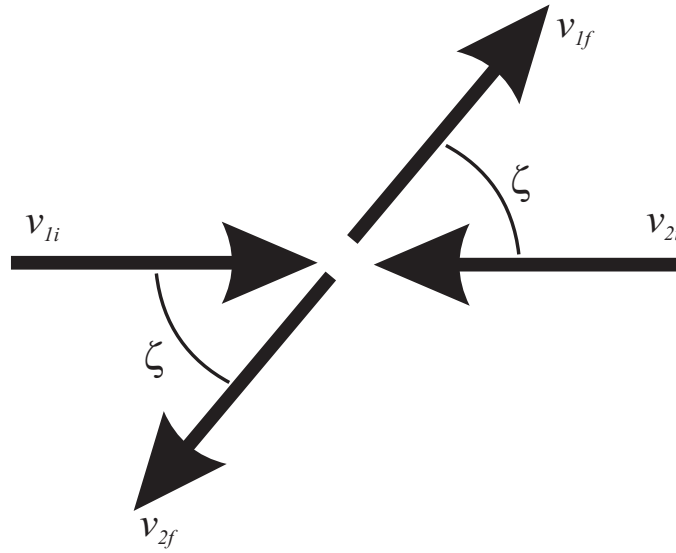
$$p_3 = \frac{p_1}{2\cos\theta} = p_4 \qquad (14.7.11)$$

Figure 14.5: Two dimensional collision from the center of momentum frame

and

$$E_1 + mc^2 = 2mc^2 + 2K, \quad K = \sqrt{\frac{p_1^2 c^2}{4\cos^2\theta} + m^2 c^4} \tag{14.7.12}$$

yields, after a little bit of algebra,

$$\cos^2\theta = \frac{E_1^2 - m^2 c^4}{c^2[(E_1 - mc^2)^2 - 4m^2 c^4]} \tag{14.7.13}$$

It is interesting to notice that in the extreme relativistic case, *i.e.,* when $E_1 \gg mc^2$, $\cos\theta \to 1$ and the separation angle approaches zero whereas, in the limit that the rest mass energy is much greater than the incident kinetic energy, $\cos\theta \to 0$ and the separation angle approaches $\pi/2$ radians. This, as we know, is the non-relativistic case.

The view from the center of momentum frame is different. Now the two particles scatter as shown in figure 14.5 because both the initial and final total spatial momentum must vanish,

$$E_1' + E_2' = E_3' + E_4' = M_{\text{cm}}c^2$$

$$\vec{p}_1' + \vec{p}_2' = 0 = \vec{p}_3' + \vec{p}_4' \Rightarrow \vec{p}_1' = -\vec{p}_2', \quad \vec{p}_3' = -\vec{p}_4' \tag{14.7.14}$$

Let the initial momenta lie along the $x-$axis and let the masses (initial and final) all be the same, say m as before. Then, because of the second equation above, $E_1' = E_2' = E_i'$

and $E'_3 = E'_4 = E'_f$ and because of energy conservation $2E'_i = 2E'_f = M_{cm}c^2$. Therefore all momenta have the same magnitude. Since the final velocities are anti-parallel, there is only one final angle, ξ, between the outgoing particles and the $x-$axis, but there is not enough information to determine it. However, we can relate ξ to the angle θ in the Laboratory frame discussed earlier by performing a Lorentz transformation.[15]

In general, a good amount of information about any collision in any frame is obtained directly from the Lorentz invariants. For example, if, in the Laboratory frame, our incoming particles had momenta p_{1i} and p_{2i} then, because p^2 is a Lorentz invariant, $p^2 = p_{cm}^2$ or

$$(p_{1i} + p_{2i})^2 = p_{1i}^2 + p_{2i}^2 + 2p_{1i} \cdot p_{2i} = p_{cm}^2 \qquad (14.7.15)$$

and therefore

$$(m_1^2 + m_2^2)c^4 + 2(E_{1i}E_{2i} - c^2\vec{p}_{1i} \cdot \vec{p}_{2i}) = M_{cm}^2 c^4. \qquad (14.7.16)$$

Using $E_i = m_i c^2 + K_i$ in each case, we arrive at

$$(m_1 + m_2)^2 c^4 + 2(m_1 c^2 K_{2i} + m_2 c^2 K_{1i} + K_{1i}K_{2i} - c^2\vec{p}_{1i} \cdot \vec{p}_{2i}) = M_{cm}^2 c^4. \qquad (14.7.17)$$

so if particle "2" (say) is initially at rest in the Laboratory frame, then $\vec{p}_{2i} = 0$, $E_{2i} = m_2 c^2$, $E_{1i} = m_1 c^2 + K_{1i}$ where K_{1i} is the initial kinetic energy of particle "1" and

$$M_{cm}^2 c^4 = (m_1 + m_2)^2 c^4 + 2m_2 c^2 K_{1i} \qquad (14.7.18)$$

Thus the available center of momentum energy increases as the *square root* of the incident kinetic energy. Equation (14.7.16) holds whenever two particles collide whether or not the collision is inelastic and no matter how many the end products of the collision.

14.8   Accelerated Observers

A question of interest is how to relate an accelerated frame to an inertial one in the context of the special theory of relativity. Naturally this cannot be done directly because the Lorentz transformations only relate inertial frames. However, it can be done by considering a special one parameter family of inertial frames each of which is at rest relative to and coincident with the accelerated frame at *one particular* instant of time. Geometrically, this is equivalent to replacing the accelerated observer's curved world line in Minkowski space by a set of infinitesimal straight line segments along her world line. Each of the infinitisimal segments corresponds to an inertial frame over an infinitesimal path length. In this section we consider this problem in general and then specialize to one particular

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[15]<u>Problem</u>: Determine the relationship between the angle of scattering, ξ, in the center of momentum frame and the angle θ in the Laboratory frame, assuming that particle "2" is initially at rest in this frame and that particles "3" and "4" leave the collision center symmetrically, as discussed.

case: the "Rindler observer". Rindler observers, named after Wolfgang Rindler who first considered this problem, undergo a constant proper acceleration (recall that the proper acceleration is the acceleration of the detector w.r.t. a frame that is instantaneously at rest relative to it).

First we analyze the problem in two dimensions. Let S be an inertial frame and let $\widetilde{S}$ be the frame of the Rindler observer. $\widetilde{S}$ is not an inertial frame and cannot be directly connected to S within the context of the special theory, so introduce a one parameter family of inertial frames, $\{\overline{S}(s)\}$, each of which is instantaneously at rest relative to $\widetilde{S}$ and coincides with it at proper time s/c. If $\widetilde{S}$ possesses an acceleration, $\alpha(s)$, at s/c relative to the frame $\overline{S}(s)$ then $\alpha(s)$ is the **proper acceleration** of the Rindler observer,

$$\alpha(s) = \frac{d^2\overline{x}}{d\overline{t}^2}. \tag{14.8.1}$$

To begin with, we'll let $\alpha(s)$ be arbitrary. Now S and every member of the family $\overline{S}(s)$ are inertial frames and therefore they are related by Lorentz transformations. For a fixed s, we have

$$\overline{t} = \gamma(t - vx/c^2)$$
$$\overline{x} = \gamma(x - vt). \tag{14.8.2}$$

where $v = v(s)$ is the velocity of frame $\overline{S}(s)$ relative to S. Defining the velocity $u = dx/dt$, we find

$$\overline{u} = \frac{d\overline{x}}{d\overline{t}} = \frac{(u-v)}{1 - uv/c^2} \tag{14.8.3}$$

and therefore (remember that we must keep $v(s)$ fixed because it represents the velocity of $\overline{S}(s)$ relative to S and $\overline{S}(s)$ is inertial)

$$d\overline{u} = \frac{du}{1 - uv/c^2} - \frac{(u-v)(-v/c^2 du)}{(1 - uv/c^2)^2} = \frac{du}{\gamma^2(1 - uv/c^2)^2}. \tag{14.8.4}$$

This gives

$$\overline{a} = \frac{d\overline{u}}{d\overline{t}} = \frac{a}{\gamma^3(1 - uv/c^2)^3} \tag{14.8.5}$$

Now, because the frame $\overline{S}(s)$ is instantaneously at rest relative to the the Rindler observer

$\widetilde{S}$ at proper time s/c, it follows that $\overline{u}(s) = 0$ and $\overline{a}(s) = \alpha(s)$. Therefore $u = v(s)$ and[16,17]

$$\frac{a}{\left(1 - \frac{u^2}{c^2}\right)^{\frac{3}{2}}} = \overline{a} = \alpha(s). \tag{14.8.6}$$

But

$$a = \frac{du}{ds}\frac{ds}{dt} = \frac{c}{\gamma}\frac{du}{ds} \tag{14.8.7}$$

Therefore, integrating we find that

$$\int_0^s \frac{du}{\left(1 - \frac{u^2}{c^2}\right)} = c\tanh^{-1}\left(\frac{u}{c}\right)\Big|_{u_0}^s = \frac{1}{c}\int_0^s ds\,\alpha(s) \tag{14.8.8}$$

so

$$u = c\tanh\left(\frac{1}{c^2}\int_0^s ds\,\alpha(s) + \tanh^{-1}\frac{u_0}{c}\right) = c\tanh\eta \tag{14.8.9}$$

where we've called the argument of the hyperolic tangent on the right η. Again

$$u = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{c}{\gamma}\frac{dx}{ds} = \frac{c}{\cosh\eta}\frac{dx}{ds} \Rightarrow \frac{dx}{ds} = \sinh\eta \tag{14.8.10}$$

therefore

$$x - x_0 = \int_0^s ds\,\sinh\eta$$

$$t - t_0 = \frac{1}{c}\int_0^s ds\,\cosh\eta \tag{14.8.11}$$

Without loss of generality choose $u_0 = 0$. Further, specialize to the case of a constant proper acceleration and let $\alpha(s) = a$, where $a = $ const., then $\eta = as/c^2$ and

$$x - x_0 = \frac{c^2}{a}\left[\cosh\frac{as}{c^2} - 1\right], \quad t - t_0 = \frac{c}{a}\sinh\frac{as}{c^2}. \tag{14.8.12}$$

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[16]Problem: Consider the same problem in four dimensions. We have seen that the relationship between the longitudinal component (*i.e.,* in the direction of the motion) of the acceleration in S and the corresponding component of the proper acceleration, is $\overline{a}_{\parallel} = \gamma^3 a_{\parallel}$. What is the relationship between the *transverse* component of the acceleration in S and the transverse component of the proper acceleration? Show that $\overline{a}_{\perp} = \gamma^2 a_{\perp}$.

[17]Problem: Using the result of the last problem together with equation (14.5.27), show that

$$\frac{d\vec{p}}{dt} = \gamma^3 m\vec{a}_{\parallel} + \gamma m\vec{a}_{\perp}$$

Then obtain this result directly by differentiating $\vec{p} = m\gamma\vec{v}$ w.r.t. t.
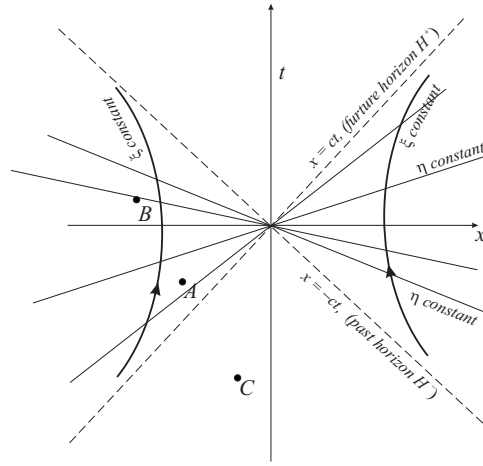
Figure 14.6: Trajectory of the Rindler observer

This gives the trajectory of the accelerated observer as viewed by the inertial observer, S. Convenient initial conditions would be $x_0 = c^2/a$ at $t_0 = 0$ and we find that the trajectory may be expressed in the form

$$x^2 - c^2 t^2 = \frac{c^4}{a^2}. \tag{14.8.13}$$

It is a hyperbola, shown in figure (14.6). Notice that the path of the accelerated observer may never cross the lines $x = \pm ct$. These lines represent "horizons" (past and future) that mark the boundaries of that portion of Minkowski space that is accessible to the accelerating observer. Not all of Minkowski space will be accessible to her, as is evident from the diagram in figure 14.6, where one sees that she can *never* receive information from events A and B and, while she will receive information from C, she will be unable to ascribe a time to it! The lines $x = \pm ct$ are called "Rindler horizons" because they apply only to accelerating observers. They divide Minkowski space into four causal wedges, called "Rindler wedges", defined by $x > c|t|$ (right), $x < c|t|$ (left), $ct > |x|$ (future) and $ct < |x|$ (past).

Let us consider one of these wedges. Define the coordinates

$$\xi = \frac{c^2}{2a} \ln\left[\frac{a^2}{c^4}(x^2 - c^2 t^2)\right], \quad \widetilde{\eta} = \frac{c}{a} \tanh^{-1} \frac{ct}{x} \tag{14.8.14}$$

then, because a is an acceleration it follows that ξ has dimension of length and $\widetilde{\eta}$ has dimension of time. Both ξ and η range over the entire real line even though these coordinates

do not cover all of Minkowski space. The inverse transformations are

$$x = \frac{c^2}{a}e^{a\xi/c^2}\cosh\frac{a\widetilde{\eta}}{c}, \quad t = \frac{c}{a}e^{a\xi/c^2}\sinh\frac{a\widetilde{\eta}}{c} \qquad (14.8.15)$$

If $a > 0$, the new coordinates $(\widetilde{\eta}, \xi)$ cover only the right wedge in the (t, x) plane, *i.e.*, $x > c|t|$. They are called "Rindler" coordinates and define the "Rindler frame". In this frame, lines of constant ξ are hyperbolæ in the Minkowski frame and represent curves of constant proper acceleration equal to $ae^{-a\xi/c^2}$. The hyperbola $\xi = 0$ describes the trajectory of our particular accelerating observer and lines of constant $\widetilde{\eta}$ (time) are straight lines through the origin as shown in figure 14.6. The metric is

$$ds^2 = e^{2a\xi}(c^2d\widetilde{\eta}^2 - d\xi^2), \quad \widetilde{\eta} \in (-\infty, \infty), \quad \xi \in (-\infty, \infty) \qquad (14.8.16)$$

and the horizons are located at $\xi \to -\infty$. Another coordinatization that is often used is obtained by defining $y = c^2 e^{a\xi/c^2}/a$ (or, in terms of the original Minkowski coordinates: $y = \sqrt{x^2 - c^2t^2}$) then

$$ds^2 = \frac{a^2}{c^2}y^2d\widetilde{\eta}^2 - dy^2, \quad \widetilde{\eta} \in (-\infty, \infty), \quad y \in (0, \infty) \qquad (14.8.17)$$

gives another parametrization of Rindler space. In these coordinates, the Rindler observer is located on the (vertical) line $y = c^2/a$ and the horizons are located at $y = 0$. Notice that in both coordinatizations the horizons get defined by setting the time-time component of the metric to zero. This is a generic feature of time independent metrics.

But what exactly is a Rindler horizon and why does the coordinate system break down there? Notice that the definition of y is quite independent of a, but the definition of $\widetilde{\eta}$ depends on it therefore the proper time intervals of the observer will scale with her acceleration although proper distance does not. Thus consider Rindler observers with different proper accelerations living on vertical lines given by $y' = c^2/a'$. Notice that the greater the proper acceleration the smaller the value of y and, vice versa, the smaller the proper acceleration the greater the value of y. Now it should be clear that the proper distance between our observer and some other observer with $y = c^2/a$ will be fixed at $c^2|a' - a|/(aa')$. Think of this distance as the length of a rod connecting the two observers. If $a' > a$ then the observer with a' lies on the trailing end of the rod and, vice-versa, if $a' < a$ then the observer with a' is on the leading edge. Whereas in Galilean physics the two ends of a rod must have equal acceleration to keep the same length, in special relativity *the trailing end must accelerate a little bit faster to keep up!* This is because of length contraction: as the speed increases along the rod's length, its length also shrinks a little and the trailing end has to increase its velocity a little bit more in the same time interval to account for the shrinkage. Therefore observers on the "trailing end" *i.e.,* toward the horizon must accelerate more to "keep up". The horizon marks the stage at which the observer would need an *infinite* acceleration to keep up with the others.

Chapter 15

More general coordinate systems*

15.1 Introduction

As we have seen, very often the symmetries of a given physical system make Cartesian coordinates cumbersome because it may turn out to be difficult to implement the boundary conditions suited to the system in these coordinates. In that case, as we know well, we turn to coordinate systems that are better adapted to the given symmetries. The new coordinates are not usually Cartesian (for example, think of the problem of determining geodesics on a sphere). A generic feature of such systems – and one that is exploited in the physical problem – is that the coordinate surfaces are curved. They are therefore called "curvilinear" systems. In this chapter we will develop some machinery to work with such systems.

Suppose that we perform a coordinate transformation from a set of Cartesian coordinates to a set of curvilinear coordinates that are given by ξ^μ. Imagine that the Cartesian coordinates extend over the entire spacetime and let them be given by x^a. We will assume that the new coordinates are invertible functions of the Cartesian coordinates, *i.e.*, $\xi^\mu = \xi^\mu(x)$ and $x^a = x^a(\xi)$. In the Cartesian system, the distance between two spatial points is given by the 4-dimensional equivalent of Pythagoras' theorem:

$$ds^2 = -\eta_{ab}dx^a dx^b \tag{15.1.1}$$

But the distance could just as well be expressed in terms of the new coordinates, $\xi^\mu(x)$, so, exploiting Pythagoras' theorem, which is given in a Cartesian frame, we write (Einstein's summation convention used throughout)

$$ds^2 = -\eta_{ab}dx^a dx^b = -\eta_{ab}\frac{\partial x^a}{\partial \xi^\mu}\frac{\partial x^b}{\partial \xi^\nu}d\xi^\mu d\xi^\nu = -g_{\mu\nu}(\xi)d\xi^\mu d\xi^\nu \tag{15.1.2}$$

where we have made use of the chain rule:

$$dx^a = \frac{\partial x^a}{\partial \xi^\mu} d\xi^\mu \tag{15.1.3}$$

and called

$$g_{\mu\nu}(\xi) = \eta_{ab} \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial x^b}{\partial \xi^\nu} \tag{15.1.4}$$

This is the metric (it gives the distance between infinitesimally separated points) and it is, in general, a function of the new coordinates. Henceforth throughout this section, we'll use the following notation: indices from the beginning of the alphabet, $a, b, c, ...$, will represent a Cartesian basis and greek indices $\mu, \nu, ...$ will represent a general (curvilinear) basis. This will serve to distinguish indices that originate in the Cartesian system from those whose origin is in the curvilinear system.

Let us define the matrix

$$e_\mu^a(\xi) = \frac{\partial x^a}{\partial \xi^\mu} \tag{15.1.5}$$

It is a function of position and is called the "vielbein". A useful way is to think of it as a collection of four vectors, $\{\vec{e}_\mu\} = (\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$, whose Cartesian components are given by $[\vec{e}_\mu]^a = e_i^a$. These are the basis vectors of the new coordinate system $\xi^\mu(x)$. They are (generally) functions of the position because the new basis is not (generally) rigid. If we consider the identity transformations, for example, i.e., $\xi^\mu = (t, x, y, z)$ then it's easy to see that $\vec{e}_0 = (1, 0, 0, 0)$, $\vec{e}_1 = (0, 1, 0, 0)$, $\vec{e}_2 = (0, 0, 1, 0)$ and $\vec{e}_3 = (0, 0, 0, 1)$ or, said in another way, $e_\mu^a = \delta_\mu^a$.

The metric in (15.1.4) can be thought of as the matrix whose components are the (Cartesian) inner products of the $\vec{e}_\mu$, i.e.,[1]

$$g_{\mu\nu}(\xi) = \eta_{ab} e_\mu^a e_\mu^b = \vec{e}_\mu \cdot \vec{e}_\nu \tag{15.1.6}$$

and it is manifestly a scalar under Lorentz transformations. Also, g_{ij} is invertible, if the transformation $x \to \xi$ is invertible, in which case we can define the inverse metric by

$$g^{\mu\nu} g_{\nu\kappa} = \delta_\kappa^\mu \tag{15.1.7}$$

and it's easy to see that

$$g^{\mu\nu} = \eta^{ab} \frac{\partial \xi^\mu}{\partial x^a} \frac{\partial \xi^\nu}{\partial x^b} = \vec{E}^\mu \cdot \vec{E}^\nu \tag{15.1.8}$$

where $\vec{E}^\mu$ is the vector with (covariant) Cartesian components

$$[\vec{E}^\mu]_a = E_a^\mu = \frac{\partial \xi^\mu}{\partial x^a} \tag{15.1.9}$$

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[1]Problem: check that the identity transformation leads to $g_{\mu\nu} = \eta_{\mu\nu}$

It is called the *inverse* vielbein because, clearly,

$$e^a_\mu E^\mu_b = \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial \xi^\mu}{\partial x^b} = \delta^a_b, \quad e^a_\mu E^\nu_a = \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial \xi^\nu}{\partial x^a} = \delta^\nu_\mu \tag{15.1.10}$$

where we have repeatedly used the chain rule from elementary calculus.

15.2 Vectors and Tensors

Now the set $\{\vec{e}_\mu\}$ forms a complete basis in which any four vector can be expanded, *i.e.*,

$$\mathbb{A} = A^\mu \vec{e}_\mu \tag{15.2.1}$$

and A^μ are the contravariant components of the vector in the directions given by the $\vec{e}_\mu$. We can think of the r.h.s. as giving the contravariant *Cartesian* components of the vector according to

$$A^a = A^\mu e^a_\mu \tag{15.2.2}$$

in the original frame, while the contravariant components, A^μ, in the basis $\vec{e}_\mu$, are given in terms of its Cartesian components by the inverse relation

$$A^a E^\mu_a = A^\nu e^a_\nu E^\mu_a = A^\nu \delta^\mu_\nu = A^\mu \tag{15.2.3}$$

Any vector can be specified by specifying either the components A^a or the components A^μ. Naturally, there is also a description in terms of the covariant components of the vector using the inverse vielbeins:

$$\mathbb{A} = A_\mu \vec{E}^\mu \tag{15.2.4}$$

and the r.h.s. may be thought of as specifying the covariant Cartesian components of $\mathbb{A}$,

$$A_a = A_\mu E^\mu_a, \tag{15.2.5}$$

its covariant components, A_μ, in the basis $\vec{E}^\mu$ being given by

$$A_\mu = e^a_\mu A_a \tag{15.2.6}$$

in keeping with the relations for contravariant vectors.

We must next consider the transformation properties of the components, Cartesian or curvilinear. Note that there are two kinds of transformations to think about, *viz.*, Lorentz transformations, which concern the original Cartesian basis and general coordinate transformations, which concern the new (curvilinear) basis.

- Lorentz transformations act on the Cartesian coordinates, x^a, so, when $x^a \to x'^a$,

$$e'^a_\mu = \frac{\partial x'^a}{\partial \xi^\mu} = L^a{}_b \frac{\partial x^b}{\partial \xi^\mu} = L^a{}_b e^a_\mu \qquad (15.2.7)$$

showing that e^a_μ transforms as a contravariant vector w.r.t. Lorentz transformations. On the other hand, E^μ_a will transform as a covariant vector

$$E'^\mu_a = \frac{\partial \xi^\mu}{\partial x'^a} = (L^{-1})^b{}_a \frac{\partial \xi^\mu}{\partial x^b} = (L^{-1})^b{}_a E^\mu_b \qquad (15.2.8)$$

w.r.t. the same transformations

- Under general coordinate transformations, $\xi^\mu \to \xi'^\mu$,

$$e'^a_\mu = \frac{\partial x^a}{\partial \xi'^\mu} = \frac{\partial \xi^\nu}{\partial \xi'^\mu} \frac{\partial x^a}{\partial \xi^\nu} = \frac{\partial \xi^\nu}{\partial \xi'^\mu} e^a_\nu = (\Lambda^{-1})^\nu{}_\mu e^a_\nu \qquad (15.2.9)$$

or

$$\vec{e}'_\mu = \frac{\partial \xi^\nu}{\partial \xi'^\mu} \vec{e}_\nu = (\Lambda^{-1})^\nu{}_\mu \vec{e}_\nu \qquad (15.2.10)$$

and

$$E'^\mu_a = \frac{\partial \xi'^\mu}{\partial x^a} = \frac{\partial \xi'^\mu}{\partial \xi^\nu} \frac{\partial \xi^\nu}{\partial x^a} = \frac{\partial \xi'^\mu}{\partial \xi^\nu} E^\nu_a = \Lambda^\mu{}_\nu E^\nu_a \qquad (15.2.11)$$

or

$$\vec{E}'^\mu = \frac{\partial \xi'^\mu}{\partial \xi^\nu} \vec{E}^\nu = \Lambda^\mu{}_\nu \vec{E}^\nu \qquad (15.2.12)$$

Note that $\hat{\Lambda}$, unlike $\hat{L}$, is **not** necessarily a constant matrix. These transformation properties imply that the metric in (15.1.4) transforms (under coordinate transformations) as

$$g'_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu g_{\alpha\beta} \qquad (15.2.13)$$

and

$$g'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g^{\alpha\beta} \qquad (15.2.14)$$

but is a Lorentz *scalar*.

It should be clear that the Cartesian components, A^a (A_a) transform only under Lorentz transformations (they are "coordinate scalars") and the components A^μ (A_μ) transform only under general coordinate transformations (they are "Lorentz scalars"). How do they transform? They follow the same rules as the Cartesian components, but transform under $\hat{\Lambda}$ instead of $\hat{L}$. As a vector does not depend on the basis in which it is expanded,

$$\mathbb{A} = A'^\mu \vec{e}'_\mu = A'^\mu (\Lambda^{-1})^\nu{}_\mu \vec{e}_\nu = A^\nu \vec{e}_\nu \qquad (15.2.15)$$

implying obviously that

$$A'^{\mu} = \Lambda^{\mu}{}_{\nu} A^{\nu} \tag{15.2.16}$$

and a completely analogous argument shows that

$$A'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu} A_{\nu} \tag{15.2.17}$$

is the transformation property of the covariant components. The contravariant components and the covariant components transform inversely to one another, so they must be related by the metric

$$A_{\mu} = g_{\mu\nu} A^{\nu}$$

$$A^{\mu} = g^{\mu\nu} A_{\nu} \tag{15.2.18}$$

because the metric and it's inverse have precisely the transformation properties required. Moreover, it's easy to see now how we might construct scalars (under both Lorentz and general coordinate transformations):

$$\mathbb{A}^2 = \mathbb{A} \cdot \mathbb{A} = (A^{\mu} \vec{e}_{\mu}) \cdot (A^{\nu} \vec{e}_{\nu}) = A^{\mu} A^{\nu} \vec{e}_{\mu} \cdot \vec{e}_{\nu} = g_{\mu\nu} A^{\mu} A^{\nu} = A^{\mu} A_{\mu} \tag{15.2.19}$$

That $\mathbb{A} \cdot \mathbb{A}$ is really a scalar follows from

$$g_{\mu\nu} A^{\mu} A^{\nu} = \eta_{ab} e^a_{\mu} e^b_{\nu} A^{\mu} A^{\nu} = \eta_{ab} A^a A^b = A^a A_a \tag{15.2.20}$$

or directly from

$$A'^{\mu} A'_{\mu} = \Lambda^{\mu}{}_{\alpha} (\Lambda^{-1})^{\beta}{}_{\mu} A^{\alpha} A_{\beta} = \delta^{\beta}_{\alpha} A^{\alpha} A_{\beta} = A^{\alpha} A_{\alpha} \tag{15.2.21}$$

As usual we will define tensors as copies of vectors, their components in any basis being given by

$$\mathbb{T} = T^{\mu\nu\lambda\cdots} \vec{e}_{\mu} \otimes \vec{e}_{\nu} \otimes \vec{e}_{\lambda}\cdots = T_{\mu\nu\lambda\cdots} \vec{E}^{\mu} \otimes \vec{E}^{\nu} \otimes \vec{E}^{\lambda}\cdots \tag{15.2.22}$$

where $T^{\mu\nu\lambda\cdots}$ and $T_{\mu\nu\lambda\cdots}$ are the contravariant and covariant components of $\mathbb{T}$ respectively. Then their transformation properties are given by

$$T'^{\mu\nu\lambda\cdots} = \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \Lambda^{\lambda}{}_{\gamma} T^{\alpha\beta\gamma\cdots} \tag{15.2.23}$$

and

$$T'_{\mu\nu\lambda\cdots} = (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu} (\Lambda^{-1})^{\gamma}{}_{\lambda} T_{\alpha\beta\gamma\cdots} \tag{15.2.24}$$

respectively. Just as for vectors, the covariant and contravariant components of a tensor are related by the metric (tensor):

$$T^{\mu\nu\lambda\cdots} = g^{\mu\alpha} g^{\nu\beta} g^{\lambda\gamma}\cdots \ T_{\alpha\beta\gamma\cdots} \tag{15.2.25}$$

and

$$T_{\mu\nu\lambda...} = g_{\mu\alpha}g_{\nu\beta}g_{\lambda\gamma}...\ T^{\alpha\beta\gamma...} \tag{15.2.26}$$

and one can interpolate between components in the original Cartesian basis and in the curvilinear basis by simply applying the vielbein and its inverse, just as we did for vectors

$$T^{ab...} \ = \ e^a_\mu e^b_\nu...\ T^{\mu\nu...}, \qquad T^{\mu\nu...} = E^\mu_a E^\nu_b...\ T^{ab...}$$

$$T_{ab...} \ = \ E^\mu_a E^\nu_b...\ T_{\mu\nu...}, \qquad T_{\mu\nu...} = e^a_\mu e^b_\nu...\ T^{ab...} \tag{15.2.27}$$

15.3   Differentiation

In differentiating a tensor, we are generally interested in measuring the rates of change of the tensor as we move from point to point on the manifold. To do so we measure the difference between the values of the tensor at infinitesimally close points, finally taking the limit as the points approach each other. However, depending on what differences we measure, the resulting rate of change may not have definite transformation properties under general coordinate transformations. Below we will consider two ways to define the "derivative" of a tensor so that the derivative is itself a tensor.

15.3.1   Lie Derivative

Often we may be interested in how the components of a given vector or tensor or even just a function change(s) as we move along some curve, $\xi^\mu(\lambda)$, parametrized by λ, from a point p to another point p'. To define the Lie derivative, we consider a special set of curves, which are constructed from coordinate transformations. Consider a one parameter family of coordinate transformations $\xi'^\mu(\lambda, \xi)$ so that the $\lambda = 0$ transformation is just the identity transformation, $\xi'(0, \xi) = \xi$. Let the coordinates of point p be ξ^μ_p. Holding ξ_p fixed, $\xi'^\mu(\lambda, \xi_p)$ represents the a curve passing through p at $\lambda = 0$. Suppose that we have chosen our one parameter family of transformations so that the curve $\xi'^\mu(\lambda, \xi_p)$ passes through p' at $\delta\lambda$. Let $U^\mu(\lambda, \xi_p)$ be tangent to the curve. The point p' is therefore represented by

$$\xi'^\mu = \xi'^\mu(\delta\lambda, \xi_p) = \xi^\mu_p + \delta\lambda U^\mu(\xi_p) \tag{15.3.1}$$

This is the "active" view of coordinate transformations, where they are used to actually "push points around". If $\mathbb{T}$ is a tensor, it transforms as

$$T'^{\mu_1\mu_2...}{}_{\nu_1\nu_2...}(\xi') = \frac{\partial \xi'^{\mu_1}}{\partial \xi^{\alpha_1}}...\frac{\partial \xi^{\beta_1}}{\partial \xi'^{\nu_1}}...T^{\alpha_1\alpha_2...}{}_{\beta_1\beta_2...}(\xi) \tag{15.3.2}$$

and therefore the left hand side of the above should give the value of $\mathbb{T}$ at p' from its value at p. The Lie derivative of $\mathbb{T}$ is then defined as

$$[\mathcal{L}_U \mathbb{T}]^{\mu_1\mu_2...}{}_{\nu_1\nu_2...} = \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[T^{\mu_1\mu_2...}{}_{\nu_1\nu_2...}(\xi) - T'^{\mu_1\mu_2...}{}_{\nu_1\nu_2...}(\xi) \right] \tag{15.3.3}$$

It measures the rate of change of the *functional form of the components* of a tensor field by a coordinate transformation, in the direction of U.

For scalar functions, we see immediately that this is just the directional derivative, for if $\mathbb{T}$ is a scalar function, $f(\xi)$, then $f'(\xi') = f(\xi) \Rightarrow f(\xi) - f'(\xi) = \delta\lambda U^\mu \partial_\mu f$ (to order λ), therefore

$$\mathcal{L}_U f(x) = \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[f(\xi) - f'(\xi) \right] = U^\mu \partial_\mu f(\xi) \tag{15.3.4}$$

If $\mathbb{T}$ is a vector field, $V^\mu(\xi)$, then

$$
\begin{aligned}
[\mathcal{L}_U V]^\mu &= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[V^\mu(\xi) - V'^\mu(\xi) \right] \\
&= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\frac{\partial \xi^\mu}{\partial \xi'^\alpha} V'^\alpha(\xi') - V'^\mu(\xi) \right] \\
&= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\frac{\partial \xi^\mu}{\partial \xi'^\alpha} (V'^\alpha(\xi) + \delta\lambda U^\kappa \partial_\kappa V^\alpha + ...) - V'^\mu(\xi) \right] \\
&= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[(\delta^\mu_\alpha - \delta\lambda \partial_\alpha U^\mu + ...)(V'^\alpha(\xi) + \delta\lambda U^\kappa \partial_\kappa V^\alpha + ...) - V'^\mu(\xi) \right] \\
&= U^\kappa \partial_\kappa V^\mu - V^\kappa \partial_\kappa U^\mu \tag{15.3.5}
\end{aligned}
$$

Finally, for a co-vector field, $W_\mu(\xi)$

$$
\begin{aligned}
\mathcal{L}_U W &= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[W_\mu(\xi) - W'_\mu(\xi) \right] \\
&= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[\frac{\partial \xi'^\alpha}{\partial \xi^\mu} W'_\alpha(\xi') - W'_\mu(\xi) \right] \\
&= \lim_{\delta\lambda \to 0} \frac{1}{\delta\lambda} \left[(\delta^\alpha_\mu + \delta\lambda \partial_\mu U^\alpha + ...)(W'_\alpha(\xi) + \delta\lambda U^\kappa \partial_\kappa W_\alpha + ...) - W'_\mu(\xi) \right] \\
&= U^\kappa \partial_\kappa W_\mu + W_\alpha \partial_\mu U^\alpha \tag{15.3.6}
\end{aligned}
$$

and so on for tensors of higher rank.[2] If $\mathcal{L}_U \mathbb{T} = 0$, then $\mathbb{T}$ does not change as we move along the integral curve of U. In this case, the vector U is called a "symmetry" of $\mathbb{T}$. Note that the Lie derivative of a tensor field $\mathbb{T}$ is of the same rank as $\mathbb{T}$ itself.

---

[2]Obtain the Lie derivative of second rank contravariant, covariant and mixed tensors. In general, the Lie derivative of a mixed tensor takes the form

$$[\mathcal{L}_U \mathbb{T}]^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \cdots} = U^\sigma \partial_\sigma T^{\mu_1 \mu_2 \cdots}{}_{\nu_1 \nu_2 \cdots} - T^{\sigma \mu_2 \cdots}{}_{\nu_1 \nu_2 \cdots} \partial_\sigma U^{\mu_1} - \cdots$$

$$+ T^{\mu_1 \mu_2 \cdots}{}_{\sigma \nu_2 \cdots} \partial_{\nu_1} U^\sigma + \cdots$$

where the ellipsis means that we repeat the terms of each index of the same type.

15.3.2   Covariant Derivative: the Connection

The Lie derivative can be thought of as an operator that acts upon a tensor to yield another tensor of the same rank. However, when we think of a derivative, we think of the operator ∂_a (say), which has the effect of increasing the rank of the tensor. Thus, for example, if $\mathbb{T}$ is a rank (m, n) tensor (m contravariant indices and n covariant indices) then $\partial\mathbb{T}$ is a tensor of rank $(m, n+1)$, if the partial derivative is applied in a Cartesian coordinate system. However, $\partial\mathbb{T}$ is not a tensor in a general coordinate system, as we will see below. We would like to obtain a derivative operator, ∇, in general curvilinear coordinates that plays the role of ∂ in Cartesian coordinates, *i.e.*, an operator which acts on an (m, n) tensor to give an $(m, n+1)$ tensor. So let us begin with vectors. Imagine transporting a vector from some point p to some other point p'. The basis vectors (the vielbeins) and their inverses are not necessarily constant during this transport – they would be constant only if the coordinate system is not curviliear. Instead of asking about changes in the components of a vector $\mathbb{A}$, let's ask instead how the vector as a whole changes as we move from p to p'. We find

$$\delta\mathbb{A} = (\delta A^\mu)\vec{e}_\mu + A^\mu(\delta\vec{e}_\mu) \tag{15.3.7}$$

Assume that the change in $\vec{e}_\mu$ is a linear combination of the $\vec{e}_\mu$ themselves. This is a reasonable assumption because the basis at p is complete. Then

$$\delta\vec{e}_\mu = (\delta\Gamma^\nu_\mu)\vec{e}_\nu \tag{15.3.8}$$

and $\Gamma^\nu_\mu \equiv \Gamma^\nu_\mu(x)$ will be in general a function of the position. In fact we can obtain $\delta\Gamma^\nu_\mu$ in terms of $\vec{e}_\mu$ and $\vec{E}^\mu$ as follows: begin with

$$\vec{e}_\mu \cdot \vec{E}^\sigma = \delta^\sigma_\mu \Rightarrow (\delta\vec{e}_\mu) \cdot E^\sigma = -\vec{e}_\mu \cdot (\delta\vec{E}^\sigma) \tag{15.3.9}$$

but, using (15.3.8), we see that

$$\delta\Gamma^\nu_\mu(\vec{e}_\nu \cdot \vec{E}^\sigma) = \delta\Gamma^\sigma_\mu = -\vec{e}_\mu \cdot (\delta\vec{E}^\sigma) = (\delta\vec{e}_\mu) \cdot E^\sigma \tag{15.3.10}$$

We can write

$$\delta\mathbb{A} = (\delta A^\mu)\vec{e}_\mu + A^\mu(\delta\Gamma^\nu_\mu)\vec{e}_\nu \tag{15.3.11}$$

and we see that the change in $\mathbb{A}$ is made up of two parts: (i) the change in its components and (ii) the changing the basis, as we move from one point to another. The term $\delta\Gamma$ takes into account the change in basis. The difference, $\delta\mathbb{A}$, is also a vector and is expandable in the basis $\vec{e}_\mu$. If we consequently write it as $\delta\mathbb{A} = (DA^\mu)\vec{e}_\mu$, we find

$$DA^\mu = \delta A^\mu + (\delta\Gamma^\mu_\nu)A^\nu \tag{15.3.12}$$

but what does DA^μ represent? Notice that if the basis is rigid then $\delta\vec{e}_\mu = 0 = \delta\Gamma^\nu_\mu$ and there is no difference between the variations DA^μ and δA^μ. This equality fails in a general

coordinate system, however, and the second term is important. The derivative corresponding to the infinitesimal change given in (15.3.12) is called the "covariant derivative" of A^μ,

$$D_\nu A^\mu \equiv \nabla_\nu A^\mu = \partial_\nu A^\mu + (\partial_\nu \Gamma^\mu_\lambda) A^\lambda = \partial_\nu A^\mu + \Gamma^\mu_{\nu\lambda} A^\lambda \qquad (15.3.13)$$

and the 3-index object $\Gamma^\mu_{\nu\lambda} = \partial_\nu \Gamma^\mu_\lambda$ is called a "connection". Using (15.3.10), it can be written as

$$\Gamma^\mu_{\nu\lambda} = (\partial_\nu \vec{e}_\lambda) \cdot \vec{E}^\mu. \qquad (15.3.14)$$

It is interesting to see that the ordinary derivative, $\partial_\nu A^\mu$ of a contravariant vector does not transform as a (mixed) tensor, but the covariant derivative, $\nabla_\nu A^\mu$, does. The fact that the covariant derivative transforms as a tensor is of great importance. As we have mentioned, the laws of physics should not depend on one's choice of coordinates. This means that they should "look the same" in any system, which is possible only if the two sides of any dynamical equation transform in the same manner, *i.e.*, either as scalars, vectors or tensors under transformations between coordinate systems. Thus, covariant derivatives and not ordinary derivatives are more meaningful in physics.

First let's see that $\partial_\nu A^\mu$ is not a tensor:

$$\frac{\partial A'^\mu}{\partial \xi'^\nu} = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial}{\partial \xi^\lambda} \left(\frac{\partial \xi'^\mu}{\partial \xi^\kappa} A^\kappa \right) = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\kappa} \frac{\partial A^\kappa}{\partial \xi^\lambda} + \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\kappa} A^\kappa \qquad (15.3.15)$$

The first term on the r.h.s. corresponds to the tensor transformation, but the second term spoils the transformation properties of $\partial_\nu A^\mu$. Let us then examine the transformation properties of $\nabla_\nu A^\mu$:

$$\begin{aligned} \nabla'_\nu A'^\mu &= \partial'_\nu A'^\mu + \Gamma'^\mu_{\nu\kappa} A'^\kappa \\[1em] &= \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\gamma} \frac{\partial A^\gamma}{\partial \xi^\lambda} + \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\gamma} A^\gamma + \frac{\partial \xi'^\kappa}{\partial \xi^\gamma} \Gamma'^\mu_{\nu\kappa} A^\gamma \end{aligned} \qquad (15.3.16)$$

If we can show that

$$\frac{\partial \xi'^\kappa}{\partial \xi^\gamma} \Gamma'^\mu_{\nu\kappa} = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \Gamma^\sigma_{\lambda\gamma} - \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\gamma} \qquad (15.3.17)$$

then we will have

$$\nabla'_\nu A'^\mu = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \left[\frac{\partial A^\sigma}{\partial \xi^\lambda} + \Gamma^\sigma_{\lambda\gamma} A^\gamma \right] = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \nabla_\lambda A^\sigma = (\Lambda^{-1})^\lambda_{\ \nu} \Lambda^\mu_{\ \sigma} \nabla_\lambda A^\sigma \quad (15.3.18)$$

and we will have accomplished the task of showing that $\nabla_\nu A^\mu$ is a tensor. It is not so difficult to show (15.3.17). First put it in the form

$$\Gamma'^\mu_{\nu\kappa} = \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial \xi'^\mu}{\partial \xi^\sigma} \frac{\partial \xi^\gamma}{\partial \xi'^\kappa} \Gamma^\sigma_{\lambda\gamma} - \frac{\partial \xi^\lambda}{\partial \xi'^\nu} \frac{\partial^2 \xi'^\mu}{\partial \xi^\lambda \xi^\gamma} \frac{\partial \xi^\gamma}{\partial \xi'^\kappa} \qquad (15.3.19)$$

To show (15.3.19), write

$$\Gamma'^{\mu}_{\nu\kappa} = \partial'_{\nu}\Gamma'^{\mu}_{\kappa} = (\partial'_{\nu}\vec{e}'_{\kappa}) \cdot \vec{E}'^{\mu} = -\vec{e}'_{\kappa}(\partial'_{\nu}\vec{E}'^{\mu}) \tag{15.3.20}$$

where we have used the fact that $\Gamma^{\mu}_{\nu\kappa} = (\partial_{\nu}\vec{e}_{\kappa}) \cdot \vec{E}^{\mu}$, which follows from the definition of $\delta\Gamma$ in (15.3.8), and $\vec{e}'_{\kappa} \cdot \vec{E}'^{\mu} = \delta^{\mu}_{\kappa}$. Then

$$\begin{aligned}
\Gamma'^{\mu}_{\nu\kappa} &= -\frac{\partial\xi^{\gamma}}{\partial\xi'^{\kappa}}\frac{\partial\xi^{\lambda}}{\partial\xi'^{\nu}}\vec{e}_{\gamma} \cdot \frac{\partial}{\partial\xi^{\lambda}}\left(\frac{\partial\xi'^{\mu}}{\partial\xi^{\sigma}}\vec{E}^{\sigma}\right) \\
&= -\frac{\partial\xi^{\gamma}}{\partial\xi'^{\kappa}}\frac{\partial\xi^{\lambda}}{\partial\xi'^{\nu}}\frac{\partial\xi'^{\mu}}{\partial\xi^{\sigma}}\vec{e}_{\gamma} \cdot (\partial_{\lambda}\vec{E}^{\sigma}) - \frac{\partial\xi^{\gamma}}{\partial\xi'^{\kappa}}\frac{\partial\xi^{\lambda}}{\partial\xi'^{\nu}}\frac{\partial^2\xi'^{\mu}}{\partial\xi^{\lambda}\partial\xi^{\sigma}}\vec{e}_{\gamma} \cdot \vec{E}^{\sigma} \\
&= \frac{\partial\xi^{\gamma}}{\partial\xi'^{\kappa}}\frac{\partial\xi^{\lambda}}{\partial\xi'^{\nu}}\frac{\partial\xi'^{\mu}}{\partial\xi^{\sigma}}\Gamma^{\sigma}_{\lambda\gamma} - \frac{\partial\xi^{\lambda}}{\partial\xi'^{\nu}}\frac{\partial^2\xi'^{\mu}}{\partial\xi^{\lambda}\partial\xi^{\gamma}}\frac{\partial\xi^{\gamma}}{\partial\xi'^{\kappa}}
\end{aligned} \tag{15.3.21}$$

which is the desired result. Again, notice that without the second term the above would correspond to a tensor transformation, but the second term spoils the transformation properties. In fact it is precisely because of the presence of the second term that $\nabla_{\nu}A^{\mu}$ transforms as a tensor. Note also that if the unprimed coordinates were Cartesian, $(\sigma, \lambda, \gamma) \equiv (a, b, c)$, then $\Gamma^{a}_{bc} \equiv 0$ and

$$\Gamma^{\mu}_{\nu\kappa} = -\frac{\partial x^{b}}{\partial\xi^{\nu}}\frac{\partial^2\xi^{\mu}}{\partial x^{b}\partial x^{c}}\frac{\partial x^{c}}{\partial\xi^{\kappa}} \tag{15.3.22}$$

which shows that $\Gamma^{\mu}_{\nu\kappa}$ is symmetric in (ν, κ).

In the Cartesian basis, the derivative of a vector is just $\partial_{a}A^{b}$. If we now transform to the curvilinear coordinates,

$$\partial_{a}A^{b} = \frac{\partial\xi^{\mu}}{\partial x^{a}}\frac{\partial}{\partial\xi^{\mu}}(A^{\nu}e^{b}_{\nu}) = E^{\mu}_{a}(\partial_{\mu}A^{\nu})e^{b}_{\nu} + E^{\mu}_{a}A^{\nu}(\partial_{\mu}e^{b}_{\nu}) \tag{15.3.23}$$

so that

$$\begin{aligned}
e^{a}_{\sigma}E^{\lambda}_{b}\partial_{a}A^{b} &= e^{a}_{\sigma}E^{\lambda}_{b}E^{\mu}_{a}(\partial_{\mu}A^{\nu})e^{b}_{\nu} + e^{a}_{\sigma}E^{\lambda}_{b}E^{\mu}_{a}A^{\nu}(\partial_{\mu}e^{b}_{\nu}) \\
&= \partial_{\sigma}A^{\lambda} + E^{\lambda}_{b}A^{\nu}(\partial_{\sigma}e^{b}_{\nu}) = \partial_{\sigma}A^{\lambda} + \Gamma^{\lambda}_{\sigma\nu}A^{\nu}
\end{aligned} \tag{15.3.24}$$

If we think of $\partial_{a}A^{b}$ as the components of a (mixed) tensor in the Cartesian system then, in a general coordinate system, its components should be given by $e^{a}_{\sigma}E^{\lambda}_{b}\partial_{a}A^{b}$. The above equation shows that its components in the general coordinate basis are given by the components of the covariant derivative. In other words, derivatives of vectors in the Cartesian coordinate system must be replaced by covariant derivatives in general coordinate systems.

The connection measures the rate of change of the basis as we move from point to point and is computed from the metric $g_{\mu\nu}$ as we will now see. Consider the change in metric as we move from x to $x + dx$,

$$\delta g_{\mu\nu} \;=\; \delta\vec{e}_\mu \cdot \vec{e}_\nu + \vec{e}_\mu \cdot \delta\vec{e}_\nu = (\delta\Gamma^\kappa_\mu)\vec{e}_\kappa \cdot \vec{e}_\nu + \vec{e}_\mu \cdot \vec{e}_\kappa(\delta\Gamma^\kappa_\nu)$$

$$\rightarrow \quad \partial_\gamma g_{\mu\nu} \;=\; \Gamma^\kappa_{\gamma\mu} g_{\kappa\nu} + \Gamma^\kappa_{\gamma\nu} g_{\mu\kappa} \tag{15.3.25}$$

If we take the combination

$$\partial_\gamma g_{\mu\nu} + \partial_\nu g_{\gamma\mu} - \partial_\mu g_{\nu\gamma} \;=\; \Gamma^\kappa_{\gamma\mu} g_{\kappa\nu} + \Gamma^\kappa_{\gamma\nu} g_{\mu\kappa} + \Gamma^\kappa_{\nu\gamma} g_{\kappa\mu} + \Gamma^\kappa_{\nu\mu} g_{\gamma\kappa} - \Gamma^\kappa_{\mu\nu} g_{\kappa\gamma} - \Gamma^\kappa_{\mu\gamma} g_{\nu\kappa} \tag{15.3.26}$$

and use the fact that $\partial_{[\gamma}\Gamma^\kappa_{\nu]} \equiv 0$ according to (15.3.22), then

$$\partial_\gamma g_{\mu\nu} + \partial_\nu g_{\gamma\mu} - \partial_\mu g_{\nu\gamma} = 2\Gamma^\kappa_{\gamma\nu} g_{\mu\kappa} \tag{15.3.27}$$

and

$$\Gamma^\kappa_{\gamma\nu} g_{\mu\kappa} g^{\mu\rho} = \Gamma^\kappa_{\gamma\nu}\delta^\rho_\kappa = \Gamma^\rho_{\gamma\nu} = \frac{1}{2} g^{\rho\mu}\left[\partial_\nu g_{\mu\gamma} + \partial_\gamma g_{\mu\nu} - \partial_\mu g_{\gamma\nu}\right], \tag{15.3.28}$$

where we have used the symmetry of $g_{\mu\nu}$ throughout.

It should be clear that the covariant derivative of a tensor copies the covariant derivative of the vector. Setting,

$$\mathbb{T} = T^{\mu\nu\cdots}\vec{e}_\mu\vec{e}_\nu\cdots \tag{15.3.29}$$

we get

$$\delta\mathbb{T} = \delta T^{\mu\nu\cdots}\vec{e}_\mu\vec{e}_\nu\cdots + T^{\mu\nu\cdots}(\delta\vec{e}_\mu)\vec{e}_\nu\cdots + T^{\mu\nu\cdots}\vec{e}_\mu(\delta\vec{e}_\nu)\cdots + \cdots \tag{15.3.30}$$

from which it follows that

$$\nabla_\gamma T^{\mu\nu\cdots} = \partial_\gamma T^{\mu\nu\cdots} + \Gamma^\mu_{\gamma\lambda} T^{\lambda\nu\cdots} + \Gamma^\nu_{\gamma\lambda} T^{\mu\lambda\cdots} + \cdots \tag{15.3.31}$$

We have defined the covariant derivatives of a contravariant vector. How about the covariant derivative of a covector? We should find

$$\delta\mathbb{A} = (\delta A_\mu)\vec{E}^\mu + A_\mu(\delta\vec{E}^\mu) \tag{15.3.32}$$

and we want to know what $\delta\vec{E}^\mu$ is. Use the fact that

$$e^a_\mu E^\mu_b = \delta^a_b \rightarrow (\delta e^a_\mu)E^\mu_b + e^a_\mu(\delta E^\mu_b) = 0 \tag{15.3.33}$$

Therefore

$$e^a_\mu(\delta E^\mu_b) = -(\delta e^a_\mu)E^\mu_b = -(\delta\Gamma^\kappa_\mu)e^a_\kappa E^\mu_b \tag{15.3.34}$$

and, multiplying the l.h.s. by E_a^ν gives

$$E_a^\nu e_\mu^a (\delta E_b^\mu) = \delta_\mu^\nu (\delta E_b^\mu) = \delta E_b^\nu = -(\delta \Gamma_\mu^\kappa) E_a^\nu e_\kappa^a E_b^\mu = -(\delta \Gamma_\mu^\kappa) \delta_\kappa^\nu E_b^\mu = -(\delta \Gamma_\mu^\nu) E_b^\mu \quad (15.3.35)$$

Thus we have found simply that

$$\delta \vec{E}^\mu = -(\delta \Gamma_\kappa^\mu) \vec{E}^\kappa \quad (15.3.36)$$

which should be compared with (15.3.8) for the variation of $\vec{e}_\mu$. Therefore

$$\delta \mathbb{A} = (DA_\mu) \vec{E}^\mu = (\delta A_\mu) \vec{E}^\mu - (\delta \Gamma_\kappa^\mu) A_\mu \vec{E}^\kappa \quad (15.3.37)$$

and we could write the covariant derivative of the covector, A_μ

$$\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma_{\nu\mu}^\kappa A_\kappa \quad (15.3.38)$$

and of a co-tensor, $T_{\mu\nu\kappa...}$

$$\nabla_\gamma T_{\mu\nu...} = \partial_\gamma T_{\mu\nu...} - \Gamma_{\gamma\mu}^\lambda T_{\lambda\nu...} - \Gamma_{\gamma\nu}^\lambda T_{\mu\lambda...} + ... \quad (15.3.39)$$

in complete analogy with the covariant derivative of contravectors and tensors. In particular we see that

$$\nabla_\gamma g_{\mu\nu} = \partial_\gamma g_{\mu\nu} - \Gamma_{\gamma\mu}^\kappa g_{\kappa\nu} - \Gamma_{\gamma\nu}^\kappa g_{\mu\kappa} \equiv 0 \equiv \nabla_\gamma g^{\mu\nu} \quad (15.3.40)$$

by (15.3.25). This is called the "metricity" property.[3]

15.3.3   Absolute Derivative: parallel transport

Having defined the covariant derivative operator, in a general coordinate system, we may now define the *absolute derivative* of a tensor, $\mathbb{T}$, along some curve with tangent vector U as the projection of the covariant derivative on the tangent, *i.e.,* if the curve is specified by $\xi^\mu(\lambda)$,

$$\frac{D\mathbb{T}}{D\lambda} = U \cdot \nabla \mathbb{T} \quad (15.3.41)$$

The absolute derivative measures the total rate of change of the vector along the curve $\xi^\mu(\lambda)$ and is a tensor of the same rank as $\mathbb{T}$ itself. It is also called the directional derivative of $\mathbb{T}$.

---

[3]Using the Lie derivative of the metric (a rank two co-tensor) show that if U is a symmetry of the metric then it must satisfy

$$\nabla_{(\mu} U_{\nu)} = \nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$$

The symmetry vectors of the metric are called Killing vectors. In Minkowski space there are 10 of them and they generate the Poincaré group: translations, spatial rotations and boosts.

A tensor $\mathbb{T}$ will be said to be "parallel transported" along a curve $\xi^\mu(\lambda)$ if and only if

$$\frac{D\mathbb{T}}{D\lambda} = f(\lambda)\mathbb{T} \tag{15.3.42}$$

where $f(\lambda)$ is an arbitrary function of the curve's parameter. Let A^μ be parallely transported along the curve, then

$$U \cdot \nabla A^\mu = U^\sigma(\partial_\sigma A^\mu + \Gamma^\mu_{\sigma\kappa}A^\kappa) = \frac{dA^\mu}{d\lambda} + \Gamma^\mu_{\sigma\kappa}U^\sigma A^\kappa = f(\lambda)A^\mu \tag{15.3.43}$$

This is the condition for parallel transport. In particular, if A^μ is the tangent (velocity) vector of the curve itself, we see that

$$\frac{dU^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\kappa}U^\sigma U^\kappa = \frac{d^2\xi^\mu}{d\lambda^2} + \Gamma^\mu_{\sigma\kappa}\frac{d\xi^\sigma}{d\lambda}\frac{d\xi^\kappa}{d\lambda} = f(\lambda)\frac{d\xi^\mu}{d\lambda} \tag{15.3.44}$$

This is a second order equation for $\xi^\mu(\lambda)$. We notice that if the coordinates were Cartesian, the connections would vanish and with $f(\lambda) = 0$ we would find simply

$$\frac{d^2x^a}{d\lambda^2} = 0 \tag{15.3.45}$$

which we recognize to be the equation of a straight line, the shortest distance between two points (the geodesic). The equation (15.3.44) generalizes this equation for geodesics to arbitrary curved coordinate systems and is called the geodesic equation.

15.3.4   The Laplacian

A very important operator in physics is the Laplacian. It is an invariant under coordinate transformations being defined, in an arbitrary system of coordinates, as $\Box_x = \nabla_\mu\nabla^\mu$. Because it involves the covariant derivative its action will depend on whether it operates on a scalar, a vector or a tensor. Consider its operation on a scalar function, ϕ (remember that $\nabla^\mu\phi = \partial^\mu\phi$ is a vector)

$$\Box_x\phi = \nabla_\mu\nabla^\mu\phi = \partial_\mu\nabla^\mu\phi + \Gamma^\mu_{\mu\kappa}\nabla^\kappa\phi = \partial_\mu g^{\mu\nu}\partial_\nu\phi + \Gamma^\mu_{\mu\kappa}g^{\kappa\nu}\partial_\nu\phi \tag{15.3.46}$$

where we have used the fact that the covariant derivative operating on a scalar function is just the partial derivative. But

$$\Gamma^\mu_{\mu\kappa} = \frac{1}{2}g^{\mu\rho}[\partial_\kappa g_{\rho\mu} + \partial_\mu g_{\rho\kappa} - \partial_\rho g_{\mu\kappa}] \tag{15.3.47}$$

Interchanging $(\mu\rho)$ in the middle term shows that it cancels the last, so

$$\Gamma^\mu_{\mu\kappa} = \frac{1}{2}g^{\mu\rho}\partial_\kappa g_{\rho\mu} \tag{15.3.48}$$

This expression may be further simplified: let g be the determinant of $g_{\mu\nu}$, then

$$\ln g = \operatorname{tr} \ln \hat{g} \to \delta \ln g = \frac{\delta g}{g} = \operatorname{tr} \hat{g}^{-1} \delta \hat{g} = g^{\mu\rho} \delta g_{\mu\rho} \qquad (15.3.49)$$

and therefore

$$\frac{1}{g}\partial_\kappa g = g^{\mu\rho}\partial_\kappa g_{\mu\rho}, \quad \Gamma^\mu_{\mu\kappa} = \partial_\kappa \ln \sqrt{g} \qquad (15.3.50)$$

which means that

$$\Box_x \phi = \partial_\mu(g^{\mu\nu}\partial_\nu\phi) + g^{\mu\nu}(\partial_\mu \ln \sqrt{g})\partial_\nu\phi = \frac{1}{\sqrt{g}}\partial_\mu \sqrt{g}\, g^{\mu\nu}\partial_\nu\phi \qquad (15.3.51)$$

This is a very compact formula. Life is not so easy if the Laplacian, $\Box_x$, operates on a vector (worse, on a tensor), instead of a scalar. Then we have

$$
\begin{aligned}
\Box_x A^\mu &= \nabla_\nu \nabla^\nu A^\mu = g^{\nu\kappa}\nabla_\nu\nabla_\kappa A^\mu = g^{\nu\kappa}[\partial_\nu \nabla_\kappa A^\mu - \Gamma^\lambda_{\nu\kappa}\nabla_\lambda A^\mu + \Gamma^\mu_{\nu\lambda}\nabla_\kappa A^\lambda] \\
&= g^{\nu\kappa}[\partial_\nu(\partial_\kappa A^\mu + \Gamma^\mu_{\kappa\lambda}A^\lambda) - \Gamma^\lambda_{\nu\kappa}(\partial_\lambda A^\mu + \Gamma^\mu_{\lambda\gamma}A^\gamma) \\
&\qquad\qquad + \Gamma^\mu_{\nu\lambda}(\partial_\kappa A^\lambda + \Gamma^\lambda_{\kappa\gamma}A^\gamma)] \qquad (15.3.52)
\end{aligned}
$$

which is certainly more complicated. Let's see how this works through some common examples. Only the results will be given, the details are left to the reader.

15.4   Examples

Spherical Coordinates

Take the following coordinate functions: $\xi^\mu = (t, r, \theta, \phi)$ where

$$
\begin{aligned}
t &= t \\
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\
\varphi &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
\qquad (15.4.1)
$$

and the inverse transformations: $x^a = x^a(\xi)$

$$
\begin{aligned}
t &= t \\
x &= r\sin\theta\cos\varphi \\
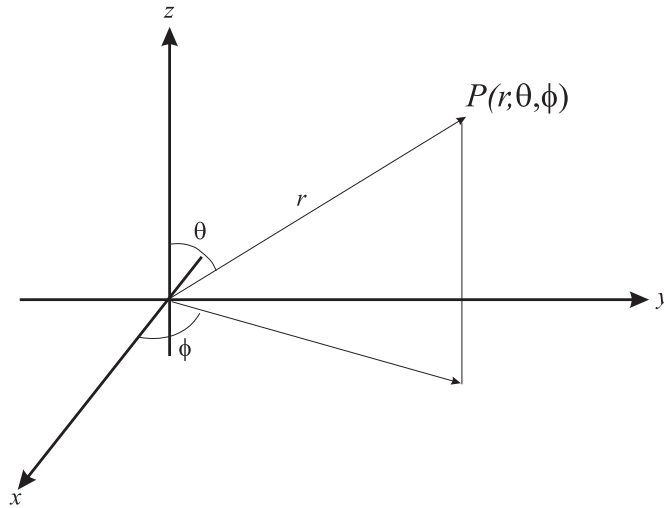y &= r\sin\theta\sin\varphi
\end{aligned}
$$

Figure 15.1: Spherical coordinates

$$z \;=\; r\cos\theta \qquad\qquad (15.4.2)$$

Let's compute the vielbein

$$\vec{e}_t \;=\; (\frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}) = (1,0,0,0)$$

$$\vec{e}_r \;=\; (\frac{\partial t}{\partial r}, \frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r}) = (0, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\vec{e}_\theta \;=\; (\frac{\partial t}{\partial \theta}, \frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}) = r(0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\vec{e}_\varphi \;=\; (\frac{\partial t}{\partial \varphi}, \frac{\partial x}{\partial \varphi}, \frac{\partial y}{\partial \varphi}, \frac{\partial z}{\partial \varphi}) = r(0, -\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) \qquad (15.4.3)$$

and its inverse

$$\vec{E}^t \;=\; (\frac{\partial t}{\partial t}, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}) = (1,0,0,0)$$

$$\vec{E}^r \;=\; (\frac{\partial r}{\partial t}, \frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}) = (0, \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

$$\vec{E}^\theta \;=\; (\frac{\partial \theta}{\partial t}, \frac{\partial \theta}{\partial x}, \frac{\partial \theta}{\partial y}, \frac{\partial \theta}{\partial z}) = \frac{1}{r}(0, \cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$$

$$\vec{E}^{\varphi} \;=\; (\frac{\partial\varphi}{\partial t},\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial y},\frac{\partial\varphi}{\partial z}) = \frac{1}{r}(0,-\sin\theta\sin\varphi,\sin\theta\cos\varphi,0) \qquad (15.4.4)$$

It is easy to check that $\vec{e}_m \cdot \vec{E}^n = \delta^n_m$ and that $e^a_m E^m_b = \delta^a_b$. Now compute the inner products to get the metric function: $g_{tt} = -1$, $g_{rr} = 1$, $g_{\theta\theta} = r^2$ and $g_{\varphi\varphi} = r^2 \sin^2\theta$ (all other components vanish). In matrix notation,

$$g_{\mu\nu} = \begin{bmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix} \qquad (15.4.5)$$

and the distance function is given explicitly by,

$$ds^2 = c^2 dt^2 - (dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2) \qquad (15.4.6)$$

Next compute the connections using either $\Gamma^{\mu}_{\nu\kappa} = (\partial_\nu \vec{e}_\kappa)\cdot\vec{E}^\mu$ or (15.3.28) to get the non-vanishing components

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \frac{1}{r}$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta, \qquad \Gamma^\varphi_{\varphi\theta} = \Gamma^\varphi_{\theta\varphi} = \cot\theta \qquad (15.4.7)$$

(all others vanish). What is the action of of the Laplacian, $\Box_x$, on a scalar function?

$$\Box_x\phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = -\frac{1}{c^2}\partial_t^2\phi + \frac{1}{r^2}\partial_r(r^2\partial_r\phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\phi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\phi$$
$$(15.4.8)$$

the spatial part of which will be recognized as the standard result from ordinary vector analysis. Its action on vectors is quite a bit more complicated but can be written out,

$$\Box_x A^0 = \left[-\frac{1}{c^2}\partial_t^2 A^0 + \frac{1}{r^2}\partial_r(r^2\partial_r A^0) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta A^0) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^0\right]$$

$$\Box_x A^r = \left[-\frac{1}{c^2}\partial_t^2 A^r + \frac{1}{r^2}\partial_r(r^2\partial_r A^r) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta A^r) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^r\right.$$
$$\left.-\frac{2}{r^2}(A^r + r\cot\theta A^\theta + r\partial_\theta A^\theta + r\partial_\varphi A^\varphi)\right]$$

$$\Box_x A^\theta = \left[-\frac{1}{c^2}\partial_t^2 A^\theta + \frac{1}{r^4}\partial_r(r^4\partial_r A^\theta) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta A^\theta) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^\theta\right.$$
$$\left.+\frac{2}{r^3}(\partial_\theta A^r - \frac{1}{2}r\cos 2\theta A^\theta - r\cot\theta\partial_\phi A^\varphi)\right]$$
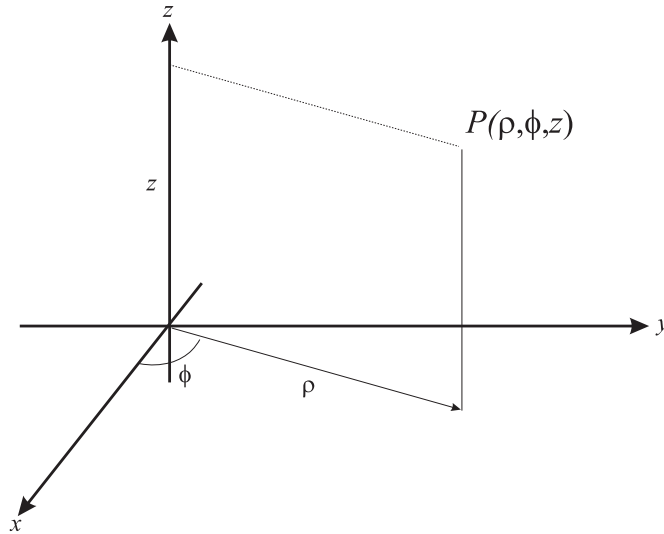
Figure 15.2: Cylindrical coordinates

$$\Box_x A^\varphi = \left[-\frac{1}{c^2}\partial_t^2 A^\varphi + \frac{1}{r^4}\partial_r(r^4\partial_r A^\varphi) + \frac{1}{r^2\sin^3\theta}\partial_\theta(\sin^3\theta\partial_\theta A^\varphi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2 A^\varphi \right.$$
$$\left. + \frac{2}{r^3\sin^3\theta}(\sin\theta\partial_\varphi A^r + r\cos\theta\partial_\varphi A^\theta) \right] \qquad (15.4.9)$$

We see that the Laplacian acts on the time component, A^0, of A^μ, just exactly as it does on a scalar. This is because the coordinate transformation was purely spatial. On the other hand, its action on the space components mixes them.

Cylindrical coordinates

Take the following coordinate functions: $\xi^\mu = (t, \rho, \varphi, z)$ where

$$\begin{aligned} t &= t \\ \rho &= \sqrt{x^2 + y^2} \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned} \qquad (15.4.10)$$

and the inverse transformations: $x^a = x^a(\xi)$

$$t = t$$

$$
\begin{aligned}
x &= \rho\cos\varphi \\
y &= \rho\sin\varphi \\
z &= z
\end{aligned}
\tag{15.4.11}
$$

Let's compute the vielbein

$$
\begin{aligned}
\vec{e}_t &= (\frac{\partial t}{\partial t}, \frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}) = (1,0,0,0) \\
\vec{e}_\rho &= (\frac{\partial t}{\partial\rho}, \frac{\partial x}{\partial\rho}, \frac{\partial y}{\partial\rho}, \frac{\partial z}{\partial\rho}) = (0, \cos\varphi, \sin\varphi, 0) \\
\vec{e}_\varphi &= (\frac{\partial t}{\partial\varphi}, \frac{\partial x}{\partial\varphi}, \frac{\partial y}{\partial\varphi}, \frac{\partial z}{\partial\varphi}) = \rho(0, -\sin\varphi, \cos\varphi, 0) \\
\vec{e}_z &= (\frac{\partial t}{\partial z}, \frac{\partial x}{\partial z}, \frac{\partial y}{\partial z}, \frac{\partial z}{\partial z}) = (0,0,0,1)
\end{aligned}
\tag{15.4.12}
$$

and its inverse

$$
\begin{aligned}
\vec{E}^t &= (\frac{\partial t}{\partial t}, \frac{\partial t}{\partial x}, \frac{\partial t}{\partial y}, \frac{\partial t}{\partial z}) = (1,0,0,0) \\
\vec{E}^\rho &= (\frac{\partial\rho}{\partial t}, \frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}) = (0, \cos\varphi, \sin\varphi, 0) \\
\vec{E}^\varphi &= (\frac{\partial\varphi}{\partial t}, \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z}) = \frac{1}{\rho}(0, -\sin\varphi, \cos\varphi, 0) \\
\vec{E}^z &= (\frac{\partial z}{\partial t}, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial z}{\partial z}) = (0,0,0,1)
\end{aligned}
\tag{15.4.13}
$$

Again, it's easy to check that $\vec{e}_m \cdot \vec{E}^n = \delta_m^n$ and that $e_m^a E_b^m = \delta_b^a$. Now compute the inner products to get the metric function: $g_{tt} = -1$, $g_{\rho\rho} = 1$, $g_{\varphi\varphi} = \rho^2$ and $g_{zz} = 1$ (all other components vanish). In matrix notation,

$$
g_{\mu\nu} =
\begin{bmatrix}
-c^2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \rho^2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\tag{15.4.14}
$$

and the distance function is given explicitly by,

$$
ds^2 = c^2 dt^2 - (d\rho^2 + \rho^2 d\varphi^2 + dz^2)
\tag{15.4.15}
$$

The non-vanishing components of the connections, obtained by using either $\Gamma_{\nu\kappa}^\mu = (\partial_\nu \vec{e}_\kappa) \cdot \vec{E}^\mu$ or (15.3.28) are just

$$
\Gamma_{\varphi\varphi}^\rho = -\rho,
$$

$$\Gamma^{\varphi}_{\rho\varphi} = \Gamma^{\varphi}_{\varphi\rho} = \frac{1}{\rho} \tag{15.4.16}$$

(all others vanish), while the action of the Laplacian, $\Box_x$, on a scalar function is

$$\Box_x \phi = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) = -\frac{1}{c^2} \partial_t^2 \phi + \frac{1}{\rho} \partial_\rho (\rho \partial_\rho \phi) + \frac{1}{\rho^2} \partial_\varphi^2 \phi + \partial_z^2 \phi \tag{15.4.17}$$

the spatial part of which being, as before, the standard result from ordinary vector analysis. Its action on vectors can be written out and we leave this as a straightforward exercise.[4],[5]

15.5 Integration: The Volume Element

When passing from Cartesian coordinates to general coordinates one must also take care to account for the change in the integration measure, which follows the usual rule,

$$\int d^4x \rightarrow \int d^4\xi \left\| \frac{\partial x}{\partial \xi} \right\| \tag{15.5.1}$$

where $\|\partial \xi/\partial x\|$ represents the Jacobian of the transformation. Now notice that under the coordinate transformation that took $x^a \rightarrow \xi^\mu$, the metric also underwent a transformation

$$\eta_{ab} \rightarrow g_{\mu\nu} = \eta_{ab} \frac{\partial x^a}{\partial \xi^\mu} \frac{\partial x^a}{\partial \xi^\mu} = \eta_{ab} e^a_\mu e^b_\mu \tag{15.5.2}$$

It follows, upon taking determinants, that

$$\|\hat{g}\| = \|\hat{\eta}\| \left\| \frac{\partial x}{\partial \xi} \right\|^2 = -c^2 \left\| \frac{\partial x}{\partial \xi} \right\|^2 \tag{15.5.3}$$

where we have used $\|\hat{\eta}\| = -c^2$. Therefore,

$$\left\| \frac{\partial x}{\partial \xi} \right\| = \frac{1}{c} \sqrt{-\|\hat{g}\|} \tag{15.5.4}$$

We have previously used the notation g for the determinant of the metric, $\hat{g}$. Continuing with this notation we notice that (15.5.1) can be written as

$$\int d^4x \rightarrow \frac{1}{c} \int d^4\xi \sqrt{-g} \tag{15.5.5}$$

---

[4]Problem: Write out $\Box_x A^\mu$ for each component of A^μ in cylindrical coordinates.
[5]Problem: Work out the details of the Rindler spacetime.

Again, we see that if the transformation is just a Lorentz transformation,

$$\int d^4x \rightarrow \int d^4x'$$

(15.5.6)

because the metric $g_{\mu\nu}$ then continues to be just the Lorentz metric. Thus we are led to define

$$\frac{1}{c} \int d^4\xi \sqrt{-g}$$

(15.5.7)

as the correct volume integration measure in *any* system of coordinates. Simple examples are spherical coordinates:

$$\frac{1}{c} \int d^4\xi \sqrt{-g} = \int dt \int dr\ r^2 \int_0^\pi d\theta\ \sin\theta \int_0^{2\pi} d\varphi$$

(15.5.8)

and cylindrical coordinates:

$$\frac{1}{c} \int d^4\xi \sqrt{-g} = \int dt \int dz \int d\rho\ \rho^2 \int_0^{2\pi} d\varphi$$

(15.5.9)

but (15.5.7) has general applicability.

Chapter 16

Ideal Fluids

16.1   Introduction

According to the atomic theory of Leucippus and his student Democritus, matter is distinguished from space by the fact that it is not infinitely divisible. If we accept this atomic theory, a truly "continuous" medium, defined as matter that is infinitely divisible, cannot really exist in nature. Yet, there are situations in which a system of particles may be treated as though it were a "continuous medium". To understand what these situations are, let R represent a typical length scale over which observations are carried out on a system of particles, let a be the characteristic size of the particles and let N be the number of particles within a volume determined by R. The length scale R could be, for example, the size of a body immersed in the system of particles or the size of an imaginary volume element within the system. Suppose we are interested in describing the motion of the immersed body, or, more generally, of an imaginary volume element, without regard to the individual motions of the molecules. We will treat a medium as "continuous" if the mean free path, $\langle l \rangle$, of the particles is much smaller than the scale R. For example, air can be considered a continuous medium for the propagation of sound waves provided that the wavelength of the wave is large compared with the mean free path of the air molecules. At shorter wavelengths the air stops behaving as a continuous medium. More precisely, in terms of the **Knudsen number**

$$K = \frac{\langle l \rangle}{R}, \tag{16.1.1}$$

the collection of particles is deemed continuous if $K \ll 1$ at all times. The number density of particles in a spherical volume of radius R being roughly N/R^3, the mean free path of the particles is given by the Clausius relation,

$$\langle l \rangle \approx \frac{R^3}{\sqrt{2}\pi a^2 N}, \tag{16.1.2}$$

where a is a typical atomic or molecular size, from which it follows that for the system of particles to be treated as constituting a continuous medium, we should require that

$$\sqrt{N} \gg \frac{R}{a} \tag{16.1.3}$$

Typically the distance scales over which a classical experiment is carried out is very much larger than the molecular size, so the condition implies that both N and R/a are much greater than unity.

   If an equation of state is available, the condition (16.1.1) may be framed in terms of the state variables of the system. For example for a classical, ideal gas the equation of state $p = nkT$, where n is the number density of molecules and p is the pressure of the gas, implies that

$$K = \frac{kT}{\sqrt{2}\pi a^2 pR} \ll 1 \tag{16.1.4}$$

should hold at all times. An equivalent way to define the continuum approximation is to say that a typical molecule undergoes a very large number of collisions in traversing the experimental distance scale R.

   When (16.1.1) holds it it no longer necessary to concern oneself with the individual motions of the molecules. One concentrates instead on the motion of the medium as a whole, thereby gaining relative simplicity but losing the ability to describe the medium microscopically. Thus we consider volumes of the fluid that are "microscopically large but macroscopically small" *i.e.,* small enough to be treated as infinitesimal volume elements but large enough to contain a very large number of particles. In the process the molecular size is essentially shrunk to zero and the molecules are treated as if they are point-like. This is the **continuum approximation**.

   Continuous media can be divided into solids, liquids and gases, the latter two being generally classified as "fluids". The essential difference between them is the strength of the interaction between the particles (molecules) that consitute the medium. These interactions are very strong in solids and weak in differing degrees for fluids, being weakest for gases. When we say "strong" and "weak" we are of course referring to the ratio between the binding energy of the constituents, $\langle E_b \rangle$, and the typical thermal excitation energy kT. If $\langle E_b \rangle/kT \gg 1$ the medium is solid, but if $\langle E_b \rangle/kT \lesssim 1$ the medium is a liquid or gas because thermal excitations or relatively weak mechanical forces are able to break the inter-particle bonds and cause the medium to flow. For this reason, solids turn into liquids and gases at high enough temperatures, according to our common experience. In this chapter we concentrate on fluids.

   A fluid is described by five fields, *viz.,*

   • The three components of a velocity vector field, $\vec{v}(\vec{r}, t)$, which gives the *average* velocity of a molecule at any point, $\vec{r}$, within the fluid at any time, t,
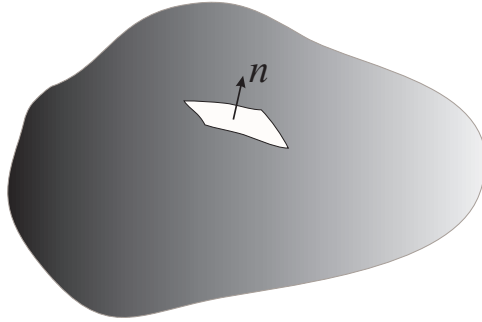
Figure 16.1: A fluid volume element

- the pressure field, $p(\vec{r}, t)$, which gives the pressure within the fluid at any point in space and at any time, and

- the density field, $\rho(\vec{r}, t)$, which gives the density of the fluid at any point and at any time.

Thus we need five equations to completely determine the evolution of the fluid. Notice that the last two variables are thermodynamic in character leading us to expect that the thermodynamic equation of state is one of the required equations. Another equation will express the conservation of mass (in a relativistic treatment, this turns into the conservation of mass-energy) and the remaining equations will be the analogues of the Newtonian equations of motion. We will address these equations in the following sections.

16.2   Equation of Continuity

The principle of mass conservation is expressed by the equation of continuity. If $\rho(\vec{r}, t)$ represent the density of a fluid at a given time t then, by definition, the mass contained within a volume V of the fluid at any time t is given by

$$M(t) = \int_V d^3\vec{r}\,\rho(\vec{r}, t) \qquad (16.2.1)$$

Let the volume be bounded by the closed surface S as shown in figure (16.1), where $\widehat{n}$ represents the normal to an infinitesimal surface element. Consider the flow of fluid particles into and out of the surface S. To do so, first imagine the simple situation illustrated in figure (16.2), in which a beam of particles, each of velocity $\vec{v}$, is incident perpendicularly on a surface of area A. In an interval of time δt a volume $vA\delta t$ of fluid crosses the surface A, so that *per unit time* the volume of fluid that has crossed the area
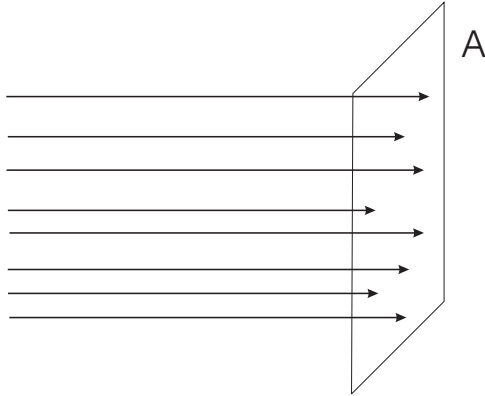
Figure 16.2: Fluid flow perpendicular to an area A

A is simply vA. Now if A is infinitesimal, say dS then the infinitesimal mass $\delta m = \rho v dS$ crosses the surface element per unit time. Again, if $\vec{v}$ is not perpendicular to the surface, but makes an angle of θ with the normal $\widehat{n}$ then the fluid mass that crossed the surface would be $\rho \vec{v} \cdot \widehat{n} dS$. Returning to the our volume V, shown in figure 16.1, we see that the mass that enters the volume by crossing the surface element shown must be $-\rho \vec{v} \cdot \widehat{n} dS$ per unit time. The negative sign is necessary because of our convention that $\widehat{n}$ is the *outward* normal to the surface. The fluid entering our volume must contribute to the increase of mass within it. Integrating over the entire surface,

$$\frac{d}{dt} \int_V d^3 \vec{r} \rho(\vec{r}, t) = - \oint_S \rho \vec{v} \cdot \widehat{n} dS \qquad (16.2.2)$$

Using Gauss' theorem to convert the integral on the right to a volume integral, we find that

$$\int_V d^3 \vec{r} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) \right] \equiv 0 \qquad (16.2.3)$$

and, since our volume is arbitrary it must hold that

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) \equiv 0. \qquad (16.2.4)$$

This is the **continuity equation**. It upholds the conservation of mass (matter cannot be created nor destroyed) and is not an equation of motion but a conservation law. It is kinematical in nature and is therefore true for all fluids and in all flows. The quantity $\vec{j} = \rho \vec{v}$ is called the three current density.

16.3   Ideal Fluids

An ideal fluid will be one in which there is no dissipation of energy within the fluid. Microscopically, internal energy dissipation occurs via inelastic collisions between the molecules of the fluid, so an ideal fluid is one in which the intermolecular collisions are elastic. We will also assume for the present that there is no energy transfer between the fluid and its environment through its boundaries. The absence of heat exchange between the various parts of the fluid and between the fluid and its boundaries implies that the flow is isentropic, *i.e.,* that the flow is such that the entropy per particle is constant throughout its evolution. The entropy, like the pressure, will depend on the spatial position within the fluid as well as on time. Let s be the entropy per unit mass. An isentropic flow requires that

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + (\vec{v} \cdot \vec{\nabla})s = 0 \tag{16.3.1}$$

and if we combine this with the continuity equation, we find that

$$
\begin{aligned}
\frac{\partial}{\partial t}(\rho s) &= \frac{\partial \rho}{\partial t} s + \rho \frac{\partial s}{\partial t} \\
&= -\vec{\nabla} \cdot (\rho \vec{v}) s - \rho (\vec{v} \cdot \vec{\nabla}) s \\
&= -\vec{\nabla} \cdot (\rho s \vec{v}) \tag{16.3.2}
\end{aligned}
$$

where we have used (16.2.4). Thus,

$$\frac{\partial (\rho s)}{\partial t} + \vec{\nabla} \cdot (\rho s \vec{v}) \equiv 0, \tag{16.3.3}$$

which is the continuity equation expressing entropy conservation in the same terms as (16.2.4) expresses mass conservation. The quantity ρs is the entropy density and $\vec{\sigma} = \rho s \vec{v}$ is the entropy current three density.

16.4   Euler's equation for an Ideal Fluid

Euler's equations are the precise analogues for a fluid of Newton's equations for a particle. They therefore define the dynamics of the ideal fluid and will need to be modified when we want to take into account energy dissipation and heat transfer within the fluid and between the fluid and its environment.

Consider a volume V of fluid as shown in (16.1) and recall that this volume lies within the fluid. Due to the surrounding medium, which exerts a pressure on the surface bounding the volume, there will be a net force on the volume which is given by

$$\vec{F} = -\oint_S p \widehat{n} \cdot dS \tag{16.4.1}$$

Again, it can be expressed in terms of a volume integral by Gauss' theorem

$$\vec{F} = -\int_V d^3\vec{r}\, \vec{\nabla} p. \tag{16.4.2}$$

The total force on the volume V is the force due to the rest of the fluid as well as the force exerted by the environment on the volume, which we will simply refer to as $\vec{F}^{\text{ext}}$. It is preferable, however, to work in terms of force densities, so let us define $\vec{f}^{\text{ext}}$ as

$$\vec{F}^{\text{ext}} = \int_V d^3\vec{r}\, \vec{f}^{\text{ext}} \tag{16.4.3}$$

To apply Newton's second law, we must ask what constitutes the left hand side. For the rate of change of momentum associated with the volume we have

$$\frac{d}{dt}\int_V d^3\vec{r}(\rho\vec{v}). \tag{16.4.4}$$

but we must add to this a term that accounts for the rate of change of momentum associated with the surface deformations. This is[1]

$$\oint_S dS\, \rho\vec{v}(\hat{n}\cdot\vec{v}) \tag{16.4.5}$$

Newton's second law of motion then takes the form

$$\frac{d}{dt}\int_V d^3\vec{r}(\rho\vec{v}) + \oint_S dS\, \rho\vec{v}(\hat{n}\cdot\vec{v}) = -\int_V d^3\vec{r}\, \vec{\nabla} p + \int_V d^3\vec{r}\, \vec{f}^{\text{ext}} \tag{16.4.6}$$

where $\rho\vec{v}$ is the momentum density of the fluid in our volume element. The first term on the left hand side is

$$\int_V d^3\vec{r}\left[\rho\frac{\partial\vec{v}}{\partial t} + \vec{v}\frac{\partial\rho}{\partial t}\right] = \int_V d^3\vec{r}\left[\rho\frac{\partial\vec{v}}{\partial t} - \vec{v}\vec{\nabla}\cdot(\rho\vec{v})\right] \tag{16.4.7}$$

where we used the equation of continuity. Let us consider the second term in components (Einstein's summation convention is assumed throughout). We have

$$-v_i\partial_j(\rho v_j) = -\partial_j(\rho v_i v_j) + \rho v_j\partial_j v_i$$

$$\Rightarrow -\vec{v}\vec{\nabla}\cdot(\rho\vec{v}) = -\vec{\nabla}\cdot(\vec{v}\otimes\vec{v}) + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} \tag{16.4.8}$$

Upon integrating over the volume V and applying Gauss' theorem, the first term on the right turns into

$$-\oint_S dS\, \rho\vec{v}(\hat{n}\cdot\vec{v}) \tag{16.4.9}$$

---

[1]This comes from the theory of elasticity and here we shall take it as given.

and cancels the second term on the left of (16.4.6) leaving

$$\int_V d^3\vec{r} \left[\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} \right] = -\int_V d^3\vec{r}\,\vec{\nabla}p + \int_V d^3\vec{r}\,\vec{f}^{\text{ext}}. \tag{16.4.10}$$

Again, because the volume we are considering is arbitrary, it follows that

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \vec{f}^{\text{ext}} \tag{16.4.11}$$

These are Euler's equations for an ideal fluid. Notice that the left hand side is nothing but the total derivative of $\vec{v}(\vec{r},t)$, in terms of which the equations could be written as

$$\rho\frac{d\vec{v}}{dt} = -\vec{\nabla}p + \vec{f}^{\text{ext}}. \tag{16.4.12}$$

The left hand side is called the "total" acceleration of the fluid. It is made up of two parts, one of which, $\partial\vec{v}/\partial t$, measures the rate of change of velocity at a fixed point and is called the **local acceleration**. The other is the term $(\vec{v}\cdot\vec{\nabla})\vec{v}$ and represents a contribution to the rate of change of velocity from a change in location within the fluid. It is called the **convective** acceleration.

The external force in equation (16.4.12) is expected from a naïve application of Newton's second law to a unit volume of fluid, without explicitly taking into account the surface terms. As an example of an external force, consider a fluid in a constant gravitational field $\vec{g}(\vec{r},t)$. Euler's equations would read

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \rho\vec{g}. \tag{16.4.13}$$

The five equations governing the motion of an ideal fluid are therefore

- an equation of state, $f(p,T,\rho) = 0$,

- the continuity equation in (16.2.4) and

- Euler's equations in (16.4.11)

For an isentropic flow one can trade the pressure for the enthalpy per unit mass. This simplifies Euler's equations. Because the enthalpy,

$$H = U + pV, \tag{16.4.14}$$

where U, the internal energy of the system, is an extensive quantity. Define the enthalpy per unit mass by $h = u + p/\rho$, where u is the internal energy per unit mass and $v = 1/\rho$ is the volume occupied by a unit mass. Then

$$dh = du + \frac{dp}{\rho} - \frac{p}{\rho^2}d\rho \tag{16.4.15}$$

and because the internal energy per unit mass obeys the first law of thermodynamics

$$du = Tds - pd\left(\frac{1}{\rho}\right) = Tds + \frac{p}{\rho^2}d\rho \tag{16.4.16}$$

where s is the entropy per unit mass of the fluid, it follows that

$$dh = Tds + \frac{p}{\rho^2}d\rho + \frac{dp}{\rho} - \frac{p}{\rho^2}d\rho = Tds + \frac{dp}{\rho} \tag{16.4.17}$$

i.e., the enthalpy is a function of (s, p). Since $ds = 0$ for an isentropic flow, we have $dh = dp/\rho \Rightarrow \vec{\nabla}p = \rho\vec{\nabla}h$ and Euler's equations can be written in terms of h

$$\frac{d\vec{v}}{dt} = -\vec{\nabla}h + \frac{\vec{f}^{\text{ext}}}{\rho}. \tag{16.4.18}$$

This is the second form of Euler's equations. Taking the curl of this equation we get

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{v}) + \vec{\nabla} \times (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} \times \left(\frac{\vec{f}^{\text{ext}}}{\rho}\right) \tag{16.4.19}$$

where we have used the fact that the curl of a gradient vanishes identically. Now[2]

$$\vec{\nabla} \times [\vec{v} \times (\vec{\nabla} \times \vec{v})] = \vec{\nabla} \times \left(\frac{1}{2}\vec{\nabla}\vec{v}^2 - (\vec{v} \cdot \vec{\nabla})\vec{v}\right) = -\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{16.4.20}$$

and therefore

$$\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{v}) = \vec{\nabla} \times [\vec{v} \times (\vec{\nabla} \times \vec{v})] + \vec{\nabla} \times \left(\frac{\vec{f}^{\text{ext}}}{\rho}\right), \tag{16.4.21}$$

or

$$\frac{\partial\vec{\omega}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{\omega}) = \vec{\nabla} \times \left(\frac{\vec{f}^{\text{ext}}}{\rho}\right), \tag{16.4.22}$$

where $\omega = \vec{\nabla} \times \vec{v}$ is called the **vorticity** of the flow. This is sometimes referred to as the third form of Euler's equations.

16.5   Waves in Fluids

The set of five equations, *viz.,* Euler's equations, the equation of continuity and the equation of state imply the existence of (sound) waves in fluids. Consider a simplified situation

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[2]<u>Problem</u>: Prove this.

in which the fluid is static, *i.e.*, $\vec{v} = 0$ and of uniform density ($\rho = \rho_0$) and pressure ($p = p_0$). Now examine small perturbations about this equilibrium,

$$\vec{v} = \delta\vec{v}(\vec{r}, t), \quad \rho = \rho_0 + \delta\rho(\vec{r}, t), \quad p = p_0 + \delta p(\vec{r}, t), \tag{16.5.1}$$

then by the continuity equation and to first order

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \vec{\nabla} \cdot \delta\vec{v} = 0. \tag{16.5.2}$$

Likewise, by Euler's equation,

$$\frac{\partial \delta\vec{v}}{\partial t} = -\frac{1}{\rho_0}\vec{\nabla}\delta p \tag{16.5.3}$$

To complete the set we need to exploit the equation of state. Imagine that the propagation of the perturbations occurs under certain conditions (eg. isothermal or adiabatic), and take it to be of the form $p = p(\rho)$, so that $p_0 = p(\rho_0)$ and $\delta p = p'(\rho_0)\delta\rho$. Taking a second time derivative of the continuity equation and using Euler's equations,

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c^2\vec{\nabla}^2\delta\rho = 0, \quad c^2 = p'(\rho_0), \tag{16.5.4}$$

which is the equation for sound waves traveling at

$$c = \sqrt{\left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}}. \tag{16.5.5}$$

For example, consider an ideal gas in which waves propagate under isothermal conditions. Then $p'(\rho_0) = kT/m$, where m is the molecular mass of the gas, and we recover (8.2.22). If we consider adiabatic propagation instead then the equation of state, $p = \text{const.} \times \rho^\gamma$, implies that $p'(\rho_0) = \gamma kT/m$, where γ is the ratio of heat capacities, $\gamma = C_p/C_V$, and this reproduces (8.2.21).

16.6   Special Flows

We now consider some special flows for which Euler's equations become greatly simplified.

16.6.1   Hydrostatics

For static fluids the velocity field is vanishing everywhere at all times and Euler's equations reduce to

$$\vec{\nabla}p = \vec{f}^{\text{ext}} \tag{16.6.1}$$

This is the "**equation of hydrostatics**". It must be supplemented by the condition (from the continuity equation) that ρ does not depend explicitly on time, *i.e.*, $\partial\rho/\partial t = 0$.

16.6.2   Steady Flows

A **steady flow** is one for which the velocity does not depend explicitly on time, *i.e.,* one for which $\partial \vec{v}/\partial t = 0$. In this case Euler's equations become

$$\rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\vec{\nabla}p + \vec{f}^{\text{ext}} \qquad (16.6.2)$$

and using

$$\vec{v}\times(\vec{\nabla}\times\vec{v}) = \frac{1}{2}\vec{\nabla}\vec{v}^2 - (\vec{v}\cdot\vec{\nabla})\vec{v} \qquad (16.6.3)$$

we find

$$\frac{1}{2}\vec{\nabla}\vec{v}^2 - \vec{v}\times(\vec{\nabla}\times\vec{v}) + \vec{\nabla}h = \frac{\vec{f}^{\text{ext}}}{\rho} \qquad (16.6.4)$$

Taking the inner product with the unit vector that points in the direction of $\vec{v}$ we get

$$\widehat{v}\cdot\vec{\nabla}\left[\frac{1}{2}\vec{v}^2 + h\right] = \frac{\widehat{v}\cdot\vec{f}^{\text{ext}}}{\rho} \qquad (16.6.5)$$

where we made use of the fact that the term $\vec{v}\times(\vec{\nabla}\times\vec{v})$ is perpendicular to both $\vec{v}$ and $\vec{\nabla}\times\vec{v}$. Now if the force per unit mass is derivable from a potential per unit mass according to the usual relation

$$\frac{\vec{f}^{\text{ext}}}{\rho} = -\vec{\nabla}\Phi \qquad (16.6.6)$$

then (16.6.5) turns into

$$\widehat{v}\cdot\vec{\nabla}\left[\frac{1}{2}\vec{v}^2 + h + \Phi\right] \equiv 0 \qquad (16.6.7)$$

This takes a bit of interpreting. Notice that $\widehat{v}\cdot\vec{\nabla}$ represents the directional derivative along a curve whose tangent at any point is the velocity. Such curves are called streamlines and (16.6.7) says that the directional derivative along a streamline vanishes, or along streamlines

$$\frac{1}{2}\vec{v}^2 + h + \Phi = \text{const.} \qquad (16.6.8)$$

This is **Bernoulli's equation** and expresses the conservation of energy. It is very important to emphasize that this holds *only along streamlines* because the constant on the right hand side may vary from streamline to streamline. To appreciate the significance of a streamline, notice that because the velocity has no explicit dependence on t therefore at any point $P(\vec{r}_P)$ it is always constant. The typical (average) molecule at P would have velocity $\vec{v}_P$, so its displacement in a time interval δt would be

$$\delta\vec{r}_P = \vec{v}_P\delta t \qquad (16.6.9)$$

However, at point $\vec{r}_P + \delta\vec{r}_P$ its velocity is once again fixed, which would then determine its further development. Thus molecules that find themselves at P would, on the average, follow the same trajectory. This trajectory is *unique* to molecules that pass through P and is called a streamline. Two streamlines may not cross one another. If they did then, at the intersection point, the velocity field would not be single valued. Neither can a streamline bifurcate at any point. If it did then, at the bifurcation point, the acceleration would not be single valued. Mathematically, streamlines are the integral curves of the velocity vector field,

$$\frac{d\vec{r}}{dt} = \vec{v} \implies \frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}. \qquad (16.6.10)$$

When $\vec{v}$ does depend explicitly on t the concept of a streamline is no longer useful as the molecules passing through P would have, on the average, different velocities depending on the time at which they passed through P.

Bernoulli's equation is further simplified when the fluid is **incompressible**, *i.e.,* when $\rho(\vec{r}, t)$ is constant. In this case, the continuity equation requires that

$$\vec{\nabla} \cdot \vec{v} = 0 \qquad (16.6.11)$$

Furthermore, $dh = dp/\rho = d(p/\rho)$ allows us to write Bernoulli's equation in the form

$$\hat{v} \cdot \vec{\nabla} \left[\frac{1}{2}\rho\vec{v}^2 + p + \rho\Phi \right] \equiv 0 \qquad (16.6.12)$$

or

$$p + \frac{1}{2}\rho\vec{v}^2 + \rho\Phi = \text{const.} \qquad (16.6.13)$$

along streamlines. For example, if Φ represents a constant gravitational potential then

$$p + \frac{1}{2}\rho\vec{v}^2 + \rho g z = \text{const.} \qquad (16.6.14)$$

where we have assumed that the gravitational field is oriented in the negative z direction.

In general incompressibility implies that the velocity field is derivable from a vector potential, $\vec{v} = \vec{\nabla} \times \vec{V}$, because the divergence of a curl is identically zero. However, in two dimensional flows incompressibility implies that $\vec{v}$ is derivable from a scalar potential, $\psi(x, y)$, according to

$$v_x = \partial_y\psi, \quad v_y = -\partial_x\psi. \qquad (16.6.15)$$

The equation $\psi(x, y) = \text{const.}$ gives the streamlines.[3]

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[3]This follows because, according to (16.6.10),

$$v_x dy - v_y dx = \partial_x\psi dx + \partial_y\psi dy = 0$$

$$\Rightarrow d\psi = 0 \Rightarrow \psi(x, y) = \text{const.}$$

16.6.3   Irrotational or Potential Flows

The **vorticity** $\vec{\omega}$ of a flow is defined by $\vec{\omega} = \vec{\nabla} \times \vec{v}$. If the vorticity is everywhere zero then the flow is said to be "irrotational". Irrotational flows are also called **potential flows** because the condition $\vec{\nabla} \times \vec{v} = 0$ implies that the velocity vector is derivable from a potential, *i.e.,* $\vec{v} = \vec{\nabla}\psi$ for some scalar function $\psi(\vec{r}, t)$. Let us see what Euler's equations turn into when we demand that the flow be irrotational. From the third form of Euler's equations we see immediately that irrotational flows can exist only if the external force per unit mass is also irrotational. Let Φ be the potential energy per unit mass, then inserting $\vec{v} = \vec{\nabla}\psi$ in the first form of the equations we have

$$\vec{\nabla}\frac{\partial \psi}{\partial t} + (\vec{\nabla}\psi \cdot \vec{\nabla})\vec{\nabla}\psi = -\vec{\nabla}(h + \Phi) \tag{16.6.16}$$

Now in components

$$[(\vec{\nabla}\psi \cdot \vec{\nabla})\vec{\nabla}\psi]_i = \partial_j \psi \partial_j (\partial_i \psi) = \partial_j \psi \partial_i (\partial_j \psi), \tag{16.6.17}$$

where we have applied the integrability condition, so that

$$(\vec{\nabla}\psi \cdot \vec{\nabla})\vec{\nabla}\psi = \frac{1}{2}\vec{\nabla}(\vec{\nabla}\psi)^2 \tag{16.6.18}$$

and therefore

$$\vec{\nabla}\left[\frac{\partial \psi}{\partial t} + \frac{1}{2}(\vec{\nabla}\psi)^2 + h + \Phi\right] \equiv 0 \tag{16.6.19}$$

or

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}(\vec{\nabla}\psi)^2 + h + \Phi = f(t) \tag{16.6.20}$$

where $f(t)$ is an arbitrary function of t. This function can be set to be constant, however, because from the definition of $\vec{v}$ in terms of ψ it is clear that

$$\psi \rightarrow \psi + \int f(t)dt + \text{const.} \tag{16.6.21}$$

does not change $\vec{v}$ but does eliminate $f(t)$ in (16.6.20). After all, it is not ψ that is measured but $\vec{v}$. As for any potential, ψ is arbitrary up to some redefinition. This is Bernoulli's equation for irrotational flows, with one important distinction. Whereas the constant on the right hand side of (16.6.8) is different in principle from streamline to streamline it is the same for *all* streamlines in (16.6.20).

16.6.4 Incompressible Flows

An incompressible flow is one for which $d\rho/dt = 0$, *i.e.,* by the equation of continuity, $\vec{\nabla} \cdot \vec{v} = 0$. The third form of Euler's equation is seen to be an equation for the vorticity of a flow,

$$\frac{\partial \vec{\omega}}{\partial t} - \vec{\nabla} \times (\vec{v} \times \vec{\omega}) = \vec{\nabla} \times \left(\frac{\vec{f}^{\text{ext}}}{\rho} \right) \tag{16.6.22}$$

Expanding the left hand side of this equation,

$$\frac{\partial \vec{\omega}}{\partial t} - (\vec{\omega} \cdot \vec{\nabla})\vec{v} + (\vec{v} \cdot \vec{\nabla})\vec{\omega} + \vec{\omega}(\vec{\nabla} \cdot \vec{v}) - \vec{v}(\vec{\nabla} \cdot \vec{\omega}) = \vec{\nabla} \times \left(\frac{\vec{f}^{\text{ext}}}{\rho} \right) \tag{16.6.23}$$

and since $\vec{\nabla} \cdot \vec{\omega} \equiv 0$, we find a simplified equation when the fluid is also incompressible $(\vec{\nabla} \cdot \vec{v} = 0)$,

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} + \vec{\nabla} \times \left(\frac{\vec{f}^{\text{ext}}}{\rho} \right) \tag{16.6.24}$$

or just

$$\frac{d\vec{\omega}}{dt} = (\vec{\omega} \cdot \vec{\nabla})\vec{v} \tag{16.6.25}$$

in the absence of external force fields. If an incompressible fluid undergoes irrotational flow then $\vec{v} = \vec{\nabla}\psi$ and Euler's equations are automatic. Moreover the incompressibility condition, $\vec{\nabla} \cdot \vec{v} = 0$, turns into the equation $\vec{\nabla}^2 \psi = 0$.

16.7 Elementary Applications

16.7.1 Hydrostatics

We will now look at some simple applications of the above types of flow, beginning with the simplest of all: hydrostatics. In this case, because $\vec{v}(\vec{r}, t)$ is constrained to be vanishing we have just

$$\vec{\nabla}p = \vec{f}^{\text{ext}} \tag{16.7.1}$$

which expresses the fact that the force per unit volume is balanced by the pressure within the fluid. If the external force density is due to a homogeneous external gravitational field $\vec{g}$ in, say, the negative z direction then

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\rho g \tag{16.7.2}$$

and the first two equations imply only that $p = p(z)$. The last equation will have different solutions depending on whether or not the fluid is compressible.

If the fluid is incompressible then the solution is trivially

$$p(z) = p_0 - \rho g(z - z_0) = p_0 + \rho g(z_0 - z) \qquad (16.7.3)$$

where p_0 is the pressure at $z = z_0$. The quantity $z - z_0$ will be referred to as the "height" and $z_0 - z$ as the "depth". The formula gives the variation of pressure with height (depth) assuming that the fluid is effectively incompressible. This approximation is generally reasonable in liquids up to a certain depth (depending on the liquid) but is not very useful in treating gases.

If the fluid is compressible, the density is a non-trivial function of the pressure and temperature. Consider a column of an ideal gas at a fixed absolute temperature T. Using the equation of state

$$p = nkT = \frac{\rho kT}{m} \Rightarrow \rho = \frac{mp}{kT} \qquad (16.7.4)$$

where m is the molecular mass, we find

$$\frac{dp}{p} = -\frac{mg}{kT}dz \Rightarrow p = p_0 e^{-\frac{mg}{kT}(z - z_0)} \qquad (16.7.5)$$

where p_0 is the pressure at z_0. Now if the absolute temperature is also a function of z, the solution will be more complex. For example, if $T(z) = T_0 - \gamma(z - z_0)$ where γ is a positive constant called the "**lapse rate**" the equation to be solved is

$$\frac{dp}{p} = -\frac{mg}{k[T_0 - \gamma(z - z_0)]}dz \Rightarrow p = p_0 \left[1 - \frac{\gamma(z - z_0)}{T_0}\right]^{\frac{mg}{k\gamma}} \qquad (16.7.6)$$

In this solution T_0 and p_0 represent the temperature and pressure respectively at $z = z_0$.

It may also turn out that the fluid is not in a constant, homogeneous external gravitational field as we have examined. For example a star can be thought of as a very large mass of gass that is held together by its own gravitational attraction. In the simplest case of a star that is not rotating we can think of it as made up of spherical shells, each shell being subject to the gravitational potential due to all the shells within it. The force density is then $\vec{f}^{\text{ext}} = -\rho\vec{\nabla}\Phi$ where Φ represents this potential energy and from Gauss' law

$$\vec{\nabla}^2\Phi = -4\pi G\rho \qquad (16.7.7)$$

Inserting Φ, dividing throught by the density and taking a divergence of the hydrostatic equation (16.6.1) gives

$$\vec{\nabla} \cdot \left(\frac{\vec{\nabla}p}{\rho}\right) = -\vec{\nabla}^2\Phi = 4\pi G\rho \qquad (16.7.8)$$

This equation must then be solved subject to some equation of state relating (p, ρ, T) and some assumptions regarding the variation of T with the radius r.
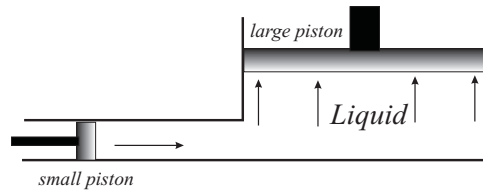
Figure 16.3: A hydraulic press

The hydraulic press is an application of the hydrostatic equation that is of considerable engineering importance. This is a device whose function is to increase an applied force many fold. To see how this comes about, consider the force exerted by the fluid on any external bounding surface,

$$\vec{F} = \int_S p\, \widehat{n} dS \qquad (16.7.9)$$

where $\widehat{n}$ is the outward normal to the surface. This force depends on the area of the surface and this feature is exploited by hydraulics to generate huge forces by the actual application of comparatively small ones as shown in the figure (16.3) of a basic hydraulic press. Imagine that the space between the pistons is filled with an incompressible fluid, at least at the pressures for which the press is functional. Let the area of the small piston be A_1 and of the large piston be A_2, so that $A_2 \gg A_1$. Suppose that the small piston is pushed forward with a force F_1 and the piston displaced by Δx_1. Now the pressure in the fluid, due to an application of the force F_1 will be $p_1 = F_1/A_1$ and, in equilibrium, this pressure will be constant throughout the fluid. As a consequence a force $F_2 = p_1 A_2$ is exercised on the large piston by this pressure and we find that this force

$$F_2 = \frac{A_2}{A_1} F_1 \qquad (16.7.10)$$

may be many times the original applied force, depending on the ratio A_2/A_1. However, the work done by the applied force is the same as the work done by the large piston. To show that this must be so, note that the volume of liquid displaced at the small piston is $A_1 \Delta x_1$ and, because the fluid is incompressible, this volume of fluid will displace the large piston by a distance $\Delta x_2 = A_1 \Delta x_1 / A_2$. The work done by F_2 is therefore

$$W_2 = F_2 \Delta x_2 = \left(\frac{A_2}{A_1} F_1 \right) \times \left(\frac{A_1}{A_2} \Delta x_1 \right) = F_1 \Delta x_1 = W_1 \qquad (16.7.11)$$

In this example, the calculation of the force exercised on the large piston was simple because the fluid was considered to be incompressible and the surface of the piston was
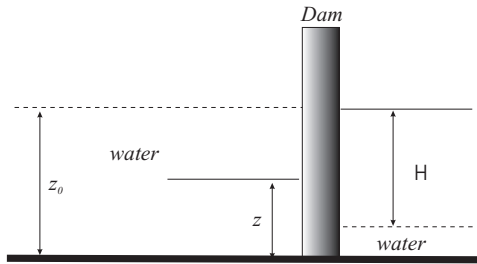
Figure 16.4: Schematic of a Dam

planar. We will now consider two further examples. In the first the pressure will not be constant and in the second, the surface will not be planar.

Consider a dam of height H, *i.e.*, the height of the water on one side of it is a height H above the water on the other side as shown in figure (16.4). Let's calculate the net force on the dam. To a very good approximation over the height of a normal dam, the water may be considered to be incompressible and we can write the pressure as a function of depth,

$$p(z) = p_0 + \rho g(z_0 - z) \tag{16.7.12}$$

so that the force on the dam itself to the right, due to the water on the left, is

$$F_R = \int_0^L dx \int_0^{z_0} dz[p_0 + \rho g(z_0 - z)] \tag{16.7.13}$$

The force to the left, due to the water on the right is

$$F_L = \int_0^L dx \int_0^{z_0-H} dz[p_0 + \rho g(z_0 - H - z) + \int_0^L dx \int_{z_0-H}^{z_0} dz\, p_0 \tag{16.7.14}$$

With $z_1 = z_0 - H$, the net force on the dam (clearly, to the right) is therefore

$$F = \rho g L \left[\int_0^{z_0} dz\, (z_0 - z) - \int_0^{z_1} dz\, (z_1 - z) \right] = \frac{1}{2}\rho g L H(H + 2z_0) \tag{16.7.15}$$

In this example, the surface over which the integration is carried out is trivial.

Let us now consider an example in which the surface is not planar. Suppose we want to to determine the force on the side ABC of a tank shaped as shown in figure (16.5). Assume that the tank is rectangular, apart from the curved portion of the base. Again, we will treat the water as incompressible. It is a simple affair to calculate the force on the portion BC of the tank wall, which can be written as

$$\vec{F}_{\text{BC}} = \int_0^L dx \left[\int_R^H dz[p_0 + \rho g(H - z)] - \int_R^H dz p_0 \right] \hat{y}$$
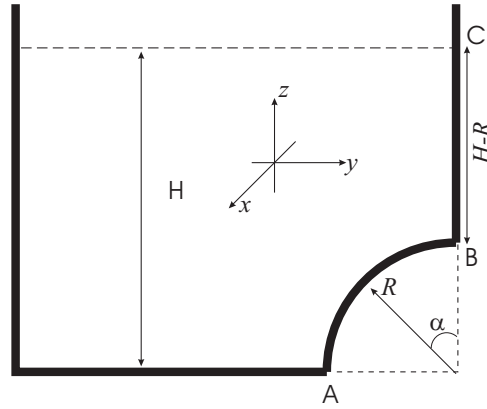
Figure 16.5: Force on one wall of a water tank

$$= \frac{1}{2}\rho g L (H - R)^2 (0, 1, 0) \tag{16.7.16}$$

directed to the right. Now to calculate the force on the curved portion, AB of the tank we must first recognize that the pressure depends only on the height and the height can be written as $z = R\cos\alpha$. The unit normal is $-\hat{r}$ ($\hat{r}$ is the unit radial vector) since it must be *outgoing*. Therefore

$$
\begin{aligned}
\vec{F}_{\mathrm{AB}} &= -\int_0^L dx \int_0^R dz \, \rho g (H - z) \, \hat{r}(z) \\
&= -\rho g L R \int_{\pi/2}^0 d\alpha (H - R\cos\alpha)(0, -\sin\alpha, \cos\alpha) \\
&= \rho g L R \left(0, H - \frac{R}{2}, \frac{\pi R}{4} - H\right) \tag{16.7.17}
\end{aligned}
$$

Combining both results we find

$$\vec{F}_{\mathrm{ABC}} = \rho g L \left[\frac{1}{2}(H - R)^2 + R\left(H - \frac{R}{2}\right)\right] \hat{y} + \rho g L R \left[\frac{\pi R}{4} - H\right] \hat{z} \tag{16.7.18}$$

It should be obvious that realistic problems, even in hydrostatics, can quickly get very complicated.

Archimedes' principle is a direct consequence of the hydrostatic equations. It states that when a body is immersed in a fluid it suffers a loss of weight equal to the weight of the fluid displaced by it. Equivalently, the body experiences an upward force equal to the

weight of the fluid displaced. The upward force is called the **buoyancy** and is the reason boats and ships float. To understand how this comes about, suppose that the density of the fluid is ρ_f and that the density of the body itself is ρ. Let S be the surface bounding the body, so that the net force on the body is

$$\vec{F} = - \oint_S p d\vec{S} + \int_V d^3\vec{r} \vec{f}^{\text{ext}} \tag{16.7.19}$$

where $\vec{f}^{\text{ext}}$ is the external force density which we take to be due to an external, homogeneous gravitational field, *i.e.*, $\vec{f}^{\text{ext}} = \rho g(0,0,-1)$ directed in the negative z direction. Now, if the fluid is incompressible then $p = p_0 + \rho_f g(H - z)$ where $H - z$ represents the depth from the surface of the fluid and p_0 represents the atmospheric pressure. Thus

$$\vec{\nabla} p = (0,0,-\rho_f g) \tag{16.7.20}$$

and therefore, applying Gauss' theorem,

$$\vec{F} = - \int_V d^3\vec{r} [0,0,(\rho - \rho_f)g] = (\rho_f - \rho)gV\hat{z} \tag{16.7.21}$$

Now $\rho_f V$ is the mass of liquid having the same volume as the immersed body itself and is therefore the mass of the fluid displaced by the body. The first term therefore represents the buoyancy. Naturally a body will float so long as the buoyancy can overcome its weight *i.e.*, $\rho_f \geq \rho$. If $\rho_f > \rho$ only a portion of the body, of volume $V' < V$, is immersed. The fraction of the body's volume that is immersed is given by the condition $\rho_f V' = \rho V$ or

$$V' = \frac{\rho}{\rho_f} V \tag{16.7.22}$$

16.7.2   Steady Flows

Let us now examine some examples of steady flows. In this case, Euler's equation reduces to

$$\rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \vec{f}^{\text{ext}} \tag{16.7.23}$$

a first integral of which is the Bernoulli equation

$$\frac{1}{2}\vec{v}^2 + h + \Phi = \text{const.} \tag{16.7.24}$$

along any streamline. Let us assume that ρ also does not depend explicitly on time. In other words, that the system is in equilibrium. By continuity, this is only possible if

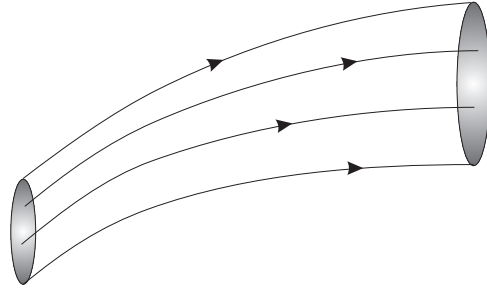$$\vec{\nabla} \cdot (\rho\vec{v}) = 0 \tag{16.7.25}$$

Figure 16.6: Streamlines

and therefore, integrating over an arbitrary closed volume and applying Gauss' theorem,

$$\oint_S d\vec{S} \cdot (\rho\vec{v}) = 0 \tag{16.7.26}$$

Now, take a special volume made up of a collection of streamlines and let the edges of the tube be sections of streamlines (*i.e.,* everywhere perpendicular to them) as shown in figure (16.6). Because the velocity is everywhere tangent to the streamlines, it is perpendicular to the sections and we have (keeping in mind the directions of the outgoing normals)

$$\oint_S d\vec{S} \cdot (\rho\vec{v}) = -\int_{S_1} \rho v dS + \int_{S_2} \rho v dS = 0 \tag{16.7.27}$$

The integrals on the right are called the **mass flux**es of the fluid across the surfaces A_1 and A_2 respectively. They represent the quantity of matter entering the volume (this can only happen via the end sections because the walls of our volume are streamlines). The condition says that the matter entering our volume equals the matter leaving, which is necessary for the density of fluid within the tube to stay constant.

There are several simple yet powerful applications of Bernoulli's equation, some of which we will now consider. Consider the lift experienced by an airfoil (an airplane wing, for example) that is positioned at an angle relative to the direction of fluid flow as shown in figure (16.7). The velocity of the air at the bottom of the wing is less than that at the top. The reason is simply that molecules of air collide with the airfoil losing kinetic energy in the process. Because the change in gravitational potential between the streamlines passing over the wing and those passing under it is negligible (the airfoil is "thin"), Bernoulli's equation asserts that

$$p_A + \frac{1}{2}\rho_A v_A^2 = p_B + \frac{1}{2}\rho_B v_B^2 \tag{16.7.28}$$

From this we find

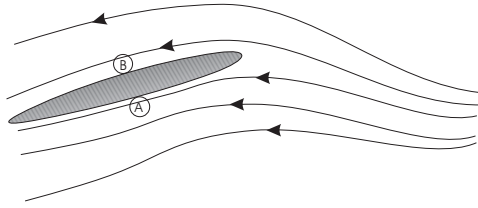$$\Delta p = p_A - p_B = \frac{1}{2}\rho_B v_B^2 - \frac{1}{2}\rho_A v_A^2 \tag{16.7.29}$$

Figure 16.7: An airfoil positioned at an angle relative to the airflow
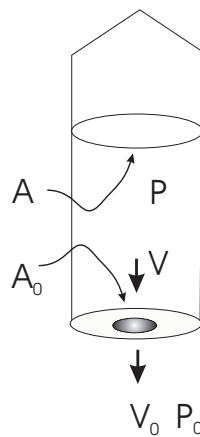

Figure 16.8: A simplified rocket

To a good approximation we may set $\rho_A \approx \rho_B \approx \rho_{\text{air}}$. Then the pressure differential between the lower and upper surfaces of the wing approximates to

$$\Delta p \approx \frac{1}{2}\rho_{\text{air}}(v_B^2 - v_A^2) \tag{16.7.30}$$

The "**lift**" is the total (upward) force

$$F = \Delta p A_{\text{wing}} \approx \frac{1}{2}\rho_{\text{air}} A_{\text{wing}}(v_B^2 - v_A^2) \tag{16.7.31}$$

where A_{wing} is the "**wingspan**", *i.e.,* the area of the wing.

   Another example of a direct use of Bernoulli's equation is thrust of a rocket which, in our simplified considerations, will consist of a gas confined within a chamber. The gas is able to escape through a small hole at one end. Let v be the speed of the gas inside the chamber abd v_0 its speed outside. Once more neglecting the gravitational potential

difference, Bernoulli's equation reads

$$p + \frac{1}{2}\rho v^2 = p_0 + \frac{1}{2}\rho v_0^2 \tag{16.7.32}$$

where p_0 is the atmospheric pressure. Furthermore, if A is the cross-section of the chamber and A_0 is the cross-sectional area of the hole, continuity implies that

$$\rho A v = \rho A_0 v_0 \Rightarrow v = \frac{A_0}{A} v_0 \tag{16.7.33}$$

The pressure differential is

$$p - p_0 = \frac{1}{2}\rho \left[1 - \left(\frac{A_0}{A}\right)^2 \right] v_0^2 \tag{16.7.34}$$

whence it follows that

$$v_0 = A \sqrt{\frac{2(p - p_0)}{\rho(A^2 - A_0^2)}} \tag{16.7.35}$$

Now if the rocket is designed so that $A_0 \ll A$ then

$$v_0 \approx \sqrt{\frac{2(p - p_0)}{\rho}} \tag{16.7.36}$$

The mass of gas flowing out per unit time is $\rho A_0 v_0 = dm/dt$ so, according to (2.10.5), the upward thrust on the rocket is

$$v_0 \frac{dm}{dt} = 2A_0(p - p_0) \tag{16.7.37}$$

16.7.3   Potential flows of Incompressible fluids

We have seen that if the flow is irrotational then by definition $\vec{\nabla} \times \vec{v} = 0$ and therefore the velocity may be expressed as the gradient of a scalar potential, ψ. The velocity vector is everywhere normal to the equipotential surfaces. If, moreover, the flow is incompressible then $\vec{v}$ must also satisfy $\vec{\nabla} \cdot \vec{v} = 0$ and therefore the scalar function ψ obeys Laplace's equation

$$\vec{\nabla}^2 \psi = 0 \tag{16.7.38}$$

As is customary, this equation must be solved subject to externally specified conditions at the boundaries of the fluid. The boundary conditions are fairly straightforward to determine. If the boundary is fixed then the component of the velocity normal to the boundary must vanish there. On the other hand, if the boundary is moving in a specified

way then the normal component of the fluid velocity must equal the normal component of the velocity of the bounding surface. In either case, the normal component of $\vec{v}$ is a specified function at the boundaries. Notice that (16.7.38) does not contain a time derivative so that the time dependence of ψ enters the solution *only through the boundary conditions*.

The simplest possible flow occurs if the velocity is constant everywhere, then the streamlines are parallel straight lines and the potential is time independent and given by

$$\psi(t, x, y, z) = \sum_i v_{0i} x_i + \text{const.} \tag{16.7.39}$$

More intersting potential flows occur when sources are present. Consider first a point-like source and assume the trivial boundary condition that the velocity vanishes at infinity. Spherical symmetry requires that the velocity is radially directed, $\vec{v} = v_r(r)\hat{r}$ and depends only on r. In spherical coordinates,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\psi}{dr}\right) = 0 \tag{16.7.40}$$

gives

$$\psi(r) = -\frac{Q}{r} + \text{const.}, \quad \vec{v} = \vec{\nabla}\psi(r) = \frac{Q}{r^2}\hat{r} \tag{16.7.41}$$

where Q is a constant which determines both the flow rate and its direction.

Again, because Laplace's equation is linear, for the same boundary conditions we may treat extended sources by summing over infinitesimal point like sources. For instance for a one dimensional source of arbitrary shape

$$\psi(r) = -\int_C \frac{Q(s)ds}{|\vec{r} - \vec{r}'(s)|} + \text{const.} \tag{16.7.42}$$

where the integral is over the source, parameterized by s as shown in figure (16.9).

As an example, consider a line source of length L located along the $x-$axis, as shown in figure (16.10). We have

$$\psi(x) = -Q\int_{-L/2}^{L/2} \frac{dx'}{\sqrt{(x-x')^2 + y^2 + z^2}} \tag{16.7.43}$$

assuming that Q is constant over the length of the source. Now, letting $(x - x') = \sqrt{y^2 + z^2}\sinh\eta$ and performing the resulting integration, we find

$$\psi(x, y, z) = Q\left[\sinh^{-1}\frac{x - L/2}{\sqrt{y^2 + z^2}} - \sinh^{-1}\frac{x + L/2}{\sqrt{y^2 + z^2}}\right] \tag{16.7.44}$$
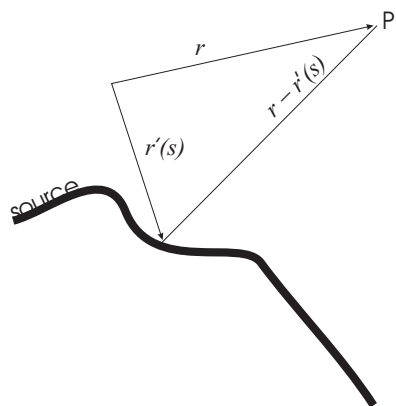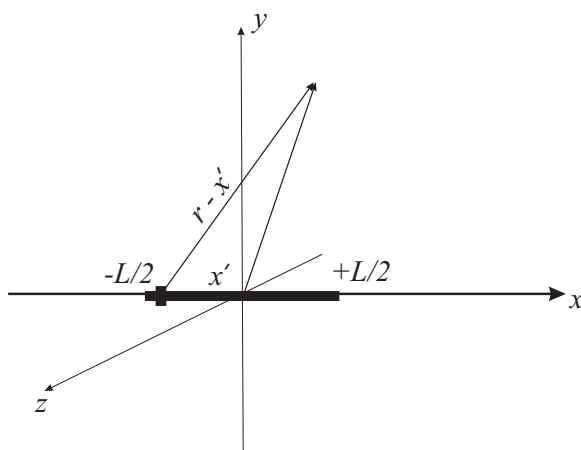
Figure 16.9: An extended source
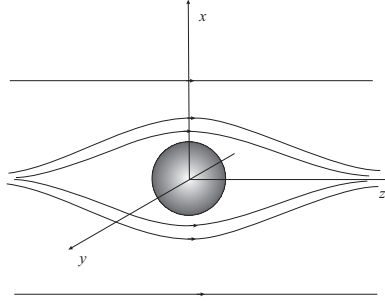

Figure 16.10: A line source

Figure 16.11: Potential flow past a sphere

The potential is singular on the $x-$axis and the velocity components are given by $\vec{v} = \vec{\nabla}\psi$.

Consider the potential flow around a sphere of radius R held fixed with its center at the origin of coordinates. The equation of motion for an incompressible fluid, $\vec{\nabla}^2\psi = 0$ must be solved subject to certain boundary conditions. As $x \to \infty$ we we will require that $\vec{v}_\infty = (0,0,v_\infty)$, where v_∞ is a constant. At the surface of the sphere, we require that the normal component of the velocity vanishes. Now in spherical coordinates

$$\partial_r r^2 \partial_r \psi + \frac{1}{\sin\theta}\partial_\theta \sin\theta\partial_\theta\psi + \frac{1}{\sin^2\theta}\partial_\varphi^2\psi = 0 \qquad (16.7.45)$$

Let us assume a solution in which the variables are separated, *i.e.,* take

$$\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi). \qquad (16.7.46)$$

The separation leads in the usual way to the radial equation

$$r^2\frac{d^2 R_l}{dr^2} + 2r\frac{dR_l}{dr} - l(l+1)R_l = 0 \qquad (16.7.47)$$

and gives the solution as a linear combination of

$$\psi(r,\theta,\varphi) = R_l(r)Y_{lm}(\theta,\varphi) \qquad (16.7.48)$$

where Y_{lm} are the spherical harmonics ($l = 0, 1, 2, \ldots$ and $m = \pm 1, \pm 2, \ldots, \pm l$). The radial equation is furthermore a linear superposition of the solutions

$$R_l(r) = r^{\omega_\pm}, \quad \omega_\pm = \frac{1}{2}\left[-1 \pm \sqrt{1 + 4l(l+1)},\right] \qquad (16.7.49)$$

so[4]

$$\psi(r,\theta,\varphi) = A_0 + B_0 r^{-1} + (A_1 r + B_1 r^{-2})\cos\theta + (A_2 r^2 + B_2 r^{-3})(3\cos^2\theta - 1) + \dots \quad (16.7.50)$$

Now if our solution is to reduce to $(0,0,v_\infty)$ as r approaches infinity, we should have $A_1 = v_\infty$ and $A_j = 0 \ \forall \ j \geq 2$. Furthermore, the term $B_0 r^{-1}$ would represent a sphere that is a source, so we must also set $B_0 = 0$ and, finally, a vanishing radial velocity at $r = R$ is possible if and only if $B_j = 0 \ \forall \ j \geq 2$. We conclude that

$$\psi(r,\theta,\varphi) = A_0 + v_\infty r \cos\theta + B_1 r^{-2}\cos\theta \quad (16.7.51)$$

Applying the boundary condition at $r = R$,

$$v_r(R) = \partial_r \psi|_{r=R} = 0 \Rightarrow v_\infty = 2B_1 R^{-3} \quad (16.7.52)$$

we find the solution

$$\psi(r,\theta,\varphi) = v_\infty \left[1 + \frac{R^3}{2r^3}\right] r \cos\theta \quad (16.7.53)$$

which, taking derivatives, gives the velocity field everywhere.

16.8   The Circulation

Define the **circulation** of a flow, Γ, by the line integral

$$\Gamma = \oint_C \vec{v} \cdot d\vec{r} \quad (16.8.1)$$

about an arbitrary contour drawn in the fluid at some instant. A **vortex** is a flow in which the circulation is non-vanishing in some region. The circulation of an isentropic flow is independent of time! To prove note that using Euler's equation we have

$$\frac{d\Gamma}{dt} = \oint_C \frac{d\vec{v}}{dt} \cdot d\vec{r} = -\oint_C \frac{\vec{\nabla}p}{\rho} \cdot d\vec{r} \quad (16.8.2)$$

and, if the flow is isentropic, $\vec{\nabla}p/\rho = \vec{\nabla}h \Rightarrow$

$$\frac{d\Gamma}{dt} = -\oint_C \vec{\nabla}h \cdot d\vec{r} \equiv 0 \quad (16.8.3)$$

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[4]Note that

$$l = 0 \Rightarrow \omega_\pm = 0, -1$$
$$l = 1 \Rightarrow \omega_\pm = 1, -2$$
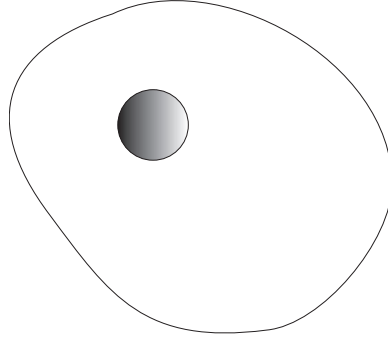$$l = 2 \Rightarrow \omega_\pm = 2, -3$$
$$\dots$$

Figure 16.12: Two dimensional space is not simply connected when an object is present.

and Γ is a conserved quantity. Note that an *ideal* fluid is one for which all flows are isentropic and therefore the circulation is *always* conserved for ideal fluid flows. Moreover, the circulation is clearly related to the vorticity by Stokes theorem,

$$\Gamma = \oint_C \vec{v} \cdot d\vec{r} = \int_S (\vec{\nabla} \times \vec{v}) \cdot d\vec{S} \tag{16.8.4}$$

It follows that if the flow is irrotational then the circulation must be zero provided that the space is simply connected. In two dimensional flows, the presence of any object makes the space not simply connected. As shown in figure (16.12), the presence of an object prevents the closed curve surrounding it to be shrunk to zero. In three dimensions the curve can be shrunk to zero by deforming it along the third dimension as shown in figure (16.13). However, situations in which space is not simply connected because of the presence of an object do occur in three dimensions. An example is provided in figure (16.14). When the space is not simply connected, Stokes theorem does not apply and one may have circulation even though the flow itself is irrotational.

An example in two dimensions by is the vortex given by the potential flow[5]

$$\psi(r,\theta) = \frac{\Lambda}{2\pi}\theta \tag{16.8.5}$$

where Λ is constant. We find

$$v_\theta = \frac{1}{r}\partial_\theta \psi(r,\theta) = \frac{\Lambda}{2\pi r}, \quad v_r = 0 \tag{16.8.6}$$

so, taking a curve around the origin,

$$\Gamma = \oint_C v_\theta r d\theta = \Lambda, \tag{16.8.7}$$

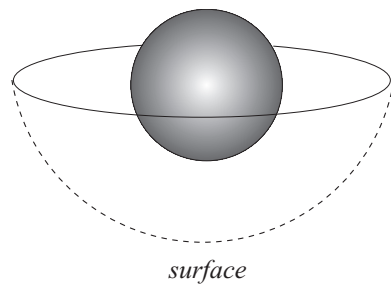[5]Problem: Verify that this solves Laplace's equation.

surface

Figure 16.13: In three dimensions the surface can be deformed in the third dimension.
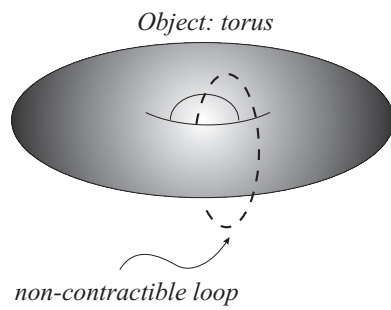

Object: torus

non-contractible loop

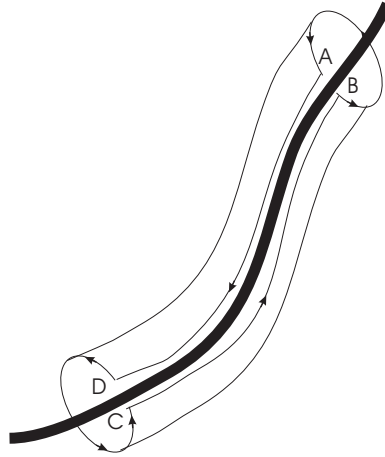Figure 16.14: Example of a non-simply connected space in three dimensions.

Figure 16.15: A line vortex.

whereas $\vec{\nabla} \times \vec{v} \equiv 0$. Therefore, the circulation does not vanish even though the vorticity does. This is because the potential is not defined at the origin, where θ is meaningless.

In three dimensions, one can have a **line vortex**, *i.e.,* a line about which the circulation is non-zero. If the flow is irrotational, the line must extend from boundary to boundary, for if not then by deforming the surface as before, one can apply Stoke's theorem to show that the circulation is also vanishing. We can show that the circulation is constant along the line, for consider the line integral $ADCBA$, as shown in figure (16.15). The curve is obviously closed and if, moreover, tythe flow is irrotational then

$$\oint \vec{v} \cdot d\vec{r} = \int \vec{\nabla} \times \vec{v} \cdot d\vec{S} \Rightarrow$$

$$\int_A^D \vec{v} \cdot d\vec{r} + \int_D^C \vec{v} \cdot d\vec{r} + \int_C^B \vec{v} \cdot d\vec{r} + \int_B^A \vec{v} \cdot d\vec{r} = 0 \qquad (16.8.8)$$

But now, bringing the split pieces close together gives

$$\int_A^D \vec{v} \cdot d\vec{r} + \int_C^B \vec{v} \cdot d\vec{r} = 0 \Rightarrow$$

$$\oint_{CD} \vec{v} \cdot d\vec{r} = \oint_{AB} \vec{v} \cdot d\vec{r} = \Gamma \qquad (16.8.9)$$

i.e., the circulation, taken anywhere along the line vortex, is the same.

Chapter 17

Energy and Momentum in Fluids

We now turn to the conservation of energy and momentum in fluids. This will lead us to the concept of the stress energy tensor, which will, in turn, serve as our guide as we move to modify Eulerian dynamics in such a way as to take into account non-ideal fluids in which energy and momentum is lost by the effects of friction. For incompressible fluids, Euler's equations will then turn into the Navier-Stokes equations.

17.1   The Energy Flux Density Vector

If u represent the internal energy of a unit mass of fluid and $\vec{v}$ its velocity, then one can write the energy per unit volume (or energy density) of fluid as

$$\varepsilon = \frac{1}{2}\rho\vec{v}^2 + \rho u \tag{17.1.1}$$

and thus its rate of change with time will be

$$\frac{\partial \varepsilon}{\partial t} = \frac{\partial \rho}{\partial t}\left(\frac{1}{2}\vec{v}^2 + u\right) + \rho\left(\vec{v}\cdot\frac{\partial\vec{v}}{\partial t} + \frac{\partial u}{\partial t}\right) \tag{17.1.2}$$

Now we can use the equation of continuity to replace $\partial\rho/\partial t$ by spatial gradients of the density and velocity and we can use the Euler's equation to do the same with $\partial\vec{v}/\partial t$. When this is done, the above equation will take the form

$$\frac{\partial \varepsilon}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{v})\left(\frac{1}{2}\vec{v}^2 + u\right) - \rho\left(\vec{v}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{v}\cdot\vec{\nabla}p - \frac{\partial u}{\partial t}\right). \tag{17.1.3}$$

However, because

$$\vec{v}\cdot(\vec{v}\cdot\vec{\nabla})\vec{v} = \frac{1}{2}(\vec{v}\cdot\vec{\nabla})\vec{v}^2$$

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this may be re-expressed as

$$\frac{\partial \varepsilon}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v}) \left(\frac{1}{2} \vec{v}^2 + u \right) - \rho \left(\frac{1}{2} (\vec{v} \cdot \vec{\nabla}) \vec{v}^2 + \frac{1}{\rho} \vec{v} \cdot \vec{\nabla} p - \frac{\partial u}{\partial t} \right). \tag{17.1.4}$$

Now apply the first law of thermodynamics to a unit mass of fluid,

$$du = T ds - p dV = T ds + \frac{p}{\rho^2} d\rho \tag{17.1.5}$$

which gives

$$\frac{\partial u}{\partial t} = T \frac{\partial s}{\partial t} + \frac{p}{\rho^2} \frac{\partial \rho}{\partial t} = T \frac{\partial s}{\partial t} - \frac{p}{\rho^2} \vec{\nabla} \cdot (\rho \vec{v}) \tag{17.1.6}$$

upon using the equation of continuity. Further,

$$dh = T ds + \frac{dp}{\rho} \Rightarrow \vec{\nabla} p = \rho \vec{\nabla} h - \rho T \vec{\nabla} s \tag{17.1.7}$$

and inserting (17.1.6) and (17.1.7) into the time evolution equation for the energy density, we find

$$\frac{\partial \varepsilon}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v}) \left(\frac{1}{2} \vec{v}^2 + u + \frac{p}{\rho} \right) - \rho \vec{v} \cdot \vec{\nabla} \left(\frac{1}{2} \vec{v}^2 + h \right) \tag{17.1.8}$$

and, since $h = u + p/\rho$,

$$\frac{\partial \varepsilon}{\partial t} = -\vec{\nabla} \cdot (\rho \vec{v}) \left(\frac{1}{2} \vec{v}^2 + h \right) - \rho \vec{v} \cdot \vec{\nabla} \left(\frac{1}{2} \vec{v}^2 + h \right) = -\vec{\nabla} \cdot \left(\rho \vec{v} \left[\frac{1}{2} \vec{v}^2 + h \right] \right) \tag{17.1.9}$$

and integrating both sides over a volume of the fluid enclosed by a surface S, we find using Gauss' law,

$$\frac{d}{dt} \int d^3 \vec{r} \, \varepsilon(t, \vec{r}) = - \oint_S \rho \vec{v} \left[\frac{1}{2} \vec{v}^2 + h \right] \cdot d\vec{S} \tag{17.1.10}$$

The left hand side of the equation represents the change of energy of the volume V, so the right hand side must represent the energy flux density across the bounding surface S. Thus we call

$$\vec{\mathcal{E}} = \rho \vec{v} \left[\frac{1}{2} \vec{v}^2 + h \right] \tag{17.1.11}$$

the **energy flux density vector** of the flow.

17.2 Momentum Flux Density Tensor

Let us follow the same procedure we applied to the energy density, now applying it to the momentum density, which can be defined as $\vec{P} = \rho\vec{v}$. Its time rate of change will obey

$$\frac{\partial \vec{P}}{\partial t} = \frac{\partial \rho}{\partial t}\vec{v} + \rho\frac{\partial \vec{v}}{\partial t} \qquad (17.2.1)$$

and, using the continuity equation together with Euler's equations we find

$$\frac{\partial \vec{P}}{\partial t} = -\vec{\nabla}\cdot(\rho\vec{v})\vec{v} - \rho(\vec{v}\cdot\vec{\nabla})\vec{v} - \vec{\nabla}p \qquad (17.2.2)$$

It is better to rewrite this equation in components as follows

$$\begin{aligned}
\partial_t \vec{P}_i &= -\partial_j(\rho v_j)v_i - \rho v_j\partial_j v_i - \partial_i p \\[2mm]
&= -\partial_j\left(\rho v_i v_j + p\delta_{ij}\right) \qquad (17.2.3)
\end{aligned}$$

and integrating over a volume bounded by the surface S we find

$$\frac{d}{dt}\int d^3\vec{r}\, P_i = -\oint_S \left(\rho v_i v_j + p\delta_{ij}\right)dS_j \qquad (17.2.4)$$

where we once again used Gauss' law. As the right hand side represents the rate at which the momentum of volume V of the fluid is changing, the right hand side must represent the rate at which momentum is carried enters (or leaves) the volume by flowing into (or out of) the surface S. The tensor

$$\Pi_{ij}^{(0)} = p\delta_{ij} + \rho v_i v_j \qquad (17.2.5)$$

is the **momentum flux density tensor** of the flow.

17.3 The Stress Tensor

Let us note that (17.2.3) is just another way to express Euler's equations. This follows, of course, by symply performing the derivatives,

$$(\partial_t\rho)v_i + \rho(\partial_t v_i) = -\partial_i p - \rho v_i\partial_j v_j - \rho v_j\partial_j v_i - v_i v_j\partial_j\rho \qquad (17.3.1)$$

or, in vector notation,

$$\frac{\partial \rho}{\partial t}\vec{v} + \rho\frac{\partial \vec{v}}{\partial t} + \vec{\nabla}p + \rho(\vec{\nabla}\cdot\vec{v})\vec{v} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla}\rho)\vec{v} \qquad (17.3.2)$$

Now the first term together with the fourth and sixth give the combination

$$\left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v})\right] \vec{v}, \tag{17.3.3}$$

which vanishes by the continuity equation and we are left with

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\vec{\nabla} p \tag{17.3.4}$$

which is simply (16.4.11) in the absence of external forces.

We have considered ideal fluids, *i.e.,* fluids in which there is no energy loss by friction and no heat transfer either within the fluid or between the fluid and its environment. Unfortunately, the ideal fluid does not exist in nature and it is necessary to think of how Euler's equations must be modified so as to allow for both of the above possibilities. One approach would be to modify the right hand side of (17.2.3) by adding a term to Π_{ij}

$$\Pi_{ij}^{(0)} \to \Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \sigma'_{ij} \stackrel{\text{def}}{=} \rho v_i v_j - \sigma_{ij} \tag{17.3.5}$$

The tensor

$$\sigma_{ij} = \sigma'_{ij} - p\delta_{ij} \tag{17.3.6}$$

is called the **stress tensor** of the fluid and σ'_{ij} is the **viscosity tensor**. Thus σ_{ij} gives the portion of momentum flux that is *not* due to the direct transfer of momentum with the fluid mass.

The viscosity tensor must be obtained phenomenologically.

- First we note that internal friction between the molecules will occur when there is an average relative velocity between different portions of the fluid, specifically when $\partial_i v_j \neq 0$.

- Next, if the velocity gradients are not too large we may suppose that the effects of viscosity depend only linearly upon them.

- Thirdly, *all* terms in the viscosity tensor *must* depend on velocity gradients, and

- finally, the stress tensor must vanish when the fluid as a whole rotates, *i.e.,* when $\vec{v}$ is given as $\vec{v} = \vec{\omega} \times \vec{r}$.

For a velocity flow whose components are of the form $v_i = [\vec{\omega} \times \vec{r}]_i = \epsilon_{ilm}\omega_l x_m$, its spatial gradients would be $\partial_j v_i = -\epsilon_{ijl}\omega_l$, which is antisymmetric in (i, j). Therefore the sum

$$\lambda_{ij} = \partial_j v_i + \partial_i v_j \tag{17.3.7}$$

vanishes identically and a viscosity tensor defined as

$$\sigma'_{ij} = a\lambda_{ij} + b\delta_{ij}\partial_k v_k \tag{17.3.8}$$

would satisfy all the stated conditions. We write the stress tensor as

$$\sigma_{ij} = -p\delta_{ij} + \xi(\vec{\nabla} \cdot \vec{v})\delta_{ij} + \eta\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}(\vec{\nabla} \cdot \vec{v})\delta_{ij}\right) \tag{17.3.9}$$

after rearranging terms and redefining constants so that the term proportional to η is traceless. The coefficients η and ξ could in principle depend on position within the fluid, but we will take them to be constant. η is called the **shear viscosity** and ξ the **bulk viscosity**. Now putting everything together,

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \xi(\vec{\nabla} \cdot \vec{v})\delta_{ij} - \eta\left(\partial_j v_i + \partial_i v_j - \frac{2}{3}(\vec{\nabla} \cdot \vec{v})\delta_{ij}\right) \tag{17.3.10}$$

and the modified Euler equations now read

$$\partial_t(\rho v_i) + \partial_j \Pi_{ij} = 0, \tag{17.3.11}$$

which, in the more traditional form, reads

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \eta\vec{\nabla}^2\vec{v} + \left(\xi + \frac{\eta}{3}\right)\vec{\nabla}(\vec{\nabla} \cdot \vec{v}). \tag{17.3.12}$$

If the fluid is incompressible so that $\vec{\nabla} \cdot \vec{v} = 0$,

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \eta\vec{\nabla}^2\vec{v}. \tag{17.3.13}$$

External forces may be taken into account by adding $\vec{f}^{\text{ext}}$ to the right hand side of this equation,

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \eta\vec{\nabla}^2\vec{v} + \vec{f}^{\text{ext}}. \tag{17.3.14}$$

These are the **Navier-Stokes equation** for an incompressible fluid, a very good approximation when working with liquids to the extent that they can be treated as incompressible, but not so good for gases. On the other hand, for gases the shear viscosity can often be so low as to be negligible and only the bulk term survives. Sometimes the ratio η/ρ is denoted by ν and called the **kinematic viscosity**, as opposed to "dynamic" or "absolute" viscosity.

The equation may be reformulated in terms of the vorticity directly if we take the curl of (17.3.14),

$$\rho\frac{\partial \vec{\omega}}{\partial t} + \rho\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla})\vec{v} = \eta\vec{\nabla} \times \vec{\nabla}^2\vec{v} + \vec{\nabla} \times \vec{f}^{\text{ext}}. \tag{17.3.15}$$

Then using

$$\vec{v} \times (\vec{\nabla} \times \vec{v}) = \frac{1}{2}\vec{\nabla}\vec{v}^2 - (\vec{v} \cdot \vec{\nabla})\vec{v}, \tag{17.3.16}$$

we see that

$$\vec{\nabla} \times (\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} \times (\vec{v} \times (\vec{\nabla} \times \vec{v})) = -\vec{\nabla} \times (\vec{v} \times \vec{\omega}) \tag{17.3.17}$$

and therefore

$$\rho\frac{\partial\vec{\omega}}{\partial t} - \rho\vec{\nabla} \times (\vec{v} \times \vec{\omega}) = \eta\vec{\nabla}^2\vec{\omega} + \vec{\nabla} \times \vec{f}^{\text{ext}}, \tag{17.3.18}$$

which is the desired equation.

17.4   Energy Dissipation

Because of the viscosity, a flowing fluid will eventually lose its mechanical energy to thermal energy. For a fixed volume, V, of the fluid the kinetic energy content may be given as

$$K = \frac{1}{2}\int_V d^3\vec{r}\rho\vec{v}^2 \tag{17.4.1}$$

so we consider the kinetic energy density, $k = \rho\vec{v}^2/2$ and its time derivative, assuming that the fluid density is constant throughout,

$$\frac{\partial k}{\partial t} = \rho\vec{v} \cdot \frac{\partial\vec{v}}{\partial t}. \tag{17.4.2}$$

Then using the Navier-Stokes equation we find

$$\frac{\partial k}{\partial t} = \rho\vec{v} \cdot \left[-(\vec{v} \cdot \vec{\nabla})\vec{v} - \frac{\vec{\nabla}p}{\rho} + \nu\vec{\nabla}^2\vec{v}\right], \tag{17.4.3}$$

which may be put in the form $((\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{1}{2}\vec{\nabla}\vec{v}^2)$

$$\frac{\partial k}{\partial t} = -\rho\vec{v} \cdot \vec{\nabla}\left(\frac{1}{2}\vec{v}^2 + \frac{p}{\rho}\right) + \eta\vec{v} \cdot \vec{\nabla}^2\vec{v} \tag{17.4.4}$$

or, using the condition for incompressibility, $\vec{\nabla} \cdot \vec{v} = 0$,

$$\frac{\partial k}{\partial t} = -\vec{\nabla} \cdot \left(\frac{1}{2}\rho\vec{v}^2 + p\right)\vec{v} + \eta\vec{v} \cdot \vec{\nabla}^2\vec{v}. \tag{17.4.5}$$

It's worth examining the last term a bit more. Since $\vec{v} \cdot \vec{\nabla}^2\vec{v} = v_i\partial_j^2 v_i$, consider

$$\frac{1}{2}\partial_j^2\vec{v}^2 = \partial_j(v_i\partial_j v_i) = v_i\partial_j^2 v_i + (\partial_j v_i)^2 \tag{17.4.6}$$

and so

$$\vec{v} \cdot \vec{\nabla}^2 \vec{v} = \frac{1}{2} \vec{\nabla}^2 \vec{v}^2 - (\partial_j v_i)^2$$

$$\Rightarrow \frac{\partial k}{\partial t} = -\vec{\nabla} \cdot \left[\left(\frac{1}{2} \rho \vec{v}^2 + p \right) \vec{v} - \frac{1}{2} \eta \vec{\nabla} \vec{v}^2 \right] - \eta (\partial_j v_i)^2 \qquad (17.4.7)$$

Integrating over the volume, V,

$$\frac{dK}{dt} = -\int \left[\left(\frac{1}{2} \rho \vec{v}^2 + p \right) \vec{v} - \frac{1}{2} \eta \vec{\nabla} \vec{v}^2 \right] \cdot d\vec{S} - \eta \int_V d^3\vec{r} \, (\partial_j v_i)^2 \qquad (17.4.8)$$

The first term represents an energy flux across the bounding surface. It is made up of two parts, the first corresponding to that of an ideal fluid and the second proportional to η and representing the flux due to processes of internal friction. The last term represents the rate of decrease in kinetic energy due to **dissipation**. If the integral is extended over the entire volume of the fluid then the surface integral vanishes and we see that the rate of change of total kinetic energy is just

$$\frac{dK}{dt} = -\eta \int_V d^3\vec{r} \, (\partial_i v_j)^2 \qquad (17.4.9)$$

17.5 Boundary Conditions

Ideal fluids are required to satisfy the requirement that the normal component of the velocity field vanishes at the boundaries. For real fluids, depending on the situation encountered, various other conditions may be appropriate. The most common of these are the following

- **No-slip walls:** If a boundary is rigid and if the forces between the molecules of the fluid and the molecules of the walls of the boundary are strong enough to stop the tangential motion of the fluid molecules closest to the walls then not only is the normal component of the velocity field vanishing but also its tangential component, assuming that the boundary is static. The two components (normal and tangential) can only vanish if the velocity itself vanishes at the boundary, *i.e., for a real fluid $\vec{v} = 0$ at the (static) boundaries*. More generally, if such a "sticky boundary" moves with a velocity $\vec{u}$ then the fluid velocity would equal $\vec{u}$ at the boundary in this case. Rigid, "sticky" boundaries are called "no-slip" walls.

- **Fluid-Fluid interface:** To motivate the boundary conditions at the interface of two immiscible fluids (eg., oil and water) it is worth asking what force is exerted

by a fluid (assumed incompressible) on its boundary. If S represents the bounding surface, the total force will be

$$F_i = \oint_S \Pi_{ik} dS_k \tag{17.5.1}$$

so that the force per unit area on the boundary is

$$f_i = \Pi_{ik} n_k, = \rho(\widehat{n} \cdot \vec{v}) v_i - \sigma_{ik} n_k, \tag{17.5.2}$$

where $\widehat{n}$ is the outgoing normal to the surface. If in addition the fluid velocity vanishes at the boundary then the only contribution to the force per unit area comes from the second term above,

$$f_i = -\sigma_{ik} n_k = p n_i - \sigma'_{ik} n_k. \tag{17.5.3}$$

The first term on the right hand side is the fluid pressure and acts normal to the surface. The second term, due to viscosity, is the force of friction acting on the surface. Now, another condition that should be met at the interface is Newton's third law of action and reaction, in other words, the force per unit area due to one fluid should exactly balance out the force per unit area due to the other, or

$$n_k^{(1)} \sigma_{ik}^{(1)} + n_k^{(2)} \sigma_{ik}^{(2)} = 0 \tag{17.5.4}$$

where $n_k^{(1,2)}$ are the unit outgoing normals for the two fluids, labeled by (1) and (2) and satisfy $n_i^{(1)} = -n_i^{(2)}$.

- **Free boundaries:** If one of the fluids is inviscid ($\eta = 0$), or if the boundary is a free surface of the fluid, shear cannot be supported and we must require the stress to vanish there. This is the condition that

$$p n_i - \sigma'_{ik} n_k = 0, \tag{17.5.5}$$

at a free boundary, where n_k is the (outgoing) normal.

17.6   Reynolds and Froude Numbers

It is worthwhile reconsidering the significance of each term in the Navier-Stokes equation,

$$\rho \frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla} p + \eta \vec{\nabla}^2 \vec{v} + \vec{f}^{\text{ext}}. \tag{17.6.1}$$

Beginning with the left hand side, the first term defines a "local" acceleration of the fluid and the second a "convective" acceleration depending, as it does, on the spatial change in the velocity field. Combined the two terms form the "total" derivative

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} \tag{17.6.2}$$

in terms of which the Navier-Stokes equations is sometimes written as

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla}p + \eta \vec{\nabla}^2 \vec{v} + \vec{f}^{\text{ext}}. \tag{17.6.3}$$

The first term on the right hand side represents a normal surface force due to pressure. The second term is the "viscous" force and is of great importance, being the single property unique to the concept of a "flow" and therefore to fluids. We have already seen that viscosity is connected with a transport of momentum between adjoining layers of a fluid, moving at different velocities and that because of it a viscous fluid dissipates energy at a rate that is proportional to η, according to (17.4.9). The interaction between adjoining layers in a fluid arises either (i) because of the intermolecular forces (which are weak in gases) or (ii) by thermal agitation that causes diffusion of molecules from faster moving layers to slower moving ones. In either case the net result is that mechanical energy is dissipated in the fluid and ends up reappearing as thermal energy. As a result, the flow of real fluids is never isentropic – the entropy rises – and always results in the loss of usable energy. Therefore external forces are always required to maintain fluid motion.

We have five non-linear, coupled differential equations *viz.,* the Navier-Stokes equations above, the continuity equation and an equation of state, with which to determine the flow velocity, the pressure and the density. This is an extremely difficult system to solve and therefore not many exact solutions are actually available (we will obtain the simplest of them here). However, progress in understanding the physical properties of solutions can be made by some fairly simple considerations based in dimensional analysis.

Now every set of equations will involve a set of variables and a set of parameters, and these will, in general, be dimensionful. For instance, the independent variables appearing in the Navier-Stokes equations are $\vec{r}$ and t, having mechanical dimensions of length and time respectively. The mechanical dimension of the dependent variables are

$$[\vec{v}] = \frac{l}{t}, \quad [\rho] = \frac{m}{l^3}, \quad [p] = \frac{m}{lt^2} \tag{17.6.4}$$

and the mechanical dimension of the viscosity coefficient is

$$[\eta] = \frac{m}{lt}. \tag{17.6.5}$$

(There may be other parameters present in the external force, eg. the acceleration due to gravity or Newton's universal constant of gravitation).

As in any mechanical system, the number of fundamental physical quantities or dimensions is three, *viz.,* mass, length and time. We would like to have "typical" values for these quantities, that is, values that characterize the system we are studying. What is "typical" would depend on the system of course; for example, if the problem is flow through a pipe then we may want to set the "typical" length scale to be the diameter or radius, R, of the pipe. While the length scale is often obvious from the geometry and symmetries of a given problem, it is difficult to determine a time scale directly. Instead one picks a "typical" or characteristic *velocity* scale. Thus, continuing with the example of flow in a pipe, we may take the "centerline velocity" *i.e.,* the speed in the center of the pipe, v_c, to set the velocity scale. Having a length scale and a velocity scale implies that we have a time scale according to

$$t_c = R/v_c, \tag{17.6.6}$$

but what are we to do about a "mass" scale? We will deal with this indirectly as before: if we had a characteristic pressure, p_c, instead, then a characteristic mass would be (from dimensional analysis)

$$m_c = p_c R t_c^2 = \frac{p_c R^3}{v_c^2} \tag{17.6.7}$$

and a characteristic density,

$$\rho_c = \frac{m_c}{R^3} = \frac{p_c}{v_c^2}, \tag{17.6.8}$$

Now let's define the dimensionless quantities

$$\tau = \frac{t}{t_c}, \quad \vec{R} = \frac{\vec{r}}{R}, \quad \vec{u} = \frac{\vec{v}}{v_c}, \quad \wp = \frac{p}{p_c}, \quad \varrho = \frac{\rho}{\rho_c} \tag{17.6.9}$$

and re-express the Navier-Stokes equations in terms of them. We find

$$\varrho \frac{d\vec{u}}{d\tau} = -\widetilde{\nabla} \wp + \frac{\eta v_c}{p_c R} \widetilde{\nabla}^2 \vec{u} \tag{17.6.10}$$

where $\widetilde{\nabla}$ is the gradient operator defined with respect to the scaled, dimensionless coordinates and

$$\frac{d\vec{u}}{d\tau} = \frac{\partial \vec{u}}{\partial \tau} + (\vec{u} \cdot \widetilde{\nabla})\vec{u}. \tag{17.6.11}$$

The dimensionless quantity

$$\mathfrak{R} = \frac{p_c R}{\eta v_c} \tag{17.6.12}$$

is called the **Reynolds number** of the flow. It is sometimes expressed in terms of the kinematic viscosity, ν, as

$$\mathfrak{R} = \frac{R v_c}{\nu}, \tag{17.6.13}$$

where we used the relation $\rho_c v_c^2 = p_c$ and the fact that $\vec{u}$ is defined to be η/ρ_c. All solutions will now depend on the dimensionless independent variables and this single dimensionless parameter, *i.e.*, in particular, $\vec{u} = \vec{u}(\vec{R}, \tau, \mathfrak{R})$, $\wp = \wp(\vec{R}, \tau, \mathfrak{R})$ and $\varrho = \varrho(\vec{R}, \tau, \mathfrak{R})$. Had we included an external force term of the form $\vec{f}^{\text{ext}} = \rho\vec{g}$, describing a fluid in a constant gravitational field, then we would find another dimensionless parameter, [1]

$$\mathfrak{F} = \frac{v_c}{\sqrt{gR}}, \tag{17.6.14}$$

which is called the **Froude number** and we find

$$\varrho \frac{d\vec{u}}{d\tau} = -\widetilde{\nabla}\wp + \frac{1}{\mathfrak{R}}\widetilde{\nabla}^2\vec{u} - \frac{1}{\mathfrak{F}^2}\varrho\,\hat{g} \tag{17.6.15}$$

where $\hat{g}$ is the unit vector pointing in the direction of $\vec{g}$. The Reynolds number is relevant only for viscous flows whereas the Froude number is relevant only for flows occurring in a (constant) gravitational field. In general, in any particular problem, if a length scale L and a velocity scale V can be identified then the Reynolds and Froude numbers are respectively

$$\mathfrak{R} = \frac{VL}{\nu} \quad \text{and} \quad \mathfrak{F} = \frac{V}{\sqrt{gL}}. \tag{17.6.16}$$

All solutions of the Navier-Stokes equations would depend on the geometry, the boundary and initial conditions and these two dimensionless parameters. In particular, two systems having the same geometry, obeying the same boundary conditions and possessing identical Reynolds and Froude numbers would possess identical scaled velocity, pressure and density fields and therefore behave identically, their actual velocity fields, pressures and density differing only in scale from one another. Such flows are said to be similar and this is called the **law of similarity**.

So what do the Reynolds number and the Froude number tell us and how are we to interpret them? Consider the Reynolds number and notice that it is proportional to the characteristic speed and length scales but inversely proportional to the kinematic viscosity. Now suppose we wish to test out a new design for an airplane wing with the help of a "wind tunnel". Of course we will want to scale down the wing so that it may fit into our wind tunnel, so we construct a smaller but geometrically similar model of the real airplane wing. However, scaling down the wing (*i.e.*, decreasing the characteristic length, L) will lead to a different behavior from the one that describes the real wing unless we keep $\mathfrak{R}$ the same. Therefore, to truly replicate the behavior of the original wing, the scaling down of the wing span must be accompanied by a scaling *up* of the ratio V/ν. This can be accomplished (i) by increasing the characteristic flow speed in the wind tunnel commensurately, while keeping the kinematic viscosity the same or (ii) decreasing the kinematic viscosity while

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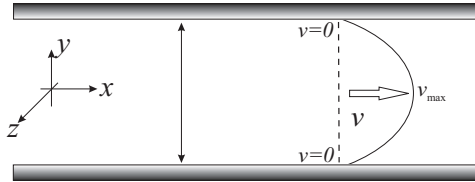[1] <u>Problem</u>: Show this and check that $\mathfrak{F}$ is indeed dimensionless.

Figure 17.1: Plane Poiseuille Flow

keeping the flow velocity the same or (iii) a combination of both so as to keep the expected value of $\Re$ for the conditions under which the original wing is required to function. Similar considerations apply to the Froude number.

17.7   Applications of the Navier-Stokes equation

We shall now consider some examples of the use of the Navier-Stokes equation. These examples are simple enough to possess analytical solutions, but it should be kept in mind that this is rare in fluid dynamics and that the most interesting solutions are generally obtained via the application of computational methods ("Computational Fluid Dynamics" or CFD) that are far beyond the scope of these notes. Furthermore, the mere existence of solutions, whether analytical or computational, does not guarantee their stability and, indeed, stability must be dealt with as a separate issue *à posteriori*.

17.7.1   Plane Poiseuille Flow

Plane Poiseuille flow is a steady flow occurring between two infinite planes, which we take to be parallel to the $x - z$ plane and situated at $y = 0$ and $y = h$ respectively. This represents is pressure driven flow in a duct and we will assume that it is defined by the following conditions:

- the density of the fluid is constant,

- the velocity field varies only in the y direction (since the x and z directions are allowed to be infinite in extent there is no way to impose boundary conditions that can support dependencies on x and z),

- the pressure is independent of z and

- the velocity field vanishes on the planes (no slip boundary conditions).

Accordingly, the continuity equation, $\vec{\nabla} \cdot \vec{v} = 0$, implies that $\partial_y v_y = 0$ and therefore v_y must be a constant, which we must take to be zero so as to satisfy the boundary conditions. Under these conditions the Navier-Stokes equations collapse into the **Stokes** equations,

$$\eta \vec{\nabla}^2 \vec{v} = \vec{\nabla} p \tag{17.7.1}$$

and its $y-$component tells us that $\partial_y p = 0$, so the pressure does not depend on y. Its $z-$component implies that $\partial_y^2 v_z = 0$ and therefore $v_z = ay+b$ where a and b are constants. However, requiring v_z to vanish at the boundaries forces both a and b to vanish as well. The $x-$component of the Stokes equations is

$$\eta \frac{d^2 v_x}{dy^2} = \frac{dp}{dx} \tag{17.7.2}$$

Now we notice that the left hand side of the above equation depends only on y whereas the right hand side depends at most on x. This is only possible if each side is equal to the same constant,

$$\frac{dp}{dx} = \lambda, \quad \eta \frac{d^2 v_x}{dy^2} = \lambda, \tag{17.7.3}$$

where λ represents the constant pressure gradient in the direction of the flow and the pressure as a function of position is

$$p(x) = \lambda x + p_0 \tag{17.7.4}$$

where p_0 represents the pressure at $x = 0$. The second equation just as easily solved and subjected to the requirement that v_x vanish at the boundaries and we find

$$v_x = \frac{\lambda}{2\eta}(y^2 - hy) \tag{17.7.5}$$

which is positive for $\lambda < 0$ because $y < h$. This implies that the pressure gradient is negative in the direction of the flow. The velocity profile is parabolic with its maximum in the middle of the flow, at $h/2$.

17.7.2   Couette Flow

We can introduce a non-trivial moving boundary into the plane Poiseuille flow by requring the no slip conditions $\vec{v}(0) = 0$ and $\vec{v}(h) = V\hat{x}$, where V is the assumed velocity of the plane situated at $y = h$.[2] Now we consider a flow which is not be pressure driven but shear-driven, produced by the plate at $y = h$ moving in the positive x direction with speed

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[2]This is generally called the Couette Flow after Maurice Couette, who discovered the solution in the late nineteenth century.
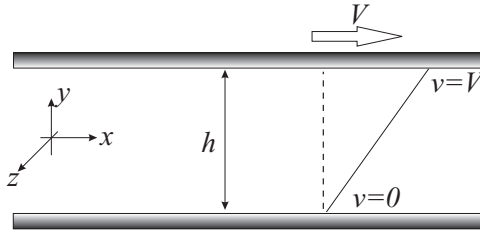
Figure 17.2: Couette Flow

V. We will retain all the assumptions of the Poiseuille flow, but one: instead of requiring that p remain independent only of z we also require that it remain independent of x, since both the x and z directions are of infinite extent and the flow is not driven by the pressure. Once again the continuity equation will require that v_y is constant, which can only satisfy the boundary conditions if it vanishes. The $y-$component of the effective Stokes equation in (17.7.1) then reads $\partial_y p = 0$, which of course implies that p is independent of y as well and therefore it is constant throughout the fluid. The $z-$component of the same equation yields $\partial_y^2 v_z = 0$, whose solution, as before, only satisfies the boundary conditions if $v_z = 0$. Finally the x component of the Stokes equation reads

$$\eta \frac{d^2 v_x}{dx^2} = 0 \Rightarrow v_x = \frac{V}{h} y \tag{17.7.6}$$

is the solution satisfying the no-slip conditions. Here the velocity profile is linear and, not surprisingly, the maximum fluid velocity occurs in the layer adjoining the moving boundary. It is independent of the viscosity. This simple flow is particularly useful in the analysis of lubrication.

17.7.3 Hagen-Poiseuille Flow

We consider the conditions of the Poiseuille flow, but this time in a cylindrical pipe of infinite length and radius R, whose axis we take to be the $z-$axis. The symmetry makes it most convenient to work in cylindrical coordinates. Thus we take ρ to be constant as before, we assume the flow is steady, depending only on the radial coordinate and we impose no slip boundary conditions, $\vec{v}(R) = 0$. Moreover, we assume that the pressure does not depend on the azimuthal coordinate so that $p = p(r, z)$.

This provides an excellent opportunity to use the methods of Chapter 15. Thus the

continuity equation reads[3]

$$\vec{\nabla} \cdot \vec{v} = 0 = \nabla_j v^j = \partial_j v^j + \Gamma^j_{jl} v^l \tag{17.7.7}$$

and using the Christoffel symbols derived in (15.4.16) we find

$$\partial_r v^r + \frac{v^r}{r} = 0 \Rightarrow v^r = \frac{a}{r}, \tag{17.7.8}$$

which can only vanish because of the boundary conditions. Next, the Navier-Stokes equations read

$$\rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p + \eta \vec{\nabla}^2 \vec{v} \tag{17.7.9}$$

which, in components, becomes

$$\rho(v^j \partial_j v^i + \Gamma^i_{jk} v^j v^k) = -g^{ij} \partial_j p + \eta g^{jk} \nabla_j \nabla_k v^i \tag{17.7.10}$$

The first term on the left vanishes because all velocity components depend only r and $v^r = 0$. Thus

$$g^{ij} \partial_j p = \eta g^{jk} \nabla_j \nabla_k v^i - \rho \Gamma^i_{jk} v^j v^k \tag{17.7.11}$$

This turns into a complicated expression, but it is immensely simplified if we recall that the flow velocity may depend only on r. One eventually finds the following equations

$$\partial_r p = \rho r v^{\varphi 2}$$

$$0 = \eta \left(\frac{d^2 v^\varphi}{dr^2} + \frac{3}{r} \frac{dv^\varphi}{dr} \right)$$

$$\partial_z p = \eta \left(\frac{d^2 v^z}{dr^2} + \frac{1}{r} \frac{dv^z}{dr} \right) \tag{17.7.12}$$

The second equation above has the solution

$$v^\varphi = \frac{a}{2} \frac{R^2 - r^2}{r^2 R^2} \tag{17.7.13}$$

but, even though it satisfies the boundary conditions, it is ill behaved on the axis and so must be set to zero by taking $a = 0$. By the first equation of (17.7.12) this means that the pressure is a function only of z. The last equation of (17.7.12) can be written as

$$\frac{dp}{dz} = \eta \left(\frac{d^2 v^z}{dr^2} + \frac{1}{r} \frac{dv^z}{dr} \right) = \lambda \tag{17.7.14}$$

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[3]We must be careful to distinguish between covariant and contravariant quantitites since we are now working in a changing basis.

where λ is a constant representing the pressure gradient in the the $z-$direction. The solution that obeys the boundary conditions *and* is well behaved on the axis is

$$p = \lambda z + p_0, \quad v^z = v_z = \frac{\lambda}{4\eta}(r^2 - R^2) \qquad (17.7.15)$$

The velocity profile, as in the planar case, is parabolic and its maximum is achieved when $r = 0$, at which $v_{z,\text{max}} = -\lambda R^2/4\eta$. Blood flow through the arteries, for example, can be modeled by the Hagen-Poiseuille solution above. The volume of fluid delivered can be shown to be proportional to the fourth power of the pipe radius,[4] which means that a condition like arteriosclerosis, which is the accumulation of fatty material on the walls of the arteries, can very rapidly block the flow of blood to cells.

17.8   Relativistic Fluids

We have seen that the dynamics of ideal fluids in the absence of external forces is suitably expressed by the continuity equation, the three Euler equations,

$$\partial_t(\rho v_i) + \partial_j \Pi_{ji}^{(0)} = 0, \qquad (17.8.1)$$

where $\Pi_{ji}^{(0)} = p\delta_{ji} + \rho v_j v_i$ represents the momentum flux density tensor and one equation of state relating the pressure to the energy density. Clearly, the continuity and Euler equations are valid only in the regime in which Galilean relativity applies, *i.e.*, when the flow velocity is much less than the speed of light. We seek a generalization of this equation that is valid even when the flow velocity is close to the speed of light. This essentially means that we want to construct a second rank tensor, $T_{\mu\nu}$, that reduces to the momentum flux density tensor at low flow velocity. We go about this in the same way as we went about deducing the structure of the relativistic force four vector in Chapter 13.

17.8.1   Perfect Fluids

Consider therefore a frame in which the fluid is instantaneously at rest, *i.e.*, $v_i = 0$. In this frame $\Pi^{ij} = p\delta^{ij}$, and we must identify this with the spatial part of $\overline{T}^{\mu\nu}$,[5] but what can be said about the temporal part? Since all terms in the trace of $\overline{T}$

$$\eta_{\mu\nu}\overline{T}^{\mu\nu} = \overline{T}^{\mu}{}_{\mu} = \overline{T}^{0}{}_{0} + 3p \qquad (17.8.2)$$

---

[4] Problem: Compute the total volume of fluid that flows through a cross section of the pipe per second.

[5] We write everything in contravariant form for ease of calculation. In a Cartesian frame there is no difference between covariant and contravariant three tensors.

must have the same mechanical dimension, it follows that

$$[\overline{T}^0{}_0] = [p] = \frac{m}{lt^2} \tag{17.8.3}$$

and again, because $\overline{T}^{00} = \eta^{00}\overline{T}^0{}_0$, the mechanical dimension of $\overline{T}^{00}$ must be

$$[\overline{T}^{00}] = \frac{m}{l^3} \tag{17.8.4}$$

Thus we take $\overline{T}^{00} = \rho$, the mass density of the fluid. Finally, because we are in the instantaneous rest frame we set $\overline{T}^{0i} = \overline{T}^{i0} = 0$ and we get the **energy-momentum flux density** tensor – or energy-momentum tensor for short – in the instantaneous rest frame,

$$\overline{T}^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{17.8.5}$$

To find the energy momentum tensor in an arbitrary frame, we simply perform a Lorentz transformation with boost parameter v_i,

$$T^{\mu\nu} = L^\mu{}_\alpha L^\nu{}_\beta \overline{T}^{\alpha\beta} \tag{17.8.6}$$

and get the expressions

$$T^{00} = L^0{}_\alpha L^0{}_\beta \overline{T}^{\alpha\beta} = \gamma^2\left(\rho + \frac{\vec{v}^2}{c^4}p\right) = -\frac{p}{c^2} + \left(\rho + \frac{p}{c^2}\right)\gamma^2$$

$$T^{0i} = L^0{}_\alpha L^i{}_\beta \overline{T}^{\alpha\beta} = L^0{}_0 L^i{}_0 \overline{T}^{00} + L^0{}_j L^i{}_k \overline{T}^{jk} = \left(\rho + \frac{p}{c^2}\right)\gamma^2 v^i$$

$$T^{ij} = L^i{}_\alpha L^j{}_\beta \overline{T}^{\alpha\beta} = L^i{}_0 L^j{}_0 \overline{T}^{00} + L^i{}_m L^j{}_n \overline{T}^{mn} = p\delta^{ij} + \left(\rho + \frac{p}{c^2}\right)\gamma^2 v^i v^j$$

$$\tag{17.8.7}$$

where we have used the expressions in (14.3.42). Recall that $U^0 = \gamma$ and $U^i = \gamma v^i$ are the components of the four velocity vector as defined in (14.5.15). With these definitions, the components above can be integrated into the single symmetric tensor

$$T^{\mu\nu} = p\eta^{\mu\nu} + \left(\rho + \frac{p}{c^2}\right)U^\mu U^\nu \tag{17.8.8}$$

Just as the conservation of fluid momentum yields Euler's equations for a non-relativistic fluid, so also the motion of the relativistic fluid is governed by energy-momentum conservation,

$$\partial_\mu T^{\mu\nu} = 0. \tag{17.8.9}$$

This is a set of *four* equations so let's see what they mean, first in the non-relativistic limit: we are interested in the limit of small velocities so take $U^0 = \gamma \approx 1$ and $U^i \approx v^i$, and find that indeed

$$T^{00} \approx \rho$$

$$T^{0i} \approx \rho v^i$$

$$T^{ij} \approx p\delta^{ij} + \rho v^i v^j \tag{17.8.10}$$

where we have taken $c \to \infty$. The time component of the conservation of energy and momentum then gives in this limit

$$\partial_t T^{00} + \partial_i T^{i0} = 0 = \partial_t \rho + \partial_i(\rho v^i) \tag{17.8.11}$$

or, in vector form,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v}) = 0, \tag{17.8.12}$$

which we recognize as the continuity equation. Likewise the space components of the equation yield

$$\partial_t T^{0i} + \partial_j T^{ji} = 0 = \partial_t(\rho v^i) + \partial_i p + \partial_j(\rho v^i v^j) \tag{17.8.13}$$

This can be rewritten as

$$(\partial_t \rho)v^i + \rho\partial_t v^i + \partial_i p + \partial_j(\rho v^j)v^i + \rho v^j \partial_j v^i = 0 \tag{17.8.14}$$

and, using the continuity equation above, it can be expressed in vector form as

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho(\vec{v} \cdot \vec{\nabla})\vec{v} = -\vec{\nabla}p, \tag{17.8.15}$$

which, of course, are Euler's equations.

Energy and momentum conservation yields more complicated equations at high flow velocities. The continuity equation in vector form is now

$$\frac{\partial}{\partial t}\left(\frac{p\vec{v}^2/c^4 + \rho}{1 - \vec{v}^2/c^2}\right) + \vec{\nabla} \cdot \left(\frac{(p/c^2 + \rho)\vec{v}}{1 - \vec{v}^2/c^2}\right) = 0 \tag{17.8.16}$$

and, using this condition, the relativistic Euler equations get simplified to[6]

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1 - \vec{v}^2/c^2}{p/c^2 + \rho}\left[\vec{\nabla}p + \frac{v}{c^2}\frac{\partial p}{\partial t}\right] \tag{17.8.17}$$

---

[6]To arrive at the Euler equations from energy momentum conservation, begin with the space components of the conservation law,

$$\partial_t\left[\left(\frac{p}{c^2} + \rho\right)\gamma^2 v^i\right] + \partial_j\left[p\delta^{ji} + \left(\frac{p}{c^2} + \rho\right)\gamma^2 v^j v^i\right] = 0$$

17.8.2 Conserved Currents

A fluid may carry other conserved quantities (such as electric charge) besides energy and momentum, so what does the conservation law for such quantities look like? In this section, we address this question for conserved scalars. Let A be one such conserved quantity and let a represent its comoving density, *i.e.*, its density in a frame that moves with the fluid at each point (the comoving frame). We define a current density, A^μ, associated with this quantity, by requiring it to have the form

$$\overline{A}^0 = a, \quad \overline{A}^i = 0 \tag{17.8.18}$$

in the comoving frame. In any other frame, its form will be given by a boost

$$A^0 = L^0{}_0\overline{A}^0 = \gamma a, \quad A^i = L^i{}_0\overline{A}^0 = \gamma a v^i \tag{17.8.19}$$

or, more concisely, $A^\mu = aU^\mu$. The statement that the quantity whose comoving density is "a" is conserved is then equivalent to the Lorentz covariant statement that the associated current density is divergence free,

$$\partial_\mu A^\mu = 0. \tag{17.8.20}$$

This equation generalizes (16.2.4). It can be put into integral form if we integrate the equation over a volume V bounded by the surface S,

$$\int_V d^3\vec{r}\,\partial_\mu A^\mu = \frac{d}{dt}\int_V d^3\vec{r}A^0 + \int_V d^3\vec{r}\,\vec{\nabla}\cdot\vec{A}, \tag{17.8.21}$$

and use Gauss' theorem for the second term on the right to get

$$\frac{d}{dt}\int_V d^3\vec{r}A^0 = -\oint_S \vec{A}\cdot d\vec{S}, \tag{17.8.22}$$

which is a generalization of (16.2.2).

For example, if the total number, N, of fluid particles in a given volume is conserved, let n represent the number density of particles. The conservation of particle number can

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Expanding and collecting terms, it is straightforward to put this equation in the form

$$\partial_t \vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{1-\vec{v}^2/c^2}{p/c^2+\rho}\left[\vec{\nabla}p + \vec{v}\left(\partial_t\left[\frac{(p/c^2+\rho)}{1-\vec{v}^2/c^2}\right] + \vec{\nabla}\cdot\left[\frac{(p/c^2+\rho)\vec{v}}{1-\vec{v}^2/c^2}\right]\right)\right]$$

Now

$$p\vec{v}^2/c^2 + \rho = (p/c^2+\rho) - p/(\gamma^2 c^2) \Rightarrow (p/c^2+\rho)\gamma^2 = (p\vec{v}^2/c^2+\rho)\gamma^2 + p/c^2$$

and so

$$\partial_t[(p/c^2+\rho)\gamma^2] = \partial_t[(p\vec{v}^2/c^2+\rho)\gamma^2] + 1/c^2\partial_t p$$

<u>Problem</u>: Complete the steps leading to the Euler equations.

be expressed as the statement that the number density current, $N^\mu = nU^\mu$ is divergence free,

$$\partial_\mu(nU^\mu) = 0. \tag{17.8.23}$$

If the fluid is charged then, letting N^+ represent that number of positively charged particles and N^- the number of negatively charged particles, conservation of charge would require that $N^+ - N^-$ remains constant throughout the motion. We can construct the charge density, $\rho_q = q(n^+ - n^-)$, where q is the magnitude of the elementary charge, and the charge current density, $j_q^\mu = \rho_q U^\mu$. Conservation of charge is then embodied in the requirement that $\partial_\mu j_q^\mu = 0$.

The conservation of energy and momentum together with the conservation of particle number leads naturally to the second law of thermodynamics as we now show. Energy-momentum conservation requires that

$$U_\beta(\partial_\alpha T^{\alpha\beta}) = 0 = U^\beta \left[\partial_\beta p + \partial_\alpha\left(\frac{p}{c^2} + \rho\right)U^\alpha U_\beta\right] \tag{17.8.24}$$

and, using $U_\beta U^\beta = -c^2$, we may re-express this in the form

$$U^\beta \partial_\beta p - \partial_\beta[(p + \rho c^2)U^\beta] = 0 \tag{17.8.25}$$

or

$$-U \cdot \partial(\rho c^2) - (p + \rho c^2)\partial \cdot U = 0 \tag{17.8.26}$$

but, if the particle number is also conserved, then

$$\partial_\beta(nU^\beta) = 0 \Rightarrow \partial \cdot U = \frac{U \cdot \partial n}{n} \tag{17.8.27}$$

and, inserting this relation into (17.8.26), we find

$$U \cdot \left[-\partial(\rho c^2) + \frac{1}{n}p\partial n + \frac{\rho c^2}{n}\partial n\right] = 0 \tag{17.8.28}$$

and therefore

$$-nU \cdot \left[p\partial\left(\frac{1}{n}\right) + \partial\left(\frac{\rho c^2}{n}\right)\right] = 0 \tag{17.8.29}$$

or

$$pd\left(\frac{1}{n}\right) + d\left(\frac{\rho c^2}{n}\right) = 0. \tag{17.8.30}$$

Now $1/n$ is simply the specific volume, v, or the volume per particle of the fluid and $\rho c^2/n$ is the energy per particle, which we call ε, so the conservation of energy, momentum and particle number for a perfect fluid imply the relation

$$d\varepsilon + pdv = 0. \tag{17.8.31}$$

This has the form of the first law of thermodynamics,

$$T d\sigma = d\varepsilon + p\,dv, \tag{17.8.32}$$

where σ represents the specific entropy (or entropy per particle) and says that the flow is isentropic, *i.e.*, $d\sigma = 0$. Since $n\sigma$ represents the comoving entropy density of the fluid, we can form the entropy current four vector $S^\mu = n\sigma U^\mu$, which is conserved during the flow.

17.8.3 Imperfect Fluids

The energy momentum tensor describing non-ideal fluids must include terms involving the shear and bulk viscosity. We assume that these terms add to the stress energy tensor and collect all of them in the tensor $\Delta T_{\mu\nu}$ writing

$$T^{\mu\nu} = T^{\mu\nu}_{(0)} + \Delta T^{\mu\nu} \tag{17.8.33}$$

where $T^{(0)}_{\mu\nu}$ represents the ideal fluid tensor and $\Delta T^{\mu\nu}$ purports to contain the effects of dissipation. Likewise, for the number density current we write

$$N^\mu = N^\mu_{(0)} + \Delta N^\mu = nU^\mu + \Delta N^\mu, \tag{17.8.34}$$

where ΔN^μ represents the particle **drift**. However, correction terms of this kind introduce an ambiguity in the definition of the fluid four velocity because heat transfer makes it necessary to specify whether U^μ represents the four velocity of energy transport (the energy frame) or of particle transport (the particle frame).[7]

Before commiting ourselves to a particular frame and considering what specific form $\Delta T^{\mu\nu}$ might take, let's examine some of the consequences of dissipation. As before, energy conservation and the balance equation for the particle number will take the form[8]

$$\partial_\alpha T^{\alpha\beta} = 0 = \partial_\alpha N^\alpha \tag{17.8.35}$$

Following the same steps as for a perfect fluid, energy momentum conservation is seen to require that

$$U_\beta(\partial_\alpha T^{\alpha\beta}) = -nU \cdot \left[p\partial\left(\frac{1}{n}\right) + \partial\left(\frac{\rho c^2}{n}\right) \right] + \frac{p + \rho c^2}{n}\partial \cdot \Delta N + U_\beta \partial_\alpha \Delta T^{\alpha\beta} = 0 \tag{17.8.36}$$

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[7]The energy frame is attributed to Landau and Lifshitz, *Fluid Mechanics*, 2nd ed., Pergamon Press, NY (1987), and the particle frame is attributed to Eckart, Phys. Rev. D **58** (1940) 919.

[8]Here, we will always assume that there are no external sources or sinks and no particle creation or annihilation by chemical processes within the fluid. If this does not hold then the particle number conservation law must be modified by the addition of a term which allows for a change in the fluid composition,

$$\partial_\alpha N^\alpha = \Psi.$$

Now the first term is given in terms of the specific entropy according to the first law of thermodynamics, so

$$- TnU^\mu\partial_\mu\sigma + \frac{p + \rho c^2}{n}\partial \cdot \Delta N + U_\nu\partial_\mu\Delta T^{\mu\nu} = 0 \qquad (17.8.37)$$

and using the conservation of particle number, $\partial_\alpha(nU^\alpha + \Delta N^\alpha) = 0$, we find

$$\partial_\mu(n\sigma U^\mu) = \frac{\mu}{T}\partial \cdot \Delta N + \frac{1}{T}U_\nu\partial_\mu\Delta T^{\mu\nu}. \qquad (17.8.38)$$

where

$$\mu = \left(\frac{p + \rho c^2}{n} - \sigma T\right) \qquad (17.8.39)$$

is **Euler's relation**. In the absence of the dissipation terms we had identified $S^\mu = n\sigma U^\mu$ as the entropy current four vector but (17.8.38) says that when dissipation is present this current is not conserved. This is to be expected in the presence of dissipation, so let us evaluate the right hand side of this equation. If we continue to interpret ρ and n as the energy density and particle number density respectively in the comoving frame, then $\overline{\Delta T}^{00} = 0 = \overline{\Delta N}^0$. With this interpretation, evaluating the right hand side of (17.8.38) in the comoving frame[9] we find

$$\partial_\mu(n\sigma U^\mu) = \frac{\mu}{T}\partial_i\overline{\Delta N}^i - \frac{c^2}{T}\partial_i\overline{\Delta T}^{i0}, \qquad (17.8.40)$$

which may be negative for arbitrary flows and could violate the second law of thermodynamics if we continued to interpret $S^\mu = n\sigma U^\mu$ as the entropy current four vector. This is not surprising since we must allow for the possibility that S^μ is modified by the dissipation terms.

How then should the entropy current four vector be defined? One proceeds as follows: first define S^μ so that (i) its divergence contains no gradients of $\Delta T^{\mu\nu}$ and (ii) the entropy density is $\overline{S}^0 = n\sigma$ in the local rest frame, then (iii) determine $\Delta T^{\mu\nu}$ and ΔN^μ by requiring that the divergence of the entropy current is non-negative in *any* process. The last condition is justified by the second law of thermodynamics and can be accomplished by setting

$$S^\mu = n\sigma U^\mu - \frac{\mu}{T}\Delta N^\mu - \frac{1}{T}U_\nu\Delta T^{\mu\nu}, \qquad (17.8.41)$$

which leads to

$$\partial_\mu S^\mu = -\partial_\mu\left(\frac{\mu}{T}\right)\Delta N^\mu - \partial_\mu\left(\frac{U_\nu}{T}\right)\Delta T^{\mu\nu}. \qquad (17.8.42)$$

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[9]It is most straightforwardly evaluated in a comoving frame. Because we are evaluating a scalar we expect that the result will hold in any frame.

We take (17.8.41) as a *definition* of the entropy current and determine $\Delta T^{\mu\nu}$ by requiring that its divergence never decreases for *any* flow. Then (17.8.42) gives the rate of entropy production per unit volume in the presence of dissipation.

Now we must address the question of what form ΔN^μ and $\Delta T^{\mu\nu}$ can take. Since we have required $\partial_\mu S^\mu$ to be non-negative, let us expand the right hand side of (17.8.42) in a local comoving frame, keeping in mind our interpretation of n and ρ and setting U^i and all gradients of U^0 equal to zero,[10]

$$\partial_\mu S^\mu = -\partial_i \left(\frac{\mu}{T}\right) \overline{\Delta N}^i - \left(\frac{1}{T}\dot{U}_i + \frac{c^2}{T^2}\partial_i T\right) \overline{\Delta T}^{i0} - \frac{1}{T}(\partial_i U_j)\overline{\Delta T}^{ij} \qquad (17.8.43)$$

In order for the right hand side to be non-negative for all fluid configurations each term must be a perfect square. This leads to

$$\overline{\Delta N}^i = -\alpha \eta^{ij}\left(T\partial_j \mu - \mu \partial_j T\right),$$

$$\overline{\Delta T}^{i0} = \overline{\Delta T}^{0i} = -\chi\left(T\dot{U}^i + c^2 \eta^{ij}\partial_j T\right) \qquad (17.8.44)$$

and

$$\overline{\Delta T}_{ij} = -\eta\left(\partial_i U_j + \partial_j U_i - \frac{2}{3}(\partial \cdot U)\eta_{ij}\right) - \xi(\partial \cdot U)\eta_{ij} \qquad (17.8.45)$$

where α, χ, η and ξ are non-negative, possibly temperature dependent, coefficients. By direct comparison with the non-relativistic momentum flow tensor we immediately identify η and ξ with the shear and bulk viscosity coefficients respectively. The new quantities are the **diffusion coefficient**, α, and the **coefficient of heat conduction**, χ.[11]

17.9   Scaling behavior of fluid flows

A class of particular solutions to the Hydrodynamic equations have the form $f(\vec{r}, t) = g(t)F[\vec{r}/h(t)]$, where $g(t)$ and $h(t)$ are time dependent scales of the spatial coordinate and $f(\vec{r}, t)$ is any of the unknown functions appearing in the Navier-Stokes equations. Such solutions are called **self-similar** because the spatial distribution of these solutions remains similar to itself at all times during the motion.

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[10]<u>Problem</u>: Show this.

[11]The distinction between the Eckart and Landau-Lifshitz pictures arises as follows: If U^μ is taken to be the velocity of particle transport (Eckart) then comoving observers do not see any particle drift, $\overline{\Delta N}^\mu = 0$, which implies that the diffusion coefficient vanishes. If U^μ is taken to be the velocity of energy transport (Landau and Lifshitz) then comoving observers do not observe any contribution to the energy flux, so $\overline{\Delta T}^{0\mu} = 0$, *i.e.*, the coefficient of heat conduction must vanish.

To understand how such solutions come about, let us return to the non-relativistic Navier-Stokes equation in (17.3.14) and ask how the variables and parameters behave under a multiplicative change in the spatial and temporal scales,

$$\vec{r} \to \vec{r}' = \sigma \vec{r}$$

$$t \to t' = \sigma^\alpha t \tag{17.9.1}$$

where σ is the "scale factor". The scaling dimension of a variable or parameter is the power of σ in the transformation

$$f \to f' = \sigma^b f. \tag{17.9.2}$$

Thus $\vec{r}$ has scaling dimension unity and t has scaling dimension α. If the Navier-Stokes equation is to be scale invariant then all terms in the equation must have the same scaling dimension. Dividing throughout by ρ we find that for the two terms on the left hand side of the equation to have the same scaling dimension the velocity must scale as

$$\vec{v} \to \vec{v}' = \sigma^{1-\alpha} \vec{v} \tag{17.9.3}$$

and the scaling dimension of the left hand side becomes $1 - 2\alpha$. This requires p/ρ to have scaling dimension $2(1 - \alpha)$ and the kinematic viscosity to have scaling dimension $2 - \alpha$, i.e.,

$$p/\rho \to p'/\rho' = \sigma^{2(1-\alpha)} p/\rho$$

$$\nu \to \nu' = \sigma^{2-\alpha} \nu \tag{17.9.4}$$

Suppose that the viscosity has scaling dimension β, then we will have the following scaling behavior for all our variables

$$\vec{r} \to \vec{r}' = \sigma \vec{r}$$

$$t \to t' = \sigma^\alpha t$$

$$\vec{v} \to \vec{v}' = \sigma^{1-\alpha} \vec{v}$$

$$\eta \to \eta' = \sigma^\beta \eta$$

$$\rho \to \rho' = \sigma^{\beta+\alpha-2} \rho$$

$$p \to p' = \sigma^{\beta-\alpha} p \tag{17.9.5}$$

and we recognize that the Reynolds number remains invariant provided that the characteristic values of the parameters and variables that determine $\mathfrak{R}$ also scale as above.

The self-similarity relations above may be regarded as special cases of a more general relation of the form

$$f(\vec{r}, t) \to f(\sigma\vec{r}, \sigma^\alpha t) = \sigma^b f(\vec{r}, t) \qquad (17.9.6)$$

where σ is a function of t. Now we have the following theorem: a necessary and sufficient condition for a function $f(\vec{r}, t)$ to satisfy the above condition is that

$$f(\vec{r}, t) = t^{b/\alpha} F(\vec{r}/t^{1/\alpha}) \qquad (17.9.7)$$

That it is necessary follows because if we take $\sigma = t^{-1/\alpha}$ then

$$f(\sigma\vec{r}, \sigma^\alpha t) = \sigma^b f(\vec{r}, t) \Rightarrow f(\vec{r}/t^{1/\alpha}, 1) = t^{-b/\alpha} f(\vec{r}, t) \Rightarrow f(\vec{r}, t) = t^{b/\alpha} F(\vec{r}/t^{1/\alpha}), \quad (17.9.8)$$

where $F(x) = f(x, 1)$. Of course, σ is constant in (17.9.5), therefore the condition is not in fact a necessary one in our particular case. However, that it is sufficient even for a constant scaling parameter is straightforward since

$$f(\vec{r}, t) = t^{b/\alpha} F(\vec{r}/t^{1/\alpha}) \Rightarrow f(\sigma\vec{r}, \sigma^\alpha t) = \sigma^b t^{b/\alpha} F(\vec{r}/t^{1/\alpha}) = \sigma^b f(\vec{r}, t) \qquad (17.9.9)$$

and so, according to (17.9.5), the velocity function

$$\vec{v}(\vec{r}, t) = t^{(1-\alpha)/\alpha} \vec{v}(\vec{r}/t^{1/\alpha}) \qquad (17.9.10)$$

will represent a self-similar flow. We therefore have a one parameter family of self-similar solutions. For example, the solution with $\alpha = 1$ will have

$$\vec{v}(\vec{r}, t) = \vec{u}(\vec{r}/t) \qquad (17.9.11)$$

and with $\alpha = 2$ we should find

$$\vec{v}(\vec{r}, t) = \frac{1}{\sqrt{t}} \vec{u}(\vec{r}/\sqrt{t}). \qquad (17.9.12)$$

Self-similar solutions are interesting from a theoretical point of view because the Navier Stokes equations are reduced to ordinary differential equations. However, the equations obtained are non-linear and so it is often not clear that much by way of simplification has in fact been gained. In practical terms these solutions appear to be applicable when the effects of boundaries can be ignored or when the boundaries have only a local effect so that in most of the fluid domain the self-similar solution remains valid. This must be verified independently before a given self-similar solution is taken to be relevant to any real situation.

17.10   An Example

Similar arguments can be made for relativistic hydrodynamics as well and we leave that as an exercise for the reader.[12] Let us close our discussion of fluid dynamics by considering a special self-similar solution to the hydrodynamic equations for a relativistic, imperfect fluid, first in two and then in four space-time dimensions. This will serve to put together the concepts of the last two sections in a single problem, which also turns out to be physically interesting in its own right because it is believed to describe the evolution of very hot (ultra-relativistic) hadron matter produced in extreme relativistic nucleus-nucleus collisions as well as in the very early universe.

The solutions we seek will be of the following form: if $q_i \in \{1, 2, 3\}$ compose some arbitrary, orthogonal coordinate system in which h_i are the corresponding scale factors so that

$$ds^2 = c^2 dt^2 - h_i^2 dq^{i^2}, \tag{17.10.1}$$

then we assume that the velocity along any non-compact dimension is

$$v_i(q_i, t) = \frac{s_i}{h_i t} \tag{17.10.2}$$

where s_i is the spatial distance,

$$s_i = \int h_i dq_i, \tag{17.10.3}$$

and it is zero along any compact dimension.

We will begin with a two dimensional version of this problem. In order to generate such a flow in two dimensions with

$$ds^2 = c^2 dt^2 - dx^2 \tag{17.10.4}$$

and $v(x, t) = x/t$, we first seek the local comoving frame of the fluid. Let (τ, y) represent the coordinates in this frame, then by definition $U^\tau = 1$ and $U^y = 0$. To determine (τ, y) we simply note that

$$U^\tau = \frac{\partial \tau}{\partial t} U^t + \frac{\partial \tau}{\partial x} U^x = 1, \quad U^y = \frac{\partial \tau}{\partial t} U^t + \frac{\partial \tau}{\partial x} U^x = 0 \tag{17.10.5}$$

Now because $v(t, x) = x/t$

$$U^x = \frac{cx}{\sqrt{c^2 t^2 - x^2}}, \quad U^t = \frac{ct}{\sqrt{c^2 t^2 - x^2}} \tag{17.10.6}$$

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[12] Problem: Repeat the scaling arguments above for the relativistic hydrodynamic equations. Argue that the only Lorentz invariant scaling solution has $\alpha = 1$.

and it is easy to see that the solutions of the two linear, first order equations for τ and y are

$$\tau = \frac{1}{c}\sqrt{c^2t^2 - x^2} + f\left(\frac{x}{t}\right),$$
$$y = y\left(\frac{x}{t}\right) \tag{17.10.7}$$

where f and y are arbitrary functions of their argument. Given this coordinate transformation, we can construct the inverse vielbein as follows

$$\begin{aligned}
E_t^\tau &= \frac{\partial \tau}{\partial t} = \frac{ct}{\sqrt{c^2t^2 - x^2}} - \frac{x}{t^2}f'(x/t) \\
E_x^\tau &= \frac{\partial \tau}{\partial x} = -\frac{x}{c\sqrt{c^2t^2 - x^2}} + \frac{1}{t}f'(x/t) \\
E_t^y &= \frac{\partial y}{\partial t} = -\frac{x}{t^2}y'(x/t) \\
E_x^y &= \frac{\partial y}{\partial x} = \frac{1}{t}y'(x/t)
\end{aligned} \tag{17.10.8}$$

where the prime refers to a derivative with respect to the argument x/t and we find the metric components in the new system according to

$$g^{\mu\nu} = \eta^{ab} E_a^\mu E_b^\nu \tag{17.10.9}$$

Carrying out the algebra yields

$$\begin{aligned}
g^{\tau\tau} &= -\frac{1}{c^2} + \frac{c^2t^2 - x^2}{c^2t^4}f'(x/t)^2 \\
g^{\tau y} &= \frac{c^2t^2 - x^2}{c^2t^4}f'(x/t)y'(x/t) \\
g^{yy} &= \frac{c^2t^2 - x^2}{c^2t^4}y'(x/t)^2
\end{aligned} \tag{17.10.10}$$

Since f and y are arbitrary functions and y is not allowed to vanish, we choose $f = 0$. This rids us of the off-diagonal component of $g^{\mu\nu}$. We see also that

$$g^{yy} = \frac{\tau^2}{t^4}\, y'(x/t)^2 \Rightarrow y'(x/t)^2 = \frac{t^4}{\tau^2}g^{yy} \tag{17.10.11}$$

and, because y' must also be a function of x/t, we take $g^{yy} = 1/\tau^2$. This gives

$$y'(v) = \frac{t^2}{\tau^2} = \frac{c^2}{c^2 - v^2} \tag{17.10.12}$$

where $v = x/t$, along with the solution

$$y(x/t) = c \tanh^{-1}\left(\frac{x}{ct}\right) + \text{const.} \tag{17.10.13}$$

$y(x/t)$ is called the "rapidity".

To sumarize our progress so far: for the fluid flow $v(t,x) = x/t$, we have determined a comoving frame, in which $U^\tau = 1$ and $U^y = 0$, whose space-time metric is

$$ds^2 = c^2 d\tau^2 - \tau^2 dy^2. \tag{17.10.14}$$

In this frame the ideal fluid energy momentum tensor has the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 \\ 0 & \frac{p}{\tau^2} \end{pmatrix} \tag{17.10.15}$$

and it is a simple matter to solve the Euler equations,

$$\nabla_\mu T^{\mu\tau} = \frac{\partial \rho}{\partial \tau} + \frac{\rho}{\tau} + \frac{p}{c^2 \tau} = 0$$

$$\nabla_\mu T^{\mu y} = \frac{\partial p}{\partial y} = 0. \tag{17.10.16}$$

The last equation says that $p = p(\tau)$ and the first can only be solved after specifying equation of state. Let us use the equation of state for an extremely relativistic (flow velocity close to the speed of light) bosonic fluid, which we know from statistical mechanics to be

$$p = \frac{1}{3}\rho c^2 = \frac{1}{3}\gamma T^4, \tag{17.10.17}$$

where T is the fluid temperature and γ is the Stefan-Boltzmann constant. With this, Euler's equations are solved by

$$\rho = \rho_0 \left(\frac{\tau_0}{\tau}\right)^{4/3}, \tag{17.10.18}$$

where ρ_0 is the mass density at the time τ_0, which we understand as the time required to achieve the initial thermal equilibrium. The temperature behavior as a function of τ then follows directly.

We'll now include the effects of dissipation, but take the fluid to be incompressible so that $\xi = 0$. Let U^μ represent the velocity of particle transport. To simplify matters still further, suppose also that the temperature dependence of both χ and η are given by the power laws,

$$\begin{aligned} \eta(T) &= \eta_0 T^a \\ \chi(T) &= \chi_0 T^b \end{aligned} \tag{17.10.19}$$

and that the flow is extremely relativistic so that (17.10.17) holds. The components of the full energy momentum tensor in our comoving frame are

$$T^{00} = \rho$$

$$T^{0i} = T^{i0} = -\frac{c^2 \chi_0 T^b}{\tau^2} \partial_y T$$

$$T^{ij} = \frac{p}{\tau^2} - \frac{4\eta_0 T^a}{3\tau^3} \qquad (17.10.20)$$

where the prime refers to a derivative with respect to y. The conservation equation

$$\nabla_\mu T^{\mu y} = \partial_y p - \frac{4\eta_0}{3\tau} \partial_y T^a - \chi_0 c^2 \left[\frac{T^b}{\tau} \partial_y T + \partial_\tau (T^b \partial_y T) \right] = 0, \qquad (17.10.21)$$

once again implies that the temperature does not depend on the rapidity, *i.e.*, $T = T(\tau)$ and, after dropping all derivatives with respect to the rapidity, the other equation becomes

$$\nabla_\mu T^{\mu \tau} = \partial_\tau \rho + \frac{\rho}{\tau} + \frac{p}{c^2 \tau} - \frac{4\eta_0 T^a}{3c^2 \tau^2} = 0, \qquad (17.10.22)$$

which is seen to be a Bernoulli equation when rewritten in terms of the temperature,

$$\partial_\tau T + \frac{T}{3\tau} - \frac{\eta_0}{3\gamma} \frac{T^{a-3}}{\tau^2} = 0. \qquad (17.10.23)$$

It has the solutions

$$T = T_0 \left(\frac{\tau_0}{\tau} \right)^{1/3} \exp \left[-\frac{\eta_0}{3\gamma} \left(\frac{1}{\tau} - \frac{1}{\tau_0} \right) \right], \quad \text{if } a = 4$$

$$T = T_0 \left(\frac{\tau_0}{\tau} \right)^{1/3} \left[1 - \frac{\eta_0}{\gamma} \frac{T_0^{-3}}{\tau_0} \ln \left(\frac{\tau_0}{\tau} \right) \right]^{1/3}, \quad \text{if } a = 1 \qquad (17.10.24)$$

and

$$T = T_0 \left[\frac{\eta_0}{\gamma} \left(\frac{a-4}{a-1} \right) T_0^{a-4} \left(\frac{\tau^{-\frac{(a-1)}{3}} - \tau_0^{-\frac{(a-1)}{3}}}{\tau^{-\frac{(a-4)}{3}}} \right) + \left(\frac{\tau_0}{\tau} \right)^{-\frac{a-4}{3}} \right]^{-\frac{1}{a-4}} \qquad (17.10.25)$$

otherwise. Thus there is no energy loss via heat conduction for this type of flow.

For completeness, let us also compute the entropy production for this one dimensional flow. Continuing in the comoving frame we have,

$$S^\tau = n\sigma, \quad S^y = 0 \qquad (17.10.26)$$

and (17.8.43), which reduces to

$$\partial_\tau(n\sigma) + \frac{n\sigma}{\tau} = \frac{4\eta_0}{3}\frac{T^{a-1}}{\tau^2}. \tag{17.10.27}$$

Once again we arrive at a Bernoulli equation, this time for the rate of entropy production. With $Q(\tau) = 4\eta_0 T^{a-1}/3\tau^2$, we find the entropy density to be

$$S(\tau) = \frac{\tau_0}{\tau}\left[S(\tau_0) + \frac{1}{\tau_0}\int_{\tau_0}^\tau d\tau'\tau'Q(\tau')\right] \tag{17.10.28}$$

Now the entropy contained in any interval dy will be

$$d\mathcal{S} = S(\tau)\tau dy \tag{17.10.29}$$

and therefore

$$\partial_\tau(\partial_y\mathcal{S}) = \tau Q(\tau) \tag{17.10.30}$$

showing that the rapidity density of the entropy is not a constant of the motion but increases as a consequence of the dissipation.

Four dimensional expansion with the velocity distribution specified in the introduction to this section may similarly be described in arbitrary, orthogonal frames. An interesting feature of these solutions is that the effects of dissipation vanish exactly. This is a consequence of the velocity distribution and the fact that there are no constraints on the system due to external boundaries, hence no shear viscosity. We will consider separately the cases of co-ordinate systems with no compact co-ordinate and one compact co-ordinate.

If none of the co-ordinates are compact (for example Cartesian or parabolic cylindrical co-ordinates) we can generate a self-similar solution for the velocity distribution (17.10.2) in a manner similar to that employed before *i.e.* by introducing comoving coordinates. It can easily be shown that these coordinates are quite generally given by

$$\tau = \sqrt{c^2t^2 - \sum_{i=1}^3 s_i^2}$$

$$u = \frac{1}{2}\ln\left[\frac{\sqrt{c^2t^2 - \sum_{i=1}^2 s_i^2} + s_3}{\sqrt{c^2t^2 - \sum_{i=1}^2 s_i^2} - s_3}\right]$$

$$v = \frac{1}{2}\ln\left[\frac{\sqrt{c^2t^2 - s_1^2} + s_3}{\sqrt{c^2t^2 - s_1^2} - s_3}\right]$$

$$y = \frac{1}{2}\ln\left[\frac{ct + s_1}{ct - s_1}\right] \tag{17.10.31}$$

With the same equation of state that we have been using, the temperature will be a function only of τ and we find a remarkably simple equation for the temperature evolution

$$\partial_\tau \epsilon + \frac{3}{4\tau}(\epsilon + p) = 0 \tag{17.10.32}$$

whose solution is trivially

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right). \tag{17.10.33}$$

and so, as mentioned before, the dissipation terms play no role in the evolution. The same holds when there is one or more compact coordinate. In the case that only one of the coordinates is compact, for definiteness say q_3, we let

$$\tau = \sqrt{c^2 t^2 - \sum_{i=1}^{2} s_i^2}$$

$$u = \frac{1}{2} \ln \left[\frac{\sqrt{c^2 t^2 - s_1^2} + s_2}{\sqrt{c^2 t^2 - s_1^2} - s_2}\right]$$

$$v = \frac{1}{2} \ln \left[\frac{ct + s_1}{ct - s_1}\right]$$

$$y = q_3 \tag{17.10.34}$$

The fact that $y = q_3$ is compact means that $\partial_\tau h_3 = h_3/\tau$ which implies

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right) \tag{17.10.35}$$

and all dissipation effects vanish exactly, as before. We leave it as exercise to show that, in four dimensions, the entropy density is also a function only of τ

$$s(\tau) = s(\tau_0) \left(\frac{\tau_0}{\tau}\right)^3 \tag{17.10.36}$$

and that the rapidity density of entropy is a constant of the motion.

Appendix A

The $\delta-$function

A.1 Introduction

Let $x \in (a, b)$ and define the object $\delta(x' - x)$ by

$$\int_a^b dx' \ \delta(x' - x) = \begin{cases} 0 & x \notin (a, b) \\ \\ 1 & x \in (a, b) \end{cases}$$

$$\int_a^b dx' \ f(x')\delta(x' - x) = \begin{cases} 0 & x \notin (a, b) \\ \\ f(x) & x \in (a, b) \end{cases} \qquad (A.1.1)$$

Normally, we'll be concerned with all of space, so the limits will be from $-\infty$ to ∞. Let us consider some examples.

A.1.1 An example

Consider the sequence of functions (see figure A1)

$$\delta_n(x) = \begin{cases} 0 & x < -\frac{1}{2n} \\ \\ n & x \in (-\frac{1}{2n}, \frac{1}{2n}) \\ \\ 0 & x > \frac{1}{2n} \end{cases} \qquad (A.1.2)$$

and note that

$$\int_{-\infty}^{\infty} \delta_n(x)dx = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} ndx = 1, \quad \forall \ n \in \mathbb{N} \qquad (A.1.3)$$

i
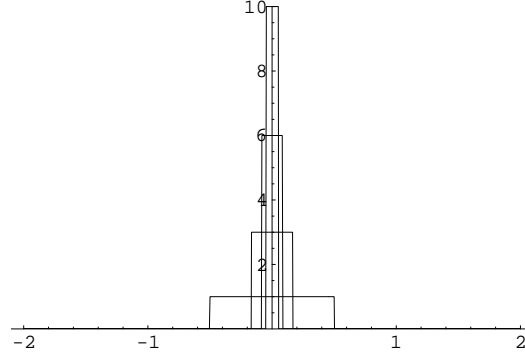
Figure A.1: The sequence of functions in (A.1.2)

while

$$\int_{-\infty}^{\infty} dx \ \delta_n(x) f(x) = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} n f(x) dx = \frac{[g(1/2n) - g(-1/2n)]}{1/n} \tag{A.1.4}$$

where $g(x)$ is the primitive of $f(x)$. Calling $\epsilon = \frac{1}{n}$ and taking the limit as $n \to \infty$ ($\epsilon \to 0$) we have

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} dx \ \delta_n(x) f(x) = \lim_{\epsilon\to 0} \frac{[g(\epsilon/2) - g(-\epsilon/2)]}{\epsilon} = g'(0) = f(0) \tag{A.1.5}$$

Thus we may define the $\delta-$function as

$$\delta(x) = \lim_{n\to\infty} \delta_n(x) \tag{A.1.6}$$

because in this limit both conditions in (A.1.1) are obeyed.

A.1.2   Another example

Consider a slightly more complicated sequence of functions

$$\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2} \tag{A.1.7}$$

and again note that

$$\int_{-\infty}^{\infty} dx \ \delta_n(x) = 1, \quad \forall \ n \in \mathbb{N} \tag{A.1.8}$$

Now let us consider

$$\int_{-\infty}^{\infty} dx \ \delta_n(x) f(x) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \ e^{-n^2 x^2} f(x). \tag{A.1.9}$$
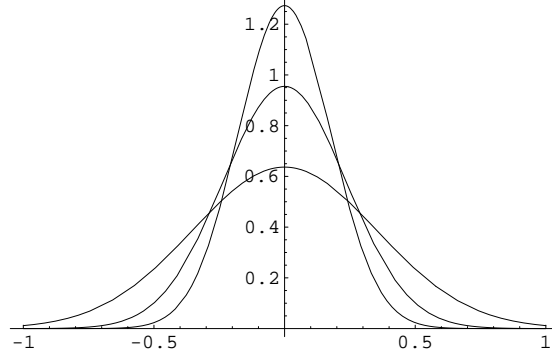
Figure A.2: The sequence of functions in (A.1.7)

Expanding $f(x)$ about the origin in a Taylor series

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)} x^j}{j!} \tag{A.1.10}$$

our integral becomes

$$\frac{n}{\sqrt{\pi}} \sum_j \frac{f^{(j)}}{j!} \int_{-\infty}^{\infty} dx \; x^j e^{-n^2 x^2}. \tag{A.1.11}$$

where $f^{(j)}$ is the jth derivative of $f(x)$ at $x = 0$. Clearly, the only non-vanishing contributions come from even j.

$$\int_{-\infty}^{\infty} dx \; \delta_n(x) f(x) = \frac{n}{\sqrt{\pi}} \sum_j \frac{f^{(2j)}}{(2j)!} \int_{-\infty}^{\infty} dx \; x^{2j} e^{-n^2 x^2}$$

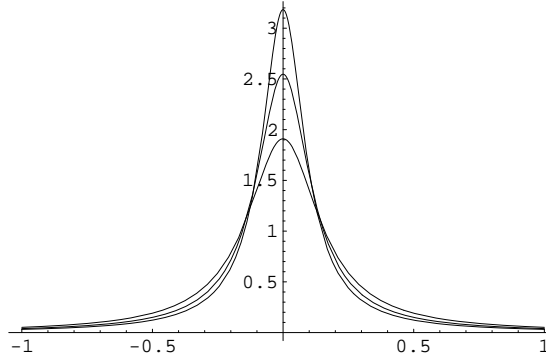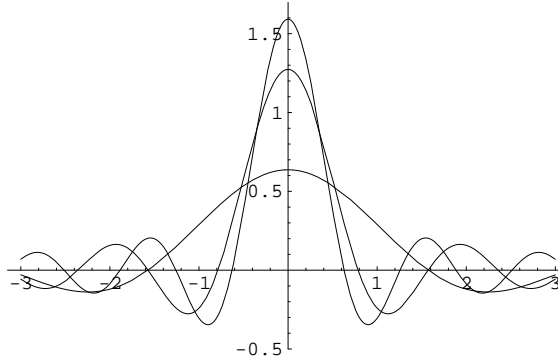$$= \sum_j \frac{f^{(2j)}}{(2j)!} \frac{\Gamma(j + \frac{1}{2})}{\sqrt{\pi}} n^{-2j}. \tag{A.1.12}$$

We see that the limit as $n \to \infty$ of the r.h.s. is just $f(0)$. Thus we also have

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} dx \; \delta_n(x) f(x) = f(0) \tag{A.1.13}$$

and we could define

$$\delta(x) = \lim_{n \to \infty} \delta_n(x), \quad \delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}, \tag{A.1.14}$$

thereby getting an alternative representation for the $\delta-$function.

Figure A.3: $\delta_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$

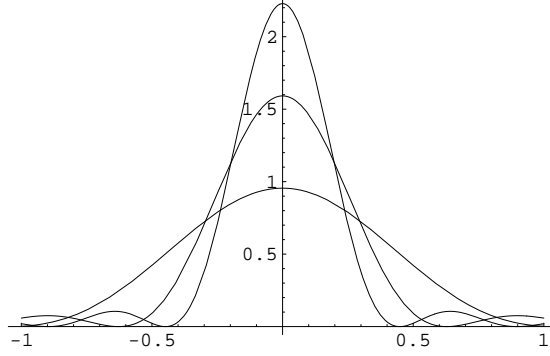Figure A.4: $\delta_n(x) = \frac{\sin nx}{\pi x}$

A.1.3   Properties

We may likewise analyze expressions such as

$$\delta(x) = \lim_{n\to\infty}\delta_n(x), \quad \delta_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$$

$$\delta(x) = \lim_{n\to\infty}\delta_n(x), \quad \delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi}\int_{-n}^{n}dt\,e^{ixt}$$

$$\delta(x) = \lim_{n\to\infty}\delta_n(x), \quad \delta_n(x) = \frac{\sin^2 nx}{n\pi x^2} \tag{A.1.15}$$

with the same results.

Here is a list of some of the more interesting properties of the $\delta-$function. They can be proved by simply applying the defining equations.

Figure A.5: $\delta_n(x) = \frac{\sin^2 nx}{n\pi x^2}$

1. $\delta(cx) = \frac{1}{|c|}\delta(x)$ (therefore, $\delta(-x) = \delta(x)$), or more generally,

$$\delta(g(x)) = \sum_j \frac{\delta(x - x_j)}{|g'(x_j)|} \qquad\qquad (A.1.16)$$

where x_j is a simple zero of the function $g(x)$, *i.e.*, $g(x_j) = 0$ and $g'(x_j) \neq 0$,

2. $g(x)\delta(x - x_o) = g(x_o)\delta(x - x_o)$,

3. $\int_{-\infty}^{\infty} dx\ \delta(x - y)\delta(x - z) = \delta(y - z)$ and

4. $\Theta'(x) = \delta(x)$, where $\Theta(x)$ is the Heaviside Θ−function.

Note, however, that the limits of the defining sequences themselves do not exist on the real line, *i.e.*, the δ−function has no meaning independently. The only meaning that can be given the object is via its defining integrals. It is a distribution.

A.2   The δ−function in curviliear coordinates

Beginning with the δ−function in a cartesian system, we can deduce its form in a general curvilinear coordinate system by using its defining properties. By definition

$$\int d^n x' \delta^n(x' - x) \;\;=\;\; 1 = \int d^n \xi' \left\|\frac{\partial x'}{\partial \xi'}\right\| \delta^n(x' - x)$$

$$\int d^n x' \delta^n(x' - x) f(x') \;\;=\;\; f(x) = \int d^n \xi' \left\|\frac{\partial x'}{\partial \xi'}\right\| \delta^n(x' - x) f(x') \qquad (A.2.1)$$

where $\|\|$ is the Jacobian of the transformation, f is a scalar function and, on the r.h.s.,
$x = x(\xi)$. But we have seen in Chapter 15 that the Jacobian can be replaced by the
determinant of the metric, *i.e.*,

$$1 = \int d^n\xi' \left\|\frac{\partial x'}{\partial \xi'}\right\| \delta^n(x' - x) = \frac{1}{c}\int d^n\xi'\sqrt{-g(\xi')}\delta^n(x' - x). \qquad (A.2.2)$$

Now we *define* the n dimensional $\delta-$function in a general coordinate system in the same
way as we had before:

$$\int d^n\xi' \delta^n(\xi' - \xi) \;\; = \;\; 1$$

$$\int d^n\xi' f(\xi')\delta^n(\xi' - \xi) \;\; = \;\; f(\xi). \qquad (A.2.3)$$

Then, simply comparing the expressions above, we find

$$\delta^n(x' - x) = \frac{c}{\sqrt{-g(\xi)}}\delta^n(\xi' - \xi) \qquad (A.2.4)$$

Thus $cg^{-\frac{1}{2}}(\xi)\delta^n(\xi' - \xi)$ is a scalar under general coordinate transformations and $\delta^n(\xi' - \xi)$
is a scalar *density*. As examples let's write the three dimensional $\delta-$function in spherical
coordinates (in three dimensions, the factor of c does not appear of course):

$$\delta^{(3)}(\vec{r}' - \vec{r}) = \frac{1}{r^2\sin\theta}\delta(r' - r)\delta(\theta' - \theta)\delta(\varphi' - \varphi) \qquad (A.2.5)$$

and in cylindrical coordinates:

$$\delta^{(3)}(\vec{r}' - \vec{r}) = \frac{1}{\rho}\delta(\rho' - \rho)\delta(\varphi' - \varphi)\delta(z' - z). \qquad (A.2.6)$$

Notice the density factors in each case.